M.Sc. Previous Year

Chemistry, MC-05 (A-1)

MATHEMATICS FOR CHEMISTS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

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Published by Registrar, MP Bhoj (open) University, Bhopal in 2020

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E-28, Sector-8, Noida - 201301 (UP)
Phone: 0120-4078900 • Fax: 0120-4078999
Regd. Office: A-27, 2nd Floor, Mohan Co-operative Industrial Estate, New Delhi 1100 44
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SYLLABI-BOOK MAPPING TABLE

Mathematics for Chemists

Syllabi	Mapping in Book
Unit I: Vectors and Matrix Algebra **A. Vectors:** Vectors, Dot, Cross and Triple Products, etc. The Gradient, Divergence and Curl. Vector Calculus, Gauss' Theorem, Divergence Theorem, etc. **B. Matrix Algebra:** Addition and Multiplication; Inverse, Adjoint and Transpose of Matrices, Special Matrices (Symmetric, Skew-Symmetric, Hermitian, Skew-Hermitian, Unit, Diagonal, Unitary, etc.) and Their Properties. Matrix Equations: Homogeneous, Non-Homogeneous Linear Equations and Conditions for the Solution, Linear Dependence and Independence. Introduction to Vector Spaces, Matrix Eigenvalues and Eigenvectors, Diagonalization, Determinants (examples from Hückel Theory). Introduction to Tensors; Polarizability and Magnetic Susceptibility as Examples.	**Unit-1:** Vector and Matrix Algebra **(Pages 3-114)**
Unit II: Differential Calculus Functions, Continuity and Differentiability, Rules for Differentiation, Applications of Differential Calculus including Maxima and Minima (examples related to Maximally Populated Rotational Energy Levels, Bohr's Radius and most Probable Velocity from Maxwells' Distribution, etc.), Exact and Inexact Differentials with Their Applications to Thermodynamic Properties. Integral Calculus, Basic Rules for Integration, Integration by Parts, Partial Fraction and Substitution. Reduction Formulae, Applications of Integral Calculus. Functions of Several Variables, Partial Differentiation, Co-Ordinate Transformation (e.g., Cartesian to Spherical Polar), Curve Sketching.	**Unit-2:** Differential Calculus **(Pages 115-190)**
Unit III: Elementary Differential Equations Variables-Separable and Exact First-Order Differential Equations, Homogeneous, Exact and Linear Equations. Applications to Chemical Kinetics, Secular Equilibria, Quantum Chemistry, etc., Solutions to Differential Equations by the Power Series Method, Fourier Series, Solutions of Harmonic Oscillator and Legendre Equation, etc., Spherical Harmonics, Second Order Differential Equations and Their Solutions.	**Unit-3:** Elementary Differential Equations **(Pages 191-263)**
Unit IV: Permutation and Probability Permutations and Combinations, Probability and Probability Theorems, Probability Curves, Average, Root Mean Square and Most Probable Errors, Examples from the Kinetic Theory of Gases, etc., Curve Fitting (including Least Squares Fit, etc.) with a General Polynomial Fit.	**Unit-4:** Permutation and Probability **(Pages 265-352)**

CONTENTS

INTRODUCTION

NOTES

Mathematics includes the study of topics, such as numbers (arithmetic and number theory), formulas and related structures (algebra), shapes and spaces in which they are contained (geometry), and quantities and their changes (calculus and analysis). Most of mathematical activity consists of discovering and proving (by pure reasoning) properties of abstract objects. Mathematics is essential in various fields, including natural sciences, engineering, physics, chemistry, medicine, finance, computer science and social sciences. Mathematics is, therefore, the most important subject for achieving excellence in any field of Science and Engineering.

Mathematical chemistry studies the applications of mathematics to chemistry, it is principally concerned with the mathematical modeling of chemical phenomena. Major areas of research in mathematical chemistry include chemical graph theory, which deals with topology, such as the mathematical study of isomerism and the development of topological descriptors or indices which find application in quantitative structure-property relationships; and chemical aspects of group theory, which finds applications in stereochemistry and quantum chemistry.

This book, *Mathematics For Chemists*, is divided into four units, dealing in introduction vectors and matrix algebra, differential calculus, elementary differential equations, and permutation and probability. The book follows the self-instruction mode or the SIM format wherein each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The content is presented in a simple and structured form interspersed with Answers to 'Check Your Progress' for better understanding. A list of 'Summary' along with a 'Key Terms' and a set of 'Self-Assessment Questions and Exercises' is provided at the end of each unit for effective recapitulation.

UNIT 1 VECTOR AND MATRIX ALGEBRA

NOTES

Structure

1.0 INTRODUCTION

Vector, in mathematics, a quantity that has both magnitude and direction but not position. Algebraically, the dot product is the sum of the products of the corresponding entries of the two sequences of numbers. The cross product a × b is defined as a vector c, i.e., Perpendicular (orthogonal) to both a and b, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span. In geometry and algebra, the triple product is a product of three-dimensional vectors, usually Euclidean vectors.

Vector calculus, or vector analysis, is concerned with differentiation and integration of vector fields, primarily in 3-dimensional Euclidean space $\mathbb{R}^3$. The Gauss divergence theorem states that the vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface. In mathematics, physics, and engineering, a vector space is a set of objects called vectors, which may be added together and multiplied by numbers called scalars.

NOTES

In mathematics, a matrix is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to represent a mathematical object or a property of such an object. In mathematics, particularly in linear algebra, a skew-symmetric (or antisymmetric) matrix is a square matrix whose transpose equals its negative. On the main diagonal of a skew-Hermitian matrix have to be pure imaginary, i.e., on the imaginary axis (the number zero is also considered purely imaginary).

Diagonalizable matrices hold only over an algebraically closed field (such as, the complex numbers). In this case, diagonalizable matrices are dense in the space of all matrices, which means any defective matrix can be deformed into a diagonalizable matrix by a small perturbation. Whereas eigenvalues are the special set of scalar values that is associated with the set of linear equations most probably in the matrix equations. The eigenvectors are also termed as characteristic roots. It is a non-zero vector that can be changed at most by its scalar factor after the application of linear transformations.

In mathematics, the determinant is a scalar value that is a function of the entries of a square matrix. It allows characterizing some properties of the matrix and the linear map represented by the matrix.

In mathematics, a tensor is an algebraic object that describes a multilinear relationship between sets of algebraic objects related to a vector space.

In this unit, you will study about the dot, cross and triple product of vector, gradient, divergence and curl gradient, vector calculus, Gauss or divergence theorem, vector space, matrix algebra, eigenvalues, eigenvectors and diagonalizable, determinant, tensor.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the dot, cross and triple product of vector
- Analyse the gradient, divergence and curl
- Discuss about the vector calculus
- State the Gauss or divergence theorem
- Elaborate on the vector space
- Know about the matrix algebra
- Comprehend the special matrix
- Calculate the homogeneous and non-homogeneous linear equations
- Define the eigenvalues, eigenvectors and diagonalizable
- Explain about the determinant
- Analyse the tensor

1.2 INTRODUCTION TO VECTORS

NOTES

Since a vector is the result of abstraction, its magnitude and direction may be represented by a line *OP* directed from the *initial point O* to the *terminal point P* and denoted by $\overrightarrow{OP}$. Here the length of vector $\overrightarrow{OP}$ denoted by $|\overrightarrow{OP}| = OP$ is called *magnitude* or *module* or *modulus* of the vector and the direction in space is indicated by an arrow head on the line.

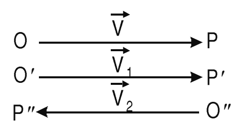

Fig. 1.1 *Vector* $\overrightarrow{OP}$ *has been Shown by* $\vec{V}$

In Figure 1.1, the vector $\overrightarrow{OP}$ has been shown by $\vec{V}$ (or in Clarendon print by **V**) while its scalar magnitude is stated by *V*. Thus *OP* is the length of the vector **V**, while the line of indefinite length of which the directed line segment $\overrightarrow{OP}$ is only a part is the support of **V** and the sense is from *O* to *P*.

It should be noted that formulation of a law of physics in terms of vectors is however independent of the choice of axes of reference, *i.e.*, the vector representation has a physical content without reference to any coordinate system.

Kinds of Vectors

Equal vectors: Two given vectors may be equal only when they have the same magnitude and the same direction, *i.e.*, the given two vectors are equal if and only if they have the same or parallel support with equal length and the same sense. For example in Fig. 1.1, we have

$$\mathbf{V}\ (=\overrightarrow{OP}) = \mathbf{V}_1\ (=\overrightarrow{O'P'}) = -\mathbf{V}_2\ (=\overrightarrow{O''P''})$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ have the same scalar magnitude as **V** but $\mathbf{V}_1$ has the same and $\mathbf{V}_2$ the opposite sense to that of **V**.

Null vector: A vector having the initial and the terminal points coincident is termed as a *zero vector* or a *null vector*. Thus a null vector has its module zero.

Unit vector: A vector having its modulus as unity is called a *unit vector.*

If **a** is a vector and '*a*' its modulus, then unit vector **a** is denoted by $\hat{\mathbf{a}}$ (read as 'a hat' or 'a caret') defined as

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{a}$$

Polar vectors: The line vectors representing the quantities like force, velocity, etc., in which merely a linear action in a particular direction is involved, are termed as *polar* vectors.

Axial vectors: The line vectors representing the quantities like angular velocity, angular acceleration, etc., in which some rotational action is involved about an axis and which are drawn parallel to the axis of rotation in order that the magnitude of the quantity is determined by the length of the vector and the direction by the rule of right handed screw (*i.e.*, rotation being considered in clockwise direction), are termed as *axial* vectors.

NOTES

Free vector: Evidently a vector can be represented by an infinite number of equal vectors by drawing parallel supports. Such a vector which can be transported from place to place such that it remains of the same magnitude and keep up the same direction is termed as a *free* vector. In fact a free vector is assumed to remain the same through transportation, irrespective of its position in space.

Localised or Line vector: We have defined that the value of a free vector depends only on its length and direction, but if it depends also on its position in space, *i.e.*, if a vector is restricted to pass through a given origin, then it is termed as a *localised* vector.

Collinear vectors: The vectors parallel to the same line, regardless of their magnitudes and sense of directions are termed as *collinear* vectors. In other words the vectors having the same or parallel supports are known as collinear vectors. Such vectors are parallel to each other and they may coincide in a special case. As such there exists a scalar ratio say λ between any two collinear vectors **a** and **b** of the form

$$\mathbf{b} = \lambda\, \mathbf{a}$$

which follows that one of the two collinear vectors can be expressed as the scalar multiple of the other.

Non-collinear vectors: The vectors whose directions are neither parallel nor coincident are said to be *non-collinear.*

Like vectors or co-directional vectors. The vectors which are collinear and have the same sense of directions, *i.e.*, the vectors directed in the same sense irrespective of their magnitudes are termed as *like* vectors.

Unlike vectors. The vectors which are collinear but have opposite sense of directions from each other are termed as *unlike* vectors.

Coplanar vectors. A system of vectors lying in the parallel planes or which can be made to lie in the same plane are said to be *coplanar* vectors. Evidently any two vectors are always coplanar.

Non-coplanar vectors. A system of vectors consisting of three or more vectors which cannot be made to lie in the same plane are called *non-coplanar* vectors.

Reciprocal vector. Any vector having its direction the same as that of a given vector **a**, but its magnitude as the reciprocal of the magnitude of **a** is termed as the *reciprocal* vector of **a** and written as $\mathbf{a}^{-1}$ or $\frac{1}{\mathbf{a}}$. As such

$$\mathbf{a}^{-1} = \frac{1}{a}\,\hat{\mathbf{a}} = \frac{\mathbf{a}}{a^2}\,\hat{\mathbf{a}} = \frac{\mathbf{a}}{a^2} \text{ (by definition of a unit vector).}$$

In this connection it is notable that the magnitude and so the reciprocal of the magnitude of a unit vector being unity, the unit vector is reciprocal to itself and it is said to be *self-reciprocal.*

Negative vector. The vector having the same magnitude as the vector **a** but opposite direction, is known as the *negative* of **a** and written as –**a**.

Position vector. If a vector $\vec{OP}$ specifies the position of a point relative to an arbitrarily chosen point O, then $\vec{OP}$ is called the *Position* vector of P with respect to O, the origin of vectors.

NOTES

Example 1.1: *If* {**a**, **b**, **c**} *is a right handed set, which of the following sets are right handed?*

(*i*) **a**, **c**, **b**; (*ii*) **b**, **c**, **a**; (*iii*) **b**, **a**, **c**; (*iv*) **c**, **a**, **b**; (*v*) **c**, **b**, **a.**

It is clear that the sets (*ii*) and (*iv*), *i.e.*, **b**, **c**, **a**; and **c**, **a**, **b** are right handed.

Example 1.2: *Discuss the geometrical significance of* $a\mathbf{A} + b\mathbf{B} = 0$.

We have $a\mathbf{A} + b\mathbf{B} = 0$, a, b being scalars.

This can be written as $\mathbf{A} = -\frac{b}{a}\mathbf{B}$

$$= \lambda\,\mathbf{B} \quad \text{if} \quad \lambda = -\frac{b}{a}$$

i.e., **A** is expressible as a scalar multiple of **B** so that the vectors **A** and **B** are parallel or collinear.

Addition of Vectors

The charactertisation of process of summation is inherited in the composition of two or more displacements of a point. Suppose that we have two vectors **a** and **b** acting at a point O as shown in Figure. 1.2. Let $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.

Clearly the resultant effect of the vectors **a** and **b** is the same as that of their vector sum **v** obtained by setting off the vector **b** at the end of **a** and then joining the beginning of **a** to the end of **b**. This geometrical construction utilised to find the vector sum of two vectors **a** and **b** is known as *the parallelogram law of addition of vectors* (Refer Figure 1.2).

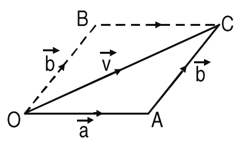

Fig. 1.2 *Representation Addition of a Vector by Parallelogram Law*

Thus $$\mathbf{v} = \vec{OC} = \vec{OA} + \vec{AC} = \mathbf{a} + \mathbf{b} \qquad \text{... (1.1)}$$

A similar result follows by starting with **b** and setting off the vectors **a** on **b**, *i.e.*,

$$\mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC} = \mathbf{b} + \mathbf{a} \qquad \text{... (1.2)}$$

Conclusively the result of adding two co-initial vectors is the vector represented by the diagonal of the parallelogram having the two given vectors as its adjacent sides.

From Equations (1.1) and (1.2), it follows that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

i.e., the two vectors obey the **commutative law of addition**, according to which the vector sum of two vectors is independent of their order.

We now propose to find the sum of three vectorssay **a**, **b**, **c**. Let $\vec{OA}$ = **a**, $\vec{AB}$ = **b**, $\vec{BC}$ = **c** as shown in Figure 1.3. Then

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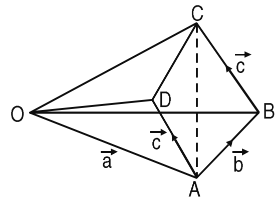


Fig. 1.3 *Representation Sum of Vector*

$$\mathbf{v} = \vec{OC} = \vec{OA} + \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b} + \mathbf{c} \qquad \text{.... (1.3)}$$

Also $\mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC}$

$= (\vec{OA} + \vec{AB}) + (\vec{BC})$

$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$. ... (1.4)

Similarly $\mathbf{v} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ ... (1.5)

and $\mathbf{v} = (\mathbf{a} + \mathbf{c}) + \mathbf{b}$ ... (1.6)

It follows from Equations (1.3), (1.4), (1.5) and (1.6) that

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{c}) + \mathbf{b}.$$

i.e., the three vectors obey the **associative law of addition**, *according to which the vector sum of three vectors is independent of the mode in which component vectors are associated in different groups.*

In general, if there are *n* vectors **a**, **b**, **c**...**n**, then their vector sum **v** is given by

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} + ... + \mathbf{n}$$

Subtraction of Vectors

If there are two vectors **a** and **b**, then

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}),$$

i.e., the subtraction of **b** from **a** may be regarded as the addition of **–b** to **a**. Thus, to substract **b** from **a**, reverse the direction of **b** and add to **a** (Refer Figure 1.4).

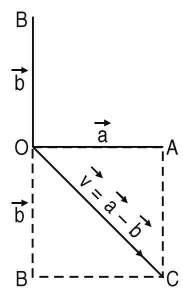


Fig. 1.4 *Subtraction of a Vector*

NOTES

Multiplication of a Vector By a Scalar

If **a** be any given vector and s a given scalar, then $s\mathbf{a}$ or $\mathbf{a}s$ is defined as a vector whose magnitude is $|s|$ times the magnitude of the vector **a**, *i.e.*, $|s|$ times the length of **a**, the support being the same or parallel to that of **a** and direction being the same or opposite to that of **a**, according as s is positive or negative. We thus have

(*i*) $s(-\mathbf{a}) = (-s)\mathbf{a} = -s\mathbf{a}$.

(*ii*) $(-s)(-\mathbf{a}) = s\mathbf{a}$.

(*iii*) $(s+t)\,\mathbf{a} = s\mathbf{a} + t\mathbf{a}$, t being another scalar.

(*iv*) $(st)\,\mathbf{a} = s\,(t\mathbf{a}) = t\,(s\mathbf{a})$

(*v*) $0\mathbf{a} = \mathbf{0}$, **0** being the null vector.

(*vi*) If two non-zero vectors **a** and **b** are collinear, then there exists a non-zero scalar m, such that

$$\mathbf{a} = m\mathbf{b}.$$

Conversely the relation of this type implies that **b** is parallel or collinear to **a**.

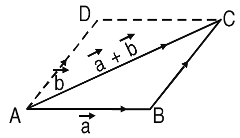
(*vii*) If $\hat{\mathbf{a}}$ is the unit vector co-directional with **a** while $a = |\mathbf{a}|$, then

$$\mathbf{a} = a\hat{\mathbf{a}} \quad \text{or} \quad s\mathbf{a} = s(a\hat{\mathbf{a}}) = sa\hat{\mathbf{a}}.$$

Also as $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}}{a}$ and if **b** is parallel to **a**, then $\mathbf{b} = \pm b\dfrac{\mathbf{a}}{a}$ according as **b** and **a** have the same or opposite directions.

Note. Division of a vector **a** by a non-zero scalar s is regarded as the multiplication of the vector **a** by a scalar $1/s$.

Example 1.3: If there are two vectors **a** and **b** represented by the sides AB and BC of a triangle, then show that their resultant is represented by the third side AC. Why is this method equivalent to the parallelogram law of addition?



Solution: As shown in Figure, the vectors **a** and **b** are represented by the sides AB and BC of the triangle. Here $\vec{AC}$ is a vector drawn between the initial point of **a** and terminal point of **b** and thus may be obtained by parallelogram law of addition, for if we complete the parallelogram $ABCD$, then $\vec{AC}$ represents a vector along the diagonal of the parallelogram and passing through the common point of the adjacent sides AB and AD representing the vectors **a** and **b**. As such the vector addition obeys the parallelogram law of forces.

Example 1.4: What vector must be added to the two vectors $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$, so that the resultant may be a unit vector along the x-axis?

Solution: Suppose that

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Then $\quad \mathbf{a} + \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}.$

Hence, in order that the resultant of **a** and **b**, *i.e.*, **a + b** be **i**, we have to add a vector.

$$\mathbf{i} - (3\mathbf{i} - \mathbf{j} + \mathbf{k}), \text{ i.e., } -2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

NOTES

Vector Space or Linear Space

A vector space (or linear space) over a field F is a set V of elements called vectors which may be combined by two primary operations—addition and scalar multiplication; such that

(*A*) (*i*) If the vectors **a** and **b** belong to V, then $\mathbf{a} + \mathbf{b}$ also belongs to V. (This is known as **closure property**).

(*ii*) The vector sum of two vectors **a** and **b** belonging to V, is **commutative**, *i.e.*,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

(*iii*) The vector sum of three vectors **a**, **b**, **c** belonging to V, is **associative**, *i.e.*,

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = (\mathbf{a} + \mathbf{c}) + \mathbf{b} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

(*iv*) In vector addition there exists an **additive identity vector** known as null vector, such that

$$\mathbf{a} + 0 = \mathbf{a}.$$

(*v*) To every vector **a** in V, there corresponds a vector $-\mathbf{a}$ known as **additive inverse vector**, such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

(*B*) (*i*) If m, n are any two scalars and **a** is a vector in V, then **distributive law** holds, *i.e.*,

$$(m + n)\,\mathbf{a} = m\mathbf{a} + n\mathbf{a}.$$

(*ii*) If m is any scalar and **a**, **b** are two vectors belonging to V, then **distributive law of scalar multiplication** holds, *i.e.*,

$$(\mathbf{a} + \mathbf{b})\,m = \mathbf{a}m + \mathbf{b}m.$$

(*iii*) If m, n are any two scalars and **a**, is a vector belonging to V, then **associative law** holds, *i.e.*,

$$m\,(n\mathbf{a}) = (mn)\,\mathbf{a} = n\,(m\mathbf{a}).$$

(*iv*) To every vector **a** in V, there corresponds a **multiplicative identity scalar**, such that

$$1\mathbf{a} = \mathbf{a}.$$

Note. In case of *scalar quantities* m, n, p, we have the following laws of combination:

(*i*) The *addition is commutative*, *i.e.*,

$$m + n = n + m.$$

(*ii*) The *addition is associative*, *i.e.*,

$$m + n + p = (m + n) + p = (m + p) + n = m + (n + p).$$

(*iii*) There exists *an additive identity scalar* 0, which when added to another scalar, leaves it unchanged, such as

$$m + 0 = m.$$

(*iv*) To every scalar m, there corresponds an *additive inverse scalar* $-m$, such that

$$m + (-m) = 0.$$

In fact m and $-m$ are inverse of each other as their sum is zero (identity scalar).

(*v*) The *multiplication is distributive*, *i.e.*,

$$m \cdot (n + p) = m \cdot n + m \cdot p.$$

(*vi*) The *multiplication is commutative*, *i.e.*,

$$m \cdot n = n \cdot m.$$

(*vii*) The *multiplication is associative*, *i.e.*,

$$m \cdot (n \cdot p) = (m \cdot n) \cdot p = n \cdot (m \cdot p).$$

(*viii*) There exists a *multiplicative identity scalar* 1, such that

$$m.1 = m.$$

(*ix*) To every non-zero scalar m, there corresponds a *multiplicative inverse scalar* $\frac{1}{m}$, such that

$$m \cdot \frac{1}{m} = 1 \text{ (the identity scalar).}$$

NOTES

Interpretation: Due to directional properties of vectors these laws cannot be applied to vectors and laws for vectors are consistent with the physical problems in which vector quantities occur.

1.2.1 Dot, Cross and Triple Product of Vectors

A careful observation of the ways in which two vector quantities enter into combinations in various branches of mathematics and mechanics leads us to define two well marked and distinct kinds of products, one being called **scalar** or **dot product** and other being called **vector** or **cross product**. The former yields a number (scalar) while the latter, a vector quantity. In either case the product is jointly proportional to the Modules (moduli) of the two vectors.

Conventionally, the scalar or dot product of two vectors **a** and **b** is denoted by **a** · **b** or (**a, b**) and their vector or cross product by **a** × **b** or [**ab**].

(1) The Scalar or Dot Product of Two Vectors

Definition: The scalar or dot product of two vectors **a** and **b**, with modules a and b respectively and their directions being inclined at an angle θ, is defined to be the real number $ab \cos \theta$, *i.e.*,

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta.$$

Characteristics of dot product: (*i*)The dot product of two vectors **a** and **b** is independent of their order

i.e., $\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = \mathbf{b} \cdot \mathbf{a}$

NOTES

(*ii*) The dot product of two vectors **a** and **b** may be expressed as the product of two numbers, one being the length of one vector and the other resolute of the second in the direction of the first, *i.e.*,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \text{(length of } \mathbf{a}\text{) times (scalar projection of } \mathbf{b} \text{ onto } \mathbf{a}\text{)} \\ &= \text{(length of } \mathbf{b}\text{) times (scalar projection of } \mathbf{a} \text{ onto } \mathbf{b}\text{).} \end{aligned}$$

(*iii*) If $\mathbf{a} \cdot \mathbf{b} = 0$, then either of the two vectors is a null vector or the vectors **a** and **b** are mutually perpendicular, *i.e.*

$$\mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0} \text{ or } \theta = \tfrac{1}{2}\pi.$$

In particular, $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$; **i**, **j**, **k** being mutually perpendicular unit vectors.

(*iv*) The vectors **a** and **b** are parallel if $\theta = 0$ or π, *i.e.*, if $\mathbf{a} \cdot \mathbf{b} = \pm ab$, where a, b are modules of **a** and **b** respectively.

(*v*) The scalar product of two equal vectors **a**, **a** is given by

$$\mathbf{a} \cdot \mathbf{a} = a \cdot a \cos 0° = a^2, \text{ since then } \theta = 0$$

In case **a** is a unit vector *i.e.*, $\mathbf{a} = \hat{\mathbf{a}}$ then $|\hat{\mathbf{a}}| = 1$ so that $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1^2 = 1$

In particular $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

(*vi*) The scalar product of two unit vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ is given by

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta \text{ since then } |\hat{\mathbf{a}}| = 1 = |\hat{\mathbf{b}}|$$

(*vii*) *The scalar product is associative i.e.* if **a**, **b** be any two vectors and m, n be any two scalars, then

$$(m\mathbf{a}) \cdot (n\mathbf{b}) = mn\,(\mathbf{a} \cdot \mathbf{b}) = mn\,\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot mn\mathbf{b} = n\mathbf{a} \cdot m\mathbf{b}$$

(*viii*) The scalar product being a number, can occur as the numerical coefficient of any vector, *e.g.*, $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ represents a vector parallel to **c** and whose module is $(\mathbf{a} \cdot \mathbf{b})$ times that of **c**.

(*ix*) In the case of scalar product, the distributive law of multiplication holds *i.e.* if **a**, **b**, **c** be three vectors, then

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Referred to Figure 1.5, let $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$, $\vec{OC} = \mathbf{c}$ and projections of OB and BC on OA be respectively OM and MN, so that

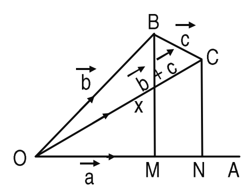


Fig. 1.5 *Projection of OB and BC on OA*

$$ON = OM + MN.$$

It is also clear that

$$\vec{OC} = \vec{OB} + \vec{BC} = \mathbf{b} + \mathbf{c}$$

NOTES

Now, $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \overrightarrow{OC}$ = (length of **a**) times (scalar projection of $\overrightarrow{OC}$ onto **a**)

$$\begin{aligned} &= a\,(ON),\ a \text{ being module of } \mathbf{a} \\ &= a\,(OM + MN) \\ &= a\,(OM) + a\,(MN) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \text{ by } (ii) \end{aligned}$$

In general, we have

$(\mathbf{a} + \mathbf{b} + \mathbf{c}...) \cdot (\mathbf{l} + \mathbf{m} + \mathbf{n}\ ...) = \mathbf{a} \cdot \mathbf{l} + \mathbf{a} \cdot \mathbf{m} + ... + \mathbf{b} \cdot \mathbf{l} + \mathbf{b} \cdot \mathbf{m} + ...$ and *in particular*,

$$\begin{aligned} (\mathbf{a} \pm \mathbf{b}) \cdot (\mathbf{a} \pm \mathbf{b}) &= (\mathbf{a} \pm \mathbf{b})^2 \\ &= \mathbf{a}^2 \pm 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b}^2 \\ \text{and} \quad (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a}^2 - \mathbf{b}^2. \end{aligned}$$

(x) If θ be the angle between two vectors **a** and **b** whose orthogonal projections (components) in the directions of axes of x, y, z are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively and if **i**, **j**, **k** be the unit vectors along the axes, then

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

so that $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{ab} = \dfrac{(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})}{|(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}| \cdot |(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{j})|}$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}.$$

(2) The Vector or Cross Product of Two Vectors

Definition: Given two vectors **a** and **b** whose directions are inclined at an angle θ, their vector product is defined to be the vector **r**, whose module is *ab* in sin θ and whose direction is perpendicular to both **a** and **b**, being positive relative to a rotation from **a** to **b**, *i.e.*,

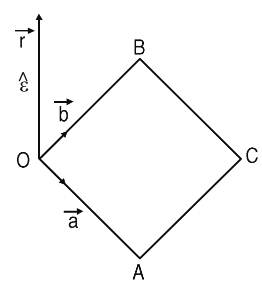


Fig. 1.6 *Representation of the Rotation from a Tab*

$$\mathbf{r} = \mathbf{a} \times \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\sin\theta\,\hat{\varepsilon}$$
$$= ab\sin\theta\,\hat{\varepsilon}$$

where $\hat{\varepsilon}$ is a unit vector perpendicular to the plane of **a** and **b**, and has the same direction as is obtained by the motion of a right handed screw due to rotation from **a** to **b**, and a, b are the modules of **a** and **b** respectively.

NOTES

Characteristics of vector product. (*i*) *The vector product is not commutative*, *i.e.*, by reversing the order of the factors, the sign of the product is reversed, *e.g.*,

$$\mathbf{b} \times \mathbf{a} = ba \sin(-\theta)\ \hat{\varepsilon} = -ab \sin\theta\ \hat{\varepsilon} = -\mathbf{a} \times \mathbf{b}$$

(*ii*) The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram of which $\mathbf{a}$ and $\mathbf{b}$ are adjacent sides, *i.e.*,

$$|\mathbf{a} \times \mathbf{b}| = |ab \sin\theta\ \hat{\varepsilon}| = ab \sin\theta, \text{ as } |\hat{\varepsilon}| = 1$$

$= OA$ multiplied by the perpendicular distance of OA from B.

$=$ Area of the parallelogram $OACB$.

(*iii*) *The vector product is associative*, *i.e.*, if m be a scalar $\mathbf{a}$, $\mathbf{b}$ be two vectors, then

$$(m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) = m(\mathbf{a} \times \mathbf{b}) = m(ab \sin\theta\ \hat{\varepsilon})$$

(*iv*) *The vector* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel,* if the angle θ included between their directions is 0 or π *i.e.*, if $\theta = 0$ or π so that

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \text{ as } \sin\theta = 0 \text{ for } \theta = 0 \text{ or } \pi$$

which follows that the vector product of two parallel vectors is a null vector.

(*v*) *The vector product of two equal vectors* $\mathbf{a}$, $\mathbf{a}$ *is given by*

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}.$$

Since the two vectors are equal if they are either collinear or parallel. So that the angle θ between them being 0 or π, $\sin\theta = 0$ and hence the result follows.

In particular, if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the principal axes, then

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

(*vi*) The two vectors $\mathbf{a}$ and $\mathbf{b}$ with modules a and b respectively, will be mutually perpendicular if the angle θ between their directions is 90°, so that $\sin\theta = 1$.

As such if $\mathbf{a}$, $\mathbf{b}$ are at right angles, then $\mathbf{a} \times \mathbf{b} = ab\ \hat{\varepsilon}$, $\hat{\varepsilon}$ being a unit vector normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

In case $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then $|\hat{\mathbf{a}}| = 1$ and $|\hat{\mathbf{b}}| = 1$, therefore $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\varepsilon}$, which shows that the cross product of two mutually perpendicular unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is a unit vector $\hat{\varepsilon}$ normal to the plane of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

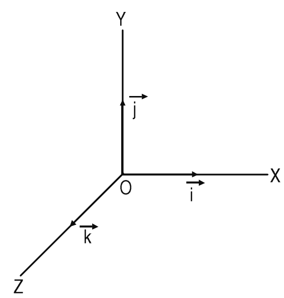


Fig. 1.7

Hence, in particular if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the unit vectors along the principal axes, then

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\,\mathbf{i} \times \mathbf{k}$$

(*vii*) The vector product of two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is given by

$$\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \sin\theta\,\hat{\varepsilon} \text{ as } |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1,$$

NOTES

where θ is the angle between their directions and $\hat{\varepsilon}$ is the unit vector normal to the plane of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

(*viii*) *The distributive law holds, i.e.* in case of vector product if **a**, **b**, **c** are three vectors, then

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Let the components of **a**, **b**, **c** along the principal axes be (a_1, a_2, a_3); (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively. Then if **i**, **j**, **k** be the unit vectors along the axes, we have

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\ \mathbf{c} &= c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \end{aligned}$$

So that, $\mathbf{b} + \mathbf{c} = (b_1 + c_1)\,\mathbf{i} + (b_2 + c_2)\,\mathbf{j} + (b_3 + c_3)\,\mathbf{k}$

$\therefore$ $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times \{(b_1 + c_1)\,\mathbf{i} + (b_2 + c_2)\,\mathbf{j} + (b_3 + c_3)\,BF\}$

$$\begin{aligned} &= \{a_2(b_3 + c_3) - a_3(b_2 + c_2)\}\,\mathbf{i} + \{a_3(b_1 + c_1) - a_1(b_3 + c_3)\}\,\mathbf{j} \\ &\qquad + \{a_1(b_2 + c_2) - a_2(b_1 + c_1)\}\,\mathbf{k} \\ &\qquad\qquad \text{as } \mathbf{i} \times \mathbf{j} = \mathbf{k} \text{ etc.} \\ &= \{(a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}\} \\ &\qquad + \{(a_2c_3 - a_3c_2)\,\mathbf{i} + (a_3c_1 - a_1c_3)\,\mathbf{j} + (a_1c_2 - a_2c_1)\,\mathbf{k}\} \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}. \end{aligned}$$

[Since $\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

Similarly $\mathbf{a} \times \mathbf{c} = (a_2c_3 - a_3c_2)\,\mathbf{i} + (a_3c_1 - a_1c_3)\,\mathbf{j} + (a_1c_2 - a_2c_1)\,\mathbf{k}$]

In general, $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \ldots) \times (\mathbf{l} + \mathbf{m} + \mathbf{n} + \ldots)$

$$= \mathbf{a} \times \mathbf{l} + \mathbf{a} \times \mathbf{m} + \ldots + \mathbf{b} \times \mathbf{l} + \mathbf{b} \times \mathbf{m} + \ldots + \ldots$$

(*ix*) *Vector product in terms of components.* Consider two vectors **a** and **b** whose components are (a_1, a_2, a_3), (b_1, b_2, b_3) along the principal axes. Then if **i**, **j**, **k** are the unit vectors along these axes, we have

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \qquad \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k} \\ &\qquad \text{as } \mathbf{i} \times \mathbf{j} = \mathbf{k} = -\,\mathbf{j} \times \mathbf{i} \text{ etc.} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

Now if θ be the angle between the directions of **a** and **b** and $\hat{\varepsilon}$, a unit vector normal to the plane of **a** and **b**, then

$(\mathbf{a} \times \mathbf{b})^2 = (ab \sin\theta\ \hat{\varepsilon})^2 = \{(a_2b_3 - a_3b_2)\ \mathbf{i} + (a_3b_1 - a_1b_3)\ \mathbf{j} + (a_1b_2 - a_2b_1)\ \mathbf{k}\}^2$

i.e., $a^2 b^2 \sin^2\theta = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$ as $\hat{\varepsilon}^2 = 1$

NOTES

or $\sin^2\theta = \dfrac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_3^2 + b_3^2)}$

as $\mathbf{a}^2 = a^2 = a_1^{\ 2} + a_2^{\ 3} + a_3^{\ 2}$ etc.

