M.Sc. Previous Year

Mathematics, MM-05

DIFFERENTIAL EQUATIONS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

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SYLLABI-BOOK MAPPING TABLE

Differential Equations

Syllabi	Mapping in Book
Unit I Homogenous Linear Equation with Variable coefficient Simultaneous differential equation, Total differential Equation.	**Unit-1:** Homogenous Linear Equations and Total Differential Equations **(Pages 3-48)**
Unit II Picard's Method of Integration, successive Approximation, Existence Theorem, Uniqueness Theorem. Existence & Uniqueness theorem (All Proof by Picard's method).	**Unit-2:** Picard's Method of Integration and Successive Approximation **(Pages 49-84)**
Unit III Dependence on initial conditions and parameters; Preliminaries. Continuity. Differentiability. Higher Order Differentiability. Poincare-Bendixson Theory-Autonomous systems. Umlaufsatz. Index of a stationary point. Poincare-Bendixson theorem. Stability of periodic solutions, rotation point, foci, nodes and saddle points.	**Unit-3:** Dependence on Initial Conditions and Parameters **(Pages 85-105)**
Unit IV Linear second order equations-Preliminaries, Basic facts. Theorems of Sturm.Sturm-Liouville Boundary Value Problems·. Numbers of zeros. Nonoscillatory equations and principal solutions. Nonoscillation theorems.	**Unit-4:** Linear Second Order Equations **(Pages 107-143)**
Unit V Partial differential Equation of first & Second order. linear partial differential Equation with constant coefficient.	**Unit-5:** Partial Differential Equation of First and Second Order **(Pages 145-190)**

CONTENTS

INTRODUCTION

NOTES

The subject of differential equations is built upon the subject of calculus. Differential equations occur frequently in many branches of science and, in both pure and applied mathematics. One possible explanation for this is to remember that a derivative describes a rate of change, so anytime it is used to describe how changes in one thing depend on changes in some other thing, differential equations are lurking in the background. Differential equations allow us to model changing patterns in both physical and mathematical problems.

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics and other disciplines. The term *differential equation* was coined by Leibniz in 1676 for a relationship between the two differentials dx and dy for the two variables x and y. Soon after the first usage of this term, differential equations quickly became understood as any algebraic or transcendental equation which involved derivatives. Differential equations are specifically used whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated.

Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere. The study of the stability of solutions of differential equations is known as stability theory. Both ordinary and partial differential equations are broadly classified as linear and nonlinear. A differential equation is linear if the unknown function and its derivatives appear to the power 1 (products are not allowed) and nonlinear otherwise. The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations. Homogeneous linear differential equations are a further subclass for which the space of solutions is a linear subspace, i.e., the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables; if these coefficients are constants then one speaks of a constant coefficient linear differential equation. Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions.

This book is divided into five units. The topics discussed is designed to be a comprehensive and easily accessible book covering the basic concepts of homogeneous linear equation with variable coefficient, total differential equation, Picard's method of integration, existence theorem, uniqueness theorem, dependence on initial conditions and parameters, continuity differentiability, higher order differentiability, Poincare-Bendixson theory, Umlaufsatz, stability of a periodic solution, linear second order equations, theorems of strum, strum-Liouville boundary

NOTES

value problem, non-oscillatory equations and principle solutions, non-oscillation theorems, partial differential equation of first and second order and linear partial differential equation with constant coefficient.

The book follows the Self-Instructional Mode (SIM) wherein each unit begins with an 'Introduction' to the topic. The 'Objectives' are then outlined before going on to the presentation of the detailed content in a simple and structured format. 'Check Your Progress' questions are provided at regular intervals to test the student's understanding of the subject. 'Answers to Check Your Progress Questions', a 'Summary', a list of 'Key Terms', and a set of 'Self-Assessment Questions and Exercises' are provided at the end of each unit for effective recapitulation.

NOTES

UNIT 1 HOMOGENOUS LINEAR EQUATIONS AND TOTAL DIFFERENTIAL EQUATIONS

Structure

1.0 INTRODUCTION

In mathematics, a differential equation is homogeneous if it is a homogeneous function of the unknown function and its derivatives. In the case of linear differential equations, this means that there are no constant terms. The solutions of any linear ordinary differential equation of any order may be deduced by integration from the solution of the homogeneous equation obtained by removing the constant term.

In simultaneous differential equations we'll look at systems of simultaneous linear differential equations with one independent variable and two or more dependent variables next. In general, the number of equations equals the number of dependent variables, hence there will be *n* equations if there are n dependent variables.

An exact differential equation or total differential equation is a certain kind of ordinary differential equation which is widely used in physics and engineering. The single equations with one independent variable and several dependent variables. These equations have the differential coefficients of dependent variables with respect to one independent variable. Such equations are called total differential equations. We learn those differential equations which contain one independent variable and two or more than two dependent variables. The equation may be ordinary or partial depending upon the ordinary or partial derivatives.

In this unit, you will learn about the homogeneous linear equation with variable coefficient, simultaneous differential equation and total differential equation.

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1.1 OBJECTIVES

After going through this unit, you will be able to:

- Learn about the homogeneous linear equation with variable coefficient
- Explain the simultaneous differential equation
- Analysis the total differential equation

1.2 HOMOGENEOUS LINEAR EQUATION

Any homogeneous differential equation of the from.

$$x^n \frac{d^n y}{dx^n} + p_1 x^n + \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^n - 2} ......p_n y_z = \phi$$

is called a homogeneous linear differential equation of nth order, where $p_1, p_2 - p_n$ are caustants and Q is a function of x.

Solution of homogeneous linear equation

Homogeneous linear differential equation is reducible to linear differntial equation with constant coefficient by subsituation

$$x = e^z \therefore z = \log x$$

$$or \frac{dz}{dn} = \frac{1}{x}$$

Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dx}$

or $x\frac{dy}{dx} = \frac{dy}{dz}$

or $xD = D_1$ Where $D_1 = \frac{d}{dz}$

Also $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$$= \frac{-1}{x_2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{1}{x^2}\left[-\frac{dy}{dz} + \frac{d^2 y}{dz^2}\right]$$

$$x^2 D^2 = D_1 (D_1 - 1)$$

Proceeding in the same way

$$x^n \mathrm{D}^n = \mathrm{D}_1(\mathrm{D}_1 - 1) - (\mathrm{D}_1 - n + 1)$$

NOTES

Now equation is reducible to linear equation with constant coefficient and may be solved by previously defined methods.

Example 1.1: Solve $\left(x^2D^2 + xD - 4\right)y = x^2$

Solution: Let $x = e^z$

$\therefore$ Equation becomes $[D_1(D_1\ 1)+D_1\ 4]\ y = e^{2z}$

or $\left(D_1^2 - 4\right)y = c^{2z}$

$\therefore$ Auxiliary equation is $m^2 - 4 = 0$

or $m = \pm 2$

$\therefore$ C.F $= c_1e^{2z} + c_2 - e^{-2z}$

and P.I $= \dfrac{1}{D_1^2 - 4}e^{2z} = e^{2z}\dfrac{1}{\left(D_1^2 + 2\right)^2 - 4}$

$$= e^{2z}\frac{1}{D_1^2 + 4D_1}.1 = e^{2z}\frac{1}{4D_1}\left(1 + \frac{D_1}{4}\right)^{-1} 1$$

$$= \frac{e^{2z}}{4}.\frac{1}{D}\left(1 - \frac{D_1}{4}\right).1 = \frac{e^{2z}}{4}\frac{1}{D_1}(1)$$

$$= \frac{e^{2z}}{4}.z$$

$$y = \text{C.F} + \text{P.I} = c_1e^{2z} + c_2e^{-22} + \frac{e^{2z} - z}{4}$$

$$= c_1x^2 + c_2x^{-2} + \frac{x^2}{4}\log x$$

Example 1.2: Solve $\left(x^3D^3 + 2x^2D^2 + 3xD - 3\right)y = x^2 + x$

Solution: Let $x = e^2 \therefore z = \log x$ and equation reduces to

$$\left[D_1CD_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 3D_1 - 3\right]y = e^{2z} + e^z$$

or $\left(D_1^3 - D_1^2 + 3D_1 - 3\right)y = e^{2z} + e^z$

A.E is $m^3\ m^2 + 3m\ \ 3 = 0$

or $(m^3 + 3)(m-1) = 0$

or $m = 1, +i\sqrt{3}$

C.F $= c_1e^z + c_2\cos\left(\sqrt{3}\,z\right) + c_3\sin\left(\sqrt{3}z\right)$

and P.I $= \dfrac{1}{D_1^3 - D_1^2 + 3D_1 - 3}\left(e^{2z} + e^z\right)$

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$$= \frac{1}{D_1^3 - D_1^2 + 3D_1 - 3} e^{2z} + \frac{1}{D_1^3 - D_1^2 + 3D_1 - 3} e^z$$

$$= \frac{1}{8 - 4 + 6 - 3} e^{2z} + e^z \frac{1}{(D_1 + 1)^3 - (D_1 + 1)^2 + 3(D_1 + 1) - 3} .1$$

$$= \frac{1}{7} e^{2z} + e^z \frac{1}{D_1^3 + 2D_1^2 + 4D_1} (1)$$

$$= \frac{1}{7} e^{2z} + e^z \frac{1}{4D_1\left(1 + \frac{D^1}{2} + \frac{D_1{}^2}{4}\right)} (1)$$

$$= \frac{1}{7} e^{2z} + e^z \frac{1}{4D_1} \left[1 + \frac{D^1}{2} + \frac{D_1{}^2}{4}\right]^{-1} (1)$$

$$= \frac{1}{7} e^{2z} + e^z . \frac{1}{4D_1} (1)$$

$$= \frac{1}{7} e^{2z} + e^z . \left(\frac{1}{4} z\right)$$

$$y = C \cdot F + P \cdot \mathrm{I} = c_1 e^z + c_2 \cos\left(\sqrt{3}z\right) + c_3 \sin\left(\sqrt{3}z\right) + \frac{1}{7} e^{2z} + \frac{e^z}{4} .z$$

$$= c_1 x + c_2 \cos\left(\sqrt{3} \log x\right) + c_2 \sin\left(\sqrt{3} \log x\right) + \frac{1}{7} x^2 \frac{1}{4} x \log x$$

Example 1.3: Solve $\left(x^2 D^2 + 7xD + 13\right) y = \log x$

Solution: Let $x = e^z \therefore z = \log x$ and equation reduces to

$$\left[D_1^2 (D_1 + 7D_1 + 13\right] y = z$$

or $\left(D_1{}^2 + 6D_1 + 13\right) y = z$

$\therefore$ A.E is $m^2 + 6m + 13 = 0$

i.e., $m = \;\; 3 + 2i$

C.F = $e^{-3z} \left[c_1 \cos 2z + c_2 \sin 2z\right]$

and P. I $= \frac{1}{D_1^2 + 6D_1 + 13} . z = \frac{1}{13}\left(1 + \frac{6}{13} D_1 + \frac{1}{13} D_1^2\right)^{-1} . z$

$$= \frac{1}{13}\left(z - \frac{6}{13}\right) = \frac{1}{169}(13z - 6)$$

$$\therefore \qquad y = e^{-3z}\left[c_1 \cos 2z + c_2 \sin 2z\right] + \frac{1}{169}(13z - 6)$$

$$= x^{-3}\left[c_1 \cos(2\log x) + c_2 \sin(2\log x) + \frac{1}{169}(13\log x - 6)\right]$$

Example 1.4: Solve $\left(x^2 D^2 - xD - 3\right) y = x^2 \log x$

Solution: Let $x = e^z \therefore z = \log x$ and equation reduces to

$$\left[D_1(D_1 - 1) - D_1 - 3\right] y = e^{2z}.z$$

or $\qquad \left(D_1^2 - 2D_1 - 3\right) y = ze^{zz}$

$\therefore \qquad$ A.E is $m^2 = 2m \;\; 3 = 0$ i e., $m = \;\; 1, 3$

$\therefore \qquad$ $\text{C.F} = c_1 e^{-z} + c_2 e^{3z}$

and $\qquad \text{P.I} = \dfrac{1}{D_1^2 2D_1 - 3} ze^{2z} = e^{2z} \dfrac{1}{(D_1 + 2) - (2D_1 + 2) - 3} z$

$$= e^{2z} - \frac{1}{D_1^2 + 2D_1 - 3} z = -\frac{e^{2z}}{3}\left[1 - \frac{2}{3}D_1 - \frac{1}{3}D_1^2\right] z$$

$$= -\frac{1}{3}e^{2z}\left[1 + \frac{2}{3}D_1\right] z = \frac{1}{3}e^{2z}\left(z + \frac{2}{3}\right)$$

$\therefore \qquad$ Solution is $y = \text{C.F} + \text{P.I} = c_1 e^{-z} + c_2 e^{3z} - \dfrac{e^{2z}}{3}\left(z + \dfrac{2}{3}\right)$

$$= \frac{c_1}{x} c_2 x^3 - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right)$$

Example 1.5: Solve $(x + a)^2 \dfrac{d^2 y}{dx^2} - 4(x + a)\dfrac{dy}{dx} + 6y = x$

Solution: Let $(x + a) = e^z$ or $z = \log(x + a)$

$\therefore \qquad \dfrac{dz}{dx} = \dfrac{1}{x + a}$

and $\qquad \dfrac{dy}{dx} = \dfrac{dy}{dz} = \dfrac{1}{x + a}\dfrac{dy}{dz}$

or $\qquad (x + a)\dfrac{d}{dx} = \dfrac{d}{dz} = D_1$

NOTES

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[\frac{1}{(x+a)}\frac{dy}{dz}\right]$$

$$=-\frac{1}{(x+a)^2}\frac{dy}{dz}+\frac{1}{x+a}\cdot\frac{1}{(x+a)}\frac{d}{dz}\left(\frac{dy}{dz}\right)$$

$$=\frac{1}{(x+a)^2}\left[\frac{d^2y}{dz^2}-\frac{dy}{dz}\right]$$

or $$(x+a)^2\frac{d^2}{dx^2}=\mathrm{D}_1^2\mathrm{D}_1=\mathrm{D}_1(\mathrm{D}_1-1)$$

Substituting the value in the given equation, we get

$$\left[\mathrm{D}_1\left(\mathrm{D}_1^2-1\right)-4\mathrm{D}_1+6\right]y=e^z-a$$

or $$\left(\mathrm{D}_1^2-5\mathrm{D}_1+6\right)y=e^z a$$

A.E m^2 $5m+6=0$ i.e., $m=2,3$

$\therefore$ C.F $=c_1\mathrm{e}^{2z}+c_2\mathrm{e}^{3z}$

and $$\text{P.I}=\frac{1}{\mathrm{D}_1^2-5\mathrm{D}+6}\left(e^z-a\right)$$

$$=\frac{1}{\mathrm{D}_1^2-5\mathrm{D}_1+6}e^z-\frac{1}{\mathrm{D}_1^2-5\mathrm{D}_1+6}.a$$

$$=\frac{e^z}{2}-\frac{a}{6}\left(1-\frac{5}{6}\mathrm{D}_1+\frac{\mathrm{D}_1{}^2}{6}\right)^{-1}.1$$

$$=\frac{e^z}{2}-\frac{a}{6}\left(1+\frac{5}{6}\mathrm{D}_1\right)1$$

$$=\frac{e^z}{z}-\frac{a}{6}$$

$\therefore$ $$y=c_1e^{2z}+c_2e^{3z}+\frac{e^z}{2}-\frac{9}{6}$$

$$=c_1(x+a)^2+c_2(x+a)^3+\left(\frac{x+a}{2}\right)-\frac{9}{6}$$

Example 1.6: Solve $(1+x)^2\,y^{11}+(1+x)y^1+y=4\cos\log(1+x)$

Solution: Let $1+x=e^z$ or $z=\log(1+x)$

Proceeding as in example 1.4 4 we have

$$(1+x)\frac{dy}{dx}=\frac{dy}{dz}\text{ and}\left(1+x^2\right)^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}$$

And given equation become

$$\left[D_1\left(D_1-1\right)+D_1+1\right]y=4\cos z$$

or $\quad \left(D_1{}^2+1\right)y=4\cos z$

A.E. is $m^2+1=0$ or $m=\pm i$

$\therefore \quad$ C.F $= c_1\cos z+c_2\sin z$

and $\quad$ P.I $=\dfrac{1}{D_1^2+1}\left(4\cos z\right)=4\dfrac{1}{D_1^2+1}\cos z$

$$=4\left[\frac{z}{2}\sin z\right] \qquad \because \frac{1}{D^2+a^2}\cos ax=\frac{x}{2a}\sin ax$$

$$=2\,z\sin z$$

and $\quad y=c_1\cos z+c_2\sin z+2z\sin z$ and

$$=c_1\cos\left[\log\left(1+x\right)\right]+c_2\sin\left[\log\left(1+x\right)\right]+2\log\left(1+x\right)\sin\left[\log\left(1+x\right)\right]$$

Example 1.7: Solve $(x^2D^2 \; 3x\,D+5)y=x^2\sin(\log x)$

Solution: Let $x=e^z$ or $z=\log x$ and equation reduces to

$$\left[D_1\left(D_1-1\right)-3D_1+5\right]y=e^{2z}\sin z$$

or $\quad \left(D_1^2-4D_1+5\right)y=e^{2z}\sin z$

A.E is $m^2 \; 4m+5=0$ i.e., $m=2+1$

$\therefore \quad$ C.F $=e^{2z}\left[c_1\cos z+c_2\sin z\right]$

and $\quad$ P.I $=\dfrac{1}{D_1^2-4+5}e^{2z}\sin z$

$$=e^{2z}\frac{1}{\left(D_1+2\right)^2-4\left(D_1+2\right)+5}\sin z$$

$$=e^{2z}\frac{1}{D_1^2+1}\sin z=e^{2z}\left[\frac{-z}{2}\cos z\right]$$

$$=\frac{-1}{2}z\,e^{2z}\cos z$$

$\therefore \quad y=e^{2z}\left[c_1\cos z+c_2\sin z\right]-\dfrac{z}{2}e^{2z}\cos z$

$$= x^2\left[c_1\cos(\log x) + c_2\sin(\log x)\right] - \frac{x^2}{2}\log x.\cos(\log x)$$

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Example 1.8: $\dfrac{\left[\left(x^2D^2 - 3xD + 1\right)y = \log x.\sin\left(\log x\right) + 1\right]}{x}$

Solution: Let $x = e^z$ or $z = \log x$, equation reduces to

$$\left[\left(D_1D_1 - 1\right) - 3D_1 + 1\right]y = \left(z\sin z + 1\right)e^{-z}$$

or $\quad \left(D_1^2 - 4D_1 + 1\right)y = e^{-z}\left(z\,\text{sinz}+1\right)$

$\therefore \quad$ A.E is $m^2 \quad 4\,m + 1 = 0$ i.e., $m = 2 + \sqrt{3}$

$\therefore \quad$ C.F $= e^{2z}\left[c_1\cos h\left(\sqrt{3}z\right) + c_2\sin h\left(\sqrt{3}z\right)\right]$

and $\quad$ P.I $= \dfrac{1}{D_1^2 - 4D_1 + 1}\left(z\sin z + 1\right)e^{-z}$

$$= e^{-z}\frac{1}{\left(D_1 - 1\right)^2 - 4\left(D_1 - 1\right) + 1}\left(2\sin z + 1\right)$$

$$= e^{-z}\frac{1}{D_1^2 - 6D_1 + 6}\left(z\sin z + 1\right)$$

$$= e^{-z}\left[\frac{1}{D_1^2 - 6D_1 + 6}z\sin z + \frac{1}{D_1^2 - 6D_1 + 6}e^{0.z}\right]$$

$$= e^{-z}\left[\frac{1}{D_1^2 - 6D_1 + 6}z\sin z + \frac{1}{6}\right]$$

$$= e^{-z}\left[\frac{1}{D_1^2 - 6D_1 + 6}z\sin z + \frac{1}{6}\right]$$

Now $\dfrac{1}{D_1^2 - 6D_1 + 6}z\sin z = z\dfrac{-1}{D_1^2 - 6D_1 + 6}\sin z\dfrac{-2D_1 - 6}{\left(D_1^2 + 6D_1 + 6\right)^2}\sin z$

$$= z\frac{1}{-1 - 6D_1 + 6}\sin z - \frac{\left(2D_1 - 6\right)}{\left(-1 - 6D_1 + 6\right)^2}\sin z$$

$$= z\frac{\left(5 + 6D_1\right)}{25 - 36D_1^2}\sin z - \frac{\left(2D_1 - 6\right)}{25 + 36D_1^2 - 60D_1}\sin z$$

$$= z\frac{\left(5 + 6D_1\right)}{61}\sin z + \frac{\left(2D_1 - 6\right)}{11 + 60D_1}\sin z$$

$$= \frac{z}{61}\left(5+6D_1\right)\sin z + \frac{\left(2D_1-6\right)\left(11-60D_1\right)}{121-3600{D_1}^2}\sin z$$

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$$= \frac{z}{61}\left(5\sin z+6\cos z\right) + \frac{\left(-120D_1^2-66+382D_1\right)\sin z}{3721}$$

$$= \frac{z}{61}\left(5\sin z+6\cos z\right) + \frac{1}{3721}\left(54\sin z+382\cos z\right)$$

$\therefore \qquad y = \text{C.F} + \text{P.I}$

$$= x^2\left[c_1\cos h\left(\sqrt{3}\log x\right)+c_2\sin h\left(\sqrt{3}\log x\right)+\frac{\log x}{61}5\sin\left(\log x\right)+6\cos\left(\log x\right)\right]$$

$$+\frac{1}{3721}\left[54\sin\left(\text{og}\,x\right)+382\cos\left(\log x\right)\cos(\log x)\right]$$

1.2.1 Homogeneous Differential Equation with Variable Coefficients

A differential equation of the form

$$\frac{d^2y}{dx^2}+a_1\left(x\right)\frac{dy}{dx}+a_2\left(x\right)y=0$$

Is said to be linear homogeneous differential equation with variable coefficients where $a_1\,(x)$ and $a_2\,(x)$ are continuous function in the interval $[a, b]$.

For second order homogeneous differential equation there is no general method for finding a particular solution. While few solution on can be guessed by using a particular solution. It $y_1\,(x) \neq 0$ is a particular solution of homogeneous linear second order equation then the original equation can be converted to a first order linear equation by substitution $y = y_1\,(x)\,z\,(x)$ and the subsequent replacement $z_1\,(x) = \text{u}$

This method is known as method of reduction of order.

Another method is called method of variation of parameter.

Variation of Parameters

Here we shall explain the method of finding the complete primitive of a linear equation whose C.F is known.

1. To find particular integral of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad (1.1)$$

Let C.F $A\text{y}_1, + By_2$

Then $\text{P.I} = uy_1 + \text{v}y_2$ where

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$$u = \int -\frac{y_2 R}{y_1 y_2' - y_1' y_2} dx$$

and $v = \dfrac{y_1 R}{y_1 y_2' - y_1' y_2} dx$

General solution = C.F + P.I

2. Let $y = A\phi(x) + B\psi(x)$ be the C.F where A and B are constants and $\phi(x)$ and $\psi(x)$ function of x

as $y = A\,\phi(x) + B\psi(x)$ satisfies the equation

$$\frac{d^2y}{dx^2} + P\,\frac{dx}{dx} + Q\,y = 0$$

$\therefore$ $[A\phi''(x) + B\psi''(x)] + P\,[A\phi'(x) + B\psi'(x)] + Q\,[A\phi(x) + B\psi(x)] = 0$

or $A\,[\phi''(x) + P\phi'(x) + Qd(x)] + B[\psi''(x) + P\psi'(x) + Q\psi'(x)] = 0$

$\therefore$ $\phi''(x) + P\phi'(x) + Q\phi(x) = 0$ (1.2)

and $\psi''(x) + P\psi'(x) + Q\psi(x) = 0$ (1.3)

Now let us assume that

$y = A\phi(x) + B\psi(x)$ (1.4)

is complete primitive of (1.1) where A and B are functions of x, so chosen that (1.1) will be satisfied.

$$\therefore \frac{dy}{dx} = A\phi'(x) + B\psi'(x) + \frac{dA}{dx}Q(x) + \frac{dB}{dx}\psi(x)$$

Let A and B satisfy the equation.

$$\phi'(x)\,\frac{dA}{dx} + \psi(x)\frac{dB}{dx} = 0 \tag{1.5}$$

$$\therefore \quad \frac{dy}{dx} = A\phi'(x) + B\psi'(x)$$

and $\dfrac{d^2y}{dx^2} = A\phi''(x) + B\psi''(x) + \dfrac{dA}{dx}\phi'(x) + \dfrac{dB}{dx}\psi'(x)$

Substituting in equation (1.1)

$$\left[A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)\right]$$

$$+P\left[A\phi'(x) + B\psi'(x)\right] + Q[A\phi(x) + B\psi(x)] = R$$

Or $A\left[\phi''(x) + P\phi'(x) + Q\phi(x)\right] + B[\psi''(x) + P\psi'(x) + Q\psi(x)]$

$$+\phi'(x)\frac{dA}{dx}+\psi'(x)\frac{dB}{dx}=R \quad (1.6)$$

As coefficients of A and B are zero by Equations (1.2) and (1.3) from Equations (1.5) and (1.6)

NOTES

$$\frac{dA}{dx}=\left[\phi(x)\psi^{'}(x)-\phi'(x)\psi(x)\right]=-R\psi(x)$$

or $$\frac{dA}{dx}=\frac{R\psi(x)}{\psi(x)-\phi(x)\psi'(x)}$$

or integration we can find the value of A similarly B can be determined from Equations (1.5) and (1.6) as the solution is obtained by varying the arbitrary constants of the complementary function the method is known as variation of particular .

Working Rule

1. Find the C.F of the Equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+Qy=R$

   Let C.F $=c_1\phi(x)+c_2\psi(x)$

   Where c_1 and c_2 are arbitrary constants and $\phi(x)$, $\psi(x)$ are functions of x.
2. Replacing c_1,c_2 by A and B which are functions of x, taken the general solution of Equation on (1.1) as

$$y=A\phi(x)+B\psi(x) \quad ...(1.7)$$

3. Differencing Equation (1.7) we have

$$\frac{dy}{dx}=A\varphi'(x)+B\psi'(x)+\frac{dA}{dx}\phi(x)+\frac{dB}{dx}\psi(x)$$

   Now choose Equations (A) and (B) such that

$$\frac{dA}{dx}\phi(x)+\frac{dB}{dx}\psi(x)=0 \quad \ldots(1.8)$$

$$\therefore \frac{dy}{dx}=A\phi'(x)+B\psi'(x)$$

4. $$\therefore \frac{d^2y}{dx^2}=A\phi''(x)+B\psi''(x)+\frac{dA}{dx}\phi'(x)+\frac{dB}{dx}\psi'(x)$$

   Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in Equation (1.1) reduces to

$$\frac{dA}{dx}\phi'(x)+\frac{dB}{dx}\psi'(x)=R \quad \ldots(1.9)$$

Solving Equations (1.8) and (1.9) we can find $\frac{dA}{dB}$ *and* $\frac{dB}{dx}$, and integration gives equations (1.7) and (1.8)

NOTES

5. Substitute these values in equations (1.7) to get the general solution of the given equation,

Example 1.9: Solve $y''+y$ = cosec x

Solution: Given $(D^2+1)\,y = \text{cosec}\,x$

A.E is $m^2+1=0$, i.e., $M = \neq 1$

$\therefore$ C·F = $c_1\cos x + c_2\sin x$ hence $y_1 = \cos x$, $y_2 = \sin x$

Let P.I = $uy_1 + vy_2$

$= u\cos x + \sin x$

where $u = \int -\frac{y_2 \text{cosec}\, x\, dx}{y_1y_2' - y_1'y_2}$

$$= \int -\frac{\sin x \times \text{cosec}\, x\, dx}{\cos x(\cos x)-(-\sin x)(\sin x)} = -\int \frac{dx}{\cos^2 x + \sin^2 x}$$

$$= -\int dx = -x$$

and $v = \int \frac{y_1 \text{cosec}\, x\, dx}{y_1y_2' - y_1'y_2}$

$$= \int \frac{\cos x\, \text{cosec}\, x\, dx}{\cos^2 x + \sin^2 x} = \int \cot x\, dx$$

$= \log \sin x$

$\therefore$ P·I = $(-x)\cos x + (\log \sin x)\sin x$

$\therefore$ y = C·F + P.I = $c_1\cos x + c_2\sin x - x\cos x + \sin x\ \log(\sin x)$

Alter) C.F = $c_1\cos x + c_2\sin x$

Let y = A cos$x$ + B sin x where A and B are functions

$$\therefore \frac{dy}{dx} = -A\sin x + B\cos x + \frac{dA}{dx}\cos x + \frac{dB}{dx}\sin x$$

choose A and B such that

$$\frac{dA}{dx}\cos x + \frac{dB}{dx}\sin x = 0 \qquad \ldots(1)$$

$$\therefore \frac{dy}{dx} = -A\sin x + B\cos x$$

also $\frac{d^2y}{dx^2} = -\frac{dA}{dx}\sin x + \frac{dB}{dx}\cos x - A\cos x - B\sin y$ substituting in given equations, equation becames

$$\frac{-dA}{dx}\sin x + \frac{dB}{dx} = \text{coses } x \qquad \ldots(2)$$

Solving Equations (1) and (2),we have

NOTES

$$\frac{dA}{dx} = -1, \qquad \therefore \text{ A} = -x + \text{c}_1$$

and $\frac{dB}{dx} = \cot x \qquad \therefore \text{ B} = \log \sin x + \text{c}_2$

$\therefore$ general solution $y = c_1\cos x + c_2\sin x - x\cos x + \sin x \log \sin x$

Example 1.10: $(\text{D}^2-1)\, y = \dfrac{2}{1+e^x}$

Solution: A·E $\quad m^2-1 = 0 \qquad$ i.e., $m = \pm 1$

$\therefore \quad$ C·F $= c_1e^x + c_2e^{-x}$

$y_1 = e^x, \qquad y_2 = e^{-x}$

Let $\quad$ P·I $= uy_1 + vy_2$

where $u = \displaystyle\int \frac{y_2 \cdot \frac{2}{1+e^x}}{y_1y_2' - y_1'y_2} = -\int \frac{\frac{2e^{-x}}{1+e^x}}{e^x(-e^{-x}) - e^x(e^{-x})}dx$

$$= -2\int \frac{\frac{e^{-x}}{1+e^x}dx}{-2}$$

$$= \int \frac{e^{-x}}{1+e^x}dx = \int \frac{1}{e^x(1+e^x)}dx$$

$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= \int e^{-x}dx - \int \frac{e^{-x}}{e^{-x}+1}dx$$

$$= -e^{-x} + \log(e^{-x}+1)$$

and $\quad V = \displaystyle\int \frac{y_1 \cdot \frac{2}{1+e^x}}{y_1y_2' - y_1'y_2} = \int \frac{\frac{2e^x}{1+e^x}}{-2}dx$

$$= -\int \frac{e^x}{1+e^x}dx = -\log(1+e^x)$$

NOTES

$\therefore \quad P\cdot I = [(-e^{-x} + \log(e^{-x}+1)]\, e^{x} - e^{-x}\log(1+e^{x})]$

$\quad = -1+e^{x} + \log(e^{-x}+1) - e^{-x}\log(e^{x}+1)$

$y = c_1 e^{x} + c_2\, e^{-x} + e^{x}\log(e^{-x}+1) - e^{-x}\log(e^{x}+1) - 1$

Alter $\quad C.F = c_1 e^{x} + c_2 e^{-x}$

Now let $y = Ae^{x} + Be^{-x}$ where A and B are functions of x

$$\therefore \quad \frac{dy}{dx} = Ae^{x} - Be^{-x} + \frac{dA}{dx}e^{x} + \frac{dB}{dx}e^{-x}$$

Choose A and B such that

$$\frac{dy}{dx}e^{x} + \frac{dB}{dx}e^{-x} = 0 \qquad \dots (1)$$

$$\therefore \quad \frac{dy}{dx} = Ae^{x} - Be^{-x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{dA}{dx}e^{x} - \frac{dB}{dx}e^{-x} + Ae^{x} + Be^{-x}$$

Substituting in given equation, we have

$$\frac{e^{x}dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{1+e^{x}} \qquad \dots (2)$$

Solving Equations (1) and (2), we have

$$2e^{x}\frac{dA}{dx} = \frac{2}{1+e^{x}}$$

i.e., $\quad \dfrac{dA}{dx} = \dfrac{e^{-x}}{1+e^{x}}$ or $A = -\, e^{-x}\log(1+e^{-x})$

and $\quad \dfrac{dB}{dx} = \dfrac{-e^{x}}{1+e^{x}}$ $\; B = -\log(1+e^{x})$

$\therefore \quad y = c_1 e^{x} + c_2 e^{-x} + e^{x}\log(1+e^{-x}) - e^{x}\log(1+e^{x}) - 1$

To find one integral in c·f· by Inspection

It given equation is $\dfrac{d^2y}{dx^2} + \dfrac{\text{pdy}}{dx} + Qy = R \dots \qquad (1.10)$

(i) $y = e^{x}$ is a solution of (1.10) if $1 + P + Q = 0$

(ii) $y = e^{-x}$ is a solution of (1.10) if $1 - P + Q = 0$

(iii) $y = e^{mx}$ is a solution of (1.10) if $m^2 + Pm + Q = 0$

(iv) $y = x$ is a solution of (1.10) if $P + Qx = 0$

(v) $y = x^{m}$ is a solution of (1.10) if $2 + 2\,Px + Qx^2 = 0$

(vi) $y = x^2$ is a solution of (1) if $m(m-1) + pm^{x} + Qx^2 = 0$

Example 1.11: Solve $\frac{x^2 d^2 y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$ by the method of variation of parameters.

NOTES

Solution: Given equation is $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$

or $\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = e^x$ … (1)

To find C·F $\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$

Here $P + Qx = 0$

$\therefore$ $y = x$ is a part of C·F

Let $y = vx$

So that $\frac{dy}{dx} = \frac{dv}{dx}x + v$

And $\frac{d^2 y}{dx^2} = \frac{d^2 y}{dx^2}x + 2\frac{dv}{dx}$

Putting in Equation (1) we have $\frac{d^2 v}{dx^2} + \frac{3}{x}\frac{dv}{dx} = 0$

$$\therefore \quad \frac{\frac{d^2 v}{dx^2}}{\frac{dv}{dx}} = -\frac{3}{x}$$

Integrating $\log\frac{dv}{dx} = -3\log x + \log c$

$$\frac{dv}{dx} = \frac{c}{x^3}$$

Integrating $v = \frac{-c}{2x^2} + c_1$

$\therefore$ C·F of the equation is $y = vx = c_1 x - \frac{c}{2x}$

or $y = c_1 x + \frac{c_2}{x}$

Now let $y = Ax + \frac{B}{x}$ … (A)

be the complete positive of the given equation, where (A) and (B) are function of *x*.

NOTES

$$\therefore \qquad \frac{dy}{dx} = A - B + \frac{dA}{dx}x + \frac{1}{x}\frac{dB}{dx}$$

Now choosing A and B such that

$$x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} = 0$$

We have $\dfrac{dy}{dx} = A - \dfrac{B}{x^2}$ …(2)

and $\dfrac{d^2y}{dx^2} = \dfrac{dA}{dB} - \dfrac{1}{x^2}\dfrac{dB}{dx} + \dfrac{2}{x^3}B$

Substituting in the given equation, we get

$$\frac{dA}{dB} - \frac{1}{x}\frac{dB}{dx} = e^x \qquad \text{…(3)}$$

Solving Equations (2) and (3), we get

$$\frac{dA}{dB} = \frac{e^x}{2} \qquad\qquad \therefore\ A = \frac{e^x}{2} + c_1$$

and $\dfrac{dB}{dx} = -\dfrac{1}{2}x^2 ex$

$$\therefore \qquad B = -\frac{1}{2}\int x^2 e^x dx$$

$$= -\frac{1}{2}x^2 e^x + xe^x - e^x + c_2$$

Substituting in (A)

$$y = c_1 x + \frac{c_2}{x} + \frac{1}{2}xe^x + e^x - \frac{1}{x}e^x$$

$$= c_1 x + \frac{c_2}{x} + e^x - \frac{1}{x}e^x$$

Example 1.12: Solve $\dfrac{d^2y}{dx^x} + n^2 y = \sec nx$

Solution: A·E is $m^2 + n^2 = 0$ i.e., $m = \underline{+}1\, n$

or $\quad C{\cdot}F = c_1 \cos nx + c_2 \sin nx$

where c_1 and c_2 are arbitrary constants

let $y = A\cos nx + B \sin nx$ … (A) be the complete primitive of the given equation where A and B are function of x.

$$\frac{dy}{dx} = An\sin nx + Bn\cos nx + \frac{dA}{dx}\cos nx + \frac{dB}{dx}\sin nx$$

NOTES

Choose A and B such that

$$\frac{dA}{dx}\cos nx + \frac{dB}{dx}\sin nx = 0 \quad \ldots (1)$$

we have $\frac{dy}{dx} = -An\cos nx + Bn\sin nx$

$$\therefore \quad \frac{d^2y}{dx^2} = -An^2\cos nx - Bn^2\sin nx - n\frac{dA}{dx}\sin nx + n\frac{dB}{dx}\cos nx$$

Substituting in the given equation we have

$$-n\frac{dA}{dx}\sin nx + n\frac{dB}{dx}\cos nx = \sec nx \quad \ldots (2)$$

Solving Equations (1) and (2), we have

$$n\frac{dA}{dx} = -\tan nx \qquad \therefore A = \frac{1}{n_2}\log \cos nx + c_1$$

and $\quad n\frac{dB}{dx} = 1 \qquad \therefore B = \frac{x}{n} + c_2$

substituting the values in A

$$y = c_1\cos nx + c_2\sin nx + \frac{1}{n_2}\cos nx \cdot \log\cos nx + \frac{x}{n}\sin nx$$

Example 1.13: By the method of variation of parameters, solve

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(x+1)y = x^3$$

Solution: Given equation is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x$$

Here $\quad p = -\frac{2(1+x)}{x}, \; Q = \frac{2(x+1)}{x^2}$

or $\quad$ P + Qx= 0

$\therefore \quad y = x$ is a part of C.F

Let $\quad y = vx$ so that $\frac{dy}{dx} = \frac{dv}{dx}x + v$

NOTES

and $$\frac{d^2y}{dx^2}=\frac{d^2v}{dx^2}x+2\frac{dv}{dx}$$

substituting in $$\frac{d^2y}{dx}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(x+1)}{x^2}y=0 \quad (1)$$

we have $$x\frac{d^2v}{dx}+2\frac{dv}{dx}-\frac{2(1+x)}{x}\left(x\frac{dv}{dx}+v\right)+\left(\frac{2x+1}{x^2}\right)vx=0$$

or $$\frac{d^2v}{dx^2}-\frac{2dv}{dx}=0$$

A·E = m^2–$2m$ = 0 i.e., m = 0,2

$\therefore \quad v=c_1+\ c_2e^{2x}$

and solution of Equation (1) is $y=vx=c_1x\ +c_2xe^{2x}$

Now let $y=Ax+Bxe^{2x}$ ...(2)

Be the complete primitive of the given equation ,where A and B are function of x

$$\frac{dy}{dx}=A+B(e^{2x}+2xe^{2x})+x\frac{dA}{dx}+xe^{2x}\frac{dB}{dx}$$

Now choosing A and B such that

$$x\frac{dA}{dx}+xe^{2x}\frac{dB}{dx}=0 \quad \ldots(3)$$

$$\therefore \quad \frac{dy}{dx}=\text{A}+\text{B}\ (e^{2x}+2xe^{2x})$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d\text{A}}{dx}+e^{2x}\frac{d\text{B}}{dx}(1+2x)+2\text{B}\ e^{2x}+2\text{B}(1+2x)e^{2x}$$

Substituting in the given equation, we have

$$\frac{d\text{A}}{dx}+e^{2x}\frac{d\text{B}}{dx}(1+2x)+2\text{B}\ e^{2x}+2\text{B}(1+2x)e^{2x}$$

$$-2\frac{(1+x)}{x}\left[\text{A}+\text{B}(1+2x)e^{2x}\right]+2\left(\frac{x+1}{x^2}\right)\left[\text{A}x+\text{B}x\,e^{2x}\right]=x$$

or $$\frac{d\text{A}}{dx}+e^{2x}(1+2x)\frac{d\text{B}}{dx}=x \quad (4)$$

Solving Equations (3) and (4)

$$\frac{d\text{A}}{dx}=\frac{-1}{2} \qquad \therefore \text{A}=\frac{-x}{3}+c_1$$

and $$\frac{d\text{B}}{dx}=\frac{1}{2}e^{-2x} \qquad \therefore \text{B}=\frac{-1}{4}e^{-2x}+c_2$$

$\therefore$ Requird solution is $y = c_1x + c_2xe^{2x} - \frac{x^2}{2}\frac{-x}{4}$

Example 1.14: Apply the method of variation of parameters to solve the equation.

NOTES

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)$$

Solution: Given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = (1-x)$$

Here P + Qx = 0 $\therefore$ $y = x$ is a part of C.F

Now to find the C.F of given equation, i.e., the solution of

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{y}{1-x} = 0 \quad (1)$$

Let $y = vx$ then equation c_1 reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x} + \frac{2}{x}\right)\frac{dv}{dx} = 0 \qquad \text{Let } \frac{dv}{dx} = P$$

or $$\frac{dP}{dx} + \left(\frac{x}{1-x} + \frac{2}{x}\right)P = 0$$

or $$\frac{dP}{dx} + \left(-1 - \frac{1}{x-1} + \frac{2}{x}\right)P = 0$$

or $$\frac{dP}{P} = \left(1 + \frac{1}{x-1} + \frac{2}{x}\right)dx$$

Integrating log P = x log (x-1) -2 log x+4 log c_1

or $$P = \frac{dv}{dx} = \frac{c_1(x-1)e^x}{x^2}$$

or $$dv = c_1\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)dx$$

Integrating $$v = c_1\left[\int\frac{e^x}{x}dx - \int\frac{e^x}{x^2}dx\right]$$

$$= \frac{c_1}{x}e^x + c_2$$

$\therefore$ C.F of the given equation is $y = vx$

or $y = c_1e^x + c_2x$

Now let $y = Ae^x + Bx$ be the complete soluation of the given equation where A and B are functions of x

$$\therefore \frac{dy}{dx} = Ae^x + B + e^x \frac{dA}{dx} + x\frac{dB}{dx}$$

Choose A and B such that,

NOTES

$$\therefore \frac{dy}{dx} = Ae^x + B$$

and $$\frac{d^2y}{dx^2} = e^x \frac{dA}{dx} + \frac{dB}{dx} + e^x A$$

Substituting in the given equation, we have

$$e^x \frac{dA}{dx} + \frac{dB}{dx} = 1 - x \qquad (3)$$

Solving Equations (2) and (3) we have

$$-(1-x)e^x \frac{dA}{dx} = (1-x)x$$

or $$\frac{dA}{dx} = -xe^{-x}$$

or $$A = -\int xe^{-x}dx = xe^{-x} + e^{-x} + C$$

and $$(1-x)\frac{dB}{dx} = 1\ x \text{ or } \frac{dB}{dx} = 1$$

$\therefore$ $$B = x + c_2$$

$\therefore$ $$y = c_1e^x + c_2x + x + 1 + x^2$$

Example 1.15: Solve $\frac{d^2y}{dx^2} + (1-\cos x)\frac{dy}{dx} - y\cot x = \sin^2 x$

Solution: Here 1 P + Q = 0 $\qquad \therefore y = e^{-x}$ is a part of the C.F

Putting $y = ve^x$ in the given equation.

$$\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = 0$$

or $$\frac{dP}{dx} - (1+\cot x)P = 0 \qquad \text{where } P = \frac{dv}{dx}$$

or $$\int \frac{dP}{P} = \int (1 + \cos - x)\,dx$$

i.e., $$\log P = x + \log(\sin x) + \log c_1$$

or $$P = \frac{dv}{dx} = c_1e^x \sin x$$

i.e $$v = c_1 \int e^x \sin x\,dx$$

$$= c_1\frac{e^x}{2}(\sin x - \cos x) + c_2$$

NOTES

$\therefore$ C.F of the given equation $= ve^x = \frac{c_1}{2}\left(\sin x - \cos x\right) + c_2 e^{-x}$

Let $y = A[\sin x \;\; \cos x] + Be^x$ be the complete solution of the given equation where A and B are functions of x.

$$\therefore \quad \frac{dy}{dx} = A(\cos x + \sin x) \quad Be^x + \frac{dA}{dx}(\sin x - \cos x) + \frac{dB}{dx}e^{-x}$$

A and B are choosen such that

$$\frac{dA}{dx}(\sin x - \cos x) + \frac{dB}{dx}e^{-x} = 0$$

$$\therefore \quad \frac{dy}{dx} = A(\cos x + \sin x) - Be^{-x}$$

and $\quad \frac{d^2y}{dx^2} = \frac{dA}{dx}(\cos x + \sin x) - \frac{dB}{dx}e^{-x} + A(-\sin x + \cos x) + Be^{-x}$

Putting in the given equation, we have

$$\frac{dA}{dx}(\cos x + \sin x) - \frac{dB}{dx}e^{-x} = \sin^2 x \qquad (2)$$

Solving Equations (1) and (2) we get

$$\frac{dA}{dx} = \frac{1}{2}\sin x, \quad A = -\frac{1}{2}\cos x + c_1$$

and $\quad \frac{dB}{dx} = \frac{e^x}{2}\left(\sin x \cos x - \sin^2 x\right)$

i.e., $\quad B = \frac{1}{4}\int e^x(\sin 2x - 1 + \cos 2x)dx + C_2$

$$= \frac{1}{4}\frac{e^x}{5}(\sin 2x - 2\cos 2x) - \frac{e^x}{4} + \frac{1}{4}\cdot\frac{e^x}{5}(\cos 2x + 2\sin 2x) + c_2$$

$$= \frac{e^x}{20}(3\sin 2x - \cos 2x) - \frac{e^x}{4} + c_2$$

$$\therefore \quad y = a(\sin x - \cos x) + c_2 e^{-x} - \frac{1}{10}(\sin 2x - 2\cos 2x)$$

Check Your Progress

1. Solve by the method of variation of parameter $(b^2 + 1)\, y = \tan x$
2. Solve by the method of variation of parameter $(D^2 + 1)\, y$-$\sec x$
3. Solve by the method of variation of parameter $(D^2$-$4^2)\, y = e^2 x$
4. Solve by the method of variation of parameter $(D^2 - 3D + 2)y = \sin x$
5. Solve by the method of variation of parameter $(D^2 - 3D + 2)y = \sec x \tan x$
6. Solve by the method of variation of parameter $(D^2 + 1)y = \sec x \tan x$
7. Solve by the method of variation of parameter $(D^2 + 9)\, y = \sec 3x$

NOTES

1.3 SIMULTANEOUS DIFFERENTIAL EQUATION

An ordinary differential equation (ODE) is a relation that contains function of only one indepnedent, and one or more of then derivin a

Methods of Solving Simultaneous Linear Differential Equations with Constant Coefficients

In this section, we shall discuss two methods for solving the simultaneous linear differential equation where x, y are two dependent variables and t is the independent variable.

Using Operator D

Let the symbolic form of the equations be $F_1(D)x + F_2(D)y = T_1$ … (1.11)

and $\phi_1(D)x + \phi_2(D)y = T_2$ … (1.12)

where D denotes $\frac{d}{dt}$. Also, T_1 and T_2 are functions of independent variable t and $F_1(D)$, $F_2(D)$, $\phi_1(D)$ and $\phi_2(D)$ are all rational integral functions of D with constant coefficients.