(*x*) *Cross product in terms of dot product.* By definition,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b})^2 &= (ab \sin\theta\ \hat{\varepsilon})^2\\ &= a^2b^2 \sin^2\theta,\\ &= a^2b^2(1 - \cos^2\theta)\\ &= a^2b^2 - a^2b^2\cos^2\theta\\ &= \mathbf{a^2b^2} - (\mathbf{a}\cdot\mathbf{b})^2,\ \because\ \mathbf{a}^2 = a^2, \mathbf{b}^2 = b^2, \mathbf{a}\cdot\mathbf{b} = ab\cos\theta\end{aligned}$$

Example 1.5: If **a** and **b** are unit vectors and θ is the angle between them, show that

$$\sin\frac{\theta}{2} = \frac{1}{2}|\mathbf{a} - \mathbf{b}|.$$

Solution: We have $|\ \mathbf{a} - \mathbf{b}\ |^2 = (\mathbf{a} - \mathbf{b})^2$

$$\begin{aligned} &= a^2 + b^2 - 2\mathbf{a}\cdot\mathbf{b},\\ \because\quad \mathbf{a}^2 = a^2 = 1 \text{ and } \mathbf{b}^2 = b^2 = 1\\ &= 2 - 2\cos\theta\\ &= 4\sin^2\theta/2,\end{aligned}$$

i.e. $|\ \mathbf{a} - \mathbf{b}\ | = 2\sin\theta/2,$

so that $\sin\theta/2 = \frac{1}{2}|\ \mathbf{a} - \mathbf{b}\ |.$

Example 1.6: From the relations (Lorentz transformation equations in theory of relativity).

$$\begin{cases}\mathbf{r}' = \mathbf{r} + \left[\dfrac{\vec{\gamma} - 1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r} - \vec{\gamma}\,t\right]\mathbf{v},\\[2ex] t' = \vec{\gamma}\left[t - \dfrac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right],\end{cases}$$

where $\vec{\gamma} = \dfrac{c}{\sqrt{(c^2 - \mathbf{v}^2)}}$, prove the reciprocal relations

$$\begin{cases}\mathbf{r} = \mathbf{r}' + \left[\dfrac{\vec{\gamma} - 1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r}' + \vec{\gamma}t'\right]\mathbf{v},\\[2ex] t' = \vec{\gamma}\left[t' + \dfrac{\mathbf{v}\cdot\mathbf{r}'}{c^2}\right],\end{cases}$$

Solution: Here, $\mathbf{r}' + \left[\dfrac{\vec{\gamma} - 1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r}' + \vec{\gamma}t'\right]\mathbf{v}$

$$= \mathbf{r} + \left[\frac{\vec{\gamma} - 1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r} - \vec{\gamma}t\right]\mathbf{v} + \left[\frac{\vec{\gamma} - 1}{\mathbf{v}^2}\mathbf{v}\cdot\left\{\mathbf{r} + \left(\frac{\vec{\gamma} - 1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r} - \vec{\gamma}t\right)\mathbf{v}\right\} + \vec{\gamma}^2\left(t - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right)\right]\mathbf{v}$$

$$= \mathbf{r} + \frac{\gamma - 1}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - \gamma t\,\mathbf{v} + \frac{\gamma-1}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} + \left(\frac{\gamma-1}{\mathbf{v}^2}\right)^2(\mathbf{v}\cdot\mathbf{r})\mathbf{v}^2\,\mathbf{v}$$

NOTES

$$-\frac{\gamma-1}{\mathbf{v}^2}\gamma\, t\,\mathbf{v}^2\mathbf{v} + \gamma^2\, t\,\mathbf{v} - \frac{\gamma^2}{c^2}\mathbf{v}\cdot\mathbf{r}\mathbf{v}$$

$$= \mathbf{r} + 2\frac{(\gamma-1)}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\,\mathbf{v} - \gamma\, t\,\mathbf{v} + \frac{(\gamma-1)^2}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - (\gamma-1)\,\gamma\, t\,\mathbf{v}$$

$$+\ \gamma^2\, t\,\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\mathbf{v} \quad \left[\because \gamma^2 = \frac{c^2}{c^2-\mathbf{v}^2}\right]$$

$$= \mathbf{r} + \frac{\gamma-1}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\,\mathbf{v}\,(2+\gamma-1) - \gamma\, t\,\mathbf{v} - \gamma^2 t\,\mathbf{v} + \gamma\, t\,\mathbf{v} + \gamma^2 t\,\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\,\mathbf{v}$$

$$= \mathbf{r} + \frac{\gamma^2-1}{\mathbf{v}^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\,\mathbf{v}$$

$$= \mathbf{r} + \frac{1}{\mathbf{v}^2}\left(\frac{c^2}{c^2-\mathbf{v}^2} - 1\right)(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\,\mathbf{v}$$

$$= \mathbf{r} + \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\,\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2}\,\mathbf{v} = \mathbf{r}.$$

Again $\vec{\gamma}\left[t' + \dfrac{\mathbf{v}\cdot\mathbf{r}'}{c^2}\right]$

$$= \vec{\gamma}\left[\vec{\gamma}\left(t' - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right) + \frac{1}{c^2}\mathbf{v}\cdot\left\{\mathbf{r} + \left(\frac{\vec{\gamma}-1}{v^2}\mathbf{v}\cdot(\mathbf{r} - \vec{\gamma}\, t)\mathbf{v}\right)\right\}\right]$$

$$= \vec{\gamma}^2\, t - \vec{\gamma}^2\,\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} + \vec{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} - \frac{\vec{\gamma}}{c^2}\cdot\frac{\vec{\gamma}-1}{\mathbf{v}^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v}^2 - \frac{\vec{\gamma}\, t}{c^2}\mathbf{v}^2$$

$$= \frac{c^2\, t}{c^2-\mathbf{v}^2} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2} + \vec{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} + \frac{\vec{\gamma}}{c^2}\mathbf{v}\cdot\mathbf{r} - \vec{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} - \frac{\mathbf{v}^2 t}{c^2-\mathbf{v}^2}$$

$$= \frac{(c^2-\mathbf{v}^2)t}{c^2-\mathbf{v}^2} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2} + \frac{\mathbf{v}\cdot\mathbf{r}}{c^2-\mathbf{v}^2} \qquad \left[\text{since } \frac{\vec{\gamma}^2}{c^2} = \frac{1}{c^2-\mathbf{v}^2}\right]$$

$$= t.$$

Example 1.7: What is the meaning of $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ for the case where $\mathbf{a}^2 = \mathbf{b}^2$?

Solution: Here $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}^2$

$= \mathbf{a}^2 - \mathbf{b}^2$ as $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

$= \mathbf{0}$ as $\mathbf{a}^2 = \mathbf{b}^2$.

This shows that either $\mathbf{a}+\mathbf{b} = \mathbf{0}$, $\mathbf{a}-\mathbf{b} = \mathbf{0}$ or the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are mutually at right angles.

In the former case when $\mathbf{a}+\mathbf{b} = \mathbf{0}$, or $\mathbf{a}-\mathbf{b} = \mathbf{0}$, we have $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, *i.e.*, both the vectors **a** and **b** are null vectors.

NOTES

Conclusively, either both the vectors **a** and **b** are null vectors or the angle between the vectors **a** + **b** and **a** – **b** is $\frac{1}{2}\pi$.

Example 1.8: What is the unit vector perpendicular to each of the vectors 2**i** – **j** + **k** and 3**i** + 4**j** – **k**? Calculate the sine of the angle between these vectors.

Solution: Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

If $\hat{\varepsilon}$ be a unit vector perpendicular to the plane of **a** and **b**, then since **a** × **b** is also a vector perpendicular to the plane of **a** and **b**, we have

$$\hat{\varepsilon} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}. \qquad \text{... (1)}$$

Now $\mathbf{a}\times\mathbf{b} = (2\mathbf{j} - \mathbf{j} + \mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$$

$\therefore$ $|\mathbf{a}\times\mathbf{b}| = |-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}| = \sqrt{(9 + 25 + 121)} = \sqrt{(155)}$.

Hence $\hat{\varepsilon} = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \dfrac{-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}}{\sqrt{(155)}}$.

Again $|\mathbf{a}| = |2\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{(4 + 1 + 1)} = \sqrt{6}$

$|\mathbf{b}| = |3\mathbf{i} + 4\mathbf{j} - \mathbf{k}| = \sqrt{(9 + 16 + 1)} = \sqrt{26}$.

Thus if θ is the angle between the directions of **a** and **b**, then

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \hat{\varepsilon}$$

i.e. $\sin\theta = \dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|\hat{\varepsilon}} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\ |\mathbf{b}|}$ by (1)

$$= \frac{\sqrt{(155)}}{\sqrt{6}\ \sqrt{26}} = \sqrt{\left(\frac{155}{156}\right)}.$$

Example 1.9: If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, then calculate

(*i*) the module of each,

(*ii*) the scalar product **a** · **b**,

(*iii*) the vector sum and difference **a** + **b** and **a** – **b**.

Solution: (*i*) We have $|\mathbf{a}| = |3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}|$

$= \sqrt{(9 + 16 + 25)} = 5\sqrt{2}$

and $|\mathbf{b}| = |-\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}|$

$= \sqrt{(1 + 4 + 36)} = \sqrt{41}$.

(*ii*) $\mathbf{a}\cdot\mathbf{b} = (3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})\cdot(-\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$= 3(-1) + 4 . 2 + (-5)\cdot 6$

$= -3 + 8 - 30 = -25$.

(*iii*) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + (-\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$= 2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$,

$\mathbf{a} - \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} - (-\mathbf{i} + 2\mathbf{i} + 6\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$.

1.2.2 Gradient, Divergence and Curl

NOTES

Consider a scalar function *i.e.*, a function whose value depends upon the values of coordinates (x, y, z). Being a scalar its value is constant at a fixed point in space.

The gradient of any scalar function ϕ is defined as

$$\text{grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x}\right)\phi \quad \text{... (1.7)}$$

$$= \nabla\phi$$

where operator ∇ is generally known as '**Del**' or '**Nabla**' operator and read as 'Gradient' or 'Grad' in short. We have already mentioned that a scalar field is the region in which the scalar point function specifies the scalar physical quantity like temperature, electric potential, density, etc. It is represented by a continuous scalar function giving the value of the quantity at each point. In scalar field all the points having same value of ϕ can be connected by means of surfaces, which are called *equal* or *level* surfaces.

Consider a co-ordinate system with axes such that any level surface lies in x-y plane while z-axis is along the normal to that level surface. Since the value of ϕ does not change along the level surface, *i.e.*,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0,$$

therefore $\qquad \text{grad } \phi = \mathbf{k}\frac{\partial \phi}{\partial z}. \qquad$ ... (1.8)

Clearly grad ϕ is directed along z-axis, *i.e.*, along the normal to the level surface. Therefore Equation (1.8) may be written as

$$\text{grad } \phi = \frac{\partial \phi}{\partial n}\mathbf{n} \quad \text{... (1.9)}$$

where **n** is unit vector along the normal to the level surface at any point.

From Equation (1.9) we may state, "*The magnitude of grad ϕ at any point is rate of change of function ϕ with distance along the normal to the level surface at the point and is directed along unit vector* **n**."

Note. It is to be noted that gradient of any scalar quantity is a vector.

Example 1.10: Prove $\nabla r^n = nr^{n-2}\mathbf{r}$.

Solution: $\qquad$ L.H.S. $= \nabla r^n = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(r^n)$

$$= \mathbf{i}\frac{\partial r^n}{\partial x} + \mathbf{j}\frac{\partial r^n}{\partial y} + \mathbf{k}\frac{\partial r^n}{\partial z}$$

$$= \mathbf{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \mathbf{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \mathbf{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right]$$

since $r^2 = x^2 + y^2 + z^2$; $\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial x} = \frac{z}{r}$

$$= nr^{n-2} (\mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z) = nr^{n-2}\, \mathbf{r}.$$

NOTES

Example 1.11: If **r** is the position vector of a point, deduce the value of grad (1/*r*).

Solution: As given, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

So that $\frac{1}{r} = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$.

$\therefore$ grad $(1/r) = \nabla\, (1/r)$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right\}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right] + \mathbf{j}\frac{\partial}{\partial y}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$+ \mathbf{k}\frac{\partial}{\partial z}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

$$= \mathbf{i}\left[-\frac{1}{2}\cdot\frac{2x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \mathbf{j}\left[-\frac{1}{2}\cdot\frac{2y}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$+ \mathbf{k}\left[-\frac{1}{2}\cdot\frac{2z}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\mathbf{r}}{(r^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}.$$

The Gradient of a Scalar-Point Function

If $\phi\,(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space specified as a scalar field, we have

$$= \left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}\right) = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right) \quad ...\ (1.10)$$

Its R.H.S. is the scalar product of two vectors $(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and $\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$, where the vector $(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ is a unit vector along a line whose direction cosines are l, m, n and the second vector depends only on the point (x, y, z) and not on any direction. Thus we conclude that directional derivative along any line can be obtained by multiplying the vector

$\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$ scalarly with the unit vector $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

The vector function $\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$ is called *the gradient of a scalar-point function* ϕ and is written as grad ϕ or $\nabla\phi$. Thus,

$$\nabla\phi = \text{grad } \phi \equiv \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \equiv \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

It is clear that the gradient of a scalar-point function is a vector.

NOTES

In case, ϕ is a constant, grad $\phi = 0$ since $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$, all will be zero in that case. Its converse is also true.

The Gradient or Sum of Two Scalar-Point Functions

If u and v are two differentiable scalar functions of x, y, z, then the gradient of their sum is given by

$$\begin{aligned}
\nabla (u + v) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial x}\right)(u+v) \\
&= \mathbf{i}\frac{\partial}{\partial x}(u+v) + \mathbf{j}\frac{\partial}{\partial y}(u+v) + \mathbf{k}\frac{\partial}{\partial z}(u+v) \\
&= \mathbf{i}\frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{j}\frac{\partial v}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\,\mathbf{k}\frac{\partial v}{\partial z} \\
&= \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right) + \left(\mathbf{i}\frac{\partial v}{\partial x} + \mathbf{j}\frac{\partial v}{\partial y} + \mathbf{k}\frac{\partial v}{\partial z}\right) \\
&= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)u + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)v \\
&= \nabla u + \nabla v.
\end{aligned}$$

Showing that the gradient of sum of two scalar-point functions is equal to the sum of their gradients.

This rule may be generalised for any number of scalar-point functions.

The Gradient of Product of Two Scalar-Point Functions

If u and v be two differentiable scalar-point functions of x, y, z, then the gradient of their product is given by

$$\begin{aligned}
\nabla (uv) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(uv) \\
&= \mathbf{i}\frac{\partial}{\partial x}(uv) + \mathbf{j}\frac{\partial}{\partial y}(uv) + \mathbf{k}\frac{\partial}{\partial z}(uv) \\
&= \mathbf{i}\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) + \mathbf{j}\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) + \mathbf{k}\left(u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z}\right) \\
&= u\left[\mathbf{i}\frac{\partial v}{\partial x} + \mathbf{j}\frac{\partial v}{\partial y} + \mathbf{k}\frac{\partial v}{\partial z}\right] + v\left[\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right] \\
&= u\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)v + v\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)u \\
&= u\,\nabla\, v + v\,\nabla\, u.
\end{aligned}$$

Showing that the gradient of the product of two scalar-point functions is obtained by the same rule as is valid for derivatives of the algebraic functions.

The Divergence of a Vector-Point Function

If $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ *be a continuous differentiable vector-point function specified in a vector field, then the divergence of* $\mathbf{V}$ *is defined as:*

NOTES

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}$$

and is written as $\nabla \cdot \mathbf{V}$ *or div* $\mathbf{V}$ *and read as divergence* $\mathbf{V}$.

$$\therefore \qquad \nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

which is clearly a scalar quantity.

Note. If $\nabla \cdot \mathbf{V} = 0$ then $\mathbf{V}$ is known as *Solenoidal Vector.*

The Divergence of Sum of two Vector Functions

If $\mathbf{U}$ and $\mathbf{V}$ be two vector-point functions expressed as

$$\mathbf{U} = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$$
$$\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}.$$

Then

$$\nabla \cdot (\mathbf{U} + \mathbf{V}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(U_1 + V_1)\,\mathbf{i} + (U_2 + V_2)\,\mathbf{j} + (U_3 + U_3)\,\mathbf{k}]$$

$$= \frac{\partial}{\partial x}[U_1 + V_1] + \frac{\partial}{\partial y}[U_2 + V_2] + \frac{\partial}{\partial z}[U_3 + V_3]$$

$$= \left(\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial U_3}{\partial z}\right) + \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k})$$

$$+ \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \nabla \cdot \mathbf{U} + \nabla \cdot \mathbf{V}.$$

$$= \text{div}\,\mathbf{U} + \text{div}\,\mathbf{V}.$$

Showing that the divergence of the sum of two vector functions is equal to the sum of their divergences.

This rule may be generalised for any number of vector functions.

The Divergence of Product

If the vector point function $\mathbf{U}$ is expressed as

$\mathbf{U} = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$ and V is a scalar point-function.

Then $\nabla \cdot (\mathbf{U}V) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k})V]$

NOTES

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot[VU_1\mathbf{i} + VU_2\mathbf{j} + VU_3\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(VU_1) + \frac{\partial}{\partial y}(VU_2) + \frac{\partial}{\partial z}(VU_3)$$

$$= U_1\frac{\partial V}{\partial x} + V\frac{\partial U_1}{\partial x} + U_2\frac{\partial V}{\partial y} + V\frac{\partial U_2}{\partial y} + U_3\frac{\partial V}{\partial z} + V\frac{\partial U_3}{\partial z}$$

$$= \left(U_1\frac{\partial V}{\partial x} + U_2\frac{\partial V}{\partial y} + U_3\frac{\partial V}{\partial z}\right) + V\left(\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} + \frac{\partial U_3}{\partial z}\right)$$

$$= \left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}\right)\cdot(U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k})$$

$$+ V\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k})$$

$$= (\nabla V)\cdot\mathbf{U} + V(\nabla\cdot\mathbf{U})$$

i.e., $\text{div}(\mathbf{U}V) = (\text{grad } V)\cdot\mathbf{U} + V \text{ div } \mathbf{U}$.

The Curl or Rotation of a Vector Point Function

Let $\mathbf{f}(x, y, z) = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ be a continuous differentiable vector-point function; then the curl of **f** or rotation of **f** is given by

$$\left(\mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{f}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{f}}{\partial z}\right)$$

and is written as curl **f** or $\nabla\times\mathbf{f}$ or rot **f**.

i.e., $$\nabla\times\mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k})$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

It is clear that **curl f** or **rotatioin f** is a vector quantity and read as **del cross f**.

Note. If curl = 0, f is known as *Irrotatinal Vector*.

Interpretation of the Curl f: *If a rigid body is in motion, the curl of its linear velocity at any point gives twice its angular velocity.*

Consider the motion of a rigid body rotating with angular velocity ω about an axis *OA*; *O*, being a fixed point in the body. Let **r** be the position vector of any point *P* of the body. Draw *PQ* perpedicular from *P* to the axis *OA*. Then,

Linear velocity *V* of *P* due to circular motion

NOTES

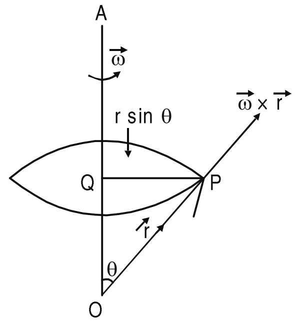


Fig. 1.8 *Rotation of Rigid Body on Fixed Point*

$$= |\mathbf{V}|$$
$$= \omega QP = \omega r \sin\theta = |\vec{\omega} \times \mathbf{r}|$$

i.e., $\mathbf{V} = \vec{\omega} \times \mathbf{r}$

where, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

and $\vec{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$.

But we know that curl $\mathbf{V}$ = $\nabla \times \mathbf{V} = \nabla \times (\vec{\omega} \times \mathbf{r})$

$$= \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(\omega_2 z - \omega_3 y)\,\mathbf{i} + (\omega_3 x - \omega_1 z)\,\mathbf{j} + (\omega_1 y - \omega_2 x)\,\mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 z - \omega_2 x \end{vmatrix}$$

$$= 2\,[\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}]$$

$= 2\vec{\omega}$ which proves the proposition.

Curl of the Sum of Two Vector-Point Functions

If $\mathbf{u}$ and $\mathbf{v}$ be two vector-point functions given by

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$$
$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

then $\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times [(u_1 + v_1)]\mathbf{i} + (u_2 + v_2)\,\mathbf{j} + (u_3 + v_3)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(u_3 + v_3) - \frac{\partial}{\partial z}(u_2 + v_2)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(u_1 + v_1) - \frac{\partial}{\partial x}(u_3 + v_3)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(u_2 + v_2) - \frac{\partial}{\partial y}(u_1 + v_1)\right]$$

NOTES

$$= \mathbf{i}\left[\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right]+\mathbf{j}\left[\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right]+\mathbf{k}\left[\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right]$$

$$+\,\mathbf{i}\left[\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right]+\mathbf{j}\left[\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right]+\mathbf{k}\left[\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \nabla \times \mathbf{u} + \nabla \times \mathbf{v}. \qquad \text{(by the definition)}$$

i.e., curl $(u + v) =$ curl $\mathbf{u}$ + curl $\mathbf{v}$.

Hence curl of sum of two vector point functions is equal to the sum of their curls.

The result may be generalised for any number of vector-point functions.

Note. If $\mathbf{r}$ is the position vector of a variable point with respect to a fixed origin such that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then curl $\mathbf{r} = \mathbf{0}$.

Since curl $\mathbf{r} = \left(\mathbf{i}\times\dfrac{\partial}{\partial x}+\mathbf{j}\times\dfrac{\partial}{\partial y}+\mathbf{k}\times\dfrac{\partial}{\partial z}\right) \times (x\mathbf{i} + y\mathbf{i} + z\mathbf{k})$

$$= \left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]+\mathbf{j}\left[\frac{\partial}{\partial z}(x)-\frac{\partial}{\partial x}(z)\right]+\mathbf{k}\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$$= \mathbf{0}.$$

Curl of the Product of two Vector-Point Functions

We have to consider the curl of the forms $u\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ where u is a scalar and $\mathbf{u}$, $\mathbf{v}$ vector point functions.

Suppose, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

and u is a scalar point function.

Then, curl $(u\mathbf{v}) = \nabla \times (u\mathbf{v}) = \left(\mathbf{i}\times\dfrac{\partial}{\partial x}+\mathbf{j}\times\dfrac{\partial}{\partial y}+\mathbf{k}\times\dfrac{\partial}{\partial z}\right) (uv_1\mathbf{i} + uv_2\mathbf{j} + uv_3\mathbf{k})$

$$= \nabla \times (uv_1\mathbf{i} + uv_2\,\mathbf{j} + uv_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ uv_1 & uv_2 & uv_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(uv_3)-\frac{\partial}{\partial z}(uv_2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(uv_1)-\frac{\partial}{\partial x}(uv_3)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(uv_2)-\frac{\partial}{\partial y}(uv_1)\right]\mathbf{k}$$

$$= \left[u\frac{\partial v_3}{\partial y} + v_3\frac{\partial u}{\partial y} - u\frac{\partial v_3}{\partial z} - v_2\frac{\partial u}{\partial z}\right]\mathbf{i} + \left[u\frac{\partial v_1}{\partial z} + v_1\frac{\partial u}{\partial z} - u\frac{\partial v_3}{\partial x} - v_3\frac{\partial u}{\partial x}\right]\mathbf{j}$$

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$$+ \left[u\frac{\partial v_2}{\partial x} + v_2\frac{\partial u}{\partial x} - u\frac{\partial v_1}{\partial y} - v_1\frac{\partial u}{\partial y}\right]\mathbf{k}$$

$$= u\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial y_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}\right]$$

$$+ \left[\left(\frac{\partial u}{\partial y}v_3 - \frac{\partial u}{\partial z}v_2\right)\mathbf{i} + \left(\frac{\partial u}{\partial z}v_1 - \frac{\partial u}{\partial x}v_3\right)\mathbf{j} + \left(\frac{\partial u}{\partial x}v_2 - \frac{\partial u}{\partial y}v_1\right)\mathbf{k}\right]$$

$$= u\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = u \text{ curl } \mathbf{v} + (\text{grad } u) \times \mathbf{v}$$

i.e., $\nabla \times (uv) = u\,\nabla \times \mathbf{v} + (\nabla u) \times \mathbf{v}$.

Again $\text{curl}(\mathbf{u} \times \mathbf{v}) = \nabla \times (\mathbf{u} \times \mathbf{v})$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \nabla \times [(v_3u_2 - v_2u_3)\,\mathbf{i} + (v_1u_3 - v_3u_1)\,\mathbf{j} + (v_2u_1 - v_1u_2)\,\mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (v_3u_2 - v_2u_3) & (v_1u_3 - v_3u_1) & (v_2u_1 - v_1u_2) \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(v_2u_1 - v_1u_2) - \frac{\partial}{\partial z}(v_1u_3 - u_1v_3)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(v_3u_2 - v_2u_3) - \frac{\partial}{\partial x}(v_2u_1 - v_1u_2)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(v_1u_3 - v_3u_1) - \frac{\partial}{\partial y}(v_3u_2 - v_2u_3)\right]$$

$$= \mathbf{i}\left[u_1\frac{\partial v_2}{\partial y} + v_2\frac{\partial u_1}{\partial y} - u_2\frac{\partial v_1}{\partial y} - v_1\frac{\partial u_2}{\partial y} - u_3\frac{\partial v_1}{\partial z} - v_1\frac{\partial u_3}{\partial z} + u_1\frac{\partial v_3}{\partial z} + v_3\frac{\partial u_1}{\partial z}\right]$$

$$+ \mathbf{j}\left[u_2\frac{\partial v_3}{\partial z} + v_3\frac{\partial u_2}{\partial z} - u_3\frac{\partial v_2}{\partial z} - v_2\frac{\partial u_3}{\partial z} - u_1\frac{\partial v_2}{\partial x} - v_2\frac{\partial u_1}{\partial x} + v_1\frac{\partial v_2}{\partial x} + u_2\frac{\partial v_1}{\partial x}\right]$$

$$+ \mathbf{k}\left[u_3\frac{\partial v_1}{\partial x} + v_1\frac{\partial u_3}{\partial x} - v_3\frac{\partial u_1}{\partial x} - u_1\frac{\partial v_3}{\partial x} - u_2\frac{\partial v_3}{\partial y} - v_3\frac{\partial u_2}{\partial y} + v_2\frac{\partial u_3}{\partial y} + u_3\frac{\partial v_2}{\partial y}\right]$$

$$= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\right]$$

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$$- \ (v_1\mathbf{i} + v_2\mathbf{j} + v_3\ ) \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}\right]$$

$$+ \left[(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} +\right)\right](u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})$$

$$- \left[(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} +\right)\right](v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \mathbf{u} \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \mathbf{u} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{v}.$$

Aliter $\quad$ $\text{curl } \mathbf{f} = \mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{f}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{f}}{\partial z} = \Sigma\,\mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x}.$

Now $\text{curl } (\mathbf{u}\times\mathbf{v}) = \Sigma\,\mathbf{i}\times\frac{\partial[\mathbf{u}\times\mathbf{v}]}{\partial x}$

$$= \Sigma\,\mathbf{i}\times\left[\mathbf{u}\times\frac{\partial \mathbf{v}}{\partial x} + \mathbf{v}\times\frac{\partial \mathbf{u}}{\partial x}\right]$$

$$= \Sigma\,\mathbf{i}\times\left(\mathbf{u}\times\frac{\partial \mathbf{v}}{\partial x}\right) + \Sigma\,\mathbf{i}\times\left(\mathbf{v}\times\frac{\partial \mathbf{u}}{\partial x}\right)$$

$$= \Sigma\left(\mathbf{i}\cdot\frac{\partial \mathbf{v}}{\partial x}\right)\mathbf{u} - \Sigma(\mathbf{i}\cdot\mathbf{u})\frac{\partial \mathbf{v}}{\partial x} + \Sigma(\mathbf{i}\cdot\mathbf{v})\frac{\partial \mathbf{u}}{\partial x} - \Sigma\left(\mathbf{i}\cdot\frac{\partial \mathbf{u}}{\partial x}\right)\mathbf{v}$$

[by vector triple product]

$$= \left[\left(\Sigma\,\mathbf{i}\cdot\frac{\partial \mathbf{v}}{\partial x}\right)\right]\mathbf{u} - [\Sigma(\mathbf{i}\cdot\mathbf{u})]\frac{\partial \mathbf{v}}{\partial x} + [\Sigma(\mathbf{i}\cdot\mathbf{v})]\frac{\partial \mathbf{u}}{\partial x} - \left[\Sigma\left(\mathbf{i}\cdot\frac{\partial \mathbf{u}}{\partial x}\right)\right]\mathbf{v}$$

$$= \mathbf{u} \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \mathbf{u} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{v}.$$

To Express Gradient of Scalar Product in Terms of Curl

We have to show that

$\text{grad } (\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\times\text{curl } \mathbf{v} + \mathbf{v}\times\text{curl } \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{u}$

We know that

$$\text{grad } (\mathbf{u}\cdot\mathbf{v}) = \Sigma\,\mathbf{i}\frac{\partial}{\partial x}\ (\mathbf{u}\cdot\mathbf{v})$$

$$= \Sigma\,\mathbf{i}\left[\mathbf{u}\cdot\frac{\partial \mathbf{v}}{\partial x} + \mathbf{v}\cdot\frac{\partial \mathbf{u}}{\partial x}\right]$$

$$= \Sigma\,\mathbf{i}\left(\mathbf{u}\cdot\frac{\partial \mathbf{v}}{\partial x}\right) + \Sigma\,\mathbf{i}\left(\mathbf{v}\cdot\frac{\partial \mathbf{u}}{\partial x}\right) \qquad \text{... (1.11)}$$

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And $\quad \mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) = \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} - (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$

or $\quad \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \left[\mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) \right] + (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$

$$\therefore \quad \Sigma \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \Sigma \left[\mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) \right] + \Sigma (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$$

$$= \mathbf{u} \times \text{curl } \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} \qquad \text{... (1.12)}$$

Similarly $\Sigma \left(\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} \qquad \text{... (1.13)}$

Substituting values of Equations (1.12) and (1.13) in (1.11) we find

$$\text{grad } (\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times \text{curl } \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u}$$
$$= \mathbf{u} \times \text{curl } \mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}.$$

To Express Divergence of Vector Product in Terms of Curl

We have to show that $\quad \text{div } (\mathbf{u} \times \mathbf{v}) = \text{curl } \mathbf{u} \cdot \mathbf{v} - \text{curl } \mathbf{v} \cdot \mathbf{u}$.

We know that,

$$\text{div } (\mathbf{u} \times \mathbf{v}) = \Sigma\, \mathbf{i} \cdot \frac{\partial}{\partial x} (\mathbf{u} \times \mathbf{v})$$
$$= \Sigma\, \mathbf{i} \cdot \left[\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v} + \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x} \right]$$
$$= \left[\Sigma\, \mathbf{i} \cdot \left(\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v} \right) + \Sigma\, \mathbf{i} \cdot \left(\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x} \right) \right]$$
$$= \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{v} + \Sigma\, \mathbf{i} \cdot \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}$$

(interchanging dot and cross)

$$= \left(\Sigma\, \mathbf{i} \times \frac{\partial \mathbf{u}}{\partial x} \right) \cdot \mathbf{v} + \left(\Sigma\, \mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) \cdot \mathbf{u}$$
$$= \text{curl } \mathbf{u} \cdot \mathbf{v} - \text{curl } \mathbf{v} \cdot \mathbf{u}.$$

1.3 VECTOR CALCULUS

In this section, scalar product and vector product are discussed.

Scalar Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ in this order and is written as $\left(\vec{a} \times \vec{b} \right) . \vec{c}$ or $\left[\vec{a} \vec{b} \vec{c} \right]$ or $\left[\vec{a}, \vec{b}, \vec{c} \right]$.

Geometrical Representation of Scalar Triple Product

Geometrically, the scalar product represents the volume of parallelopiped having $\vec{a}, \vec{b}$ and $\vec{c}$ as its coterminous edges.

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Consider a parallelopiped with $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ as coterminous edges (Refer Figure 1.9).

$\vec{a} \times \vec{b}$ is the vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Let $\hat{u}$ is a unit vector along $\vec{a} \times \vec{b}$ and θ be the angle between $\hat{u}$ and $\vec{c}$.

Now, $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = (Area of parallelogram OADB) $\hat{u} \,.\, \vec{c}$

= (Area of parallelogram OADB) $\left|\hat{u}\right| \left|\vec{c}\right| \cos\theta$

= (Area of parallelogram OADB) OC cosθ

= (Area of parallelogram OADB) OL

= (Area of parallelogram OADB) × Height

= Volume of parallelopiped

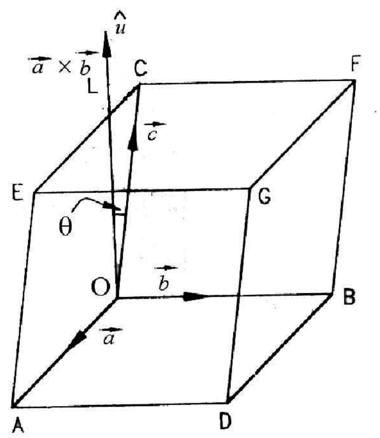

Fig. 1.9 *Parallelopiped*

Properties of Scalar Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors and *m* be a scalar.