Now, eliminate x from (1.11) and (1.12) by operating on both sides of (1.11) by $\phi_1(D)$ and (1.12) by $F_1(D)$, we get $F_1(D)\phi_1(D)x + F_2(D)\phi_1(D)y = \phi_1(D)T_1$

$$\phi_1(D)F_1(D)x + \phi_2(D)F_1(D)y = F_1(D)T_2$$

On subtracting these equations, we get

$$F_2(D)\phi_1(D)y - \phi_2(D)F_1(D)y = \phi_1(D)T_1 - F_1(D)T_2$$

$$\Rightarrow \quad g_1(D)\, y = T \text{ (say)}$$

which is a linear equation in y and t. This equation can be solved to get the value of y.

Now, by putting this value of y in (1.11) or (1.12), we get the value of x.

Note: Similarly, we can also eliminate y and get a linear differential equation in x and t which can be solved to get the value of x in terms of t. Further the value of y can be obtained from (1.11) or (1.12) by putting the value of x.

Method of differentiation

Sometimes, by differentiating one of the equations (1.11) or (1.12) or both, we can easily eliminate x or y. From resulting equation, after eliminating one dependent variable, x or y can be solved to give the other dependent variable and then the value of the other variable can be obtained by putting these values in equation (1.11) or (1.12).

Example 1.16: Solve the simultaneous equations

$$\frac{dx}{dt} + 4x + 3y = t \text{ and } \frac{dy}{dt} + 2x + 5y = e^t$$

NOTES

Solution: The given equations are $\frac{dx}{dt} + 4x + 3y = t$ … (1)

and $\frac{dy}{dt} + 2x + 5y = e^t$ … (2)

By putting $\frac{d}{dt} = \text{D}$ in equations (1) and (2), we get

$$(\text{D}+4)x + 3y = t \quad \text{... (3)}$$

and $2x + (\text{D}+5)y = e^t$ ...(4)

Eliminating y, we get $\left[(\text{D}+4)(\text{D}+5)-6\right]x = (\text{D}+5)t - 3e^t$

$\Rightarrow$ $(\text{D}^2 + 9\text{D} + 14)x = 1 + 5t - 3e^t$.

$\therefore$ Its A.E. is $\text{D}^2 + 9\text{D} + 14 = 0 \Rightarrow \text{D} = -2, -7$

$\therefore$ C.F. $= c_1e^{-2t} + c_2e^{-7t}$

$$\text{P.I.} = \frac{1}{14 + 9\text{D} + \text{D}^2}.(1+5t) - \frac{1}{14 + 9\text{D} + \text{D}^2}.3e^t$$

$$= \frac{1}{14}\left(1 + \frac{9}{14}\text{D} + \frac{1}{14}\text{D}^2\right)^{-1}.(1+5t) - \frac{3e^t}{14+9+1}$$

$$= \frac{1}{14}\left(1 - \frac{9}{14}\text{D} + ...\right)(1+5t) - \frac{1}{8}e^t$$

$$= \frac{1}{14}\left(1 + 5t - \frac{9}{14}.5\right) - \frac{1}{8}e^t = \frac{1}{14}\left(5t - \frac{31}{14}\right) - \frac{1}{8}e^t$$

$\therefore$ $$x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{1}{8}e^t - \frac{31}{196}$$

$\Rightarrow$ $$\frac{dx}{dt} = -2c_1e^{-2t} - 7c_2e^{-7t} + \frac{5}{14} - \frac{1}{8}e^t$$

By putting the values of x and $\frac{dx}{dt}$ in equation (1), we get

$$3y = -2c_1e^{-2t} + 3c_2e^{-7t} - \frac{10}{7}t + t - \frac{5}{14} + \frac{31}{49} + \frac{1}{8}e^t + \frac{1}{2}e^t$$

$\Rightarrow$ $$y = \frac{1}{3}\left[-2c_1e^{-2t} + 3c_2e^{-7t} - \frac{3}{7}t + \frac{27}{98} + \frac{5}{8}e^t\right]$$

and $$x = c_1e^{-2t} + c_2e^{-2t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t.$$

Example 1.17: Solve the simultaneous equations

$$\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 2y = 3e^t \text{ and } 3\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 4e^{2t}$$

NOTES

Solution: The given equations are $\frac{dx}{dt}+2\frac{dy}{dt}-2x+2y=3e^t$ … (1)

and $3\frac{dx}{dt}+\frac{dy}{dt}+2x+y=4e^{2t}$ … (2)

By putting $\frac{d}{dt}=\text{D}$ in the equations (1) and (2), we get

$$\text{D}x+2\text{D}y-2x+2y=3e^t$$

$\Rightarrow$ $(\text{D}-2)x+2(\text{D}+1)y=3e^t$ … (3)

and $(3\text{D}+2)x+(\text{D}+1)y=4e^{2t}$ … (4)

To eliminate y from Equations (3) and (4), multiply Equation (4) by 2 and subtract from Equation (3).

$$(\text{D}-2)x+2(\text{D}+1)y-2(3\text{D}+2)x-2(\text{D}+1)y=3e^t-8e^{2t}$$

$\Rightarrow$ $\left[\text{D}-2-6\text{D}-4\right]x=3e^t-8e^{2t}$

$\Rightarrow$ $(-5\text{D}-6)x=3e^t-8e^{2t}$

$\Rightarrow$ $(5\text{D}+6)x=8e^{2t}-3e^t \Rightarrow \frac{dx}{dt}+\frac{6}{5}x=\frac{8}{5}e^{2t}-\frac{3}{5}e^t$ … (5)

which is a linear differential equation of the form $\frac{dx}{dt}+\text{P}x=\text{Q}$

where $\text{P}=\frac{6}{5}$ and $\text{Q}=\frac{8}{5}e^{2t}-\frac{3}{5}e^t$

$\therefore$ $\text{I.F.}=e^{\int \text{P}dt}=e^{\int \frac{6}{5}dt}=e^{\frac{6}{5}t}$

Thus, the solution of Equation (5) is $x\left(e^{\frac{6}{5}t}\right)=\int\left(\frac{8}{5}e^{2t}-\frac{3}{5}e^t\right)e^{\frac{6}{5}t}dt+c_1$

$\Rightarrow$ $xe^{\frac{6}{5}t}=\int\left(\frac{8}{5}e^{\frac{16}{5}t}-\frac{3}{5}e^{\frac{11}{5}t}\right)dt+c_1 =\frac{8}{5}\frac{5}{16}e^{\frac{16}{5}t}-\frac{3}{5}\frac{5}{11}e^{\frac{11}{5}t}+c_1$

$\Rightarrow$ $x=e^{-\frac{6}{5}t}\left[\frac{1}{2}e^{\frac{16}{5}t}-\frac{3}{11}e^{\frac{11}{5}t}\right]+c_1e^{-\frac{6}{5}t}$

$\Rightarrow$ $x=\frac{1}{2}e^{2t}-\frac{3}{11}e^t+c_1e^{-\frac{6}{5}t}$ … (6)

$\Rightarrow$ $\frac{dx}{dt}=\frac{1}{2}(2)e^{2t}-\frac{3}{11}e^t+c_1\left(-\frac{6}{5}\right)e^{-\frac{6}{5}t}$ … (7)

From Equation (1), $\frac{dx}{dt}-2x+2\frac{dy}{dt}+2y=3e^t$

Using Equations (6) and (7), we get

NOTES

$$e^{2t} - \frac{3}{11}e^t - \frac{6}{5}c_1 e^{-\frac{6}{5}t} - 2\left[\frac{1}{2}e^{2t} - \frac{3}{11}e^t + c_1 e^{-\frac{6}{5}t}\right] + 2\frac{dy}{dt} + 2y = 3e^t$$

$\Rightarrow$
$$2\frac{dy}{dt} + 2y = \frac{30}{11}e^t + \frac{16}{5}c_1 e^{-\frac{6}{5}t} \Rightarrow \frac{dy}{dt} + y = \frac{15}{11}e^t + \frac{8}{5}c_1 e^{-\frac{6}{5}t} \quad \ldots (8)$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int 1dt} = e^t$$

Thus the solution of Equation (8) is

$$ye^t = \int\left(\frac{15}{11}e^t + \frac{8}{5}c_1 e^{-\frac{6}{5}t}\right)e^t dt + c_2$$

$$= \int\left(\frac{15}{11}e^{2t} + \frac{8}{5}c_1 e^{-\frac{1}{5}t}\right)dt + c_2 \quad = \frac{15}{22}e^{2t} - 8c_1 e^{-\frac{1}{5}t} + c_2$$

$\Rightarrow$
$$y = \frac{15}{22}e^t - 8c_1 e^{-\frac{6}{5}t} + c_2 e^{-t}$$

Hence the required solutions of given equations are

$$x = \frac{1}{2}e^{2t} - \frac{3}{11}e^t + c_1 e^{-\frac{6}{5}t};\ y = \frac{15}{22}e^t - 8c_1 e^{-\frac{6}{5}t} + c_2 e^{-t}$$

Example 1.18: Solve the simultaneous equations $t\frac{dx}{dt} + y = 0, t\frac{dy}{dt} + x = 0$ given that $x(1) = 1,\ y(-1) = 0$.

Solution: The given equations are $t\frac{dx}{dt} + y = 0$ … (1)

and
$$t\frac{dy}{dt} + x = 0 \quad \ldots (2)$$

Differentiating Equation (1) with respect to t, we have $t\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$

Multiplying by t, we get $t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$ … (3)

Subtracting Equation (2) from (3), we get $t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ … (4)

which is an homogeneous linear equation.

Put $t = e^z \Rightarrow \log t = z$

$\therefore$
$$t\frac{d}{dt} = \frac{d}{dz} = \text{D} \text{ and } t^2\frac{d^2}{dt^2} = \text{D}(\text{D}-1)$$

$\therefore$ Equation (4) becomes $[\text{D}(\text{D}-1) + (\text{D}-1)]x = 0$

$\Rightarrow$
$$[\text{D}^2 - 1]x = 0$$

NOTES

$\therefore$ Its A.E. is $D^2 - 1 = 0 \Rightarrow D^2 = 1 \Rightarrow D = \pm 1$

Thus the solution is $x = c_1 e^z + c_2 e^{-z}$

$$\Rightarrow \qquad x = c_1 t + c_2 t^{-1} = c_1 t + \frac{c_2}{t} \qquad \dots (5)$$

Differentiating Equation (5) with respect to t, we get $\frac{dx}{dt} = c_1 - \frac{c_2}{t^2}$

By putting this value of $\frac{dx}{dt}$ in Equation (1), we get $t\left[c_1 - \frac{c_2}{t^2}\right] + y = 0$

$$\Rightarrow \qquad c_1 t - \frac{c_2}{t} + y = 0 \Rightarrow y = -c_1 t + \frac{c_2}{t} \qquad \dots (6)$$

Given, $x(1) = 1; y(-1) = 0$

Putting $t = 1, x = 1$ in Equation (5), we have $1 = c_1 + c_2$ … (7)

Putting $t = -1, y = 0$ in Equation (6), we have $0 = c_1 - c_2$ … (8)

Solving Equations (7) and (8), we get $c_1 = c_2 = \frac{1}{2}$

Thus, the required solutions are $x = \frac{1}{2}\left(t + \frac{1}{t}\right); y = \frac{1}{2}\left(-t + \frac{1}{t}\right)$.

1.3.1 Simultaneous Equations in a Different Form

If the equations are given in the form $P_1 dx + Q_1 dy + R_1 dz = 0$

… (1.13)

and $P_2 dx + Q_2 dy + R_2 dz = 0$ … (1.14)

where $P_1, P_2, Q_1, Q_2, R_1, R_2$ are all function of x, y, z.

Dividing Equations (1.13) and (1.14) by dz, we get

$$P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0 \qquad \dots (1.15)$$

$$\text{and} \qquad P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0 \qquad \dots (1.16)$$

Solving Equations (1.15) and (1.16), by cross-multiplication method, we get

$$\frac{\frac{dx}{dz}}{Q_1 R_2 - Q_2 R_1} = \frac{\frac{dy}{dz}}{R_1 P_2 - R_2 P_1} = \frac{1}{P_1 Q_2 - P_2 Q_1}$$

$$\Rightarrow \qquad \frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - R_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1}$$

which is of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ … (1.17)

where P, Q, R are functions of x, y and z.

Thus, simultaneous Equation of the type (1.13) and (1.14) can always be put in the form Equation (1.15).

NOTES

Methods for solving the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

First method: Let the multipliers l, m, n be such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

Choose l, m, n such that $lP + mQ + nR = 0$, and hence $ldx + mdy + ndz = 0$

If it is an exact differential equation say du, then on integrating, we get, one part of the complete solution of Equation (1.15).

Again, if we choose another set of multipliers l', m', n' such that $l'P + m'Q + n'R = 0$ we get $l'dx + m'dy + n'dz = 0$

Then, on integration, it will give another equation. The two equations thus obtained by using two sets of multipliers will form the complete solutions of given simultaneous equations.

Note: Sometimes it may also happen that we choose multipliers l, m, n such that $\frac{ldx + mdy + ndz}{lP + mQ + nR}$ is of the form that numerator is the exact differential coefficient of the denominator.

Second method: The given equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ … (1.18)

First take any two members of $\frac{dx}{P} = \frac{dy}{Q}$ (say) and integrate it to get one of the equation of the complete solution.

Again, take other two members $\frac{dy}{Q} = \frac{dz}{R}$ (say) and integrate it also to get another equation of the complete solution. These two equations so obtained form the complete solution.

Example 1.19: Solve the simultaneous equations $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z + \frac{1}{z}}$.

Solution: The given equations are $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z + \frac{1}{z}}$ … (1)

Choosing 1, 1, 0 as multipliers, we get

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z + \frac{1}{z}} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)}$$

$$\Rightarrow \quad \frac{zdz}{z^2 + 1} = \frac{dx + dy}{\sin(x+y) + \cos(x+y)}$$

$$\Rightarrow \quad \frac{2zdz}{2(z^2 + 1)} = \frac{d(x+y)}{\sqrt{2}\sin\left(x + y + \frac{\pi}{4}\right)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\left[\frac{2zdz}{z^2+1}\right] = \operatorname{cosec}\left[(x+y)+\frac{\pi}{4}\right]d(x+y)$$

Integrating both sides, we get

NOTES

$$\frac{1}{\sqrt{2}}\log(z^2+1) = \log\tan\frac{1}{2}\left((x+y)+\frac{\pi}{4}\right)+\log c_1$$

$$\Rightarrow \quad \log(z^2+1)^{1/\sqrt{2}} - \log\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = \log c_1$$

$$\Rightarrow \quad \log\frac{(z^2+1)^{1/\sqrt{2}}}{\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)} = \log c_1 \quad \Rightarrow \frac{(z^2+1)^{1/\sqrt{2}}}{\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)} = c_1 \qquad \ldots (2)$$

Now, choosing 1, 1, 0 and 1, –1, 0 as multipliers in Equation (1), we get

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\Rightarrow \quad \frac{[\cos(x+y)-\sin(x+y)]}{\cos(x+y)+\sin(x+y)}(dx+dy) = dx-dy$$

$$\Rightarrow \quad \frac{[\cos(x+y)-\sin(x+y)]}{\sin(x+y)+\cos(x+y)}d(x+y) = d(x-y)$$

Integrating both sides, we have $\log|\sin(x+y)+\cos(x+y)| = x-y+\log c_2$

$$\Rightarrow \quad \log|\sin(x+y)+\cos(x+y)| + (y-x) = \log c_2$$

$$\Rightarrow \quad \log|\sin(x+y)+\cos(x+y)|\,e^{y-x} = \log c_2$$

$$\Rightarrow \quad |\sin(x+y)+\cos(x+y)|\,e^{y-x} = c_2 \qquad \ldots (3)$$

Thus, Equations (2) and (3) together form the complete solution of the given equations.

Example 1.20: Solve the simultaneous equations $\dfrac{xdx}{z^2-2yz-y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$.

Solution: The given equations are $\dfrac{xdx}{z^2-2yz-y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$ … (1)

Choosing Equation (1), y, z as multipliers, we get

$$\frac{xdx}{z^2-2yz-y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$$

$$= \frac{xdx+ydy+zdz}{z^2-2yz-y^2+y(y+z)+z(y-z)} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \quad xdx + ydy + zdz = 0$$

Integrating both sides, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = \dfrac{c_1}{2}$, where c_1 is any arbitrary constant.

NOTES

$\Rightarrow \quad x^2 + y^2 + z^2 = c_1$ … (2)

From last two fractions of Equation (1), we have $\dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

$\Rightarrow \quad (y - z)\, dy = (y + z)\, dz \Rightarrow ydy - (zdy + ydz) - zdz = 0$

Integrating both sides, we get $\dfrac{y^2}{2} - yz - \dfrac{z^2}{2} = \dfrac{c_2}{2}$, where c_2 is any arbitrary constant.

$\Rightarrow \quad y^2 - 2yz - z^2 = c_2$ … (3)

Thus, equations (2) and (3) together form the complete solution of the given equations.

Example 1.21: Solve $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$.

Solution: The given equations are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$ … (1)

From Equation (1), we have

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

Choosing the first two members, we have $\dfrac{dx-dy}{y-x} = \dfrac{dy-dz}{z-y}$

On integrating both sides, we get $\log(y-x) = \log(z-y) + \log c_1$

$\therefore \quad \dfrac{y-x}{z-y} = c_1 \Rightarrow x - y = c_1(y-z)$ … (2)

Again choosing the first and the last members, we have

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) - \log c_2$$

$\Rightarrow \quad (x-y)^2(x+y+z) = c_2$ …(3)

Thus, equations (2) and (3) together form the complete solution of the given equations.

1.4 TOTAL DIFFERENTIAL EQUATION

Let a relation be $f(x, y, z) = c$ where x, y, z are variables and c is a constant.

Differentiating this relation, we get

$df = 0$ Or $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz = 0$ …(1.19)

The general form of the Equation (1.19) in three variables can be written as

Pdx + Qdy + Rdz = 0 …(1.20)

where P, Q and R are functions of x, y and z, respectively.

The equations of the form Equation (1.20) are known as the total differential equations or the single equations in three variables x, y, z.

Also, the equations of the form $P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \ldots + P_n dx_n = 0$ where $P_1, P_2, P_3, \ldots, P_n$ are functions of $x_1, x_2, x_3, \ldots, x_n$, respectively are known as the total differential equations in n variables.

1.4.1 Condition for Integrability

Theorem 1.1: The necessary and the sufficient condition for the integrability of the total differential equation $Pdx + Qdy + Rdz = 0$ is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Proof: Necessary Condition:

The given total differential equation is $Pdx + Qdy + Rdz = 0$ …(1.21)

Let the integral of equation Equation (1.21) be $f(x, y, z) = c$.

So, we have $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$ …(1.22)

Comparing Equation (1.21) and (1.22), we get $\frac{\frac{\partial f}{\partial x}}{P} = \frac{\frac{\partial f}{\partial y}}{Q} = \frac{\frac{\partial f}{\partial z}}{R} = \lambda$ (say)

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \lambda P, \frac{\partial f}{\partial y} = \lambda Q, \frac{\partial f}{\partial z} = \lambda R$$

$$\text{As } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \left[\because \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}\right]$$

$$\therefore \quad \frac{\partial}{\partial y}(\lambda P) = \frac{\partial}{\partial x}(\lambda Q)$$

$$\Rightarrow \quad \lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

$$\Rightarrow \quad \lambda\frac{\partial P}{\partial y} - \lambda\frac{\partial Q}{\partial x} = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \Rightarrow \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \quad \ldots(1.23)$$

Similarly, we can get

$$\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \quad \ldots(1.24)$$

And $\lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x}$ …(1.25)

Multiplying the Equations (1.23), (1.24) and (1.25) by R, P and Q, respectively and then adding we get

$$\lambda\left[R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\right] = 0$$

$$\Rightarrow \qquad P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad \ldots(1.26)$$

which is the necessary condition for the Equation (1.26) to possess an integral Equation (1.22).

This condition can also be written as $\begin{vmatrix} P & Q & R \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0$

Sufficient Condition

Let the coefficients P, Q and R of the Equation (1.21) satisfy the condition (1.26).

Consider the equation Pdx + Qdy = 0 where P and Q are the functions of x and y, respectively.

If this equation is not an exact differential equation, then we can found an integrating factor λ, by which the equation can be multiplied to make the equation exact.

Now, we have Pdx + Qdy = dV

$$\Rightarrow \qquad P\,dx + Q\,dy = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$\therefore \qquad P = \frac{\partial V}{\partial x} \text{ and } Q = \frac{\partial V}{\partial y}$$

$$\Rightarrow \qquad \frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z \partial x} \text{ and } \frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z \partial y}$$

$$\text{Also,} \qquad \frac{\partial P}{\partial y} = \frac{\partial^2 V}{\partial y \partial x} \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial^2 V}{\partial x \partial y}$$

$$\therefore \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \left[\because \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y}\right]$$

Substituting the above values in Equation (1.26), we get

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z \partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial Q}{\partial x}\right)=0$$

$$\Rightarrow \qquad \frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z \partial x}\right)=0$$

$$\Rightarrow \qquad \frac{\partial V}{\partial x}.\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}-R\right)-\frac{\partial V}{\partial y}.\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}-R\right)=0$$

which implies that there exists a relation independent of x and y between V and $\left(\frac{\partial V}{\partial z}-R\right)=R_1$. Thus, R_1 can be expressed as a function of z and V.

$$\therefore \qquad \frac{\partial V}{\partial z} - R = R_1 \Rightarrow \frac{\partial V}{\partial z} - R_1 = R$$

$$\text{Now, } P dx + Q dy + R dz = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \left(\frac{\partial V}{\partial z} - R_1\right) dz$$

$$= dV - R_1 dz$$

$$= dV - f(z, V) dz$$

[$\because$ R_1 is a function of z and V.]

Thus, the Equation (1.21) reduces to $dV - f(z, V)dz = 0$.

Since this is an equation in two variables (z , V) and the solution is of the form $f(z, V) = 0$, thus the condition is sufficient.

Condition to be Satisfied for Exactness

The conditions for the differential equation to be exact are

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

1.4.2 Methods to Solve Total Differential Equations

Inspection method

In this method, first check if $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$ to see whether the condition for integrability is satisfied and then rearrange the terms to make the equation exact. Finally, find the solution.

Example 1.22: Solve the differential equation

$$\left(3x^2 + 2xy - y^2 + z\right)dx + \left(x^2 - 2xy - 3y^2 + z\right)dy + \left(x + y\right)dz = 0$$

Solution: The given differential equation is

$$\left(3x^2 + 2xy - y^2 + z\right)dx + \left(x^2 - 2xy - 3y^2 + z\right)dy + \left(x + y\right)dz = 0 \qquad \ldots(1)$$

Comparing Equation (1) with $P dx + Q dy + R dz = 0$, we get

$$P = 3x^2 + 2xy - y^2 + z, \; Q = x^2 - 2xy - 3y^2 + z \text{ and } R = x + y$$

$$\therefore \qquad \frac{\partial P}{\partial y} = 2x - 2y,\; \frac{\partial Q}{\partial x} = 2x - 2y,\; \frac{\partial Q}{\partial z} = 1,\; \frac{\partial R}{\partial y} = 1,\; \frac{\partial R}{\partial x} = 1 \text{ and } \frac{\partial P}{\partial z} = 1$$

Putting these values in $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$, we get

$$\left(3x^2 + 2xy - y^2 + z\right)(1 - 1) + \left(x^2 - 2xy - 3y^2 + z\right)(1 - 1) + \left(x + y\right)$$
$$\{(2x - 2y) - (2x - 2y)\} = 0$$

$\therefore$ The condition of integrability is satisfied.

Now, Equation (1) can be written as

$$\left(3x^2dx - 3y^2dy\right) + \left(2xydx + x^2dy\right) - \left(y^2dx + 2xydy\right) + zd\left(x+y\right) + \left(x+y\right)dz = 0$$

$$\Rightarrow \qquad d\{x^3 - y^3 + x^2y - xy^2 + z\left(x+y\right)\} = 0$$

NOTES

Integrating, we get $x^3 - y^3 + x^2y - xy^2 + z\left(x+y\right) = c$

which is the solution of the given equation.

Example 1.23: Solve the differential equation $yz\log z\,dx - zx\log z\,dy + xy\,dz = 0$.

Solution: The given differential equation is

$$yz\log z\,dx - zx\log z\,dy + xy\,dz = 0 \qquad ...(1)$$

Comparing Equation (1) with $\text{P}dx + \text{Q}dy + \text{R}dz = 0$, we get

$$\text{P} = yz\log z, \text{Q} = -zx\log z \text{ and } \text{R} = xy$$

$$\therefore \qquad \frac{\partial \text{P}}{\partial y} = z\log z,\ \frac{\partial \text{Q}}{\partial x} = -z\log z,\ \frac{\partial \text{Q}}{\partial z} = -x\log z - x,\ \frac{\partial \text{R}}{\partial y} = x,\ \frac{\partial \text{R}}{\partial x} = y$$

$$\text{and} \qquad \frac{\partial \text{P}}{\partial z} = y\log z + y$$

Putting these values in $\text{P}\left(\frac{\partial \text{Q}}{\partial z} - \frac{\partial \text{R}}{\partial y}\right) + \text{Q}\left(\frac{\partial \text{R}}{\partial x} - \frac{\partial \text{P}}{\partial z}\right) + \text{R}\left(\frac{\partial \text{P}}{\partial y} - \frac{\partial \text{Q}}{\partial x}\right)$, we get

$$\left(yz\log z\right)\left(-x\log z - x - x\right) - zx\log z\left(y - y\log z - y\right) + xy\left(z\log z + z\log z\right)$$

$$= xyz\log z\left(-\log z - 2\right) - xyz\log z\left(-\log z\right) + 2xyz\log z = 0$$

$\therefore$ The condition of integrability is satisfied.

Dividing Equation (1) by $xyz\log z$, we get

$$\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z\log z} = 0$$

Integrating, we get $\log x - \log y + \log\log z = \log c$

$$\Rightarrow \qquad \log\frac{x\log z}{y} = \log c$$

$$\Rightarrow \qquad \frac{x\log z}{y} = c \Rightarrow x\log z = cy$$

which is the solution of the given equation.

Taking one variable as constant from three variables

Step 1: Check if $\text{P}\left(\frac{\partial \text{Q}}{\partial z} - \frac{\partial \text{R}}{\partial y}\right) + \text{Q}\left(\frac{\partial \text{R}}{\partial x} - \frac{\partial \text{P}}{\partial z}\right) + \text{R}\left(\frac{\partial \text{P}}{\partial y} - \frac{\partial \text{Q}}{\partial x}\right) = 0$ to see whether the condition for integrability is satisfied.

Step 2: Take one variable z (say) as constant out of three variables in Pdx + Qdy + Rdz = 0 and differentiate it to get $dz = 0$.

Step 3: Let the solution of Pdx + Qdy = 0 be $u = f(z)$, where $f(z)$ is a function of z and considered as constant with respect to variables x and y.

Step 4: Differentiate $u = f(z)$.

Step 5: Compare the result obtained in Step 4 with $Pdx + Qdy + Rdz = 0$.

Step 6: Eliminate the functions of x and y if the coefficients of df or dz contain functions of x and y. Thus, we obtain $\frac{\partial f}{\partial z}$ which is independent of x and y.

Step 7: Integrate to obtain f, which is the solution of the given equation.

Example 1.24: Solve the total differential equation

$$2yz\,dx + zx\,dy - xy(1+z)\,dz = 0.$$

Solution: The given differential equation is

$$2yz\,dx + zx\,dy - xy(1+z)\,dz = 0 \qquad \ldots(1)$$

Comparing Equation (1) with $Pdx + Qdy + Rdz = 0$, we get

$$P = 2yz, Q = zx \text{ and } R = -xy(1+z)$$

$$\therefore \quad \frac{\partial P}{\partial y} = 2z,\ \frac{\partial Q}{\partial x} = z,\ \frac{\partial Q}{\partial z} = x,\ \frac{\partial R}{\partial y} = -x - xz,\ \frac{\partial R}{\partial x} = -y - yz \text{ and } \frac{\partial P}{\partial z} = 2y$$

Putting these values in $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$, we get

$$2yz(x + x + xz) + zx(-y - yz - 2y) - xy(1+z)(2z - z)$$

$$= 2yz(2x + xz) + zx(-3y - yz) - (xy + xyz)(z)$$

$$= 4xyz + 2xyz^2 - 3xyz - xyz^2 - xyz - xyz^2 = 0$$

$\therefore$ The condition of integrability is satisfied.

Taking z as constant $\Rightarrow dz = 0$

Now, Equation (1) can be written as

$$2yz\,dx + zx\,dy = 0 \Rightarrow 2y\,dx + x\,dy = 0$$

Dividing both sides by xy, we get $2\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating, we get

$$2\log x + \log y = \text{ constant which contains terms of } z$$

$$\therefore \quad \log x^2 + \log y = \log \phi(z)$$

$$\Rightarrow \quad \log x^2 y = \log \phi(z) \Rightarrow x^2 y = \phi(z) \qquad \ldots(2)$$

Differentiating it, we get

$$2xy\,dx + x^2 dy = \phi'(z)\,dz \Rightarrow 2xy\,dx + x^2 dy - \phi'(z)\,dz = 0$$

Multiplying both sides by $\frac{z}{x}$, we get

$$2yz\,dx + xz\,dy - \frac{z}{x}\phi'(z)\,dz = 0 \qquad \ldots(3)$$

Comparing Equations (1) and (3), we get

$$\frac{z}{x}\phi'(z) = xy(1+z)$$

$$\Rightarrow \phi'(z) = x^2y\left(\frac{1+z}{z}\right) \Rightarrow \phi'(z) = \phi(z)\left(\frac{1+z}{z}\right) \qquad \text{[From Equation (2)]}$$

$$\Rightarrow \frac{\phi'(z)}{\phi(z)} = \left(\frac{1}{z}+1\right)$$

Integrating, we get $\log\phi(z) = \log z + z + \log c$

$$\Rightarrow \log\phi(z) = \log z + \log e^z + \log c$$

$$\Rightarrow \phi(z) = cze^z \Rightarrow x^2y = cze^z$$

which is the solution of the equation.

Homogeneous Equations

Step 1: Check if $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$ to see whether the condition for integrability is satisfied.

Step 2: If in the total differential equation $Pdx + Qdy + Rdz = 0$, the P, Q and R are homogeneous functions of x, y and z, then separate one variable z (say) from the other two variables by putting $x = uz$ and $y = vz$, i.e., $dx = zdu + udz$ and $dy = zdv + vdz$.

Step 3: Integrate the reduced equation to find the solution of the given equation.

Example 1.25: Solve the total differential equation $yz(y+z)dx + zx(x+z)dy + xy(x+y)dz = 0$.

Solution: The given differential equation is

$$yz(y+z)dx + zx(x+z)dy + xy(x+y)dz = 0$$

$$\Rightarrow \quad (y^2z + yz^2)dx + (x^2z + xz^2)dy + (x^2y + xy^2)dz = 0 \qquad \ldots(1)$$

Comparing Equation (1) with $Pdx + Qdy + Rdz = 0$, we get

$$P = y^2z + yz^2,\ Q = x^2z + xz^2 \text{ and } R = x^2y + xy^2$$

$$\therefore \quad \frac{\partial P}{\partial y} = 2yz + z^2,\ \frac{\partial Q}{\partial x} = 2xz + z^2,$$

$$\frac{\partial Q}{\partial z} = x^2 + 2xz,\ \frac{\partial R}{\partial y} = x^2 + 2xy,\ \frac{\partial R}{\partial x} = 2xy + y^2$$

and $\quad \frac{\partial P}{\partial z} = y^2 + 2yz$

NOTES

Putting these values in $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$, we get

$$\left(y^2z+yz^2\right)\left(x^2+2xz-x^2-2xy\right)+\left(x^2z+xz^2\right)\left(2xy+y^2-y^2-2yz\right)$$
$$+\left(x^2y+xy^2\right)\left(2yz+z^2-2xz-z^2\right)$$

$$=\left(y^2z+yz^2\right)\left(2xz-2xy\right)+\left(x^2z+xz^2\right)\left(2xy-2yz\right)+\left(x^2y+xy^2\right)\left(2yz-2xz\right)$$

$$=2xy^2z^2+2xyz^3-2xy^3z-2xy^2z^2+2x^3yz+2x^2yz^2$$

$$-2x^2yz^2-2xyz^3+2x^2y^2z+2xy^3z-2x^3yz-2x^2y^2z=0$$

$\therefore$ The condition of integrability is satisfied.

The equation is homogeneous function in x, y and z.

Taking $x=uz$ and $y=vz$

$\Rightarrow dx=z\,du+u\,dz$ and $dy=z\,dv+v\,dz$

Putting these values in Equation (1), we get

$$v(v+1)z^3(z\,du+u\,dz)+u(u+1)z^3(z\,dv+v\,dz)+uvz^3(u+v)dz=0$$

$$\Rightarrow\quad v(v+1)(z\,du+u\,dz)+u(u+1)(z\,dv+v\,dz)+uv(u+v)dz=0$$

$$\Rightarrow\quad v(v+1)z\,du+u(u+1)z\,dv+2uv(u+v+1)dz=0$$

Dividing by $uv(u+v+1)z$, we get

$$\frac{v+1}{u(u+v+1)}du+\frac{u+1}{v(u+v+1)}dv+2\frac{dz}{z}=0$$

$$\Rightarrow\quad \frac{(u+v+1)-u}{u(u+v+1)}du+\frac{(u+v+1)-v}{v(u+v+1)}dv+2\frac{dz}{z}=0$$

$$\Rightarrow\quad \left(\frac{1}{u}-\frac{1}{u+v+1}\right)du+\left(\frac{1}{v}-\frac{1}{u+v+1}\right)dv+2\frac{dz}{z}=0$$

$$\Rightarrow\quad \frac{du}{u}+\frac{dv}{v}-\left(\frac{du+dv}{u+v+1}\right)+2\frac{dz}{z}=0$$

Integrating, we get $\log u+\log v-\log(u+v+1)+2\log z=\log c$

$$\Rightarrow\quad \log u+\log v-\log(u+v+1)+\log z^2=\log c$$

$$\Rightarrow\quad \log\frac{uvz^2}{u+v+1}=\log c$$

$$\Rightarrow\quad uvz^2=c(u+v+1)$$

$$\Rightarrow\quad \frac{x}{z}.\frac{y}{z}.z^2=c\left(\frac{x}{z}+\frac{y}{z}+1\right)\Rightarrow xyz=c(x+y+z)$$

which is the solution of the given equation.

NOTES

Auxiliary Equation

This method is used in case when the differential equation $Pdx + Qdy + Rdz = 0$ is not exact and the methods discussed above are not convenient.

Step 1: Check if $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$ …(1.27)

to see whether the condition for integrability is satisfied.

Step 2: Compare Equation (1.27) with Pdx + Qdy + Rdz = 0 to obtain the auxiliary equations

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

Step 3: Solve the auxiliary equation obtained in Step 2 just like simultaneous equations.

Step 4: Let the integrals of the auxiliary equations be $u = a$ and $v = b$.

Step 5: Compare $Adu + Bdv = 0$ with the given differential equation to find the values of A and B.

Step 6: Put the values of A and B in $Adu + Bdv = 0$.

Step 7: Integrate the equation obtained in Step 6 to find the solution of the given equation.

Example 1.26: Solve the differential equation $z(z-y)dx + (z+x)zdy + x(x+y)dz = 0$

Solution: The given differential equation is

$$z(z-y)dx + (z+x)zdy + x(x+y)dz = 0 \quad ...(1)$$

Comparing Equation (1) with $Pdx + Qdy + Rdz = 0$, we get

$$P = z(z-y) = z^2 - zy,\; Q = (z+x)z = z^2 + xz \text{ and } R = x(x+y) = x^2 + xy$$

$$\therefore \quad \frac{\partial P}{\partial y} = -z,\; \frac{\partial Q}{\partial x} = z,\; \frac{\partial Q}{\partial z} = 2z + x,\; \frac{\partial R}{\partial y} = x,\; \frac{\partial R}{\partial x} = 2x + y \text{ and } \frac{\partial P}{\partial z} = 2z - y$$

Putting these values in $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$, we get …(2)

$$(z^2 - zy)(2z + x - x) + (z^2 + xz)(2x + y - 2z + y) + (x^2 + xy)(-z - z)$$

$$= (z^2 - zy)(2z) + (z^2 + xz)(2x + 2y - 2z) - (x^2 + xy)(2z)$$

$$= 2z\left[z^2 - yz + x^2 - z^2 + xy + yz - x^2 - xy\right] = 0$$

$\therefore$ The condition of integrability is satisified.

Comparing Equation (2) with $Pdx + Qdy + Rdz = 0$, we get

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

NOTES

$$\Rightarrow \quad \frac{dx}{2z} = \frac{dy}{2x+2y-2z} = \frac{dz}{-2z}$$

$$\Rightarrow \quad \frac{dx}{z} = \frac{dy}{x+y-z} = \frac{dz}{-z} \qquad \ldots(3)$$

From first and last members, we get $\frac{dx}{z} = \frac{dz}{-z}$

$\Rightarrow \quad dx + dz = 0$

Integrating, we get $x + z = u$

From Equation (3), $\frac{dx+dy}{x+y} = \frac{dz}{-z}$

Integrating, we get $\log(x+y) + \log z = \text{constant}$

$\Rightarrow \quad \log(x+y)z = \log v$

$\Rightarrow \quad (x+y)z = v$

If $\text{A}\,du + \text{B}\,dv = 0$ is identical with Equation (1), then

$\text{A}(dx+dz) + \text{B}\left[(x+y)dz + (dx+dy)z\right] = 0$ is identical with Equation (1)

$\Rightarrow \quad (\text{A} + z\text{B})dx + \text{B}z\,dy + \left[\text{A} + \text{B}(x+y)\right]dz = 0$ is identical with Equation (1)

Comparing it with Equation (1), we get

$\text{A} + z\text{B} = z(x-y)$, $\text{B}z = (z+x)z$ and $\text{A} + \text{B}(x+y) = x(x+y)$

$\therefore \quad \text{B} = (z+x) = u$ and

$\text{A} = (x+y)(x-\text{B}) = (x+y)(x-z-x) = -(x+y)z = -v$

By putting the value of A, B in $\text{A}du + \text{B}dv = 0$, we get

$\Rightarrow \quad -v\,du + u\,dv = 0 \Rightarrow -\frac{du}{u} + \frac{dv}{v} = 0$

Integrating, we get

$$-\log u + \log v = \log c$$

$$\Rightarrow \quad \log\frac{v}{u} = \log c$$

$$\Rightarrow \quad \frac{v}{u} = c \Rightarrow v = cu \Rightarrow (x+y)z = c(x+z)$$

which is the solution of the given equation.

1.4.3 Solution of Exact and Homogeneous Total Differential Equations

Theorem 1.2: The solution of the total differential equation Pdx + Qdy + Rdz = 0 when it is exact and homogeneous of degree $n \neq -1$ is $x\text{P} + y\text{Q} + z\text{R} = c$.

Proof: Consider the total differential equation

$$\text{P}dx + \text{Q}dy + \text{R}dz = 0 \qquad \ldots(1.28)$$

where coefficients P, Q and R are homogeneous functions of x, y and z of degree $n \neq -1$.

Now,

$$\left.\begin{aligned} x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z}=nP \\ x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}+z\frac{\partial Q}{\partial z}=nQ \\ x\frac{\partial R}{\partial x}+y\frac{\partial R}{\partial y}+z\frac{\partial R}{\partial z}=nR \end{aligned}\right\} \quad \ldots(1.29)$$

[By Euler's Theorem]

Suppose the solution of equation (1.28) is given by

$x\mathrm{P} + y\mathrm{Q} + z\mathrm{R} = c$ …(1.30)

Differentiating Equation (1.30), we get

$$\left(P+x\frac{\partial P}{\partial x}+y\frac{\partial Q}{\partial x}+z\frac{\partial R}{\partial x}\right)dx+\left(x\frac{\partial P}{\partial y}+Q+y\frac{\partial Q}{\partial y}+z\frac{\partial R}{\partial y}\right)dy+\left(x\frac{\partial P}{\partial z}+y\frac{\partial Q}{\partial z}+R+z\frac{\partial R}{\partial z}\right)dz=0 \quad \ldots(1.31)$$

As equation (1.28) is exact, so the conditions for exactness must be satisfied, i.e.,

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$$

Putting these values in equation (1.31), we get

$$\left(P+x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z}\right)dx+\left(x\frac{\partial Q}{\partial x}+Q+y\frac{\partial Q}{\partial y}+z\frac{\partial Q}{\partial z}\right)dy+\left(x\frac{\partial R}{\partial x}+y\frac{\partial R}{\partial y}+R+z\frac{\partial R}{\partial z}\right)dz=0$$

$$\Rightarrow\ (P+nP)dx+(Q+nQ)dy+(R+Rn)dz=0 \qquad \text{[Using Equation (1.29)]}$$

$$\Rightarrow\ (n+1)(P\,dx+Q\,dy+R\,dz)=0 \Rightarrow\ P\,dx+Q\,dy+R\,dz=0$$

So, the assumption is true and hence the solution of the Equation (1.28) is $x\mathrm{P} + y\mathrm{Q} + z\mathrm{R} = c$.

Example 1.27: Solve the differential equation

$$\left(y^2+z^2+2xy+2xz\right)dx+\left(x^2+z^2+2xy+2yz\right)dy+(x^2+y^2+2xz+2yz)dz=0$$

Solution: The given differential equation is

$$\left(y^2+z^2+2xy+2xz\right)dx+\left(x^2+z^2+2xy+2yz\right)dy+(x^2+y^2+2xz+2yz)dz=0 \quad \ldots(1)$$

Comparing Equation (1) with $\mathrm{P}dx+\mathrm{Q}dy+\mathrm{R}dz=0$, we get

$$P=y^2+z^2+2xy+2xz,\ \ Q=x^2+z^2+2xy+2yz \text{ and}$$

$$R=x^2+y^2+2xz+2yz$$

$$\therefore \qquad \frac{\partial P}{\partial y}=2y+2x,\ \frac{\partial Q}{\partial x}=2y+2x,\ \frac{\partial Q}{\partial z}=2z+2y,\ \frac{\partial R}{\partial y}=2z+2y,$$

$$\frac{\partial R}{\partial x}=2x+2z \text{ and } \frac{\partial P}{\partial z}=2x+2z$$

NOTES

$\therefore$ The given equation is exact and P, Q and R are homogeneous functions in x, y and z.

The solution of the given equation is $x\text{P} + y\text{Q} + z\text{R} = c$

$$\therefore \; x\left(y^2 + z^2 + 2xy + 2xz\right) + y\left(x^2 + z^2 + 2xy + 2yz\right) + z(x^2 + y^2 + 2xz + 2yz) = c$$

$$\Rightarrow \; xy^2 + xz^2 + 2x^2y + 2x^2z + x^2y + z^2y + 2xy^2 + 2y^2z + x^2z + y^2z + 2xz^2 + 2yz^2 = c$$

$$\Rightarrow \; 3xy^2 + 3xz^2 + 3x^2y + 3x^2z + 3y^2z + 3yz^2 = c$$

$$\Rightarrow \; xy^2 + xz^2 + x^2y + x^2z + y^2z + yz^2 = c_1$$

$$\Rightarrow \; x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2) = c_1$$

which is the solution of the given equation.

Check Your Progress

8. $\dfrac{dx}{dt} - 7x + y = 0$ and $\dfrac{dy}{dt} - 2x - 5y = 0$.
9. $\dfrac{dx}{dt} + 5x + y = e^t$ and $\dfrac{dy}{dt} - x + 3y = e^{2t}$
10. $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x$.
11. $\dfrac{adx}{(b - c)yz} = \dfrac{bdy}{(c - a)zx} = \dfrac{cdz}{(a - b)xy}$
12. $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z(x^3 - y^3)}$
13. $\dfrac{dx}{x+y} = \dfrac{dy}{-(x+y)} = \dfrac{dz}{z}$
14. $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2y^2z^2}$
15. Solve the following differential equations by the inspection method:
    (a) $\left(yz + xyz\right)dx + \left(zx + xyz\right)dy + \left(xy + xyz\right)dz = 0$
    (b) $z\left(1 - z^2\right)dx + zdy - \left(x + y + xz^2\right)dz = 0$.
16. Solve the differential equation $3x^2dx + 3y^2dy - \left(x^3 + y^3 + e^{2z}\right)dz = 0$ by regarding one variable as constant.

1.5 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $y = c_1\cos x + c_2 \sin x - \cos x \log (\sec x + \tan x)$
2. $y = c_1\cos x + c_2 \sin x + x \sin - x + \cos x \log(\cos x)$
3. $- y = c_1e^{2x} + c_2e^{-2x} + e^2x$-
4. $y = c_1\cos x + c_2 \sin x$-$\cos x + \sin x$

5. $y = c_1e^x + c_2e^{2x} + (3\cos x + \sin x)$
6. $y = c_1\cos x + c_2\sin x + x\cos x + \sin x \log(\sec x) - \sin x$
7. $y = c_1\cos 3x + c_2 \sin 3x + \cos 3x \log(\cos 3x) + \sin 3x$
8. $x = e^{6t}(c_1\cos t + c_2\sin t)$; $y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$
9. $x = (c_1 + c_2t)e{-}^4 + \frac{4}{25}e^t - \frac{e^{2t}}{36}$; $y = -\{c_1 + c_2(t+1)\}e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}$
10. $x = c_1e^{2t} + c_2e^{-t}\cos(\sqrt{3}t + c_3)$;

$y = c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t + c_3 + \frac{2\pi}{3}\right)$ and

$z = c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t + c_3 + \frac{4\pi}{3}\right)$

15. (a) $\log xyz + (x+y+z) = c$ (b) $\frac{x+y}{z} - xz = c$
16. $x^3 + y^3 = e^{2z} + ce^z$

1.6 SUMMARY

- Any homogeneous differential equation of the from.

$$x^n \frac{d^n y}{dx^n} + p_1x^n + \frac{d^{n-1} y}{dx^{n-}1} + p_2x^{n-2}\frac{d^{n-2} y}{dx^n - 2}......p_n y_z = \phi$$

is called a homogeneous linear differential equation of nth order, where p_1, p_2 - p_n are caustants and Q is a function of x.

- Homogeneous linear differential equation is reducible to linear differntial equation with constant coefficient by subsituation

$$x = e^z \therefore z = \log x$$

$$or \frac{dz}{dn} = \frac{1}{x}$$

- A differential equation of the form

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

Is said to be linear homogeneous differential equation with variable coefficients where $a_1(x)$ and $a_2(x)$ are continuous function in the interval $[a, b]$.

- For second order homogeneous differential equation there is no general method for finding a particular solution.

- The equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved by using two sets of multipliers to get the complete solutions. This equation can also be solved by taking any

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two members and then integrating them. Again we need to take other two members and integrate. These two equations are thus obtained form the complete solution.

- x or y can also be eliminated by differentiating one of the given equations or both. Then, we can solve the resulting equation to get other variables.
- Simultaneous equations of the type $P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0$ and $P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0$ can always be put in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where P, Q, R are functions of x, y and z.
- By eliminating x, we can get a linear differential equation in y and t which when solved gives the value of y in terms of t. Further, the value of y can be obtained by putting the value of x in the given equation.
- We need as many numbers of simultaneous differential equations as are the number of dependent variables to solve such type of equation.
- The solution of the total differential equation $Pdx + Qdy + Rdz = 0$, which is exact and homogeneous of degree $n \neq -1$, is $xP + yQ + zR = c$.
- There are various methods to solve total differential equations, such as inspection method, taking one variable as constant out of three variables in $Pdx + Qdy + Rdz = 0$, in case of homogeneous equations and auxiliary equations. In all these methods, first of all we need to verify the condition of integrability and then follow the steps specific to each method.
- The conditions for the differential equation to be exact are

  $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$
- The necessary and the sufficient condition for the integrability of the total differential equation $Pdx + Qdy + Rdz = 0$ is

  $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

1.7 KEY TERMS

- **Homogeneous linear equation:** Any homogeneous differential equation of the from.

  $$x^n \frac{d^n y}{dx^n} + p_1 x^n + \frac{d^{n-1} y}{dx^{n-}1} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^n - 2} ......p_n y_z = \phi$$

  is called a homogeneous linear differential equation of nth order, where p_1, p_2 - p_n are caustants and Q is a function of x.
- **Linear differential equation:** It is a differential equation in which the dependent variable and all its derivatives appear only in the first degree and are not multiplied together.