(i) The cyclic permutation of three vectors does not change the value of scalar product.

$$\left(\vec{a} \times \vec{b}\right).\vec{c} = \left(\vec{b} \times \vec{c}\right).\vec{a} = \left(\vec{c} \times \vec{a}\right).\vec{b} \text{ or } \left[\vec{a}\vec{b}\vec{c}\right] = \left[\vec{b}\vec{c}\vec{a}\right] = \left[\vec{c}\vec{a}\vec{b}\right]$$

(ii) The change in the cyclic order of three vectors changes the sign of the scalar triple product but not the magnitude.

$$\left[\vec{a}\vec{b}\vec{c}\right] = -\left[\vec{b}\vec{a}\vec{c}\right] = -\left[\vec{c}\vec{b}\vec{a}\right] = -\left[\vec{a}\vec{c}\vec{b}\right]$$

(iii) In a scalar triple product, the dot and cross can be interchanged provided that the cyclic order of the vectors remains the same.

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\vec{a}.\left(\vec{b}\times\vec{c}\right)$$

(iv) The scalar triple product of three vectors is zero if any two of them are equal.

$$\left[\vec{a}\vec{b}\vec{c}\right]=0 \text{ if } \vec{a}=\vec{b} \text{ or } \vec{b}=\vec{c} \text{ or } \vec{c}=\vec{a}$$

(v) $\left[m\vec{a}\vec{b}\vec{c}\right]=m\left[\vec{a}\vec{b}\vec{c}\right]$

(vi) The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

$$\left[\vec{a}\vec{b}\vec{c}\right]=0 \text{ if } \vec{a}=m\vec{b} \text{ or } \vec{b}=m\vec{c} \text{ or } \vec{c}=m\vec{a}$$

Note: $\left[\hat{i}\hat{j}\hat{k}\right]=\left(\hat{i}\times\hat{j}\right).\hat{k}=\hat{k}.\hat{k}=1$, where $\hat{i}, \hat{j}$ and $\hat{k}$ stands for the unit vectors along the axes. Similarly, $\left[\hat{j}\hat{k}\hat{i}\right]=\left[\hat{k}\hat{i}\hat{j}\right]=1$ and thus $\left[\hat{i}\hat{j}\hat{k}\right]=\left[\hat{j}\hat{k}\hat{i}\right]=\left[\hat{k}\hat{i}\hat{j}\right]=1$.

Coplanarity of Three Vectors

The necessary and sufficient condition for three non-zero non-collinear vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is that $\left[\vec{a}\vec{b}\vec{c}\right]=0$, i.e,

$$\vec{a}, \vec{b}, \vec{c} \text{ are coplanar} \Leftrightarrow \left[\vec{a}\vec{b}\vec{c}\right]=0$$

Proof: Condition is necessary: Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$. This means that

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=0 \text{ or } \left[\vec{a}\vec{b}\vec{c}\right]=0$$

Condition is sufficient: Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero non-collinear vectors such that $\left[\vec{a}\vec{b}\vec{c}\right]=0$ or $\left(\vec{a}\times\vec{b}\right).\vec{c}=0$.

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero non-collinear vectors; thus, $\vec{a}\times\vec{b}\neq 0$ and $\vec{c}\neq 0$. This means that $\left(\vec{a}\times\vec{b}\right)\perp\vec{c}$. But $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ lies in the same plane, i.e., they are coplanar.

Scalar Triple Product in terms of Components

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three vectors, then,

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

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Volume of a Tetrahedron

Let ABCD be a tetrahedron, and its three edges AB, AC and AD represent three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively (Refer Figure 1.10).

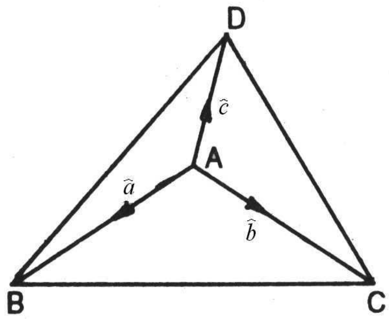

Fig. 1.10 *Tetrahedron*

Volume of tetrahedron $= \frac{1}{3}$ (area of ΔABC) $\times$ (height of vertex D above the plane ABC)

$= \frac{1}{3}\left(\frac{1}{2}\text{area of parallelogram whose adjacent edges AB and AC}\right) \times$ (height of vertex D above the plane ABC)

$= \frac{1}{6}$ (volume of parallelepiped having AB, AC and AD as coterminous edges)

$= \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = \frac{1}{6}\left[\vec{a}\vec{b}\vec{c}\right]$

Thus, the volume of tetrahedron with three edges AB, AC and AD representing three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $\frac{1}{6}\left[\vec{a}\vec{b}\vec{c}\right]$.

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the vertices A, B, C and D of the tetrahedron ABCD, then $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$ and $\overrightarrow{AD} = \vec{d} - \vec{a}$.

$\therefore$ Volume of tetrahedron $= \frac{1}{6}\left[\vec{b} - \vec{a} \quad \vec{c} - \vec{a} \quad \vec{d} - \vec{a}\right]$

Four points A, B, C and D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if the volume of tetrahedron ABCD is 0, i.e.,

$$\frac{1}{6}\left[\vec{b}-\vec{a} \quad \vec{c}-\vec{a} \quad \vec{d}-\vec{a}\right] = 0 \Rightarrow \left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right].\left(\vec{d}-\vec{a}\right) = 0$$

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$$\Rightarrow \left[\vec{b}\vec{c}\vec{d}\right] - \left[\vec{b}\vec{c}\vec{a}\right] + \left[\vec{b}\vec{d}\vec{a}\right] - \left[\vec{c}\vec{d}\vec{a}\right] = 0$$

Example 1.12: Find the volume of a parallelepiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}, -5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

Solution: Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$. We know that the volume of a parallelepiped whose three adjacent edges are $\vec{a},\vec{b},\vec{c}$ is equal to $\left|\left[\vec{a}\vec{b}\vec{c}\right]\right|$.

We have,

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$= -3\ (-21-15) -7\ (15 + 21) + 5\ (25 - 49)$
$= 108 - 252 - 120 = -264$

So, required volume of the parallelepiped $= \left|\left[\vec{a}\vec{b}\vec{c}\right]\right| = |-264|$
$= 264$ cubic units

Example 1.13: Find the value of λ for which the four points with position vectors $3\hat{i}-2\hat{j}-\hat{k}, 2\hat{i}+3\hat{j}-4\hat{k}, -\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$ are coplanar.

Solution: Let A, B, C, D be the given points. Then,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A
$= \left(2\hat{i}+3\hat{j}-4\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right)=-\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC}$ = Position vector of C – Position vecor of A
$= \left(-\hat{i}+\hat{j}+2\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right)=-4\hat{i}+3\hat{j}+3\hat{k}$

$\overrightarrow{AD}$ = Position vector of D – Position vecor of A
$= \left(4\hat{i}+5\hat{j}+\lambda\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right)=\hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

The given points are coplanar iff vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar.

$\therefore$ Points A, B, C, D are coplanar.

$\Leftrightarrow$ $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar,

$\Leftrightarrow$ $\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = 0$

$$\Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

$\Leftrightarrow \quad -1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-28 - 3) = 0$

$\Leftrightarrow \quad -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Leftrightarrow \quad 17\lambda + 146 = 0$

$\Rightarrow \quad \lambda = -\dfrac{146}{17}$

NOTES

Vector Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the vector products of $\vec{a}\times\vec{b}$ with $\vec{c}$ and $\vec{a}$ with $\vec{b}\times\vec{c}$ are called the **vector triple products** of $\vec{a}, \vec{b}$ and $\vec{c}$. These products are written as $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$.

Expansion Formula

For three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Let us prove this formula.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

$$\therefore \vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\hat{j} + [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\hat{k}$$

Now, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$[\because \vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $\vec{a}.\vec{c} = a_1c_1 + a_2c_2 + a_3c_3]$

$$= (a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1)\hat{i} + (a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2)\hat{j} + (a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3)\hat{k}$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} +$$

$$\left[a_3\left(b_2c_3-b_3c_2\right)-a_1\left(b_1c_2-b_2c_1\right)\right]\hat{j}+$$

$$\left[a_1\left(b_3c_1-b_1c_3\right)-a_2\left(b_2c_3-b_3c_2\right)\right]\hat{k}$$

NOTES

Thus, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Similarly, it can be shown that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a}$

Note: (i) The vector triple product $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ is a linear combination of those two vectors which are within brackets.

(ii) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ is perpendicular to $\vec{a}$ and $\vec{b}\times\vec{c}$.

(iii) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$, i.e, vectors, triple product is not associative.

$\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ is a vector which lies in the plane of $\vec{b}$ and $\vec{c}$ whereas $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ is a vector which lies in the plane of $\vec{b}$ and $\vec{a}$.

$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\vec{c}\times\left(\vec{a}\times\vec{b}\right)=$

$$-\left[\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}\right]=\left(\vec{c}.\vec{a}\right)\vec{b}-\left(\vec{c}.\vec{b}\right)\vec{a}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a}$$

Example 1.14: If $\vec{a},\vec{b},\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then find the angles, which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.

Solution : We have, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$$\Rightarrow(\vec{a}.\vec{c})\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$\Rightarrow\left(\vec{a}.\vec{c}-\frac{1}{\sqrt{2}}\right)\vec{b}-\left(\vec{a}.\vec{b}+\frac{1}{\sqrt{2}}\right)\vec{c}=\vec{0}$$

$$\Rightarrow\vec{a}.\vec{c}-\frac{1}{\sqrt{2}}=0 \text{ and } \vec{a}.\vec{b}+\frac{1}{\sqrt{2}}=0 \qquad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow\vec{a}.\vec{c}=\frac{1}{\sqrt{2}} \text{ and } \vec{a}.\vec{b}=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow\cos\alpha=\frac{1}{\sqrt{2}}$$

and $\cos\beta=-\frac{1}{\sqrt{2}}$, where α and β are the angles made by $\vec{a}$ with $\vec{b}$ and $\vec{c}$, respectively.

$\Rightarrow \alpha = \frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$.

NOTES

Scalar Product of Four Vectors

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are any four vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c} \times \vec{d}$ is called the **scalar product** of $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. This product is written as $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$.

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \left[(\vec{a} \times \vec{b}) \times \vec{c}\right].\vec{d}$ [interchanging the dot and cross product]

$$= \left[(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}\right].\vec{d}$$

$$= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$ is known as Lagrange's identity. It can be expressed in the form of determinant as

$$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}$$

Vector Product of Four Vectors

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, then the vector product of $\vec{a} \times \vec{b}$ with $\vec{c} \times \vec{d}$ is called the **vector product** of $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. This product is written as $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

Vector product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ being a vector perpendicular to $\vec{a} \times \vec{b}$ is coplanar with $\vec{a}$ and $\vec{b}$. Also, it being a vector perpendicular to $\vec{c} \times \vec{d}$ is coplanar with $\vec{c}$ and $\vec{d}$. Thus, we can look upon $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ as a vector triple product in two ways by putting $\vec{a} \times \vec{b} = \vec{q}$ and $\vec{c} \times \vec{d} = \vec{p}$.

(i) Expressing $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{a}$ and $\vec{b}$.

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{p}$, where $\vec{c} \times \vec{d} = \vec{p}$

$$= (\vec{a}.\vec{p})\vec{b} - (\vec{b}.\vec{p})\vec{a}$$

$$= (\vec{a}.\vec{c} \times \vec{d})\vec{b} - (\vec{b}.\vec{c} \times \vec{d})\vec{a}$$

$$= \left[\vec{a}\vec{c}\vec{d}\right]\vec{b} - \left[\vec{b}\vec{c}\vec{d}\right]\vec{a}$$

Here, the vector product appears as the linear combination of $\vec{a}$ and $\vec{b}$.

(ii) Expressing $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{c}$ and $\vec{d}$.

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{q} \times (\vec{c} \times \vec{d})$, where $\vec{a} \times \vec{b} = \vec{q}$

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$$= (\vec{q}.\vec{d})\vec{c} - (\vec{q}.\vec{c})\vec{d}$$

$$= (\vec{a}\times\vec{b}.\vec{d})\vec{c} - (\vec{a}\times\vec{b}.\vec{c})\vec{d}$$

$$= [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$$

Here, the vector product appears as the linear combination of $\vec{c}$ and $\vec{d}$.

Thus, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a} = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$

Example 1.15: Prove that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$.

Solution: We know that

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) = (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{b}.\vec{d})(\vec{c}.\vec{a})$$

$$(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) = (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{c}.\vec{d})(\vec{a}.\vec{b})$$

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

Adding we get,

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$$

Reciprocal System of Vectors

If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, i.e, $[\vec{a}\vec{b}\vec{c}] \neq 0$ and if $\vec{a}',\vec{b}'$ and $\vec{c}'$ are three other vectors such that

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

then $\vec{a}',\vec{b}'$ and $\vec{c}'$ are called the **reciprocal system** to the vectors $\vec{a},\vec{b}$ and $\vec{c}$.

Theorem 1.1: If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a}',\vec{b}'$ and $\vec{c}'$ form a reciprocal system of vectors, then $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

Theorem 1.2: If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a}',\vec{b}'$ and $\vec{c}'$ form a reciprocal system of vectors, then $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$

Theorem 1.3: If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a}',\vec{b}'$ and $\vec{c}'$ form a reciprocal system of vectors, then $[\vec{a}'\vec{b}'\vec{c}'] = \frac{1}{[\vec{a}\vec{b}\vec{c}]}$

Proof: $[\vec{a}'\vec{b}'\vec{c}'] = (\vec{a}'\times\vec{b}').\vec{c}' = \left[\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]} \times \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}\right].\vec{c}'$

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$$= \left[\lambda\left(\vec{b}\times\vec{c}\right)\times\lambda\left(\vec{c}\times\vec{a}\right)\right].\vec{c}' \qquad \left(\text{Let } \lambda = \frac{1}{\left[\vec{a}\vec{b}\vec{c}\right]}\right)$$

$$= \lambda^2\left[\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)\right].\vec{c}'$$

$$= \lambda^2\left[\left\{\left(\vec{b}\times\vec{c}\right).\vec{a}\right\}\vec{c} - \left\{\left(\vec{b}\times\vec{c}\right).\vec{c}\right\}\vec{a}\right].\vec{c}'$$

$$= \lambda^2\left[\left[\vec{b}\vec{c}\vec{a}\right]\vec{c} - \left[\vec{b}\vec{c}\vec{c}\right]\vec{a}\right].\vec{c}'$$

$$= \lambda^2\left[\left[\vec{b}\vec{c}\vec{a}\right]\vec{c}\right].\vec{c}' \qquad \left(\because \left[\vec{b}\vec{c}\vec{c}\right] = 0\right)$$

$$= \lambda^2\left[\left[\vec{b}\vec{c}\vec{a}\right]\left(\vec{c}.\vec{c}'\right)\right]$$

$$= \lambda^2\left[\vec{b}\vec{c}\vec{a}\right] \qquad \left(\because \vec{c}.\vec{c}' = 1\right)$$

$$= \frac{1}{\left[\vec{a}\vec{b}\vec{c}\right]^2}\left[\vec{b}\vec{c}\vec{a}\right]$$

$$= \frac{1}{\left[\vec{a}\vec{b}\vec{c}\right]}$$

Theorem 1.4: The orthonormal vector triads $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a self-reciprocal system, i.e.,

$$\hat{i}' = \hat{i},\ \hat{j}' = \hat{j} \text{ and } \hat{k}' = \hat{k}$$

Proof: Let $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ be the system of vectors parallel to the system of $\hat{i}$, $\hat{j}$ and $\hat{k}$. Then, $\hat{i}' = \dfrac{\hat{j}\times\hat{k}}{\left[\hat{i}\hat{j}\hat{k}\right]} = \hat{i}$.

Theorem 1.5: If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, i.e, $\left[\vec{a}\vec{b}\vec{c}\right] \neq 0$ and $\vec{a}',\vec{b}'$ and $\vec{c}'$ constitute the reciprocal system of vectors, then any vector $\vec{r}$ can be expressed as $\vec{r} = \left(\vec{r}.\vec{a}'\right)\vec{a} + \left(\vec{r}.\vec{b}'\right)\vec{b} + \left(\vec{r}.\vec{c}'\right)\vec{c}$

Proof: Since $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, $\vec{r}$ can be expressed as the linear combination in the form

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}\text{, where } x, y \text{ and } z \text{ are scalars} \qquad ...(1.14)$$

Multiplying both sides by $\vec{b}\times\vec{c}$,

$$\vec{r}.\left(\vec{b}\times\vec{c}\right) = x\vec{a}.\left(\vec{b}\times\vec{c}\right) + y\vec{b}.\left(\vec{b}\times\vec{c}\right) + z\vec{c}.\left(\vec{b}\times\vec{c}\right)$$

$$\Rightarrow \vec{r}.\left(\vec{b}\times\vec{c}\right) = x\left[\vec{a}\vec{b}\vec{c}\right] \Rightarrow x = \frac{\vec{r}.\left(\vec{b}\times\vec{c}\right)}{\left[\vec{a}\vec{b}\vec{c}\right]} = \vec{r}.\vec{a'}$$

NOTES

Similarly, $y = \vec{r}.\vec{b'}$ and $z = \vec{r}.\vec{c'}$

Substituting the values of x, y and z in Equation (1.14), we get

$$\vec{r} = \left(\vec{r}.\vec{a'}\right)\vec{a} + \left(\vec{r}.\vec{b'}\right)\vec{b} + \left(\vec{r}.\vec{c'}\right)\vec{c}$$

Example 1.16: Given $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k},\ \vec{b} = 2\hat{i} + \hat{j} - \hat{k},\ \vec{c} = \hat{i} + 3\hat{j} - k$, find the reciprocal triads $\vec{a'}, \vec{b'}, \vec{c'}$ and verify that $\left[\vec{a}, \vec{b}, \vec{c}\right]\left[\vec{a'}, \vec{b'}, \vec{c'}\right] = 1$.

Solution : $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 2 & -1 & 3 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2(-1+3) + 1(-2+1) + 3(6-1)$

$$= 4 - 1 + 15 = 18$$

Now, $\vec{a'} = \dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \dfrac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix}}{18} = \dfrac{1}{18}\left[2\hat{i} + \hat{j} + 5\hat{k}\right]$

$$\vec{b'} = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}}{18} = \frac{1}{18}\left[8\hat{i} - 5\hat{j} - 7\hat{k}\right]$$

$$\vec{c'} = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix}}{18} = \frac{1}{18}\left[-2\hat{i} + 8\hat{j} + 4\hat{k}\right]$$

Now, $\left[\vec{a'}, \vec{b'}, \vec{c'}\right] = \dfrac{1}{(18)^3}\begin{vmatrix} 2 & 1 & 5 \\ 8 & -5 & -7 \\ -2 & 8 & 4 \end{vmatrix}$

$$= \frac{1}{(18)^3}\left[2\left(-20+56\right) - \left(32-14\right) + 5\left(64-10\right)\right]$$

$$= \frac{1}{18\times18\times18}\left[72 - 18 + 270\right]$$

$$= \frac{324}{18\times18\times18} = \frac{1}{18}$$

$\therefore\ \left[\vec{a}\,\vec{b}\,\vec{c}\right]\left[a'\,b'\,c'\right] = 18\times\dfrac{1}{18} = 1$

NOTES

1.3.1 The Gauss or Divergence Theorem

The Gauss theorem states that the outward flux of a vector field through a surface to the behaviour of the vector field inside the surface.

The Gauss (or Divergence) Theorem

The Gauss theorem demonstrates equality between triple integral (volume integral) of a function over a region of three-dimensional space and double integral (surface integral) of the function over the surface that bounds the corresponding region. In vector calculus, the Gauss theorem is also known as Divergence Theorem (Refer Figure 1.11).

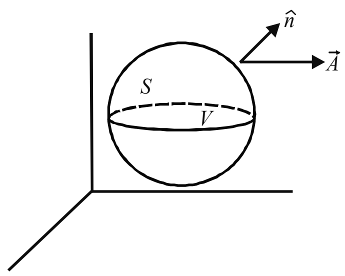

Fig. 1.11 *Region of Three-Dimensional Space*

Let $\vec{A}$ be a vector point function that is continuously differentiable on a closed-space region, V bounded by a closed surface S. Then,

$$\iiint_V \nabla.\vec{A}\, dV = \iint_S \vec{A}.\hat{n}\, dS$$

where $\hat{n}$ is the outwardly drawn unit normal vector to the surface *S*.

If we take $\hat{n}$ as direction cosines, i.e., $\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$,

then, $\vec{A}.\hat{n} = A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma$

∴ Gauss theorem can also be expressed as:

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma)\, dS$$

Check Your Progress

1. What is polar vector?
2. Differentiate between like and unlike vector.
3. Define the dot product.
4. What do you understand by gradient of any scalar function?
5. Define the vector triple product.
6. State the Gauss theorem.

1.4 VECTOR SPACES

NOTES

The motivating factor in rings was set of integers and in groups the set of all permutations of a set. A vector space originates from the notion of a vector that we are familiar with in mechanics or geometry. You would recall that a vector is defined as a directed line segment, which in algebraic terms is defined as an ordered pair (a, b) being coordinates of the terminal point relative to a fixed coordinate system. Addition of vectors is given by the rule:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

You can easily verify that set of vectors under this forms an abelian group. Also, scalar multiplication is defined by the rule $\alpha (a, b) = (\alpha a, \alpha b)$ which satisfies certain properties. This concept is extended similarly to three dimensions. You can generalize the whole idea through the definition of a vector space and vary the scalars not only in the set of reals but in any field F. A vector space thus differs from groups and rings in as much as it also *involves* elements from outside itself.

Definition: Let $< V, + >$ be an abelian group and $< F, +, \cdot >$ be a field. Define a function $\times$ (called scalar multiplication) from $F \times V \to V$, such that, for all $\alpha \in F$, $v \in V$, $\alpha \cdot v \in V$. Then V is said to form a *vector space* over F if for all $x, y \in V$, $\alpha, \beta \in F$, the following hold

(*i*) $(\alpha + \beta)\, x = \alpha x + \beta x$

(*ii*) $\alpha\, (x + y) = \alpha x + \alpha y$

(*iii*) $(\alpha\beta)\, x = \alpha\, (\beta x)$

(*iv*) $1 \cdot x = x$, 1 being unity of F.

Also then, members of F are called *scalars* and those of V are called *vectors*.

Note: You can use the same symbol + for the two different binary compositions of V and F, for convenience. Similarly, the same symbol, is used for scalar multiplication and product of the field F.

Since $< V, + >$ is a group, its identity element is denoted by 0. Similarly, the field F would also have zero element which will also be represented by 0. In case of doubt, you can use different symbols like 0_v and 0_F, etc.

Since you generally work with a fixed field, you would only be writing V as a vector space (or sometimes $V(F)$ or V_F). It would always be understood that it is a vector space over F (unless stated otherwise).

You have defined the scalar multiplication from $F \times V \to V$. You can also define it from $V \times F \to V$ and have a similar definition. The first one is called a left vector space and the second a right vector space. It is easy to show that if V as a left vector space over F, then it is a right vector space over F and conversely. In view of this result, it becomes redundant to talk about left or right vector spaces. We will consider about only vector spaces over F.

You can also talk about the above system when the scalars are allowed to take values in a ring instead of a field, which leads to the definition of modules.

Theorem 1.6: In any vector space V(F), the following results hold:

(*i*) $0.x = 0$

(*ii*) $\alpha.0 = 0$

(*iii*) $(-\alpha)x = -(\alpha x) = \alpha(-x)$

(*iv*) $(\alpha - \beta)x = \alpha x - \beta x,\ \alpha, \beta \in F,\ x \in V$

NOTES

Proof: (*i*) $0.x = (0 + 0).x = 0.x + 0.x$

$\Rightarrow 0 + 0.x = 0.x + 0.x$

$\Rightarrow 0 = 0.x$ (cancellation in V)

(*ii*) $\alpha.0 = \alpha.(0 + 0) = \alpha.0 + \alpha.0 \Rightarrow \alpha.0 = 0$

(*iii*) $(-\alpha)x + \alpha x = [(-\alpha) + \alpha]x = 0 . x = 0$

$\Rightarrow (-\alpha x) = -\alpha x$

(*iv*) follows from above.

The following examples illustrate Theorem 1.6

(i) If $< F, +, . >$ be a field, then F is a vector space over F as $< F, + > = < V, + >$ is an additive abelian group. Scalar multiplication can be taken as the product of F. All properties are seen to hold. Thus $F(F)$ is a vector space.

(ii) Let $< F, +, . >$ be a field

Let $V = \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in F\}$

Define + and . (scalar multiplication) by

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$$
$$\alpha(\alpha_1, \alpha_2) = (\alpha\alpha_1, \alpha\alpha_2)$$

You can check that all conditions in the definition are satisfied. Here $V = F \times F = F^2$

One can extend this to F^3 and so on. In general we can take n–tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in F$ and define F^n or $F^{(n)} = \{(\alpha_1, \alpha_2, ..., \alpha_n) \mid \alpha_i \in F\}$ as a vector space over F.

(iii) If $F \subseteq K$ be two fields then $K(F)$ will form a vector space, where addition of $K(F)$ is + of K and for any $\alpha \in F$, $x \in K$, $\alpha . x$ is taken as product of α and x in K.

Thus **C**(**R**), **C**(**C**), **R**(**Q**) would be some examples of vector spaces, where **C** = complex numbers., **R** = reals and **Q** = rationals.

(iv) Let V = set of all real valued continuous functions defined on [0, 1]. Then V forms a vector space over the field **R** of reals under addition and scalar multiplication defined by:

$$(f + g)x = f(x) + g(x) \quad f, g \in V$$
$$(\alpha f)x = \alpha f(x) \quad \alpha \in \mathbf{R} \quad \text{for all } x \in [0, 1]$$

It may be recalled here that sum of two continuous functions is continuous and scalar multiple of a continuous function is continuous.

(v) The set $F[x]$ of all polynomials over a field F in an indeterminate x forms a vector space over F with respect to, the usual addition of polynomials and the scalar multiplication defined by:

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For $\qquad f(x) = a_0 + a_1x + ... + a_nx^n \in F[x], \quad \alpha \in F$

$$\alpha.(f(x)) = \alpha a_0 + \alpha a_1 x + ... + \alpha a_n x^n.$$

(vi) $M_{m \times n}(F)$, the set of all $m \times n$ matrices with entries from a field F forms a vector space under addition and scalar multiplication of matrices.

We use the notation $M_n(F)$ for $M_{n \times n}(F)$.

(vii) Let F be a field and X a non-empty set.

Let $F^X = \{f \mid f : X \to F\}$, the set of all mappings from X to F. Then F^X forms a vector space over F under addition and scalar multiplication defined as follows:

For $\qquad f, g \in F^X, \alpha \in F$

Define $\qquad f + g : X \to F, \alpha F : X \to, F$ such that

$$(f + g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x) \qquad \forall\, x \in X$$

(viii) Let V be the set of all vectors in three dimensional space. Addition in V is taken as the usual addition of vectors in geometry and scalar multiplication is defined as:

$\alpha \in \mathbf{R}$, $\vec{v} \in V \Rightarrow \alpha \vec{v}$ is a vector in V with magnitude $|\alpha|$ times that of V. Then V forms a vector space over $\mathbf{R}$.

1.5 INTRODUCTION TO MATRIX

A set of mn numbers, real or complex, arranged in a rectangular array of m rows and n columns, such as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a **matrix** of order $m \times n$.

In other words, a scheme of detached coefficients a_{ij} arranged in m rows and n columns is called a **matrix of order** m by n or an $m \times n$ **matrix** or a **matrix of type** $m \times n$.

In case $m = n$, the rectangular array becomes a square and so the matrix having number of rows and number of columns equal is called a **Square Matrix** of order n.

Any matrix obtained by deleting any number of rows and any number of columns from a given matrix is said to be a **Sub-Matrix** of the given matrix.

The mn numbers a_{ij}, $(i = 1, 2, \ldots m; j = 1, 2, \ldots n, i \neq j)$ constituting the $m \times n$ matrix are called its **elements** or **constitutents**. The elements a_{ij} $(i = j)$ of a square matrix **A** are called its diagonal elements and their sum as **trace** of **A** denoted by $tr.\ \mathbf{A} = \sum_{i=1}^{n} a_{ii}$.

A matrix is usually denoted by capital letters like **A** (in clarendon type) or $[a_{ij}]$, where a_{ij} represents the (i, j)th element, *i.e.*, the element in the ith row and jth column of the matrix.

Thus an $m \times n$ matrix may be expressed as

$$\mathbf{A} = [a_{ij}] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{array}{l} \text{where } 1 \leq i \leq m \\ \text{and } 1 \leq j \leq n \\ \text{but } i \neq j \end{array}$$

We have so far used only a pair of brackets, *i.e.*, [ ] to denote a matrix, but a pair of parenthess**,** *i.e.*, ( ) and double bars, *i.e.*, $\| \; \|$, are also sometimes used to indicate a matrix.

A matrix having all of its elements zero is said to be a **null matrix** and denoted by **O**, *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix of order n having all its diagonal elements unity and zero elements everywhere else is called a **unit matrix** or an **identity matrix** and denoted by $\mathbf{I}_n$. Thus

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

It is possible that a matrix may have only a single row or a single column such as

$$\begin{bmatrix} a_1, & a_2, & \ldots & a_p \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix}$$

the first one being a matrix of order $1 \times p$ is called a **row matrix** and the second one being a matrix of order $q \times 1$ is called a **column matrix.**

A single element constitutes a matrix of order 1×1.

In relation to matrices, the numbers are usually known as scalars; for they behave as operators exactly like ordinary numbers as multipliers and hence are called scalars.

1.5.1 Addition of Matrices

[1] The Commutative Law

If **A** *and* **B** *are two matrices of the same order, say; $m \times n$, then*

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, $i = 1, 2, ...m$; $j = 1, 2, ...n$

We have

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]\\ &= [b_{ij} + a_{ij}] \text{ since } b_{ij} \text{ and } a_{ij} \text{ are scalars}\\ &= [b_{ij}] + [a_{ij}]\\ &= \mathbf{B} + \mathbf{A}\end{aligned}$$

i.e., the commutative law of addition holds.

[2] The Associative Law

If **A**, **B**, **C** *are three matrices of the same order, say*, $m \times n$, *then*

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$, and $\mathbf{C} = [c_{ij}]$, $i = 1, 2, ...m, j = 1, 2, ...n$

$$\begin{aligned}\text{We have}(\mathbf{A} + \mathbf{B}) + \mathbf{C} &= ([a_{ij}] + [b_{ij}) + [c_{ij}]\\ &= ([a_{ij} + b_{ij}] + [c_{ij}])\\ &= [(a_{ij} + b_{ij}) + c_{ij}]\\ &= [a_{ij} + (b_{ij} + c_{ij})],\ a_{ij}, b_{ij}, c_{ij}, \text{ being scalars}\\ &= [a_{ij}] + ([b_{ij} + c_{ij}])\\ &= \mathbf{A} + (\mathbf{B} + \mathbf{C})\end{aligned}$$

i.e., the associative law of addition holds.

[3] The Distributive Law

If **A** *and* **B** *are two matrices of the same order, say,* $m \times n$ *and* k *is a scalar, then*

$$k\,(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$$

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} =[b_{ij}]$, $i = 1, 2, ...m$; $j = 1, 2, ...n$

$$\begin{aligned}\text{We have, } k(\mathbf{A} + \mathbf{B}) &= k[a_{ij} + b_{ij}]\\ &= [k\,(a_{ij} + b_{ij})]\\ &= [ka_{ij}] + [kb_{ij}]\\ &= k[a_{ij}] + k[b_{ij}],\ k \text{ being a scalar}\\ &= k\mathbf{A} + k\mathbf{B}.\end{aligned}$$

i.e., the distributive law of addition holds.

[4] Existence of Additive Identity

If **A** *be a matrix of any order, say,* $m \times n$ *and* **O** *a null matrix of the same order such that when it is added to* **A**, *leaves it unchanged.*

i.e., $\mathbf{A} + \mathbf{O} = \mathbf{A}$

then **O** *is said to be the additive identity of* **A**.

Its proof immediately follows from the fact that if $A = [a_{ij}]$ and $\boldsymbol{O}$ is a null matrix *i.e.,* a matrix having each of its elements zero, then

$$\begin{aligned}\mathbf{A} + \mathbf{O} &= [a_{ij} + 0]\\ &= [a_{ij}] \text{ since a zero added to any scalar leaves it unchanged.}\\ &= \mathbf{A}\end{aligned}$$

Because of this fact **O** is said to be an additive identity of **A**.

[5] Existence of Additive Inverse

If **A** *be a matrix of any order say m × n, and there exists a matrix* –**A** *of the same order such that if it is added to* **A**, *gives a null matrix* **O**.

i.e. $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$

then (–**A**) *is said to be the additive inverse of* **A**.

$$\begin{aligned} \text{Let} \quad \mathbf{A} &= [a_{ij}] \\ \text{Then,} \quad -\mathbf{A} &= -[a_{ij}] = [-a_{ij}] \\ \text{So that} \quad \mathbf{A} + (-\mathbf{A}) &= [a_{ij}] + [-a_{ij}] \\ &= [a_{ij} - a_{ij}] \\ &= \mathbf{O} \end{aligned}$$

Because of this fact (–**A**) is said to be an additive inverse of **A**.

[6] **The Cancellation Law**

If **A**, **B**, **C** *are three matrices conformable for addition then the relation*

$$\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$$

holds if and only if $\mathbf{B} = \mathbf{C}$

Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$, and $\mathbf{C} = [c_{ij}]$, $i = 1, 2, ...m$

$j = 1, 2, ...n$

Then the relation **A** + **B** = **A** + **C** follows that (i, j)th elements on either side are identically equal, *i.e.,*

$$a_{ij} + b_{ij} = a_{ij} + c_{ij}$$

which yields $b_{ij} = c_{ij}$ since a_{ij}, b_{ij}, c_{ij} all are scalars.

i.e., (i, j)th element of **B** = (i, j)th element of **C** for all values of i and j.