- **Total differential equations in n variables:** The equations of the form $P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \ldots + P_n dx_n = 0$, where $P_1, P_2, P_3, \ldots, P_n$ are functions of $x_1, x_2, x_3, \ldots, x_n$, respectively, are known as the total differential equations in n variables.
- **Total differential equations:** The equations of the form $Pdx + Qdy + Rdz = 0$, where P, Q and R are functions of x, y and z, respectively, are known as the total differential equations or the single equations in three variables x, y, z.

1.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is homogeneous linear equation?
2. Give the solution of homogeneous linear equation.
3. Define the method of variation of parameters.
4. Solve the following differential equations by the homogeneous equation method.

   (i) $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$

   (ii) $(x^2 y - y^3 - y^2 z)dx + (xy^2 - x^2 z - x^3)dy + (xy^2 + x^2 y)dz = 0$

   (iii) $yz^2(x^2 - yz)dx + zx^2(y^2 - zx)dy + xy^2(z^2 - xy)dz = 0$

   (iv) $z^2 dx + (z^2 - 2yz)dy + (2y^2 - yz - xz)dz = 0$

   (v) $(2z^2 - xy + y^2)zdx + (2z^2 + x^2 - xy)zdy - (x + y)(xy + z^2)dz = 0$

5. Solve the following differential equations by taking one variable as constant from the three variables method:

   (i) $yzdx + (x^2 y - zx)dy + (x^2 z - xy)dz = 0$

   (ii) $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$

   (iii) $z(x^2 - yz - z^2)dx + (x + z)dy + x(z^2 - x^2 - xy)dz = 0$

   (iv) $(y + b)(z + c)dx + (x + a)(z + c)dy + (x + a)(y + b)dz = 0$

   (v) $(e^x y + e^z)dx + (e^y z + e^x)dy + (e^y - e^x y - e^y z)dz = 0$

5. Solve the following differential equations by the inspection method:

   (i) $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$

   (ii) $(y^2 + z^2 - x^2)dx - 2xy\,dy - 2xz\,dz = 0$

   (iii) $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$

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(iv) $\left(x^2y - y^3 - y^2z\right)dx + \left(xy^2 - x^2z - x^3\right)dy + \left(xy^2 + x^2y\right)dz = 0$

(v) $\left(y+z\right)\left(x^2y^2z^2 - 1\right)dx + x^2y^2z^2\left(xy + xz\right)\left(\dfrac{dy}{y} + \dfrac{dz}{z}\right) + x\left(dy + dz\right) = 0$

Long-Answer Questions

1. Discuss briefly about the homogeneous linear equation with the help of giving examples.
2. Solve the following simultaneous equations:

(i) $\dfrac{dx}{dt} = 3x + 2y; \dfrac{dy}{dt} + 5x + 3y = 0$

(ii) $\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} - x = e^t \cos t; \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} - y = e^t \sin t$

(iii) $\dfrac{dx}{dt} = ax + by; \dfrac{dy}{dt} = a'x + b'y$

(iv) $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2y = 2\cos t - 7\sin t; \dfrac{dx}{dt} - \dfrac{dy}{dt} + 2x = 4\cos t - 3\sin t$

(v) $\dfrac{dx}{dt} + 5x - 2y = t, \dfrac{dy}{dt} + 2x + y = 0$; given that $x = y = 0$ when $t = 0$

(vi) $2\dfrac{d^2y}{dx^2} - \dfrac{dz}{dx} - 4y = 2x; 2\dfrac{dy}{dx} + 4\dfrac{dz}{dx} - 3z = 0$

(vii) $\dfrac{d^2x}{dt^2} + 4x + 5y = t^2; \dfrac{d^2y}{dt^2} + 5x + 4y = t + 1$

(viii) $\dfrac{ldx}{(m-n)yz} = \dfrac{mdy}{(n-l)zx} = \dfrac{ndz}{(l-m)xy}$

(ix) $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)}$

(x) $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

(xi) $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$

(xii) $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2(x^2-y^2)}$

(xiii) $\dfrac{-dx}{x(x+y)} = \dfrac{dy}{y(x+y)} = \dfrac{dz}{(x-y)(2x+2y+z)}$

(xiv) $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2 \sin(y+2x)}$

3. Solve the following differential equations by the auxiliary equation method:

   (i) $\left(2xyz + y^2 z + yz^2\right)dx + \left(x^2 z + 2xyz + xz^2\right)dy + \left(x^2 y + xy^2 + 2xyz\right)dz = 0$

   (ii) $z^2 dx + \left(z^2 - 2yz\right)dy + \left(2y^2 - yz - xz\right)dz = 0$

   (iii) $\left(z + z^2\right)\cos x \dfrac{dx}{dt} - \left(z + z^2\right)\dfrac{dy}{dt} + \left(1 - z^2\right)\left(y - \sin x\right)\dfrac{dz}{dt} = 0$

   (iv) $\left(2x^2 y + 2xy^2 + 2xyz + 1\right)dx + \left(x^3 + x^2 y + x^2 z + 2xyz + 2y^2 z + 2yz^2 + 1\right)dy$
   $+ \left(xy^2 + y^3 + y^2 z + 1\right)dz = 0$

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UNIT 2 PICARD'S METHOD OF INTEGRATION AND SUCCESSIVE APPROXIMATION

Structure

2.0 INTRODUCTION

In mathematics, Picard's method of integration for solving differential equations in many of the Engineering problems, we are often confronted with the differential equations whose solution cannot be obtained by standard techniques. For such problems, we find an approximation solution using the Picard's iteration method which gives an approximation solution of the initial value problem. The existence and uniqueness theorems are specifically used for solving differential equations when any differential equation cannot be solved using standard methods. The system of differential equations for local and nonlocal existence theorems for nth order equations. There are many instances where a physical problem is represented by differential equations with initial or boundary conditions. Existence of solutions in the large is also known as nonlocal existence. Approximate solutions are arrived at using approximations. Approximate solutions of differential equations can be formulated by obtaining the analytic expressions (formulas) or numerical values that approximate the desired solution of a differential equation to some degree of accuracy. If a solution is represented by means of an infinite series, a finite portion of the series can be taken as the approximate solution.

An existence theorem is a theorem which asserts the existence of a certain object. It might be a statement which begins with the phrase 'There exist(s)', or it might be a universal statement whose last quantifier is existential. In the formal terms of symbolic logic, an existence theorem is a theorem with a prenex normal form involving the existential quantifier, even though in practice, such theorems are usually stated in standard mathematical language. A uniqueness theorem is a mathematical theorem that asserts the uniqueness of an entity that meets particular circumstances, or the equivalence of all objects that meet those conditions. The Picard–Lindelöf theorem, which proves the uniqueness of solutions to first-order differential equations, is an example of a uniqueness theorem.

In this unit, you will learn about the Picard's method of integration, existence theorem, uniqueness theorem and existence and uniqueness theorem proof's by Picard's method.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain Picard's iteration method
- Define the significance of successive approximations
- Understand the various methods of successive approximations
- Find solution of a given differential equation using Picard's iteration method
- Describe the existence and uniqueness of initial value problems
- Identify the Lipschitz condition
- State the existence and uniqueness theorems
- Understand the system of differential equations
- Define the equations for local and nonlocal existence theorems for nth order equations
- Approximate the error using the error approximation theorem
- Find the existence and uniqueness solutions for linear systems
- Explain about the uniqueness theorem
- Discuss about the existence and uniqueness theorem - proofs by Picard's method

2.2 PICARD'S METHOD OF INTEGRATION

Picard's method of solving a differential equation (initial value issues) is an iterative method in which the numerical answers grow more and more accurate the more times it is employed. Finding the solution to a differential equation might be challenging at times. The approximate solution of a given differential equation can be obtained in such instances.

Picard's iteration method was first used to show that an initial value problem exists. Despite the fact that this approach is not practical and is rarely utilized for actual determination of a solution to the initial value problem (due to sluggish convergence and difficulties with doing explicit integrations), there are known enhancements in this procedure that make it feasible. Picard's iteration approach is significant because it yields an equivalent integral formulation that may be used to build a variety of numerical algorithms.

When a user applies Picard's iteration method in a computer, it aids in the development of algorithmic thinking. It was the first method for solving nonlinear differential equations analytically. Everyone can improve their computing skills by working with Picard's iterations and revisions.

2.2.1 Successive Approximation

In many of the Engineering problems, we are often confronted with the differential equations whose solution cannot be founded by standard methods.

In such problems, it is sufficient to obtain an approximation solution only, We shall mention here the Picards iteration method for giving an approximation solution of the initial value problem of the form,

$$\frac{dy}{dx} = f\left(x_1 y\right), \quad y(x_0) = y_0 \tag{2.1}$$

By the initeration method we mean a method which consists of repeated application of exactly the same type of steps where in each step is picards method. By integration, we may write Equation (2.1) in the form.

$$y(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y(t)\right] dt \tag{2.2}$$

Where t is the variable of integration. It is easy to check that the integral is zero when $x = x_0$, so that $y = y_0$. Thus Equation (2.2) satisfies the intial condition in Equation (2.1). Also if we differentiale Equation (2.2), we obtained the given differential equation.

In order to obtain a solution y (x) of Equation (2.2), we proceed stepwise as follows:

Put $y = y_0$ = Constant. on the right. This gives,

$$y_1(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y(t)\right] dt$$

We now substitute $y_1(x)$ in the same manner and get

$$y_2(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_1(t)\right] dt .$$

Continuing in this way at the 4th step of itegration process, we get,

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_{n-1}(t)\right] dt \tag{2.3}$$

Thus we obtain a sequence of approximation solutions.

$y_1(x)_1 = y_2(x)_1 .... y_n(x)$

Example 2.1: Apply Picard's iteration method of intial value problem.

$$\frac{dy}{dx} = y_1 \quad t(0) = 1$$

and show that the successive approximations tends to the limit $y = e^x$, the exact solution.

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Solution: Here $x_0 = 0, y_0 = 1, \quad f(x_1 y) = y$

So that Equation $y_n(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_{n-1}(t)\right] dt$ of prceeding section becomes,

$$y_n(x) = y_0 + \int_{x_0}^{x} y_{n-1}(t)\, dt$$

Taking $y_0 = 1,$ we obtain

$$y_1(x) = y_0 + \int_{x_0}^{x} y_0(x)\ dt = 1 + \int_0^x | \cdot dt = 1 + x$$

$$y_2(x) = y_0 + \int_{x_0}^{x} y_1(t)\ dt = 1 + \int_0^x (1+t)\ dt = 1 + x + \frac{x^2}{2}$$

$$y_3(x) = y_0 + \int_{x_0}^{x} y_2(t)\ dt = 1 + \int_0^x (1 + t + \frac{t^2}{2})\ dt$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

In general, we get,

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Thus the successive approximation tend to the limit $y = e^x$ as $n \to \infty$, which is the exact solution.

Example 2.2: Apply Picard's method of the initial value problem.

$$\frac{dy}{dx} = xy + 1, \quad y(0) = 0$$

and find the successive approximation.

Solution: Here $x_0 = 0, \quad y_0 = 0, \quad f(x_1\ y) = xy + 1,$ So that

Equation $y_n(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_{n-1}(t)\right] dt$ of preceeding section becomes,

$$y_n(x) = y_0 + \int_{x_0}^{x} \left[t \cdot y_{n-1}(t) + 1\right] dt$$

Starting from $x_0 = 0$ and $y_0 = 0$, we get

$$y_1(x) = y_0 + \left[t\ y_0(t) + 1\right] dt = \int_0^x dt = x$$

$$y_2(x) = y_0 + \left[t\ y_1(t) + 1\right] dt = \int_0^x (t^2 + 1) dt$$

$$= \frac{x^3}{3} + x$$

$$y_3(x) = y_0 + \left[t\ y_2(t) + 1\right] dt$$

$$= \int_0^x \left[t\left[t + \frac{t^3}{3}\right] + 1\right] dt$$

$$= x + \frac{x^3}{3} + \frac{x^5}{3.5}$$

Hence, we have,

$$y_0 = 0,\ y_1 = x,\ y_2 = x + \frac{x^3}{3},\ y_3 = x + \frac{x^3}{3} + \frac{x^5}{3.5} \ etc.$$

Example 2.3: Using Picard's method find the third approximation of the solution of the equation,

$$\frac{dy}{dx} = 2y - 2x^3 - 3 \text{ where } y = 2 \text{ when } x = 0$$

Solution: Here $x_0 = 0,\ y_0 = 2$

And $f(x, y) = 2u - 2x^2 - 3$

We have $y_n(x) = y_0 + \int_{x_0}^x \left[2y_{n-1}(t) - 2t^2 - 3\right] dt$

First appoximation : Taking $x_0 = 0$ and $y_0 = 2$

$$y_1(x) = y_0 + \int_0^x (2y_0 - 2t^2 - 3) dt$$

$$= 2 + \int_0^x (4 - 2t^2 - 3) dt = 2 + x - \frac{2}{3}x^3$$

Second appoximation:

$$y_2(x) = 2 + \int_0^x (2y_1 - 2t^2 - 3)\, dt$$

$$2 + \int_0^x \left(4 + 2t + 2t^2 - \frac{2}{3}t^4 - \frac{4}{3}t^3 - 2t^3 - 3\right) dt$$

$$= 2 + x + x^3 - \frac{1}{3}x^4 - \frac{2}{15}x^5$$

Example 2.4: Find the third approximation of the solution of the equation,

$$\frac{dy}{dx} = 2 - \frac{y}{x}$$

By Picard's method, where $y = 2$ when $x = 1$

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Solution: $x_0 = 1,\ y_0 = 2$ and $f(x, y) = 2 - \dfrac{y}{x}$

We have $y_n(x) = y0 + \int_{x_0}^{x}\left(2 - \dfrac{y_{n-1}(t)}{t}\right)dt_1$

First approximation: Taking $x_0 = 1$ and $y_0 = 2$

$$y_1(x) = y_0 + \int_{x_0}^{x}\left(2 - \frac{y_0}{t}\right)dt = 2 + \int_{1}^{x}\left(2 - \frac{2}{t}\right)dt$$

$$= 2 + [2x - 2\log t]_1^x = 2x - 2\log x$$

Second approximation:

$$y_2(x) = 2 + \int_1^x\left[2 - \frac{1}{t}y_1(t)\right]dt$$

$$= 2 + \int_1^x\left[2 - \frac{1}{t}(2t - 2\log t)\right]dt$$

$$= 2 + \int_1^x \frac{2}{t}\log t\ dt = 2 + (\log x)^2$$

Third approximation:

$$y_3(x) = 2 + \int_1^x\left(2 - \frac{1}{t}y_2(t)\right)dt = 2 + \int_1^x\left[2 - \frac{1}{t}\left\{2 + (\log t)^2\right\}\right]dt$$

$$= 2 + \int_1^x\left[2 - \frac{2}{t} - \frac{1}{t}\left\{(\log t)^2\right\}\right]dt$$

$$= 2x - 2\log x - \frac{1}{3}(\log x)^3$$

Example 2.5: Apply Picard's method to find third approximation of the solution of the equation,

$$\frac{dy}{dx} = x + y^2,\ \text{where } y = 0,\ \text{when } x = 0$$

Solution: Here $x_0 = 0,\ y_0 = 0$ and $f(x, y) = x + y^2$

We have $y_n(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_{n-1}(t)\right]dt$

$$\therefore \qquad y_n(x) = \int_0^x\left[t + y_{n-1}^2(t)\right]dt \qquad ...(1)$$

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First approximation: From Equation (1), we have

$$y_1(x) = \int_0^x \left[t + y_0^2(t)\right] = \int_0^x t\, dt = \frac{x^2}{2}$$

Second approximation: From Equation (1), we have

$$y_2(x) = \int_0^x \left[t + y_1^2(t)\right] dt = \int_0^x \left(t + \tfrac{1}{4}t^4\right) dt$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

Third approximation: From Equation (1), we have

$$y_3(x) = \int_0^x \left[t + y_2^2\ (t)\right] dt = \int_0^x \left[t + \left(\frac{t^2}{2} + \frac{t^5}{20}\right)\right] dt$$

$$= \int_0^x \left[t + \frac{t^4}{4} + \frac{t^7}{20} + \frac{t^{10}}{400}\right] dt$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

Example 2.6: Find the third approximation of the solution of the equation,

$$\frac{dy}{dx} = Z_1 \quad \frac{dz}{dx} = x^2 z + x^4 y$$

By Picard's method, $y = 5$ and $z = 1$ when $x = 0$

Solution: Here the given simultaneous equations are,

$$\frac{dy}{dx} = Z = f(x, y, z); \quad \frac{dz}{dx} = x^2 2 + x^4 y = g(x, y, z)$$

And $x_0 = 0,\ y_0 = 5,\ z_0 = 1$

$$y_n = y_0 + \int_{x_0}^x f\left[t,\ y_{n-1}(t),\ z_{n-1}(t)\right] dt$$

$$= y_0 + \int_{x_0}^x Z_{n-1}(t)\ dt$$

And $Z_n = Z_0 + \int_{x_0}^x g\left[t,\ y_{n-1}(t)\right] Z_{n-1}(t)\ dt$

$$= Z_0 + \int_{x_0}^x \left[t^2,\ Z_{n-1}(t) + t^4\ \ y_{n-1}(t)\right] dt$$

First approximation: Taking $y_0 = 5,\ Z_0 = 1$ and $x_0 = 0$

$$y_1 = y_0 + \int_{x_0}^x Z_0(t)\ dt = 5 + \int_0^x 1 \cdot dt = 5 + x$$

$$Z_1 = Z_0 + \int_{x_0}^x \left[t^2 Z_0(t) + t^4 \cdot y_0(t)\right] dt$$

$$= 1 + \int_0^x \left(t^2 \cdot 1 + t^4 \cdot 5\right) dt = 1 + \frac{x^3}{3} + x^5$$

NOTES

Second approximation:

$$y_2 = y_0 + \int_0^x Z_1(t)\,dt = 5 + \int_0^x \left(1 + \frac{t^3}{3} + t^5\right) dt$$

$$= 5 + x + \frac{x^4}{12} + \frac{x^6}{6}$$

$$Z_2 = Z_0 + \int_0^x \left[t^2 Z_1\ (t) + t^4\ y_1(t)\right] dt$$

$$= 1 + \int_0^x \left[t^2\left(1 + \frac{t^3}{3} + t^5\right) + t^4(5+t)\right] dt$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8}$$

Third approximation:

$$y_3 = y_0 + \int_0^x Z_2(t)\,dt = 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2}{63}x^7 + \frac{x^9}{72}$$

$$Z_3 = Z_0 + \int_0^x \left[t^2 Z_2(t) + t^4 y_4(t)\right] dt$$

$$= 1 + \int_0^x \left[t^2\left(1 + \frac{t^3}{3} + \frac{t^5}{1} + \frac{2t^6}{9} + \frac{t^8}{8}\right) + t^4\left(5 + t + \frac{t^4}{12} + \frac{t^6}{6}\right)\right] dt$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8} + \frac{11}{224}x^9 + \frac{7}{264}x^{11}$$

Example 2.7: Find the third approximation of the solution of the equation,

$$\frac{dy}{dx}Z, \quad \frac{dz}{dx} = x^3(x+2)$$

By Picard's method where $y = 1, Z = ½$, when $x = 0$.

Solution: Here the given simultaneous equations are,

$$\frac{dy}{dx} = Z\, f(x, y, z)$$

$$\frac{dy}{dx} = x^3(y+z) = g(x, y, z) \text{ and } x = 0,\ y_0 = 1,\ Z_0 = ½$$

$$y_n = y_0 + \int_{x_0}^x f\left[t,\ y_{n-1}(t),\ Z_{n-1}(t)\right] dt$$

$$= y_0 + \int_{x_0}^x Z_{n-1}(t)\ dt$$

And $Z_n = Z_0 + \int_{x_0}^x g\left[t,\ y_{n-1}(t), Z_{n-1}(t)\right] dt$

$$= Z_0 + \int_{x_0}^x t^3\left[y_{n-1}(t) + 2_{n-1}(t)\right] dt$$

NOTES

First approximation: Taking $x_0 = 0,\ y_0 = 1,\ Z_0 = x/2$

$$Z_1 = Z_0 + \int_{x_0}^{x} t^3 (y_0 + Z_0)\, dt = \frac{1}{2} + \int_0^x t^3 (1 + 1/2)\, dt$$

$$= \frac{1}{2} + \frac{3}{8}x^4$$

Second approximation:

$$y_2 = y_0 + \int_{x_0}^{x} Z_1(t)\, dt = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8}t^4\right) dt$$

$$= 1 + \frac{1}{2}x + \frac{3}{40}x^5$$

$$Z_2 = Z_0 + \int_0^x \left[t^3 y_,(t) + Z_1(t)\right] dt$$

$$= \frac{1}{2} + \int_0^x \left[t^3 y_1(t) + Z_1(t)\right] dt$$

$$= \frac{1}{2} + \int_0^x t^3 \left(\frac{3}{2} + \frac{t}{2} + \frac{3}{8}t^4\right) dt$$

$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3}{64}x^8$$

Third approximation:

$$y_3 = y_0 + \int_{x0}^{x} Z_2(t)\, dt$$

$$= 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8}t^4 + \frac{1}{10}t^5 + \frac{3}{64}t^8\right) dt$$

$$= 1 + \frac{1}{2}x + \frac{3}{40}x^5 + \frac{1}{60}x^6 + \frac{1}{192}x^9$$

$$Z_3 = Z_0 + \int_{x_0}^{x} \left[t^3 y_2(t) + Z_2(t)\right] dt$$

$$= \frac{1}{2} + \int_0^x t^3 \left(\frac{3}{2} + \frac{t}{2} + \frac{3}{8}t^4 + \frac{7}{40}t^5 + \frac{3}{64}t^8\right) dt$$

$$= \frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{64}x^8 + \frac{7}{360}x^9 + \frac{1}{256}x^{12}$$

Example 2.8: Find the third approximation of the solution of the equation,

$$\frac{d^2 y}{dx^2} = x^2 \frac{dy}{dx} + x^2 y,$$

Where $y = 5$, and $\frac{dy}{dx} = 1$ when $x = 0$

Solution: Let $\frac{dy}{dx} = Z$

NOTES

$$\therefore \quad \frac{dz}{dx} = \frac{d^2y}{dx^2} = x^2z + x^4y$$

And $x_0 = 0$, $y_0 = 5$, and $Z_0 = 1$

Which is the same problem as Example 7.

Third approximation:

$$y_3(x) = 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72}$$

Existence and Uniqueness of Solutions

It may happen that an intial value problem has no solution or it may have exactly one solution or it may have more than one solution. Our aim in this section is to find under what condition an intial value problem has at least one solution and under what conditions does that problem have one and only one solution, that is, a unique solution. This lead us to the existence theorem and uniqueness theorem, respectively. These existence and uniqueness theorems play an important role when a differential equation cannot solved by elementary standard methods.

The Lipschitz Condition

If $f(x, y)$ be a function defined for (x, y) is a domain D in $x - y$ plane, then the function $f(x, y)$ is said to satisfy the Lipschitz condition in D if there exists a positive constant K such that,

$$| f(x, y_2) - f(x, y_1) | \leq K | y_2 - y_1 |$$

for every pair of points $(x, y_1), (x, y_2) \in D$. The constant K being independent of x, y_1 and y_2 and is called the Lipschitz constant.

Existence Theorem

The intial value problem,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad ...(2.4)$$

has at least one solution $y(x)$ provided the function $f(x, y)$ is continuous and bounded for the values of x, in a domain D and there exists positive constants M and K such that

$$\left| f(x, y) \leq M \right| \qquad ...(2.5)$$

Which satisfies the Lipschitz condition.

$$\left| f(x, y_2) - f(x, y_1) \right| \leq K \left| y_2 y_1 \right| \qquad ...(2.6)$$

For all points in domain D.

Proof by Picard's Method: Consider the iterative Equence.

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left[t_1 y_{n-1}(t)\right] dt \qquad ...(2.7)$$

$n = 1, 2....$

NOTES

With $y_0(t) = y_0$ for the intial value problem of Equation (2.4). In order that the intial value problem of Equation (2.4) may have a solution, it is necessary that the sequence $\{y_n(x)\}$ of functions converges to a limiting function $y(x)$ which is a solution of of Equation (4) or of the equivalent integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f\left[t_1 y(t)\right] dt \qquad ...(2.8)$$

To ensure the existence of the limiting function.

$$y(x) = \lim y_n(x) \qquad ...(2.9)$$

We use the fact that y_n may be written as a sum of successive differeness:

$$y_n = y_0 + \sum_{i=0}^{n-1}\left(y_{i+1} - y_i\right) \qquad ...(2.10)$$

This follows that the sequence $\{y_n - y_i\}$ converges.

From Equation (2.7) we have,

$$y_i(x) = y_0 + \int_{x_0}^{x} f\left[t_1 y_{i-1}(t)\right] dt$$

And $y_{i+1}(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_i(t)\right] dt$

$$\therefore \quad y_{i+1}(x) - y_i(x) = \int_{x_0}^{x} \left\{ f\left[t_1\ y_i(t)\right] - \left[t_1\ y_{i-1}(t)\right]\right\} \qquad ...(2.11)$$

The Equation (2.11) is true for all intiger $i = 1, 4$

Also from Equation (2.7) $y_1(\boldsymbol{x}) = y_0 + \int_{x_0}^{x} f\left[t_1\ y_0\right] dt$

$$\therefore \quad y_1(x) - y_0 = \int_{x_0}^{x} f\left[t_1 y_0\right] dt \qquad ...(2.12)$$

The condition in Equation (2.5) ensure the existence of integrals in Equations (2.11) and (2.12), Considering Equation (2.12), we have,

$$\left|y_1(x) - y_0\right| \le \int_{x_0}^{x} \left|f\left[t_1\ y_0\right]\right| |dt|$$

$$\le \int_{x_0}^{x} M|dt| \qquad \text{by Equation (2.5)}$$

$$M|x - x_0| \qquad ...(2.13)$$

Again making use of Lipschitz condition in Equation (2.6), we get from Equation (2.11),

$$\left|y_2(x) - y_1(x)\right| \le \int_{x_0}^{x} \left|f\left\{t_1\ y_1(t)\right\} - f\left(t_1\ y_0\right)\right| |dt|$$

$$\le \int_{x_0}^{x} K\left|g_1(t) - y_0\right| |dt| \qquad \text{by Equation (2.6)}$$

NOTES

$$\leq \int_{x_0}^{x} K.M\,|t - x_0||dt| \qquad \text{by Equation (2.13)}$$

$$= KM\frac{|x - x_0|^2}{2!} \qquad ...(2.14)$$

Similarly $|y_3(x) - y_2(x)| \leq \dfrac{K^2 M|x - x_0|}{3!}$

In general, we shall have

$$|y_i(x) - y_{i-1}(x)| \leq M\ K^{i-1}\frac{|x - x_0|^i}{i!} \qquad ...(2.15)$$

The truthness of result of Equation (2.15) for all values of i can be established by the mathematical induction method.

We must show that the identify Equation (2.15) holds when i is replaced by $i + 1$. For this purpose, we again make use of Equation (2.11) and Equation (2.6), we have.

$$|y_{i+1}(x) - y_i(x)| \leq \int_{x_0}^{x} \left|f[t, y_i(t)] - f[t, y_{i-1}(t)]\right||dt|$$

$$\leq \int_{x_0}^{x} K|y_i(t) - y_{i-1}(t)||dt|$$

$$\leq \int_{x_0}^{x} K.M^{i-1}\frac{|x - x_0|}{i!}|dt| = MK^i\frac{|x - x_0|}{(i+1)!} \qquad ...(2.16)$$

The relation Equation (2.16) establishes the validity of Equation (2.15) for all values of *i*.

From Equation (2.16), we see that absolutes values of terms in the series Equation (2.10) are term by term smaller than the corresponding terms in the series.

$$y_0 + M|x - x_0| + M.K\frac{|x - x_0|^2}{2!} + M.K^2\frac{|x - x_0|^3}{3!} + ...$$

Whose sum is $y_0 + \dfrac{M}{K}\left[\exp\{K|x - x_0|\} - 1\right]$

Now the above Taylor's series converges for all values of $(x - x_0)$, and so the function $y(x)$ for all values of x in any finite interval, i.e., $\lim y_n(x) \to y(x)$.

Now proceeding to the limit as $n \to \infty$, we get from Equation (12).

$$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f[t_1\ y_{n-1}(t)]\,dt$$

Or
$$y(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f[t_1\ y_{n-1}(t)\ dt] \qquad ..(2.17)$$

Since $f(x, y)$ is continous function of both x and y in the range of values considered and hence $y_n(x)$ converges to $g(x)$ uniformly over the interval the following interchanges of limiting operations are justified.

$$\lim_{n\to\infty}\int_{x_0}^{x} f\left[t_1\ y_{n-1}(t)\ dt\right] = \int_{x_0}^{x}\lim_{n\to\infty} f\left[t_1\ y_{n-1}(t)\right]dt$$

$$= \int_{x_0}^{x} f\left[t_1\ \lim_{n\to\infty} y_{n-1}(t)\right]dt = \int_{x_0}^{x} f\left[t_1\ y(t)\right]dt$$

This shows that interative sequence Equation (2.7) converges to a solution of the differential equation, problem in Equation (2.4) for all values of $x \in D$ under the given conditions thus the theorem is completely established.

Uniquesness Theorem

The intial value problem,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad ...(2.18)$$

has a unique solution provided the function $f(x, y)$ is continuous and bounded for all values of x in a domain D and there exist positive constants M and K such that,

$$|f(x, y)| \le M$$

And satisfy the Lipschitz condition,

$$|f(x, y_2) - f(x, y_1)| \le K|y_2 - y_1|$$

For all points in domain D.

Proof by Picard's Method: Suppose if possible the intial value problem of Equation (2.18) has two district solution $y(x)$ and $u(x)$.

Then,

$$u(x) = y_0 + \int_{x_0}^{x} f\left[t_1\ u(t)\right]dt$$

and $$y(t) = y_0 + \int_{x_0}^{x} f\left[t_1\ y(t)\right]dt$$

$\therefore$ $$y(x) - u(x) = \int_{x_0}^{x}\left[t\{t,\ y(t)\} - f\{t, u(t)\}\right]dt \qquad ...(2.19)$$

Since $f(x, y)$ is bounded and satisfies Lipschitz condition, we have,

$$|f(x, y)| \le M \qquad ...(2.20)$$

And $$|f(x, y_2) - f(x, y_1)| \le K|y_2 - y_1| \qquad ...(2.21)$$

Using Equations (2.19) and (2.20), we have

$$|y(x) - 4(x)| \le \int_{x_0}^{x}\left|f\left[t,\ y(t)\right] - f\left[t,\ 4(t)\right]\right||dt|$$

$$\le \int_{x_0}^{x}\left[\left|f\left[t,\ y(t)\right]\right| + \left|-f\left[t,\ 4(t)\right]\right|\right]|dt|$$

$$\le \int_{x_0}^{x}(M + M)|dt| = 2M|x - x_0| \qquad ...(2.22)$$

NOTES

Again using Equations (2.19) and (2.21), we have

$$|y(x) - 4(x)| \le \int_{x_0}^{x} K|y(t) - 4(t)||dt| \qquad ...(2.23)$$

Combining Equations (2.22) and (2.23), we obtain

$$|y(x) - 4(x)| \le \int_{x_0}^{x} K \cdot 2M|t - x_0||dt|$$

$$= 2MK\frac{|x - x_0|}{2!} \qquad ...(2.24)$$

Employing inquality in Equation (2.24) on the right hand side of Equation (2.23), we have

$$|y(x) - 4(x)| \le \int_{x_0}^{x} K \cdot 2MK\frac{|t - x_0|^2}{2!}|dt|$$

$$= 2MK^2\frac{|t - x_0|^3}{3!} \qquad ...(2.25)$$

Continuing in this way, we shall obtain.

$$|y(x) - 4(x)| \le 2\frac{MK^{n-1}(x - x_0)^4}{n!}, n = 1, 2, 3... \qquad ...(2.26)$$

Now the right-hand side of Equation (2.26) tends to zero as n tends to infinity for all finite values of x. Thus,

$$|y(x) - 4(x)| = 0 \Rightarrow y(x) = 4(x)$$

for all finite values of x. This shows that the solution is unique.

Existence and Uniqueness Theorem (In General Case)

The intial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution for all values of x in the range

$$|x - x_0| \le a,$$

Provided the function $f(x, y)$ is continuous and satisfy the conditions.

(i) $|x - y| \le M$

(ii) $|f(x, y_2) - f(x, y_1)| \le K|y_2 - y_1|$ (Lipschitz condition)

For all values x and y. $M.K$ being positive constants, in the rectangle R defined by,

$|x - x_0| \le a$ and $|y - y_0| \le Ma$

Proof by Picard's Method: The point of difference in two theorem is that here we are considering the limited range.

$$|x - x_0| \le a \qquad ...(2.27)$$

Instead of considering all values of x so that theorem may be applicable to wider class of function $f(x, y)$, we have seen that the proof of the existence theorem depended upon obtaining the inequality.

$$|y_{i+1}(x) - y_i(x)| \le MK^i \frac{|x - x_0|^{i+1}}{(i+1)!} \quad ...(2.28)$$

For all valees of x, here our aim is to esbablish the inequality Equation (2.28) for limited Equation (2.27).

For this we again consider the relation,

$$y_{i+1}(x) - g_i(x) = \int_{x_0}^{x} \left[f\{t_1\ g_i(t)\} - f\{t_1\ y_{i+1}(t)\} \right] dt$$

$$i = 1, 2, ... \quad ...(2.29)$$

Since we do not require the conditions given in Equations (2.27) and (2.28) to be applicable for all values of y but merely in a suibable neighourhood of y_0, we shall consider the possibility of obtaining bounds not for $y_{i+1} - y_i$ but for $y_i - y_0$.

Now $$y_i - y_0 = \int_{x_0}^{x} f[t,\ y_{i-1}(t)]\, dt \quad ...(2.30)$$

As long as $|f(t, y)| \le M$ for $|t - x0| \le a$ and $y = y_0$

We get from Equation (2.30), $|y,(x) - y_0| \le \int_{x_0}^{x} |f(t_1\ y)||dt|$

$$\Rightarrow |y_1(x) - y_0(x)| \le M|x - x_0| \le Ma \quad ...(2.31)$$

For $|x - x_0| \le a$ and $|y - y_0| \le Ma$,

We conclude from,

$$y_2(x) - y_0 = \int_{x_0}^{x} f[t_1\ y_1(t)]\, dt$$

$$|y_2(x) - y_0| \le \int_{x_0}^{x} |f[t,\ y_1(t)]||dt|$$

$$\le \int_{x_0}^{x} M|dt| = M|x - x_0|$$

$$\le Ma \quad ...(2.32)$$

Then by induction, we get

$$|y_i(x) - y_0| \le Ma \quad ...(2.33)$$

For $|x - x_0| \le a$ and all t.

After finding the bound of Equation (2.33) for y_i we formulate our Lipschitz condition for Equation (2.29) as follows:

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1| \quad ...(2.34)$$

NOTES

In the range $|x - x_0| \le a$ and $|y - y_0| \le Ma$, we then have

$$|y_{i+1}(x) - y_i(x)| \le \int_{x_0}^{x} K|y_i(t) - y_{i-1}(t)||dt| \qquad ...(2.35)$$

Using Equation (2.31) in Equation (2.35) with $i = 1$, we obtain

$$|y_2(x) - y_1(x)| \le \int_{x_0}^{x} MK|t - x_0||dt| = MK\frac{|x - x_0|^2}{2!} \qquad ...(2.36)$$

Continuing in this way, we shall write as,

$$|y_{i+1}(x) - y_i(x)| \le MK^i \frac{|x - x_0|^{i+1}}{(i+1)!}$$

Valid in the interval $|x - x_0| \le a$

Note: If $f(x, y)$ satisfies the condition $\left|\frac{\partial f}{\partial y}\right| \le M$ ...(2.37)

For all values of x, y in the given range, then the Lipschitz condition is also satisfied with the same constant M. For, we have by the mean value theorem of differential calculus.

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left|\frac{\partial f}{\partial y}\right| y = \bar{y}, y_1 < \bar{y} <_2 \qquad ...(2.38)$$

Where (x, y_1) and (x, y_2) are assumed in the given range. From Equations (2.37) and (2.38), we have,

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1|$$

Which is Lipschitz condition. Thus the Lipschitz condition can be replaced by the stronger condition Equation (2.37).

Example 2.9: Show that for the problem $\frac{dy}{dx} = y_1\ y(0) = 1,$ the constant a in Picard's theorem must be smaller than unity.

Solution: Here the condition for boundness of f, i.e.,

$|f(x, y)| \le M$ for $|y, y_0| \le Ma$ take the form

$|y| \le M$ for $|y - 1| \le Ma$

And if we choose $M \ge 1$, Lipschitz condition is also satisfied since in this case of Lipschitz condition assumes the form.

$$|f(x, y_2) - f(x, y_1)| = |y_2 - y_1| \le M|y_2 - y_1| \ \therefore M \ge 1$$

Now $|y - 1| \le Ma$ implies that $|y| - 1 \le Ma$

NOTES

Hence the inequlity $|y| \leq M$ will be satisfied for all values of $|y-1| \leq Ma$ provided it is satisfied for,

$$|y| = 1 + Ma$$

According, we must have,

$$1 + Ma \leq \text{ or } a \leq \frac{M-1}{M} = 1 - \frac{1}{M} \Rightarrow a < 1$$

(Since M is a positive finite constant)

Example 2.10: For the initial value problem,

$$\frac{dy}{dx} = e^y,\ y(0) = 0$$

Find the largest interval $|x| \leq a$ in which the Picard's theorem gurantees existence of a unique solution.

Solution: The condition for boundedness of $f(x, y)$, i.e., $|f(x, y| \leq M$ for $|y - y_0| \leq Ma$ takes the form $e^y \leq M$, for $|y - 0| \leq Ma$

If y_1, y_2 with $y_1 < y_2$ lies in the range $|y| \leq Ma$, we have by the mean value theorem, $e^{y_2} - e^{y_1} = (y_2 - y_1)\left|\frac{\partial}{\partial y} e^y\right|_{y=\overline{y}}$ where $y_1 < \overline{y} < y_2$ or

$e^{y_2} - e^{y_1} \leq (y_2 - y_1)M \quad [\therefore e^y \leq M]$

So that Lipschitz condition is also satisfied. Now the inequality $e^y \leq M$ will be satisfied for all values of $|y| \leq Ma$

Provided it is satisfied if $y = Ma$. Accordingly, we must have,

$$e^{Ma} \leq M \text{ or } a \leq \frac{\log M}{M}$$

Since a is positive, M lies in the range $1 \leq M < \infty$. It is easy to see that $\frac{\log M}{M}$ is maximum when $M = e = 2.718$.

The Picard's theorem then assure existence of a unique solution in the interval $|x| \leq a$

Where $a = \frac{1}{e} = 0.308$

Example 2.11: If S is defined by the rectangle $|x| \leq a, |y| \leq b$, show that the function $f(x, y) = x^2 + y^2$, satisfies the Lipschitz condition. Find the Lipschitz constant.

NOTES

Solution: (x, y_1) and (x, y_2) be two points in the rectangle S, then

$$|f(x, y_2) - f(x, y_1)| = |(x^2 + y_2^2) - (x^2 + y_1^2)|$$

$$= |y_2^2 - y_1^2| = |y_2 + y_1||y_2 - y_1|$$

$$\Rightarrow \ |f(x, y_2) - f(x, y_1)| \le 2b|y_2 - y_1|$$

$\Rightarrow f(x, y)$ satisfy the Lipschitz condition, and the Lipschitz constant $K = 2b$.

Aliter: Here $f(x, y) = x^2 + y^2$

$$\left|\frac{\partial}{\partial y} f(x, y)\right| = |2y| \le 2b, \text{ for } (x, y) \in S$$

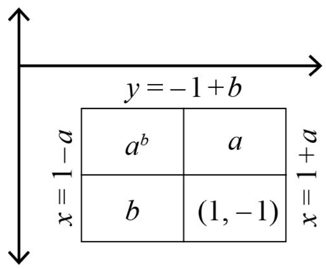
Thus $\frac{\partial}{\partial y}$ exists is continuous and bounded for all $(x, y) \in S$. Hence $f(x, y)$ satisfies Lipschitz condition and Lipschitz constant is $2b$.

Example 2.12: Examine existence and uniqueness of the solution of the initial value problem.

$$\frac{dy}{dx} = y^2, \ y(1) = -1$$

Solution: Here $f(x, y) = \frac{dy}{dx} = y^2$ and $\frac{df}{dy} = 2y$ obviously f and $\frac{df}{dy}$ are both continous for all (x, y). We consider the rectangle R.

$$|x - 1| \le a, \ (y + 1) \le b, \text{ about the point } (1, -1)$$



Obviously in this rectangle,

$$|f(x, y_2) - f(x, y_1)| = |y_2^2 - y_1^2|$$

$$= |y_2 + y_1||y_2 - y_1|$$

$$\le (2 + 2b)\,|y_2 - y_1|$$

Thus the Lipschitz condition is satisfied in the rectangle R.

Now let $M = \text{Max } |f(x, y)|$ for $x, y \in R$ and $h = \text{Min } (a, b/m)$

Then the given problem possesses a unique solution for $|x-1| \leq h$.

In this case,

$$M = \text{Max } |f(x,y)| = \text{Max. } |y^2| = |(-1-b)^2| = (1+b)^2$$

$$\therefore \quad h = \text{Min } \{a, b/m\} = \text{Min}\{a, b/(1+b)^2\}$$

Now let $b/(1+b)^2 = \phi(b)$

$$\therefore \quad h = \text{Min } \{a, b/m\} = \text{Min}\{a, b/(1+b)^2\}$$

Now let $b/(1+b)^2 = \phi(b)$

$$\therefore \quad \phi'(b) = \frac{1-b}{(1+b)^3} \text{ and } \phi''(b) = \frac{2b-4}{(1+b)^4}$$

For Max or Min of $\phi(b), \phi'(b) = 0 \Rightarrow b = 1$ and Max. $\phi(b) = \phi(1) = 1/4$

$$\therefore \quad \text{if } a \geq 1/4, \phi(b) = \frac{b}{(1+b)^2} \leq a, \text{ for all } b > 0$$

$$\Rightarrow h = \frac{b}{(1+b)^2} \leq \frac{1}{4}$$

And if $a < \frac{1}{4}$, then $h < \frac{1}{4}$, $\therefore$ Max $\frac{b}{(1+b)^2}$ is $\frac{1}{4}$.

Hence the given problem possesses a unique solution when $|x-1| \leq \frac{1}{4}$ *i.e,* in the interval $\frac{3}{4} \leq x \leq \frac{5}{4}$.

Example 2.13: Illustrate by an example that a continous function may not satisfy a Lipschitz condition on a rectangle.

Solution: Let us consider the function,

$f(x, y) = y^{2/3}$, on the rectangle $|x| \leq 1, |y| \leq 1$ obviously $f(x, y)$ is continous in the rectangle as it is a polynomial in y.

$$\text{But} \quad \left|\frac{\partial}{\partial y} f(x,y)\right| = \left|\frac{2}{3y^{1/3}}\right|$$

NOTES

Which does not exist at $y = 0$, which is a point of the rectangle. Hence the Lipschitz condition is not satisfied on the rectangle.

Check Your Progress

1. Why is Picard's iteration method used?
2. When is an approximation solution required?
3. What does iteration method specify?
4. Define the term unique solution.
5. Explain the term Lipschitz condition.

2.3 EXISTENCE THEOREM

System of Ordinary Differential Equations: First let us consider a system of n-ordinary differential equations of first order where the derivatives $y_1', y_2', \cdots y_n'$ appear explicity,

$$y_1' = f_1(x, y_1, y_2, \cdots y_n)$$
$$y_2' = f_2(x, y_1, y_2, \cdots y_n)$$
$$\vdots$$
$$y_n' = f_n(x, y_1, y_2, \cdots y_n)$$

We can write it in a vector form as,

$$y' \equiv \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \equiv \underline{f}(x, \underline{y})$$

It is the analogue of the single variable case:

$y' = f(x, y)$

Here $f_1, f_2, ..., f_n$ are given complex-valued functions defined in some set R. In the $(x, y_1, ... y_n)$-space, where x is real and $y_1, y_2, ... y_n$ are complex.

Now, we have to find 'n' differentiable functions.

$\phi_1, ... \phi_n$ on some interval I such that,

(i) $(x, \phi_1(x), \phi_2(x) \ldots \phi_n(x)) \in R$, for $x \in I$

(ii) $\phi_1'(x) = f_1(x, \phi_1(x), ..., \phi_n(x))$

$$\vdots$$

$\phi_n'(x) = f_n(x, \phi_1(x), ..., \phi_n(x))$, for all $x \in I$

If 'n' such functions exist we say $\phi = (\phi_1, \phi_2, ... \phi_n)$ is a solution of the system given earlier on the interval I.

Local Existence Theorem: Let $\underline{f}$ be a continuous vector-valued function defined on

NOTES

$$R: \left|x - x_0\right| \le a, \left|\underline{y} - \underline{y}_0\right| \le b, (a, b > 0)$$

And suppose $\underline{f}$ satisfies a Lipschitz condition on R. If M is the constant such that,

$$\left|\underline{f}(x, y)\right| \le M \ \forall (x, \underline{y}) \in R$$

The successive approximations $\{\underline{\phi}_k\}$ where, $K = 0, 1, 2, \ldots$ given by

$$\{\underline{\phi}_0\}(x_0) = \underline{y}_0$$

$$\underline{\phi}_k(x) = \underline{y}_0 + \int_{x_0}^{x} (t, \underline{\phi}_{k-1}(t))dt, k - 1, 2, 3 \ldots$$

Converge on the interval,

$$I: \left|x - x_0\right| \le \alpha = \text{Min}\left\{a, \frac{b}{M}\right\},$$

To a solution $\underline{\phi}$ of the initial value problem,

$$\underline{y}' = \underline{f}(x, y), \underline{y}(x_0) = \underline{y}_0, \text{ on } I$$

Error Approximation Theorem: If $\underline{f}$ satisfies the same conditions as defined in the previous local existence theorem and K is a Lipschitz constant for $\underline{f}$ in R, then

$$\left|\underline{\underline{\phi}}(x)\underline{\phi}_k(x)\right| \le \frac{M}{K} \frac{(K\alpha)^{K+1}}{(K+1)!} e^{K\alpha}, \text{ for all } x \in I$$

Theorem of Non-Local Existence: Let $\underline{f}$ be a continuous vector-valued function defined on the strip.

$$S. \left|x - x_0\right| \le a, \left|\underline{y}\right| < \infty, (a > 0),$$

which satisfy that there is a Lipschitz condition, then the successive approximation $\{\underline{\phi}_k\}$ for the problem,

$$\underline{y}' \underline{f}(x, \underline{y}), \underline{y}(x_0) = \underline{y}_0, (|\underline{y}_0| < \infty),$$

exist on $|x - x_0| \le a$ and converges there to a solution $\underline{\underline{\phi}}$ of this problem.

Corollary: Suppose $\underline{f}$ is a continuous vector-valued function defined on $|x| < \infty, |y| < \infty$, and satisfy a Lipschitz condition on each strip $|x| \le a, |\underline{y}| < \infty$, where a is any positive number. Then every initial value problem: $\underline{y}' = \underline{f}(x, \underline{y}), \underline{y}(x_0) = y_0$ has a solution which exists $\forall x \in R$.

Uniqueness Theorem: Let $\underline{f}, \underline{g}$ be two continuous vector-valued functions defined on $R: |x - x_0| \le a, |\underline{y} - \underline{y}_0| \le b, (a, b > 0)$, and suppose $\underline{f}$ satisfied a Lipschitz condition on R with Lipschitz constant K. Suppose $\underline{\phi}, \underline{\psi}$ are solutions of the problem:

$$\underline{y}' = \underline{f}(x, y), \underline{y}(x_0) = \underline{y}_1$$
$$\underline{y}' = \underline{g}(x, y), \underline{y}(x_0) = \underline{y}_2,$$

NOTES

respectively, on same interval I containing x_0. If for $\in, \delta \geq 0$

$$|\underline{y}(x, y) - \underline{g}(x, y)| \leq \varepsilon \text{ and } |\underline{y}_1 - \underline{y}_2)| \leq \delta, \text{ then}$$

$$\left|\underline{\underline{\phi}}(x) - \underline{\underline{\psi}}(x)\right| \leq e^{K|x-x_0|} + \frac{\varepsilon}{K}(e^{|x-x_0|} - 1) \text{ for all } x \in I.$$

In particular, the problem $\underline{y}' = \underline{f}(x, y), \underline{y}(x_0) = \underline{y}_0$, has at most one solution on any interval containing x_0.

Existence and Uniqueness Theorems for Linear Systems

Case I: Let us consider a linear system

$$\underline{y}' = \underline{f}(x, \underline{y}) \text{ with}$$

$$f_1(x, \underline{y}) = \left[\sum_{i=1}^{n} a_{1i}(x) y_i\right] + b_1(x)$$

$$\vdots$$

$$f_n(x, \underline{y}) = \left[\sum_{i=1}^{n} a_{ni}(x) y_i\right] + b_n(x)$$

Here $\{a_{ij}\}$ and $\{b_j\}$ are complex-valued functions defined for real x in some interval I.

If for all the $\{a_{ij}\}$ are continous on an interval,

$|x - x_0| \leq a, (a > 0)$, then the corresponding vector-valued function $\underline{y}$ satisfy a Lipschitz condition on the strip

$$S: |x - x_0| \leq a, |\underline{y}| < \infty:$$

$$\left|\frac{\partial f}{\partial y_k}(x, \underline{y})\right| = \left|(a_{1k}(x), \ldots, a_{nk}(x))\right| = \sum_{j=1}^{n} \left|a_{jk}(x)\right| \leq K$$

Case II: Consider a linear system,

$Y' = f(x, y)$ where the components of f are given by,

$$f_j(x, y) = \sum_{k=1}^{n} a_{jk}(x) y_k + b_j(x)$$

($j = 1, 2, ..., n$), and the function $\{a_{jk}\}$, $\{b_j\}$ are continuous on an interval I containing x_0. If $\underline{y}_0$ is any vector in $\not\subset^n$, $\exists$ unique solution $\underline{\underline{\phi}}$ of the initial value problem:

$$\underline{y}' = \underline{f}(x, \underline{y}), \underline{y}(x_0) = \underline{y}_0, \text{ on } I.$$

Equation of Order n: An n–m order equation $y^n = f(x, y, y' \, ..., y^{n-1})$ may be viewed as a system of n equations of the first order.

Define $y_1 = y \Rightarrow y_1' = y' = y_2$

$y_2 = y' \Rightarrow y_2' = y'' = y_3$

$\vdots$

$y_n = y^{n-1} \Rightarrow y_n' = y^n = f$

Now the system is,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ f \end{pmatrix}$$

In the vector form we write,

$$\underline{y'} = \underline{f}(x, y_1, ..., y_n)$$

$$= \begin{pmatrix} f_1(x, y_1, ... y_n) \\ \vdots \\ f_n(x, y_1, ... y_n) \end{pmatrix}$$

Here $f_1 (x, y_1, y_2 ... y_n) = y_2$

$f_2 (x, y_1, y_2 ... y_n) = y_3$

$\vdots$

$f_n (x, y_1, y_2 ... y_n) = f(x, y_1, ... y_n)$

Moreover if ϕ is a solution of the nth order equation then the vector,

$\underline{\phi} = (\phi, \phi', ..., \phi^{n-1})$ is a solution of the vector equation.

Conversely, if $\underline{\underline{\phi}} = (\phi_1, \phi_2, ... \phi_n)$ is the solution of the vector equation then the first component ϕ_1 is solution of the nth order equation.

$y^{(n)} = f(x, y, y' ... y^{n-1})$

Since we have,

$\phi_1' = \phi_2$

$\phi_1'' = \phi_2' = \phi_3$

$\vdots$

$\phi_1^{(n-1)} = \phi_n$

$\phi_1^{(n)}(x) = \phi_n' = f(x, \phi_1(x), ..., \phi_1^{(n-1)}(x))$

Now all the above results are proved for first order system may be applied to give results for nth order equations.

$y^n = f(x, y, y', ... y^{n-1})$

Theorem 2.1: Let f be a complex-valued continuous function defined on,

$R: |x - x_0| \le a, |\underline{y} - \underline{y}_0| \le b, (a, b > 0)$

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Such that $|f(x, y)| \leq N$ for all $(x, \underline{y}) \in R$. Suppose there exists a constant $L < 0$ such that,

$$|f(x, \underline{y}) - f(x, \underline{z})| \leq L\,|\underline{y} - \underline{z}|$$

For all $x, \underline{y})$ and $(x, \underline{z}) \in R$

Then $\exists$ unique solution ϕ of $y^{(n)} = f(x, y, y' \ldots y^{n-1})$ on the interval

$$I{:}\,|x - x_0| \leq \text{Min}\left\{a, \frac{b}{M}\right\}$$

$(M = N + b + |y_0|)$, which satisfies $\phi(x_0) = \alpha_1$, $\phi'(x_0) = \alpha_2$, ..., $\phi^{n-1}(x_0) = \alpha_n$, $(\underline{y}_0 = (\alpha_1, \alpha_2, \ldots \alpha_n))$

Proof: Consider the system: $\underline{y}' = f(x, \underline{y})$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ f(x, y_1, y_2 \ldots y_n) \end{pmatrix} = \underline{f}$$

and observe the continuity and the Lipschitz continuity of $\underline{f}$.

$$|\underline{f}(x, \underline{y})| = |y_2| + |y_3| + \cdots + |y_n| + |f(x, \underline{y})|$$

$$\leq |y| + |f(x, \underline{y})|$$

$$\leq |y_0| + b + M$$

Since $\quad |y| - |y_0| \leq |y - y_0| \leq b.$

Then prove the continuity of $\underline{f}$

Now Lipschitz continuity,

$$|\underline{f}(x, \underline{y}) - \underline{f}(x, \underline{z})| = |y_2 - z_2| + |y_3 - z_3| + \cdots + |y_n - z_n|$$

$$+ |f(x, \underline{y}_n) - f(x, \underline{z})|$$

$$\leq |\underline{y} - \underline{z}| + L\,|\underline{y} - \underline{z}|$$

$$= (1 + L)\,|\underline{y} - \underline{z}|$$

Thus $\underline{f}$ satisfy a Lipschitz condition on R with Lipschitz constant $K = 1 + L$

Theorem 2.2: Let $a_1, \ldots, a_n, b$ be continuous complex-valued function on an interval contuning a point x_0. If $\alpha_1, \ldots \alpha_n$ are any n constants, $\exists$ unique solution ϕ of the equation.

$$y^n + a_1(x)y^{n+1} + \ldots + a_n(x)y = b(x)$$

on I satisfying,

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \ldots, \phi^{(n-1)}(x_0) = \alpha_n$$

Proof: Let $\underline{y}_0 = (\alpha_1, ..., \alpha_n)$ and consider a linear system,

$$y_1' = y_2,$$
$$y_2' = y_3,$$
$$\vdots$$
$$y_{n-1}' = y_n,$$
$$y_n' = -a_n(x)y_1 - a_{n-1}(x)y_2 ... a_1(x)y_n + b(x)$$

By the existence and uniqueness theorem there is a unique solution.

$\underline{\phi} = (\phi_1, ...\phi_n)$ of this system on I satisfying

$\phi_1(x_0) = \phi_2(x_0) = \alpha_2, ...\phi_n(x_0) = \alpha_n$

But since $\phi_2 = \phi_1', \phi_3 = \phi_2'', ..., \phi_n = \phi_1^{(n+1)}]$

The $f_n \cdot \phi_1$ is a required solution on I.

2.4 UNIQUENESS THEOREM

The 'Cauchy Problem' is exactly the initial value problem or IVP and is used to solve x'(t)= f(t,x) with the condition $x(t_0)= x_0$. Picard's theorem is explained for given any point in the plane, (x_0, y_0) and a function f(x,y), continuous on some neighborhood of (x_0, y_0) and Lipschitz in y on that neighborhood, then there exist a unique function y(x) satisfying y'= f(x,y) and $y(x_0)= y_0$. A 'neighborhood' of a point is an open set containing that point. A function, f(x), is 'Lipschitz' on a set if and only if there exist a positive number C such that for any x, y in that set, |f(x)–f(y)|< C|x–y|.

If f(x) is Lipschitz on a set then it is continuous at every point of that set. The mean value theorem can be used to show that if a function is differentiable at every point of a set, then it is Lipschitz on the set while 'continuous' and 'differentiable' are defined at points. If f(x,y) is continuous but not Lipschitz on a set, then there may be many functions satisfying the differential equation and 'initial condition'. The Picard's method for solving an initial value problem is considered as the basis for his proof.

Uniqueness

The system is equivalent to the integral equation. If we have a Lipschitz condition, then we can use the Picard iterates method on the integral equation to get a unique solution. We define,

$$y_0(x) = y_0$$
$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

As we commented above, this converges to a unique solution if *f* is Lipschitz in *y*.

Alternately, we could use Gronwall's Inequality

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Gronwall's Inequality

Let u, v be nonnegative continuous functions $[a, b]$ such that

$$v(t)C + \int_a^t v(s)u(s)ds,\ a \le t \le b,$$

then

$$v(t) \le Ce^{\int_a^t u(s)ds}$$

In particular, if $C = 0$, then $v = 0$.

Proof. Let $h(t) := C + \int_a^t v(s)u(s)ds$ Therefore,

$$h'(t) := v(t)u(t) \le h(t)u(t)$$

This reduces to the differential inequality

$$h' - uh \le 0$$

Multiplying the LHS by

$$e^{-\int_a^t u(s)ds},$$

we get

$$\left(h(t)e^{-\int_a^t u(s)ds}\right)' \le 0$$

And integrate from 0 to x to get

$$h(x)e^{-\int_a^x u(s)ds} - h(a) \le 0$$

$$v(x) \le h(a)e^{\int_a^x uds}$$

Finally,

$$v(x) \le \ h(x) \le Ce^{\int_a^x u(s)ds}$$

This allows us to state a new uniqueness theorem.

Theorem Uniqueness of Solutions to IVPs

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on

$$Q := \{(x, y) : |x - x_0| \le a, |y - y_0| \le a\}$$

and satisfies

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|.$$

Then the solution to the IVP exists on $[x_0 - \alpha, x_0 + \alpha]$, where $\alpha := \dfrac{\alpha}{M}$, and the solution is unique.

NOTES

Proof. Existence follows,

If there exists two solutions $\phi_1(t)$ and $\phi_2(t)$ then define

$$w(t) := \phi_1(t) - \phi_2(t)$$

Then, $\quad w'(t) = \phi_1'(t) - \phi_2'(t)$, and

$$\int_{x_0}^{x} w'(t)dt = w(x) - w(x_0) = \int_{x_0}^{x} \left[f(t,\phi_1(t)) - f(t,\phi_2(t))\right]dt$$

$$w(x_0) = \phi_1(x_0) - \phi_2(x_0) = 0$$

So, we get the following for w:

$$w(x) = \int_{x_0}^{x} \left[f(t,\phi_1(t) - f(t,\phi_2(t))\right]dt$$

Therefore,

$$|w(s)| \le \left|\int_{x_0}^{x} f(t,\phi_1) - f(t,\phi_2)dt\right|$$

$$\le \int_{x_0}^{x} |f(t,\phi_1) - f(t,\phi_2)|\, dt \le K \int_{x_0}^{x} |\phi_1(t) - \phi_2(t)|\, dt$$

$$= K \int_{x_0}^{x} |w(t)|\, dt$$

Thus, from Gronwell's Inequality with $u(t) := K,\ v(t) := |w(t)|$, and $C = 0$, we get $|w(t)|$. Thus, $\phi_1 = \phi_2$, and the uniqueness is shown.

2.4.1 Existence and Uniqueness Theorem for Proof's by Picard's Method

The Peano theorem can be compared with another existence result in the same context, the Picard–Lindelöf theorem. The Picard–Lindelöf theorem both assumes more and concludes more. It requires Lipschitz continuity, while the Peano theorem requires only continuity; but it proves both existence and uniqueness where the Peano theorem proves only the existence of solutions. To illustrate, consider the ordinary differential equation, $y' = |y|^{\frac{1}{2}}$ on the domain $[0,1]$.

According to the Peano theorem, this equation has solutions, but the Picard-Lindelöf theorem does not apply since the right hand side is not Lipschitz continuous in any neighbourhood containing 0. Thus we can conclude existence but not uniqueness. It turns out that this ordinary differential equation has two kinds of solutions when starting at $y(0) = 0$, either $y(x) = 0$ or $y(x) = x^2 / 4$. The transition between $y = 0$ and $y = (x - C)^2 / 4$ can happen at any C.

The Carathéodory existence theorem is a generalization of the Peano existence theorem with weaker conditions than continuity. In mathematics, in the study of differential equations, the Picard–Lindelöf theorem, Picard's existence theorem or Cauchy–Lipschitz theorem is an important theorem on existence and uniqueness of solutions to first order equations with a given initial value problems.

NOTES

Picard–Lindelöf Theorem

Consider the initial value problem,

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_{0,} \quad t \in [t_0 - \varepsilon, t_0 + \varepsilon].$$

Suppose f is Lipschitz continuous in *y* and continuous in *t*. Then, for some value $\varepsilon > 0$, there exists a unique solution *y*(*t*) to the initial value problem within the range $[t_0 - \varepsilon, t_0 + \varepsilon]$.

Proof

The proof relies on transforming the differential equation, and applying fixed-point theory. By integrating both sides, any function satisfying the differential equation must also satisfy the integral equation,

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))ds.$$

A simple proof of existence of the solution is obtained by successive approximations. In this context, the method is known as Picard iteration.

Set

$$\varphi_0(t) = y_0$$

And

$$\varphi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, \varphi_k(s))ds.$$

It can then be shown, by using the Banach fixed point theorem, that the sequence of Picard iterates is convergent and that the limit is a solution to the problem.

Analysis of Proof

Let be the compact cylinder where f is defined, this is and . Let , this is, the maximum slope of the function in modulus. Finally, let L be the Lipschitz constant of f with respect to the second variable.

Analysis of Proof

Let $C_{a,b} = \overline{I_a(t_0)} \times \overline{B_b(x_0)}$ be the compact cylinder where *f* is defined and is represented as $t \in \overline{I_a(t_0)} = [t_0 - a, t_0 + a]$ and $\overline{B_b(x_0)} = [x_0 - b, x_0 + b]$. Let $M = \sup_{C_{a,b}} \| f \|$, which is considered as the maximum slope of the function in modulus. Finally, let *L* be the Lipschitz constant of *f* with respect to the second variable.

An operator between two functional spaces of continuous functions, Picard's operator, is defined as follows:

$$\Gamma : C(I_\alpha(t_0), B_b(x_0)) \to C(I_\alpha(t_0), B_b(x_0))$$

It can defined by:

$$\Gamma\varphi(t) = x_0 + \int_{t_0}^{t} f(s,\varphi(s))ds.$$

We presume that it is well defined and that its image must be a function taking values on $B_b(x_0)$ or equivalently that the norm of $\Gamma\varphi(t) - x_0$ is less than b.

$$\| \Gamma\varphi(t) - x_0 \| = \| \int_{t_0}^{t} f(s,\varphi(s))ds \| \leq | \int_{t_0}^{t} \| f(s,\varphi(s)) \| \, ds | \leq M \, | t - t_0 | \leq M\alpha \leq b$$

The last step is the imposition, hence we require $\alpha < b/M$. Let, the Picard's operator to be contractive under certain hypothesis over α that later on we will be able to omit.

Given two functions $\varphi_1, \varphi_2 \in C(I_\alpha(t_0), B_b(x_0))$ we want:

$$\| \Gamma\varphi_1 - \Gamma\varphi_2 \| = \| \int_{t_0}^{t} (f(s,\varphi_1(s)) - f(s,\varphi_2(s))ds) \|$$

$$\leq | \int_{t_0}^{t} \| f(s,\varphi_1(s)) - f(s,\varphi_2(s)) \| \, ds | \, .$$

Then since f is Lipschitz with respect to the second variable, we have that:

$$L \, | \int_{t_0}^{t} \| \varphi_1(s) - \varphi_2(s) \| \, ds | \leq L\alpha \, \| \varphi_1 - \varphi_2 \|$$

This is contractive if $\alpha < 1/L$ or equivalently, in order to have equality, if α 1/(2L).

Therefore, since the Picard's operator is an operator between Banach spaces (in particular, metric spaces induced by the norm) and contractive, by means of the Banach fixed point theorem there exists a unique function $\varphi \in C(I_\alpha(t_0), B_b(x_0))$ such that $\Gamma\varphi = \varphi$ is, solution of the initial value problem defined on $I\alpha$ where α must satisfy the condition given above, $\alpha = \min\{a, b/M, 1/(2L)\}$.

Optimization of the Solution's Interval

There is a corollary of the Banach fixed point theorem that states that if an operator T^n is contractive for some $n \in \mathbb{N}$ then T has a unique fixed point. This theorem is applied to the Picard's operator. For this, let us use the following lemma that will be very useful for this situation.

Lemma: $\| \Gamma^m\varphi_1 - \Gamma^m\varphi_2 \| \leq \dfrac{L^m\alpha^m}{m!} \| \varphi_1 - \varphi_2 \|$

This can be checked by induction as follows:

For $m = 1$, let us assume that it is true for m " 1 and let us check it for m:

$$\| \Gamma^m\varphi_1 - \Gamma^m\varphi_2 \| = \| \Gamma\Gamma^{m-1}\varphi_1 - \Gamma\Gamma^{m-1}\varphi_2 \| \leq | \int_{t_0}^{t} \| f((s, \Gamma^{m-1}\varphi_1(s)) - f(s, \Gamma^{m-1}\varphi_2(s)) \| \, ds |$$

$$\leq L \, | \int_{t_0}^{t} \| \Gamma^{m-1}\varphi_1(s) - \Gamma^{m-1}\varphi_1(s) \| \, ds | \leq \frac{L^m\alpha^m}{m!} \| \varphi_1 - \varphi_2 \|$$

Now taking into account this inequality we can guarantee that for some m large enough, the quantity $\frac{L^m\alpha^m}{m!} < 1$ and hence Γ^m will be contractive. So using the previous corollary Γ will have a unique fixed point. Thus, the interval of the solution can be optimized by taking $\alpha = \min\{a, b/M\}$.

The importance of this consequence is that the interval of definition of the solution does eventually not depend on the Lipschitz constant of the field, but essentially depends on the interval of definition of the field and its maximum absolute value of it.

The Picard-Lindelöf theorem shows that the solution exists and that it is unique. The Peano existence theorem shows only existence and not uniqueness, but it assumes only that f is continuous in y instead of Lipschitz continuous. For example, the right-hand side of the equation $y' = y^{1/3}$ with initial condition $y(0) = 0$ is continuous but not Lipschitz continuous. In fact, the solution of this equation is not unique; two different solutions are given besides the trivial one $y(t) = 0$

$$y(t) \pm \left(\frac{2}{3}t\right)^{3/2}.$$

Check Your Progress

6. Give the system of ordinary differential equations and then write its vector form.
7. Define the term 'equation of order n' for existence and uniqueness.
8. What is uniqueness?
9. Define Picard-Lindelöf theorem.

2.5 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Picard's iteration method is used for giving an approximation solution of the initial value problem.
2. In many of the Engineering problems, we are often confronted with the differential equations whose solution cannot be obtained by standard methods. In such problems, it is must to obtain an approximation solution only.
3. The iteration method specifies a method which consists of repeated application of exactly the same type of steps where in each steps is Picard's method.
4. An initial value problem has no solution or it may have exactly one solution or it may have more than one solution. To find under what condition an initial value problem has at least one solution and under what conditions does that problem have one and only one solution, that is, a unique solution.
5. If $f(x, y)$ be a function defined for (x, y) is a domain D in $x - y$ plane, then the function $f(x, y)$ is said to satisfy the Lipschitz condition in D if there exists a positive constant K.

6. First let us consider a system of n-ordinary differential equations of first order where the derivatives $y_1', y_2', \cdots y_n'$ appear explicity,

$$y_1' = f_1(x, y_1, y_2, \cdots y_n)$$
$$y_2' = f_2(x, y_1, y_2, \cdots y_n)$$
$$\vdots$$
$$y_n' = f_n(x, y_1, y_2, \cdots y_n)$$

   We can write it in a vector form as,

$$y' \equiv \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \equiv \underline{f}(x, \underline{y})$$

7. An $n–m$ order equation $y^n = f(x, y, y' \ldots, y^{n-1})$ may be viewed as a system of n equations of the first order.
8. The system is equivalent to the integral equation. If we have a Lipschitz condition, then we can use the Picard iterates method on the integral equation to get a unique solution. We define,

$$y_0\,(x) = y_0$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

   As we commented above, this converges to a unique solution if f is Lipschitz in y.
9. The Picard-Lindelöf theorem requires Lipschitz continuity to prove both existence and uniqueness of solutions. It shows that the solution exists and that it is unique. It guarantees a unique solution on some interval containing t_0 if f is continuous on a region containing t_0 and y_0 and satisfies the Lipschitz condition on the variable y.

2.6 SUMMARY

- In many of the Engineering Problems, we are often confronted with the differential equation whose solution cannot be obtained by standard methods.
- In such problems, it is must to obtain an approximation solution only.
- The Picard's iteration method is used for giving an approximation solution of the initial value problem.
- The iteration method specifies a method which consists of repeated applications of exactly the same type of steps where in each steps is Picard's method.
- An initial value problem has no solution or it may have exactly one solution or it may have more than one solution.
- To find under what condition an initial value problem has at least one solution and under what conditions does that problem have one and only one solution, that is, a unique solution.

NOTES

- The existence and uniqueness theorems plays an important role in solving differential equations when any differential equation cannot be solved by elementary standard methods.
- If f(x, y) be a function defined for (x, y) is a domain D in x – y plane, then the function f(x, y) is said to satisfy the Lipschitz condition in D if there exists a positive constant K.
- $f_1, f_2, ..., f_n$ are given complex-valued functions defined in some set R. In the $(x, y_1, ... y_n)$-space, where x is real and $y_1, y_2, ... y_n$ are complex.
- Let $\underline{f}$ be a continuous vector-valued function defined on the strip.

  $S.\ |x - x_0| \leq a, |\underline{y}| < \infty, (a > 0),$

  which satisfy that there is a Lipschitz condition, then the successive approximation $\{\underline{\phi}_k\}$ for the problem,

  $\underline{y}' \underline{f}(x, \underline{y}), \underline{y}(x_0) = \underline{y}_0, (|\ \underline{y}_0\ | < \infty),$

  exist on $|\ x - x_0\ | \leq a$ and converges there to a solution $\underline{\underline{\phi}}$ of this problem.
- Suppose $\underline{f}$ is a continuous vector-valued function defined on

  $|\ x\ | < \infty, |\ y\ | < \infty,$ and satisfy a Lipschitz condition on each strip $|\ x\ | \leq a, |\ \underline{y}\ | < \infty,$ where a is any positive number. Then every initial value problem: $\underline{y}' = \underline{f}(x, \underline{y}), \underline{y}(x_0) = y_0$ has a solution which exists $\forall x \in R.$
- Linear system,

  $Y' = f(x, y)$ where the components of f are given by,

  $$f_j(x, y) = \sum_{k=1}^{n} a_{jk}(x) y_k + b_j(x)$$

  $(j = 1, 2, ..., n)$, and the function $\{a_{jk}\}, \{b_j\}$ are continuous on an interval I containing x_0. If $\underline{y}_0$ is any vector in $\not\subset^n, \exists$ unique solution $\underline{\underline{\phi}}$ of the initial value problem:

  $\underline{y}' = \underline{f}(x, \underline{y}), \underline{y}(x_0) = \underline{y}_0,$ on I.
- The Picard-Lindelöf theorem requires Lipschitz continuity to prove both existence and uniqueness of solutions. It shows that the solution exists and that it is unique. It guarantees a unique solution on some interval containing t_0 if f is continuous on a region containing t_0 and y_0 and satisfies the Lipschitz condition on the variable y.

2.7 KEY TERMS

- **Picard's iteration method:** Picard's iteration method is used for giving an approximation solution of the initial value problem.

- **Iteration method:** Iteration method specifies a method which consists of repeated applications of exactly the same type of steps where in each steps is Picard's method.
- **Initial value problem:** Initial value problem has no solution or it may have exactly one solution or it may have more than one solution.
- **Lipschitz condition:** If f(x, y) be a function defined for (x, y) is a domain D in x – y plane, then the function f(x, y) is said to satisfy the Lipschitz condition in D if there exists a positive constant K.
- **Uniqueness theorem:** Let $\underline{f}, \underline{g}$ be two continuous vector-valued functions defined on *R:* $|x - x_0| \leq a, |\underline{y} - \underline{y}_0| \leq b, (a, b > 0),$ and suppose $\underline{f}$ satisfied a Lipschitz condition on *R* with Lipschitz constant *K*.
- **Equation of order *n*:** An *n–m* order equation $y^n = f(x, y, y' \ldots, y^{n-1})$ may be viewed as a system of *n* equations of the first order.

2.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Why and when the Picard's iteration method used?
2. Explain the significance of various methods of successive approximations.
3. How many different types of successive approximations can be obtained in a solution?
4. Define the equations that are used in Picard's iteration method to give an approximation solution.
5. Explain the method involved for the existence and uniqueness solutions of initial value problems.
6. What do you mean by Lipschitz condition and Lipschitz constant?
7. State the existence and uniqueness theorems.
8. Define the importance of nonlocal existence method of finding solutions.
9. Differentiate between approximation to solutions and uniqueness of solutions.
10. How is existence and uniqueness of solutions obtained for the systems of nth order equations?
11. State the following theorems:
    - Local existence theorem
    - Error approximation theorem
    - Nonlocal existence theorem
    - Uniqueness theorem
    - Existence and uniqueness for linear systems

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12. What do you understand by the uniqueness theorem?
13. How will you define the Picard-Lindelof theorem?

Long-Answer Questions

1. Apply Picard's method to the following initial value problems and find the succsessive approximation:

   (i) $\frac{dy}{dx} = 1 + y^L,\ y(0) = 0$

   (ii) $\frac{dy}{dx} = xy,\ y(0) = 2$

   (iii) $\frac{dy}{dx} = x + y,\ y(0) = -1$

   (iv) $\frac{dy}{dx} = x + y,\ y(0) = 1$

   (v) $\frac{dy}{dx} 2xy - 1,\ y(0) = 0$

   (vi) $\frac{dy}{dx} = y^2, y(0) = 1$

   (vii) $\frac{dy}{dx} = 3e^x + 2y,\ y(0) = 0$

   (viii) $\frac{dy}{dx} = 2x - y^2$ where $y = 0$ at $x = 0$.

2. Find the third approximation of the solution for the following equation.

   $$\frac{dy}{dx} = 2x + 2, \frac{d^2}{dx} = 3xy + x^2 z$$

   Where $y = 2$ and $z = 0$ when $x = 0$

3. Find the first three approximations in the solution of the following equation.

   $$\frac{dy}{dx} = 1 + xy,\ y(0) = 2$$

4. Apply Picard's method to find the solutions of the problem.

   $$\frac{dy}{dx} = y - x,\ y(0) = 2$$

   Show that the iterative solution approaches the exact solution.

5. Apply Picard's method up to third approximation to solve the equation.

   $$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$$

Given that $y = 2, z = 1$ when $x = 0$

6. Solve the differential equation $\frac{dy}{dx} = x - y$ with the condition $y = 1$ when $x = 0$, and show that the sequence of approximation given by Picard's method tends to the exact solution as a limit.

7. Use Picard's method to obtain a solution of the differential equation,

   $\frac{dy}{dx} = x^2 - y,\ y(0) = 0$. Find at least the fourth approximation to each solution.

8. Obtain the solution of the equation $\frac{dy}{dx} = x^2 + y^2;\ y(0) = 1$ by Picard's method, the term involving x^4.

9. Use Picard's method of approximation to find the solution of the equation

   $\frac{dy}{dx} = 2xy^2 = 0$ with $y = 1$ when $x = 0$ and hence show that $y = 1/(1 + x^2)$.

10. If $(x, y) = y^{2/3}$, show that the Lipschitz condition is not satisfied in any containing the origin and that the solution of the differential equation,

$$\frac{dy}{dx} = f(x, y)$$

   satisfying the initial condition $y = 0$ when $x = 0$ is not unique.

11. If S is defined by the rectangle $|x| \leq a, |y| \leq b,$ show that the function.

$$f(x, y) = x \text{ Sin } y + y \text{ Cos } x$$

   satisfy the Lipschitz condition. Find Lipschitz constant.

12. Examine the existence and uniqueness of solution of the intial value problem,

$$\frac{dy}{dx} = y^{1/3},\ y(0) = 0$$

13. Show that the Picard's theorem, ensure existence of a unique solution in the interval $|x| \leq \frac{1}{2}$ for the initial value problem.

   $\frac{dy}{dx} = x + y^2,\ y(0) = 0 \qquad 2.$

14. Discuss the conditions when nonlocal existence theorem is used for finding solutions for the systems of nth order equations.

15. Prove that approximation method for finding solutions using existence and uniqueness theorems gives accurate system of required ordinary differential equations.

16. Consider an initial value problem:

$$x' = f(t, x), \quad x(t_0) = x_0$$

Give proofs and system of equations reference to the local and nonlocal existence theorems.

17. Denote the column vector x with components $x_1, x_2, \ldots, x_n$ and vector f with components $f_1, f_2, \ldots, f_n$ for the system of equations of the form $x' = f(t, x)$.
18. Briefly explain about the uniqueness theorem with the help of giving examples.
19. Explain Picard-Lindelof existence and uniqueness theorem for solving differential equations with the help of examples.

2.9 FURTHER READING

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UNIT 3 DEPENDENCE ON INITIAL CONDITIONS AND PARAMETERS

Structure

3.0 INTRODUCTION

In mathematics, one of the most essential topics in mathematics is continuity and differentiability, which helps concepts like as continuity at a point, continuity on an interval, derivative of functions, and many more. Continuity and Differentiability of functional parameters, on the other hand, are extremely difficult to achieve. To explain the concept of higher order differentiability, consider a function $y = f(x)$ differentiable in the interval (a, b). For defining the first-order differential of the function at the point $x \in (a, b)$ the formula is $dy = f'(x)\, dx$.

Poincare-Bendixson theorem gives the complete determination of the asymptotic behaviour of a large class of flows on the plane and cylinder. An autonomous system is one that does not depend on the independent variable. The critical point of a function of a real variable is any value in the domain where either the function is not differentiable or its derivative is zero.

A stationary point of a differentiable function of one variable is a point on the graph of the function where the function's derivative is zero. Informally, it is a point where the function 'Stops' increasing or decreasing (hence the name). For a differentiable function of several real variables, a stationary point is a point on the surface of the graph where all its partial derivatives are zero (equivalently, the gradient is zero). Stationary points are easy to visualize on the graph of a function of one variable: they correspond to the points on the graph where the tangent is horizontal (i.e., parallel to the x-axis). For a function of two variables, they correspond to the points on the graph where the tangent plane is parallel to the xy plane.

NOTES

Periodic solutions of equations are solutions that describe regularly repeating processes. Floquet theory is a theory concerning the structure of the space of solutions and the properties of solutions, of a linear system of differential equations with periodic coefficients for periodic systems. The limit cycle is an isolated closed trajectory that occurs only in nonlinear systems.

The critical point provides useful information about the behaviour of the system and hence is considered important.

In this unit, you will learn about the continuity and differentiability, Poincare-Bendixson theorem, higher order differentiability, autonomous system, Umlaufsatz, index of a stationary point, stability of periodic solutions, rotational point, foci, nodes and saddle points.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic concept of continuity and differentiability of solutions
- Explain about the higher order differentiability
- Define Poincare-Bendixson theory
- Analysis the autonomous systems
- Learn about the term Umlaufsatz
- Elaborate on the index of a stationary point
- Define periodic solutions and Floquet theory for periodic systems
- Define and classify critical points
- Know stability of critical points

3.2 CONTINUITY AND DIFFERENTIABILITY OF SOLUTIONS

The dependence of solutions of initial value problems on the initial values and on parameters in the differential equation can be studied with the knowledge of a fundamental estimate.

Definition: $x(t)$ is defined as an ρ-approximate solution of the DE $dx/dt = f(t, x)$ on an interval I if

$$|x(t) - f(t, x(t))| \leq \rho \qquad \forall\ t \in I.\ (\rho)$$

Fundamental Estimate

Let $f(t, x)$ be continuous in t and x, and uniformly Lipschitz continuous in x with Lipschitz constant L. Suppose $x'(t)$ is an ρ_1-approximate solution and $x''(t)$ is an ρ_2-approximate solution of $dx/dt = f(t, x)$ on an interval I with $t_0 \in I$, and suppose $|x'(t_0) - x'(t_0)| \leq \delta$. Then

$$\left|x'(t0) - x''(t0)\right| \leq \delta e^{L|t-t_0|} + \frac{\hat{I}_1 + \hat{I}_2}{L}\left(e^{L|t-t_0|} - 1\right) \qquad \forall\ t \in I.$$

Continuity with Respect to Panameters and Initial Conditions

Let us consider initial value problems as:

$$Dx/dt = f(t, x, \mu), \qquad x(t_0) = y,$$

where εμ is a vector of parameters and εy belongs to n Euclidian space. . Assuming that for each value of μ, $f(t, x, \mu)$ is continuous in t and x and Lipschitz in x with Lipschitz constant L locally independent of μ. For each fixed μ, y, this is a standard intial value problem, which has a solution on some interval about t_0 as t $x(t, \mu, y)$.

Theorem 3.1: If f is continuous in t, x, μ and Lipschitz in x with Lipschitz constant independent of t and μ, then $x(t, \mu, y)$ is continuous in (t, μ, y) jointly.

Differentiability

Dependence on parameters can be transformed into initial conditions, it will suffice to prove the following.

Suppose f is continuous in t, x and C in x, and x(t, y) is the solution of the Initial Value Problem dx/dt= f(t, x), x(t0) = y (say on an interval [a, b] containing t0 for y in some closed ball B = {y ∈ F n : |y – x0| ≤ r}). Then x is a C function of t and y on [a, b] ×B.

3.3 HIGHER ORDER DIFFERENTIABILITY

To explain the concept of higher order differentiability, consider a function $y = f(x)$ differentiable in the interval (a, b). For defining the first-order differential of the function at the point $x \in (a, b)$ the formula is $dy = f'(x)\,dx$.

The differential dy depends on the following two quantities:

1. The variable x (through the derivative $y = f'(x)$).
2. The differential of the independent variable dx.

On fixing the increment dx, it is assumed that dx is constant. Then the differential dy becomes a function only of the variable x for which the differential can also be defined by taking the same differential dx as the increment Δx. Consequently, the second differential or differential is obtained of the second order, denoted as d^2y or $d^2f(x)$. Therefore, by definition,

$$d^2y = d(dy) = d\left[f'(x)dx\right] = df'(x)dx = f''(x)dxdx = f''(x)(dx)^2.$$

Normally it is denoted as $(dx)^2 = dx^2$. Hence, we have:

$$d^2y = f''(x)dx^2.$$

Similarly, it can be established that the third differential or differential of the third order has the form,

$$d^3y = f'''(x)dx^3.$$

Usually, the differential of an arbitrary order n is given by,

$$d^ny = f^{(n)}(x)dx^n,$$

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This can be meticulously proved by means of mathematical induction. This formula indicates the following expression for the nth order derivative:

$$f^{(n)}(x) = \frac{d^n y}{dx^n}.$$

Remember that in case of the linear function $y = ax + b$, the second and subsequent higher-order differentials are zero. Certainly,

$$d^2(ax+b) = (ax+b)'' dx^2 = 0 \cdot dx^2 = 0, \ldots, d^n(ax+b) = 0.$$

In this instance, it is obvious that,

$$d^n x = 0 \text{ for } n > 1.$$

Properties of Higher-Order Differentials

If the functions u and v have the nth order derivatives, then the following properties are valid:

- $d^n(\alpha u + \beta v) = \alpha d^n u + \beta d^n v;$
- $d^n(uv) = \sum_{i=0}^{n} C_n^i d^{n-i} u d^i v.$

The last equality follows directly from the Leibniz formula.

Higher Order Differential of a Composite Function

Consider the composition of two functions such that $y = f(u)$ and $u = g(x)$. In this instance, y is a composite function of the independent variable x.

$$y = f(g(x))$$

The first differential of y can be written as,

$$dy = [f(g(x))]' dx = f'(g(x)) g'(x) dx.$$

Compute the second differential d^2y (assuming dx is constant by definition). Using the product rule, the equation becomes:

$$d^2y = [f'(g(x))g'(x)]' dx^2 = \left[f''(g(x))(g'(x))^2 + f'(g(x))g''(x)\right] dx^2 = f''(g(x))(g'(x)dx$$

Consider that,

$$g'(x)dx = du \text{ and } g''(x)dx^2 = d^2u.$$

Consequently,

$$d^2y = f''(u)du^2 + f'(u)d^2u$$

Or,

$$d^2y = y'' du^2 + y' d^2u.$$

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Similarly, we can obtain the expression for the third order differential of a composite function of the form:

$$d^3y = f'''(u)du^3 + 3f''(u)dud^2u + f'(u)d^3u.$$

It follows from the above that the higher order differentials $d^2y, d^3y, \ldots, d^ny$ are generally not invariant.

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

3.4 POINCARE-BENDIXSON THEORY

It is a part of the qualitative theory of differential equations and theory of dynamical systems involving the limiting (when $t \to \pm\infty$) behaviour of trajectories of autonomous systems of two differential equations of the first order:

$$\dot{x}_i = f_i(x_1, x_2), \qquad i = 1, 2 \qquad \ldots(3.1)$$

In the most important case when the system has only a finite number of equilibrium positions in a bounded part of the plane, the basic result of H. Poincare and I. Bendixson is that any bounded semi-trajectory (positive or negative) either tends to an equilibrium position or coils round (like a spiral) to a limit cycle, or in an analogous way coils to a closed separatrix or separatrix contour consisting of several separatrices joining certain equilibrium positions, or is itself an equilibrium position or a closed trajectory. The corollary used most often is: If the semi-trajectory does not leave a given compact domain not containing an equilibrium position, then there is a closed trajectory in this domain. For cases when there are an infinite number of equilibrium positions or when the semi-trajectories are not bounded, there is also a fairly complete, although more complicated, description. Finally one can consider a continuous flow in the plane without assuming that it is given by the differential equations (3.1), because in this case it is still possible to use the basic technical premises of the Poincare–Bendixson theory: the Jordan theorem and the Poincare return map for local cross-sections which are homeomorphic to a segment.

The Poincare–Bendixson theory borders on: the connection, discovered by Poincare, between the rotation of a certain field on the boundary of a domain and the indices of the equilibrium positions inside it; results of Bendixson and L.E.J. Brouwer on the possible types of behaviour of trajectories near equilibrium positions; results making the role of singular trajectories (equilibrium positions, limit cycles and separatrices) more precise in the qualitative picture arising on the phase plane.

Although the general theory gives complete information about the possible types of behaviour of the phase trajectories for the system (3.1), this does not answer the question of which type is realized for a certain actual system.

Theorem

Let $S+$ be a positive semi orbit contained in a closed subset K of an open subset D of the real (x, y) plane. If $L(S+)$ consists of regular points only, then either

a. $S+$ (= $L(S+)$ is a periodic orbit, or

b. $L(S+)$ is a periodic orbit.

A periodic orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time.

3.5 AUTONOMOUS SYSTEMS

Autonomous system of ordinary differential equations is a system of ordinary differential equations which does not explicitly contain the independent variable t (time). The general form of a first-order autonomous system in normal form is:

$$\dot{x}_j = f_j(x_1, \cdots x_n), \qquad j = 1, \cdots, n, \text{ or, in vector notation,}$$

$$\dot{x} = f(x). \qquad \text{....(3.2)}$$

A non-autonomous system $\dot{x} = f(t, x)$ can be reduced to an autonomous one by introducing a new unknown function $x_n + 1 = t$. Historically, autonomous systems first appeared in descriptions of physical processes with a finite number of degrees of freedom. They are also called dynamical or conservative systems.

A complex autonomous system of the form Equation (3.2) is equivalent to a real autonomous system with $2n$ unknown functions

$$\frac{d}{dt}(\text{Re } x) = \text{Re } f(x), \qquad \frac{d}{dt}(\text{Im } x) = \text{Im } f(x).$$

The essential contents of the theory of complex autonomous systems — unlike in the real case — is found in the case of an analytic $f(x)$.

Consider an analytic system with real coefficients and its real solutions. Let $x = \phi(t)$ be an (arbitrary) solution of the analytic system Equation (3.2), let $\Delta = (t_-, t_+)$ be the interval in which it is defined and let $x(t, t_0, x^0)$ be the solution with initial data $x|t = t_0 = x^0$. Let G be a domain in R^n and $f \in C^1(G)$ The point $x^0 \in G$ is said to be an equilibrium point, or a point of rest, of the autonomous system Equation (3.2) if $f(x^0) \equiv 0$. The solution $\phi(t) \equiv x^0$, $t \in \mathbf{R} = (-\infty, +\infty)$, corresponds to such an equilibrium point.

Local properties of solutions:

(1) If $\phi(t)$ is a solution, then $\phi(t + c)$ is a solution for any $c \in \mathbf{R}$.

(2) Existence: For any $t_0 \in \mathbf{R}$, $x^0 \in \boldsymbol{G}$, a solution $x(t, t^0, x^0)$ exists in a certain interval $\Delta . \in t$.

(3) Smoothness: If $f \in C^p(G)$, $p \geq 1$, then $\phi(t)$ Î $C^{p+1}(\Delta)$.

(4) Dependence on parameters: Let $f = f(x, \alpha)$, $\alpha \in G_\alpha \subset \mathbf{R}^m$, where G_α is a domain; if $f \in C^p(G \times G_\alpha)$, $p \geq 1$, then $x(t, t_0, x^0, \alpha) \in C^p(\Delta \times G_\alpha)$.

(5) Let x^0 be a non-equilibrium point then there exist neighbourhoods V, W of the points x^0, $f(x^0)$, respectively, and a differentiable homeomorphism $y = h(x) : V \to W$ such that the autonomous system has the form $\dot{y} = \text{const}$ in W.

A substitution of variables $x = \phi(y)$ in the autonomous system Equation (3.2) yields the system

$$\dot{y} = \left(\phi'(y)\right)^{-1} f\left(\phi(y)\right), \qquad \text{....(3.3)}$$

where $\phi'(y)$ is the Jacobi matrix.

Global properties of solutions:

(1) Any solution $x = \phi(t)$ of the autonomous system Equation (3.2) may be extended to an interval $\Delta = (t_-, t_+)$. If $\Delta = \mathbf{R}$, the solution is said to be unboundedly extendable; if $t_+ = +\infty$, $t_- > -\infty$, the solution is said to be unboundedly extendable forwards in time (and, in a similar manner, backwards in time). If $t_+ < +\infty$ then, for any compact set $K \subset \Omega$, $x^0 \in K$, there exists a $\tau = \tau(K) < t_+$ such that the point $x(t, t_0, x^0)$ is outside K for $t < \tau(K)$ and analogously, for $t_- > -\infty$;

(2) The extension is unique in the sense that any two solutions with common initial data are identical throughout their range of definition.

(3) Any solution of an autonomous system belongs to one of the following three types: a) aperiodic, with $\phi(t_1) \neq \phi(t_2)$ for all $t_1 \neq t_2$, $t_j \in \mathbf{R}$; b) periodic, non-constant; or c) $\phi(t) \equiv \text{const}$.

To each solution $x = \phi(t)$ is assigned a corresponding curve $\Gamma: x = \phi(t)$, $t \in \Delta$, inside the domain ***G***. ***G*** is then said to be the phase space of the autonomous system, is its trajectory in the phase space and the solution is interpreted as motion along this trajectory in the phase space. The mapping $g^t : G \to G$ defined by the formula $g^t x^0 = x(t, 0, x^0)$ (i.e., each point moves along the phase trajectory during time t) is called the phase flow. In its domain of definition the phase flow satisfies the following conditions: (1) $g^t x$ is continuous in (t, x); and 2) the group property $g^{t_1+t_2}\, x = g^{t_1} g^{t_2} x$.

The Liouville theorem is valid: Let $D \subset G$ be a domain with a finite volume and let v_t be the volume of the domain $g^t D \subset G$, then

$$\left.\frac{dv_t}{dt}\right|_{t=0} = \int_D \operatorname{div} f(x) dx. \qquad \text{...(3.4)}$$

For a Hamiltonian system, a consequence of Equation (3.4) is the conservation of the phase volume by the phase flow. A second variant of Equation (3.4) is obtained as follows. Let $x = \phi(t, \alpha)$ be a family of solutions of Equation (3.2), $\alpha = (\alpha_1, \cdots, \alpha_{n-1}) \in G_\alpha$, let G_α be a domain and let $\phi \in C^1(\Delta \times G_\alpha)$, then

$$\frac{d}{dt} \text{In } I(t, \alpha) = \text{div } f(x), \qquad \ldots(3.5)$$

where $I(t, \alpha) = \det \partial x / \partial(t, \alpha)$.

NOTES

Structure of phase trajectories:

(1) Any two phase trajectories have either no point in common or coincide.

(2) Any phase trajectory belongs to one of the following types: (a) a smooth, simple, non-closed Jordan arc; (b) a cycle, i.e., a curve diffeomorphic (differentiable homeomorphism) to a circle; or (c) a point (an equilibrium point). The local structure of phase trajectories in a small neighbourhood of a point other than an equilibrium point is trivial: The family of phase trajectories is diffeomorphic to a family of parallel straight lines. For a linear autonomous system the structure of phase trajectories in a neighbourhood of an equilibrium point is known, since the autonomous system is integrable. For non-linear autonomous systems this problem has not yet been completely solved, even for $n = 2$. One aspect of this problem is the question of stability of an equilibrium point.

Let x^0, y^0 be equilibrium points of the system Equation (3.2).

Let $\dot{y} = g(y)$ ...(3.6)

and let U, V be neighbourhoods of the points x^0, y^0. The systems (3.3) and (3.7) are said to be equivalent in neighbourhoods of their equilibrium points x^0, y^0 if there exist neighbourhoods U, V and a bijective mapping $h: U \to V$ such that $(h \circ f^t)x = (g^t \circ h)x$ (for $x \in U$, f^t $x \in U$, $(g^t \circ f^t)x \in V$), i.e., as a result of the substitution $y = h(x)$ the trajectories of the autonomous system Equation (3.2) go into trajectories of the autonomous system Equation (3.6). The equivalence is said to be differentiable (topological) if ***h*** is a diffeomorphism (homeomorphism). Let x^0 be an equilibrium point of the autonomous system Equation (3.2), let the matrix $f'(x^0)$ be non-degenerate, and let it not possess any pure imaginary eigen values. Then the autonomous system Equation (3.2) in a neighbourhood of x^0 is topologically equivalent to its linear part $\dot{y} = f'(x^0)y$. An important example is the autonomous system $\dot{x} = Ax$, $\dot{y} = By$ where A, B are constant matrices with pure imaginary eigen values and $n > 2$; it is not known when these autonomous systems are topologically equivalent. One of the most fundamental problems in the theory of autonomous systems is that of the structure of the entire family of phase trajectories. The most complete results have been obtained for $n = 2$, but even in this case the solution is far from complete.