As such **B** = **C**

Hence the relation **A** + **B** = **A** + **C** holds if and only if **B** = **C**.

1.5.2 Multiplication of Matrices

Two matrices **A** *and* **B** *are conformable for multiplication if and only if the number of columns in* **A** *is equal to the number of rows in* **B**. *The product of the two matrices* **A** *and* **B** *denoted by* **AB** *is then defined as the matrix whose elements in the ith row and jth column is the algebraic sum of the products of the elements in the ith row of* **A** *by the corresponding elements in the jth column of* **B**.

In other words, the product **AB** of two matrices conformable for multiplication, is the matrix whose element in ith row and jth column is the inner or scalar product of the ith row of **A** by the jth column of **B**, while the *inner product* or *scalar product* of two numbers x and y with components $x_1, x_2, ...x_n$ and $y_1, y_2, ...y_n$ is equal to $x_1y_1 + x_2y_2 + ... + x_ny_n$

It should be noted that inner product of two numbers with unequal number of components is not defined.

As an illustrative example if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

NOTES

then it is clear that the two matrices are conformable for multiplication since the number of columns in **A** is equal to the number of rows in **B**.

$$\therefore\ \mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

It is worth noting that the product **BA** is not defined, since the number of columns in **B** is not equal to the number of rows in **A**.

In the product **AB**, the matrix **A** is known as *Prefactor* and **B** as *Post factor*.

As an illustration in generalized form if

$\mathbf{A} = [a_{ij}]$, a matrix of order $m \times n$

$\mathbf{B} = [b_{ik}]$, a matrix of order $n \times p$

then $\mathbf{AB} = \mathbf{C}$ (say) is a matrix of order $m \times p$

i.e. $\mathbf{C} = [c_{ik}]$ is a matrix of order $m \times p$ such that

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

$$= a_{i1} + b_{1k} + a_{i2}\, b_{2k} + \cdots + a_{in}\, b_{nk}; i = 1, 2, ..., m \text{ and } j = 1, 2, ..., p$$

Thus

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

Corollary. *tr* (**AB**) = *tr* (**BA**), *all matrices being square of order n.*

If $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$ then $\mathbf{AB} = \mathbf{C}$ say $= [c_{ij}]$

where $$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

Let $\mathbf{BA} = \mathbf{D} = [d_{ij}]$ where $d_{ij} = \sum_{k=1}^{n} b_{ik}\, a_{kj}$

NOTES

Thus $$tr.\ (\mathbf{AB}) = \sum_i c_{ii} = \sum_i \left(\sum_k a_{ik} b_{ki} \right)$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} b_{ki}$$

(on interchanging the order of summation)

$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right)$$

$$= \sum_{k=1}^{n} d_{kk} = tr.\ (\mathbf{BA})$$

which proves the proposition.

Properties of Matrix-Multiplication

[1] The Commutative Law of Multiplication

Consider the matrices,

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ -3 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

These are conformable for multiplication and so

$$\mathbf{AB} = \begin{bmatrix} 1+0 & 0-2 & 2-6 \\ 2+0 & 0+3 & 4+9 \\ -3+0 & 0+1 & -6+3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 3 & 13 \\ -3 & 1 & -3 \end{bmatrix}$$

And $$\mathbf{BA} = \begin{bmatrix} 1+0-6 & -2+0+2 \\ 0+2-9 & 0+3+3 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ -7 & 6 \end{bmatrix}$$

It is apparent that the product matrix **AB** is of order 3×3 while the product matrix **BA** is of order 2×2 and therefore the two product matrices are quite different, *i.e.,*

$$\mathbf{AB} \neq \mathbf{BA}$$

This follows that the commutative law of multiplication does not hold.

Had the order of the matrices **AB** and **BA** been the same, then it would be possible that **AB** ≠ **BA** if every or at least one element in **AB** would differ from the corresponding element in **BA**. Though there are a few exceptions in which case the commutative law holds good. Such cases will be considered in the discussion of special matrices.

In fact, for a given pair of matrices **A** and **B** it is possible that the products **AB** and **BA** may not be conformable. For example if **A** is of order $m \times n$ and **B** of order $n \times p$, then the product **AB** is conformable and will be of order $m \times p$ while the product **BA** is not conformable since number of columns in **B** is not equal to number of rows in **A**. Thus the product **AB** exists while **BA** does not.

[2] The Distributive Law for Multiplication

NOTES

If **A**, **B**, **C** *be three matrices of suitable orders such that the products* **A**(**B** + **C**) *and* **AB**, **AC** *are conformable then*

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}.$$

Suppose that

$$\mathbf{A} = [a_{ij}] \text{ is of order } m \times n$$
$$\mathbf{B} = [b_{jk}] \text{ is of order } n \times p$$
and $$\mathbf{C} = [c_{jk}] \text{ is of order } n \times p$$

Then, (**B** + **C**) is of order $n \times p$ and **A** is of order $m \times n$ so that **A** (**B** + **C**) is conformable and of order $m \times p$. Also **AB** and **AC** both are of order $m \times p$ so that the sum matrix (**AB** + **AC**) is of order $m \times p$. Hence the matrices **A** (**B** + **C**) and **AB** + **AC** are of the same order so that they are comparable for equality.

Now,

$$(i, k)\text{th element of } \mathbf{A}(\mathbf{B}+\mathbf{C}) = \sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk})$$

$$= \sum_{j=1}^{n} a_{ij}\, b_{jk} + \sum_{j=1}^{n} a_{ij}\, c_{jk}$$

$= (i, k)$th element of **AB** + (i, k)th element of **AC**

$= (i, k)$th element of (**AB** + **AC**)

for all $i = 1, 2, ...m$ and $k = 1, 2, ...p$

$$\therefore \quad \mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$

A similar procedure will show that

$$(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$$

where **B**, **C** and **A** are of orders $m \times n$, $m \times n$ and $n \times p$ respectively.

Hence the matrix multiplication is distributive with respect to addition.

[3] The Associative Law for Multiplication

If **A**, **B**, **C** *be three matrices of suitable orders such that the products* (**AB**)**C** *and* **A**(**BC**) *are conformable then*

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

Suppose that

$$\mathbf{A} = [a_{ij}] \text{ is of order } m \times n$$
$$\mathbf{B} = [b_{jk}] \text{ is of order } n \times p$$
and $$\mathbf{C} = [c_{kl}] \text{ is of order } p \times q$$

Then, (**AB**) is of order $m \times p$ and **C** is of order $p \times q$ so that (**AB**)**C** is of order $m \times q$. Also (**BC**) is of order $m \times q$ and **A** is of order $m \times n$ so that **A**(**BC**) is of order $m \times q$. Hence the matrices (**AB**)**C** and **A**(**BC**) are of the same order so that they are comparable for equality.

Now, $$(i, k)\text{th element of } \mathbf{AB} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

so that, (i, l)th element of $(\mathbf{AB})\mathbf{C} = \sum_{k=1}^{p}\left\{\sum_{j=1}^{n} a_{ij}\, b_{jk}\right\} c_{kl}$

$$= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl}$$

NOTES

Also, (j, i)th element of $(\mathbf{BC}) = \sum_{k=1}^{p} b_{jk} c_{kl}$

so that (i, l)th element of $\mathbf{A}(\mathbf{BC}) = \sum_{j=1}^{n}\left\{\sum_{k=1}^{p} b_{jk}\, c_{kl}\right\} a_{ij}$

$$= \sum_{j=1}^{n}\ \sum_{k=1}^{p} a_{ij}\, b_{jk}\, c_{kl}$$

$$= \sum_{k=1}^{p}\ \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl}$$

$= (i, l)$th element of $(\mathbf{AB})\mathbf{C}$
for all $i = 1, 2, ...m$, $l = 1, 2,....p$

$$\therefore \qquad \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

Hence the matrix multiplication is associative.

[4] *If* **A** *be a matrix of order $m \times n$ and* **O** *a null matrix of order $n \times p$ then the product* **AO** *is another null matrix of order $m \times p$, i.e.,*

$$\mathbf{AO}_{n,\,p} = \mathbf{O}_{m,\,p}$$

Also if **O** be an $m \times n$ null matrix and **A** a matrix of order $n \times p$ then their product is a null matrix of order $m \times p$, *i.e.,*

$$\mathbf{O}_{m,\,n}\,\mathbf{A} = \mathbf{O}_{m,\,p}$$

Conclusively, if **A** be an n-rowed square matrix and **O** an n-rowed null matrix, then

$$\mathbf{AO} = \mathbf{OA} = \mathbf{O}$$

[5] *If the product of two matrices* **A** *and* **B** *is a null matrix then it is not essential that either of them is a null matrix, i.e.,*

If **AB** = **O**, it does not necessarily mean that at least one of **A** and **B** is a null matrix,

As an illustrative example if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

then $$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+1 & 1-1 \\ -2+2 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$$

i.e. the product of two non-zero matrices **A** and **B** is a zero matrix.

NOTES

[6] *If* **I** *be an m-rowed unit matrix and* **A** *an* $m \times n$ *matrix, then,*

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}$$

Suppose that $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$.

Then, $\mathbf{I}_m\mathbf{A}$ is a matrix of order $m \times n$ and so is comparable to $\mathbf{A}$.

Now, (i, j)th element of $\mathbf{I}_m\mathbf{A} = \sum_{k=1}^{m} (i, k)\text{th element of } \mathbf{I}_m \cdot a_{kj}$

But the (i, k)th element of $\mathbf{I}_m$ is zero except when $k = i$; therefore the right hand sum of the above expression will have only one term different from zero and that is the ith term which is equal to

$$(i, i)\text{th element of } \mathbf{I}_m.a_{ij} = 1.a_{ij}$$
$$= a_{ij}$$

$$\therefore (i, j)\text{th element of } \mathbf{I}_m\mathbf{A} = a_{ij}$$
$$= (i, j)\text{th element of } \mathbf{A}$$

from which it follows that

$$\mathbf{I}_m\,\mathbf{A} = \mathbf{A}$$

Similarly it may be shown that if $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{I}_n$ an n-rowed unit matrix, then

$$\mathbf{A}\,\mathbf{I}_n = \mathbf{A}$$

Note. This result shows the existence of a multiplicative identity.

[7] Positive Integral Powers of Square Matrices

If **A** *is a square matrix of order n* (*say*), *then*

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}$$

and the associative law of multiplication leads

$$\mathbf{A}^2\mathbf{A} = (\mathbf{A}\mathbf{A})\,\mathbf{A} = \mathbf{A}\,(\mathbf{A}\mathbf{A}) = \mathbf{A}\mathbf{A}^2$$

or

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A} = \mathbf{A}^2\mathbf{A} = \mathbf{A}\mathbf{A}^2$$

In the generalised form if p, q are two positive integers, then

$$\begin{aligned} \mathbf{A}^p\mathbf{A}^q &= (\mathbf{A}\mathbf{A} \,.........\, \mathbf{A}, \textit{p times})\,(\mathbf{A}\mathbf{A} \,.........\, \mathbf{A}, \textit{q times}) \\ &= \mathbf{A}\mathbf{A} \,..........\, \mathbf{A}, (p + q) \text{ times} \\ &= \mathbf{A}^{p+q} \end{aligned}$$

and

$$\begin{aligned} (\mathbf{A}^p)^q &= (\mathbf{A}\mathbf{A}.......\mathbf{A}, p \text{ times})^q \\ &= \mathbf{A}^q\mathbf{A}^q \,.......\, \mathbf{A}^q, p \text{ times} \\ &= \mathbf{A}^{pq} \end{aligned}$$

Corollary. *If* **I** *is a unit matrix of any order, then*

$$\mathbf{I}^2 = \mathbf{I}^3 = ... = \mathbf{I}^p = \mathbf{I}.$$

1.5.3 Inverse of a Matrix

Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation.

Proof: Let A be an $n \times n$ matrix with characteristic equation

$$|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0 \qquad \ldots(1.15)$$

NOTES

Here, we have to prove that A satisfies Equation (1.15), which means

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_{n-1}A + a_nI = O \qquad \ldots(1.16)$$

As the element of $(A-\lambda I)$ are atmost of the first degree in λ, thus the elements of adj.$(A-\lambda I)$ are atmost of degree $(n-1)$ in λ and hence adj.$(A-\lambda I)$ can be written as a matric polynomial of degree $(n-1)$ in λ, i.e,

$$\text{adj.}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-2}\lambda + B_{n-1} \qquad \ldots(1.17)$$

where $B_0, B_1, B_2, B_3, \ldots, B_{n-2}, B_{n-1}$ are $n \times n$ matrices.

We know that

$$(A-\lambda I)\,\text{adj.}(A-\lambda I) = |A-\lambda I|\,I \qquad \ldots(1.18)$$

Substituting the values from Equations (1.15) and (1.17) in (1.18), we get

$$(A-\lambda I)\left(B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-2}\lambda + B_{n-1}\right)$$
$$= \left(a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n\right)I \quad \ldots(1.19)$$

Comparing the coefficients of like powers of λ on both the sides of Equation (1.19), we get

$$-B_0 = a_0I;\ \ AB_0 - B_1 = a_1I;\ \ AB_1 - B_2 = a_2I;$$

....

$$AB_{n-2} - B_{n-1} = a_{n-1}I;\ AB_{n-1} = a_nI$$

Pre-multiplying the above equations by $A^n, A^{n-1}, A^{n-2}, \ldots, A^2, A$, I respectively and adding we get

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_{n-1}A + a_nI = O$$

This equation is same as Equation (1.16) and hence establishes the theorem.

Inverse of a Matrix using Cayley-Hamilton Theorem

According to Cayley-Hamilton Theorem, the square matrix A satisfies its characteristic equation

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_{n-1}A + a_nI = O$$

Pre-multiplying both sides by A^{-1}, we get

$$a_0A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_{n-1}I + a_nA^{-1} = O$$

$$\Rightarrow \qquad A^{-1} = \frac{-1}{a^n}\left(a_0A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_{n-1}I\right)$$

Example 1.17: Prove that $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ satisfies its characteristic equation.

Also find its inverse.

NOTES

Solution: The characteristic equation of given matrix A is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$$

By Cayley-Hamilton theorem,

$$A^3 - 6A^2 + 7A + 2I = 0$$

Now, $A^2 = A.A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$

and $A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$

$$\therefore A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, $\quad A^3 - 6A^2 + 7A + 2I = O \qquad \ldots(1.20)$

and hence A satisfies its characteristic equation.

To find the inverse, multiplying Equation (1.20) by A^{-1}, we get

$$A^2 - 6A + 7I + 2A^{-1} = O$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}\left[-A^2 + 6A - 7I\right]$$

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$$\Rightarrow \quad A^{-1} = \frac{1}{2}\left\{-\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} -5+6-7 & 0 & -8+12+0 \\ -2+0+0 & -4+12-7 & -5+6+0 \\ -8+12+0 & 0 & -13+18-7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

1.5.4 Adjoint of Matrix

Definition: The *adjoint* or *adjugate* of a square matrix is the transpose of a matrix whose elements are the cofactors of the original matrix. The adjoint of a matrix A is denoted by adj A.

$$\text{Thus, if } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example 1.18: Find adj A where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution: Here

$$\text{matrix of cofactors of } A = \begin{bmatrix} +\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -18 & 4 \\ -2 & 7 & -1 \\ 2 & 8 & -4 \end{bmatrix}.$$

$$\text{Hence adj } A = \begin{bmatrix} -2 & -2 & 2 \\ -18 & 7 & 8 \\ 4 & -1 & -4 \end{bmatrix}.$$

1.5.5 Transpose of Matrices

NOTES

If a matrix A is of the type $m \times n$, then the matrix obtained by interchanging the rows and the columns of A will be of the type $n \times m$ and is known as the **transpose** of the matrix A, denoted by A' or A^T. If A', B' be the transpose matrices of A, B respectively, then

(i) $(A')' = A$

(ii) $(A+B)' = A' + B'$ where A and B are conformable for addition.

(iii) $(kA)' = kA'$, where k is a scalar.

(iv) $(AB)' = B'A'$, where A and B are conformable for multiplication.

Check Your Progress

7. When V is said to form a vector space over F?
8. Define the square matrix of order n.

1.6 SPECIAL MATRICES

Matrices arise in many situations of mathematics. In representing rotation analytically, in solving systems of linear equations, in the study of extrema of functions of two or more variables and in many other ways matrices come up as natural habitats of the domain of mathematics.

Some Definitions

Definition 1: A matrix is a rectangular array (*i.e.,* arrangement) of objects. The number of rows and the number of columns are called its dimensions. The objects are called its entries. If the objects are real number, it is called a *real matrix*; if the objects are complex numbers, it is called a *complex matrix*. If the objects are polynomials then it is called a *polynomial matrix*. It is to be noted that the entries themselves can be matrices also.

For example, $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -3 \end{bmatrix}$ is a matrix with 2 rows and three columns. It is normally referred to as a 2 × 3-matrix, read as 2 by 3 matrix, by convention the number of rows coming first followed by the number of columns. The above matrix can be called an integral matrix as all the entries are integers. The matrix $\begin{bmatrix} x^2+1 & 2x-1 & x^3 \\ 3x+2 & 5 & x^4-1 \end{bmatrix}$ is a polynomial matrix as the entries are polynomials.

Of special significance are the matrices like [1, 2, 0], $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}$.

The first one has only one row and three columns. Such a matrix is called a *row matrix*. The second one is likewise a *column matirx* having 2 rows and 1 column. The third one is a square matrix having the same number of rows as the number of columns.

NOTES

Definition 2: A matrix whose number of rows equals the number of columns is called a *square matrix*.

A convention has it that for square matrices, we don't refer to the dimensions but to the order which is, by definition, is the number rows and is therefore also the number of columns.

Thus $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ is a square matrix of order 2.

In general a matrix will be referred to by capital letters such as A, B, U, V, etc. and their corresponding entries by a_{ij}, b_{ij}, u_{ij}, v_{ij}, etc. Thus $A = [a_{ij}]_{m \times n}$ stands for the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note a_{32} is the element which belongs to the 3rd row and the 2nd column. Similarly, a_{ij} is the element at the crossing of the i^{th} row and the j^{th} column.

Thus a square matrix A of order n should look like

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The elements $a_{11}, a_{22}, ..., a_{ii}, ..., a_{nn}$ are said to form the *principal diagonal* of the matrix and the elements $a_{1n}, a_{2n-1}, ..., a_{n1}$ are said to form its *second diagonal* of the square matrix A. The sum of the elements in the principal diagonal is called the *trace* of A and is denoted by tr A. Thus

$$\text{tr } A = a_{11} + a_{22} + ... + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

For the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -1 \\ 3 & 2 & 2 \end{bmatrix}$, tr $A = 2 + 4 + 2 = 8$.

Definition 3: A matrix is called a *zero matrix*, if every element of it is zero.

Thus $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of dimensions 2×3.

A square matrix is called an *identity matirx* if all its principal diagonal elements are 1 and its non-diagonal elements are zeros.

Thus $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

A square matrix is called a *diagonal matrix* if all elements other than the principal diagonal elements are zero and at least one of the principal diagonal elements is non-zero.

Thus $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix; so also $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

NOTES

Definition 4: Two matrices A and B are said to be equal if they have the same dimensions and the corresponding elements are equal.

For example, $\begin{bmatrix} 2 & a & 3 \\ 4 & 0 & b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & 1 \end{bmatrix}$ implies $a = 0, b = 1$.

A square matrix is called a *scalar matrix* if every non-principal diagonal elements is zero and all principal diagonal elements are non-zero and equal. Thus

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix but $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ is not a scalar matrix.

Note that the identity matrix is also a scalar matrix.

A square matrix is called an *upper triangular* if all the elements below the principal diagonal are zero and at least one above the principal diagonal is non-zero.

Note $A = [a_{ij}]$ is uppertriangular if $a_{ij} = 0$ for all $i > j$

Clearly $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ is an upper triangular matrix.

1.6.1 Symmetric and Skew-Symmetric Matrices

A square matrix is called a *lower traingular* matrix if all the elements above the principal diagonal are zero and at least one below it is non-zero.

Note $A = [a_{ij}]$ is lower triangular if $a_{ij} = 0$ for all $i < j$.

Thus $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 2 & 3 \end{bmatrix}$ is a lower triangular matrix.

A square matrix is called *symmetric* if the elements situated symmetrically with respect to the principal diagonal are equal.

Note a square matrix which is also, a zero matrix is symmetric.

Thus $A = [a_{ij}]$ is symmetrix if $a_{ij} = a_{ji}$ for all i and j.

Clearly $A = \begin{bmatrix} 4 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{bmatrix}$ is symmetric, as the elements $a_{12} = a_{21} = 3$, $a_{31} = a_{13} = 5$, $a_{23} = a_{32} = -1$, the elements $\{a_{12}, a_{21}\}$, $\{a_{31}, a_{13}\}$ and $\{a_{23}, a_{32}\}$ being symmetrically situated with respect to the principal diagonal.

A square matrix is called *skew symmetric* if the elements situated symmetrically with respect to the principal diagonal have opposite signs but the same magnitude and the principal diagonal elements are zero. Thus a matrix $A = [a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

Clearly, $\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & 3 \\ 5 & -3 & 0 \end{bmatrix}$ is skew-symmetric since $a_{12} = -a_{21}$, $a_{13} = -a_{31}$, $a_{31} = -a_{13}$ and $a_{11} = a_{22} = a_{33} = 0$.

Note a square matrix which is also a zero matrix is skew symmetric.

Operations on Matrix

With matrices we can perform a number of operations, defined below:

Transposing: If A is an $m \times n$-matrix, then its transpose is an $n \times m$-matrix obtained by converting rows into corresponding columns. The transpose of A is denoted by A^t. Clearly, if $A = [a_{ij}]_{m \times n}$, then $A^t = [a_{ji}]_{n \times m}$.

Thus if $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ 3 & -2 \end{bmatrix}$

Note:

(1) $(A^t)^t = A$.

(2) The tranpose of an upper triangular matrix is a lower triangular matrix.

(3) The transpose of a symmetric matrix is the matrix itself, *i.e.*, $S^t = S$ if S is symmetric.

(4) The trace of a square matrix remains invariant order transposition *i.e.*, tr (A^t) = tr (A).

(5) The transpose of an indentity matrix is the matrix itself.

Addition: If A and B have the same dimensions, then the sum $A + B$ is defined as a matrix of the same dimensions obtained by adding the corresponding elements. Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 8 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 & -1 \\ 1 & 3 & 2 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 8+(-3) & 2+2 & 3+(-1) \\ 4+1 & 0+3 & 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 2 \\ 5 & 3 & 7 \end{bmatrix}.$$

Observe that (1) $A + B = B + A$ where A and B are real matrices.

(2) $(A + B)^t = A^t + B^t$.

(3) $A + 0 = A = 0 + A$ where 0 is the zero matrix of the same dimensions of those of A.

Subtraction: If A and B are of the same dimensions, then $A - B$ is a matrix of the same dimensions, obtained by subtracting the elements of B from the corresponding elements of A.

Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, then $A - B = [a_{ij} - b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then $A - B = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$.

Observe that (1) $A - B \neq B - A$.

(2) $(A - B)^t = A^t - B^t$.

(3) $A - 0 = A$ where 0 is the zero matrix.

NOTES

Scalar Multiplication: If real numbers are taken as scalar and $\lambda \in \mathbb{R}$, then the scalar multiplication λA is a matrix of the same dimensions of that of $\mathbb{A}$ and is obtained by multiplying each element of A by λ. Thus, if $A = [a_{ij}]_{m \times n}$, then $\lambda A = [\lambda a_{ij}]_{m \times n}$

For example, if $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$, then $2A = \begin{bmatrix} 4 & 2 & 6 \\ 0 & 8 & 4 \end{bmatrix}$.

Observe that (1) $\lambda(A \pm B) = \lambda A \pm \lambda B$ where A and B are two matrices.

(2) $(\lambda A)^t = \lambda A^t$.

(3) Every scalar matric is of the form λI.

(4) The transpose of a skew-symmetric matrix is negative of the matrix, *i.e.*, if T is skew-symmetric, $T^t = -T = (-1)T$.

A result of importance is the following:

Theorem 1.7: Every square matrix can be expressed as the sum of two matrices, one of which is symmetric and the other skew-symmetric.

Proof: Let A be a square matrix.

Then we can write

$$A = \frac{1}{2}.2A = \frac{1}{2}[A + A] = \frac{1}{2}(A + A^t + A - A^t)$$

i.e., $$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

$$= S + T, \text{ writing } S = \frac{1}{2}(A + A^t), T = \frac{1}{2}(A - A^t).$$

If now suffices to prove that S is symmetric and T is skew-symmetric. To this end, we observe

$$S^t = \left[\frac{1}{2}(A + A^t)\right]^t = \frac{1}{2}(A + A^t)^t = \frac{1}{2}(A^t + A) = \frac{1}{2}(A + A^t) = S.$$

So S is symmetric

$$T^t = \left[\frac{1}{2}(A - A^t)\right]^t = \frac{1}{2}(A - A^t)^t = \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t) = -T.$$

So T is skew-symmetric.

Hence the proof.

Example 1.19: If $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, verify

(*i*) $(A + B)^t = A^t + B^t$ (*ii*) $(A - B)^t = A^t - B^t$

(*iii*) $(3A)^t = 3A^t$ (*iv*) $(2A + 3B)^t = 2A^t + 3B^t$

(*v*) $A + B = B + A$.

Solution: (*i*) and (*ii*) Here

$$A + B = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix}, A - B = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & 2 \end{bmatrix}, A^t = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}, B^t = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

NOTES

$$\therefore \quad (A+B)^t = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, (A-B)^t = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 3 & 2 \end{bmatrix},$$

$$A^t + B^t = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, A^t - B^t = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 3 & 2 \end{bmatrix}.$$

Clearly, $(A+B)^t = A^t + B^t$ and $(A-B)^t = A^t - B^t$.

(*iii*) Now, $3A = 3\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 6 & -3 & 9 \end{bmatrix}$

$$\therefore \quad (3A)^t = \begin{bmatrix} 3 & 6 \\ 3 & -3 \\ 6 & 9 \end{bmatrix}, 3A^t = 3\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & -3 \\ 6 & 9 \end{bmatrix}$$

Clearly, $(3A)^t = 3A^t$.

(*iv*) Now, $2A + 3B = 2\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 & 4 \\ 4 & -2 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & -3 \\ 3 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 & 1 \\ 7 & 4 & 9 \end{bmatrix}$$

$$\therefore (2A+3B)^t = \begin{bmatrix} 11 & 7 \\ 2 & 4 \\ 1 & 9 \end{bmatrix}$$

Again, $2A^t + 3B^t = 2\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} + 3\begin{bmatrix} 9 & 3 \\ 0 & 6 \\ -3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 11 & 7 \\ 2 & 4 \\ 1 & 9 \end{bmatrix}$$

Clearly, $(2A+3B)^t = 2A^t + 3B^t$.

(*v*) Now $B + A = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \text{evidently } A+B = B+A.$$

Example 1.20: Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

$$(i) \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \quad (ii) \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (iii) \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} \quad (iv) \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}.$$

Solution: (*i*) Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$.

$$\therefore \quad A^t = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}.$$

NOTES

Hence $\frac{1}{2}(A+A^t) = \frac{1}{2}\begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} = S$, say and

$\frac{1}{2}(A-A^t) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T$, say

Clearly, $A = S + T$, where S is symmetric and T is skew-symmetric.

(*ii*) Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then $B^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

Hence $\frac{1}{2}(B+B^t) = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} = P$, say.

and $\frac{1}{2}(B-B^t) = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = Q$, say.

Clearly $B = P + Q$, where P is symmetric and Q is skew-symmetric.

(*iii*) Let $C = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$. Then $C^t = \begin{bmatrix} 0 & 4 & -3 \\ -4 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$.

Hence $\frac{1}{2}(C+C^t) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$, say.

and $\frac{1}{2}(C-C^t) = \frac{1}{2}\begin{bmatrix} 0 & -8 & 6 \\ 8 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} = V$, say.

Clearly, $C = U + V$, where U is symmetric and V is skew-symmetric.

(*iv*) Let $D = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}$. Then $D^t = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}$.

Hence $\frac{1}{2}(D+D^t) = \frac{1}{2}\begin{bmatrix} 6 & -2 & 10 \\ -2 & 4 & 8 \\ 10 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix} = S$, say.

$\frac{1}{2}(D-D^t) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T$, say.

Clearly, $D = S + T$, where S is symmetric, and T is skew-symmetric.

Example 1.21: Given an example of a matrix which is,

(*i*) symmetric but not skew-symmetric

(*ii*) skew-symmetric but not symmetric

(*iii*) both symmetric and skew-symmetric

(*iv*) neither symmetric nor skew-symmetric.

Solution:

$(i)\begin{bmatrix}2&1&3\\1&0&2\\3&2&5\end{bmatrix}$ $(ii)\begin{bmatrix}0&3&5\\-3&0&4\\-5&-4&0\end{bmatrix}$ $(iii)\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $(iv)\begin{bmatrix}2&7&4\\4&1&3\\6&2&3\end{bmatrix}$

NOTES

Product of Matrices: If A and B are matrix, the product AB is defined only when the number of columns of A equals the number of rows of B. This condition is called the conformabitity condition for multiplication. Thus, if A and B are conformable for the product AB, then the product is a matrix obtained by the following rule:

If $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$, then $AB = [c_{ik}]$ where $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$

For example, if

$A = \begin{bmatrix}2&1&3\\4&0&2\end{bmatrix}$, $B = \begin{bmatrix}1&1&0\\2&-1&2\\1&3&4\end{bmatrix}$, A and B are conformable for the product AB and clearly AB has the dimensions 2×3.

$$\text{Now } AB = \begin{bmatrix}2\times1+1\times2+3\times1 & 2\times1+1\times(-1)+3\times3 & 2\times0+1\times2+3\times4\\ 4\times1+0\times2+2\times1 & 4\times1+0\times(-1)+2\times3 & 4\times0+0\times2+2\times4\end{bmatrix}$$

$$= \begin{bmatrix}7&10&14\\6&10&8\end{bmatrix}.$$

Observe that to obtain the element c_{23}, the 2nd row of A and 3rd column of B have been multiplied by coordinate-wise and then added (*i.e.* dot product has been taken). Similarly other entries of C are calculated.

Observe that for any three matrices A, B and C and a scalar λ:

(1) $AB \neq BA$. [*Non-commutativity*]

(2) $A(B \pm C) = AB \pm AC$ [*Left Distributivity*]

(3) $\lambda(AB) = (\lambda A)B = A(\lambda B)$ [*Homogeneity*]

(4) $(A + B)C = AC + BC$ [*Right Distributivity*]

(5) $AB = BA \Leftrightarrow A^tB^t = B^tA^t$

(6) $AI = IA = A$ [*Identity Property*]

(7) $(AB)C = A(BC)$ [*Associativity*]

(8) $(AB)^t = B^tA^t$.

Example 1.22: If A and B be two symmetric matrices of the same order, then prove that

(*i*) $A + B$ is symmetric.

(*ii*) AB is symmetric if and only if $AB = BA$.

Solution: Since A and B are symmetric matrices, then $A^t = A$, $B^t = B$

(*i*) Now $(A + B)^t = A^t + B^t = A + B$

Hence $A + B$ is symmetric.

(*ii*) Let AB be symmetric, then $(AB)^t = AB$

$\therefore \quad AB = (AB)^t = B^tA^t = BA$

Conversely let $AB = BA$, then $(AB)^t = B^tA^t = BA = AB$

Hence AB is symmetric.

Example 1.23: If A and B be commuting matrices, then prove that A^t and B^t commute.

NOTES

Solution: Since A and B commute, then $AB = BA$.

Now $\quad A^tB^t = (BA)^t = (AB)^t \quad (\because AB = BA)$

$\qquad = B^tA^t$

Hence A^t and B^t commute.

Example 1.24: If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, then prove that $A^2 - 2A + I_2 = 0$. Hence find A^{50}.

Solution: Now $A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

$$\therefore A^2 - 2A + I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Here $\quad A^2 = 2A - I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$$\therefore A^3 = A.A^2 = A(2A - I_2) = 2A^2 - A = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$A^4 = A^2.A^2 = A^2(2A - I_2) = 2A^3 - A^2 = \begin{pmatrix} 2 & 0 \\ -6 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

and so on.

Hence $\quad A^{50} = \begin{pmatrix} 1 & 0 \\ -50 & 1 \end{pmatrix}$.

Example 1.25: (*i*) If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, then find AB or BA if they exist.

(*ii*) If A and B are square matrices of the same order, does the equality $(A + B) \times (A - B) = A^2 - B^2$ hold good? Give reasons.

Solution: (*i*) Since A is a 3×4 matrix and B is a 3×3 matrix, then the number of columns of $A \neq$ number of rows of B.

Hence AB is not defined.

But the number of columns of B = number of rows of A.

Hence BA is defined.

NOTES

$$\therefore \quad BA = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+0 & 4+0+0 & 6+1+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 9+2+0 & 12+4+5 \\ 1+0+3 & 2+0+1 & 3+0+0 & 4+0+5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{pmatrix}.$$

(*ii*) Now $(A + B)(A – B) = A(A – B) + B(A – B) = AA – AB + BA – BB$

$= A^2 – AB + BA – B^2$

If $AB = BA$, then only the equality $(A + B)(A – B) = A^2 – B^2$ holds good otherwise not.

Example 1.26: (*i*) If A, B and C are matrices of appropriate order with $AB = AC$, then does it imply that $B = C$? Give an example in support of your conclusion.

(*ii*) If A, B and C are three matrices of order n such that $AB = I_n$ and $BC = I_n$, then prove that $A = C$.

Solution: (*i*) If A, B and C three matrices of appropriate order with $AB = AC$, then it does not necessarily imply that $B = C$.

For example, let $A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

and $\quad C = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix}$

Now $AB = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 11 & 0 & -5 \end{pmatrix}$

and $AC = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 11 & 0 & -5 \end{pmatrix}$

$\therefore \quad AB = AC$ But $B \neq C$.

(*ii*) Now $AB = I_n$

$\therefore \quad (AB)C = (I_n)C$

$\Rightarrow \quad A(BC) = C$ [$\because$ matrix product is associative and $I_nC = C$]

$\Rightarrow \quad AI_n = C$

$\Rightarrow \quad A = C \qquad [\because AI_n = A]$

1.6.2 Hermitian And Skew-Hermitian Matrix

Hermitian Matrix

A square matrix $A = [a_{ij}]$ over the complex numbers is said to be **Hermitian** if the transposed conjugate of the matrix is equal to the matrix itself i.e., $A^{\theta} = A$.

Suppose A = $[a_{ij}]$ is of the type $m\times n$, then $A^{\theta} = [\alpha_{ij}]$ will be of the type $n\times m$ where $\alpha_{ij} = \overline{\alpha}_{ji}$

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So, for the matrix A to be Hermitian, $m = n$ and $a_{ij} = \overline{\alpha_{ji}}$ for all i and j, i.e., (i, j)th element of A = complex conjugate (j, i) th element of A.

For example $\begin{bmatrix} 0 & 7-2i \\ 7+2i & 4 \end{bmatrix}$, $\begin{bmatrix} 12 & 1+i & i \\ 1-i & 9 & 5+4i \\ -i & 5-4i & 0 \end{bmatrix}$ are Hermitian matrices.

Corollary: A Hermitian matrix has all its diagonal elements as real numbers.

Proof: Let A be a Hermitian matrix.

$\therefore\ a_{ij} = \overline{a_{ji}}$ for all i and j.

Putting $j = i$ for the diagonal elements, we have

$a_{ii} = \overline{a_{ii}}$ for all i

$\Rightarrow \alpha + i\beta = \alpha - i\beta\ [\because\ a_{ii} = \alpha + i\beta \Rightarrow \overline{a_{ii}} = \alpha - i\beta\ ]$

$\Rightarrow 2i\beta = 0 \Rightarrow \beta = 0$

$\therefore\ a_{ii} = \alpha + i \times 0 = \alpha$

Thus, the diagonal elements of a Hermitian matrix are real numbers.

Note: If a Hermitian matrix has all its elements as real, then it becomes a symmetric matrix.

Theorem 1.8: The necessary and sufficient condition for a matrix A to be Hermitian is that $A^{\theta} = A$.

Proof: Condition is necessary: Let A = $[a_{ij}]$ be a Hermitian matrix of order n.

$\therefore\ a_{ij} = \overline{a_{ji}}$ for all i and j

Since A is of the type $n\times n$, thus A^{θ} is of the type $n\times n$.

$\Rightarrow$ A and A^{θ} are of the same type.

Now, (i, j)th element of A = (j, i)th element of $\overline{A}$

= (i, j)th element of $(\overline{A})'$ = (i, j)th element of A^{θ}

$\therefore$ By equality of matrices A = A^{θ}

Condition is sufficient: Let us assume that A = A^{θ}

$\therefore (i, j)$th element of A = (i, j)th element of A^{θ}

$\Rightarrow\ a_{ij} = \overline{a_{ji}}$ for all i and j

Thus, A is a Hermitian matrix.

Skew-Hermitian Matrix

A square matrix A = $[a_{ij}]$ over the complex numbers is said to be **Skew-Hermitian** if the transposed conjugate of the matrix is equal to the negative of the matrix itself i.e., $A^{\theta} = -A$.

Suppose A = $[a_{ij}]$ is of the type $m \times n$, then $A^{\theta} = [\alpha_{ij}]$ will be of the type $n \times m$ where $\alpha_{ij} = -\overline{a_{ji}}$

So, the matrix to be Skew-Hermitian we must have $m = n$ and $\overline{a_{ji}} = -a_{ij}$ for all i and j.

i.e., (i, j)th element of A = negative of the complex conjugate (j, i) th element of A.

Corollary: Skew-Hermitian matrix has all its diagonal elements as either purely imaginary or zero.

Proof: Let A be a Skew-Hermitian matrix.

$\therefore\ a_{ij} = -\overline{a_{ji}}$ for all i and j.

Putting $j = i$ for the diagonal elements, we have

$a_{ii} = -\overline{a_{ii}}$ for all i

$\Rightarrow \alpha + i\beta = -(\alpha - i\beta)$ [$\because$ If $a_{ii} = \alpha + i\beta$ then $\overline{a_{ii}} = \alpha - i\beta$]

$\Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

$\therefore\ a_{ii} = 0 + i\beta = i\beta$

Thus, the diagonal elements of a Skew-Hermitian matrix are purely imaginary or zero.

Note: If a Skew-Hermitian matrix has all its elements as real, then it becomes a skew-symmetric matrix.

Theorem 1.19: The necessary and sufficient condition for a matrix A to be Skew-Hermitian is that $A^{\theta} = -A$.

Proof: Condition is necessary: Let A = $[a_{ij}]$ be a Skew-Hermitian matrix of order n

$\therefore\ a_{ij} = -\overline{a_{ji}}$ for all i and j

$\because$ A is of the type $n \times n \Rightarrow -A$ is of the type $n \times n$

And A^{θ} is of the type $n \times n$

$\Rightarrow -A$ and A^{θ} are of the same type.

NOTES

Now, (i, j)th element of A^θ = complex conjugate of (i,j)th element of A'

$= \overline{a_{ji}}$ for all i and j

$= -a_{ij}$ for all i and j = (i, j)th element of $-A$

$\therefore$ By equality of matrices $A^\theta = -A$

Condition is sufficient: Let us assume that $A^\theta = -A$

$\therefore (i, j)$th element of A = $-(i, j)$th element of A^θ

$= -(j,i)$th element of $\overline{A} = -\overline{a_{ji}}$ for all i and j

$\Rightarrow a_{ij} = -\overline{a_{ji}}$ for all i and j

Thus, A is a Skew-Hermitian matrix.

Example 1.27: If A is a square matrix then prove that

(i) $A + A^\theta$ is Hermitian matrix.

(ii) $A - A^\theta$ is Skew-Hermitian matrix.

Solution: (i) Consider $\left(A + A^\theta\right)^\theta = A^\theta + \left(A^\theta\right)^\theta$

$= A^\theta + A \left[\because \left(A^\theta\right)^\theta = A\right]$

$= A + A^\theta$

Thus, $A + A^\theta$ is Hermitian matrix.

(ii) Consider $\left(A - A^\theta\right)^\theta = A^\theta - \left(A^\theta\right)^\theta$

$= A^\theta - A \left[\because \left(A^\theta\right)^\theta = A\right]$

$= -\left(A - A^\theta\right)$

Thus, $A - A^\theta$ is Skew-Hermitian matrix.

Example 1.28: Every square matrix A can be expressed in one and only one way as P + iQ, where P and Q are Herimitian matrices.

Solution. We have $A = \frac{1}{2}(2A) = \frac{1}{2}\left[A + A^\theta + A - A^\theta\right]$

$$= \frac{1}{2}\left(A + A^\theta\right) + \frac{1}{2}\left(A - A^\theta\right)$$

$$= \frac{1}{2}\left(A + A^\theta\right) + i.\frac{1}{2i}\left(A - A^\theta\right) = P + iQ$$

where $P = \frac{1}{2}\left(A + A^{\theta}\right)$ and $Q = \frac{1}{2i}\left(A - A^{\theta}\right)$

Now, $P^{\theta} = \frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left(A^{\theta} + A\right) = P$

NOTES

And $Q^{\theta} = -\frac{1}{2i}\left(A - A^{\theta}\right)^{\theta} = -\frac{1}{2i}\left(A^{\theta} - A\right) = \frac{1}{2i}\left(A - A^{\theta}\right) = Q$

Thus both P and Q are Hermitian.

Hence, A can be expressed as P +iQ, where P and Q are Hermitian matrices

Now, let us prove that this expression of A is unique.

Suppose, if possible A= R + iS be antother expression for A where R and S both are Hermitian.

We shall prove that R = P and S = Q

Now, $A^{\theta} = (R + iS)^{\theta} = R^{\theta} + \bar{i}S^{\theta} = R^{\theta} - iS^{\theta} = R - iS$

$\left[\because \text{R and S are Hermitian} \Rightarrow R^{\theta} = R; S^{\theta} = S\right]$

$\therefore A + A^{\theta} = (R + iS) + (R - iS) = 2R \Rightarrow R = \frac{1}{2}\left(A + A^{\theta}\right) = P$

Also, $A - A^{\theta} = (R + iS) - (R - iS) = 2iS \Rightarrow S = \frac{1}{2i}\left(A - A^{\theta}\right) = Q$

Hence, the expression for A is unique.

Example 1.29: Prove that every Hermitian matrix A can be written as $A = B + iC$, where B is real and symmetric and C is real and skew-symmetric.

Solution. We know that, if $z = a + ib$ is a complex number, then

$\frac{1}{2}\left(z + \bar{z}\right)$ and $\frac{1}{2i}\left(z - \bar{z}\right)$ are both real.

Similarly, for Hermitian matrix A, $\frac{1}{2}\left(A + \bar{A}\right)$ and $\frac{1}{2i}\left(A - \bar{A}\right)$ are real matrices.

Now, $A = \frac{1}{2}(2A) = \frac{1}{2}\left(A + \bar{A}\right) + \frac{1}{2}\left(A - \bar{A}\right)$

$= \frac{1}{2}\left(A + \bar{A}\right) + i.\frac{1}{2i}\left(A - \bar{A}\right) = B + iC$

where $B = \frac{1}{2}\left(A + \bar{A}\right), C = \frac{1}{2i}\left(A - \bar{A}\right)$

Now, to show that B is symmetric and C is skew-symmetric.

We have, $B' = \left[\frac{1}{2}\left(A + \bar{A}\right)\right]' = \frac{1}{2}\left(A + \bar{A}\right)'$

NOTES

$$= \frac{1}{2}\left[A' + \left(\overline{A}\right)'\right] = \frac{1}{2}\left(A' + A^{\theta}\right) = \frac{1}{2}\left[\left(A^{\theta}\right)' + A\right] \qquad \left[\because A^{\theta} = A\right]$$

$$= \frac{1}{2}\left[\left\{\left(\overline{A}\right)'\right\}' + A\right] = \frac{1}{2}\left(\overline{A} + A\right) = B$$

Thus, B is symmetric.

$$\text{And } C' = \left[\frac{1}{2i}\left(A - \overline{A}\right)\right]' = \frac{1}{2i}\left(A - \overline{A}\right)' = \frac{1}{2i}\left[A' - \left(\overline{A}\right)'\right]$$

$$= \frac{1}{2i}\left(A' - A^{\theta}\right) = \frac{1}{2i}\left[\left(A^{\theta}\right)' - A\right] \left[\because A^{\theta} = A\right]$$

$$= \frac{1}{2i}\left(\overline{A} - A\right) = -\frac{1}{2i}\left(A - \overline{A}\right) = -C$$

$\therefore$ C is skew symmetric.

1.7 HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR EQUATION

A system of m linear equations in n variables is given by,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots \qquad \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m.$$

Or $\quad AX = B$ in matrix form where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The system is called *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$. Therefore the system is called *non-homogeneous* if it is not homogeneous. *i.e.* at least one of $b_1, b_2, \ldots, b_m$ is non-zero.

A system of non-homogeneous linear equations is said to be *consistent* if the system has at least one solution. *i.e.*, there exists a set of values of $x_1, x_2, \ldots, x_n$ which satisfies all the equations of the system. A system of non-homogeneous linear equations is called *inconsistent* if the system has no solution.

For example, the system

$$2x + 3y - 2z = 5$$
$$x - y + 4z = 0$$

is a consistent system of non-homogeneous linear equations but the system

$$x - y + 2z = 2$$

$$2y - z = 1$$
$$x + y + z = 4$$

is inconsistent as the system has no solution. Note that the former system has a solution (1, 1, 0). Indeed, the system has infinitely many solutions.

NOTES

The following observations are immediate.

Observation 1: Every homogeneous system of linear equations is consistent.

Observation 2: A system of linear non-homogeneous equations has either no solution or one solution or infinitely many solutions. Indeed, if such a system has two solutions, then it can be proved easily that it has infinitely many solutions.

Observation 3: The set of solutions of a system of homogeneous linear equations is a vector sapce over $\mathbb{R}$, called the solution space of the system.

We state here three very important results without proof.

Theorem 1.10: A system of non-homogeneous linear equations $AX = B$ is consistent if and only if rank (A) = rank ([A, B]) where [A, B], called the *augmented matrix*, stands for the matrix formed of the columns of A and B; the matrix A is called the *coefficient matrix*.

Theorem 1.11: If $AX = 0$ represent a system of homogeneous linear equations, then rank (A) + dim (S) = n

where S denotes the solution space of the system.

Theorem 1.12: (*a*) A system of homogeneous linear equations $AX = 0$ containing n equations in n unknowns has a non-zero solution if and only if rank (A) < n.

(*b*) A system of non-homogeneous linear equations $AX = B$ containing n equations in n unknowns has a unique solution if and only if det $A \neq 0$.

Example 1.30: Discuss the consistency of the following systems of equations

(*i*) $x - y = 1$, $2x + y = 5$ (*ii*) $x - y = 1$, $2x - 2y = 2$

(*iii*) $x - y = 1$, $-3x + 3y = 3$.

Solution: (*i*) Writing the system in the matrix form $AX = B$ we get

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } \overline{A} = [A, B] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

Clearly, rank (A) = rank($\overline{A}$) = 2.

Hence the system is consistent.

[Note that det $A \neq 0$ implies the solution is unique].

(*ii*) Here

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \quad \text{and } \overline{A} = [A, B] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}.$$

Clearly, rank(A) = rank ($\overline{A}$) = 1.

Hence the system is consistent.

NOTES

[Note that det $A = 0$ here implies that there are infinitely many solutions].

(*iii*) Here

$$A = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix},\ \bar{A} = [A, B] = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & 3 \end{bmatrix}$$

Clearly rank $(A) = 1$ but rank $(\bar{A}) = 2$.

Hence the system is inconsistent.

Example 1.31: Determine the number of independent solutions in the following systems.

$$(i) \quad \begin{aligned} x - y - z &= 0 \\ 2x + z &= 0 \\ x + y + 2z &= 0 \end{aligned} \qquad (ii) \quad \begin{aligned} x - 2y + z &= 0 \\ 3x - 6y + 3z &= 0 \\ -2x + 4y - 2z &= 0. \end{aligned}$$

Solution: (*i*) The given system consists of 3 equations in 3 unknowns. Writing the system in the matrix from $AX = 0$ we get

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and rank } (A) = 2$$

Let S denote the solution space.

Hence dim $(S) = 1$, as rank (A) + dim $(S) = 3$.

This implies that the basis of S contains only one independent vector as solution.

(*ii*) The given system consists of 3 equations in 3 unknowns. Writing the system as $AX = 0$ we get

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -6 & 3 \\ -2 & 4 & -2 \end{bmatrix} \quad \text{and rank } (A) = 1.$$

Hence dim $(S) = 2$. This implies that the basis of S contains two independent solutions.

Example 1.32: Solve, if possible.

$$(i) \quad \begin{aligned} x + 4y &= 2 \\ 2x + 7y &= 5 \\ 4x + 9y &= 10 \end{aligned} \qquad (ii) \quad \begin{aligned} x + 4y &= 2 \\ 2x + 7y &= 5 \\ 4x + 9y &= 15 \end{aligned}$$

Solution: (*i*) The coefficient matrix A of the given system is

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 4 & 9 \end{bmatrix} \text{ and the augmented matrix } \bar{A} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & 9 & 10 \end{bmatrix}$$

Let us apply the elementary row operation on $\bar{A}$, we get

$$\bar{A} \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & -7 & 2 \end{bmatrix} \xrightarrow{R_2' = -R_2} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & -7 & 2 \end{bmatrix} \xrightarrow[R_3' = R_3 - 7R_2]{R_1' = R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \end{bmatrix}$$

Therefore rank of $\bar{A} = 3$ and rank of $A = 2$

So the system is inconsistent.

(*ii*) Here $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 4 & 9 \end{bmatrix}$ and $\overline{A} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & 9 & 15 \end{bmatrix}$

NOTES

Let us apply the elementary row operation on $\overline{A}$, we get

$$\overline{A} \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & -7 & 7 \end{bmatrix} \xrightarrow{R_2' = -R_2}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & -7 & 7 \end{bmatrix} \xrightarrow[R_3' = R_3 + 7R_2]{R_1' = R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent system has two non-zero rows: Hence the rank of $\overline{A}$ is 2 and rank of $A = 2$. So the system is consistent. The system is equivalent to

$$x = 6, \; y = -1$$

The system has unique solution.

Example 1.33: Solve the system of equations,

$$x + 2y + z - 3w = 1$$
$$2x + 4y + 3z + w = 3$$
$$3x + 6y + 4z - 2w = 4$$

Solution: Here the co-efficient matrix is $A = \begin{bmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{bmatrix}$ and the augmented matrix is $\overline{A} = \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{pmatrix}$

Let us apply the elementary row operations on $\overline{A}$, then we get

$$\overline{A} \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7 & 1 \end{bmatrix} \xrightarrow[R_3' = R_3 - R_2]{R_1' = R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This equivalent system has two-non-zero rows. Hence rank of $\overline{A} = 2$ and rank of $A = 2$ $\left(\because \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0\right)$. So the system is consistent. The given system is equivalent to

$$x + 2y - 10w = 0$$
$$z + 7w = 1$$

Now we take $w = a$, $y = b$ where a, b are arbitrary real numbers, then $x = 10a - 2b$, $y = b$, $z = 1 - 7a$, $w = a$ is the solution of the given system. Since a, b are arbitrary, the given system has infinitely many solutions.

Example 1.34: For what values of a the system of equations is consistent? Solve completely in each consistent case.

$$x - y + z = 1$$
$$x + 2y + 4z = a$$
$$x + 4y + 6z = a^2$$

NOTES

Solution: Here the co-efficient matrix is $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{bmatrix}$ and the augmented matrix is $\overline{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & a \\ 1 & 4 & 6 & a^2 \end{bmatrix}$.

Now det A $= \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix} = 1(12 - 16) - (-1)(6 - 4) + 1(4 - 2)$

$= -4 + 2 + 2 = 0$

Hence rank of A is less than 3.

Now $\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 + 1 \neq 0$

Hence rank of $A = 2$

Let us apply the elementary row operation of $\overline{A}$, then we get

$$\overline{A} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & a-1 \\ 0 & 5 & 5 & a^2-1 \end{pmatrix} \xrightarrow{R_2' = 1/3\, R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 5 & 5 & a^2-1 \end{pmatrix}$$

$$\xrightarrow[R_3' = R_3 - 5R_2]{R_1' = R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 & (a+2)/3 \\ 0 & 1 & 1 & (a-1)/3 \\ 0 & 0 & 0 & (a^2-1) - 5(a-1)/3 \end{pmatrix}$$

If $a^2 - 1 - \frac{5(a-1)}{3} = 0$, then rank of $\overline{A} = 2 =$ rank of A and therefore system is consistent.

Now $a^2 - 1 - \frac{5}{3}(a-1) = 0$ gives $3a^2 - 3 - 5a + 5 = 0$ or $3a^2 - 5a + 2 = 0$

or $3a^2 - 3a - 2a + 2 = 0$ or $3a(a - 1) - 2(a - 1) = 0$ or $(a - 1)(3a - 2) = 0$

$\therefore\ a = 1, \frac{2}{3}$

$\therefore$ For $a = 1, \frac{2}{3}$, the system is consistent.

If $a = 1$, then the equivalent system become

$$x + 2z = 1$$
$$y + z = 0$$

Let $z = c$ be any arbitrary real number, then $x = 1 - 2c$, $y = -c$, $z = c$ *i.e.* $(1 - 2c, -c, c)$ is the solution of the given system and since c is arbitrary, the system has infinitely many solutions for $a = 1$.

If $a = \frac{2}{3}$, then equivalent system become

$$x + 2z = \frac{8}{9}$$
$$y + z = -\frac{1}{9}$$

NOTES

Let $z = c$ be any arbitrary real number, then $x = \frac{8}{9} - 2c, y = -\frac{1}{9} - c, z = c$

i.e. $\left(\frac{8}{9} - 2c, -\frac{1}{9} - c, c\right)$ is the solution of the given system and since c is arbitrary, the system has infinitely many solutions for $a = \frac{2}{3}$.

Example 1.35: Determine the conditions for which the system

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (*i*) only one solution (*ii*) no solution (*iii*) many solutions.

Solution: Here the coefficient matrix is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{bmatrix}$ and the augmented matrix is $\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & a & b^2 \end{bmatrix}$

$$\therefore \det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{vmatrix} = 1(2a + 7) - 1\,(a + 5) + 1.(7 - 10)$$
$$= 2a + 7 - a - 5 - 3 = a - 1$$

If $\det A \neq 0$ *i.e.*, if $a - 1 \neq 0$, *i.e.*, if $a \neq 1$, then the system has only one solution.

When $a = 1$, then $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & 1 \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & 1 & b^2 \end{bmatrix}$

Let us apply the elementary row operation on $\bar{A}$, we get

$$\bar{A} \xrightarrow[R_3' = R_3 - 5R_1]{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & -4 & b^2-5 \end{bmatrix} \xrightarrow[R_3' = R_3 - 2R_2]{R_1' = R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 & 2-b \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & 0 & b^2-5-2b+2 \end{bmatrix}$$

If $b^2 - 2b - 3 = 0$, then rank of $\bar{A}$ = rank of $A = 2$ and therefore the system is consistent.

If $b^2 - 2b - 3 \neq 0$, then rank of $\bar{A} = 3$, rank of $A = 2$ and therefore the system is inconsistent.

(*i*) If $a \neq 1$, the the system has only one solution.

(*ii*) If $a = 1, b \neq -1, 3(b^2 - 2b - 3 = 0 \Rightarrow b^2 - 3b + b - 3 = 0 \Rightarrow b(b - 3) + 1(b - 3) = 0 \Rightarrow (b - 3)\,(b + 1) = 0)$ the system has no solution.

(*iii*) If $a = 1, b = -1$, or $a = 1, b = 3$ the system has many solutions.

Example 1.36: Determine the value of a and b so that the following system of equation may have (i) unique solution (ii) many solutions (iii) no solution.

NOTES

$$2x + 3y + 4z = 9$$
$$x - 2y + az = 5$$
$$3x + 4y + 7z = b$$

Solution: Here the co-efficient matrix is $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & a \\ 3 & 4 & 7 \end{bmatrix}$ and the augmented matrix is $\bar{A} = \begin{bmatrix} 2 & 3 & 4 & 9 \\ 1 & -2 & a & 5 \\ 3 & 4 & 7 & b \end{bmatrix}$

$$\therefore \det A = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & a \\ 3 & 4 & 7 \end{vmatrix} = 2(-14 - 4a) - 3(7 - 3a) + 4(4 + 6)$$

$$= -28 - 8a - 21 + 9a + 40 = a - 9$$

If $\det A \neq 0$, *i.e.* if $a - 9 \neq 0$, *i.e.*, if $a \neq 9$, then system has only one solution.

When $a = 9$, then $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & 9 \\ 3 & 4 & 7 \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 2 & 3 & 4 & 9 \\ 1 & -2 & 9 & 5 \\ 3 & 4 & 7 & b \end{bmatrix}$

Let us apply elementary row operation on $\bar{A}$, we get

$$\bar{A} \xrightarrow{R_{12}} \begin{bmatrix} 1 & -2 & 9 & 5 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 7 & b \end{bmatrix} \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & -2 & 9 & 5 \\ 0 & 7 & 14 & -1 \\ 0 & 10 & 20 & b-15 \end{bmatrix} \xrightarrow{R_2' = \frac{1}{7}R_2} \begin{bmatrix} 1 & -2 & 9 & 5 \\ 0 & 1 & 2 & -\frac{1}{7} \\ 0 & 10 & 20 & b-15 \end{bmatrix}$$

$$\xrightarrow[R_3' = R_3 - 10R_2]{R_1' = R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 13 & \frac{33}{7} \\ 0 & 1 & 2 & -\frac{1}{7} \\ 0 & 0 & 0 & b - 15 + \frac{10}{7} \end{bmatrix}$$

If $b - 15 + \frac{10}{7} = 0$ or if $b = \frac{95}{7}$, then rank of $\bar{A}$ = rank of $A = 2$ and therefore the system is consistent.

If $b - 15 + \frac{10}{7} \neq 0$ *i.e.* if $b \neq \frac{95}{7}$, then rank of $\bar{A} = 3$, the rank of $A = 2$ and therefore the system is inconsistent.

Hence (i) if $a \neq 9$, then system has only one solution.

(ii) If $a = 9$, $b = \frac{95}{7}$, then the system has many solutions.

(iii) if $a = 9$, $b \neq \frac{95}{7}$, then the system has no solution.

Example 1.37: Solve by matrix method the system of equations

$$x + y - z = 6$$
$$2x - 3y + z = 1$$
$$2x - 4y + 2z = 1$$

Solution: The above system of equations can be written in the form

$$AX = B$$

where $$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & -4 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & -4 & 2 \end{vmatrix} = 1(-6+4) - 1(4-2) + (-1)(-8+6)$$

$$= -2 - 2 + 2 = -2 \neq 0$$

Since $\det A \neq 0$, then A^{-1} exist. Hence the unique Solution is given by $X = A^{-1} B$.

$$\therefore \quad \text{adj } A = \begin{pmatrix} +\begin{vmatrix} -3 & 1 \\ -4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -4 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \\ +\begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} \end{pmatrix}^t$$

$$= \begin{pmatrix} -2 & -2 & -2 \\ +2 & 4 & 6 \\ -2 & -3 & -5 \end{pmatrix}^t = \begin{pmatrix} -2 & 2 & -2 \\ -2 & 4 & -3 \\ -2 & 6 & -5 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{\det A} = -\frac{1}{2}\begin{pmatrix} -2 & 2 & -2 \\ -2 & 4 & -3 \\ -2 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & \frac{3}{2} \\ 1 & -3 & \frac{5}{2} \end{pmatrix}$$

$$\therefore \quad X = A^{-1} B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & \frac{3}{2} \\ 1 & -3 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ \frac{11}{2} \\ \frac{11}{2} \end{pmatrix}$$

$\therefore$ The required solution is $x = 6, y = \frac{11}{2}, z = \frac{11}{2}$.

1.8 EIGENVALUES, EIGENVECTORS AND DIAGONALIZATION OF MATRICES

NOTES

Similarity transformation. If **A**, **B** are two non-singular matrices and $\exists$ two non-singular matrices **P** and **Q** s.t. $\mathbf{B} = \mathbf{QAP}$ with $\mathbf{Q} = \mathbf{P}^{-1}$

so that $$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP} \quad ...(1.21)$$

then this transformation of matrix **A** into matrix **B** is termed as *similarity transformation* and matrices **A** and **B** are known as similar matrices. Now, (1.21) $\Rightarrow \mathbf{PBP}^{-1} = \mathbf{PP}^{-1}\mathbf{APP}^{-1} \Rightarrow \mathbf{IAI} = \mathbf{A}$ which is also a similarity transformation **B** into **A**.

A *matrix equation* $\mathbf{AX} = \mathbf{B}$ preserves its structure (form) under *similarity transformation*, since

$\mathbf{P}^{-1}(\mathbf{AX})\mathbf{P} = \mathbf{P}^{-1}\mathbf{BP} \Rightarrow \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{XP} = \mathbf{P}^{-1}\mathbf{BP}$ as $\mathbf{PP}^{-1} = \mathbf{I}$

$\Rightarrow (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{XP}) = (\mathbf{P}^{-1}\mathbf{BP})$

$\Rightarrow \mathbf{CY} = \mathbf{D}$ (say) by Equation (1.23) $\Rightarrow$ The form $\mathrm{A}\mathbf{X} = \mathbf{B}$.

Unitary transformation. If **A** be a *unitary transformation* of order $n \times n$ and **X**, **x** are column vectors of order $n \times 1$, then the linear transformation

$$\mathbf{X} = \mathbf{Ax} \quad ... (1.22)$$

is known as *unitary transformation*. Since

$$\mathbf{A}^{\Theta}\mathbf{A} = \mathbf{AA}^{\Theta} = \mathbf{I}, \text{ therefore}$$

$$\mathbf{X}^{\Theta}\mathbf{X} = (\mathbf{AX})^{\Theta}(\mathbf{Ax}) = \mathbf{x}^{\Theta}\mathbf{A}^{\Theta}\mathbf{Ax} = \mathbf{x}^{\Theta}\mathbf{x}$$

$\Rightarrow$ *the norm of vectors is invariant under similarity transformation*.

In (1.21), if **P** be unitary, *i.e.,* $\mathbf{PP}^{\Theta} = \mathbf{P}^{\Theta}\mathbf{P} = \mathbf{I} = \mathbf{PP}^{-1} = \mathbf{P}^{-1}\mathbf{P}$.

or $\mathbf{P}^{-1} = \mathbf{P}^{\Theta}$, then the transformation $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ is also unitary.

Orthogonal transformation. Any transformation $\mathbf{x} = \mathbf{AX}$ that transforms $\Sigma\mathbf{x}^2$ into $\Sigma\mathbf{X}^2$ is said to be an *orthogonal transformation* and the matrix **A** is known as *orthogonal matrix*.

If $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12}\cdots a_{1n} \\ a_{21} & a_{22}\cdots a_{2n} \\ \hdashline a_{n1} & a_{n2}\cdots a_{nn} \end{bmatrix}$$

and its transpose $$\mathbf{A}' = \begin{bmatrix} a_{11} & a_{21}\cdots a_{n1} \\ a_{12} & a_{22}\cdots a_{n2} \\ \hdashline a_{1n} & a_{2n}\cdots a_{nn} \end{bmatrix}$$

then $\mathbf{AA}' = \mathbf{I}$ which is the necesary and sufficient condition for a square matrix **X** to be orthogonal

since if $$\mathbf{x}_r = a_{r1}\mathbf{X}_1 + a_{r2}\mathbf{X}_2 + ... + a_{rn}\mathbf{X}_n,$$

then $\sum_{r=1}^{n} \mathbf{X}_r^2 = \sum_{r=1}^{n} \mathbf{x}_r^2 = \sum_{r=1}^{n} (a_{r1}\mathbf{X}_1 + a_{r2}\mathbf{X}_2 + ...a_{rn}\mathbf{X}_n)^2$ gives

NOTES

and $$\left.\begin{aligned} a_{1i}^2 + a_{2i}^2 + ... + a_{ni}^2 &= 1 \\ a_{1i}a_{ij} + a_{2i}a_{2j} + ... + a_{ni}a_{nj} &= 0 \end{aligned}\right\}$$

for $i = 1, 2...n$, $j = 1, 2...n$ and $i \neq j$.

Now $\mathbf{AA'} = \mathbf{I}$ gives $|\mathbf{A}|\,|\mathbf{A'}| = 1$.

where $|\mathbf{A}| = |\mathbf{A'}|$ as interchange of rows and columns does not alter the value of the determinant.

$$\therefore \qquad |\mathbf{A}|^2 = 1, \text{ i.e., } |\mathbf{A}| = \pm 1$$

Evidently the product of two orthogonal transformations is an orthogonal transformation. For if $\mathbf{x} = \mathbf{AX}$ and $\mathbf{X} = \mathbf{BY}$ be two orthogonal transformations then $\mathbf{AA'} = \mathbf{I}$, $\mathbf{BB'} = \mathbf{I}$.

$$\begin{aligned} \therefore \qquad [\mathbf{AB}]\,[\mathbf{AB}]' &= \mathbf{ABB'A'} \text{ by the law of reversal transpose} \\ &= \mathbf{AIA'} \\ &= \mathbf{AA'} = \mathbf{I}. \end{aligned}$$

In (1.21) if $\mathbf{P}$ is orthogonal, then $\mathbf{P}^{-1} = \mathbf{P'}$ and (1.21) is an orthogonal transformation.

Orthogonal Set. *A set of vectors is said to be an* **orthogonal set** *of vectors if*

(*i*) *each vector of the set is a normal vector*

(*ii*) *any two vectors of the set are orthogonal*

While a complex n-vector $\mathbf{X}$ is said to be orthogonal to another complex n-vector $\mathbf{Y}$ if $(\mathbf{X}, \mathbf{Y}) = \mathbf{O}$ *i.e.*, if $\mathbf{X}^{\Theta}\,\mathbf{Y} = \mathbf{O}$.

The relation of orthogonality in the set of all complex n-vectors is *symmetric*. The positive square root of $\mathbf{X}^{\Theta}\mathbf{X}$ is known as the length of $\mathbf{X}$.

In other words, using the Kronecker delta symbol $\delta_{ij} = 0$ for $i \neq j$.

$= 1$ for $i = j$.

a set S of complex n-vectors $\mathbf{X}_1, \mathbf{X}_2, ...\mathbf{X}_k$ is termed as an orthogonal set if $(\mathbf{X}_i, \mathbf{X}_j) = \delta_{ij}$, $i = 1, 2,....k$, $j = 1, 2,....k$.

Theorem 1.13: Every orthonormal set of vectors is linearly independent.

Proof: Suppose that the vectors $\mathbf{X}_1, \mathbf{X}_2, ...\mathbf{X}_k$ form an orthonormal set of vectors.

Thus (*i*) $\mathbf{X}_i$ is a normal vector for every i, *i.e.,* $\mathbf{X}_i^{\Theta}\mathbf{X}_j = \mathbf{I}$

(*ii*) $\mathbf{X}_i, \mathbf{X}_j$ are orthogonal vectors for every i, j such that $i \neq j$

i.e., $\mathbf{X}_i^{\Theta}\mathbf{X}_j = \mathbf{O}$ for every i and j, $i \neq j$ ...(1.23)

Consider a relation $a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + ... + a_k\mathbf{X}_k = \mathbf{O}$

where $a_1, a_2, ...a_k$ are scalars.

Premultiplying Equation (1.23) by $\mathbf{X}_1^{\Theta}$ and applying the above condition (*i*) and (*ii*), we find

$$a_1\mathbf{I} = \mathbf{O} \text{ or } a_1 = \mathbf{O} \text{ as } \mathbf{I} \neq \mathbf{O}.$$

NOTES

Similarly premultiplying (3) by $\mathbf{X}_2^\Theta$, $\mathbf{X}_3^\Theta$,........ successively, we may get $a_2 = 0$, $a_3 = 0$........, $a_k = 0$.