A plane autonomous system is an autonomous system for which $n = 2$. It follows that the general equation is given by

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Or

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$

The solutions, $P(x, y) = 0$, of this latter equation are called the characteristics of the system. Note that the two representations are not necessarily equivalent since the case $f(x, y) = 0$ presents no problems in the time-domain, but generates an ill-defined problem in phase space.

The topological properties of the orbits are generally determined by the nature of the critical points, $f = g = 0$, and can be discussed through studying the properties of the equation

$$\frac{dy}{dx} = \frac{Ax + By}{Cx + Dy}$$

which is obtained by linearizing $g(x, y)$ and $f(x, y)$ and shifting the coordinate of origin. Alternatively, the phase-plane topology can be determined by considering the nature of the eigenvalues, (λ_1, λ_2), of the linearized time-domain system.

Check Your Progress

1. Give the equation of continuity and differentiability of solutions.
2. How do you find the differential of a function with two variables?
3. State Bendixson non-existence theorem.
4. Define a plane autonomous system.

3.6 UMLAUFSATZ

Let Ω be an open connected subset of the plane $\mathbf{R}^2$, and let $\eta = (\eta_1, \eta_2)$ be a C^0 non-vanishing vector field defined in Ω. For $z \in \Omega$, define a real number $\zeta_\eta(z)$ which represents the angle between $\eta(z)$ and the positive x"direction.

An appropriate approach to do this is by using the complex variables. The positive x"direction is represented by the complex number 1 (or the real vector (1, 0)), and let,

$$\eta_1(z) = \frac{\eta(z)}{|\eta(z)|}$$

denote the unit vector in the direction of $\eta(z)$.

Let $t \in \mathbf{R}$ be any real number such that $e^{it} = \eta_1(z)$. It can be said that t is an angle between $\eta(z)$ and the positive x"direction. This is also an angle between $\eta(z)$ and (1, 0).

Consider any other real number θ such that θ"$t = 2\pi n$ for some integer n also gives us an angle between $\eta(z)$ and the positive x"direction. Consequently, this angle really is an element in the circle $\frac{\mathbf{R}}{2\pi\mathbf{Z}}$; i.e., it is well-defined up to an integral multiple of 2π.

Definition. A region Ω is ***simply connected*** if every closed curve in Ω is homotopic to a constant curve.

Consequently, the region Ω is simply connected if and only if, for every continuous function $\gamma : [0, 1] \to \Omega$ such that $\gamma(1) = \gamma(0)$, there is a continuous function F : $[0, 1] \times [0, 1] \to \Omega$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = \gamma(0) = \gamma(1)$ for all $t \in [0, 1]$.

There is one more significant criterion for simply connectivity. A region Ω is simply connected if and only if every continuous function from the unit circle S^1 in $\mathbf{R}^2$ extends to a continuous function on the closed unit disk D^2 in R^2.

Proposition

Let γ_1 and γ_2 be two Jordan curves which can be continuously deformed into one another without passing through a singularity of the vector field *f*. Then, $jf(\gamma_1) = jf(\gamma_2)$.

Theorem (Umlaufsatz)

Let γ be a C^1 positively oriented Jordan curve in the plane and let γ' be its tangent vector field. Then,

$$j_{\gamma'}(\gamma) = 1$$

Proof.

The result is evidently independent of the position of the curve γ in the plane. Therefore, translate the curve γ so that it is above and tangent to the *x*–axis.

Let the curve be given by $s \to \gamma(s) = (x(s), y(s))$ with $0 \le s \le 1$, $\gamma(0) = \gamma(1)$ and $\gamma(s)$ $\gamma(t)$ for $s < t < 1$.

Consider the triangle $\Delta = \{(s, t) : 0 \le s \le t \le 1\}$, and the subset $\Delta_0 = \{(s, t) : 0 \le s < t \le 1\}$.

3.7 INDEX OF A STATIONARY POINT

Let f(x,y,z) be the objective function for an unconstrained optimization problem then the index of a stationary point can be defined as the number of negative eigen values of the hessian matrix of f(x,y,z).

Stationary Points

A point is said to be stationary point of a differentiable function of one variable is a point of the function where the function's derivative is zero. At the stationary point, the function stops increasing or decreasing, therefore the name stationary is given. To find a stationary point of a function f(x) mathematically, it is defined as a point where the derivative of f(x) is equal to 0 . Graphically, this corresponds to points on the graph of f(x) where the tangent to the curve is a horizontal line.

Stationary points are categorised as: maximum point, minimum point and point of inflection

Maximum Point

The point at which function attaints its maximum value, is said to be its maximum point. At this point gradient of the function is positive just before the maximum point, zero at the maximum point, then negative just after the maximum point.

Mathematically, dy/dx is decreasing with respect to x at this point; i.e. d^2y/dx^2 is negative at maximum point.

Minimum Point

The point at which function attaints its minimum value, is said to be its minimum point. At this point gradient of the function is negative just before the minimum point, zero at the minimum point, then positive just after the minimum point. Mathematically, dy/dx is increasing with respect to x at this point; i.e. d^2y/dx^2 is positive at minimum point.

Point of Inflection

The point at which function is neither maximum not minimum, is said to be point of inflection. Just before a minimum point the gradient is negative, at the minimum the gradient is zero and just after the minimum point it is positive. d^2y/dx^2 is zero at the point of inflection.

The stability of solutions of Ordinary Differential Equations is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the eigenvalues of the equilibrium. An equilibrium point of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is defined as a solution that does not change with time. For an ordinary differential equation, $x2 = f(x)$ an equilibrium solution is defined as $x(t)=x^*$, if $f(x^*)=0$.

Jacobian matrix for a system of ODE can be defined as

$$J = \frac{f_i}{xj} = \begin{bmatrix} \quad \end{bmatrix}$$

where all the derivatives are defined at equilibrium point. While solving linear differential equations, behaviour of solutions can be understood by studying various points. An equilibrium is said to be asymptotically stable if all eigenvalues have negative real parts and it is said to be unstable if at least one eigenvalue has positive real part. The behaviour of solutions near a saddle point is explained by the eigenvalues of the Jacobian matrix. the eigenvalues of a 2×2-matrix can also be both negative or both positive. When both factors will either both decrease in time or both increase in time, then equilibrium points are called *nodes.* An equilibrium point is called a *saddle point* if the Jacobian matrix J has one negative and one positive eigenvalue.

3.8 STABILITY OF PERIODIC SOLUTIONS

NOTES

Periodic solution of an ordinary differential equation or system is the one that periodically depends on the independent variable t. For a periodic solution $x(t)$ (in the case of a system, x is a vector), there is a number $T \neq 0$ such that $x(t + \mathrm{T}) = x(t)$ for $t \in \mathrm{R}$.

All possible such T are called periods of this periodic solution; the continuity of $x(t)$ implies that either $x(t)$ is independent of t or that all possible periods are integral multiples of one of them — the minimal period $T_0 > 0$. When one speaks of a periodic solution, it is often understood that the second case applies, and T_0 is simply termed the period.

A periodic solution is usually considered for a system of ordinary differential equations where the right-hand sides either are independent of:

$$\dot{x} = f(x), \quad x \in U, \qquad \ldots(3.7)$$

where is a region in , or else periodically depend on t:

$$\dot{x} = f(t, x), \quad f(t + T_1, x) = f(t, x), \quad x \in U. \qquad \ldots(3.8)$$

(In a system with a different type of dependence on ***t*** for the right-hand sides there is usually no periodic solution.) In Equation (3.8) the period T_0 of the periodic solution usually coincides with the period T_1 of the right-hand side or is an integer multiple of T_1; other T_0 are possible only in exceptional cases. Periodic solutions with periods $T_0 = kT_1$, $k > 1$, describe subharmonic oscillations and therefore are themselves sometimes called subharmonic periodic solutions.

System Equation (3.8) determines the Poincare return map F (dependent on the choice of the initial moment t_0): If $x(t, \xi)$ is the solution to Equation (3.8) with initial value $x(t_0, \xi) = \xi$ then

$F(\xi) = x(t_0 + T_1, \xi)$.

The properties of Equation (3.8) are closely related to those of $\boldsymbol{F}$; in particular, the value at $t = t_0$ for the periodic solution with period kT_1 is a fixed point of $\boldsymbol{F}$ for $k = 1$, while for $k > 1$ it is a periodic point with period k, i.e., a fixed point for the iterate F^k . The research on periodic solutions reduces to a considerable extent to examining the corresponding fixed or periodic points of the Poincare return map.

The following modification of this construction is used for an autonomous system Equation (3.7): One takes some local section in the phase space at some point on the trajectory of the periodic solution (which is a closed curve), i.e., one takes a smooth manifold $\prod$ of codimension 1 transversal to this trajectory, and considers the mapping that converts a point $\xi \in \prod$ to the point of intersection of the trajectory of Equation (3.7) through ξ with $\prod$ that is first in time.

The behaviour of solutions close to a given periodic solution is described in linear approximation by the corresponding variational system. The coefficients in this linear system in that case periodically depend on ***t***, and therefore one can

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speak of the corresponding monodromy operator and multipliers. The latter are also termed multipliers for the given periodic solution. The linear approximation determines the properties of the periodic solution to the same extent as for an equilibrium solution.

The periodic solutions to system Equation (3.7) have some specific features: one of the multipliers is always one (if the periodic solution does not reduce to a constant), which in particular has to be borne in mind when examining the stability of these periodic solutions, and the period may change in response to small perturbations, which has to be borne in mind in perturbation theory.

The search for periodic solutions and the examination of their behaviour are of interest not only from the purely mathematical point of view but also because the periodic regimes of real physical systems usually correspond to periodic solutions in the mathematical description of these systems. However, this is a very difficult problem, since there are no general methods for establishing whether periodic solutions exist for a particular system. Various arguments and methods are used in different cases.

3.9 ROTATION POINT, FOCI, NODES AND SADDLE POINTS

A critical point is any point $x = c$ for which $F(c) = 0$. Clearly, any such point is then a solution of

$$\frac{d\mathbf{x}}{dt} = \mathrm{F}(\mathbf{x}) \text{ for all } t\ (-\infty < t < +\infty).$$

Example 3.1: Suppose we have the autonom system

$$\begin{aligned} x_1' &= +3x_1 - 5x_1^2x_2 \equiv F_1(x_1, x_2) \\ x_2' &= -2x_2^2 + 6x_1x_2 \equiv F_2(x_1, x_2) \end{aligned}$$

Find the critical points of this system.

Solution: For the critical points, we look for the points for which $F_1(x_1, x_2) = 0$ and $F_2(x_1, x_2) = 0$.

For $F_1(x_1, x_2) = 0$ we have $x_1(3 - 5x_1x_2) = 0$. Thus, we have two cases: either $x_1 = 0$ or $3 - 5x_1x_2 = 0$.

Case I: $x_1 = 0$

We require that $F_2(x_1, x_2) \equiv -2x_2^2 + 6x_1x_2 = 0$ when $x_1 = 0$. Thus we must have $x_2 = 0$.

Case II: $3 - 5x_1x_2 = 0$

We require that $F_2(x_1, x_2) \equiv -2x_2^2 + 6x_1x_2 = 0$ when $3 - 5x_1x_2 = 0$. Thus we must have $x_1x_2 = 3/5$ so that

$$-2x_2^2 + 6x_1x_2 = 0 \rightarrow -2x_2^2 + \frac{18}{5} = 0$$

Or

$$x_2 = \pm\sqrt{\frac{9}{5}}, \quad x_1 = \pm\frac{3}{5}\sqrt{\frac{5}{9}} = \pm\frac{1}{\sqrt{5}}.$$

3.9.1 Stability of Critical Points

We will now discuss the concepts of stability, asymptotic stability and instability of a solution of the autonomous system

$$dx/dt = F(x,y), \quad dy/dt = G(x,y) \qquad \text{….(3.9)}$$

Here we will give a precise mathematical meaning to these con-cepts.

A critical point $x = x_0$, $y = y_0$ (an equilibrium solution $x = x_0$, $y = y_0$) of the autonomous system (3.9) is said to be a *stable critical point* if, given any $\varepsilon > 0$, it is possible to find a δ such that every solution $x = \phi(t)$, $y = \psi(t)$ of the system (3.9), which at $t = 0$ satisfies

$$\{[\phi(0) - x_0]^2 + [\Psi(0) - y_0]^2\}^{1/2} < \delta, \qquad \text{…(3.10)}$$

and satisfies

$$\{[\phi(t) - x_0]^2 + [\Psi(t) - y_0]^2\}^{1/2} < \varepsilon, \qquad \text{…(3.11)}$$

for all $t > 0$. These mathematical statements say that all solutions that start sufficiently close to (x_0, y_0) stay close to (x_0, y_0).

A critical point (x_0, y_0) is said to be *asymptotically stable* if it is stable and if there exists a δ_0, $0 < \delta_0 < \delta$, such that if a solution $x = \theta(t)$, $y = \psi(t)$ satisfies

$$\{[\phi(0) - x_0]^2 + [\Psi(0) - y_0]^2\}^{1/2} < \delta_0, \qquad \text{...(3.12)}$$

Then

$$\lim_{t\to\infty} \phi(t) = x_0, \qquad \lim_{t\to\infty} \psi(t) = y_0. \qquad \text{….(3.13)}$$

Trajectories that start sufficiently close to (x_0, y_0) must not only stay close but must eventually approach (x_0, y_0) as t approaches infinity. Asymptotic stability is a stronger requirement than stability, since a critical point must be stable before we can even talk about whether it is asymptotically stable. On the other hand, the limit condition, which is an essential feature of asymp-totic stability, does not by itself imply even ordinary stability. Geometrically, all that is needed is a family of trajectories having members that start arbitrarily close to $(x_0. y_0)$, then recede an arbitrarily large distance before eventually approaching $(x_0. y_0)$ as t approaches infinity. For the linear system,

$$dx/dt = ax + by, \qquad dy/dt = cx + dy \qquad \text{…(3.14)}$$

with $ad - bc \neq 0$, the type and stability of the critical point (0, 0) as a function of the roots $r_1 \neq 0$ and $r_2 \neq 0$ of the characteristic equation,

$$r^2 - (a+d)r + ad - bc = 0 \qquad \text{…(3.15)}$$

are summarized in the following theorem:

Theorem 3.2: The critical point (0, 0) of the linear system (3.15) is

(i) asymptotically stable if the roots r_1, r_2 of the characteristic Equation (3.15) are real and negative or have negative real parts.

(ii) stable, but not asymptotically stable, if r_1 and r_2 are pure imaginary.

(iii) unstable if r_1 and r_2 are real and either is positive, or if they have positive real parts.

Notice that if a critical point of the linear system (3.13) is asymptotically stable then not only do trajectories that start close to the critical point approach the critical point, but, in fact, since every solution is a linear combination $e^{r_1 t}$ of and $e^{r_2 t}$ *every* trajectory approaches the critical point. In this case the critical point is said to be *globally asymptotically stable.* This property of linear systems is not, in general, true for nonlinear systems. Often an important practical problem in considering an asymptotically stable critical point of a nonlinear system is to estimate the set of initial conditions for which the critical point is asymptotically stable. This set of initial points is called the *region of asymptotic stability* for the critical point.

We now want to relate the results for the linear system (3.13) to the nonlinear system.

$$\frac{dx}{dt} = ax + by + F_1(x, y),$$

$$\frac{dy}{dt} = cx + dy + G_1(x, y), \qquad \ldots(3.16)$$

We assume that (0,0) is a critical point of the system (3.16) and that $ad - bc \neq 0$. Also we assume that F_1 and G_1 have continuous first partial derivatives and are small near the origin in the sense that $F_1(x, y)/r \to 0$ and $G_1(x, y)/r \to 0$ as $r \to 0$, where $r = (x^2 + y^2)^{1/2}$. Recall that such a system is said to be almost linear in the neighborhood of the origin. As an example, the system

$$\frac{dx}{dt} = x - x^2 - xy,$$

$$\frac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y_2 - \frac{3}{4}xy, \qquad \ldots(3.17)$$

satisfies the stated conditions. Here $a = 1$, $b = 0$, $c = 0$, $d = 1/2$. $F_1(x, y) = -x^2 - xy$, and $G_1(x,y) = -¼\, y^2 - ¾\, xy$. To show that $F_1(x,y)/r \to 0$ as $r \to 0$, let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{F_1(x, y)}{r} = \frac{-r^2\cos^2\theta - r^2\sin\theta\cos\theta}{r} = -r(\cos^2\theta + \cos\theta\sin\theta) \to 0 \quad \ldots(3.18)$$

as $r \to 0$. The argument that $G_1(x, y)/r \to 0$ as $r \to 0$ is similar.

Theorem 3.3: Let r_1 and r_2 be the roots of the characteristic Equation (3.15) of the linear system (3.13) corresponding to the almost linear system (3.16). Then the type and stability of the critical point (0, 0) of the linear system (3.14) and the almost linear system (3.17) are as shown in the table below:

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$r_1 r_2$	Linear System		Almost Linear System	
	Type	*Stability*	*Type*	*Stability*
$r_1 > r_2 > 0$	*IN*	*Unstable*	*IN*	*Unstable*
$r_1 < r_2 < 0$	*IN*	*Asymptotically stable*	*IN*	*Asymptotically stable*
$r_2 < 0 < r_1$	*SP*	*Unstable*	*SP*	*Unstable*
$r_1 = r_2 > 0$	*PN or IN*	*Unstable*	*PN, IN, or SpP*	*Unstable*
$r_1 = r_2 < 0$	*PN or IN*	*Asymptotically stable*	*PN, IN or SpP*	*Asymptotically stable*
$r_1 = r_2 = \lambda \pm i\mu$				
$\lambda > 0$	*SpP*	*Unstable*	*SpP*	*Unstable*
$\lambda > 0$	*SpP*	*Asymptotically stable*	*SpP*	*Asymptotically stable*
$r_1 = i\mu,\ r_2 = -i\mu$	*C*	*Stable*	*C or SpP*	*Indeterminate*

IN = Improper node, PN = Proper node; SP = Saddle point;
SpP = Spiral point; C = Center

Theorem 3.3 says that for x and y near zero the nonlinear terms $F_1(x, y)$ and $G_1(x, y)$ are small and do not affect the stability and type of critical point as determined by the linear terms except in two sensitive cases: r_1 and r_2 pure imaginary, and r_1 and r_2 real and equal. Small pertur-bations in the coefficients of the linear system (3.14), and hence in the roots r_1 and r can alter the type and stability of the critical point only in these two sensitive cases. When r_1 and r_2 are pure imaginary, a small perturbation can change the stable center into either an asymptotically stable or an unstable spiral point or even leave it as a center. When $r_1 = r_2$ a small perturbation does not affect the stability of the critical point, but may change the node into a spiral point. It is reasonable to expect that the small nonlinear terms in Equation (3.16) might have a similarly substantial effect, at least in these two sensitive cases. This is so, but the main significance of Theorem 3.3 is that in *all other cases* the small nonlinear terms do not alter the type or stability of the critical point. Thus, except in the two sensitive cases, the type and stability of the critical point of the nonlinear system (3.16) can be determined from a study of the much simpler linear system (3.14).

Even if the critical point is of the same type as that of the linear system, the trajectories of the almost linear system may be considerably different in appear-ance from those of the corresponding linear system.

Classification of Critical Points

1. Nodes $\lambda_1 \lambda_2 > 0$
   - If the eigenvalues are both negative, then we have a stable node.
   - If the eigenvalues are both positive, then we have an unstable node.
2. Saddle points $\lambda_1 \lambda_2 < 0$
   - Since one eigenvalue is necessarily positive, then the critical point is necessarily unstable.
3. Spiral Point or Focus $\lambda_1 = \alpha \pm i\beta$
   - The spiral point is stable if $\alpha < 0$ and unstable if $\alpha > 0$.
4. Centre: $\lambda = \pm i\beta$

- The centre point is said to be uniformly stable and the phase-plane orbits are circles or ellipses.

Check Your Progress

5. Give definition of Umlaufsatz.
6. How will you define the index of stationary point?
7. What do you mean by period of the periodic solution?
8. When is a critical point stable?
9. What is region of asymptotic stability?
10. Name the various types of critical points.

3.10 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $x(t)$ is defined as an ρ-approximate solution of the DE dx/dt= f (t, x) on an interval I if |x (t) – f (t, x(t))| ≤ r ∀ t Î I. (r)
2. The differential dy depends on the following two quantities:
   - The variable x (through the derivative $y = f'(x)$).
   - The differential of the independent variable dx.
3. Bendixson's criteria states that if D is a simply connected open subset of R^2 and if the expression is not identically zero and does not change sign in D then there are no periodic orbits of the autonomous system in D.
4. A plane autonomous system is an autonomous system for which n = 2. It follows that the general equation is given by

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

5. A region Ω is simply connected if every closed curve in Ω is homotopic to a constant curve.
6. Let f(x,y,z) be the objective function for an unconstrained optimization problem then the index of a stationary point can be defined as the number of negative eigen values of the hessian matrix of f(x,y,z).
7. For a periodic solution $x(t)$ (in the case of a system, x is a vector), there is a number $T \neq 0$ such that $x(t + \text{T}) = x(t)$ for $t \in$ R.
8. A critical point $x = x_0, y = y_0$ (an equilibrium solution $x = x_0, y = y_0$) of the autonomous system $dx/dt = F(x,y)$, $dy/dt = G(x,y)$ is said to be a stable critical point if, given any $\varepsilon > 0$, it is possible to find a δ such that every solution $x = \Phi(t), y = \psi(t)$ of the system, which at $t = 0$ satisfies

   $\{[\phi(0) - x_0]^2 + [\Psi(0) - y_0]^2\}^{1/2} < \delta$ and satisfies

   $\{[\phi(t) - x_0]^2 + [\Psi(t) - y_0]^2\}^{1/2} < \varepsilon$

   for all $\boldsymbol{t} > 0$.

NOTES

9. Often an important practical problem in considering an asymptotically stable critical point of a nonlinear system is to estimate the set of initial conditions for which the critical point is asymptotically stable. This set of initial points is called the region of asymptotic stability for the critical point.
10. Nodes, saddle points, focus and centre are the various types of critical points.

3.11 SUMMARY

- The dependence of solutions of initial value problems on the initial values and on parameters in the differential equation can be studied with the knowledge of a fundamental estimate.
- Dependence on parameters can be transformed into initial conditions
- To explain the concept of higher order differentiability, consider a function $y = f(x)$ differentiable in the interval (a, b). For defining the first-order differential of the function at the point $x \in (a, b)$ the formula is $dy = f'(x)$ dx.
- On fixing the increment dx, it is assumed that dx is constant. Then the differential dy becomes a function only of the variable x for which the differential can also be defined by taking the same differential dx as the increment Δx.
- Bendixson's criteria and Dulac's criteria give the sufficient conditions that rule out the possibility of periodic solutions.
- The basic result of H. Poincare and I. Bendixson is that any bounded semi-trajectory (positive or negative) either tends to an equilibrium position or coils round (like a spiral) to a limit cycle, or in an analogous way coils to a closed separatrix or separatrix contour consisting of several separatrices joining certain equilibrium positions, or is itself an equilibrium position or a closed trajectory.
- The general form of a first-order autonomous system in normal form is:

  $\dot{x}_j = f_j\left(x_1, \cdots x_n\right), \qquad j = 1, \cdots, n,$ or, in vector notation, $\dot{x} = f\left(x\right)$.
- The composition of two functions such that $y = f(u)$ and $u = g(x)$. In this instance, y is a composite function of the independent variable x.
- Let Ω be an open connected subset of the plane $\mathbf{R}^2$, and let $\eta = (\eta_1, \eta_2)$ be a C^0 non-vanishing vector field defined in Ω. For $z \in \Omega$, define a real number $\zeta_\eta(z)$ which represents the angle between $\eta(z)$ and the positive x"direction.
- A region Ω is simply connected if every closed curve in Ω is homotopic to a constant curve.
- Let $f(x,y,z)$ be the objective function for an unconstrained optimization problem then the index of a stationary point can be defined as the number of negative eigen values of the hessian matrix of $f(x,y,z)$.

- A point is said to be stationary point of a differentiable function of one variable is a point of the function where the function's derivative is zero.
- The point at which function attaints its maximum value, is said to be its maximum point. At this point gradient of the function is positive just before the maximum point, zero at the maximum point, then negative just after the maximum point.
- The point at which function attaints its minimum value, is said to be its minimum point. At this point gradient of the function is negative just before the minimum point, zero at the minimum point, then positive just after the minimum point.
- The point at which function is neither maximum not minimum, is said to be point of inflection. Just before a minimum point the gradient is negative, at the minimum the gradient is zero and just after the minimum point it is positive.
- Periodic solution of an ordinary differential equation or system is the one that periodically depends on the independent variable t.
- A critical point can be a node, saddle point, focus and centre.

3.12 KEY TERMS

- **Autonomous system:** Autonomous system of ordinary differential equations is a system of ordinary differential equations which does not explicitly contain the independent variable.
- **Stationary points:** A point is said to be stationary point of a differentiable function of one variable is a point of the function where the function's derivative is zero.
- **Maximum point:** The point at which function attaints its maximum value, is said to be its maximum point. At this point gradient of the function is positive just before the maximum point, zero at the maximum point, then negative just after the maximum point.
- **Minimum point:** The point at which function attaints its minimum value, is said to be its minimum point. At this point gradient of the function is negative just before the minimum point, zero at the minimum point, then positive just after the minimum point.
- **Point of inflection:** The point at which function is neither maximum not minimum, is said to be point of inflection. Just before a minimum point the gradient is negative, at the minimum the gradient is zero and just after the minimum point it is positive.
- **Periodic solution:** Periodic solution of an ordinary differential equation or system is the one that periodically depends on the independent variable.
- **Index of a critical point:** For a function of n variables, the number of negative eigenvalues of a critical point is called its index.

3.13 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Define the continuity and differentiability of solutions.
2. What do you mean by the higher order differentiability?
3. State the Poincare-Bendixson theory.
4. Give the local properties of autonomous systems.
5. What is Umlaufsatz?
6. How will you define the index of a stationary point?
7. Define asymptotic stability of a critical point.
8. When is a critical point said to be stable?

Long-Answer Questions

1. Discuss the continuity with respect to parameters and initial conditions with the help of giving examples.
2. What do you understand by the higher order differentiability? Discuss the properties of higher-order differentials with the help of relevant examples.
3. For the following system of equations, use the Poincare-Bendixson theorem to show that at least one limit cycle solution exists:
   a. $\dot{x} = 2x + 2y - x(2x^2 + y^2), \quad \dot{y} = -2x + 2y - y(2x^2 + y^2).$
   b. $\dot{x} = x - y - x(x^2 + \frac{3}{2}y^2), \quad \dot{y} = x + y - y(x^2 + \frac{1}{2}y^2).$
4. Briefly explain about the autonomous systems with the help of giving examples.
5. Discuss the definition of Umlaufsatz. Give appropriate examples.
6. Explain about the index of stationary point with the help of giving examples.
7. Determine the periodic solution of

$$\dot{x} = x - y - x(x^2 + y^2)$$
$$\dot{y} = x + y - y(x^2 + y^2).$$

8. Find the critical points of $f(x) = |x^2 - x|$.
9. Find the critical points of the function,

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$$

10. Find the critical points and trajectories of the following system,

$$\frac{dx}{dt} = 4 - 2y$$
$$\frac{dy}{dt} = 12 - 3x^2$$

3.14 FURTHER READING

NOTES

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UNIT 4 LINEAR SECOND ORDER EQUATIONS

NOTES

Structure

4.0 INTRODUCTION

A linear differential equation or a system of linear equations with constant coefficients for the associated homogeneous equations can be solved using quadrature, which means the solutions can be represented in terms of integrals. This is also true for a non-constant coefficient linear equation of order one. In general, quadrature cannot solve an equation of order two or higher with non-constant coefficients. For order two, Kovacic's approach allows determining whether there are integral solutions and, if so, computing them. Holonomic functions are the solutions to linear differential equations with polynomial coefficients. Many common and special functions, such as exponential, logarithm, sine, cosine, inverse trigonometric functions, error function, Bessel functions, and hypergeometric functions, belong to this class of functions, which are stable under sums, products, differentiation, and integration. Most calculus operations, such as computation of antiderivatives, limits, asymptotic expansion, and numerical evaluation to any precision with a certified error bound, can be made algorithmic (on these functions) thanks to their representation by the defining differential equation and initial conditions.

Sturm sequence of a univariate polynomial p is a sequence of polynomials associated with p and its derivative by a variant of Euclid's algorithm for polynomials. Sturm's theorem expresses the number of distinct real roots of p located in an interval in terms of the number of changes of signs of the values of the Sturm sequence at the bounds of the interval. Applied to the interval of all the real numbers, it gives the total number of real roots of p.

In differential equations, a boundary value problem is a differential equation together with a set of additional restraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. A large class of important boundary value problems include the Sturm-Liouville problems. The analysis of these problems involves the eigen functions of a differential operator. In mathematical applications, a boundary value problem should be well established. This means that given the input to the problem there exist a unique solution, which depends continuously on the input.

Oscillation theory was initiated by Jacques Charles François Sturm in his investigations of Sturm–Liouville problems from 1836. There he showed that the nth eigen function of a Sturm–Liouville problem has precisely n-1 roots. For the one-dimensional Schrödinger equation the question about oscillation/nonoscillation answers the question whether the eigenvalues accumulate at the bottom of the continuous spectrum.

In this unit, you will learn about the linear differential equations of second order, theorems of Strum, Sturm–Liouville boundary value problem, non-oscillatory equations and principle solutions, nonoscillation theorems and numbers of zeros in second order linear equation.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic concept of linear differential equations of second order
- Discuss about the theorems of strum
- Define Lagrange's identity and Green's formula for second order differential equations
- Learn about the nonoscillation theorem
- Analysis the nonoscillatory equations and principle solutions
- Explain about the numbers of zeros

4.2 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

Linear differential equation of second order is an equation of the form

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = \mathrm{R}$$

where P, Q and R are the functions of x.

NOTES

4.2.1 Solution by Changing the Dependent Variable when One Integral Belonging to the C.F. is Known

Let the linear differential equation of second order be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(4.1)$$

where P, Q and R are the functions of x only.

Let $y = u$ be a known integral belonging to the C.F. of the equation (4.1). Thus, its solution is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ...(4.2)$$

Taking $y = uv$ and differentiating with respect to x, we get

$$\frac{dy}{dx} = u_1v + uv_1 \text{ and } \frac{d^2y}{dx^2} = u_2v + 2u_1v_1 + uv_2$$

Putting these values in Equation (4.1), we get

$$(u_2v + 2u_1v_1 + uv_2) + P(u_1v + uv_1) + Q(uv) = R$$

$$\Rightarrow \quad uv_2 + (2u_1 + Pu)v_1 + (u_2 + Pu_1 + Qu)v = R \qquad ...(4.3)$$

Since $y = u$ is the solution of Equation (4.2), thus Equation (4.2) can be written as

$$u_2 + Pu_1 + Qu = 0$$

Using this value in equation (4.3), we get

$$uv_2 + (2u_1 + Pu)v_1 + 0 = R$$

$$\Rightarrow \quad v_2 + \left(\frac{2}{u}u_1 + P\right)v_1 = \frac{R}{u}$$

Now, putting $v_1 = p$ and $v_2 = \frac{dp}{dx}$, we get

$$\frac{dp}{dx} + \left(\frac{2}{u}u_1 + P\right)p = \frac{R}{u} \qquad ...(4.4)$$

This is a linear equation in p. Thus, its

$$\text{I.F.} = e^{\int\left(\frac{2}{u}u_1 + P\right)dx} = e^{2\log u}.e^{\int P dx} = e^{\log u^2}.e^{\int P dx} = u^2e^{\int P dx}$$

Now, we have the solution of equation (4.4) as

$$p.u^2e^{\int P dx} = \int\left(\frac{R}{u}.u^2e^{\int P dx}\right)dx + c_1$$

$$\Rightarrow p = u^{-2}e^{-\int P dx}\int\left(Rue^{\int P dx}\right)dx + c_1u^{-2}e^{-\int P dx}$$

$$\Rightarrow \frac{dv}{dx} = u^{-2}e^{-\int P dx}\int\left(Rue^{\int P dx}\right)dx + c_1u^{-2}e^{-\int P dx} \qquad \left[\because p = \frac{dv}{dx}\right]$$

NOTES

Integrating both sides, we get

$$v = \int \left(u^{-2} e^{-\int \mathrm{P}dx} \int \mathrm{R}ue^{\int \mathrm{P}dx} dx \right) dx + c_1 \int u^{-2} e^{-\int \mathrm{P}dx} dx + c_2$$

The complete solution of Equation (4.1) is given by $y = uv$

$$\Rightarrow y = u \int \left(u^{-2} e^{-\int \mathrm{P}dx} \int \mathrm{R}ue^{\int \mathrm{P}dx} dx \right) dx + c_1 u \int u^{-2} e^{-\int \mathrm{P}dx} dx + c_2 u$$

where c_1 and c_2 are two arbitrary constants.

Determining the particular integral of $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = 0$

In case the integral of C.F. is not known while solving linear differential equations of second degree, one of the following rules helps us in finding the particular integral of

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = 0 \qquad ...(4.5)$$

Rule 1: Let $y = e^{mx}$ be the solution of Equation (4.5).

Differentiating with respect to x, we get

$$\frac{dy}{dx} = me^{mx} \text{ and } \frac{d^2y}{dx^2} = m^2 e^{mx}$$

Putting these values in Equation (4.5), we get

$$m^2 e^{mx} + \mathrm{P}me^{mx} + \mathrm{Q}e^{mx} = 0$$

$$\Rightarrow \left(m^2 + \mathrm{P}m + \mathrm{Q}\right)e^{mx} = 0$$

$$\Rightarrow m^2 + \mathrm{P}m + \mathrm{Q} = 0$$

Thus, $y = e^{mx}$ is the solution of Equation (4.5) if $m^2 + \mathrm{P}m + \mathrm{Q} = 0$.

Corollary: Taking $m = 1$, $y = e^x$ is a solution of Equation (4.5) if $1 + \mathrm{P} + \mathrm{Q} = 0$.

Taking $m = -1$, $y = e^{-x}$ is a solution of Equation (4.5) if $1 - \mathrm{P} + \mathrm{Q} = 0$.

Taking $m = a$, $y = e^{ax}$ is a solution of Equation (4.5) if $a^2 + a\mathrm{P} + \mathrm{Q} = 0$.

Rule 2: Let $y = x^m$ be the solution of Equation (4.5).

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = mx^{m-1} \text{ and } \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Putting these values in Equation (4.5), we get

$$m(m-1)x^{m-2} + \mathrm{P}mx^{m-1} + \mathrm{Q}x^m = 0$$

$$\Rightarrow m(m-1) + \mathrm{P}mx + \mathrm{Q}x^2 = 0$$

Thus, $y = x^m$ is the solution of equation (4.5) if $m(m-1) + \mathrm{P}mx + \mathrm{Q}x^2 = 0$.

Corollary: Taking $m = 1, y = x$ is a solution of Equation (4.5) if $P + Qx = 0$.

Taking $m = 2, y = x^2$ is a solution of Equation (4.5) if $2 + 2Px + Qx^2 = 0$.

Example 4.1: Solve the differential equation

$$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)\, y = e^x \sin x.$$

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)\, y = e^x \sin x \qquad ...(1)$$

Comparing it with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$ we get

$$P = -\cot x, Q = -1 + \cot x, R = e^x \sin x$$

Now, $P + Q + 1 = -\cot x - 1 + \cot x + 1 = 0$

Since P + Q + 1= 0, thus $y = e^x$ is a part of C.F.

Taking $y = ve^x$,

$$\Rightarrow \quad \frac{dy}{dx} = e^x\frac{dv}{dx} + e^x v \text{ and } \frac{d^2y}{dx^2} = e^x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.e^x + ve^x$$

Substituting these values in Equation (1), we get

$$e^x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.e^x + ve^x - \cot x\left(e^x\frac{dv}{dx} + e^x v\right) - (1 - \cot x)ve^x = e^x \sin x$$

$$\Rightarrow \frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x$$

$$\Rightarrow \frac{dp}{dx} + (2 - \cot x)\, p = \sin x \text{ where } p = \frac{dv}{dx} \text{ and } \frac{dp}{dx} = \frac{d^2v}{dx^2} \qquad ...(2)$$

This is a linear differential equation in p i.e., $\frac{dp}{dx} + P'p = Q'$, where $P' = 2 - \cot x$ and $Q' = \sin x$. Thus, its I.F.

$$= e^{\int P'dx} = e^{\int (2-\cot x)dx} = e^{2x - \log\sin x} = \frac{e^{2x}}{e^{\log\sin x}} = \frac{e^{2x}}{\sin x}$$

Now, the solution of Equation (2) is given by

$$p(\text{I.F}) = \int Q'(\text{I.F})dx + c_1$$

$$\Rightarrow \quad p.\frac{e^{2x}}{\sin x} = \int \sin x.\frac{e^{2x}}{\sin x}dx + c_1$$

$$\Rightarrow \quad p.\frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow \quad p = \frac{e^{2x}}{2}.\frac{\sin x}{e^{2x}} + c_1\frac{\sin x}{e^{2x}}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{\sin x}{2} + c_1 e^{-2x}\sin x \qquad \left[\because p = \frac{dv}{dx}\right]$$

$$\Rightarrow \qquad dv = \left(\frac{\sin x}{2} + c_1 e^{-2x}\sin x\right)dx$$

NOTES

Integrating both sides, we get $\int dv = \int \left[\frac{\sin x}{2} + c_1 e^{-2x}\sin x\right]dx$

$$\Rightarrow v = -\frac{\cos x}{2} + \frac{c_1 e^{-2x}}{5}\left[-2\sin x - \cos x\right] + c_2$$

Thus, the complete solution of Equation (1) is given by $y = ve^x$

$$\Rightarrow y = -\frac{1}{2}e^x \cos x - \frac{c_1 e^{-x}}{5}\left[2\sin x + \cos x\right] + c_2 e^x$$

4.2.2 Solution by Removing the First Derivative and Changing the Dependent Variable

In case the integral of the C.F. is neither known nor can be found using the rules, there is a need of other method to find the solution of linear differential equation of second order. Here, we will learn the method which is independent of integral of C.F.

Consider the linear differential equation of second order

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R} \qquad ...(4.6)$$

Change the dependent variable in the Equation (4.6) by putting $y = uv$, where u and v are the functions of x.

Now, $\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$ and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

Putting these values in Equation (4.6), we get

$$\left(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + \mathrm{P}\left(u\frac{dv}{dx} + \frac{du}{dx}v\right) + \mathrm{Q}(uv) = \mathrm{R}$$

$$\Rightarrow u\frac{d^2v}{dx^2} + \left(\mathrm{P}u + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + \mathrm{P}\frac{du}{dx} + \mathrm{Q}u\right)v = \mathrm{R}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\mathrm{P} + \frac{2}{u}.\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{1}{u}.\frac{d^2u}{dx^2} + \frac{\mathrm{P}}{u}.\frac{du}{dx} + \mathrm{Q}\right)v = \frac{\mathrm{R}}{u} \qquad ...(4.7)$$

Taking u such that the coefficient of first derivative $\frac{dv}{dx} = 0$ (i.e., removing first derivative from Equation (4.7)), we get

$$\mathrm{P} + \frac{2}{u}\frac{du}{dx} = 0 \Rightarrow \frac{du}{u} = \frac{-\mathrm{P}}{2}dx$$

Integrating both sides, we get

$$\int \frac{du}{u} = \frac{-1}{2}\int \mathrm{P}dx$$

NOTES

$$\Rightarrow \log u = \frac{-1}{2}\int \text{P}dx \Rightarrow u = e^{\frac{-1}{2}\int \text{P}dx} \qquad \ldots(4.8)$$

Since $\text{P} + \frac{2}{u}\frac{du}{dx} = 0$, Equation (4.8) becomes

$$\frac{d^2v}{dx^2} + \left(\frac{1}{u}.\frac{d^2u}{dx^2} + \frac{\text{P}}{u}.\frac{du}{dx} + \text{Q}\right)v = \frac{\text{R}}{u} \qquad \ldots(4.9)$$

From Equation (7.8),

$$\frac{du}{dx} = e^{\frac{-1}{2}\int \text{P}dx}\left(\frac{-1}{2}\text{P}\right) = \frac{-1}{2}\text{P}u$$

and $$\frac{d^2u}{dx^2} = \frac{-1}{2}\left(\frac{\text{P}du}{dx} + u\frac{d\text{P}}{dx}\right) = \frac{-1}{2}\left(\frac{-1}{2}\text{P}^2u + u\frac{d\text{P}}{dx}\right) = \frac{1}{4}\text{P}^2u - \frac{1}{2}u\frac{d\text{P}}{dx}$$

Putting these values in Equation (4.9), we get

$$\frac{d^2v}{dx^2} + \left[\frac{1}{u}\left(\frac{1}{4}\text{P}^2u - \frac{1}{2}u\frac{d\text{P}}{dx}\right) + \frac{\text{P}}{u}\left(\frac{-1}{2}\text{P}u\right) + \text{Q}\right]v = \text{R}e^{\frac{1}{2}\int \text{P}dx}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{1}{4}\text{P}^2 - \frac{1}{2}\frac{d\text{P}}{dx} - \frac{1}{2}\text{P}^2 + \text{Q}\right]v = \text{R}e^{\frac{1}{2}\int \text{P}dx}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\text{Q} - \frac{1}{2}\frac{d\text{P}}{dx} - \frac{1}{4}\text{P}^2\right]v = \text{R}\,e^{\frac{1}{2}\int \text{P}dx} \Rightarrow \frac{d^2v}{dx^2} + \text{P}'v = \text{Q}' \qquad \ldots(4.10)$$

where $\text{P}' = \text{Q} - \frac{1}{2}.\frac{d\text{P}}{dx} - \frac{1}{4}\text{P}^2, \text{Q}' = \text{R}\,e^{\frac{1}{2}\int \text{P}dx}$

The Equation (4.10) is called the normal form of the Equation (4.6). Equation (4.10) can easily be integrated and then can be solved for v.

Thus, the general solution of Equation (4.6) is $y = uv$, which contains two arbitrary constants.

Example 4.2: Solve the differential equation $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + y\cos^2 x = 0$.

Solution: The given differential equation is $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + y\cos^2 x = 0$

$$\Rightarrow \qquad \cos^2 x\frac{d^2y}{dx^2} + (-2\sin x\cos x)\frac{dy}{dx} + y\cos^2 x = 0$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + y = 0 \qquad \ldots(1)$$

Comparing Equation (1) with $\frac{d^2y}{dx^2} + \text{P}\frac{dy}{dx} + \text{Q}y = \text{R}$, we get

$\text{P} = -2\tan x$, $\text{Q} = 1$ and $\text{R} = 0$

Putting $y = uv$ in Equation (1), the equation is transformed into

$$\frac{d^2v}{dx^2} + \text{P}'v = \text{Q}' \qquad \ldots(2)$$

NOTES

where $u = e^{-\frac{1}{2}\int \mathrm{P}dx} = e^{-\frac{1}{2}\int -2\tan x\, dx} = e^{\log \sec x} = \sec x$

$$\mathrm{P}' = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2 = 1 - \frac{1}{2}\frac{d}{dx}(-2\tan x) - \frac{1}{4}(4\tan^2 x)$$

$$= 1 + \sec^2 x - \tan^2 x = 1 + (1 + \tan^2 x) - \tan^2 x = 2$$

and $\quad \mathrm{Q}' = \mathrm{R}\, e^{\frac{1}{2}\int \mathrm{P}dx} = 0$

Putting these values in Equation (2), it gets transformed into $\dfrac{d^2v}{dx^2} + 2v = 0$ …(3)

Symbolic form of this equation is $(\mathrm{D}^2 + 2)v = 0$

Its auxiliary equation is $\mathrm{D}^2 + 2 = 0 \Rightarrow \mathrm{D}^2 = -2 \;\Rightarrow\; \mathrm{D} = \pm\sqrt{2}i$

Now, the solution of Equation (3) is given by $v = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

Thus, the solution of Equation (1) is given by $y = uv$

$$\Rightarrow \quad y = \sec x\left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$$

4.2.3 Solution by Changing the Independent Variable

Consider the linear differential equation of second degree

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R} \qquad \text{...(4.11)}$$

Changing the independent variable x to z with the help of relation $z = f(x)$, we get

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}.\frac{dz}{dx}\right) = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}.\frac{d^2z}{dx^2}$$

Substituting these values in the Equation (4.11), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}.\frac{d^2z}{dx^2} + \mathrm{P}\left(\frac{dy}{dz}.\frac{dz}{dx}\right) + \mathrm{Q}y = \mathrm{R}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \frac{\dfrac{d^2z}{dx^2} + \mathrm{P}\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{\mathrm{Q}y}{\left(\dfrac{dz}{dx}\right)^2} = \frac{\mathrm{R}}{\left(\dfrac{dz}{dx}\right)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \mathrm{P}_1\frac{dy}{dz} + \mathrm{Q}_1 y = \mathrm{R}_1 \qquad \text{...(4.12)}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}, Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}, R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Here, P_1, Q_1 and R_1 are the functions of x but can be expressed as the functions of z with the help of the relation between z and x.

1. Choosing z such that the coefficient of P_1 is zero.

$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = 0$$

$$\Rightarrow \qquad \frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \Rightarrow \frac{d^2z}{dx^2} = -P\frac{dz}{dx} \Rightarrow \frac{\dfrac{d^2z}{dx^2}}{\dfrac{dz}{dx}} = -P$$

Integrating both sides, we get

$$\log\frac{dz}{dx} = -\int P dx \Rightarrow \frac{dz}{dx} = e^{-\int P dx}$$

Integrating again, we get $z = \int\left(e^{-\int P dx}\right)dx$

For the relation $z = \int\left(e^{-\int P dx}\right)dx$, P_1 will be zero and Equation (4.12) reduces to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1 \qquad ...(4.13)$$

If Q_1 is constant or a constant multiplied by $\dfrac{1}{z^2}$, then Equation (4.13) can be solved easily giving the value of y in terms of z. Then, by replacing z in terms of x, we get the general solution of Equation (4.11).

2. Choosing z such that the coefficient of Q_1 is constant.

$$Q_1 = a^2 \text{ (say)}$$

$$\Rightarrow \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = a^2 \Rightarrow a^2\left(\frac{dz}{dx}\right)^2 = Q$$

$$\Rightarrow a\frac{dz}{dx} = \sqrt{Q} \Rightarrow \frac{dz}{dx} = \frac{1}{a}\sqrt{Q}$$

Integrating both sides, we get

$$z = \frac{1}{a}\int\sqrt{Q}dx$$

For the relation $z = \dfrac{1}{a}\int\sqrt{Q}dx$, $Q_1 = a^2$ and Equation (4.12) reduces to

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+a^2y=R_1 \qquad ...(4.14)$$

If P_1 is a constant, then Equation (4.14) can be solved easily giving the value of y in terms of z. Then, by replacing z in terms of x, we get the general solution of Equation (4.11).