As such all the scalars $a_2, a_2, \cdots a_k$ being zero, the relation (3) follows that $\mathbf{X}_1, \mathbf{X}_2, \ldots\ldots\mathbf{X}_k$ form an orthonormal set of linearly independent vectors.

Theorem 1.14: Show that a real matrix is unitary if and only if it is orthogonal.

Proof: If **A** is a real matrix, then $\mathbf{A}^\Theta = \mathbf{A}'$

$\because$ **A** is unitary if $\mathbf{A}^\Theta\mathbf{A} = \mathbf{I}$ or $\mathbf{A}'\mathbf{A} = \mathbf{I}$, *i.e.*, **A** is orthogonal.

Conversely, if **A** is orthogonal then $\mathbf{A}'\mathbf{A} = \mathbf{I}$

i.e., $\mathbf{A}^\Theta\mathbf{A} = \mathbf{I}$, *i.e.*, **A** is unitary.

Diagonalization of Matrices

Diagonalization of matrices. Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ be n distinct eigen values of a matrix **A** and $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \ldots \mathbf{X}_n$ be the n corresponding eigen vectors. Also let $\mathbf{X}_t$ be the column vector given by

$$\mathbf{X}_i = \begin{bmatrix} \mathbf{X}_{1i} \\ \mathbf{X}_{2i} \\ \mathbf{X}_{ni} \end{bmatrix} \qquad \ldots(1.24)$$

Consider a matrix **E** whose column vectors are the n eigen vectors such that

$$\mathbf{E} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12}\ldots\ldots\mathbf{X}_{1n} \\ \mathbf{X}_{21} & \mathbf{X}_{22}\ldots\ldots\mathbf{X}_{2n} \\ \hdashline \mathbf{X}_{n1} & \mathbf{X}_{n1}\ldots\ldots\mathbf{X}_{nn} \end{bmatrix} = [\mathbf{X}_{ij}] \text{ (say)} \qquad \ldots(1.25)$$

Suppose that **D** is a diagonal matrix such that

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0\ldots\ldots0 \\ 0 & \lambda_2\ldots\ldots0 \\ \hdashline 0 & 0\ldots\ldots\lambda_n \end{bmatrix} = \text{diag}\,[\lambda_1, \lambda_2\ldots\lambda_n] \qquad \ldots(1.26)$$

The

$$\mathbf{ED} = \begin{bmatrix} \lambda_1\mathbf{X}_{11} & \lambda_2\mathbf{X}_{12}\ldots\ldots\lambda_n\mathbf{X}_{1n} \\ \lambda_1\mathbf{X}_{21} & \lambda_2\mathbf{X}_{22}\ldots\ldots\lambda_n\mathbf{X}_{2n} \\ \vdots & \vdots \qquad\qquad \vdots \\ \lambda_1\mathbf{X}_{n1} & \lambda_2\mathbf{X}_{n2}\ldots\ldots\lambda_n\mathbf{X}_{nn} \end{bmatrix} = [\lambda_j\,\mathbf{X}_{ij}] \qquad \ldots(1.27)$$

(no summation over j)

$$= (\mathbf{AX}_1, \mathbf{AX}_2, \ldots \mathbf{AX}_n) \text{ (expressing matrix as vectors)}$$

$$= \mathbf{A}\,(\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n)$$

$$= \mathbf{AE} \qquad \ldots(1.28)$$

If **E** be a non-singular matrix, then premultiplying by $\mathbf{E}^{-1}$, we get

$$\mathbf{E}^{-1}\mathbf{AE} = \mathbf{D}.$$

Thus premultiplying **A** by $\mathbf{E}^{-1}$ and post-multiplying by **E**, we get diagonal matrix whose diagonal elements are the eigen values. *This process is called the diagonalization of the matrix* **A**.

Power of a Matrix. An integral power m of a matrix $\mathbf{A}$ is defined as

$$\mathbf{A}^m = \mathbf{AAA} \text{ ...... } m \text{ times} \qquad ...(1.29)$$

If $\mathbf{A}$ is non-singular then $\mathbf{A}^{-1}$ exists s.t.

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

NOTES

As such a negative integral power, say, $m = -n$ is defined as

$$\mathbf{A}^{-n} = \mathbf{A}^m = (\mathbf{A}^{-1})^n = \mathbf{A}^{-1}\mathbf{A}^{-1} \text{ ..... } m \text{ times}$$

and for $m = 0$, $\mathbf{A}^0 = \mathbf{I}$.

If $\mathbf{E}$ be a diagonalizing matrix for $\mathbf{A}$ so that

$$\mathbf{E}^{-1}\mathbf{AE} = \mathbf{D} \quad \text{and} \quad \mathbf{A} = \mathbf{EDE}^{-1}, \qquad ...(1.30)$$

where $\mathbf{D}$ is a diagonal matrix whose diagonal elements are the eigen values of $\mathbf{A}$, then we can write in functional form as

$$f(\mathbf{A}) = \mathbf{E}\, f(\mathbf{D})\, \mathbf{E}^{-1} \qquad ...(1.31)$$

Now, (1.24) $\Rightarrow \mathbf{A}^m = (\mathbf{EDE}^{-1})\,(\mathbf{EDE}^{-1})$ ... m times

$$= \mathbf{ED}^m\,\mathbf{E}^{-1} \text{ and so } \mathbf{D}^m = \mathbf{E}^{-1}\,\mathbf{A}^m\,\mathbf{E} \qquad ...(1.32)$$

For $m = -n$, $\mathbf{A}^n = \mathbf{A}^{-m} = \mathbf{E}(\mathbf{D}^{-1})^m\,\mathbf{E}^{-1}$ ...(1.33)

A few properties of trace in relation to eigen values of a matrix:

(*i*) tr $\mathbf{AB}$ = tr $\mathbf{BA}$, (*ii*) tr $\mathbf{ABCD}$ = tr $\mathbf{BCDA}$ = tr $\mathbf{CDAB}$,

(*iii*) tr $(\mathbf{A} + \mathbf{B})$ = tr $\mathbf{A}$ + tr $\mathbf{B}$, (*iv*) tr $(\mathbf{AB} - \mathbf{BA})$ = tr $[\mathbf{A}, \mathbf{B}] = 0$.

For Example,

If $\mathbf{A} = \begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$, then characteristic equation

$$|\mathbf{A} - \lambda\,\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} \frac{4}{3} - \lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 1)\,(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 1, 2 \text{ as eigen values}$$

$\therefore$ Diagonal matrix $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

By usual method, eigen vectors of $\mathbf{A}$ are

$(\sqrt{2}, -1)$ and $(1, \sqrt{2})$.

$$\therefore \qquad \mathbf{E} = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}, \text{ so that } \mathbf{E}^{-1}\,\mathbf{AE} = \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ with } |\mathbf{A}| \neq 0$$

For $m = 50$ (say)

$$(1.32) \Rightarrow \qquad \mathbf{A}^{50} = \mathbf{E}\mathbf{D}^{50}\mathbf{E}^{-1} = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{50} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}$$

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$$= \frac{1}{3} \begin{bmatrix} 2^{50}+2 & \left(2^{50}-1\right)\sqrt{2} \\ \left(2^{50}-1\right)\sqrt{2} & 2^{51}+1 \end{bmatrix}$$

Matrix as exponent power. Analogous to

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{\lfloor m}, \text{ we have } e^{\mathbf{A}} = \sum_{m=0}^{\infty} \frac{\mathbf{A}^m}{\lfloor m} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{\lfloor 2} + \cdots \qquad ...(1.34)$$

For diagonal matrix $\mathbf{D}$ with elements $\mathbf{D}_{ij} = \lambda_{ij}\,\delta_{ij}$, we have

$$e^{\mathbf{D}} = \sum_{m=d}^{\infty} \frac{\mathbf{D}^m}{\lfloor m} \qquad ...(1.35)$$

with (i, j)th element s.t.

$$[e^{\mathbf{D}}]_{ij} = \sum_{m=d}^{\infty} \frac{\left[\mathbf{D}^m\right]_{ij}}{\lfloor m} = \sum_{m=1}^{\infty} \frac{(\lambda_i)^m\,\delta_{ij}}{\lfloor m}$$

As such

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \hdashline 0 & 0 & \dots & \lambda_n \end{bmatrix} \Rightarrow e^{\boldsymbol{D}} = \begin{bmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \hdashline 0 & 0 & \dots & e^{\lambda_n} \end{bmatrix} \qquad ...(1.36)$$

In view of Equation (1.32), *i.e.*, $\mathbf{D}^m = \mathbf{E}^{-1}\,\mathbf{A}^m\,\mathbf{E}$, (1.28) yields

$$\mathbf{E}^{-1}\,e^{\mathbf{A}}\,\mathbf{E} = \mathbf{E}^{-1}\mathbf{I}\mathbf{E} + \mathbf{E}^{-1}\mathbf{A}\mathbf{E} + \mathbf{E}^{-1}\,\mathbf{A}^2\mathbf{E} + \cdots$$

$$= \mathbf{I} + \mathbf{D} + \mathbf{D}^2 + \dots = e^{\mathbf{D}} \qquad ...(1.37)$$

so that $$\mathbf{E}e^{\mathbf{D}}\mathbf{E}^{-1} = e^{\mathbf{A}} \qquad ...(1.38)$$

As such, on using $a^x = e^{x\,\log_e a}$, we have

$$a^{\mathbf{A}} = e^{\mathbf{A}\,\log_e a} = \mathrm{E}\; e^{\mathbf{E}(\mathbf{D}\log_e a)\mathbf{E}^{-1}} \qquad ...(1.39)$$

with $$e^{\mathbf{D}\,\log_e a} = a^{\mathbf{D}} = \begin{bmatrix} a^{\lambda_1} & 0 & \dots & 0 \\ 0 & a^{\lambda_2} & \dots & 0 \\ \hdashline 0 & 0 & \dots & a^{\lambda_n} \end{bmatrix}$$

A Method of Diagonalization in Practice: Writing the characteristic equation for a given matrix, the characteristic roots can be determined. If $\mathbf{A}$ be an n-square matrix and its characteristic roots are $\lambda_1, \lambda_2, \dots \lambda_n$, then the diagonal matrix is

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \dots 0 \\ 0 & \lambda_2 & 0 \dots 0 \\ 0 & 0 & \lambda_3 \dots 0 \\ \hdashline 0 & 0 & 0 \quad \lambda_n \end{bmatrix}$$

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i.e., if $\quad \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $\quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

i.e., $(1 - \lambda)(1 - 2\lambda \cos\theta + \lambda^2) = 0$

Characteristic roots are 1, $\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

i.e., 1, $\cos\theta \pm i \sin\theta$

i.e., 1, $e^{\pm i\theta}$

$\therefore \quad \lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}, \lambda_3 = 1$ (say)

Hence the diagonal matrix is $\begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note. Besides this method we can diagonalize a square matrix
(*i*) by orthogonally similar matrices
(*ii*) by unitarily similar matrices
but the above method is rather convenient in practice.

Check Your Progress

9. When complex number is said to be Hermition?
10. What is real matrix?
11. When square matrix is called symmetric?
12. Define the skew symmetric.
13. When system is called homogeneous?
14. Define the unitary transform.

1.9 DETERMINANTS

Determinant of a Square Matrix

The notion of determinant is fundamental in algebra and has tremendous applications in many spheres of mathematical activities. We begin with its definition.

Definition: The determinant of a square matrix with real entries, denoted as det A or $|A|$, is defined to be a real number obtained as follows:

If $A = [a_{11}]$, $\det A = a_{11}$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det A = a_{11}a_{22} - a_{12}a_{21}$.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \begin{aligned} \det A = {} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{31}a_{23} - a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{aligned}$$

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and so on.

Thus determinant of a matrix can be taken as a function from the set of square matrices to the set of real numbers. Sometimes, the determinant of $A = [a_{ij}]$ will also be denoted by $|a_{ij}|$ *i.e.*

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Note that a determinant can also be defined without reference to a matrix.

Minors and Cofactors: The *minor* of an element in a determinant is the determinant obtained by deleting the row and the column which intersect in that element. Let us consider the determinant of a matrix A as follows:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The minor of a_{11} which lies in first row and first column is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

$$\text{Similarly } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

are the minors of the elements a_{12}, a_{22}, a_{33} respectively.

The minor of an element a_{ij} is generally denoted by the capital letter M_{ij}. The cofactor of a_{ij} in det A is defined to be $(-1)^{i+j} \times$ (minor of a_{ij})

i.e., $A_{ij} = (-1)^{i+j} M_{ij}$

The cofactor of a_{ij} is generally denoted by A_{ij}. The cofactor of a_{11} is given by

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

$$\text{Similarly } A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Let us consider the determinant of a matrix $A = (a_{ij})_{n,n}$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

The value of det A can be obtained in terms of the elements of the ith row as

det $A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + A_{in}a_{in}$.

For the values of $i = 1, 2, 3 \ldots n$, there are n different expansions for det A given by

$$\begin{aligned} \det A &= a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} \quad \text{(for first row)} \\ &= a_{21}A_{21} + a_{22}A_{22} + \ldots + a_{2n}A_{2n} \quad \text{(for 2nd row)} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= a_{n1}A_{n1} + a_{n2}A_{n2} + \ldots + a_{nn}A_{nn} \quad \text{(for } n\text{th row)} \end{aligned}$$

proceeding with similar arguments in respect of columns of A and considering in particular, the elements of the jth column, the expansion of det A can be obtained as

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} \qquad (i = 1, 2, \ldots n)$$

If one row and one column be deleted from $n \times n$ matrix (a_{ij}), the determinant of the remaining $(n-1) \times (n-1)$ matrix is called a *minor* of order $(n-1)$ of A.

For example, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then

Expanding in terms of the first row, we get

$$\begin{aligned} \Delta &= a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix} \\ &= a(bc - f^2) - h(hc - gf) + g(hf - gb) \\ &= abc - af^2 - h^2c + hgf + hgf + g^2b \end{aligned}$$

Similarly expanding in terms of the 2nd row, we can write

$$\begin{aligned} \Delta &= -h\begin{vmatrix} h & g \\ f & c \end{vmatrix} + b\begin{vmatrix} a & g \\ g & c \end{vmatrix} - f\begin{vmatrix} a & h \\ g & f \end{vmatrix} \\ &= -h(hc - gf) + b(ac - g^2) - f(af - hg) \\ &= -h^2c + hgf + abc - bg^2 - af^2 + fgh \\ &= abc + 2hgf - af^2 - bg^2 - ch^2 \end{aligned}$$

Expanding in terms of the first column we can similarly get

$$\begin{aligned} \Delta &= a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & g \\ f & c \end{vmatrix} + g\begin{vmatrix} h & g \\ b & f \end{vmatrix} \\ &= a(bc - f^2) - h(hc - gf) + g(bf - bg) \\ &= abc - af^2 - ch^2 + hfg + hfg - bg^2 \\ &= abc + 2hfg - af^2 - bg^2 - ch^2. \end{aligned}$$

The **Hückel method** or **Hückel molecular** orbital theory, proposed by Erich Hückel in 1930, is a very simple linear combination of atomic orbitals molecular orbitals method for the determination of energies of molecular orbitals of π-electrons in π-delocalized molecules, such as ethylene, benzene, butadiene, and pyridine.

Within linear algebra, the secular equations in Equation (**i**) will also have a non-trivial solution, if and only if, the secular determinant is zero

$$\begin{vmatrix} H_{11} - ES_{11} & H_{12} - ES_{12} \\ H_{21} - ES_{21} & H_{22} - ES_{22} \end{vmatrix} = 0 \qquad \ldots\ldots\text{(i)}$$

Or in shorthand notation

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Everything in Equation (**i**) is a known number except ***E***. Since the secular determinant for ethylene is a 2×2 matrix, finding ***E***, requires solving a quadratic equation (after expanding the determinant)

$$(H_{11} - ES_{11})(H_{22} - ES_{22}) - (H_{21} - ES_{21})(H_{12} - ES_{12}) = 0$$

There will be two values of ***E*** which satisfy this equation and they are the molecular orbital energies. For ethylene, one will be the bonding energy and the other the antibonding energy for the π -orbitals formed by the combination of the two carbon 2pz orbitals (Equation (**i**)). However, if more than two $|\phi\rangle$ atomic orbitals were used, e.g., in a bigger molecule, then more energies would be estimated by solving the secular determinant.

Solving the secular determinant is simplified within Hückel method via the following four assumptions:

- All overlap integrals S_{ij} are set equal to zero. This is quite reasonable since the π–orbitals are directed perpendicular to the direction of their bonds. This assumption is often call Neglect of Differential Overlap (NDO).
- All resonance integrals H_{ij} between non-neighbouring atoms are set equal to zero.
- All resonance integrals H_{ij} between neighbouring atoms are equal and set to β.
- All coulomb integrals H_{ii} are set equal to α.

These assumptions are mathematically expressed as

$$H_{11} = H_{22} = \alpha$$
$$H_{12} = H_{21} = \beta$$

First assumptions means that the overlap integral between the two atomic orbitals is 0

$$S_{11} = S_{22} = 1$$
$$S_{12} = S_{21} = 0$$

1.10 INTRODUCTION TO TENSOR

In this section we will discuss about the tensor.

Cartesian Tensor

In geometry and linear algebra, a Cartesian tensor uses an orthonormal basis to represent a tensor in a Euclidean space in the form of components. Converting a tensor's components from one such basis to another is through an orthogonal transformation.

The most familiar coordinate systems are the two-dimensional and three-dimensional Cartesian coordinate systems. Cartesian tensors may be used with

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any Euclidean space, or more technically, any finite-dimensional vector space over the field of real numbers that has an inner product.

Use of Cartesian tensors occurs in physics and engineering, such as with the Cauchy stress tensor and the moment of inertia tensor in rigid body dynamics. Sometimes general curvilinear coordinates are convenient, as in high-deformation continuum mechanics, or even necessary, as in general relativity. While orthonormal bases may be found for some such coordinate systems (for example, tangent to spherical coordinates), Cartesian tensors may provide considerable simplification for applications in which rotations of rectilinear coordinate axes suffice. The transformation is a passive transformation, since the coordinates are changed and not the physical system.

Tensor analysis is the generalization of vector analysis as is evident by considering a vector function $f(\mathbf{r})$ of a vector $\mathbf{r}$. This function is continuous at $\mathbf{r} = \mathbf{r}_0$ if,

$$\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = f(\mathbf{r}_0)$$

and it is linear, if

$$f(\mathbf{r}+\mathbf{s}) = f(\mathbf{r}) + f(\mathbf{s}) \qquad \text{... (1.40)}$$

and

$$f(\lambda\mathbf{r}) = \lambda f(\mathbf{r}) \qquad \text{... (1.41)}$$

for arbitrary values of $\mathbf{r}$, $\mathbf{s}$, λ.

Now, we know that a linear vector function $f(\mathbf{r})$ is completely defined only if $f(\mathbf{a}_1)$, $f(\mathbf{a}_2)$ and $f(\mathbf{a}_3)$ are given for any three non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. In terms of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ as basis if we assume that

$$\mathbf{r} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + x_3\,\mathbf{a}_3 \qquad \text{... (1.42)}$$

then we have from Equations (1.40) and (1.41),

$$f(r) = x_1 f(\mathbf{a}_1) + x_2 f(\mathbf{a}_2) + x_3 f(\mathbf{a}_3)$$

As such Equation (1.42) yields

$$x_\alpha = \mathbf{r}\,.\,\mathbf{a}_\alpha,\ \alpha = 1, 2, 3$$

Let us put $\quad f(\mathbf{a}_\alpha) = \mathbf{b}_\alpha.$

So that
$$f(\mathbf{r}) = (\mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + \mathbf{b}_3\mathbf{a}_3)\,.\,\mathbf{r}$$
$$= \phi\,.\,\mathbf{r} \text{ (say)}$$

where the operator $\phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \mathbf{a}_3\mathbf{b}_3$ consists of nine components in three dimensional coordinate geometry and hence it is neither a scalar nor a vector quantity but is a new mathematical symbol called as the *dyadic*.

Suppose that there are two vectors $\mathbf{u}$ and $\mathbf{v}$ such that components of vector $\mathbf{v}$ are linear functions of the components of vector $\mathbf{u}$ defined as

$$\left.\begin{aligned} v_x &= a_{xx}\,u_x + a_{xy}\,u_y + a_{xz}\,u_z \\ v_y &= a_{yx}\,u_x + a_{yy}\,u_y + a_{yz}\,u_z \\ v_z &= a_{zx}\,u_x + a_{zy}\,u_y + a_{zz}\,u_z \end{aligned}\right\} \qquad \text{... (1.43)}$$

In this way the vector $\mathbf{v}$ is placed in one-to-one correspondence with the vector $\mathbf{u}$. The scheme of coefficients $a_{\alpha\beta}$ has thus an independent meaning if the correspondence is such that the passage from $\mathbf{u}$ to $\mathbf{v}$ is independent of the

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particular coordinate system in which the vectors are resolved into components. We call the coefficients $a_{\alpha\beta}$ in this case as the coefficients of a tensor.

It is observed that the nine components as mentioned above characterise the transformation of the components of one vector into those of other. The coeffcients $a_{\alpha\beta}$ in general transform u_β into one of three parts of v_α.

The Equations (1.39) are equivalent to a single vector equation,

$$\mathbf{v} = \phi\mathbf{u}$$

where the operator ϕ turns **u** into **v**. It is rather graphically known as **Tensor**.

The essential part of a tensor operation is the array of coefficients like $a_{\alpha\beta}$, written in the form of a matrix, such as

$$\phi = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}$$

As such the *dyadic* operator turns a vector **r** into the vector function $f(\mathbf{r})$ and is expressed as the sum of dyads **ab**, *i.e.*,

$$\phi = \sum_\alpha \mathbf{a}_\alpha \mathbf{b}_\alpha$$

In a similar way a *triadic* is expressed as the sum of the triads $\sum_\alpha \mathbf{a}_\alpha \mathbf{b}_\alpha \mathbf{c}_\alpha$.

Considering it as an opearor that converts vector **r** into the dyadic ϕ, we may write

$$\phi \,.\, \mathbf{r} = \sum(\mathbf{a}_\alpha \mathbf{b}_\alpha \mathbf{c}_\alpha)\,.\,\mathbf{r}$$

Similarly a *tetradic* is the sum of tetrads, $\sum \mathbf{a}_\alpha \mathbf{b}_\alpha \mathbf{c}_\alpha \mathbf{d}_\alpha$ and etc.

All such physical quantities as scalars, vectors, dyadics, triadics, tetradics etc. are collectively known as tensors of rank 1, 2, 3, 4 etc. and as such the tensors can be regarded as generalized extended form of vectors.

The examples of dyadic, i.e., tensor of rank two are: an operator relating dielectric displacement vector with the electric vector of an electro-magnetic wave in an isotropic medium; a stress tensor relating stress and strain in an isotropic medium in which case a component of stress **T** is a function of every component of strain **S**.

i.e., $$\mathrm{T}_\alpha = \sum_{\beta=1}^{3} \mathbf{a}_{\alpha\beta} S_\beta \quad \text{or} \quad \mathbf{T} = \varphi\mathbf{S}$$

where φ is a nine coefficient operator in three dimensional space.

Note. The dyadic or tensor of rank two is also known as *Stress Tensor*.

An Explanatory Note on Dyads and Dyadics

We know that the gradient of a vector **f** such as

$$\nabla\mathbf{f} = \mathbf{i}\,f_x + \mathbf{j}\,f_y + \mathbf{k}\,f_z \qquad \text{... (1.44)}$$

is meaningless as it consists of sum of three ordered pairs of vectors, but we sometimes take it to define as *dyadic* and the ordered vector pairs as *dyads*. In Equation (1.44) we regard $\nabla\mathbf{f}$ as an operator setting up a one-one correspondence between directions **e** at a point and its directional derivative $\frac{d\mathbf{f}}{ds}$, i.e., $\frac{d\mathbf{f}}{ds} = \mathbf{e} \cdot \nabla\mathbf{f}$

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Actually the dyadic $\nabla\mathbf{f}$ replaces an infinite number of vectors $\frac{d\mathbf{f}}{ds}$, so that any sum of dyads is called as dyadic, for example, the dyadic

$$\mathbf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + ... + \mathbf{a}_n\mathbf{b}_n$$

is a general dyadic in which the vectors $\mathbf{a}_\alpha$ are known as *antecedents* and $\mathbf{b}_\alpha$ as *consequents*, while the dyadic

$$\mathbf{D}_c = \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + ... \mathbf{b}_n\mathbf{a}_n$$

is said to be the conjugate of **D**, such that **D** is *symmetric* if $\mathbf{D} = \mathbf{D}_c$ and *skew* if $\mathbf{D} = -\mathbf{D}_c$.

In Equation (1.44) if **f** is replaced by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

such that

$$\mathbf{r}_x = \mathbf{i},\ \mathbf{r}_y = \mathbf{j},\ \mathbf{r}_z = \mathbf{k} \text{ then}$$

$$\nabla\mathbf{r} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathrm{I}$$

where the dyadic **I** is termed as *Idemfactor* as it transforms any vector **V** into itself, i.e.

$$\mathbf{V} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{V} = \mathbf{V} \text{ for every } \mathbf{V}$$

Transformation of Coordinates

If we focus our attention on some point of Minkowski's four dimensional world and consider the transformation from one system of coordinates (x_1, x_2, x_3, x_4) to another system (x_1', x_2', x_3', x_4'), such that

$$x_1' = f_1(x_1, x_2, x_3, x_4) \text{ etc.}$$

then we can solve x_1, x_2, x_3, x_4 in terms of x_1', x_2', x_3', x_4' such that

$$x_1 = \phi_1(x_1', x_2', x_3', x_4') \text{ etc.}$$

and the differentials dx_1, dx_2, dx_3, dx_4 are then transformed as

$$dx_1 = \frac{\partial x_1'}{\partial x_1} dx_1 + \frac{\partial x_1'}{\partial x_2} dx_2 + \frac{\partial x_1'}{\partial x_3} dx_3 + \frac{\partial x_1'}{\partial x_4} dx_4 \text{ etc.}$$

or symbolically

$$dx_\mu' = \sum_{\alpha=1}^{4} \frac{\partial x_\mu'}{\partial x_\alpha} dx_\alpha;\ (\mu = 1, 2, 3, 4) \text{ etc.}$$

The Summation Convention and Kronecker Delta Symbol

Let (x_1, x_2, x_3, x_4) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$ be the coordinates of two neighbouring events considered in Minkowski's four dimensional space. Then the interval ds between these two neighbouring events in any coordinate system, is given by

$$ds^2 = g_{11}\, dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3 + 2g_{14}dx_1dx_4 + 2g_{23}dx_2dx_3 + 2g_{24}dx_2dx_4 + 2g_{34}dx_3dx_4 \quad ... (1.45)$$

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where the coefficients $g_{\mu\nu}$ ($\mu, \nu = 1, 2, 3, 4$) are functions of x_1, x_2, x_3, x_4. This follows that ds^2 is some quadratic function of the difference of coordinates.

Adopting the convention that whenever a literal suffix appears twice in a term that term is to be summed for values of the suffix 1, 2, 3, 4; Equation (1.45) can be written as

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \;(\mu, \nu = 1, 2, 3, 4 \text{ and } g_{\mu\nu} = g_{\nu\mu}) \quad \dots (1.46)$$

Since μ and ν each appear twice, the right hand side of Equation (1.46) indicates the summation

$$\sum_{\mu=1}^{4} \sum_{\nu=1}^{4}$$

Any literal suffix appearing twice in a term is said to be a *dummy suffix* and it may be changed freely to any other letter not already used in that term. Also two or more dummy suffixes can be interchanged, for example,

$$g_{\alpha\beta} \frac{\partial^2 x_\alpha}{\partial x'_\mu \partial x'_\nu} \frac{\partial x_\beta}{\partial x'_\lambda} = g_{\alpha\beta} \frac{\partial^2 x_\beta}{\partial x'_\mu\, \partial x'_\nu} \cdot \frac{\partial^2 x_\alpha}{\partial x'_\lambda}$$

(by intercharging the dummy suffixes α and β and using $g_{\beta\alpha} = g_{\alpha\beta}$)

Illustration. *To prove that*

$$\frac{\partial x_\mu}{\partial x'_\alpha} \cdot \frac{\partial x'_\alpha}{\partial x_\nu} = \frac{\partial x_\mu}{\partial x_\nu} = 0 \quad \text{if} \quad \mu \neq \nu$$

$$= 1 \text{ if } \mu = \nu \text{ where } \alpha = 1, 2, 3, 4.$$

Here, $$\text{R.H.S.} = \frac{\partial x_\mu}{\partial x'_1} \frac{\partial x'_1}{\partial x_\nu} + \frac{\partial x_\mu}{\partial x'_2} \frac{\partial x'_2}{\partial x_\nu} + \frac{\partial x_\mu}{\partial x'_3} \frac{\partial x'_3}{\partial x_\nu} + \frac{\partial x_\mu}{\partial x'_4} \frac{\partial x'_4}{\partial x'_\nu}$$

$$= \frac{\partial x_\mu}{\partial x_\nu}$$

x_μ and x_ν being the coordinates of the same system, their variations are independent and so

$$dx_\mu = 0 \text{ when } \mu \neq \nu$$

and $$dx_\mu = dx_\nu \text{ when } \mu = \nu$$

$$\therefore \quad \frac{\partial x_\mu}{\partial x'_\nu} = \frac{\partial x_\mu}{\partial x'_\alpha} \cdot \frac{\partial x'_\alpha}{\partial x_\nu} = 0 \text{ when } \mu \neq \nu$$

$$= 1 \text{ when } \mu = \nu$$

Here the multiplier $\dfrac{\partial x_\mu}{\partial x'_\alpha} \dfrac{\partial x'_\alpha}{\partial x_\mu}$ acts as a *substitution operator*.

It is rather convenient to write

$$\frac{\partial x_\mu}{\partial x_\nu} = \delta_{\mu\nu} \text{ or } \delta^\mu_\nu$$ which is known as *Kronecker delta*.

As such the above results can be expressed as

$$\left.\begin{aligned} \delta^\mu_\nu &= 0 \text{ if } \mu \neq \nu \\ &= 1 \text{ if } \mu = \nu \end{aligned}\right\} \quad \dots (1.47)$$

Thus if $A(\mu)$ be an expression involving the suffix μ, then

$$\frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x'_\alpha}{\partial x_\nu} = A(\mu) = A(\nu) \quad \text{... (1.48)}$$

for; the summation on the left, with respect to μ gives four terms corresponding to $\mu = 1, 2, 3, 4$; one of which will agree with ν. Denoting the other three values by σ, τ, ρ, the left hand side of Equation (1.48) is

$$=1 \,.\, A\,(\nu) + 0 \,.\, A\,(\sigma) + 0 \,.\, A\,(\tau) + 0 \,.\, A\,(\rho) \text{ by Equation (1.47)}$$
$$=A\,(\nu)$$

i.e. $$\delta^\mu_\nu\, A\,(\mu) = A\,(\nu) \quad \text{... (1.49)}$$

Evidently $\delta^\mu_\rho\ \delta^\rho_\nu = \delta^\mu_\nu$

and $\delta^\mu_\mu = 4$

for, in the latter case, $\delta^\mu_\mu = \delta^1_1 + \delta^2_2 + \delta^3_3 + \delta^4_4$

$= 1 + 1 + 1 + 1 = 4$ by Equation (1.47)

1.10.1 Tensors as Classification of Transformation Laws

We have already mentioned that if we consider the transformation from one system of coordinates (x_1, x_2, x_3, x_4) to another system (x_1', x_2', x_3', x_4'), then the differentials dx_1, dx_2, dx_3, dx_4 are transformed as

$$dx_1' = \frac{\partial x_1'}{\partial x_1}dx_1 + \frac{\partial x_1'}{\partial x_2}dx_2 + \frac{\partial x_1'}{\partial x_3}dx_3 + \frac{\partial x_1'}{\partial x_4}dx_4 \text{ etc.}$$

or in short as $dx'_\mu = \sum_{\alpha=1}^{4}\frac{\partial x'_\mu}{\partial x_\alpha}dx_\alpha \quad \mu = 1, 2, 3, 4.$

Any set of four quantities transformed in accordance with this law is said to be a **Contravariant Vector**. Thus if a coordinate system (A^1, A^2, A^3, A^4) transforms to the new coordinate system (A'^1, A'^2, A'^3, A'^4) where

$$A'^\mu = \sum_{\alpha=1}^{4}\frac{\partial x'_\mu}{\partial x_\alpha}A^\alpha \quad \text{... (1.50)}$$

Then (A^1, A^2, A^3, A^4) or briefly A^μ is a contravariant vector. Hence the upper position of the suffix (which is definitely, not an exponent) is reserved to indicate contravariant vectors.

Now, if we consider an operator ϕ such that it is an invariant function of position, i.e., it has a fixed value at each point independent of the coordinate system used, then the four quantities

$\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}, \frac{\partial\phi}{\partial x_4}$ are transformed as

$$\frac{\partial\phi}{\partial x_1'} = \frac{\partial x_1}{\partial x_1'}\frac{\partial\phi}{\partial x_1} + \frac{\partial x_2}{\partial x_1'}\frac{\partial\phi}{\partial x_2} + \frac{\partial x_3}{\partial x_1'}\frac{\partial\phi}{\partial x_3} + \frac{\partial x_4}{\partial x_1'}\frac{\partial\phi}{\partial x_4}, \text{ etc.}$$

or in short $$\frac{\partial\phi}{\partial x'_\mu} = \sum_{\alpha=1}^{4}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial\phi}{\partial x_\alpha} \quad (\mu = 1, 2, 3, 4).$$

NOTES

Any set of four quantities transformed in accordance with this law is said to be a **Covariant Vector**.

Thus if A_μ be a covariant vector, its transformation law is

$$A'_\mu = \sum_{\alpha=1}^{4} \frac{\partial x_\alpha}{\partial x'_\mu} A_\alpha \qquad \text{... (1.51)}$$

where the lower position of the suffix indicates covariance.