NOTES

Example 4.3: Solve the differential equation

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x.$$

Solution: The given differential equation is

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x \qquad ...(1)$$

Comparing it with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we get

$$P=3\sin x-\cot x,\ Q=2\sin^2 x \text{ and } R=e^{-\cos x}\sin^2 x$$

Let $\frac{dz}{dx}=e^{-\int P\,dx}=e^{-\int(3\sin x-\cot x)dx}=e^{3\cos x+\log\sin x}=\sin x.e^{3\cos x}$

Integrating, we get

$$z=\int e^{3\cos x}\sin x\,dx=\frac{1}{-3}\int e^{3\cos x}(-3\sin x)dx=-\frac{1}{3}e^{3\cos x}$$

On changing the independent variable x to z by the relation $z=-\frac{1}{3}e^{3\cos x}$, Equation (1) reduces to the form

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \qquad ...(2)$$

where $P_1=0,\ Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{2\sin^2 x}{\sin^2 x\left(e^{3\cos x}\right)^2}=\frac{2}{(-3z)^2}=\frac{2}{9z^2}$

and $R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{e^{-\cos x}\sin^2 x}{\sin^2 x\left(e^{3\cos x}\right)^2}$

$$=\frac{e^{-\cos x}}{e^{6\cos x}}=e^{-7\cos x}=\left(e^{3\cos x}\right)^{-\frac{7}{3}}=(-3z)^{-\frac{7}{3}}=-\frac{1}{(3)^{7/3}}z^{-\frac{7}{3}}$$

Substituting these values in Equation (2), it becomes

$$\frac{d^2y}{dz^2}+\frac{2}{9z^2}y=-\frac{1}{(3)^{7/3}}z^{-\frac{7}{3}}$$

Multiplying by z^2, we get

$$z^2\frac{d^2y}{dz^2}+\frac{2}{9}y=-\frac{1}{(3)^{7/3}}z^{-\frac{1}{3}} \qquad ...(3)$$

Putting $z=e^t \Rightarrow \log z=t$

NOTES

$\therefore \quad z\frac{d}{dz} = \frac{d}{dt} = \text{D}$ and $z^2\frac{d^2}{dz^2} = \text{D}(\text{D}-1)$

Now, Equation (3) reduces to

$$\left[\text{D}(\text{D}-1)+\frac{2}{9}\right]y = -\frac{1}{3^{7/3}}e^{-\frac{t}{3}}$$

Its auxiliary equation is $\text{D}^2 - \text{D} + \frac{2}{9} = 0$

$\Rightarrow \ 9\text{D}^2 - 9\text{D} + 2 = 0 \ \Rightarrow \ \text{D} = \frac{2}{3}, \frac{1}{3}$

$$\text{C.F.} = c_1 e^{\frac{2}{3}t} + c_2 e^{\frac{1}{3}t} = c_1 z^{2/3} + c_2 z^{1/3}$$

$$= c_1\left(-\frac{1}{3}e^{3\cos x}\right)^{2/3} + c_2\left(-\frac{1}{3}e^{3\cos x}\right)^{1/3} = \text{A}_1 e^{2\cos x} + \text{A}_2 e^{\cos x}$$

and $$\text{P.I.} = \frac{1}{\text{D}^2 - \text{D} + \frac{2}{9}}\left[-\frac{1}{3^{7/3}}e^{-\frac{t}{3}}\right]$$

$$= \frac{1}{\frac{1}{9}+\frac{1}{3}+\frac{2}{9}}\left(-\frac{1}{3^{7/3}}\right)e^{-\frac{t}{3}} = \frac{9}{6}\left(-\frac{1}{3^{7/3}}\right)e^{-\frac{t}{3}}$$

$$= -\frac{1}{6(3)^{1/3}}(z)^{-\frac{1}{3}} = -\frac{1}{6(3)^{1/3}}\left[-\frac{1}{3}e^{3\cos x}\right]^{-\frac{1}{3}}$$

$$= \frac{1}{6(3)^{1/3}} \cdot \frac{1}{(3)^{-1/3}}e^{-\cos x} = \frac{e^{-\cos x}}{6}$$

Thus, the solution of Equation (1) is $y = \text{A}_1 e^{2\cos 2x} + \text{A}_2 e^{\cos x} + \frac{e^{-\cos x}}{6}$.

4.2.4 Solution by Using the Method of Variation of Parameters

Here, we shall learn the method to find the complete primitive of a linear equation whose C.F. is known. In this method, the constants of the C.F. are taken as the functions of independent variables.

Consider the linear differential equation of second degree

$$\frac{d^2y}{dx^2} + \text{P}\frac{dy}{dx} + \text{Q}y = \text{R} \qquad ...(4.15)$$

Let the C.F. of Equation (4.15) be

$$y = c_1 u + c_2 v \qquad ...(4.16)$$

where c_1 and c_2 are two arbitrary constants.

Clearly, u and v are the integrals of

NOTES

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

$\Rightarrow$ $u''+Pu'+Qu=0$ ...(4.17)

and $v''+Pv'+Qv=0$ ...(4.18)

Let the constants c_1 and c_2 in Equation (4.15) be the functions of x and the complete primitive of Equation (4.15) be

$$y=c_1(x)u+c_2(x)v \quad ...(4.19)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=c_1'(x)u+c_1(x)u'+c_2'(x)v+c_2(x)v'$$

$$\Rightarrow \frac{dy}{dx}=\left[c_1(x)u'+c_2(x)v'\right]+\left[c_1'(x)u+c_2'(x)v\right]$$

Let $c_1(x)$ and $c_2(x)$ satisfy the condition $c_1'(x)u+c_2'(x)v=0$ ...(4.20)

Thus, we get $\frac{dy}{dx}=c_1(x)u'+c_2(x)v'$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=c_1(x)u''+c_1'(x)u'+c_2(x)v''+c_2'(x)v'$$

Putting these values in Equation (4.15), we get

$$c_1(x)u''+c_1'(x)u'+c_2(x)v''+c_2'(x)v'+P\left[c_1(x)u'+c_2(x)v'\right]+Q\left[c_1(x)u+c_2(x)v\right]=R$$

$$\Rightarrow c_1(x)\left[u''+Pu'+Qu\right]+c_2(x)\left[v''+Pv'+Qv\right]+c_1'(x)u'+c_2'(x)v'=R$$

Substituting the values from Equation s (4.17) and (4.18), we get

$$\Rightarrow c_1(x)[0]+c_2(x)[0]+c_1'(x)u'+c_2'(x)v'=R$$

$$\Rightarrow c_1'(x)u'+c_2'(x)v'=R \Rightarrow c_1'(x)u'+c_2'(x)v'-R=0 \quad ...(4.21)$$

Solving the Equations (4.20) and (4.21) for $c_1'(x), c_2'(x)$, we get

$$\frac{c_1'(x)}{-vR}=\frac{c_2'(x)}{uR}=\frac{1}{uv'-vu'}$$

$$\Rightarrow \quad c_1'(x)=\frac{-vR}{uv'-vu'}, c_2'(x)=\frac{uR}{uv'-vu'}$$

Integrating, we get $c_1(x)=-\int\frac{vR}{W}dx+a, c_2(x)=\int\frac{uR}{W}dx+b$

where $W=uv'-vu'=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Putting these values in Equation (4.19), we get $y=-u\int\frac{vR}{W}dx+v\int\frac{uR}{W}dx$

which is the particular solution of Equation (4.15).

NOTES

Example 4.4: Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+a^2y=\text{cosec}\,ax$.

Solution: The given differential equation is $\frac{d^2y}{dx^2}+a^2y=\text{cosec}\,ax$ ...(1)

Symbolic form of the equation is $(D^2+a^2)y=\text{cosec}\,ax$

Its auxiliary equation is $D^2+a^2=0 \Rightarrow D^2=-a^2 \Rightarrow D=\pm ia$

$\therefore$ C.F. = $A\cos ax+B\sin ax$

Let the complete solution of Equation (1) be $y=u\cos ax+v\sin ax$ ...(2)

where u, v are unknown functions of x.

$\Rightarrow y=uy_1+vy_2$ where $y_1=\cos ax$ and $y_2=\sin ax$

Let $$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\therefore \quad W=\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}=a\left(\cos^2 ax+\sin^2 ax\right)=a$$

Now $$u=-\int \frac{y_2 R}{W}dx+c_1 \text{ where } R=\text{cosec}\,ax$$

$$=-\int \frac{\sin ax\,\text{cosec}\,ax}{a}dx+c_1=-\int \frac{1}{a}dx+c_1=-\frac{x}{a}+c_1$$

and $$v=\int \frac{y_1 R}{W}dx+c_2$$

$$=\int \frac{\cos ax\,\text{cos ec}\,ax}{a}dx+c_2=\int \frac{\cot ax}{a}dx+c_2=\frac{\log\sin ax}{a^2}+c_2$$

Substituting the value of u and v in Equation (2), we get

$$y=\left(-\frac{x}{a}+c_1\right)\cos ax+\left(\frac{\log\sin ax}{a^2}+c_2\right)\sin ax$$

$$\Rightarrow y=c_1\cos ax+c_2\sin ax-\frac{x}{a}\cos ax+\frac{\log\sin ax}{a^2}\sin ax$$

which is the complete solution of Equation (1).

4.3 THEOREMS OF STURM

In the field of ordinary differential equations, Sturm separation theorem describes the location of roots of homogeneous second order linear differential equations. Basically the theorem states that given two linear independent solutions of such an equation the zeros of the two solutions are alternating.

NOTES

Sturm Separation Theorem

Given a homogeneous second order linear differential equation and two continuous linear independent solutions $u(x)$ and $v(x)$ with x_0 and x_1 successive roots of $u(x)$, then $v(x)$ has exactly one root in the open interval $]x_0, x_1[$.

Proof

The proof is by contradiction. Assume that v has no zeros in $]x_0, x_1[$. Since u and v are linearly independent, v cannot vanish at either x_0 or x_1, so the quotient u / v is well defined on the closed interval $[x_0, x_1]$ and it is zero at x_0 and x_1. Hence, by Rolle's theorem, there is a point ξ between x_0 and x_1 where,

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{\left(v(x)\right)^2}$$

vanishes. Hence, $u'(\xi)v(\xi) = u(\xi)v'(\xi)$, which implies that u and v are linearly dependent. This contradicts our assumption and thus v has to have at least one zero between x_0 and x_1. On the other hand, there can be only one zero between x_0 and x_1, because otherwise v would have two zeros and there would be no zeros of u in between, and it was just proved that this is impossible.

An Alternative Proof

Since υ and ν are linearly independent it follows that the Wronskian $W[\upsilon, \nu]$ must satisfy $W[\upsilon, \nu]\,(x) \equiv W(x) \neq 0$ for all x where the differential equation is defined, say I. Without loss of generality, suppose that $W(x) < 0\ \forall\ x \in I$. Then

$$\upsilon(x)\ \nu'\ (x) - \upsilon'(x)\nu(x) \neq 0.$$

So at $x = x_0$

$$W(x_0) = -\upsilon'\ (x_0)\ \nu(x_0)$$

and either $\upsilon'\ (x_0)$ and $\nu\ (x_0)$ are both positive or both negative. Without loss of generality, suppose that they are both positive. Now, at $x = x_1$

$$W(x_1) = -\upsilon'\ (x_1)\ \nu\ (x_1)$$

and since $x = x_0$ and $x = x_1$ are successive zeros of $\upsilon(x)$ it causes $\upsilon'(x_1) < 0$. Thus, to keep $W(x) < 0$ we must have $\upsilon\ (x_1) < 0$. We see this by observing that if $\upsilon'(x) > 0\ \forall\ x \in (x_0, x_1)$ then $\upsilon(x)$ would be increasing (away from the x-axis), which would never lead to a zero at $x = x_1$. So for a zero to accur at $x = x_1$ at most $\upsilon'(x_1) = 0$ [i.e., $\upsilon'\ (x_1) \leq 0$ and it turns out, by our result from the Wronskian that $\upsilon'\ (x_1) \leq 0$]. So somewhere in the interval (x_0, x_1) the sign of $\nu(x)$ changed. By the intermediate value Theorem there exists $x^* \in (x_0, x_1)$ such that $\upsilon\ (x^*) = 0$.

By the same reasoning as in the first proof, $\nu(x)$ can have at most one zero for $x \in (x_0, x_1)$.

Sturm-Picone Comparison Theorem

In mathematics, in the field of ordinary differential equations, the Sturm-Picone comparison theorem is a classical theorem which provides criteria for the oscillation and nonoscillation of solutions of certain linear differential equations. Let,

1. $\left(p_1(x)y'\right)' + q_1(x)y = 0$

2. $\left(p_2(x)y'\right)' + q_2(x)y = 0$

NOTES

be two homogeneous linear second order differential equations in self-adjoint form with,

$$0 < p_2(x) \le p_1(x)$$

And, $0 < p_2(x) \le p_1(x)$

Let u be a non-trivial solution of case (1) with successive roots at z_1 and z_2 and let v be a non-trivial solution of case (2), then one of the following properties holds;

- There exists an x in $[z_1, z_2]$ such that $v(x) = 0$.
- There exists a λ in R such that $v(x) = \lambda u(x)$.

4.3.1 Sturm-Liouville Boundary Value Problems

In differential equations, a boundary value problem is a differential equation together with a set of additional restraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. A large class of important boundary value problems include the Sturm-Liouville problems. The analysis of these problems involves the eigenfunctions of a differential operator. In mathematical applications, a boundary value problem should be well established. This means that given the input to the problem there exist a unique solution, which depends continuously on the input.

A more mathematical way to picture the difference between an initial value problem and a boundary value problem is that an initial value problem has all of the conditions specified at the same value of the independent variable in the equation and that value is at the lower boundary of the domain, thus the term 'Initial' value. On the other hand, a boundary value problem has conditions specified at the extremes of the independent variable. For example, if the independent variable is time over the domain [0,1], an initial value problem would specify a value of $y(t)$ and $y'(t)$ at time $t = 0$, while a boundary value problem would specify values for $y(t)$ at both $t = 0$ and $t = 1$.

If the problem is dependent on both space and time, then instead of specifying the value of the problem at a given point for all time the data could be given at a given time for all space. For example, the temperature of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem.

Concretely, an example of a boundary value (in one spatial dimension) is the problem,

$$y''(x) + y(x) = 0$$

to be solved for the unknown function $y(x)$ with the boundary conditions,

$$y(0) = 0,\ y(\pi/2) = 2.$$

NOTES

Without the boundary conditions, the general solution to this equation is,

$y(x) = A\sin(x) + B\cos(x).$

From the boundary condition $y(0) = 0$ one obtains,

$0 = A \cdot 0 + B \cdot 1$

which implies that $B = 0$. From the boundary condition $y(\pi / 2) = 2$ one finds,

$2 = A \cdot 1$

and so $A = 2$. One sees that imposing boundary conditions allowed one to determine a unique solution, which in this case is,

$y(x) = 2\sin(x).$

Sturm-Liouville Theorem and Boundary Value Problem

A differential equation defined on the interval $a \le x \le b$ having the form of ,

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

and the boundary conditions,

$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases}$$

is called as Sturm-Liouville boundary value problem or Sturm-Liouville system, where $p(x) > 0$, $q(x)$; the weighting function $r(x) > 0$ are given functions; $a_1, a_2,\ b_1, b_2$ are given constants; and the eigenvalue is λ an unspecified parameter.

A special case of the Sturm-Liouville boundary value problem includes examples of generalized Fourier series found in Bessel functions, Legendre polynomials and other orthogonal polynomials such as Laguerre polynomials, Hermite polynomials and Chebyshev polynomials. Each of these polynomials represents a complete orthogonal set in different coordinates or circumstances and can be considered as a special case of the Sturm-Liouville boundary value problem.

The non-trivial (non-zero) solutions $\Phi_n(x)$, $n = 1, 2, 3,...$, of the Sturm-Liouville boundary value problem only exist at certain λ_n, $n = 1, 2, 3,...$, λ_n is called *eigenvalue* and $\Phi_n(x)$ is the *eigenfunction*.

The eigenvalues of a Sturm-Liouville boundary value problem are *non-negative real numbers*. In addition, the associated eigenfunctions $\Phi_n(x)$ are *orthogonal* to each other with respect to the weighting function $r(x)$,

$$\int_a^b r(x)\Phi_m(x)\Phi_n(x)\, dx = 0 \quad \text{if } m \ne n;\ m, n = 1, 2, 3, ...$$

The complete set of the solutions $\{\Phi_n(x) \mid a \le x \le b,\ n = 1, 2, 3, ...\}$ forms a *complete orthogonal set* of functions defined on the interval $a \le x \le b$. Therefore, a piecewise continuous function $f(x)$ can be expressed in terms of $\Phi_n(x)$, $n = 1, 2, 3, ...,$ such that

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) = \begin{cases} f(x) & \text{Where } f(x) \text{ is continuous} \\ \dfrac{f(x^-) + f(x^+)}{} & \text{at discontinuous points} \end{cases}$$

NOTES

Where,

$$c_n = \frac{\int_a^b r(x) f(x) \Phi_n(x) dx}{\int_a^b r(x) \Phi_n(x) \Phi_n(x) dx}$$

The completeness helps to *express any piecewise continuous function* in terms of these eigenfunctions while the orthogonality makes the expression *unique* and *compact* (no redundant terms). In addition, it can be shown that the orthogonal series is the *best* series available, i.e., each additional term fine tunes but not overhauls the sum of the existing terms. These properties generalize the conventional Fourier series sin $\lambda_n x$ and cos $\lambda_n x$ to any complete orthogonal series $\Phi n(x)$ and hence series is called the generalized Fourier series. The method of forming solutions by the general Fourier series is called the method of eigenfunction expansion which is an important technique in solving partial differential equations.

Sturm–Liouville Equation

A classical Sturm–Liouville equation, named after Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882), is a real second-order linear differential equation of the form,

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda w(x)y, \qquad ...(4.22)$$

where y is a function of the free variable x. Here the functions $p(x) > 0$, $q(x)$ and $w(x) > 0$ are specified at the outset. In the simplest of cases all are continuous on the finite closed interval $[a,b]$, and p has *continuous derivative*. In addition, the function y is typically required to satisfy some boundary conditions at a and b. The function $w(x)$, which is sometimes called $r(x)$, is called the 'weight' or 'density' function.

The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of satisfying the boundary conditions. Such values of λ when they exist are called the eigenvalues of the boundary value problem defined by and the prescribed set of boundary conditions. The corresponding solutions (for such a λ) are the eigenfunctions of this problem. Under normal assumptions on the coefficient functions $p(x)$, $q(x)$, and $w(x)$ above, they induce a Hermitian differential operator in some function space defined by boundary conditions. The resulting theory of the existence and asymptotic behaviour of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space became known as Sturm-Liouville theory or S-L theory. This theory is important in applied mathematics, where S-L problems occur very commonly, particularly when dealing with linear partial differential equations that are separable. Under the assumptions that the S-L problem is regular, i.e., $p(x)$, $w(x) > 0$ and $p(x)$, $p'(x)$, $q(x)$, and $w(x)$ are continuous functions over the finite interval $[a, b]$, with *separated boundary conditions* of the form,

$$y(a)\cos\alpha - p(a)y'(a)\sin\alpha = 0\,, \qquad \ldots(4.23)$$

$$y(b)\cos\beta - p(b)y'(a)\sin\beta = 0, \qquad \ldots(4.24)$$

NOTES

where $\alpha, \beta \in [0, \pi)$, the main tenet of Sturm-Liouville theory states that:

- The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ of the regular Sturm-Liouville problem (refer equations 1, 2 and 3) are real and can be ordered such that,

  $$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to \infty;$$

- Corresponding to each eigenvalue λ_n is a unique (up to a normalization constant) eigenfunction $y_n(x)$ which has exactly $n - 1$ zeros in (a, b). The eigenfunction $y_n(x)$ is called the nth *fundamental solution* satisfying the regular Sturm-Liouville problem (Refer Equations 4.22, 4.23 and 4.24).
- The normalized eigenfunctions form an orthonormal basis,

  $$\int_a^b y_n(x)y_m(x)w(x)\,dx = \delta_{mn},$$

  in the Hilbert space $L^2([a, b], w(x)\,dx)$. Here δ_{mn} is a Kronecker delta.

Note: Unless $p(x)$ is continuously differentiable and $q(x)$, $w(x)$ are continuous the equation has to be understood in a weak sense.

The differential Equation (4.22) is said to be in Sturm-Liouville form or self-adjoint form. All second order linear ordinary differential equations can be recast in the form on the left-hand side of Equation (4.22) by multiplying both sides of the equation by an appropriate integrating factor although the same is not true of second order partial differential equations or if y is a vector.

The following are significant examples for consideration:

The Bessel equation,

$$x^2y'' + xy' + \left(\lambda^2x^2 - v^2\right)y = 0$$

can be written in Sturm-Liouville form as,

$$(xy')' + \left(\lambda^2x - v^2/x\right)y = 0.$$

The Legendre equation,

$$\left(1 - x^2\right)y'' - 2xy' + v\left(v + 1\right)y = 0$$

can easily be put into Sturm-Liouville form, since $D(1 - x^2) = -2x$, so, the Legendre equation is equivalent to,

$$\left[\left(1 - x^2\right)y'\right]' + v\left(v + 1\right)y = 0$$

Less simple is such a differential equation as,

$$x^3y'' - xy' + 2y = 0.$$

Divide throughout by x^3:

$$y'' - \frac{x}{x^3}y' + \frac{2}{x^3}y = 0$$

NOTES

Multiplying throughout by an integrating factor of,

$$e^{\int -x/x^3 dx} = e^{\int -1/x^2 dx} = e^{1/x},$$

Gives,

$$e^{1/x} y'' - \frac{e^{1/x}}{x^2} y' + \frac{2e^{1/x}}{x^3} y = 0$$

which can be easily put into Sturm-Liouville form since,

$$De^{1/x} = -\frac{e^{1/x}}{x^2}$$

so the differential equation is equivalent to,

$$(e^{1/x} y')' + \frac{2e^{1/x}}{x^3} y = 0.$$

In general, given a differential equation,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

dividing by $P(x)$, multiplying through by the integrating factor,

$$e^{\int Q(x)/P(x)dx},$$

and then collecting gives the Sturm-Liouville form.

Sturm-Liouville Equations as Self-Adjoint Differential Operators

The map,

$$Lu = \frac{1}{w()x}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right)$$

can be viewed as a linear operator mapping a function u to another function Lu. This linear operator can be studied in the context of functional analysis. Actually, Equation (4.22) can be written as,

$$Lu = \lambda u.$$

This is precisely the eigenvalue problem, i.e., to find the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ and the corresponding eigenvectors $u_1, u_2, u_3, \ldots$ of the L operator. The proper setting for this problem is the Hilbert space $L^2([a, b], w(x)\, dx)$ with scalar product,

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) w(x)\, dx.$$

In this space L is defined on sufficiently smooth functions which satisfy the above boundary conditions. Moreover, L gives rise to a self-adjoint operator. This can be seen formally by using integration by parts twice, where the boundary terms vanish by virtue of the boundary conditions. It then follows that the eigenvalues of a Sturm-Liouville operator are real and that eigenfunctions of L corresponding to different eigenvalues are orthogonal. However, this operator is unbounded and hence existence of an orthonormal basis of eigenfunctions is not evident. To overcome this problem one looks at the resolvent,

NOTES

$$(L - z)^{-1}, \qquad z \in \mathbb{C},$$

where z is chosen to be some real number which is not an eigenvalue. Then, computing the resolvent amounts to solving the inhomogeneous equation, which can be done using the variation of parameters formula. This shows that the resolvent is an integral operator with a continuous symmetric kernel (the Green's function of the problem). As a consequence of the Arzelà-Ascoli theorem this integral operator is compact and existence of a sequence of eigenvalues α_n which converge to 0 and eigenfunctions which form an orthonormal basis follows from the spectral theorem for compact operators. Finally, note that $(L-z)^{-1}u = \alpha u$ is equivalent to $Lu = (z + \alpha^{-1})u$.

If the interval is unbounded, or if the coefficients have singularities at the boundary points, one calls L singular. In this case the spectrum does no longer consist of eigenvalues alone and can contain a continuous component. There is still an associated eigenfunction expansion (similar to Fourier series versus Fourier transform).

Example 4.5: Find a function u(x) which solves the following Sturm-Liouville problem:

$$Lu = \frac{d^2u}{dx^2} = \lambda u$$

where the unknowns are λ and $u(x)$. We add boundary conditions as,

$$u(0) = u(\pi)$$

Solution: Observe that if k is any integer, then the function

$$u(x) = \sin kx$$

is a solution with eigenvalue $\lambda = -k^2$. We know that the solutions of S–L problem form an orthogonal basis and from Fourier series it is considered that this set of sinusoidal functions is an orthogonal basis. Since orthogonal bases are always maximal (by definition) we conclude that the S–L problem in this case has no other eigenvectors. Given the preceding, let us now solve the inhomogeneous problem,

$$Lu = x, \qquad x \in (0, \pi)$$

with the same boundary conditions. In this case, we must write $f(x) = x$ in a Fourier series. The reader may check, either by integrating $\int \exp(ikx)x\,dx$ or by consulting a table of Fourier transforms, that we thus obtain,

$$Lu = \sum_{k=1}^{\infty} -2\frac{(-1)^k}{k} \sin kx.$$

This particular Fourier series is troublesome because of its poor convergence properties. It is not clear *a priori* whether the series converges pointwise. Because of Fourier analysis, since the Fourier coefficients are 'square-summable', the Fourier series converges in L^2 which is must for this function. Fourier's series converges at every point of differentiability and at jump points (the function x, considered as a periodic function, has a jump at π) converges to the average of the left and right limits.

NOTES

Therefore, by using the given equation we obtain that the solution is,

$$u = \sum_{k=1}^{\infty} 2\frac{(-1)^3}{k^3}\sin kx.$$

In this case, we could have found the answer using anti-differentiation. This technique yields $u = (x^3 - \pi^2 x)/6$, whose Fourier series agrees with the solution we found. The anti-differentiation technique is no longer useful in most cases when the differential equation is in many variables.

4.3.2 Eigen values and Eigen Function of the Strum-Louville Problem

To find the solution of problem

$$y'' + \lambda y = 0 \qquad \text{... (4.25)}$$

with boundary condition

$$y(0) = 0 \text{ and } y(\pi) = 0 \qquad \text{...(4.26)}$$

is not difficult to find. Boundary condition are the general solution of the eq^n. But we have to analyse the solution for all possible values of λ's. so, three cases arises as follows

Case 1: λ's negative or $\lambda < 1$

Let $\lambda = -m^2$

Then problem (4.25) with (4.26) becomes

$$y'' - m^2 y = 0 \;\ldots. \qquad (4.27)$$

$$y(o) = 0, \text{ and } y(\pi) = 0$$

So, the general solution is

$$y(x) = C_1 e^{mx} + C_2 e^{-mx}$$

$$y(o) = 0 \Rightarrow C_1 + C_2 = 0 \qquad \ldots (4.28)$$

$$y(\pi) = 0 \Rightarrow C_1 e^{m\pi} + C_L e^{-m\pi} = 0 \qquad \ldots (4.29)$$

Equations (4.26) and (4.27) give

$$C_1 \sinh m\pi = 0 \Rightarrow C_1 = 0 \text{ as } \sinh m\pi \neq 0 \; for\; m \neq 0$$

Hence $C_1 = C_2 = 0$, so we get only one trivial solution exists

Case 2 : $\lambda = 0$

The given problems (4.25) and (4.26) becomes

$$y'' = 0$$

$$y(0) = 0 \text{ and } y(\pi) = 0$$

hence the general solution is

$$y(x) = C_1 x + C_2$$

NOTES

when$y$ (o) = 0, and $C_2 = 0$

$y(x) = C_1 x$

when$y(\pi) = 0$, $C_1 = 0$

underboundary condition $C_1 = C_2 = 0$

we have trivial solution for given problem for this values of λ or $y = 0$

Case 3 : $\lambda > 0$

Let $\lambda = m^2$

The given problems (4.25) with (4.26) reduces to

$$y'' + m^2 y = 0$$

$$y(o) = 0 \text{ and } y(\pi) = 0$$

so general solution is

$$y(x) = C_1 \sin mx + C_2 \cos mx$$

For y (o) = 0, and $C_2 = 0$

$$y(x) = C_1 \sin mx + C_1 \cos mx$$

For y (o) = 0, and $C_2 = 0$

$$y(x) = C_1 \sin mx$$

$$y(\pi) = 0, y(o) = C \sin m\pi$$

since $C_1 \neq 0$ for seeking non – trivial solution

$$\sin m\pi = 0.$$

$$\sin m\pi = n\pi$$

$$m\pi = n\pi, n = 1, 2, 3 \ldots\ldots$$

hence $yn = n^2$, n = 1, 2, 3 ….. which is known as eigen values and corresponding solution is

$$y_n(x) = C_1 \sin nx ; n = 1, 2, 3 \ \ldots..$$

which is called as eigen function

Strum – Louville Problem

A boundary values problem consisting of second order homogeneous linear diff eq^n of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + [\lambda q(x) + r(x)]y = 0 \qquad \ldots. (4.30)$$

Where p, q and r are condition real valued function defined on $a \leq x \leq b$ such that p her a continuous derivatives, $P(x) > 0$ and $q(x) > 0$ and λ is a parameter independent of x and two homogeneous boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0 \qquad \ldots. (4.31)$$

$$B_1 y(b) + A_2 y'(b) = 0 \qquad \ldots(4.32)$$

NOTES

Where A_1, A_2, B_1 and B_2 are real constant such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero simultaneously is called Strum – Louville problem.

Example 4.6: Check whether the boundary values problem

$$y'' - \lambda y = 0, \text{ with } y(\text{o}) = 0 = y(\pi)$$

Sturm – Louville problem or not.

Solution: On comparing with standard form of Sturm – Louville problem, we have

$$p(x) = 1, q(x) = 1, r(x) = 0, q = 0, b = \pi$$

$$A_1 = B_1 = 1 \text{ and } A_2 = B_2 = 0$$

hence given problem is Sturm – Louville problem.

Example 4.7 : Find the eigen values and eigen function of the following Sturm – Louville problem.

$$\frac{d}{dx}\left(e^{2x}\frac{dy}{dx}\right) + (d+1)e^{2x}y = 0$$

$$y(0) = 0 = y(\pi)$$

Solution : Transform dependent variable for y to u by using transformation

$$y = e^{-x}4$$

$$\frac{dy}{dx} = e^{-x}\frac{d\dot{u}}{dx} - e^{-x}u$$

Therefore given diff eq^n reduces to

$$\frac{d}{dx}\left(e^{x}\left(e^{-x}\frac{dy}{dx} - e^{-x}y\right)\right) + (\lambda+1)e^{2x}e^{-x}u = 0$$

$$= 2e^{2x}\left(e^{-x}\frac{dy}{dx} - e^{-x}u\right) + e^{2x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{dy}{dx^2} + e^{-x}u - e^{-x}\frac{dy}{dx}\right) + \lambda e^{2x}\cdot e^{-x}u + e^{2x}e^{-x}y = 0$$

$$e^{x}\left[\frac{d^2y}{dx^2} + \lambda u\right] = 0$$

$$u'' + \lambda u = 0$$

and boundary condition reduces to

$$u(0) = 0 = u(\pi) \text{ since } e^{-x} \neq 0 \forall x \in \lambda$$

$$\lambda n = n^2 \quad \therefore \quad n = 1, 4$$

are the eigen values for reduced problem and corresponding eigen function are $u_n(x) = \sin nx$ hence $\lambda_n = n^2, n = 1, 2, 3 \ldots$ are the eigen values for given problem and corresponding eigen functions are

$$y_n(x)e^{-x}\sin x \quad \therefore \quad n \in N$$

NOTES

Check Your Progress

1. Solve the differential equation $x^2\frac{d^2y}{dx^2}-\left(x^2+2x\right)\frac{dy}{dx}+\left(x+2\right)y=x^3e^x$.
2. Solve the differential equation
$\left(x\sin x+\cos x\right)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x=\sin x\left(x\sin x+\cos x\right)^2$.
3. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9y=0$, given that $y=x^3$ is a part of solution.
4. Solve the differential equation
$\left(1+x\right)^2\frac{d^2y}{dx^2}+\left(1+x\right)\frac{dy}{dx}+y=4\cos\left\{\log\left(1+x\right)\right\}$.
5. Solve the differential
equation $\left(3x+2\right)^2\frac{d^2y}{dx^2}+3\left(3x+2\right)\frac{dy}{dx}-36y=3x^2+4x+1$.
6. Solve the differential equation $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin x^2$.
7. Solve the differential equation $x^6\frac{d^2y}{dx^2}+3x^5\frac{dy}{dx}+a^2y=x^{-2}$.
8. Apply the method of variation of parameters to solve
$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=2e^x$.
9. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+n^2y=\sec nx$.
10. Explain Sturm separation and comparison theorem.
11. Define Sturm-Liouville equation.

4.4 NONOSCILLATION THEOREMS

To prove nonoscillation theorem we have to prove two lemma as.

Lemma 1 : Consider the differential eq[n]

$$u'-\frac{A'(t)}{A(t)}u+\frac{A'(t)}{A(t)}\phi(t)=0, \qquad ...(4.33)$$

Where $\phi(t)$ is continuous on [T, ∞], A (t) is continuously differential on [T,] and

Let u (t) be the solution of Equation (4.33) on [T, ∞]

Where $\phi(t)$ is continuous on [T, ∞], A (t) is continuously differential on [T, ∞] and

$$A(t)>0, A'(t)<0, \lim_{t\to\infty} A(t)=0$$

Let u (t) be the solution of Equation (4.33) on [T, ∞]

Satisfying $u(t)=0$. Then, $\lim_{t\to\infty}\phi(t)=\infty[\text{or}-\infty]$ implies

$$\lim_{t\to\infty} u(t)=\left[\infty\left[or-\infty\right]\right.$$

Proof : The solution $u(t)$ is given by the formula

$$u(t)=-A(t)\int_r^t \frac{A'(s)}{A^2(s)}\phi(s)\,ds, t\geq T$$

If $\lim_{t\to\infty}\phi(t)=\infty[\text{or}-\infty]$, then it is obvious two

$$\lim_{t\to\infty}\left(-\int_T^t \frac{A'(s)}{A'(s)}\phi(s)\,ds\right)=\infty[\text{or}-\infty]$$

hence by L′ Hospital’s. rule

$$\lim_{t\to\infty} u(t)=\lim_{t\to\infty}\left|\left(-\int_T^t \frac{A'(S)}{A'(s)}\phi(s)ds\right)'\right|\left|\left(\frac{1}{A(t)}\right)'\right|$$

$$\lim_{t\to\infty} Q(t)=\left[\infty\left[or-\infty\right]\right.$$

Lemma 2: Let $\sigma(t)$ be continuous on [T, ∞] and Let $v(t)$ be continuously differentiable on [T, ∞]. If the $\lim_{t\to\infty}\left[(t)\upsilon'(t)+\upsilon(t)\right]$ exists in the extended real line R^*, then the $\lim_{t\to a}\upsilon(t)$ exists in R*.

Proof : If the conclusion is false, then there are numbers ζ and η such that

$$\lim_{t\to\infty}\inf\upsilon(t)<\zeta<\eta<\lim_{t\to\infty}\text{sub}\upsilon(t)$$

Now we can select an inereasing sequence $\left[t_v\right]_{v=1}^{\infty}$ With the following properties

$$\lim_{v\to 0} t_v=0,\quad v'(t_v)=0,\quad v=1,2 \qquad \ldots (4.34)$$

$$v(t_{2v-1})<\xi,\quad v(t_{2v})>\eta,\quad v=1,2 \qquad \ldots(4.35)$$

According to Equation (4.34)

$$\lim_{u\to\infty}\left[\sigma(t_v)v'(t_v)+v(t_v)\right]=\lim_{v\to\infty} v(t_v)$$

exists in R*. However, this is a contradiction, since Equation (4.35) implies that the sequence $\left\{v(t_v)\right\}_{v=1}^{\infty}$ cannot have a limit in R*

4.4.1 Nonoscillatory Equations and Principal Solutions

NOTES

Let $a^{+}(t) = \max\{a(t), 0\}$

$$a^{-}(t) = \max\{-a(t), 0\}$$

Let following condition hold.

$$\int^{\infty} A_{n-1}(t)a^{+}(t)\,dt = \infty, \qquad \ldots (4.37)$$

$$\int^{\infty} A_{n-1}(t)a^{-}(t)\,dt < \infty, \qquad \ldots (4.38)$$

$$\int^{\infty} A_{n-1}(t)\,|\,b(t)\,|\,dt\langle\infty \qquad \ldots (4.39)$$

Then, all bounded nonoscillatory solution, of

$$\left(r_{n-1}(t)\left(r_{n-L}(t)\left(\ldots\left(r_2(t)\left(r_1(t)y'(t)\right)' ---\right)'\right)'\right)'\right.$$

$$+a(t) + Ay\left(g(t)\right) = b\ (t), \qquad \ldots. (4.40)$$

Where $a(t), b(t), g(t), r_1(t)\ldots r_{n-1}(t)$ are real valued and continous on $[\tau, \infty]$ and $f(y)$ is real valued and continuous on $(-\infty, \infty)$

Proof : Let $y(t)$ be a boundary nonoscillatory solution of $b(t)$ we may suppose without loss of generality that

$y(t) > 0$ for $t \geq t_0$. By there exists $b_1 \geq$ to such that $g(t) \geq$ to for $t > t_1$. Thus $y(g(t)) > 0$ for $t \geq t$. we defend

$$G_0(t) = y(t),\ G_i(t) = r_i(t), \quad G_{i\text{-}1}(t);\ i = 1.....n - 1 \qquad ...(4.14)$$

$$u_k(t) = \int_{-c_1}^{t} A_{n-k-1}(s)C^1_{h-k-1}(s)\,ds_1, k = 0,1,\ldots n-1$$

A_n integration by parts yields

$$U_{k-1}(t) = \int_{t_1}^{t} A_{n-k}(s)G'_{\ n-k}(s)ds$$

$$= A_{n-k}(t)G_{n-k}(t) - A_{n-k}\left(t_1\right)G_{n-k}\left(t_1\right)$$

$$+\int_{t_1}^{t} \frac{A_{n-k-1}(s)}{r_{n-k}(s)} G_{n-k}(s)\,ds \qquad \ldots.. (a)$$

$$= \frac{A_{n-k}(t)r_{n-k}(t)}{A_{n-k-1}(t)} A_{n-k-1}(t)G'_{n-k-1}(t) - A_{n-k}\left(t_1\right)G_{n-k}\left(t_1\right)$$

NOTES

$$+\int_{t_1}^{t} A_{n-k-1}(s)G'_{n-k-1}(s)\,ds \qquad \text{..... } (b)$$

$$= \frac{-A_{n-k}(t)}{A'_{n-k}(t)} v'_k(t) + v_k(t) - A_{n-k}(t_1)G_{n-k}(t_1),$$

This show that $v_k(t)$ satisfies the diff' eq^n

$$\frac{A_{n-k}(t)}{A'_{n-k}(t)} v' - u + \phi_k(t) = 0 \qquad \text{... } (4.42)$$

$$u' - \frac{A'_{n-k}(t)}{A_{n-k}(t)} v + \frac{A'_{n-k}(t)}{A_{n-k}(t)} \phi_k(t) = 0 \qquad \text{...}(4.43)$$

where $\phi_k(t) = v_{k-1}(t_1) + A_{n-k}(t_1)G_{1\,n-k}(t_1)$

Since $v_k(t_1) = 0$ by Equation (4.41) and since $A_{n-k}(t) > 0$, $A'_{n-k}(t) < 0, \lim_{t\to\infty} A_{n-k}(t) = 0,$ by apply lemma 1 to Equations (4.43) to conclude that $\lim_{t\to\infty} v_{k-1}(t) = \infty[or - \infty]$ implies that $\lim_{t\to\infty} v_k(t) = \infty[or - \infty]$. More over, applying lemma 2 to Equation (4.42), we conclude that $\lim_{t\to\infty} v_k(t)$ exists in R^* wherever $\lim_{t\to\infty} v_{k-1}(t)$ exists in R^*. Now multiply both sides of Equation (4.42) by $A_{n-1}(t)$ and integrate it over $[t_1, t]$ then,

$$\int_{t_1}^{t} A_{n-1}(s)G^1_{n-1}(s)ds + \int_{t_1}^{t} A_{n-1}(s)\alpha^+(s) f(g(g(s)\,ds$$

$$= \int_{t_1}^{t} A_{n-1}(s)b(s)\,ds + \int_{t_1}^{t} A_{n-1}(s)\alpha^+(s) f\left(g(g(s))ds - (x_i)\right. \qquad \text{..... } (4.44)$$

we distinguish the following two cases

$$\int_{t_1}^{\infty} A_{n-1}(t)\alpha^+(t) f(y(g(t)))dt = \infty \qquad \text{...}(4.45)$$

$$\int_{t_1}^{\infty} A_{h-1}(t)\alpha^+(t) f(y(g(t)))dt < \infty \qquad \text{...}(4.46)$$

Suppose Equation (4.46) holds. In view of Equations (4.38) and (4.39) and the boundedner of $y(t)$ the right hand side of Equation (4.44) tends to a finite limit as $t \to \infty$, so that from (*xii*). We see that $\lim_{t\to\infty} u_0(t) = -\infty$. hence lemma 1 applied to (*xi*) with $k = 1$, we have $\lim_{t\to\infty} 2a_1(t) = -\infty$. Applying lemma 1 again

Equation (4.45) with k = 2, we have $\lim_{t\to\infty} u_2(t) = -0$. This however Contradictionthe assumption that y (t) is positive. Now letting $t \to \infty$ in Equation (4.45) is impossible. Now, letting $t \to \infty$ in Equation (4.44) and using Equation (4.44), we see that is finite. From lemma 2 applied to Equation (4.42) with $k = 1$ it follow that $\lim_{t\to\infty} v,(t)$ exists in R*. This limit must be finite, since $\lim_{t\to\infty} u_1(t) = -\infty$ would imply $\lim_{t\to\infty} y(t) = -\infty_a$ contradiction to the positivity of g(t), and $\lim_{t\to\infty} v_1(t) = \infty$ would imply $\lim_{t\to\infty} y(t) = -\infty$, a contradiction to the Boundedness of $y(t_1)$ continuing in this way, we conclude that is finite. Therefore $\lim_{n\to\infty} y(t)$ exists as a finite number, so it is easy to verify that

$$\liminf_{t\to\infty} y(g(t) = \liminf_{t\to 0} y(t) = 0$$

Thus it follow that $\lim_{t\to\infty} y(t) = 0$.

Theorem 4.1 : All bounded non-oscillatory solution of b(t) or equation (1) tunnel to zero as $t \to \infty$ if the following condition are satisfied:

$$\int^{\infty} A_{n-1}(t)L^{+t}(t)dt < \infty \tag{4.47}$$

$$\int^{\infty} A_{n-1}(t)L^{-t}(t)dt = \infty \tag{4.48}$$

$$\int^{\infty} A_{n-1}(t)\left|b(x)\right| dt < \infty \tag{4.49}$$

Proof : Let y (x) be a bounded non-oscillatory solution equation (4.40) such that $y(g(t)) > 0$ for $t > t_1$. A parallel argument hold if $y(g(t)) < 0$ for $t > t_1$. Defind the function $G_i(t)$ and $v_k(t)$ by the formula equation (4.41) Assume that

$$\int_{t-1}^{\infty} A_{n-1}(t)\alpha^{-}(t) f(y(g(t)))dt = \infty$$

Then, letting $t \to \infty$ in (a), (b) and using Equations (4.48) and (4.49) and the boundedness of g (t), we obtain $\lim_{n\to\infty} v_0(t) = \infty$, so that applying lemma 1 to equation (4.44) with $k = 1$, we see that $\lim_{t\to\infty} v_1(t) = \infty$, Repeat application of this argument show that $\lim_{t\to\infty} v_{n-1}(t) = \infty$ which implies that $\lim_{t\to\infty} y(t) = \infty$. But conducts that fact. That y(t) is bounded. Consequently, we must have

$$\int_{t_1}^{\infty} A_{n-1}(t)\alpha^{-}(t) f(y(g(t)))\Big) dt < \infty$$

The rest of the proof now proceeds exactly as in the second half of the above proof Theorem.

4.5 NUMBER OF ZEROS IN SECOND ORDER LINEAR DIFFERENTIAL EQUATION

NOTES

Second order linear equation :

$$y'' + p(t)y' + q(t)y = y(t) \quad \dots (4.50)$$

Homogenous eq^n $\Rightarrow$ If $y(t) = 0$, then equation (4.30) becomes

$$y'' + p(t)y' + q(t)y = 0.$$

It is an homogenous equation

Trivial solution: For the homogenous equation$y$ y(t) = 0 is always a solution regardless what $p(t)$ and $q(t)$ all this constant zero solution is called the trival solution of such an equation.

Second order linear Homogenous Diff. eq^n with constant coefficient

$$ay'' + by' + cy = 0$$

Where a, b and c are constant,

A very simple instance of such type of eq^n $y'' - y = 0$ the equation solution is any function satisfying the equality $y'' - y.$ Obviously $y_1 = e^t$ is the solution and S_0 is any constant multiple of it C_1e^t. Not as obvious, but still easy to see, is that $y_2 = e^{-t}$ is another solution. It can be easily verified that any function of the form

$$y = c_1e^t + c_2e^{-t}$$

if y_1 and y_2 are any two solution of the homogenous eq^n,

$$y'' + p(t)y' + q(t)y = 0$$

Then any function of the form

$$y = c_1\ y_1 + c_2\ y_2$$

is also a solution of the *eq*, for any pair of constant C_1 and C_2

For any homogeneous linear *eq*, any multiple of a solution is again a solution, any sum and difference of two solution is again a solution.

Example 4.8: Find the general solution of

$$y'' - 5y' = 0$$

Solution: If we let then substitute them into the *eq*, we get a new *eq^n*

$$u' - 5u = 0$$

Now there is first order linear eq with $p(t) = -5$ and $y(t) = 0$

The integratintg factor is $\mu = e^{-5t}$

$$u(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt\right) = c^{5t}\left[\int dt\right] = e^{rt}(c) = c_1e^{5t}$$

NOTES

The actual solution y is given by the relation $u = y'$, and can be found by integration

$$u(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt\right) = c^{5t}\left[\int dt\right] = e^{rt}(c) = c_1e^{5t} = C_1e^{5t} + C_2$$

Equation of non – constant coefficient with missing y – term

If the y – term is missing in a second order linear equation, now the equation can be readily converted into a first order linear equation and solved using the integrating factor

Example 4.9: $ty'' + uy' = t^{L}$

Solution: The standard form is

$$y'' + \frac{u}{t}y' = t$$

Substitute : $u' + \frac{u}{t}u = t \Rightarrow p(t) = \frac{u}{t}, \dot{g}(t) = t$

Integrating factor is $\mu = t^u$

$$u(t) = \frac{1}{t^u}\left(\int t^5 dt\right) = t^{-u}\left(\frac{t^6}{6} + c\right) = \frac{1}{6}t^2 + ct^{-4}$$

Finally

$$y(t) = \int u(t)dt = \frac{1}{18}t^3 - \frac{c}{3}t^{-3} + C_2 = \frac{1}{18}t^3 + c_1t^{-3} + C_2$$

In general, given a second order linear eq^n with the y – term missing

$$y'' + p(t)y' = g(t)$$

by the subsisting $u = y'$ and $u' = y''$ to change the eq^n to a first order linear eq^n. Use the integrity factor method to solve for u, and then integrate u to find y;

Substitute : $u' + \dot{p}(t)u = g(t)$

Integrity factor $\mu(t) = e^{\int P(t)dt}$

Solve for u : $u(t) = \dfrac{\int \mu(t)g(t)\,dt}{\mu(t)} + c$

Integrate : $y(t) = \int u(t)\,dt$

Characteristic polynomial

If $ar^2 + br + c$ is a characteristic polynomial of differential equation

There are 3 – possible cases of the solution found

1. If $b^2 - 4ac > 0$, There are two distinct real root r_1, r_2

2. If $b^2 - 4ac < 0$, There are two complex conjugate roots $r = \lambda \pm \mu i$

3. If $b^2 - 4ac = 0$, There is one repeated real root r.

NOTES

Case 1: $b^2 - 4ac > 0$

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example 4.10: $y'' + 5y' + 4y = 0$

Solution: The characteristic eq^n

$$r^2 + 5r + u = (r+1)(r+4) = 0$$

$$r = -1, -4$$

$$y = c_1 e^{-t} + c_2 e^{-4t}$$

Case 2: Two complex conjugate roots:

$$b^2 - 4ac < 0.$$

$$r_1 = \lambda + \mu i, r_2 = \lambda - \mu i$$

$$y_1 = e^{r_1 t},\ y_2 = e^{r_2 t}$$

Example 4.11: $y'' + uy = 0$

Solution: $y_1 = 2 \pm 2i$

$$y = c_1 \cos 2t + c_2 \sqrt{\sin 2t}$$

Example 4.12: $y'' + 2y' + 5y = 0$, $y(\text{o}) = 4, y'(\text{o}) = 6$

Solution: $r^2 + 2r + 5 = 0$

$$r = -1 \pm 2i$$

$$y = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

Case 3 : One repeated real roots :

$$b^2 - 4ac = 0$$

$$r = \frac{-b}{2a}$$

Example 4.13: $y'' - 4y' + 4y = 0, \quad y(\text{o}) = 4, \quad y'(\text{o}) = 5$

Solution : $r^2 - 4r + 4 = (r-2)^2 = 0$

$$r = 2$$

$$y = c_1 e^{2t} + c_2 t e^{2t}$$

NOTES

Check Your Progress

12. How will you define the second order differential equation?
13. Give the nonoscillation theorem equation for differential equation.
14. Write the conditions for nonoscillation theorem.
15. Name the equation of all boundary non-oscillatory solution.

4.6 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $y = x^2e^x - xe^x + c_1xe^x + c_2x$
2. $y = c_2x - c_1 \cos x + \frac{1}{4}x\cos 2x - \frac{1}{2}\sin 2x$
3. $y = kx^{-3} + c_2x^3$
4. $y = \cos x\left[c_1 + c_2x + \frac{\tan x}{2}\right] = (c_1 + c_2x)\cos x + \frac{\sin x}{2}$
5. $y = e^{(-3/4)x^{2/3}}\left(c_1x^3 + c_2x^{-2}\right)$
6. $y = c_1e^{x^2} + c_2e^{-x^2} - \sin x^2$
7. $y = c_1 \cos\frac{a}{2x^2} - c_2 \sin\left(\frac{a}{2x^2}\right) + \frac{1}{a^2x^2}$
8. $y = c_1e^{2x} + c_2e^{3x} + e^x$
9. $y = \frac{1}{n^2}(\log\cos nx)\cos nx + \frac{x}{n}\sin nx + c_1 \cos nx + c_2 \sin nx$
10. Sturm separation theorem describes the location of roots of homogeneous second order linear differential equations. Basically the theorem states that given two linear independent solutions of such an equation the zeros of the two solutions are alternating. The Sturm comparison theorem is a classical theorem which provides criteria for the oscillation and nonoscillation of solutions of certain linear differential equations.
11. A classical Sturm-Liouville equation is a real second order linear differential equation of the form,

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda w(x)y,$$

where y is a function of the free variable x. Here the functions $p(x) > 0$, $q(x)$ and $w(x) > 0$ are specified at the outset. In the simplest of cases all are continuous on the finite closed interval $[a,b]$ and p has *continuous derivative*.

12. Second order linear equation :

$$y'' + p(t)y' + q(t)y = y(t)$$

13. $\lim_{t\to\infty} Q(t) = [\infty[or - \infty]$

14. $\int^{\infty} A_{n-1}(t)a^{+}(t)\,dt = \infty,$

$$\int^{\infty} A_{n-1}(t)a^{-}(t)\,dt < \infty,$$

$$\int^{\infty} A_{n-1}(t)\,|\,b(t)\,|\,dt\langle\infty \quad \ldots (4.39)$$

15. $\left(r_{n-1}(t)\left(r_{n-L}(t)\left(\ldots\left(r_2(t)\left(r_1(t)y'(t)\right)' - - -\right)'\right)'\right)'\right)$

$$+a(t) + Ay\big(g(t)\big) = b\ (t),$$

4.7 SUMMARY

- Linear differential equation of second order is an equation of the form

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = \mathrm{R}$$

where P, Q and R are the functions of x.

- Let the linear differential equation of second order be

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$$

where P, Q and R are the functions of x only.

- In case the integral of the C.F. is neither known nor can be found using the rules, there is a need of other method to find the solution of linear differential equation of second order. Here, we will learn the method which is independent of integral of C.F.
- The constants of the C.F. are taken as the functions of independent variables.
- In the field of ordinary differential equations, Sturm separation theorem describes the location of roots of homogeneous second order linear differential equations.
- Given a homogeneous second order linear differential equation and two continuous linear independent solutions $u(x)$ and $v(x)$ with x_0 and x_1 successive roots of $u(x)$, then $v(x)$ has exactly one root in the open interval $[x_0, x_1]$.

NOTES

- The field of ordinary differential equations, the Sturm-Picone comparison theorem is a classical theorem which provides criteria for the oscillation and nonoscillation of solutions of certain linear differential equations.
- In differential equations, a boundary value problem is a differential equation together with a set of additional restraints, called the boundary conditions.
- A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.
- The eigenvalues of a Sturm-Liouville boundary value problem are non-negative real numbers.
- A classical Sturm–Liouville equation, named after Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882)
- Let $y\ (t)$ be a boundary nonoscillatory solution of $b(t)$ we may suppose without loss of generality that $y\ (t) > 0$ for $t \geq t_0$
- The homogenous equation$y$ y(t) = 0 is always a solution regardless what $p(t)$ and $q(t)$ all this constant zero solution. is called the trival solution.

4.8 KEY TERMS

- **Linear differential equations of second order:** It is an equation of the form $\frac{d^2y}{dx^2} + \text{P}\frac{dy}{dx} + \text{Q}\,y = \text{R}$, where P, Q and R are the functions of x.
- **Non-negative real number:** The eigenvalues of a Strum-Liouville boundary value problem are non-negative real number.
- **Sturm-separation theorem:** Given a homogeneous second order linear differential equation and two continuous linear independent solutions $u(x)$ and $v(x)$ with x_0 and x_1 successive roots of $u(x)$, then $v(x)$ has exactly one root in the open interval $[x_0, x_1]$.
- **Trivial solution:** For the homogenous equation$y$ y(t) = 0 is always a solution regardless what $p(t)$ and $q(t)$ all this constant zero solution. is called the trival solution.

4.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Solve $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$, given that for $t = 0$, $x = 0$ and $\frac{dx}{dt} = 12$.

2. Solve $\frac{d^2x}{dt^2} + b^2 x = k\cos bt$, given that $x = 0$ and $\frac{dx}{dt} = 0$, when $t = 0$.
3. Define theorems of strum.
4. What are Sturm-Liouville boundary value problems?
5. How will you define the eigen values and eigen functions of the Strum-Louville problem?
6. State the number of zeros in second order linear differential equation.
7. Give the nonoscillation theorem.
8. What is the boundary non-oscillatory solution?

Long-Answer Questions

1. Apply the method of variable of parameters to solve the following differential equations:

   (i) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

   (ii) $\frac{d^2y}{dx^2} + y = \tan x$

   (iii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}\sec 2x$

   (iv) $\frac{d^2y}{dx^2} - y = e^{-2x}\sin e^{-x}$

   (v) $(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1+x^2)y = x$

   (vi) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

   (vii) $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

2. Verify that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$. Find the general solution of $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2$.
3. Explain the methodology to solve boundary value problems using Sturm-Liouville, Sturms separation and comparison theorems.
4. Find eigen values and eigen function of the given Strum – Louville problem

   (a) $y'' + \lambda y = 0, \quad y'(0) = y'(\pi)$

   (b) $(xy)' + \frac{\lambda}{x}y = 0, y(1) = y(e^2) = 0$

   (c) $x'y'' - \lambda xy'' + \lambda y = 0, y(1) = y(2) = 0$

NOTES

(d) $y'' + 4y = x^2, \quad y(\text{o}) = y(1) = 0$

(e) $u'' + 4u = f(x), \quad u(o) = \alpha, \quad u'(1) = \beta$

(f) $x^2 y'' - \lambda xy' + \lambda y = 0, y(1) = y(2) = 0$

5. Discuss briefly about the number of zeros in second order linear differential equation with the help of giving examples.
6. Find the general solution of the given differential equation

(a) $y'' + 2y' - 8y = 0$

(b) $y'' - 13y' + 42y = 0$

(c) $y'' - 10y' + 25y = 0$

(d) $y'' + 2y' + 5y = 0$

(e) $y'' + 4y' + 13y = 0$

(f) $y'' = 0$

(g) $y'' + 2y' = 0$

(h) $2y'' + 5y' - 3y = 0$

(i) $y'' - 9y = 0$

(j) $y'' + 16y = 0$

(k) $y'' - 2y' + 2y = 0$

(l) $y' - y' - 30y = 0$

7. Find the differential equation $y'' + ay' + by = 0$ that satisfy by the given function

(a) $y_1(x) = e^{2x}, y_2(x) = e^{-5x}$

(b) $y(x) = 2xe^{3x}$

(c) $y(x) = \cos^{2x}$

(d) $y_1(1) = 3e^{1x}, y_2(x) = -ue^{-6x}$

(e) $y(x) = e^{-2x} \sin ux$

4.10 FURTHER READING

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NOTES

UNIT 5 PARTIAL DIFFERENTIAL EQUATION OF FIRST AND SECOND ORDER

Structure

5.0 INTRODUCTION

In mathematics, a first-order partial differential equation is a partial differential equation that involves only first derivatives of the unknown function of n variables. Such equations arise in the construction of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems, and in simple models for gas dynamics whose solution involves the method of characteristics. If a family of solutions of a single first-order partial differential equation can be found, then additional solutions may be obtained by forming envelopes of solutions in that family. In a related procedure, general solutions may be obtained by integrating families of ordinary differential equations.

A Partial Differential Equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a computer model. A special case is Ordinary Differential Equations (ODEs), which deal with functions of a single variable and their derivatives. PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, gravitation and quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

Partial differential equations are equations that involve rates of change with respect to continuous variables. The position of a rigid body is specified by six parameters, but the configuration of a fluid is given by the continuous distribution of several parameters, such as the temperature, pressure, and so forth. The dynamics for the rigid body take place in a finite-dimensional configuration space; the dynamics for the fluid occur in an infinite-dimensional configuration space. This distinction usually makes PDEs much harder to solve than ordinary differential equations, but here again, there will be simple solutions for linear problems. Classic domains where PDEs are used include acoustics, fluid dynamics, electrodynamics, and heat transfer.

In this unit, you will be study about the partial differential equations of the first order, partial differential equations of the second order and linear partial differential equation with constant coefficient.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Drive the partial differential equations of the first order
- Know solution of the partial differential equations of the second order
- Analyse the partial differential equations of second and higher orders
- Discuss the classification of partial differential equations of second order
- Classify the homogeneous and non-homogeneous equations with constant coefficients
- Briefly explain the partial differential equations reducible to equations with constant coefficients

5.2 PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Lagrange's Equation

The partial differential equation $Pp + Qq = R$, where P, Q, R are functions of x, y, z, is called **Lagrange's Linear Differential Equation**.

Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and find two indpendent solutions of the auxiliary equations say $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$, where C_1 and C_2 are constants. Then the solution of the given equation is $F(u, v) = 0$ or $u = F(v)$.

For example, solve $(y^2 + z^2)p - xyq = -xz$

The auxiliary equations are,

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-xz} \quad (5.1)$$

NOTES

Taking the last two equations, we get,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get $\log y = \log z + \text{constant}$,

$$\therefore \quad \frac{y}{z} = C_1$$

Each of the Equation (5.1) is equal to,

$$\frac{xdx + ydy + zdz}{x(y^2 + z^2) - xy^2 - xz^2}$$

i.e.,
$$\frac{xdx + ydy + zdz}{0}$$

i.e.,
$$xdx + ydy + zdz = 0$$

Hence after integration this reduces to,

$$x^2 + y^2 + z^2 = C_2$$

Hence the general solution of the equation is,

$$F\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0$$

Example 5.1: Solve $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z$

Solution: The auxiliary equations are,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x + y)z}$$

i.e.,
$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x + y)z}$$

i.e.,
$$\frac{dx - dy}{x - y} = \frac{dz}{z}$$

i.e., $\log (x - y) = \log z + \text{constant}$

$$\therefore \quad \frac{x - y}{z} = C_1$$

Also
$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Hence
$$-\frac{1}{x} = -\frac{1}{y} + \text{constant}$$

$$\therefore \quad \frac{1}{y} - \frac{1}{x} = C_2$$

Hence the solution is, $F\left(\frac{1}{y} - \frac{1}{x}, \frac{x - y}{z}\right) = 0$

NOTES

Example 5.2: Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution: The subsidiary equations are,

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx - dy}{x^2 - yz - (y^2 - zx)} = \frac{d(x-y)}{(x-y)(x+y+z)}$$

$$= \frac{d(y-z)}{(y-z)(x+y+z)}$$

$$\therefore \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating $\log(x-y) = \log(y-z) + \log C_1$

$$\therefore \quad \frac{x-y}{y-z} = C_1 \qquad (1)$$

Using multipliers x, y, z, each of the subsidiary equations,

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

And is also equal to $\dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$

$$\therefore \quad \frac{xdx + ydy + zdz}{x+y+z} = \frac{dx + dy + dz}{1}$$

$$xdx + ydy + zdz = (x+y+z)d\,(x+y+z)$$

On Integrating, we get,

$$x^2 + y^2 + z^2 = (x+y+z)^2 + C_2$$

$$\therefore \quad xy + yz + zx = C'_2 \qquad (2)$$

From Equations (1) and (2), we get the solution,

$$F\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0, \text{ where } F \text{ is arbitrary.}$$

Example 5.3: Solve $(a-x)p + (b-y)q = c-z$

Solution: The subsidiary equations are,

$$\frac{dx}{a-x} = \frac{dy}{b-y} = \frac{dz}{c-z} \qquad (1)$$

From Equation (1)

$$\frac{dy}{b-y} = \frac{dz}{c-z}$$

i.e.,
$$\frac{dy}{y-b} = \frac{dz}{z-c}$$

$$\log(y-b) = \log(z-c) + \log C_1$$

$$\therefore \quad \frac{y-b}{z-c} = C_1$$

NOTES

Also

$$\frac{dx}{a-x} = \frac{dy}{b-y}$$

$$\therefore \quad \frac{dx}{x-a} = \frac{dy}{y-b}$$

$$\therefore \quad \log(x-a) = \log(y-b) + \log C_2$$

$$\therefore \quad \left(\frac{x-a}{y-b}\right) = C_2$$

The general solution is

$$F\left(\frac{y-b}{z-c}, \frac{x-a}{y-b}\right) = 0$$

Example 5.4: Solve $(y-z)p + (z-x)q = x-y$

Solution: The auxiliary equations are,

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

$$\therefore \quad dx + dy + dz = 0$$

Integrating we get, $x+y+z = C_1$

Also each ratio,

$$= \frac{xdx + ydy + zdz}{x(y-z) + y(z-x) + z(x-y)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$$\therefore \quad xdx + ydy + zdz = 0$$

On integrating, we get,

$$x^2 + y^2 + z^2 = C_2$$

$\therefore$ The general solution is,

$$F(x+y+z, x^2+y^2+z^2) = 0$$

Example 5.5: Solve $(mz - ny)p - (nx - lz)q = ly - mx$

Solution: The auxiliary equations are,

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Using multipliers x, y, z, we get each ratio

$$= \frac{xdx + ydy + zdz}{x(mz-ny) + y(nx-lz) + z(ly-mx)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

NOTES

$\therefore \qquad x^2 + y^2 + z^2 = C_1$

Also by using multipliers l, m, n, we get each ratio,

$$= \frac{ldx + mdy + ndz}{0}$$

$\therefore \qquad lx + my + nz = C_2$

$\therefore$ The general solution is,

$$F\left(x^2 + y^2 + z^2, lx + my + nz\right) = 0$$

Example 5.6: Solve $x(y - z)p + y(z - x)q = z(x - y)$

Solution: The auxiliary equations are,

$$\frac{dx}{xy - xz} = \frac{dy}{yz - yx} = \frac{dz}{zx - zy}$$

$$= \frac{dx + dy + dz}{0}$$

$\therefore \qquad dx + dy + dz = 0$

On integrating, we get, $x + y + z = C_1$ (4)

$$\frac{\frac{dx}{x}}{y - z} = \frac{\frac{dy}{y}}{z - x} = \frac{\frac{dz}{z}}{x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$\Rightarrow \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

On integrating, $\log x + \log y + \log z = \log C_2$

$$xyz = C_2 \qquad (2)$$

From Equations (1) and (2), the general solution is, $F(x + y + z, xyz) = 0$

Example 5.7: Solve $x^2p + y^2q = z^2$

Solution: The auxiliary equations are,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\therefore \qquad \frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + C_1$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$$\frac{1}{y} - \frac{1}{x} = C_1$$

Also

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

NOTES

$$\therefore \qquad -\frac{1}{y} = -\frac{1}{z} + C_2$$

$$\frac{1}{z} - \frac{1}{y} = C_2$$

The general solution is,

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

Example 5.8: Solve $(y+z)p + (z+x)q = x+y$

Solution: The auxiliary equations are,

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

i.e.,
$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

$$= \frac{dx+dy+dz}{2(x+y+z)}$$

Considering first two members and integrating, we get,

$$\frac{x-y}{y-z} = C_1$$

Considering first and last members and integrating, we get,

$$\log(x-y) = \frac{1}{2}\log(x+y+z) + \log C_2$$

$$\log\frac{(x-y)^2}{x+y+z} = \log C_2'$$

$$\frac{(x-y)^2}{x+y+z} = \log C_2'$$

$\therefore$ The general solution is,

$$F\left(\frac{x-y}{y-z}, \frac{(x-y)^2}{x+y+z}\right) = 0$$

5.3 PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

The general form of a linear differential equation of nth order is,

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ... + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

Where $P_1, P_2 ..., P_n$ and Q are functions of x alone or constants.

The linear differential equation with constant coefficients are of the form,

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ... + P_{n-1}\frac{dy}{dx} + P_n y = Q \qquad (5.2)$$

NOTES

Where $P_1, P_2, ..., P_n$ are constants and Q is a function of x.

The equation,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_y y = 0 \quad (5.3)$$

This is then called the Reduced Equation (R.E.) of the Equation (5.2)

If $y = y_1(x), y = y_2(x), ..., y = y_n(x)$ are n-solutions of this reduced equation, then $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of the reduced equation where $c_1, c_2, ..., c_n$ are artbitrary constants.

The solution $y = y_1(x)$, $y = y_2(x)$, $y = y_3(x)$, ..., $y = y_n(x)$ are said to be linearly independent if the **Wronskian** of the functions is not zero where the Wronskian of the functions $y_1, y_2, ..., y_n$, denoted by $W(y_1, y_2, ..., y_n)$, is defined by,

$$W(y_1, y_2, ...y_n) = \begin{vmatrix} y_1 & y_2 & y_3 ... y_n \\ y_1' & y_2' & y_3' ... y_n' \\ y_1'' & y_2'' & y_3'' ... y_n'' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} ... y_n^{(n-1)} \end{vmatrix}$$

Since the general solution of a differential equation of nth order contains n arbitrary constants, $u = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is its complete solution.

Let v be any solution of the differential Equation (5.2), then,

$$\frac{d^n v}{dx^n} + P_1 \frac{d^{n-1} v}{dx^{n-1}} + P_2 \frac{d^{n-2} v}{dx^{n-2}} + \ldots + P_{n-1} \frac{dv}{dx} + P_n v = Q \quad (5.4)$$

Since u is a solution of Equation (5.3), we get,

$$\frac{d^n u}{dx^n} + P_1 \frac{d^{n-1} u}{dx^{n-1}} + P_2 \frac{d^{n-2} u}{dx^{n-2}} + \ldots + P_{n-1} \frac{du}{dx} + P_n u = 0 \quad (5.5)$$

Now adding Equations (5.4) and (5.5), we get,

$$\frac{d^n (u+v)}{dx^n} + P_1 \frac{d^{n-1}(u+v)}{dx^n} + P_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + \ldots + P_{n-1} \frac{d(u+v)}{dx} + P_n (u+v) = Q$$

This shows that $y = u + v$ is the complete solution of the Equation (5.2).

Introducing the operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$ etc. The Equation (5.2) can be written in the form,

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + ... + P_{n-1} Dy + y P_n = Q$$

Or $\quad (D^n + P_1 D^{n-1} + P_2 D^{n-2} + ... + P_{n-1} D + P_n) y = Q$

Or $\quad F(D) y = Q$ where $F(D) = D^n + P_1 D^{n-1} P_2 D^{n-2} + ... + P_{n-1} D + P_n$

From the above discussions it is clear that the general solution of $F(D)y = Q$ consists of two parts:

(*i*) The Complementary Function (C.F.) which is the complete primitive of the Reduced Equation (R.E.) and is of the form

$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ containing n arbitrary constants.

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(*ii*) The Particular Integral (P.I.) which is a solution of $F(D)\,y = Q$ containing no arbitrary constant.

Rules for Finding The Complementary Function

Let us consider the 2nd order linear differential equation,

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \qquad (5.6)$$

Let $y = A\,e^{mx}$ be a trial solution of the Equation (5.4); then the Auxiliary Equation (A.E.) of Equation (5.6) is given by,

$$m^2 + P_1 m + P_2 = 0 \qquad (5.7)$$

The Equation (5.7) has two roots $m = m_1$, $m = m_2$. We discuss the following cases:

(*i*) When $m_1 \neq m_2$, then the complementary function will be,

$y = c_1 e^{m1x} + c_2\,e^{m2x}$ where c_1 and c_2 are arbitrary constants.

(*ii*) When $m_1 = m_2$, then the complementary function will be,

$y = (c_1 + c_2\,x)\,e^{m1x}$ where c_1 and c_2 are arbitrary constants.

(*iii*) When the auxiliary Equation (5.7) has complex roots of the form $\alpha + i\beta$ and $\alpha - i\beta$, then the complementary function will be,

$y = e^{\alpha x}\,(c_1 \cos\beta\,x + c_2 \sin\beta\,x)$

Let us consider the equation of order n,

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = 0 \qquad (5.8)$$

Let $y = A\,e^{mx}$ be a trial solution of Equation (5.8), then the auxiliary equation is,

$$m^n + P_1\,m^{n-1} + P_2\,m^{n-2} + \ldots\ldots + P_{n-1}\,m + P_n = 0 \qquad (5.9)$$

Rule (1): If $m_1, m_2, m_3, \ldots, m_n$ be n distinct real roots of Equation (5.9), then the general solution will be,

$$y = c_1\,e^{m1x} + c_2\mathrm{e}^{\,m2x} + c_3 e^{m3x} + \ldots + c_n e^{mnx}$$

Where $c_1, c_2, c_3 \ldots\ldots c_n$ are arbitrary constants.

Rule (2): If the two roots m_1 and m_2 of the auxiliary equation are equal and each equal to m, the corresponding part of the general solution will be $(c_1 + c_2\,x)\,e^{mx}$ and if the three roots m_3, m_4, m_5 are equal to α the corresponding part of the solution is $(c_3 + c_4x + c_5x^2)\,e^{\alpha x}$ and others are distinct, the general solution will be,

$$y = (c_1 + c_2x)\,e^{mx} + (c_3 + c_4 + c_5x^2)\,e^{\alpha x} + c_6\,e^{m6x} + \ldots\ldots + c_n e^{mnx}$$

Rule (3): If a pair of imaginary roots $\alpha \pm i\beta$ occur twices, the corresponding part of the general solution will be,

$$e^{\alpha x}\,[(c_1 + c_2x)\cos\,\beta x + (c_3 + c_4x)\sin\beta x]$$

And the general solution will be,

$$y = e^{\alpha x}\,[(c_1 + c_2x)\cos\,\beta x + (c_3 + c_4x)\sin\beta x] + c_5 e^{m5x} + \ldots\ldots + c_n e^{mnx}$$

Where $c_1, c_2..., c_n$ are arbitrary constants and $m_5, m_6, ..., m_n$ are distinct real roots of Equation (5.8).

Rule (4): If the two roots (real) be m and $-m$, the corresponding part of the general solution will be $c_1 e^{mx} + c_2 e^{-mx}$

$$= c_1 (\cosh mx + \sinh mx) + c_2 (\cosh mx - \sinh mx)$$

$$= c_1' \cosh mx + c_2' \sinh mx \text{ where } c_1' = c_1 + c_2, c_2' = c_1 - c_2$$

And general solution will be,

$$y = c_1' \cosh mx + c_2' \sinh mx + c_3 e^{m_3 x} + c_4 e^{m_4 x} + ...+ c_n e^{mnx}$$

Where $c_1', c_2', c_3, ...c_n$ are arbitrary constants and $m_3, m_4 ... m_n$ are distinct real roots of Equation (5.9).

Rules for Finding Particular Integrals

Any particular solution of $F(D)\,y = f(x)$ is known as its Particular Integral (P.I). The P.I. of $F(D)y = f(x)$ is symbolically written as,

$$\text{P.I.} = \frac{1}{F(D)}\{f(x)\} \text{ where } F(D) \text{ is the operator.}$$

The operator $\frac{1}{F(D)}$ is defined as that operator which, when operated on $f(x)$ gives a function $\phi(x)$, such that $F(D)\,\phi(x) = f(x)$

i.e., $\frac{1}{F(D)}\{f(x)\} = \phi(x)$ (= P.I.)

$$\therefore \quad F(D)\left\{\frac{1}{F(D)} f(x)\right\} = f(x) \quad \left[\because \frac{1}{F(D)} f(x) = \phi(x)\right]$$

Obviously, $F(D)$ and $1/F(D)$ are inverse operators.

Case I: Let $F(D) = D$, then $\frac{1}{D} f(x) = \int f(x)\,dx$.

Proof: Let $y = \frac{1}{D}\{f(x)\}$, operating by D, we get $Dy = D \,.\, \frac{1}{D}\{f(x)\}$ or $Dy = f(x)$ or

$\frac{dy}{dx} = f(x)$ or $dy = f(x)\,dx$

Integrating both sides with respect to x, we get,

$y = \int f(x)\,dx$, since particular integrating does not contain any arbitrary constant.

Case II: Let $F(D) = D - m$ where m is a constant, then,

$$\frac{1}{D-m}\{f(x)\} = e^{mx} \int e^{-mx} f(x) dx.$$

Proof: Let $\frac{1}{D-m}\{f(x)\} = y$, then operating by $D - m$, we get,

$$(D-m) \,.\, \frac{1}{D-m}\{f(x)\} = (D-m)\,y$$

Or $$f(x) = \frac{dy}{dx} - my$$

Or $\frac{dy}{dx} - my = f(x)$ which is a first order linear differential equation and

$$\text{I.F.} = e^{\int -m dx} = e^{-mx}.$$

Then multiplying above equation by e^{-mx} and integrating with respect to x, we get,

$y\,e^{-mx} = \int f(x) e^{-mx} dx$, since particular integral does not contain any arbitrary constant,

Or $$y = e^{mx} \int f(x) e^{-mx} dx.$$

Note: If $\frac{1}{F(D)} = \frac{a_1}{D - m_1} + \frac{a_2}{D - m_2} + ... + \frac{a_n}{D - m_n}$ where a_i and m_i $(i = 1, 2, ..., n)$ are constants, then

$$\frac{1}{F(D)}\{f(x)\} = a_1 e^{m_1 x} \int f(x) e^{-m_1 x} dx + a_2 e^{m_2 x} \int f(x) e^{-m_2 x} dx + \dots + a_n e^{m_n x} \int f(x) e^{-m_n x} dx$$

$$= \sum_{i=1}^{n} a_i e^{m_i x} \int f(x) e^{-m_i x} dx$$

We now discuss methods of finding particular integrals for certain specific types of right hand functions

Type 1: $f(D)\, y = e^{mx}$ where m is a constant.

Then $$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}\} = \frac{e^{mx}}{F(m)} \text{ if } F(m) \neq 0$$

If $F(m) = 0$, then we replace D by $D + m$ in $F(D)$,

$$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}\} = e^{mx} \cdot \frac{1}{F(D + m)}\{1\}$$

Example 5.9: $(D^3 - 2D^2 - 5D + 6)\, y = (e^{2x} + 3)^2 + e^{3x} \cosh x.$

Solution: The reduced equation is,

$$(D^3 - 2D^2 - 5D + 6)\, y = 0 \qquad ...(1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (5.9). Then the auxiliary equation is,

$$m^3 - 2\,m^2 - 5m + 6 = 0 \text{ or } m^3 - m^2 - m^2 + m - 6m + 6 = 0$$

Or $m^2 (m - 1) - m (m - 1) - 6 (m - 1) = 0$

Or $(m - 1)(m^2 - m - 6) = 0$ or $(m - 1)(m^2 - 3m + 2m - 6) = 0$

Or $(m - 1)(m - 3)(m + 2) = 0$ or $m = 1, 3, -2$

$\therefore$ The complementary function is,

$y = c_1 e^x + c_2 e^{3x} + c_3\, e^{-2x}$ where c_1, c_2, c_3 are arbitrary constants.

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Again $(e^{2x}+3)^2 + e^{3x}\cosh x = e^{4x} + 6\,e^{2x} + 9 + e^{3x}\left(\dfrac{e^x + e^{-x}}{2}\right)$.

$$= e^{4x} + 6\,e^{2x} + 9e^{0\,.\,x} + \frac{e^{4x}}{2} + \frac{e^{2x}}{2}$$

$$= \frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}$$

$\therefore$ The particular integral is,

$$y = \frac{1}{D^3 - 2D^2 - 5D + 6}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{1}{(D-1)(D-3)(D+2)}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{3}{2}\frac{1}{(D-1)(D-3)(D+2)}e^{4x} + \frac{13}{2}\frac{1}{(D-1)(D+2)(D-3)}\{e^{2x}\}$$

$$+\ 9\frac{1}{(D-1)(D-3)(D+2)}e^{0.x}$$

$$= \frac{3}{2}\frac{e^{4x}}{(4-1)\,(4-3)\,(4+2)} + \frac{13}{2}\frac{e^{2x}}{(2-1)\,)(2+2)\,(2-3)}$$

$$+\,9\frac{e^{0.x}}{(0-1)(0-3)(0+2)}$$

$$= \frac{3}{2}\frac{e^{4x}}{3\,.1\,.\,6} + \frac{13}{2}\frac{e^{2x}}{1\,.\,4\,.\,(-1)} + 9\frac{e^{0\,.x}}{(-1)(-3)\,.\,2}$$

$$= \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + \frac{3}{2}.$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1e^x + c_2\,e^{3x} + c_3\,e^{-2x} + \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + \frac{3}{2}.$$

Notes: 1. When $F(m) = 0$ and $F'(m) \neq 0$, P.I. $= \dfrac{1}{F(D)}\{e^{mx}\} \quad = x\,\dfrac{1}{F'(D)}\{e^{mx}\}$

$$= \frac{xe^{mx}}{F'(m)}$$

2. When $F(m)= 0$ $F'(m) = 0$ and $F''(m) \neq 0$, then P.I. $= \dfrac{1}{F(D)}\{e^{mx}\}$

$$= x^2\,\frac{1}{F''(D)}\left\{e^{mx}\right\} = \frac{x^2e^{mx}}{F''(m)}$$

And so on.

Type 2: $f(x) = e^{mx}\,V$ where V is any function of x.

Here the particular integral (P.I.) of $F(D)\,y = f(x)$ is,

$$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}V\} = e^{mx}\frac{1}{F(D+m)}\{V\}.$$

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Example 5.10: Solve $(D^2 - 5D + 6)\, y = x^2 e^{3x}$

Solution: The reduced equation is,

$$(D^2 - 5D + 6)\, y = 0 \qquad (1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (1) and then auxiliary equation is

$$m^2 - 5m + 6 = 0 \text{ or } m^2 - 3m - 2m + 6 = 0$$

Or $\quad m\,(m-3) - 2\,(m-3) = 0$ or $(m-3)\,(m-2) = 0$

$\therefore \quad m = 2, 3$

$\therefore$ The complementary function is,

$y = c_1 e^{2x} + c_2 e^{3x}$ where c_1 and c_2 are arbitrary constants.

The particular integral is,

$$y = \frac{1}{D^2 - 5D + 6}\{x^2 e^{3x}\} = \frac{e^{3x}}{(D+3)^2 - 5(D+3) + 6}\{x^2\}$$

$$= e^{3x}\frac{1}{D^2 + 6D + 9 - 5D - 15 + 6}\{x^2\} = e^{3x}\frac{1}{D^2 + D}\{x^2\}$$

$$= e^{3x}\frac{1}{D(1+D)}\{x^2\} = e^{3x}\frac{1}{D}(1+D)^{-1}\{x^2\}$$

$$= \frac{e^{3x}}{D}(1 - D + D^2 - D^3 + D^4 - \ldots)\{x^2\}$$

$$= \frac{e^{3x}}{D}\{x^2 - 2x + 2\} = e^{3x}\left(\frac{x^3}{3} - x^2 + 2x\right)$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{2x} + c_2 e^{3x} + e^{3x}\left(\frac{x^3}{3} - x^2 + 2x\right).$$

Recall: (*i*) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

(*ii*) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

Type 3: (*a*) $F(D)\, y = \sin ax$ or $\cos ax$ where $F(D) = \phi(D^2)$.

Here $\quad$ P.I. $= \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{1}{\phi(-a^2)}\sin ax$ (if $\phi(-a^2) \neq 0$)

Or $\quad$ P.I. $= \dfrac{1}{F(D)}\{\cos ax\} = \dfrac{1}{\phi(-a^2)}\cos ax$ (if $\phi(-a^2) \neq 0$)

[Note D^2 has been replaced by $-a^2$ but D has not been replaced by $-a$.]

(*b*) $F(D)\, y = \sin ax$ or $\cos ax$ and $F(D) = \phi(D^2, D)$

Here $\quad$ P.I. $= \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{1}{\phi(D^2, D)}\{\sin ax\} = \dfrac{1}{\phi(-a^2, D)}\{\sin ax\}$

if $\quad \phi(-a^2, D) \neq 0$

Or $$y = \frac{1}{F(D)}\{\cos ax\} = \frac{1}{\phi(D^2, D)}\{\cos ax\} = \frac{1}{\phi(-a^2, D)}\{\cos ax\}$$

if $\phi(-a^2, D) \neq 0$

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(c) $F(D)\, y = \sin ax$ or $\cos ax$ and $F(D) = \dfrac{\psi(D)}{\phi(D^2)}$

Here P.I. $= \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{\psi(D)}{\phi(D^2)}\{\sin ax\} = \dfrac{\psi(D)}{\phi(-a^2)}\{\sin ax\}$ if $\phi(-a^2) \neq 0$

Or $$y = \frac{1}{F(D)}\{\cos ax\} = \frac{\psi(D)}{\phi(D^2)}\{\cos ax\}$$

$$= \frac{\psi(D)}{\phi(-a^2)}\{\cos ax\} \text{ if } \phi(-a^2) \neq 0$$

(d) $F(D)\, y = \sin ax$ or $\cos ax$, $F(D) = \phi(D^2)$ but $\phi(-a^2) = 0$.

Here P.I. $= \dfrac{1}{F(D)}\{\sin ax \text{ or } \cos ax\} = x\dfrac{1}{F'(D)}\{\sin ax \text{ or } \cos ax\}$

Alternatively, $\sin ax$ and $\cos ax$ can be written in the form $\sin ax = \dfrac{e^{ixa} - e^{-ixa}}{2i}$

And $\cos ax = \dfrac{e^{aix} + e^{-aix}}{2}$, then find P.I. by the method of Type 1.

Example 5.11: Solve $(D^4 + 2D^2 + 1)\, y = \cos x$.

Solution: The reduced equation is $(D^4 + 2D^2 + 1)\, y = 0$

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equaiton is,

$$m^4 + 2m^2 + 1 = 0 \text{ or } [(m^2 + 1)]^2 = 0 \text{ or } m = \pm i, \pm i$$

$\therefore$ C.F. $= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x$ where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore \quad \text{P.I.} = \frac{1}{D^4 + 2D^2 + 1}\{\cos x\}$$

$$= x\frac{1}{4D^3 + 4D}\{\cos x\}$$

$[\because \phi(D^2) = D^4 + 2D^2 + 1$

$\phi(-1^2) = 1 - 2 + 1 = 0$, then $\dfrac{1}{F(D)}\{f(x)\} = x\dfrac{1}{F'(D)}\{f(x)\}]$

$$= \frac{x}{4}\frac{1}{D^3 + D}\{\cos x\} = \frac{x}{4}.\frac{x}{3D^2 + 1}\{\cos x\}$$

$$= \frac{x^2}{4}\frac{1}{3D^2 + 1}\{\cos x\} = \frac{x^2}{4}.\frac{\cos x}{-3 + 1} = -\frac{x^2}{8}\cos x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{x^2}{8}\cos x.$$

Example 5.12: Solve $(D^2 - 4)y = \sin 2x$.

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Solution: The reduced equation is,

$$(D^2 - 4)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then auxiliary equation is,

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

The complementary function is,

$y = c_1 e^{2x} + c_2 e^{-2x}$ where c_1, c_2 are arbitrary constants.

The particular integral is,

$$y = \frac{1}{D^2 - 4}\{\sin 2x\} = \frac{1}{-2^2 - 4}\sin 2x \quad [\text{Replace } D^2 \text{ by} - 2^2]$$

$$= -\frac{1}{8}\sin 2x$$

The general solution is $y = \text{C.F.} + \text{P.I.} = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8}\sin 2x.$

Example 5.13: Solve $(3D^2 + 2D - 8)y = 5\cos x$.

Solution: The reduced equation is,

$$(3D^2 + 2D - 8)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then the auxiliary equation is,

$$3m^2 + 2m - 8 = 0 \text{ or } 3m^2 + 6m - 4m - 8 = 0$$

Or $\quad 3m(m + 2) - 4(m + 2) = 0$ or $(m + 2)(3m - 4) = 0$

Or $\quad m = -2, m = \frac{4}{3}$

$\therefore$ The complementary function is,

$y = c_1e^{-2x} + c_2e^{\frac{4}{3}x}$ when c_1 and c_2 are arbitrary constants.

The particular integral is,

$$y = \frac{1}{3D^2 + 2D - 8}\{5\cos x\} = 5\frac{1}{(3D - 4)(D + 2)}\{\cos x\}$$

$$= 5\frac{(3D + 4)(D - 2)}{(9D^2 - 16)(D^2 - 4)}\{\cos x\} = 5\frac{(3D + 4)(D - 2)}{[9(-1^2) - 16][-1^2 - 4]}\{\cos x\}$$

$[D^2$ is replaced by -1^2 in the denominator$]$ $\left[\frac{\psi(D)}{\phi(D^2)}\text{ form}\right]$

$$= \frac{5}{(-25)(-5)}[3D^2 - 6D + 4D - 8]\{\cos x\} = \frac{1}{25}[3D^2 - 2D - 8]\cos x$$

$$= \frac{1}{25}\left(3\frac{d^2}{dx^2}(\cos x) - 2\frac{d}{dx}(\cos x) - 8\cos x\right)$$

$$= \frac{1}{25}\left[-3\cos + 2\sin x - 8\cos x\right] = \frac{1}{25}(2\sin x - 11\cos x)$$

The general solution is,

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$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{-2x} + c_2 e^{4/3x} + \frac{1}{25}(2\sin x - 11\cos x).$$

Type 4: $F(D)\, y = x^n$, n is a positive integer.

Here $\quad$ $\text{P.I.} = \frac{1}{F(D)}\{x^n\} = [F(D)]^{-1}\{x^n\}$

In this case, $[F(D)]^{-1}$ is expanded in a binomial series in ascending powers of D upto D^n and then operate on x^n with each term of the expansion. The terms in the expansion beyond D^n need not be considered, since the result of their operation on x^n will be zero.

Example 5.14: Solve $D^2(D^2 + D + 1)y = x^2$.

Solution: The reduced equation is,

$$D^2(D^2 + D + 1)y = 0 \qquad (1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (1) and then the auxiliary equation is,

$$m^2(m^2 + m + 1) = 0$$

$$\therefore\ m = 0, 0 \text{ and } m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-I \pm \sqrt{3}i}{2}$$

$\therefore$ The complementary function is,

$$y = (c_1 + c_2 x)\, e^{0\cdot x} + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

$$= c_1 + c_2 x + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

Where c_1, c_2, c_3, c_4 are the arbitrary constant.

The particular integral is,

$$y = \frac{1}{D^2(D^2 + D + 1)}\{x^2\} = \frac{1}{D^2}(1 + D + D^2)^{-1}\{x^2\}$$

$$= \frac{1}{D^2}\{1 - (D + D^2) + (D + D^2)^2 - (D + D^2)^3 + ...\}\{x^2\}$$

$$= \frac{1}{D^2}\{1 - (D + D^2) + (D^2 + 2D^3 + D^4) - (D + D^2)^3 + ...\}\{x^2\}$$

$$= \frac{1}{D^2}\{x^2 - (2x + 2) + (2) + 0\}$$

$$= \frac{1}{D^2}\{x^2 - 2x\} = \frac{1}{D}\left\{\frac{x^3}{3} - x^2\right\} = \frac{x^4}{12} - \frac{x^3}{3}$$

The general solution is $y = $ C.F. + P.I.

$$= c_1 + c_2 x + e^{-x/2}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right) + \frac{x^4}{12} - \frac{x^3}{3}.$$

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Example 5.15: Solve $(D^2 + 4)y = x \sin^2 x$.

Solution: The reduced equation is,

$$(D^2 + 4)\, y = 0$$

The trial solution $y = A\, e^{mx}$ gives the auxiliary equation as,

$$m^2 + 4 = 0,\ m = \pm\, 2i$$

The complementary function is $y = c_1 \cos 2x + c_2 \sin 2x$

The particular integral is $y = \dfrac{1}{D^2+4}\{x \sin^2 x\}$

$$= \frac{1}{D^2+4}\left\{\frac{x}{2}(1-\cos 2x)\right\} = \frac{1}{D^2+4}\left\{\frac{x}{2} - \frac{x}{2}\cos 2x\right\}$$

$$= \frac{1}{D^2+4}\left\{\frac{x}{2}\right\} - \frac{1}{D^2+4}\left\{\frac{x}{2}\,\frac{(e^{2ix}+e^{-2ix})}{2}\right\}$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}\left\{\frac{x}{2}\right\} - \frac{1}{4}\,\frac{e^{2ix}}{(D+2i)^2+4}\{x\} - \frac{e^{-2ix}}{4(D-2i)^2+4}\{x\}$$

$$= \frac{1}{4}\frac{x}{2} - \frac{e^{2ix}}{4}\,\frac{1}{D^2+4Di-4+4}\{x\} - \frac{e^{-2ix}}{4}\,\frac{1}{D^2-4Di-4+4}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}\,\frac{1}{4Di\left(1+\frac{D}{4i}\right)}\{x\} - \frac{e^{-2ix}}{4\,.(-4Di)\left(1-\frac{D}{4i}\right)}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}.\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}\{x\} - \frac{e^{-2xi}}{4(-4Di)}\left(1-\frac{D}{4i}\right)^{-1}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}.\frac{1}{4Di}\left(1-\frac{D}{4i}+\frac{D^2}{-16}\ldots\right)\{x\} - \frac{e^{-2xi}}{4(-4Di)}\left(1+\frac{D}{4i}+\ldots\right)\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}.\frac{1}{4Di}\left(x-\frac{1}{4i}\right) + \frac{e^{-2xi}}{4\,.\,4Di}\left(x+\frac{1}{4i}\right)$$

$$= \frac{x}{8} - \frac{e^{2ix}}{2\,.\,8i}\left(\frac{x^2}{2}-\frac{x}{4i}\right) + \frac{e^{-2xi}}{2\,.\,8i}\left(\frac{x^2}{2}+\frac{x}{4i}\right)$$

$$= \frac{x}{8} - \frac{x^2}{2\,.\,8}\left(\frac{e^{2ix}-e^{-2xi}}{2i}\right) + \frac{x}{2\,.16\,.\,i^2}\left(\frac{e^{2ix}+e^{-2xi}}{2}\right)$$

$$= \frac{x}{8} - \frac{x^2}{2.8}\sin 2x - \frac{x}{2.16}\cos 2x$$

$$= \frac{x}{8} - \frac{x^2}{16}\sin 2x - \frac{x}{32}\cos 2x$$

Hence the general solution is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8} - \frac{x^2}{16} \sin 2x - \frac{x}{32}\cos 2x.$$

NOTES

Example 5.16: Solve $(D^4 + D^3 - 3D^2 - 5D - 2)\, y = 3xe^{-x}$.

Solution: The reduced equation is,

$$(D^4 + D^3 - 3D^2 - 5D - 2)\, y = 0 \quad (1)$$

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as,

$m^4 + m^3 - 3m^2 - 5m - 2 = 0$

Or $m^4 + m^3 - 3m^2 - 3m - 2m - 2 = 0$

Or $m^3 (m + 1) - 3m (m + 1) - 2 (m + 1)$

Or $(m + 1)(m^3 - 3m - 2) = 0$ or $(m + 1)\{m^3 + m^2 - m^2 - m - 2m - 2) = 0$

Or $(m + 1)\{m^2 (m + 1) - m (m + 1) - 2 (m + 1)\} = 0$

Or $(m + 1)(m + 1)(m^2 - m - 2) = 0$

Or $(m + 1)^2 (m^2 - 2m + m - 2) = 0$

Or $(m + 1)^2 (m + 1)(m - 2) = 0$

$\therefore\ m = -1, -1, -1, 2$

The complementary function is $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + c_4 e^{2x}$.

The particular integral is,

$$y = \frac{1}{(D+1)^3 (D-2)}\{3e^{-x} x\}$$

$$= 3e^{-x} \frac{1}{(D-1+1)^3 (D-3)}\{x\} = 3e^{-x} \frac{1}{D^3 (-3)(1 - D/3)}\{x\}$$

$$= -e^{-x} \frac{1}{D^3}\left(1 - \frac{D}{3}\right)^{-1}\{x\} = -e^{-x} \frac{1}{D^3}\left(1 + \frac{D}{3} + \frac{D^2}{9} + \dots\right)\{x\}$$

$$= -e^{-x} \frac{1}{D^3}\left(x + \frac{1}{3}\right) = -e^{-x} \frac{1}{D^2}\left(\frac{x^2}{2} + \frac{x}{3}\right) = -e^{-x} \frac{1}{D}\left(\frac{x^3}{6} + \frac{x^2}{6}\right)$$

$$= -e^{-x}\left(\frac{x^4}{24} + \frac{x^3}{18}\right)$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$= (c_1 + c_2 x + c_3 x^2) + c_4 e^{2x} - e^{-x}\left(\frac{x^4}{24} + \frac{x^3}{18}\right).$$

Type 5: (*a*) $F(D)\, y = xV$ where V is a function of x.

Here $\quad \text{P.I.} = \dfrac{1}{F(D)}\{xV\} = \left\{x - \dfrac{1}{F(D)} F'(D)\right\} \dfrac{1}{F(D)}\{V\}.$

NOTES

Example 5.17: Solve $(D^2+9)\,y = x\sin x$.

Solution: The reduced equation is $(D^2+9)\,y = 0$ (1)

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as,

$$m^2 + 9 = 0 \text{ or } m = \pm 3i$$

$\therefore$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$ where c_1 and c_2 are arbitrary constants.

And P.I. $= \dfrac{1}{F(D)}\{x\sin x\}$ where $F(D) = D^2 + 9$

$$= \left\{x - \frac{1}{F(D)}F'(D)\right\}\frac{1}{F(D)}\{\sin x\}$$

$$= \left\{x - \frac{2D}{D^2+9}\right\}\frac{1}{D^2+9}\{\sin x\}$$

$$= \left\{x - \frac{2D}{D^2+9}\right\}\frac{\sin x}{-1+9} = \left\{x - \frac{2D}{D^2+9}\right\}\left\{\frac{\sin x}{8}\right\}$$

$$= \frac{x\sin x}{8} - \frac{1}{4}\frac{1}{-1+9}D\{\sin x\} = \frac{x\sin x}{8} - \frac{1}{32}\cos x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + \frac{x\sin x}{8} - \frac{1}{32}\cos x$$

(*b*) $F(D)\,y = x^n V$ where V is any function of x.