Hence the Relations (1.50) and (1.51) give the laws of transformation of vectors. If we denote by $A_{\mu\nu}$ a quantity with 16 components by assigning μ and ν the values 1, 2, 3, 4 independently, then a generalization of these laws yields quantities classified as

Contravariant Tensors $A'^{\mu\nu} = \dfrac{\partial x'_\mu}{\partial x_\alpha} \dfrac{\partial x'_\nu}{\partial x_\beta} A^{\alpha\beta}$

Covariant Tensors $A'_{\mu\nu} = \dfrac{\partial x_\alpha}{\partial x'_\mu} \dfrac{\partial x'_\beta}{\partial x_\nu} A_{\alpha\beta}$

Mixed Tensors $A'^{\nu}_{\mu} = \dfrac{\partial x_\alpha}{\partial x'_\mu} \dfrac{\partial x'_\nu}{\partial x_\beta} A^{\beta}_{\alpha}$

These are the tensors of *second rank*.

Similarly $A_{\mu\nu\sigma}$ has 64 components and $A_{\mu\nu\sigma\tau}$ has 256 components. Thus a tensor of higher rank is of the type.

$$A'^{\tau}_{\mu\nu\sigma} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \frac{\partial x_\lambda}{\partial x'_\sigma} \frac{\partial x'_\tau}{\partial x'_\delta} A^{\delta}_{\alpha\beta\gamma}$$

Note. A vector is a tensor of *first rank* and an invariant or scalar is a tensor of *zero rank*.

Rank of a Tensor: The rank of a tensor is determined by the number of suffixes or indices attached to it. As a matter of fact the rank of a tensor when raised as power to the number of dimensions, yields the number of components of the tensor and hence the components of the matrix that represents the tensor. As such a tensor of rank n in four dimensional space has 4^n components. Consequently, the rank of a tensor gives the number of the mode of changes of a physical quantity when passing from one system to another system which is in rotation relative to the first. It is clear from this discussion that a quantity that does not change when the axes are rotated is a tensor of *zero rank*, since the number of mode of changes is then zero. These quantities named as tensor of zero rank are scalars while the tensors of rank one are vectors.

Contravariant Tensor: If n quantities A^α ($\alpha = 1, 3, \ldots n$) in a coordinate system ($x_1, x_2, \ldots\ldots x_n$) are related to n other quantities A'^α ($\alpha = 1, 2, \ldots n$) in another coordinate system ($x_1', x_2', \ldots\ldots, x_n'$) by the transformation laws

$$A'^\mu = \frac{\partial x'_\mu}{\partial x_\alpha} A^\alpha \text{ (Contravariant law)}$$

on change of the coordinate x_β to x'_β according to summation convention, then A^α are termed as the components of a contravariant vector or a contravariant tensor of the first rank.

NOTES

Covariant Tensor: If n quantities A_α ($\alpha = 1, 2,......n$) in a coordinate system ($x_1, x_2, ...x_n$) are related to n other quantities A'_α ($\alpha = 1, 2, ...n$) in another coordinate system ($x'_1, x'_2, ... x'_n$) by the transformation laws

$$A'_\mu = \frac{\partial x_\alpha}{\partial x'_\mu} A_\alpha \text{ (Covariant law)}$$

according to summation convention, then A_α are termed as the components of a covariant vector or a covariant tensor of the first rank.

Example 1.38: Show that the velocity of a fluid at any point is a contravariant vector of rank one.

Solution: Assuming that $x_\alpha(t)$ is the coordinate of a moving particle with the time t, we have

$$\nu^\alpha = \frac{dx_\alpha}{dt}$$

as the velocity of the particle.

In transformed coordinates the components of velocity are

$$\nu'^\alpha = \frac{dx'_\alpha}{dt}$$

But $$\nu^\alpha = \frac{d}{dt}x_\alpha = \frac{\partial x'_\alpha}{\partial x_\beta}\frac{dx_\beta}{dt} = \frac{\partial x'_\alpha}{\partial x_\beta}\nu_\beta$$

which follows that velocity is a contravariant vector of rank one.

Example 1.39: Show that the law of transformation for a contravariant vector is transitive.

Solution: We have $A'^\mu = \dfrac{\partial x'_\mu}{\partial x_\alpha} A^\alpha$

Let $$A''^\mu = \frac{\partial x''_\mu}{\partial x'_\alpha} A'^\alpha$$

$$\therefore \quad A''^\mu = \frac{\partial x''_\mu}{\partial x'_\beta} A^\beta = \frac{\partial x''_\mu}{\partial x'_\beta}\frac{\partial x'_\beta}{\partial x_\alpha} A^\alpha = \frac{\partial x''_\mu}{\partial x_\alpha} A^\alpha$$

which shows that contravariant law is transitive.

Example 1.40: Find the components of a vector in polar coordinates whose components in cartesian coordinates are $\dot{x}, \dot{y}$ and $\ddot{x}, \ddot{y}$.

As given, suppose that

$$x_1 = x,\ x_2 = y,$$

$$x_1' = r,\ x_2' = \theta$$

and (*i*) $A^1 = \dot{x},\ A^2 = \dot{y}$

(*ii*) $A^1 = \ddot{x},\ A^2 = \ddot{y}$.

Then, we have to find C'^1, A'^2.

Solution: We have the transformations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

giving $$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

so that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2}$ and $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2}$

Also $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2}$

NOTES

(*i*) Transformation law as already defined gives

$$A'^{\mu} = \frac{\partial x'_\mu}{\partial x_\alpha} A^\alpha \; (\alpha = 1, 2)$$

$$= \frac{\partial x'_\mu}{\partial x_1} A^1 + \frac{\partial x'_\mu}{\partial x_2} A^2$$

$$= \frac{\partial x'_\mu}{\partial x_1} \dot{x} + \frac{\partial x'_\mu}{\partial x_2} \dot{y} \text{ as } A^1 = \dot{x} \text{ and } A^2 = \dot{y}$$

$\therefore$
$$A'^1 = \frac{\partial x'_1}{\partial x_1} \dot{x} + \frac{\partial x'_1}{\partial x_2} \dot{y}$$

$$= \frac{\partial r}{\partial x} \dot{x} + \frac{\partial r}{\partial y} \dot{y} \quad \text{as} \quad x_1 = x, x_2 = y \quad \text{and} \quad x_1' = r$$

$$= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}$$

$$= \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\dot{r}}{r} = \dot{r}$$

And
$$A'^2 = \frac{\partial x'_2}{\partial x_1} \dot{x} + \frac{\partial x'_2}{\partial x_2} \dot{y}$$

$$= \frac{\partial \theta}{\partial x} \dot{x} + \frac{\partial \theta}{\partial y} \dot{y} \quad \text{as} \quad x'_2 = \theta, x_1 = x \quad \text{and} \quad x_2 = y$$

$$= -\frac{y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y}$$

$$= -\frac{y\dot{x} - y\dot{x}}{r^2} = \dot{\theta}$$

(*ii*)
$$A'^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y} \text{ as in part } (i)$$

$$= \frac{x\ddot{x} + y\ddot{y}}{r}$$

But $x\dot{x} + y\dot{y} = r\dot{r}$ gives on differentiation,

$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = r\ddot{r} + r\dot{r}^2$,

i.e.,
$$x\ddot{x} + y\ddot{y} = r\ddot{r} + \dot{r}^2 - \dot{x}^2 - \dot{y}^2$$

$$= r\ddot{r} + \dot{r}^2 - (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2 - (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2$$

$$= r\ddot{r} + \dot{r}^2 - \dot{r}^2 - r^2\dot{\theta}^2$$

$$= r\ddot{r} - r^2\dot{\theta}^2$$

Thus
$$A'^1 = \frac{r\ddot{r} - r^2\dot{\theta}^2}{r}$$

$$= \ddot{r} - r\dot{\theta}^2$$

NOTES

And $$A'^2 = \frac{\partial\theta}{\partial x}\ddot{x} + \frac{\partial\theta}{\theta y}\ddot{y}$$

$$= \frac{x\ddot{y} + y\ddot{x}}{r^2}.$$

But $x\dot{y} - y\dot{x} = r^2\dot{\theta}$ gives on differentiation

$$\dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x} = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}$$

Or $$x\ddot{y} - y\ddot{x} = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}$$

$\therefore$ $$A'^2 = \frac{r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}}{r^2}$$

$$= \ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r}$$

1.10.2 Symmetric and Anti-Symmetric Tensors

Let a tensor be such that contravariant or covariant indices of it can be interchanged without altering the value of the tensor, then the tensor is termed as symmetrical or symmetric in these indices.

If $A^{\mu\nu}$ and $A^{\nu\mu}$ be two contravariant tensors in a certain system of coordinates such that

$$A^{\mu\nu} = A^{\nu\mu}$$

then if $A^{\mu\nu}$ and $A^{\nu\mu}$ become $A'^{\mu\nu}$ and $A'^{\nu\mu}$ in another system of coordinates, the symmetry will be maintained in this system also if $A'^{\mu\nu} = A'^{\nu\mu}$.

To show it, let us consider

$$A'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta}A^{\alpha\beta}$$

$$= \frac{\partial x'_\nu}{\partial x_\beta}\frac{\partial x'_\mu}{\partial x_\alpha}A^{\alpha\beta} \text{ (on interchanging the indices)}$$

$$= A'^{\nu\mu} \text{ as } A^{\beta\alpha} = A^{\alpha\beta}$$

which shows the symmetry in the other system also.

Similarly if we consider two covariant tensors, $A_{\mu\nu}$ and $A_{\nu\mu}$ such that $A_{\mu\nu} = A_{\nu\mu}$ in one system, then if they become $A'_{\mu\nu}$ and $A'_{\nu\mu}$ in another system, we have

$$A'_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu}\cdot\frac{\partial x_\beta}{\beta x'_\nu}A_{\alpha\beta}$$

$$= \frac{\partial x_\beta}{\partial x'_\nu}\cdot\frac{\partial x_\alpha}{\partial x'_\mu}A_{\beta\alpha} \text{ (on interchanging the indices)}$$

$$= A'_{\nu\mu} \text{ as } A_{\alpha\beta} = A_{\beta\alpha}.$$

In case one index is contravariant and other covariant, the symmetry cannot be easily defined. But it is notable that Kronecker delta which is a mixed tensor is symmetrical with respect to its indices.

When $A^{\mu\nu}$ is symmetrical, we have

$$A^{11} = A^{11}, A^{22} = A^{22} \text{ etc.}$$

NOTES

In all, there are ${}^4C_2 + 4 = {}^5C_2$ components.

As an example, the components of the angular momentum of a rigid body B_μ are connected with the components of its angular velocity A_α by the relations $B_\mu = \sum_{\alpha=1}^{3} T_{\mu\alpha} A_\alpha$, where $T_{\mu\alpha}$ is the inertia tensor. This tensor is symmetric because

$$T_{\mu\alpha} = T_{\alpha\mu} \qquad \text{... (1.52)}$$

Now it has been already mentioned that a tensor can be expressed as a matrix and the columns and rows of a matrix when interchanged, the resulting tensor is the conjugate tensor. As such the conjugate of the tensor may be written as

$$\phi_c = \begin{bmatrix} a_{xx} & a_{yx} & a_{zx} \\ a_{xy} & a_{yy} & a_{zy} \\ a_{xz} & a_{yz} & a_{zz} \end{bmatrix}$$

Thus the tensor ϕ_c is the conjugate of ϕ, when

$$a_{xy} = a_{yx}, \quad a_{xz} = a_{zx} \quad \text{and} \quad a_{yz} = a_{zy} \qquad \text{... (1.53)}$$

If a single tensor satisfies this condition, it is called a **symmetric tensor**. It means that Condition (1.52) is essential for a tensor to be symmetric. It has only six independent elements in three dimensional space and may be written as

$$\phi_{sym} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}$$

The Relations (1.53) between the components of tensors follow that any symmetrical tensor corresponds with a transformation from the principal axes to another rectangular system of axes. To express the symmetrical linear functions by graphical function, let us suppose that we have two vectors **u** and **v** such that

$$\mathbf{v} = \phi_{sym} \cdot \mathbf{u} \qquad \text{... (1.54)}$$

Project **u** upon **v** so that we calculate

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z, \qquad \text{... (1.55)}$$

where u_x, u_y, u_z and v_x, v_y, v_z are the components of **u** and **v** respectively along the principal axes.

Multiplying both sides of Equation (1.54) scalarly by **u**, we get

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot \phi_{sym} \cdot \mathbf{u} \\ &= u_x v_x + u_y v_y + u_z v_z \text{ from Equation (1.55)} \\ &= u_x(a_{xx}u_x + a_{xy}u_y + a_{xz}u_z) + u_y(a_{yx}u_x + a_{yy}u_y + a_{yz}u_z) \\ &\quad + u_z(a_{zx}u_x + a_{zy}u_y + a_{zz}u_z) \end{aligned}$$

Say $S = a_{xx}\, u_x^2 + a_{yy}u_y^2 + a_{zz}\, u_z^2 + 2(a_{xy}u_x\, u_y + a_{yz}\, u_y\, u_z + a_{zx}\, u_z\, u_x)$

where S is a *scalar point function.*

We get

$$\frac{\partial S}{\partial u_x} = 2\,[a_{xx}u_x + a_{xy}u_y + a_{xz}u_z] = 2v_x \text{ by Equation (1.43)}$$

NOTES

or $$v_x = \frac{1}{2}\ \frac{\partial S}{\partial u_x}$$

Similarly, $$v_y = \frac{1}{2}\ \frac{\partial S}{\partial u_y} \text{ and } v_z = \frac{1}{2}\ \frac{\partial S}{\partial u_z}$$

So that $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors along principal axes

$$= \frac{1}{2}\left[\frac{\partial S}{\partial u_x}\mathbf{i} + \frac{\partial S}{\partial u_y}\mathbf{j} + \frac{\partial S}{\partial u_z}\mathbf{k}\right]$$

$$= \frac{1}{2}\ \text{grad } S.$$

which shows that $\mathbf{v}$ is a vector perpendicular to the surface S = const. in the direction of the outward normal. But S = const. is an equation of the second degree in the rectangular components of $\mathbf{u}$ regarding these as coordinates defining the extremity P of the vector $\mathbf{u}$ the locus of P is a conicoid.

As a particular case if $S = 1$, the surface defined under certain conditions is the *tensor ellipsoid* as shown in Figure 1.12.

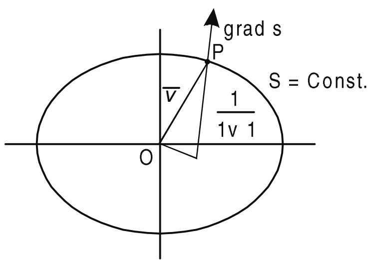

Fig. 1.12 *Tensor Ellipsoid*

Also $\mathbf{u}\,.\,\mathbf{v} = S = 1$ = resolute of $\mathbf{u}$ in the direction of $\mathbf{v}$.

In the direction of grade S, this resolute becomes $\frac{1}{|\,\mathbf{v}\,|}$.

Few other examples of symmetric tensor may be given as below

$$A_{\mu\alpha\beta} = A_{\alpha\mu\beta}$$

and $$A_{\mu\alpha\beta\gamma} = A_{\mu\beta\alpha\gamma} = A_{\alpha\mu\beta\gamma} = A_{\beta\alpha\mu\gamma} = A_{\alpha\beta\mu\gamma} = A_{\beta\mu\alpha\gamma}$$

Here the first tensor is symmetric in its first two indices and the second one is symmetric in first three indices.

If a tensor is such that two contravariant or covariant indices of it when interchanged, the components of the tensor alter in sign but not in magnitude, the tensor is said to be ***anti-symmetric*** *or* ***skew-symmetric****.*

Hence if $$A^{\mu\nu} = -A^{\nu\mu}$$

then, $$A'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta}A^{\alpha\beta}$$

$$= -\frac{\partial x'_\nu}{\partial x_\beta}\frac{\partial x'_\mu}{\partial x_\alpha}A^{\beta\alpha}$$

$$= -A'^{\nu\mu} \quad \text{as} \quad A^{\alpha\beta} = -A^{\beta\alpha}$$

NOTES

Similarly if $A_{\mu\nu} = -A_{\nu\mu}$, then $A'_{\mu\nu} = -A'_{\nu\mu}$

Here $A^{12} = -A^{21}$ etc. and $A^{11} = 0 = A^{22} = A^{33}$

As such number of numerical components is $^4C_2$ only.

Evidently, the components of an antisymmetric tensor satisfy the relations

$$A_{\mu\nu} + A_{\nu\mu} = 0 \quad \text{or} \quad A_{\mu\nu} = -A_{\nu\mu} \qquad \text{... (1.56)}$$

which follows that the tensor changes its sign when indices are interchanged. If $\mu = \nu$, then Equation (1.56) yields

$$A_{\mu\mu} + A_{\mu\mu} = 0 \quad \text{or} \quad 2A_{\mu\mu} = 0 \quad \text{or} \quad A_{\mu\mu} = 0$$

So in terms of coefficients, $a_{xx} = a_{yy} = a_{zz} = 0$

and $a_{xy} = -a_{yx}$, $a_{xz} = -a_{zx}$, $a_{yz} = -a_{zy}$

give the conditions for a tensor to be anti- or skew-symmetric.

Thus ϕ will be antisymmetric if

$$\phi_{sk} = \begin{bmatrix} 0 & a_{xy} & a_{xz} \\ -a_{xy} & 0 & a_{yz} \\ -a_{xz} & -a_{yz} & 0 \end{bmatrix}$$

The matrix has only three components. The property of having only three components is possessed by vectors. This leads to the conclusion that an operation of ϕ_{sk} on the vector **u**, is exactly equivalent to the vector product of two vectors, since the final result is itself a vector **v**. For example, consider the product of the coordinates of two points

$$A_{\mu\nu} = x_\mu\, \xi_\nu - \xi_\mu\, x_\nu \qquad \text{... (1.57)}$$

In 3-dimensional space, suffixes can have only 3 values (1, 2, 3) and so any pair of suffixes can be replaced by a single one which is not present in the pair i.e.,

$$\left.\begin{aligned} A_{23} &= -A_{32} = A_1 \\ A_{12} &= -A_{21} = A_3 \\ A_{31} &= -A_{13} = A_2 \end{aligned}\right\} \qquad \text{... (1.58)}$$

Thus the anti-symmetric system may be replaced by

$$\begin{aligned} \xi_{\mu\nu\sigma} &= +1 \text{ if } (\mu, \nu, \sigma) \text{ is an even permutation of } (1, 2, 3) \\ &= -1 \text{ if } (\mu, \nu, \sigma) \text{ is an odd permutation of } (1, 2, 3) \\ &= 0 \text{ if any of the suffixes are equal.} \end{aligned}$$

Now Equations (1.58) yield,

$$A_1 = \frac{1}{2}\,[2A_{23}],\; A_2 = \frac{1}{2}\,[2A_{13}],\; A_3 = \frac{1}{2}\,[2A_{12}]$$

i.e., $$A_1 = \frac{1}{2} \sum_{\mu,\nu=1}^{3} \xi_{\mu\nu\sigma}\, A_{\mu\nu} \qquad \text{... (1.59)}$$

There being only two terms in Sum (1.59).

Few other examples of antisymmetric tensors are

$$A_{\mu\nu\sigma} = -A_{\nu\mu\sigma}$$

and $$A_{\mu\nu\sigma\rho} = -A_{\nu\sigma\mu\rho} = A_{\sigma\mu\nu\rho} = -A_{\mu\sigma\nu\rho} = -A_{\nu\mu\sigma\rho} = -A_{\sigma\nu\mu\rho}.$$

As an illustration, *if $A_{\mu\nu}$ is antisymmetric tensor of second order and B^{μ} is a tensor of rank one, then $A_{\mu\nu} B_{\mu} B_{\nu} = 0$, summation being taken over repeated indices.*

Intrerchange of dummy suffixes gives

$$A_{\mu\nu} B^{\mu} B^{\nu} = A_{\nu\mu} B^{\nu} B^{\mu} \quad \text{... (1.60)}$$

where $A_{\mu\nu}$ being antisymmetric i.e.,

$$A_{\mu\nu} = -A_{\nu\mu}$$

renders

$$\begin{aligned} A_{\mu\nu} B^{\mu} B^{\nu} &= -A_{\nu\mu} B^{\mu} B^{\nu} \\ &= -A_{\nu\mu} B^{\nu} B^{\mu} \quad \text{... (1.61)} \end{aligned}$$

The addition of Equation (1.60) and (1.61) yields

$$A_{\mu\nu} B^{\mu} B^{\nu} = 0 \quad \text{... (1.62)}$$

1.10.3 Invariant Tensors

It is not known about any vector which has the same components in different systems of coordinates, but there exist tensors of higher ranks which have the same components in all the frames of reference. These tensors are called to have the *invariant components* or *invariant tensors* in general. One of the examples of such a tensor is Kronecker symbol defined as follows:

With respect to the old frame of the reference (i.e., before rotation)

$$\frac{\partial x_{\mu}}{\partial x_{\nu}} = \begin{cases} 0 \text{ if } \mu \neq \nu \\ 1 \text{ if } \mu = \nu \end{cases} \text{ since } x_{\mu} \text{ is independent of } x_{\nu}$$

But
$$\frac{\partial x_{\mu}}{\partial x'_{\nu}} = \frac{\partial x_{\mu}}{\partial x'_{\alpha}} \frac{\partial x'_{\alpha}}{\partial x_{\nu}}$$

Hence
$$\frac{\partial x_{\mu}}{\partial x_{\nu}} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} \frac{\partial x'_{\alpha}}{\partial x_{\nu}} = \delta^{\mu}_{\nu}$$

where
$$\delta^{\mu}_{\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$$

The symbol $\delta^{\mu}_{\nu} = \delta_{\mu\nu} = \delta^{\mu\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$ is called as **Kronecker delta symbol**. In terms of new frame of reference (i.e., after the rotation,) we may write

$$\begin{aligned} \delta'^{\mu}_{\nu} &= \frac{\partial x'_{\mu}}{\partial x'_{\nu}} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} \frac{\partial x'_{\alpha}}{\partial x'_{\nu}} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases} \\ &= \frac{\partial x'_{\mu}}{\partial x'_{\beta}} \cdot \frac{\partial x_{\alpha}}{\partial x'_{\nu}} \cdot \frac{\partial x_{\beta}}{\partial x_{\alpha}} \\ &= \frac{\partial x'_{\mu}}{\partial x_{\beta}} \frac{\partial x_{\alpha}}{\partial x'_{\nu}} \delta^{\beta}_{\alpha} \text{ since } \delta^{\beta}_{\alpha} = \frac{\partial x_{\beta}}{\partial x_{\alpha}} \end{aligned}$$

(in summation convention)

Hence δ^{μ}_{ν} is invariant and transforms as mixed tensor of rank two. Similarly $\delta_{\mu\nu}$ transforms as the components of covariant tensor of rank two while $\delta^{\mu\nu}$ transforms as a contravariant tensor of rank two.

NOTES

Kronecker symbol can be used as *substitution multiplier*.

Since $$A_\mu = \frac{\partial x_\nu}{\partial x'_\alpha}\frac{\partial x'_\alpha}{\partial x_\mu} A^\nu \quad \text{(in summation convention)}$$

$$= \delta^\mu_\nu A^\nu \text{ (by substitution of index)}$$

Similarly $$A_\nu = \frac{\partial x'_\alpha}{\partial x_\nu}\frac{\partial x_\mu}{\partial x'_\alpha} A^\mu_\alpha$$

$$= \delta^\mu_\nu A_\mu$$

and $$\delta_{\mu\nu}\ \delta^{\nu\beta} = \frac{\partial x_\mu}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\beta} = \frac{\partial x_\mu}{\partial x_\beta} = \delta^\mu_\beta$$

while $$\delta^\mu_\mu = \delta^1_1 + \delta^2_2 + \delta^3_3 \text{ (summation convention)}$$

$$= 3 \text{ (in three-dimensional geometry)}$$

$$= 4 \text{ (in four-dimensional geometry).}$$

Secondly, we define the **generalized Kronecker delta symbol**, $\delta^{\mu\nu}_{\beta\gamma}$ by

$$\delta^{\mu\nu}_{\beta\gamma} = \begin{cases} 1 \text{ if } \mu = \beta,\ \nu = \gamma \text{ and } \mu \neq \nu \ \left(\text{as } \delta^{12}_{12} = 1\right) \\ -1 \text{ if } \mu = \gamma,\ \nu = \beta \text{ and } \mu \neq \nu \ \left(\text{as } \delta^{12}_{21} = -1\right) \\ 0 \text{ for all other combinations of indices.} \end{cases}$$

Similarly we can define $\delta^{\mu\nu\sigma}_{\beta\gamma\xi}$ as an absolute (invariant) tensor of rank six.

Conclusively, if both upper and lower indices of a gernalized delta consist of the same distinct numbers chosen from 1, 2, 3, the delta is 1 of –1 according as the upper indices form an even or odd permutation of the lower; in all other pemutations the delta is zero. We have, for example,

$$\delta^{12}_{12} = 1,\ \delta^{32}_{32} = -1,\ \delta^{23}_{11} = \delta^{13}_{21} = 0$$

$$\delta^{123}_{123} = \delta^{231}_{123} = 1,\ \delta^{213}_{123} = \delta^{321}_{123} = -1,\ \delta^{312}_{123} = 0.$$

Evaluation of the various possible combinations of indices shows that

$$\left.\begin{aligned} \delta^{\mu\beta}_{\beta\nu} &= -\delta^{\mu\nu}_{\nu\beta} = 2\ \delta^\mu_\beta \\ \delta^{\mu\nu}_{\beta\xi} &= \delta^\mu_\beta\ \delta^\nu_\xi - \delta^\mu_\xi\ \delta^\nu_\beta \end{aligned}\right\}$$

Alternating or Permutation Epsilon Tensor: This tensor is also an invariant component tensor of third rank and anti-symmetric in every pair of indices. Let $\varepsilon_{\mu\nu\sigma}$ be such a tensor, then

$$\varepsilon_{\mu\nu\sigma} = -\varepsilon_{\nu\mu\sigma} = \varepsilon_{\sigma\mu\nu} = -\varepsilon_{\mu\sigma\nu} = \varepsilon_{\nu\sigma\mu} = -\varepsilon_{\sigma\nu\mu}$$

But if $$\mu = \nu \text{ then } \varepsilon_{\mu\nu\sigma} = -\varepsilon_{\nu\mu\sigma} \text{ gives } \varepsilon_{\mu\mu\sigma} = -\varepsilon_{\mu\mu\sigma}$$

or $$\varepsilon_{\mu\mu\sigma} = 0$$

It is clear that whenever two indices are equal, the component is zero. Moreover a tensor of third rank in three-dimensional geometry has 27 components. But in case of the tensor $\varepsilon_{\mu\nu\sigma}$ only 6 components are non-vanishing. All of them have the same absolute value, 3 being positive and the rest three are negative.

So

and
$$\left.\begin{aligned}\varepsilon_{123} &= \varepsilon_{xyz} = 1 = \varepsilon_{312} = \varepsilon_{231}\\ \varepsilon_{213} &= \varepsilon_{321} = \varepsilon_{132} = -1\end{aligned}\right\}$$

while all other components are zero.

Thus all the even permutations of 1, 2 and 3 correspond to the components with value +1, while all the odd permutations correspond to –1.

The transformation law for this type of tensor in three dimensional space is given by

$$\varepsilon'_{\mu\nu\sigma} = \sum_{\alpha,\beta,\gamma=1}^{3} a_{\mu\alpha} a_{\nu\beta} a_{\sigma\gamma}\, \varepsilon_{\alpha\beta\gamma} = \varepsilon_{\mu\nu\sigma}$$

i.e., $\varepsilon_{\mu\nu\sigma}$ is invariant.

Consequently,

$$\varepsilon'_{\mu\nu\sigma} = \varepsilon_{\mu\nu\sigma}\begin{cases} 1 \text{ when } \mu, \nu, \sigma \text{ are an even permutation,}\\ -1 \text{ when } \mu, \nu, \sigma \text{ are an odd permutation,}\\ 0 \text{ when } \mu, \nu, \sigma \text{ contain two or more repeated indices.}\end{cases}$$

Similarly, the contravariant components can also be discussed,

$$\left.\begin{aligned}&\varepsilon^{123} = \varepsilon^{312} = \varepsilon^{231} = 1, \varepsilon^{132} = \varepsilon^{213} = \varepsilon^{321} = -1,\\ &\varepsilon^{112} = \varepsilon^{233} = \varepsilon^{111} = \ldots = 0\end{aligned}\right\}$$

Pseudo Tensor: Let there be a tensor $\varepsilon_{\mu\sigma\tau\rho}$ of rank 4, defined such that

$$\varepsilon_{\mu\sigma\tau\rho} = \begin{cases} +1 \text{ if } \mu\sigma\tau\rho \text{ is an even permutation of } 0, 1, 2, 3\\ -1 \text{ if } \mu\sigma\tau\rho \text{ is an odd permutation of } 0, 1, 2, 3\\ 0 \text{ if two or more indices are equal}\end{cases}$$

These are termed as components of **pseudo tensor** of rank four.

In case ϕ is a scalar, the quantities $\phi\varepsilon_{\mu\sigma\tau\rho}$ are called as pseudo scalars since they have only one component.

From every antisymmetric tensor $A_{\mu\sigma}$ of the second rank a pseudo tensor $A^*_{\mu\sigma}$ of the same rank can be obtained by multiplying the former with a pseudo-tensor of rank 4.

i.e.,
$$A^*_{\mu\sigma} = \frac{1}{2}\sum_{\alpha,\beta=0}^{3} \varepsilon^{\mu\sigma\tau\rho} A_{\alpha\beta}$$

Thus the product of a tensor with a pseudo-tensor is a pseudo-tensor. It is called dual of a given tensor.

A Useful Property of ε Tensor

ε tensor can be used to write the cross-product of two vectors **A** and **B**.

Let
$$\mathbf{D} = \mathbf{A} \times \mathbf{B},$$

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then

$$D_1 = A_2B_3 - A_3B_2 = \varepsilon_{123}\, A_2B_3 + \varepsilon_{132}\, A_3B_2;$$

$A_1, A_2, A_3, B_1, B_2, B_3$ being components of **A** and **B**.

$$= \varepsilon_{1\nu\sigma}\, A_\nu B_\sigma$$

Similarly $\quad D_2 = \varepsilon_{2\nu\sigma}\, A_\nu B_\sigma$ (in summation convention)

and $\quad D_3 = \varepsilon_{3\nu\sigma}\, A_\nu B_\sigma$

or $\quad D_\mu = \varepsilon_{\mu\nu\sigma}\, A_\nu B_\sigma$

Evaluating the various possible combinations we may have

$$\varepsilon^{\mu\nu\sigma}\varepsilon_{\sigma\alpha\beta} = \delta^{\mu\nu}_{\alpha\beta} = \delta^\mu_\nu\, \delta^\nu_\beta - \delta^\mu_\beta\, \delta^\nu_\alpha$$

Thus, if $\quad r = \mathbf{C} \,.\, (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \,.\, \mathbf{D}$ for $\mathbf{D} = \mathbf{A} \times \mathbf{B}$.

then, $\quad r = C_\mu D^\mu$ (in summation convention)

or $\quad r = C_\mu{}^{\mu\nu\sigma} A_\nu\, A_\sigma$

$$= \varepsilon^{\mu\nu\sigma}\, C_\mu\, A_\nu\, B_\sigma$$

Similarly, vector triple product of three vectors can be given as

$$\mathbf{E} = \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \times \mathbf{D}$$

i.e., $\quad E^\mu = \varepsilon^{\mu\nu\sigma}\, C_\nu\, D_\sigma$

$$= \varepsilon^{\mu\nu\sigma}\, C_\nu\, (\varepsilon_{\alpha\alpha\beta}\, A^\alpha B^\beta)$$

$$= \varepsilon^{\mu\nu\sigma}\, \varepsilon_{\sigma\alpha\beta}\, C_\nu\, A^\alpha B^\beta$$

$$= \delta^{\mu\nu}_{\alpha\beta}\, C_\nu\, A^\alpha B^\beta = \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)\, C_\nu\, A^\alpha B^\beta$$

Since $\delta^\mu_\alpha\, A^\alpha = A^\mu$ etc., we get

$$E_\mu = \delta^\nu_\beta\, A^\mu C_\nu\, B^\beta - \delta^\nu_\alpha\, B^\mu\, C_\nu\, A^\alpha$$

$$= A^\mu\, (C_\beta B^\beta) - B^\mu\, (C_\alpha\, A^\alpha)$$

As such **E = A (C . B) – B (C . A).**

Evaluation of ∇ × (V × W) Using ε Tensor

Suppose, $\nabla \times (\mathbf{V} \times \mathbf{W}) = \nabla \times \mathbf{Z}$,

then $\quad (\nabla \times \mathbf{Z})_\mu = \varepsilon^{\mu\nu\sigma}\, \nabla_\nu\, Z_\sigma = \varepsilon^{\mu\nu\sigma}\, \nabla_\mu\, (\varepsilon_{\sigma\alpha\beta}\, V^\alpha\, W^\beta)$

$$\varepsilon^{\mu\nu\sigma}\, \varepsilon_{\sigma\alpha\beta}\, \nabla_\nu\, V^\alpha\, W^\beta = \delta^{\mu\nu}_{\alpha\beta}\, (V^\alpha\, \nabla_\nu\, W^\beta + W^\beta\, \nabla_\nu V^\alpha)$$

$$= \left(\delta^\mu_\nu \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)\, (V^\alpha\, \nabla_\nu\, W^\beta + W^\beta\, \nabla_\nu\, V^\alpha)$$

$$= V^\mu\, \nabla_\beta\, W^\beta - V^\nu\, \nabla_\nu\, W^\mu + W^\nu\, \nabla_\nu\, V^\mu - W^\mu\, \nabla_\alpha\, V^\alpha$$

So $\quad \nabla \times \mathbf{Z} = \nabla \times (\mathbf{V} \times \mathbf{W})$

$$= \mathbf{V}\, (\nabla \,.\, \mathbf{W}) - (\mathbf{V} \,.\, \nabla)\, \mathbf{W} + (\mathbf{W} \,.\, \nabla)\mathbf{V} - \mathbf{W}\, (\nabla \,.\, \mathbf{V})$$

Similarly all the vector relationships can be derived by using ε tensor.