Here P.I. $= \dfrac{1}{F(D)}\{f(x)\} = \dfrac{1}{F(D)}\{x^n V\} = \left\{x - \dfrac{F'(D)}{F(D)}\right\}^n \dfrac{1}{F(D)}\{V\}$

Example 5.18: Solve $(D^2-1)y = x^2 \sin x$

Solution: The reduced equation is $(D^2-1)y = 0$ (1)

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equation is,

$$m^2 - 1 = 0 \text{ or } m = \pm 1$$

$\therefore$ C.F. $= c_1e^x + c_2e^{-x}$ where c_1 and c_2 are arbitrary constants.

$\therefore$ P.I. $= \dfrac{1}{F(D)}\{x^2 \sin x\}$ where $F(D) = D^2 - 1$

$$= \left\{x - \frac{F'(D)}{F(D)}\right\}^2 \frac{1}{F(D)}\{\sin x\} = \left\{x - \frac{1}{D^2-1}2D\right\}^2 \frac{1}{D^2-1}\{\sin x\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{x - \frac{1}{D^2-1}2D\right\}\left\{\frac{1}{-1^2-1}\sin x\right\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{x - \frac{1}{D^2-1}2D\right\}\{-1/2\sin x\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{-\frac{x}{2}\sin x + \frac{1}{D^2-1}\right\}\{\cos x\}$$

NOTES

$$= \left\{x - \frac{1}{D^2 - 1} 2D\right\}\left\{-\frac{x}{2}\sin x - \frac{1}{2}\cos x\right\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2 - 1}\{D(x\sin x + \cos x)\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2 - 1}\{\sin x + x\cos x - \sin x\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2 - 1}\{x \cos x\}$$

Again $\frac{1}{D^2 - 1}\{x\cos x\} = \left\{x - \frac{1}{D^2 - 1}2D\right\}\frac{1}{D^2 - 1}\{\cos x\}$

$$= \left\{x - \frac{1}{D^2 - 1}2D\right\}\left\{\frac{1}{-1-1}\cos x\right\}$$

$$= -\frac{1}{2}x\cos x + \frac{1}{D^2 - 1}\{-\sin x\}$$

$$= -\frac{1}{2}x\cos x - \frac{\sin x}{-1^2 - 1} = -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$$

$$\therefore \text{P.I.} = -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x - \frac{x}{2}\cos x + \frac{1}{2}\sin x$$

$$= -\frac{1}{2}x^2\sin x - x\cos x + \frac{1}{2}\sin x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} - \frac{1}{2}x^2\sin x - x\cos x + \frac{1}{2}\sin x.$$

Classification of Partial Differential Equations of Second Order

Consider the following linear partial differential equation of the second order in two independent variables,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

Where A, B, C, D, E, F, and G are functions of x and y.

This equation when converted to quasi-linear partial differential equation takes the form,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

These equations are said to be of:

1. Elliptic Type if $B^2 - 4AC < 0$
2. Parabolic Type if $B^2 - 4AC = 0$
3. Hyperbolic Type if $B^2 - 4AC > 0$

NOTES

Let us consider some examples to understand this:

(i) $\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0$

$\Rightarrow u_{xx} - 2xu_{xy} + \text{x}2u_{yy} - 2u_y = 0$

Comparing it with the general equation we find that,

$A = 1, B = -2x, C = x^2$

Therefore,

$B^2 - 4AC = (-2x)^2 - 4x^2 = 0, \ \forall\ x$ and $y \neq 0$

So the equation is parabolic at all points.

(ii) $y^2u_{xx} + \text{x}^2u_{yy} = 0$

Comparing it with the general equation we get,

$A = y^2, B = 0, C= x^2$

Therefore,

$B^2 - 4AC = 0 - 4x^2y^2 < 0, \ \forall\ x$ and $y \neq 0$

So the equation is elliptic at all points.

(iii) $x^2u_{xx} - y^2u_{yy} = 0$

Comparing it with the general equation we find that,

$A = x^2, B = 0, C = -y^2$

Therefore,

$B^2 - 4AC = 0 - 4x^2y^2 > 0, \ \forall\ x$ and $y \neq 0$

So the equation is hyperbolic at all points.

Following three are the most commonly used partial differential equations of the second order:

1. Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is equation is of elliptic type.

2. One-dimensional heat flow equation,

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

This equation is of parabolic type.

3. One-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

This is a hyperbolic type.

NOTES

Check Your Progress

1. Define Lagrange's linear differential equation.
2. Write the general linear differential equation with constant coefficients.
3. What is the complementary function of 2nd order linear differential equation if the roots of equation m_1 and m_2 are equal?
4. What is the particular integral?
5. What are the three types of second order partial differential equations?

5.4 LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENT

Homogeneous Linear Equations with Constant Coefficients

Let $f(D, D')z = V(x, y)$ (5.10)

Then if,

$$f(D, D') = A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'_2 + \cdots + A_n D'^n \quad (5.11)$$

Where $A_1, A_2, \cdots, A_n$ are constants.

Then Equation (5.10) is known as Homogeneous equation and takes the form,

$$\left(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right)z = V(x, y) \quad (5.12)$$

Complementary Function

Consider the equation,

$$\left(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right)z = 0 \quad (5.13)$$

Let,

$$z = \phi(y + mx) \quad (5.14)$$

Be a solution of Equation (5.18)

Now $D^r z = m^r \phi^r (y + mx)$

$$D'^s z = \phi^{(e)}(y + mx)$$

And $D^r D'^s z = m^r \phi^{(r+s)}(y + mx)$

Therefore, on substituting Equation (5.14) in Equation (5.13), we get

$$\left(A_0 m^n + A_1 m^{n-s} + A_2 m^{n-2} + \cdots + A_n\right)\phi^{(n)}(y + mx) = 0$$

Which will be satisfied if,

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \quad (5.15)$$

Equation (5.15) is known as the auxiliary equation.

Let $m_1, m_2, \cdots, m_n$ be the roots of the Equation (5.15),

Then the following three cases arise:

NOTES

Case I: Roots $m_1, m_2, \cdots, m_n$ are Distinct.

Part of C.F. corresponding to $m = m_1$ is,

$$z = \phi_1(y + m_1 x)$$

Where 'ϕ_1' is an arbitrary function.

Part of C.F. corresponding to $m = m_2$ is,

$$z = \phi_2(y + m_2 x)$$

Where ϕ_2 is any arbitrary function.

Now since our equation is linear, so the sum of solutions is also a solution.

Therefore, our complimentary function becomes,

C.F. $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots\ldots\ldots\ldots\ldots + \phi_n(y + m_nx)$

Case II: Roots are Imaginary.

Let the pair of complex roots of the Equation (5.16) be

$u \pm iv$

Then the corresponding part of complimentary function is,

$$z = \phi_1(y + ux + ivx) + \phi_2(y + ux - ivx) \qquad \ldots(5.16)$$

Let $y + ux = P$ and $vx = Q$

Then $z = \phi_1(P + iQ) + \phi_2(P - iQ)$

Or $z = (\phi_1 + \phi_2)P + (\phi_1 - \phi_2)iQ$

If $\phi_1 + \phi_2 = \xi_1$

And $\phi_1 - \phi_2 = \xi_2$

Then,

$$\varphi_1 = \frac{1}{2}(\xi_1 + i\xi_2)$$

And

$$\varphi_2 = \frac{1}{2}(\xi_1 - i\xi_2)$$

Substituting these values in Equation (5.21), we get,

$$z = \frac{1}{2}\xi_1(P + iQ) + \frac{1}{2}i\xi_2(P + iQ) + \frac{1}{2}\xi_1(P - iQ) - \frac{1}{2}i\xi_2(P - iQ)$$

Or

$$z = \frac{1}{2}\{\xi_1(P + iQ) + \xi_1(P - iQ)\} + \frac{1}{2}i\{\xi_2(P + iQ) - \xi_2(P - iQ)\}$$

NOTES

Case III: Roots are Repeated.

Let m be the repeated root of Equation (5.15).

Then we have,

$(D - mD')(D - mD')z = 0$

Putting $(D - mD')z = U$, we get (5.17)

$(D - mD')U = 0$ (5.18)

Since the equation is linear, it has the following subsidiary equations,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dU}{0} \quad (5.19)$$

Two independent integrals of Equation (5.19) are,

$y + mx = \text{Constant}$

And $U = \text{Constant}$

$\therefore$ $U = \phi(y + mx)$

This is a solution of Equation (5.18) where ϕ is an arbitrary function.

Substituting in Equation (5.17),

$$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = \phi(y + mx) \quad (5.20)$$

Which has the following subsidiary equations,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)}$$

Two independent integrals of Equation (5.17) are,

$y + mx = \text{Constant}$

And $z = x\phi(y + mx) + \text{Constant}$

Therefore $z = x\phi(y + mx) + \psi(y + mx)$ (5.21)

This is a solution of Equation (5.20) where ψ is an arbitrary function.

Equation (5.21) is the part of C.F. corresponding to two times repeated root.

In general, if the root m is repeated r times, the corresponding part of C.F. is,

$$z = x^{r-1}\phi_1(y + mx) + x^{r-2}\phi_2(y + mx) + \cdots + \phi_r(y + mx)$$

Where $\phi_1, \phi_2, \cdots, \phi_r$ are arbitrary functions.

Example 5.19: Solve the equation, $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.

Solution: The A.E. of the given equation is,

$$m^3 - 3m^2 + 3m - 1 = 0$$

NOTES

Or $(m-1)^3 = 0$

$\Rightarrow$ $m = 1, 1, 1$

$\therefore$ C.F. = $x^2\phi_1(y+x) + x\phi_2(y+x) + \phi_2(y+x)$.

Non-Homogeneous Linear Equations with Constant Coefficients

If all the terms on left hand side of Equation (5.10) are not of same degree then Equation (5.10) is said to be **Non-Homogeneous equation**. Equation is said to be **reducible** if the symbolic function $f(D, D')$ can be resolved into factors each of which is of first degree in *D* and *D'* and irreducible otherwise.

For example, the equation,

$$f(D, D')z = (D^2 - D'^2 + 2D + 1)z = (D + D' + 1)(D - D' + 1)z = x^2 + xy$$

It is reducible while the equation,

$$f(D, D')z = (DD' + D'^3)z = D'(D + D'^2)z = \cos(x + 2y)$$

It is irreducible.

Reducible Non Homogeneous Equations

In the equation,

$$f(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\cdots(a_nD + b_nD' + c_n) \quad \ldots(5.22)$$

Where *a*'s, *b*'s and *c*'s are constants.

The complementary function takes the form,

$$(a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\cdots(a_nD + b_nD' + c_n)z = 0 \quad (5.23)$$

Any solution of the equation given by

$$(a_iD + b_iD' + c_i)z = 0 \quad (5.24)$$

This is a solution of the Equation (5.23)

Forming the Lagrange's subsidiary equations of Equation (5.24),

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_iz} \quad (5.25)$$

The two independent integrals of Equation (5.25) are,

$$b_ix - a_iy = \text{Constant}$$

And $z = \text{Constant } e^{-\frac{c_i}{a_i}x}$, if $a_i \neq 0$

Or

$z = \text{Constant } e^{-\frac{c_i}{b}y}$, if $b_i \neq 0$

Therefore,

$$z = e^{-\frac{c_i}{a_i}x}\phi_i(b_ix - a_iy), \text{ if } a_i \neq 0$$

NOTES

Or

$$z = e^{-\frac{c_i}{b_i}y}\psi_i(b_i x - a_i y) \quad \text{if } b_i \neq 0$$

This is the general solution of Equation (5.24). Here φ_i and ψ_i are arbitrary functions.

Example 5.20: Solve the differential equations,

$$\left(D^2 - D'^2 - 3D + 3D'\right)z = 0 .$$

Solution: The equation can also be written as,

$$\left(D - D'\right)\left(D + D' - 3\right)z = 0$$

$$\therefore \quad \text{C.F.} = \phi_1\left(y + x\right) + e^{3x}\phi_2\left(x - y\right)$$

Or

$$\psi_1(y + x) + e^{3y}\psi_2(x - y)$$

When the Factors are Repeated

Let the factor is repeated two times and is given by,

$$(aD + bD' + c)$$

Consider the equation,

$$\left(aD + bD' + c\right)\left(aD + bD' + c\right)z = 0 \qquad (5.26)$$

Put $\left(aD + bD' + c\right)z = U$ (5.27)

Then the Equation (5.27) reduces to,

$$\left(aD + bD' + c\right)U = 0 \qquad (5.28)$$

General solution of Equation (5.28) is,

$$U = e^{-\frac{c}{a}x}\,\phi\left(bx - ay\right) \text{ if } a \neq 0 \qquad (5.29)$$

Or

$$U = e^{-\frac{c}{b}y}\,\psi\left(bx - ay\right) \text{ if } b \neq 0 \qquad (5.30)$$

Substituting Equation (5.29) in Equation (5.27), we obtain,

$$(aD + bD' + c)z = e^{-\frac{c}{a}x}\,\phi\left(bx - ay\right) \qquad (5.31)$$

The subsidiary equations are,

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{e^{-\frac{c}{a}x}\varphi(bx - ay) - cz} \qquad (5.32)$$

The two independent integrals of Equations (5.32) are given by,

$$bx - ay = \text{Constant} = \lambda \qquad (5.33)$$

NOTES

And $\frac{dz}{dx}+\frac{c}{a}z=\frac{1}{a}e^{-\frac{c}{a}x}\varphi(bx-ay)=\frac{1}{a}e^{-\frac{c}{a}x}\varphi(\lambda)$ (5.34)

The Equation (5.34) being an ordinary linear equation has the following solution:

$$ze^{\frac{c}{a}x}=\frac{1}{a}x\varphi(\lambda)+\text{Constant}$$

Or $$ze^{\frac{c}{a}x}=\frac{1}{a}x\varphi(bx-ay)+\text{Constant}$$

Therefore, general solution of Equation (5.31) is,

$$z=\frac{x}{a}e^{-\frac{c}{a}x}\phi(bx-ay)+\phi_1(bx-ay)e^{-\frac{c}{a}x}$$

$$=e^{-\frac{c}{a}x}\{x\phi_2(bx-ay)+\phi_1(bx-ay)\} \quad \ldots(5.35)$$

Where ϕ_1 and ϕ_2 are arbitrary functions.

Similarly from Equations (5.30) and (5.27), we get

$$z=e^{-\frac{c}{b}y}\{y\psi_2(bx-ay)+\psi_1(bx-ay)\}$$

Where ψ_1 and ψ_2 are arbitrary functions.

In general, for *r* times repeated factor, $(aD+bD'+c)$

$$z=e^{-\frac{c}{a}x}\sum_{i=1}^{r}x^{i-1}\phi_i(bx-ay) \qquad \text{if } a\neq 0$$

Or

$$z-e^{-\frac{c}{b}y}\sum_{i=1}^{r}y^{i-1}\psi_i(bx-ay) \qquad \text{if } b\neq 0$$

Where $\varphi_1,\varphi_2,\ldots,\varphi_r$ and $\psi_1,\psi_2,\ldots,\psi_r$ are arbitrary functions.

Example 5.21: Solve the differential equation,

$$(2D-D'+4)(D+2D'+1^2z=0)$$

Solution: C.F. corresponding to the factor $(2D-D'+4)$ is,

$$e^{4y}\phi(x+2y)$$

C.F. corresponding to the factor $(D=2D'+1)^2$ is,

$$e^{-x}\{x\phi_2(2x-y)+\phi_1(2x-y)\}$$

Hence C.F. $=e^{4y}\phi(x+2y)+e^{-x}\{x\phi_2(2x-y)+\phi_1(2x-y)\}$

Irreducible Non-Homogeneous Equations

For solving the equation,

$$f(D, D')z = 0 \tag{5.36}$$

Substitute $z = ce^{ax+by}$ where a, b and c are constants. (5.37)

Now $D^r z = ca^r e^{ax+by}$

$D^r D'^a z = ca^r b^s e^{ax+by}$

And $D'^s z = cb^s e^{ax+by}$

Substituting Equation (5.37) in Equation (5.36), we get,

$$cf(a, b)e^{ax+by} = 0$$

Which will hold if,

$$f(a, b) = 0 \tag{5.38}$$

For any selected value of a (or b) Equation (5.38) gives one or more values of b (or a). Thus there exists infinitely many pairs of numbers (a_i, b_i) satisfying Equation (5.38).

Thus

$$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y} \tag{5.39}$$

Where $f(a_i, b_i) = 0 \ \forall \ i$, is a solution of the Equation (5.38),

If

$$f(D, D') = (D + hD' + k)g(D, D') \tag{5.40}$$

Then any pair (a, b) such that,

$$a + hb + k = 0 \tag{5.41}$$

Satisfies Equation (5.40). There are infinite number of such solutions.

From Equation (5.41),

$$a = -(hb + k)$$

Thus

$$z = \sum_{i=1}^{\infty} c_i e^{-(hb_i + k)x + b_i y}$$

$$= e^{-kx} \sum_{i=1}^{\infty} c_i e^{b_i(y - hx)} \tag{5.42}$$

This is a part of C.F. corresponding to a linear factor $(D + hD' + k)$ given in Equation (5.40).

NOTES

Equation (5.42) is equivalent to,

$$e^{-kx}\phi(y - hx)$$

Where 'ϕ' is an arbitrary function.

Equation (5.39) is the general solution if *f* (*D*, *D*′) has no linear factor otherwise general solution will be composed of both arbitrary functions and partly arbitrary constants.

Example 5.22: Solve the differential equation $\left(2D^4 + 3D^2D' + D'^2\right)z = 0$.

Solution: The given equation is equivalent to,

$$\left(2D^2 + D'\right)\left(D^2 + D'\right)z = 0$$

C.F. corresponding to the first factor,

$$= \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$$

Where a_i and b_i are related by,

$$2a_i^2 + b_i = 0$$

Or $\qquad b_i = -2a_i^2$

Therefore, part of C.F. corresponding to the first factor,

$$\sum_{i=1}^{\infty} d_i e^{e_i(x - e_i y)}$$

Where e_i and d_i are arbitrary constants.

$$\therefore \quad \text{C.F.} = \sum_{i=1}^{\infty} c_i e^{a_i(x - 2a_i y)} + \sum_{i=1}^{\infty} d_i e^{e_i(x - e_i y)}$$

Particular Integral

In the equation,

$$f(D, D')z = V(x, y) \qquad \ldots(5.43)$$

f(*D*, *D*′) is a non homogeneous function of *D* and *D*′.

$$\text{P.I.} = \frac{1}{f(D, D')}V(x, y) \qquad \ldots(5.44)$$

Here if $V(x, y)$ is of the form e^{ax+by} where '*a*' and '*b*' are constants then we use the following theorem to evaluate the particular integral:

Theorem 5.1: If $f(a, b) \neq 0$, then,

$$\frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a, b)}e^{ax+by}$$

NOTES

Proof: By differentiation,

$$D^r D'^s e^{ax+by} = a^r b^s e^{ax+by}$$

$$D^r e^{ax+by} = a^r e^{ax+by}$$

$$D'^s e^{ax+by} = b^s e^{ax+by}$$

$$\therefore \quad f(D, D')e^{ax+by} = f(a,b)e^{ax+by}$$

$$e^{ax+by} = f(a,b)\frac{1}{f(D,D')}e^{ax+by}$$

Dividing the above equation by $f(a, b)$

$$\frac{1}{f(a,b)}e^{ax+by} = \frac{1}{f(D,D')}e^{ax+by}$$

Or $$\frac{1}{f(D,D')}e^{ax+by} = \frac{1}{f(a,b)}e^{ax+by}$$

Example 5.23: Solve the equation $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$

Solution: The given equation is equivalent to,

$$(D - D')(D + D' - 3)z = e^{x-2y}$$

$$\text{C.F.} = \phi_1(y + x) + e^{3x}\phi_2(y - x)$$

$$\text{P.I.} = \frac{1}{(D - D')(D + D' - 3)}e^{x-2y}$$

$$= -\frac{1}{12}e^{x-2y}$$

Therefore, $z = \varphi_1(y + x) + e^{3x}\varphi_2(y - x) - \frac{1}{12}e^{x-2y}$

But in case $V(x, y)$ is of the form $e^{ax+by}\phi(x, y)$ where 'a' and 'b' are constants then following theorem is used to evaluate the particular integral:

Theorem 5.2: If $\phi(x, y)$ is any function, then

$$\frac{1}{f(D,D')}e^{ax+by}\phi(x,y) = e^{ax+by}\frac{1}{f(D+a, D'+b)}\phi(x,y)$$

Proof: From Leibnitz's Theorem for successive differentiation, we have

$$D^r\{e^{ax+by}\phi(x,y)\} = e^{ax+by}\{D^r\phi(x,y) + {}^r c_1 a.D^{r-1}\phi(x,y)\}$$

$$+ {}^r c_2 a^2 d^{r-2}\phi(x,y) + \cdots + {}^r c_r a^r\phi(x,y)$$

$$= e^{ax+by}\{D^r + {}^r c_1 D^{r-1} + {}^r c_2 a^2 D^{r-2} + \cdots + {}^r c^r a^r\}\phi(x,y)$$

$$= e^{ax+by}(D + a)^r\phi(x,y).$$

NOTES

Similarly,

$$D'^s\{e^{ax+by}\varphi(x,y)\} = e^{ax+by}(D'+b)^s\varphi(x,y)$$

And $D^r D'^s\{e^{ax+by}\varphi(x,y)\} = D^r[e^{ax+by}(D'+b)\varphi(x,y)]$

$$= e^{ax+by}(D+a)^r(D'+b)^s\phi(x,y)$$

So $\quad f(D,D')\{e^{ax+by}\phi(x,y)\} = e^{ax+by}f(D+a,D'+b)\phi(x,y) \qquad (5.45)$

Put $\quad f(D+a,D'+b)\phi(x,y) = \psi(x,y)$

$$\therefore \quad \phi(x,y) = \frac{1}{f(D+a,D'+b)}\psi(x,y)$$

Substituting in Equation (5.45), we get,

$$f(D,D')\left\{e^{ax+by}\frac{1}{f(D+a,D'+b)}\psi(x,y)\right\} = e^{ax+by}\psi(x,y)$$

Operating on the equation by $\dfrac{1}{f(D,D')}$

$$e^{ax+by}\frac{1}{f(D+a,D'+b)}\psi(x,y) = \frac{1}{f(D,D')}\{e^{ax+by}\psi(x,y)\}$$

Replacing $\psi(x,y)$ by $\phi(x,y)$, we have,

$$\frac{1}{f(D,D')}\left(e^{ax+by}\phi(x,y)\right) = e^{ax+by}\frac{1}{f(D+a,D'+b)}\phi(x,y)$$

Example 5.24: Solve $(D^2 - D'^2 - 3D + 3D')z = xv + e^{x+av}$.

Solution: The given equation is equivalent to,

$$(D-D')(D+D'-3)y = xy + e^{x+2y}$$

C.F. $= \phi_1(y+x) + e^{3x}\phi_2(x-y)$

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)}xy + \frac{1}{(D-D')(D+D'-3)}e^{x+2y}$$

$$= -\frac{1}{3D}\left\{1-\frac{D'}{D}\right\}^{-1}\left\{1-\frac{D+D'}{3}\right\}^{-1}xy$$

$$+ e^{x+2y}\frac{1}{(D+1-D'-2)(D+1+D'+2-3)}1.$$

$$= -\frac{1}{3D}\left\{1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right\}\left\{1+\frac{D+D'}{3}+\frac{2}{9}DD'+\cdots\right\}xy + e^{x+2y}$$

NOTES

$$\frac{1}{(D-D'-1)(D+D')}.1$$

$$=-\frac{1}{3D}\left\{1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right\}\left\{xy+\frac{x+y}{3}+\frac{2}{9}.\right\}+\ e^{x+2y}\frac{1}{(-1)(D+D')}.1$$

$$=-\frac{1}{3D}\left\{xy+\frac{2}{3}x+\frac{x^2}{2}+\frac{1}{3}y+\frac{2}{9}\right\}-xe^{x+2y}$$

$$=-\frac{1}{3}\left\{\frac{x^2y}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{1}{3}xy+\frac{2}{9}x\right\}-xe^{x+2y}.$$

$$\therefore\quad z=\phi_1(y+x)+e^{3x}\phi_2(x-y)-xe^{x+2y}-\frac{1}{6}x^2y-\frac{1}{9}x^2-\frac{x^3}{18}-\frac{1}{9}xy-\frac{2}{27}x$$

Example 5.25: Solve $(D^2-DD'+D'-1)z=\cos(x+2y)+e^y+xy+1$.

Solution: Equation is equivalent to,

$$(D-1)(D-D'+1)z=\cos(x+2y)+e^y+xy+1$$

Complementary Function $=e^x\varphi_1(y)+e^y\varphi_2(x+y)$.

Particular integral corresponding to cos $(x+2y)$ is,

$$\frac{1}{D^2-DD'+D'-1}\cos(x+2y)$$

$$=\frac{1}{(-1)-(-2)+D'-1}\cos(x+2y)$$

$$=\frac{1}{D'}\cos(x+2y)$$

$$=\frac{1}{2}\sin(x+2y)$$

Corresponding to e^y , the particular integral is,

$$=\frac{1}{D^2-DD'+D'-1}e^y$$

$$=\frac{1}{D'-1}e^y$$

$$=e^y.\frac{1}{D'}.1$$

$$=ye^y.$$

Particular Integral corresponding to the part $(xy+1)$ is,

$$=\frac{1}{(D-1)(D-D'+1)}(xy+1)$$

NOTES

$$-\{1-D\}^{-1}\{1+(D-D')\}^{-1}(xy+1)$$

$$=-\{1+D+D^2+.....\}\{1-(D-D')+(D-D')^2-.....\}(xy+1)$$

$$=-\{1+D+D^2+.....\}\{(xy+1)-(y-x)-2\}$$

$$=-\{1+D+D^2+.....\}(xy-y+x-1)$$

$$=-\{(xy-y+x-1)+(y+1)\}$$

$$=-(xy+x)$$

$$=-x(y+1)$$

$$\therefore \qquad z=e^x\varphi_1(y)+e^y\varphi_2(x+y)+\frac{1}{2}\sin(x+2y)+ye^y-x(y+1)$$

Partial Differential Equations Reducible to Equations with Constant Coefficients

The equation,

$$f(xD, yD')z = V(x, y)$$

Where $f(xD, yD') = \sum_{r,s} c_{rs}x^r y^s D^r D'^s, c_{rs} = \text{Constant.}$ (5.46)

This is reduced to linear partial differential equation with constant coefficients by the following substitution:

$u = \log x,\ v = \log y$ (5.47)

By substitution of Equation (5.47)

$$xD = x\frac{\partial}{\partial x}$$

$$= x\frac{\partial}{\partial u}\frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial u} = d \text{ (say)}$$

And,

$$x^2D^2 = x^2D\left(\frac{1}{x}\frac{\partial}{\partial u}\right)$$

$$= x^2\left(-\frac{1}{x^2}\frac{\partial}{\partial u}+\frac{1}{x^2}\frac{\partial^2}{\partial u^2}\right)$$

$$= \frac{\partial^2}{\partial u^2}-\frac{\partial}{\partial u}$$

$$= d(d-1)$$

NOTES

Therefore,

$$x^r D^r = d(d-1)(d-2).....(d-\overline{r-1})$$

And $y^s D'^s = d'(d'-1)(d'-2)...(d'-\overline{s-1})$

Hence $f(xD, yD') = \sum c_{rs} d(d-1).....(d-\overline{r-1})\, d'(d'-1).....(d'-\overline{s-1})$

$$= g(d, d')$$

Here the coefficients in $g(d, d')$ are constants.

Thus by substitution Equation (5.46) is reduced to,

$$g(d, d')z = V(e^u, e^v)$$

Or $g(d, d')z = U(u, v)$ (5.48)

Equation (5.48) can be solved by methods that have been described for solving partial differential equations with constant coefficients.

Example 5.26: Solve the differential equation,

$$(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$$

Solution: Put $u = \log x$

$$v = \log y$$

The given equation can be reduced to

$$\{d(d-1) - 4dd' + 4d'(d'-1) + 6d'\}z = e^{3u+4v}$$

Or $(d-2d')(d-2d'-1)z = e^{3u+4v}$

The complementary function is $\phi_1(2u+v) + e^u\phi_2(2u+v)$

$$= \phi_1(\log x^2y) + x\phi_2(\log x^2y)$$

$$= \psi_1(x^2y) + x\psi_2(x^2y)$$

And the particular integral is $\dfrac{1}{(d-2d')(d-2d'-1)}e^{3u+2v}$

$$= \frac{1}{30}e^{3u+4v}$$

$$= \frac{1}{30}x^3y^4$$

$$\therefore \qquad z = \psi_1(x^2y) + x\psi_2(x^2y) + \frac{1}{30}x^3y^4.$$

Example 5.27: Find the solution of, $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$

Solution: Put $u = \log x$

$$v = \log y$$

NOTES

The given differential can be reduced to,

$$\{d(d-1)-d'(d'-1)-d'+d\}z=0$$

$$\Rightarrow \quad (d^2-d'^2)z=0$$

A.E. is,

$$m^2-1=0$$

$$\Rightarrow \quad m=1,-1$$

$$\Rightarrow \quad z=\phi_1(v+u)+\phi_2(v-u)$$

$$=\phi_1(\log xy)+\phi_2\left(\log\frac{y}{x}\right)$$

$$=\Psi_1(xy)+\Psi_2\left(\frac{y}{x}\right).$$

Example 5.28: Determine the solution of the following equation,

$$(x^2D^2+2xyDD'+y^2D'^2)z+nz=n(xD+yD')z+x^2+y^2+x^3$$

Solution: Put $u=\log x$

$$v=\log y$$

The equation reduces to,

$$\{d(d-1)+2dd'+d'(d'-1)\}z-n(d+d')z+nz=e^{2u}+e^{2v}+e^{3u}$$

Or

$$\{(d+d')^2-(d+d')\}z-n(d+d')z+nz=e^{2u}+e^{2v}+e^{3u}$$

Or

$$\{(d+d')(d+d'-1)-n(d+d')+n\}z=e^{2u}+e^{2v}+e^{3u}$$

Or

$$\{(d+d')^2-(n+1)(d+d')+n\}z=e^{2u}+e^{2v}+e^{3u}$$

Or

$$(d+d'-n)(d+d'-1)z=e^{2u}+e^{2v}+e^{3u}$$

C.F. $= e^{nu}\phi_1(u-v)+e^{u}\phi_2(u-v)$

$= x^n\psi_1\left(\frac{x}{y}\right)+x\psi_2\left(\frac{x}{y}\right)$

P.I. $=\dfrac{1}{(d+d'-n)(d+d'-1)}\{e^{2u}+e^{2v}+e^{3u}\}$

$$= \frac{1}{2-n}e^{2u} + \frac{1}{2-n}e^{2v} + \frac{1}{(3-n)^2}e^{3u}$$

NOTES

$$= -\frac{x^2+y^2}{n-2} - \frac{1}{2}.\frac{1}{n-3}x^3$$

$$\therefore\ z = x^n\psi_1\left(\frac{x}{y}\right) + x\psi_2\left(\frac{x}{y}\right) - \frac{x^2+y^2}{n-2} - \frac{1}{2}\frac{x^3}{n-3}$$

Example 5.29: Solve $\left(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD'\right)z = \log\frac{y}{x} - \frac{1}{2}$

Solution: Put

$$u = \log x$$

$$v = \log y$$

Our equation reduces to,

$$\{d(d-1) - dd' - 2d'(d'-1) + d - 2d'\}z = v - u - \frac{1}{2}$$

$$\left(d^2 - dd' - 2d'^2\right)z = v - u - \frac{1}{2}$$

Or

$$(d-2d')(d+d')z = v - u - \frac{1}{2}$$

$$\text{C.F.} = \phi_1(2u+v) + \phi_2(u-v)$$

$$= \psi_1\left(x^2y\right) + \psi_2\left(\frac{x}{y}\right)$$

$$\text{P.I.} = \frac{1}{(d-2d')(d+d')}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{d-2d'}.\frac{1}{d}\left\{1 - \frac{d'}{d}\cdots\right\}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{d-2d'}.\frac{1}{d}\left\{v - u - \frac{1}{2} - u\right\}$$

$$= \frac{1}{d-2d'}.\left(uv - u^2 - \frac{1}{2}u\right)$$

$$= \frac{1}{d}\left\{1 + \frac{2d'}{d} + \frac{4d'^2}{d^2} + \cdots\right\}\left(uv - u^2 - \frac{1}{2}u\right)$$

NOTES

$$= \frac{1}{d}\left\{uv - u^2 - \frac{1}{2}u + u^2\right\}$$

$$= \frac{u^2 v}{2} - \frac{u^2}{4}$$

$$= \frac{1}{2}(\log x)^2 \log y - \frac{1}{4}(\log x)^2$$

$$\therefore \quad z = \psi_1(x^2 y) + \psi_2\left(\frac{x}{y}\right) + \frac{1}{2}(\log x)^2 \log y - \frac{1}{4}(\log x)^2.$$

Example 5.30: Solve the differential equation,

$$(x^2D^2 + 2xyDD' + y^2D'^2)z = (x^2 + y^2)^{\frac{n}{2}}$$

Solution: Put $u = \log x$

$v = \log y$

The equation is reduced to $\{d(d-1) + 2dd' + d'(d'-1)\}z = (e^{2u} + e^{2v})^{\frac{n}{2}}$

Or $\{(d + d')^2 - (d + d')\}z = (e^{2u} + e^{2v})^{\frac{n}{2}}$

Or $(d + d')(d + d' - 1)z = (e^{2u} + e^{2v})^{\frac{n}{2}}$

$$\text{C.F.} = \phi_1(u - v) + e^u \phi_2(u - v)$$

$$= \phi_1\left(\log\frac{x}{y}\right) + x\phi_2\left(\log\frac{x}{y}\right)$$

$$= \Psi_1\left(\frac{x}{y}\right) + x\Psi_2\left(\frac{x}{y}\right)$$

Particular Integral is $= \dfrac{1}{(d + d')(d + d' - 1)}(e^{2u} + e^{2v})^{\frac{n}{2}}$

Substituting $Z = \dfrac{1}{d + d' - 1}(e^{2u} + e^{2v})^{\frac{n}{2}}$

Or $\dfrac{\partial Z}{\partial u} + \dfrac{\partial Z}{\partial v} = Z + (e^{2u} + e^{2v})^{\frac{n}{2}}$

The subsidiary equations are $\dfrac{du}{1} = \dfrac{dv}{1} = \dfrac{dZ}{Z + (e^{2u} + e^{2v})^{\frac{n}{2}}}$

Two independent integrals of Equation are given by,

$u - v = \text{Constant} = a$ (say)

NOTES

And $$\frac{dZ}{dv} - Z = \left(e^{2u} + e^{2v}\right)^{\frac{n}{2}}$$

$$= e^{nv}(e^{2a}+1)^{\frac{n}{2}}$$

Since this equation is linear, therefore,

$$Ze^{-v} = \frac{e^{(n-1)v}}{(n-1)}\left(e^{2a}+1\right)^{\frac{n}{2}}$$

$$\therefore \quad Z = \frac{e^{nv}}{n-1}\left(e^{2a}+1\right)^{\frac{n}{2}}$$

$$= \frac{\left(e^{2u}+e^{2v}\right)^{\frac{n}{2}}}{(n-1)}$$

$$\therefore \quad \text{P.I.} = \frac{1}{d+d'}\left\{\frac{\left(e^{2u}+e^{2v}\right)^{\frac{n}{2}}}{n-1}\right\}$$

$$= \frac{1}{(n-1)}\int_{a=v-u}\left\{e^{2u}+e^{2a+2u}\right\}^{\frac{n}{2}} du$$

$$= \frac{1}{n-1}\left\{\int (e^{2a}+1)^{\frac{n}{2}}\int e^{nu}\,du\right\}_{a=v-u}$$

$$= \frac{1}{n(n-1)}\left\{e^{nu}\left(e^{2a}+1\right)^{\frac{n}{2}}\right\}_{a=v-u}$$

$$= \frac{1}{n(n-1)}\left(e^{2u}+e^{2v}\right)^{\frac{n}{2}}$$

$$= \frac{1}{n(n-1)}\left(x^2+y^2\right)^{\frac{n}{2}}$$

$$\therefore \quad z = \psi_1\left(\frac{x}{y}\right) + x\psi_2\left(\frac{x}{y}\right) + \frac{1}{n(n-1)}\left(x^2+y^2\right)^{\frac{n}{2}}.$$

Example 5.31: Solve $\left(x^2D^2 - 2xyDD' + y^2D'^2 - xD + 3yD'\right)z = \frac{8y}{x}$

Solution: Put $u = \log x$

$v = \log x$

NOTES

Our Equation reduces to,

$$\{d(d-1)-2dd'+d'(d'-1)-d+3d'\}z = 8e^{v-u}$$

Or $\{(d-d')^2-2(d-d')\}z = 8e^{v-u}$

Or $(d-d')(d-d'-2)z = 8e^{v-u}$

$$\text{C.F.} = \phi_1(u+v)+e^{2u}\phi_2(u+v)$$

$$= \psi_1(xy)+x^2\psi_2(xy)$$

$$\text{P.I.} = 8.\frac{1}{(d-d')(d-d'-2)}e^{v-u}$$

$$= e^{v-u}$$

$$= \frac{y}{x}$$

$\therefore$ $$z = \psi(xy)+x^2\psi_2(xy)+\frac{y}{x}.$$

Example 5.32: Solve $(x^2D^2+2xyDD'+y^2D'^2)z = x^m y^n$

Solution: Put $u = \log x$

$v = \log y$

The equation reduces to,

$$\{d(d-1)+2dd'+d'(d'-1)\}z = e^{mu+nv}$$

Or $\{(d+d')^2-(d+d')\}z = e^{mu+nv}$

Or $(d+d')(d+d'-1)z = e^{mu+nv}$

$$\text{C.F.} = \phi_1(u-v)+e^{u}\phi_2(u-v)$$

$$= \psi_1\left(\frac{x}{y}\right)+x\psi_2\left(\frac{x}{y}\right)$$

$$\text{P.I.} = \frac{1}{(d+d')(d+d'-1)}e^{mu+nv}$$

$$= \frac{1}{(m+n)(m+n-1)}e^{mu+nv}$$

NOTES

$$= \frac{1}{(m+n)(m+n-1)} x^m y^n$$

$$\therefore \qquad z = \psi_1\left(\frac{x}{y}\right) + x\psi_2\left(\frac{x}{y}\right) + \frac{1}{(m+n)(m+n-1)} x^m y^n .$$

Check Your Progress

6. What is the complementary function of the equation $\left(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right) z = 0$ if the roots are distinct?
7. Write the homogeneous linear equations with constant coefficients.
8. When is a non-homogeneous equation said to be reducible?
9. Which mathematical function is used to reduce partial differential equations to equations with constant coefficients?

5.5 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The partial differential equation $Pp + Qq = R$, where P, Q, R are functions of x, y, z is called Lagrange's linear differential equation.
2. The linear differential equation with constant coefficients are of the form,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

   Where $P_1, P_2, \ldots, P_n$ are constants and Q is a function of x.
3. When $m_1 = m_2$, then the complementary function will be,

   $y = (c_1 + c_2 x)\, e^{m1x}$ where c_1 and c_2 are arbitrary constants.
4. Any particular solution of $F(D)\, y = f(x)$ is known as its Particular Integral (P.I). The P.I. of $F(D)y = f(x)$ is symbolically written as,

$$\text{P.I.} = \frac{1}{F(D)} \{f(x)\} \text{ where } F(D) \text{ is the operator.}$$

5. The three types of equations are the elliptic type, the parabolic type and the hyperbolic type.
6. Let $m_1, m_2, ..., m_n$ be the roots of the equation then C.F. = $\phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \ldots\ldots\ldots\ldots\ldots + \phi_n(y + m_n x)$ where ϕ_i's are arbitrary functions.
7. Let $f(D, D')z = V(x, y)$

   Then if,

$$f(D, D') = A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D_2' + \cdots + A_n D'^n$$

   Where $A_1, A_2, \cdots, A_n$ are constants.

8. The equation $f(D, D')z = V(x, y)$ is said to be reducible if the symbolic function $f(D, D')$ can be resolved into factors each of which is of first degree in D and D'.
9. Logarithm function is used to reduce partial differential equations to equations with constant coefficients

5.6 SUMMARY

- The partial differential equation $Pp + Qq = R$, where P, Q, R are functions of x, y, z, is called Lagrange's linear differential equation.
- The general form of a linear differential equation of nth order is,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

- The solution $y = y_1(x), y = y_2(x), y = y_3(x), ..., y = y_n(x)$ are said to be linearly independent if the Wronskian of the functions is not zero
- The Complementary Function (C.F.) which is the complete primitive of the Reduced Equation (R.E.) and is of the form

  $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ containing n arbitrary constants.
- The Particular Integral (P.I.) which is a solution of $F(D)\, y = Q$ containing no arbitrary constant.
- If the two roots m_1 and m_2 of the auxiliary equation are equal and each equal to m, the corresponding part of the general solution will be $(c_1 + c_2 x)\, e^{mx}$ and if the three roots m_3, m_4, m_5 are equal to α the corresponding part of the solution is $(c_3 + c_4 x + c_5 x^2)\, e^{\alpha x}$ and others are distinct, the general solution will be,
  $y = (c_1 + c_2 x)\, e^{mx} + (c_3 + c_4 + c_5 x^2)\, e^{\alpha x} + c_6\, e^{m^6 x} + ... + c_n e^{mnx}$
- If a pair of imaginary roots $\alpha \pm i\beta$ occur twices, the corresponding part of the general solution will be,

  $e^{\alpha x}\, [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$
- The operator $\frac{1}{F(D)}$ is defined as that operator which, when operated on $f(x)$ gives a function $\phi(x)$, such that $F(D)\, \phi(x) = f(x)$

  the following linear partial differential equation of the second order in two independent variables,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

  Where A, B, C, D, E, F, and G are functions of x and y.
- Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  This is equation is of elliptic type.

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- One-dimensional heat flow equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

  This equation is of parabolic type.
- One-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

  This is a hyperbolic type.
- Equation is said to be reducible if the symbolic function $f(D, D')$ can be resolved into factors each of which is of first degree in *D* and *D*' and irreducible otherwise.

5.7 KEY TERMS

- **Partial differential equation:** Any equation which contains one or more partial derivatives is called a partial differential equation.
- **Reducible:** Equation is said to be reducible if the symbolic function f (D, D′) can be resolved into factors each of which is of first degree in D and D′ and irreducible otherwise.
- **Fundamental mode**: The first normal mode is referred as the fundamental mode.
- **Complementary function:** Consider the equation,

$$\left(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right) z = 0$$

  Let,

$$z = \phi(y + mx)$$

5.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define Lagrange's linear differential equation with suitable examples.
2. Define partial differential equations with suitable examples.
3. How will you identify the order of a partial differential equation?
4. How will you determine the degree of the partial differential equation?
5. Define Wronskian of functions.
6. Give the rules for finding the complementary function.
7. Explain the partial differential equation of the second order.

8. Give examples of parabolic, elliptic and hyperbolic type equations.
9. What is the difference between homogeneous and non homogeneous differential equations?
10. Explain the reducible non homogeneous equations.

Long-Answer Questions

1. Discuss the first order Lagrange's equations. Give appropriate examples.
2. Solve the equations:
   (i) $\left(D^2 + DD'^s - 1D'^3\right)z = 0$.
   (ii) $\left(D^3 + 3D^2D' - 4D'^3\right)z = 0$.
3. Solve the equations:
   (i) $\left(D^2 + 2DD' + D'^2\right)z = 12xy$.
   (ii) $\left(D^2 - 2DD' - 15D'^2\right)z = 12xy$.
   (iii) $\left(D^2 - 6DD' - 9D'^2\right)z = 12x^2 + 16xy$.
   (iv) $\left(D^3 - 7DD'^2 - 6D'^3\right)z = x^2 + xy^2 + y^3$.
   (v) $\left(D^2D' - 2DD'^2 + D'^3\right)z = \dfrac{1}{x^2}$.
4. Solve the equations:
   (i) $\left(D^2 - DD' - 2D'^2\right)z = x - y$.
   (ii) $\left(D^2 - 3DD' + 2D'^2\right)z = x + y$.
   (iii) $\left(4D^2 - 4DD' + D'^2\right)z = 16\log(x + 2y)$.
   (iv) $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x - y) + x^2 + xy^2 + y^3$.
   (v) $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \sin(x + 2y) + e^{3x+y}$.
   (vi) $\left(D^3 - 3DD'^2 + 2D'^3\right)z = \sqrt{x - 2y}$.
   (vii) $\left(D^3 - 4D^2D' + 5DD'^2 - 2D'^3\right)z = e^{y+2x} + \sqrt{y + x}$.
5. Solve the equations:
   (i) $\left(D^3 - 3DD'^2 - 2D'^3\right)z = \cos(x + 2y)$.
   (ii) $\left(D^2 + 5DD' + 5D'^2\right)z = x\sin(3x - 2y)$.
6. Solve the equations:
   (i) $\left(D^2 - Dd' - 2D'^2\right)z = (y - 1)e^x$.
   (ii) $\left(D^3 - 3DD'^2 - 2D'^3\right)z = \cos(x + 2y) - e^y(3 + 2x)$.

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7. Solve the equations:
   (i) $\left(DD' + D'^2 - 3D'\right)z = 0$.
   (ii) $(2D + D' - 1)^2 (D - 2D' + 2)^3 z = 0$.
8. Solve the equations:
   (i) $\left(2D^2 - D'^2 + D\right)z = 0$.
   (ii) $\left(D^2 + DD' + D + D' + 1\right)z = 0$.
9. Solve the equations:
   (i) $(D - D' - 1)(D + D' - 2)z = e^{2x-y}$.
   (ii) $\left(D^2 - D'\right)z = e^{x+y}$.
10. Solve the equations:
   (i) $\left(D^2 - DD' - 2D\right)z = \cos(3x + 4y)$.
   (ii) $\left(D^2 - D'\right)z = A\cos(lx + my)$, where *A*, *l*, *m* are constants.
11. Solve the equations:
   (i) $(D = D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$.
   (ii) $\left(D^3 - DD'^2 - D^2 + DD'\right)z = \dfrac{x+2}{x^3}$.
   (iii) $\left(D^2 - D'\right)y = 2y - x^2$.
12. Solve the equations:
   (i) $\left(D - D'^2\right)z = \cos(x - 3y)$.
   (ii) $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y}\sin(2x + y)$.
   (iii) $\left(D^2 + DD' + D' - 1\right)z = 4\sin\ h\ x$.
   (iv) $\left(D^2D' + D'^2 - 2\right)z =^{2y} \sin 3x - e^{\infty}\cos 2y$.
13. Solve the equations:
   (i) $\left(x^2D^3 - y^3D'^2\right)z = xy$.
   (ii) $\left(x^2D^2 + 2xyDD' + y^2D'^2\right)z = x^2y^2$.
   (iii) $\left(x^2D^2 - 2xyDD' - 3y^2D'^2 + xD - 3yD'\right)z = x^2y\cos\left(\log x^3\right)$.
14. Solve $\left(D^3 - 2D^2D' - DD'^2 + 2D'^3\right)z = e^{x+y}$.
15. Solve $\left(D^3 + D'^3 + D''^3 - 3DD'D''\right)u = x^3 = 3xyz$.

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16. Solve the following equations:
    (i) $r = x^2 e^y$.
    (ii) $xys = 1$.
17. Solve the following equations:
    (i) $t - xq = -\sin y - x\cos y$.
    (ii) $t - xq = x^2$.
    (iii) $yt - q = xy$.
18. Solve the following equations:
    (i) $xr + ys + p = 10xy^3$.
    (ii) $2yt - xs + 2q = 4yx^3$.
    (iii) $z + r = x\cos(x + y)$.
19. Solve the differential equation, $r - 2yp + y^2 z = (y - 2)e^{2x+3y}$.

5.9 FURTHER READING

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