Krutkov's Tensor: Let us consider a tensor $A^{\mu\gamma\beta\sigma}$ of fourth rank having following properties:

(1) Antisymmetric with respect to the first pair of indices

$$A^{\mu\gamma,\ \beta\alpha} = -A^{\gamma\mu,\ \beta\sigma}$$

(2) Antisymmetric in second pair of indices

$$A^{\mu\gamma,\ \beta\sigma} = -A^{\mu\gamma,\ \sigma\beta}$$

(3) Symmetric in cyclic order

$$A^{\mu\gamma,\ \beta\sigma} + A^{\mu\beta,\ \sigma\gamma} + A^{\mu\sigma,\ \gamma\beta} = 0$$

Then in terms of second derivatives of $A^{\mu\gamma,\beta\sigma}$ we can form a new tensor given by

$$B^{\mu\sigma} = \sum_{\gamma,\beta=0}^{3} \frac{\partial^2 A^{\mu\gamma,\beta\sigma}}{\partial x_\gamma\ \partial x_\beta} = 0 \qquad \text{... (1.63)}$$

This tensor is called as *Krutkov's tensor*. If we differentiate Equation (1.63) with respect to x, we have

$$\sum_{\sigma=0}^{3} \frac{\partial B^{\mu\sigma}}{\partial x_\sigma} = 0$$

This is an important property of Krutkov's tensor.

Example 1.41: Define a tensor. Prove that the Kronecker symbol δ_i^k is a tensor where components are the same in every coordinate system.

Solution: We know that the tensors are quantities obeying certain transformation laws so that tensor may be regarded as an indispensable part of study which is rather suitable for the mathematical formulation of natural laws which remain invariant when one coordinate system is changed to another. The rank of a tensor measures the number of the mode of changes of a physical quantity when passing from one system to another which is in rotation relative to the first. As such tensor of zero rank is a scalar quantity and the tensor of rank one is a vector quantity.

The laws of transformation of vector being defined by

$$A'^{\mu} = \sum_{\alpha=1}^{4} \frac{\partial x'_\mu}{\partial x_\alpha} A^{\alpha} \quad \text{(contravariant vector)}$$

and

$$A'_{\mu} = \sum_{\alpha=1}^{4} \frac{\partial x_\alpha}{\partial x'_\mu} A_{\alpha} \quad \text{(covariant vector)}$$

In Minkowski's four dimensional space, we define the tensors of rank two as follows:

Contravariant tensor: $$A'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha} \frac{\partial x'_\nu}{\partial x_\beta} A^{\alpha\beta} \qquad \text{... (1.64)}$$

Covariant tensor: $$A'_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} A_{\alpha\beta} \qquad \text{... (1.65)}$$

Mixed tensor: $$A'^{\nu}_{\mu} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x'_\nu}{\partial x_\beta} A_{\alpha\beta} \qquad \text{... (1.66)}$$

Each one having 4^2, i.e., 16 components.

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Similarly, we can define the tensors of higher ranks.

Now the Kronecker delta symbol δ_i^k is defined as

$$\delta_i^k = \frac{\partial x_k}{\partial x_i} = \frac{\partial x_k}{\partial x'_j}\frac{\partial x'_j}{\partial x_i}$$

which is easily deduced from Equation (1.66) by choosing A_α^β to be the **Kronecker delta** δ_α^β so that

$$A_\mu'^{\nu} = \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x'_\nu}{\partial x_\beta}\delta_\alpha^\beta = \frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x'_\nu}{\partial x_\alpha}\frac{\partial x_\beta}{\partial x_\alpha} = \frac{\partial x'_\nu}{\partial x'_\mu} = \delta_\mu^\nu$$

and now replacing μ by *i*, ν by *j* this gives

$$\delta_i^k = A_i'^k \frac{\partial x_\alpha}{\partial x'_i}\frac{\partial x'_k}{\partial x_\beta}A_\alpha^\beta = \frac{\partial x_\alpha}{\partial x'_i}\frac{\partial x'_k}{\partial x_\beta}\delta_\alpha^\beta$$

From the definition of mixed tensor, it follows that δ_i^k is a mixed tensor of order two with 16 components in 4-dimensional space.

In order to show that the components of the tensor δ_i^k are the same in every coordinate system, let us define the symbol δ_i^k as

$$\delta_i^k = 1 \text{ if } i = k$$
$$= 0 \text{ if } i \neq k$$

which is evident from $\delta_i^k = \dfrac{\partial x_k}{\partial x_i} = 0$ when $i \neq k$

$= 1$ when $i = k$

In terms of new frame of reference (or new coordinate system), we may have

$$\delta_i'^k = \frac{\partial x'_k}{\partial x_i'} = \frac{\partial x'_k}{\partial x_j}\frac{\partial x_j}{\partial x'_i} = 1 \text{ if } i = k$$
$$= 0 \text{ if } i \neq k$$

or
$$\delta_i'^k = \frac{\partial x'_k}{\partial x_l}\frac{\partial x_j}{\partial x'_i}\frac{\partial x_l}{\partial x_j} = \frac{\partial x'_k}{\partial x_l}\frac{\partial x_j}{\partial x'_i}\delta_j^l \text{ since } \delta_j^l = \frac{\partial x_l}{\partial x_j}$$

From which it is clear that δ_i^k is invariant and transforms as mixed tensor of rank two.

Example 1.42: Prove that Kronecker delta is a mixed tensor of rank two.

Solution: Its solution has been given in Example 1.41.

Example 1.43: Show that symmetry properties of a tensor are invariant.

Solution: If $A_{\lambda\mu\nu} = A_{\mu\lambda\nu}$ then we have to show that $A'_{\lambda\mu\nu} = A'_{\mu\lambda\nu}$

The definition follows:

$$A'_{\lambda\mu\nu} = \sum_{\alpha,\beta,\gamma=1}^{3}\frac{\partial x_\alpha}{\partial x'_\lambda}\frac{\partial x_\beta}{\partial x'_\mu}\frac{\partial x_\gamma}{\partial x'_\nu}A_{\alpha\beta\gamma}$$

and
$$A'_{\mu\lambda\nu} = \sum_{\beta,a,\gamma=1}^{3}\frac{\partial x_\beta}{\partial x'_\mu}\frac{\partial x_\alpha}{\partial x'_\lambda}\frac{\partial x_\gamma}{\partial x'_\nu}A_{\beta\alpha\gamma}$$

The given tensor being symmetrical in first two indices, we have

$$A_{\lambda\mu\nu} = A_{\mu\lambda\nu} \text{ and } A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$$

Using this relation and comparing the two equations for $A'_{\lambda\mu\nu}$ and $A'_{\mu\lambda\nu}$ we find that both the equations are identical, i.e., $A'_{\lambda\mu\nu} = A'_{\mu\lambda\nu}$.

Which follows that the tensor in the other system is also symmetrical in first two indices. Hence the properties of symmetric tensors are invariant.

1.10.4 Rules Which Govern Tensor Analysis

Rule I. *The sum and difference of two tensors of the same rank result in a third tensor of the same rank. Moreover, if $F_{\lambda\mu}$... and $G_{\lambda\mu}$... are the tensors of the same rank, then $pF_{\lambda\mu}$ ... + $qG_{\lambda\mu}$ ... is also a tensor of the same rank* (*p*, *q being numbers*).

Suppose there are two tensors $A_{\lambda\mu}$ and $B_{\lambda\mu}$, then it will be shown that

$$A_{\lambda\mu} + B_{\lambda\mu} = C_{\lambda\mu}$$

is another tensor of the same rank.

Expressing the tensor $A_{\lambda\mu}$ in the form of a matrix, we have

$$A_{\lambda\mu} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_{\lambda\mu} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and
$$C_{\lambda\mu} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

so that
$$A_{\lambda\mu} + B_{\lambda\mu} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

If the relations between the coefficient *a*'s and *b*'s be such that

$$a_{\lambda\mu} + b_{\lambda\mu} = c_{\lambda\mu}$$

then
$$A_{\lambda\mu} + B_{\lambda\mu} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = C_{\lambda\mu}$$

which is a tensor of the same rank.

Similarly
$$A_{\lambda\mu} - B_{\lambda\mu} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = D_{\lambda\mu}$$

where $a_{\lambda\mu} - b_{\lambda\mu} = d_{\lambda\mu}$.

Here $D_{\lambda\mu}$ is again a tensor of the same rank.

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Further,

$$pA_{\lambda\mu} + qB_{\lambda\mu} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = E_{\lambda\mu}$$

where $pa_{\lambda\mu} + qb_{\lambda\mu} = e_{\lambda\mu}$.
showing that $E_{\lambda\mu}$ is also a tensor of the same rank.

The rule of addition may be generalized for any number of tensors of any rank.

Suppose there are two mixed tensors T and S of rank N, having their r indices (from λ_1 to λ_r) contravariant and s indices (from μ_1 to μ_s) covariant, then laws of their transformation may be written as

$$T'^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right]\left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}}\right] T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

$$S'^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right]\left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}}\right] S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

If the sum of two tensors $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ and $S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ be a third tensor

$$U'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}, \text{ i.e., } U^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}, = T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

Then,

$$U'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} = T'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

$$= \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right]\left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}}\right]\left[T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}\right]$$

$$= \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\;\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}}\; U^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

Which is transformation equation for a tensor of rank N having r contravariant and s covariant indices and follows that the sum of two tensors of the same rank is a new tensor of the same rank.

Note. Here $\left|\frac{\partial x}{\partial x'}\right|$ is the *Jacobian of transformation* and the tensor $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ is known as *Relative tensor of weight W*. For $W = 0$, the relative tensor becomes *Absolute tensor*, whereas for $W = 1$, the relative tensor is known as *Tensor density*.

Rule II. *The direct product of two tensors gives a new tensor of rank equal to the sum of ranks of these tensors.*

Consider two tensors $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ of rank N, weight W and $S^{\sigma_1\ldots\sigma_p}_{\rho_1\ldots\mu_q}$ of rank N' weight W'.

Their transformation may be given as,

$$T'^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}} \frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}} \cdots \frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}} T^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_s}$$

$$S'^{\eta_1 \ldots \eta_r}_{\xi_1 \ldots \xi_s} = \left|\frac{\partial x}{\partial x'}\right|^{N'} \frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}} \cdots \frac{\partial x_{\rho_q}}{\partial x'_{\xi_q}} \frac{\partial x'_{\eta_1}}{\partial x_{\sigma_1}} \cdots \frac{\partial x'_{\eta_p}}{\partial x'_{\sigma_p}} S^{\sigma_1 \ldots \sigma_\rho}_{\rho_1 \ldots \mu_q}$$

Then $$T'^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} + S'^{\eta_1 \ldots \eta_\rho}_{\xi_1 \ldots \xi_q} = \left|\frac{\partial x}{\partial x'}\right|^{N+N'} \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}} \frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}} \cdots \frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}} \frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}} \cdots \frac{\partial x'_{\rho_q}}{\partial x'\xi_q}$$
$$\frac{\partial x'_{\eta_1}}{\partial x'_{\sigma_1}} \cdots \frac{\partial x'_{\eta_p}}{\partial x_{\sigma_p}} \times \left[T^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_s}\right]\left[S^{\sigma_1 \ldots \sigma_\rho}_{\rho_1 \ldots \rho_q}\right]$$

or $$U'^{\alpha_1 \ldots \alpha_s, \ldots \eta_1 \ldots \eta_p}_{\beta_1 \ldots \beta_s, \xi_1 \ldots \xi_q} = \left|\frac{\partial x}{\partial x'}\right|^{N+N'} \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}} \cdots \frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}} \cdots\cdots \frac{\partial x_{\alpha_r}}{\partial x_{\lambda_r}}$$
$$\times \frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}} \cdots \frac{\partial x_{\rho_q}}{\partial x'_{\xi_q}} \frac{\partial x'_{\eta_1}}{\partial x_{\sigma_1}} \cdots \frac{\partial x'_{\eta_p}}{\partial x_{\sigma_p}} \times U^{\lambda_1 \ldots \lambda_r, \sigma_1 \ldots \sigma_p}_{\mu_1 \ldots \mu_s, \rho_1 \ldots \rho_q} \quad \ldots (1.67)$$

where $$U^{\lambda_1 \ldots \lambda_r, \sigma_1 \ldots \sigma_p}_{\mu_1 \ldots \mu_s, \rho_1 \ldots \rho_q} = T^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_s} S^{\sigma_1 \ldots \sigma_p}_{\rho_1 \ldots \rho_q}$$

The Equation (1.67) transforms a tensor of rank $N + N'$ and weight $W + W'$.

Note: This rule may also be stated as:

The outer product of two relative tensors is itself a relative tensor of rank and weight equal to the sum of the ranks and the sum of weights of the given relative tensors respectively.

Rule III. Contraction: *The algebraic operation by which the rank of a tensor may be lowered by 2 (or by any even number) is known as contraction.*

The contraction of a tensor may be affected by adding up all the components which have equal indices in a given pair. Any two indices are converted into a pair of dummy indices.

Consider a tensor of rank 3 with one contravariant index α and two covariant indices β and γ. Then we have

$$A'^{\lambda}_{\mu\nu} = \sum_{\alpha, \beta, \gamma = 0}^{3} A^{\alpha}_{\beta\lambda} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x'_\nu} \cdot \frac{\partial x'_\lambda}{\partial x_\alpha}$$

Replacing ν by λ, we have

$$A'^{\lambda}_{\mu\nu} = \sum_{\lambda, \mu, \nu = 0}^{3} \mathrm{A}^{\alpha}_{\beta\gamma} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x'_\nu} \cdot \frac{\partial x'_\lambda}{\partial x_\alpha}$$

$$= \sum_{\alpha, \beta, \gamma = 0}^{3} \mathrm{A}^{\alpha}_{\beta\gamma} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x_\alpha}$$

But $$\frac{\partial x_\gamma}{\partial x_\alpha} = \begin{Bmatrix} 0 \text{ if } \gamma \neq \alpha \\ 1 \text{ if } \gamma = \alpha \end{Bmatrix}$$

Choosing the second condition, i.e., if $\gamma = \alpha$, $\frac{\partial x_\gamma}{\partial x_\alpha} = 1$, above relation becomes

$$A'^{\lambda}_{\mu\lambda} = \sum_{\alpha,\beta=0}^{3} A^{\alpha}_{\beta\alpha} \frac{\partial x_\beta}{\partial x'_\mu}$$

NOTES

i.e.,
$$A'_\mu = \sum_{\beta=0}^{3} \frac{\partial x_\beta}{\partial x'_\mu} A_\beta$$

Which denotes the law of transformation of tensors of rank one, i.e., vectors. As a general rule we equate a certain covariant index to a contravariant index, sum repeated indices, and obtain a new tensor of lower rank. This process is termed as **contraction**. The contraction of a tensor of rank 2 yeilds a scalar, i.e., tensor of rank zero.

Illustration: We know that the scalar product of two vectors is a scalar quantity. It follows that the scalar product of two tensors of rank one is a tensor of rank zero. As such the rank is lowered by two.

Rule IV. Extension of the Rank: *The differentiation of each component of a tensor of rank n with respect to x, y, z, gives a new tensor of rank* $(n + 1)$*, for example,*

$$\frac{\partial A_{\lambda\mu}}{\partial x_\nu} = B_{\lambda\mu\nu}$$

This rule may be proved for a simple case, where the original tensor is of rank zero, i.e., a scalar say $S(x_1, x_2, x_3, t)$ where derivatives relative to the axes K are

$$\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}.$$

In system K', the scalar is $S'(x'_1, x'_2, x'_3, t')$

$$S(x_1, x_2, x_3, t) = S'(x'_1, x'_2, x'_3, t')$$

So,
$$\left.\begin{aligned} \frac{\partial S'}{\partial x'_1} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \frac{\partial S}{\partial x_3}\frac{\partial x_{3.}}{\partial x'_1} \\ \frac{\partial S'}{\partial x'_2} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_2} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_2} + \frac{\partial S}{\partial x_3}\frac{\partial x_3}{\partial x'_2} \\ \frac{\partial S'}{\partial x'_3} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_3} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_3} + \frac{\partial S}{\partial x_3}\frac{\partial x_3}{\partial x'_3} \end{aligned}\right\}$$

which may be written as a single equation

$$\frac{\partial S'}{\partial x'_l} = \frac{\partial x_m}{\partial x'_l}\frac{\partial S}{\partial x_m} \quad (l, m = 1, 2, 3)$$

This shows that $\frac{\partial S}{\partial x_m}$ transforms like the components of a tensor of rank one, i.e., vector. Thus the differentiation of the tensor of rank zero, yields a tensor of rank one. The rank of a tensor can also be extended when a tensor depends on another tensor and a differentiation is performed. For example, consider a scalar, i.e., tensor of rank zero, say S depending on tensor $A_{\lambda\mu}$, so that

$$\frac{\partial S}{\partial A_{\lambda\mu}} = B_{\mu\nu}$$

is also a tensor of rank two. Thus the rank of the tensor is extended by two.

NOTES

Rule V. The Quotient Law: *If $A^{\lambda}B_{\mu\nu}$ is a tensor for all contravariant tensors A_{λ} then $B_{\mu\nu}$ is also a tensor.*

We have $$A'^{\lambda} B'_{\mu\nu} = A^{\alpha} B_{\beta\gamma} \frac{\partial \alpha_{\beta}}{\partial x'_{\mu}} \frac{\partial x_{\gamma}}{\partial x'_{\nu}} \frac{\partial x'_{\lambda}}{\partial x_{\alpha}}$$

$$=A'^{\lambda} B_{\beta\gamma} \frac{\partial x_{\beta}}{\partial x'_{\mu}} \frac{\partial x_{\gamma}}{\partial x'_{\nu}}$$

as $$A'^{\lambda} = A^{\alpha} \frac{\partial x'_{\lambda}}{\partial x_{\alpha}}$$

or $$A'^{\lambda}\left[B'_{\mu\nu} - B_{\mu\nu}\frac{\partial x_{\alpha}}{\partial x'_{\mu}}\frac{\partial x_{\gamma}}{\partial x'_{\nu}}\right] = 0$$

But A'^{λ} being arbitrary, $A'^{\lambda} \neq 0$ so that

$$B'_{\mu\nu} = B_{\mu\nu} \frac{\partial x_{\beta}}{\partial x'_{\mu}} \frac{\partial x_{\gamma}}{\partial x'_{\nu}}$$

which shows that $B_{\mu\nu}$ is a covariant tensor.

Note. If A be a symmetric convariant tensor of second order s.t. $\left|A_{\mu\nu}\right| = A \neq 0$, and we set,

$$A^{\mu\nu} = \frac{\text{Cofactor of } A_{\mu\nu} \text{ in } A}{A} = \frac{a_{\mu\nu}}{A} \qquad \text{... (1.68)}$$

$a_{\mu\nu}$ being cofactor of $A_{\mu\nu}$ in A, and $A_{\mu\nu}$ also being symmetric, then A and so $a_{\mu\nu}$ is symmetric. Consequently $A^{\mu\nu}$ is symmetric.

Also if B^{μ} be an arbitrary vector, then quotient law gives

$$B_{\nu} = A_{\mu\nu} B^{\mu}$$

as a convariant vector

$$\therefore \qquad B_{\nu} A^{\nu\sigma} = A_{\mu\nu} B^{\mu} A^{\mu\sigma} = A_{\mu\nu} B^{\mu} \frac{a_{\nu\sigma}}{A}$$

$$= \frac{A_{\mu\nu}\, a_{\nu\sigma}}{A} B\mu = \delta^{\sigma}_{\mu} B^{\mu} \text{ by determinant theory.}$$

or $$B_{\nu}A^{\nu\sigma} = B\sigma$$

Here Relation (1.68) $\Rightarrow$ *symmetric contravariant tensor of rank* 2, known as **conjugate** or **reciprocal tensor** of $A_{\mu\nu}$.

Example 1.44: Show that there exists no distinction between contravariant and covariant vectors if we restrict ourselves to trasformation of the type

$$x'_{\alpha} = a^{\alpha}_{\lambda}\; x_{\lambda} + b^{\alpha}$$

where b^{α} are n constant which do not necessarily form the components of a contravariant vector and a^{α}_{λ} are constant (not necessarily forming a tensor) such that

NOTES

$$a^{\alpha}_{\mu} \; a^{\alpha}_{\lambda} = \delta^{\mu}_{\lambda}$$

Solution: Given $\quad x'_{\alpha} = a^{\alpha}_{\lambda} \, x_{\lambda} + b^{\alpha} \qquad \ldots (1)$

i.e., $\quad a^{\alpha}_{\lambda} \, x_{\lambda} = x'_{\alpha} - b^{\alpha} \qquad \ldots (2)$

Multiplying Equation (2) throughout by a^{α}_{μ} and summing over the index α from 1 to n, we find

$$x_{\mu} = a^{\alpha}_{\mu} \, x'_{\alpha} - a^{\alpha}_{\mu} \, b^{\alpha} \qquad \ldots (3)$$

Now Equations (1) and (3) yield,

$$\frac{\partial x'_{\alpha}}{\partial x_{\beta}} = a^{\alpha}_{\beta} \text{ and } \frac{\partial x_{\beta}}{\partial x'_{\alpha}} = a^{\alpha}_{\beta}$$

So that $$\frac{\partial x'_{\alpha}}{\partial x_{\beta}} = \frac{\partial x_{\beta}}{\partial x'_{a}} = a^{\alpha}_{\beta}$$

This follows that the tranformation laws

$$A'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\beta}} A^{\beta} \text{ and } A'_{\mu} = \frac{\partial x_{\alpha}}{\partial x'_{\mu}} \, A_{\alpha}$$

define the same type of entity without any distinction between contravariant and covariant vectors.

1.10.5 Applications of Tensors

In many physical problems, the notion of a vector is too restricted, for example in an isotropic medium, Stress ***S*** and Strain ***X*** are related by the vector equation

$$\boldsymbol{S} = \boldsymbol{k}\,\mathbf{X}$$

X and ***S*** having the same direction.

If the medium is not isotropic, ***S*** and ***X*** are not in general in the same direction, it is then necessary to replace the scalar ***k*** by a more general mathematical construct capable, when acting on the vector ***X***, of changing its direction as well as its magnitude. Such a construct is a tensor.

As an example, let us consider the magnetization M produced in Benzene molecule in the xy-plane in a magnetic field H. When H is along the x-axis, the **diamagnetic susceptibility** results in a small magnetization in the opposite direction such that,

$$\mathrm{M}_x = \chi_1 \mathrm{H}_x \quad \text{[shown in Figure 1.13]}$$

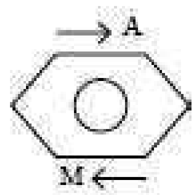

Fig. 1.13 *Small Magnetization in Opposite Direction*

Likewise, H along the y-axis produces a magnetization.

$$My = \chi_2 H_y.$$

But when H is perpendicular to the plane, the ring current leads to a greater magnetization, again in the opposite direction (such that shown in Figure 1.14.

$$M_2 = \chi_2 H_z$$

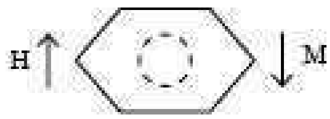

Fig. 1.14 *Ring Current Leads to Greater Magnetization*

And if H is at some arbitrary angle to the plane, then is at some different angle to the plane.

$$M = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$

$$M = \chi_1 H_x\hat{j} + \chi_2 H_y\hat{j} + \chi_3 H_3\hat{k}$$

Hence we must express the relationship between H and M by,

$$M = \chi H$$

Where M and H are three dimensional column matrices

And, $\chi = \dfrac{\delta Mj}{\delta Hj}$ is a 3 × 3 matrices

Which represent the magnetic susceptibility, therefore

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Consequently, χ is a tensor (second order tensor)

Another example of an anisotropic situation we might expert that the dipole moment induced in a Cl_2 molecule in an electric field will depend upon the orientation of the field relative to the molecular axis. The dipole moment produced when the field is perpendicular to the axis is smaller than when the field is along the axis. And for a random orientation the induced moment will not even be in the same direction as the field. Therefore we must write

$$\mu = \alpha E$$

Where α is the polarizability tensor.

Check Your Progress

15. What do you understand by determinant of a matrix?
16. Define the Cartesian tensor.

1.11 ANSWERS TO 'CHECK YOUR PROGRESS'

NOTES

1. The line vectors representing the quantities like force, velocity, etc., in which merely a linear action in a particular direction is involved, are termed as polar vectors.
2. Like vectors or co-directional vectors. The vectors which are collinear and have the same sense of directions, *i.e.*, the vectors directed in the same sense irrespective of their magnitudes are termed as like vectors. Where as unlike vector the vector which are collinear but have opposite sense of direction from each other are termed as unlike vector.
3. The scalar or dot product of two vectors *a* and *b*, with modules *a* and *b* respectively and their directions being inclined at an angle θ, is defined to be the real number *ab* cos θ, *i.e.*,

   $$\mathrm{a} \cdot \mathrm{b} = ab \cos \theta.$$
4. The gradient of any scalar function ϕ is defined as

   $$\text{grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x}\right)\phi$$

   $$= \nabla\phi$$

   where operator ∇ is generally known as '*Del*' or '*Nabla*' operator and read as 'Gradient' or 'Grad' in short.
5. If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the vector products of $\vec{a} \times \vec{b}$ with $\vec{c}$ and $\vec{a}$ with $\vec{b} \times \vec{c}$ are called the vector triple products of $\vec{a}, \vec{b}$ and $\vec{c}$.

   These products are written as $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$.
6. The Gauss theorem states that the outward flux of a vector field through a surface to the behaviour of the vector field inside the surface.
7. Let $< V, + >$ be an abelian group and $< F, +, \cdot >$ be a field. Define a function × (called scalar multiplication) from $F \times V \rightarrow V$, such that, for all $\alpha \in F$, $v \in V$, $\alpha \cdot v \in V$. Then V is said to form a *vector space* over F if for all $x, y \in V$, $\alpha, \beta \in F$, the following hold

   (*i*) $(\alpha + \beta)\, x = \alpha x + \beta x$

   (*ii*) $\alpha\,(x + y) = \alpha x + \alpha y$

   (*iii*) $(\alpha\beta)\, x = \alpha\,(\beta x)$

   (*iv*) $1 \cdot x = x$, 1 being unity of F.

   Also then, members of F are called scalars and those of V are called vectors.
8. In case $m = n$, the rectangular array becomes a square and so the matrix having number of rows and number of columns equal is called a Square Matrix of order n.

9. A square matrix $A = [a_{ij}]$ over the complex numbers is said to be Hermitian if the transposed conjugate of the matrix is equal to the matrix itself, i.e., $A^\theta = A\theta$.

10. Matrix is a rectangular array (*i.e.* arrangement) of objects. The number of rows and the number of columns are called its dimensions. The objects are called its entries. If the objects are real number, it is called a *real matrix*; if the objects are complex numbers, it is called a complex matrix.

11. A square matrix is called symmetric if the elements situated symmetrically with respect to the principal diagonal are equal.

12. A square matrix is called skew symmetric if the elements situated symmetrically with respect to the principal diagonal have opposite signs but the same magnitude and the principal diagonal elements are zero.

13. The system is called homogeneous if $b_1 = b_2 = ... = b_m = 0$. Therefore the system is called non-homogeneous if it is not homogeneous. *i.e.* at least one of $b_1, b_2, ..., b_m$ is non-zero.

14. Unitary transformation. If A be a unitary transformation of order $n \times n$ and X, x are column vectors of order $n \times 1$, then the linear transformation

    $$X = Ax$$

    is known as unitary transformation. Since

15. Determinant of a matrix can be taken as a function from the set of square matrices to the set of real numbers. Sometimes, the determinant of $A = [a_{ij}]$ will also be denoted by $|a_{ij}|$ *i.e.*

    $$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

    Note that a determinant can also be defined without reference to a matrix.

16. In geometry and linear algebra, a Cartesian tensor uses an orthonormal basis to represent a tensor in a Euclidean space in the form of components. Converting a tensor's components from one such basis to another is through an orthogonal transformation.

1.12 SUMMARY

- A vector having the initial and the terminal points coincident is termed as a zero vector or a null vector. Thus a null vector has its module zero.
- Evidently a vector can be represented by an infinite number of equal vectors by drawing parallel supports. Such a vector which can be transported from place to place such that it remains of the same magnitude and keep up the same direction is termed as a free vector.
- The value of a free vector depends only on its length and direction, but if it depends also on its position in space, i.e., if a vector is restricted to pass through a given origin, then it is termed as a localised vector.

NOTES

- A careful observation of the ways in which two vector quantities enter into combinations in various branches of mathematics and mechanics leads us to define two well marked and distinct kinds of products, one being called scalar or dot product and other being called vector or cross product.
- Scalar field is the region in which the scalar point function specifies the scalar physical quantity like temperature, electric potential, density, etc. It is represented by a continuous scalar function giving the value of the quantity at each point.
- The cyclic permutation of three vectors does not change the value of scalar product.
- The Gauss theorem demonstrates equality between triple integral (volume integral) of a function over a region of three-dimensional space and double integral (surface integral) of the function over the surface that bounds the corresponding region. In vector calculus, the Gauss theorem is also known as divergence theorem.
- A vector space originates from the notion of a vector that we are familiar with in mechanics or geometry.
- A scheme of detached coefficients a_{ij} arranged in m rows and n columns is called a matrix of order m by n or an $m \times n$ matrix or a matrix of type $m \times n$.
- A matrix whose number of rows equals the number of columns is called a square matrix.
- A matrix is called a zero matrix, if every element of it is zero.
- A system of linear non-homogeneous equations has either no solution or one solution or infinitely many solutions. Indeed, if such a system has two solutions, then it can be proved easily that it has infinitely many solutions.
- The notion of determinant is fundamental in algebra and has tremendous applications in many spheres of mathematical activities.
- The minor of an element in a determinant is the determinant obtained by deleting the row and the column which intersect in that element.
- The most familiar coordinate systems are the two-dimensional and three-dimensional Cartesian coordinate systems. Cartesian tensors may be used with any Euclidean space, or more technically, any finite-dimensional vector space over the field of real numbers that has an inner product.
- The rank of a tensor is determined by the number of suffixes or indices attached to it. As a matter of fact the rank of a tensor when raised as power to the number of dimensions, yields the number of components of the tensor and hence the components of the matrix that represents the tensor.
- The differentiation of the tensor of rank zero, yields a tensor of rank one. The rank of a tensor can also be extended when a tensor depends on another tensor and a differentiation is performed.

1.13 KEY TERMS

NOTES

- **Unit vector:** A vector having its modulus as unity is called a unit vector.
- **Coplanar vectors:** A system of vectors lying in the parallel planes or which can be made to lie in the same plane are said to be coplanar vectors. Evidently any two vectors are always coplanar.
- **Gauss theorem:** The Gauss theorem states that the outward flux of a vector field through a surface to the behaviour of the vector field inside the surface.
- **Square matrix:** A matrix whose number of rows equals the number of columns is called a square matrix.
- **Minors and cofactors:** The minor of an element in a determinant is the determinant obtained by deleting the row and the column which intersect in that element.
- **Cartesian tensor:** In geometry and linear algebra, a Cartesian tensor uses an orthonormal basis to represent a tensor in a Euclidean space in the form of components. Converting a tensor's components from one such basis to another is through an orthogonal transformation.

1.14 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the term vector.
2. Differentiate between coplanar and non- coplanar vector.
3. Give the characteristic of cross product.
4. What is divergence of vector?
5. What do you understand by vector product of four vector?
6. State the divergence theorem.
7. Define the vector space.
8. When matrix said to be sub-matrix?
9. What is product of matrix?
10. What do you understand by eigenvalues and eigenvectors?
11. Define the determinant.
12. Give the rules of govern tensor analysis.
13. What are the applications of tensors?

Long-Answer Questions

1. Discuss about the kinds of vector with their relevant examples.
2. Explain in detail about the dot, cross and triple product of vector giving examples.

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3. Illustrate the gradient, divergence and curl of scalar and vector point function.
4. Elaborate on the vector calculus with appropriate examples.
5. Briefly explain about the vector space with the help of examples.
6. Interpret the addition, multiplication, inverse and adjoint of matrix.
7. Analyse the homogeneous and non-homogeneous linear equations with appropriate examples.
8. Discuss about the eigenvalues, eigenvectors and diagonalizable with examples.
9. Explain in detail about the determinant.
10. Describe the tensor and its classification with relevant examples.

1.15 FURTHER READING

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UNIT 2 DIFFERENTIAL CALCULUS

NOTES

Structure

2.0 INTRODUCTION

Continuity of a function is the characteristic of a function by virtue of which, the graphical form of that function is a continuous wave. A differentiable function is a function whose derivative exists at each point in its domain. The process of finding derivative of a function is called differentiation.

In calculus, differentiation is one of the two important concepts apart from integration. Differentiation is a method of finding the derivative of a function. Differentiation is a process, in Maths, where we find the instantaneous rate of change in function based on one of its variables.

Maxwell-Boltzmann distribution, also called Maxwell distribution, a description of the statistical distribution of the energies of the molecules of a classical gas.

The Bohr radius is a physical constant, approximately equal to the most probable distance between the nucleus and the electron in a hydrogen atom in its ground state.

In mathematics, an integral assigns numbers to functions in a way that describes displacement, area, volume, and other concepts that arise by combining

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infinitesimal data. The process of finding integrals is called integration. In integral calculus, integration by reduction formulae is method relying on recurrence relations.

In mathematics, a partial differential equation is an equation which imposes relations between the various partial derivatives of a multivariable function. Curve, in mathematics, an abstract term used to describe the path of a continuously moving point.

In this unit, you will study about the function, continuity and differentiability, rules of differentiation, differential calculus, Maxwell distribution, Bohr radius, integral calculus, reduction formulae, partial differential equation, curve sketching.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Elaborate on the differential calculus
- Analyse the rules of differentiation
- Understand the function, continuity and differentiability
- Know about the Maxwell distribution
- Discuss the Bohr radius
- Interpret the integral calculus
- Analyse the reduction formulae
- Comprehend the partial differential equation
- Illustrate the curve sketching

2.2 DIFFERENTIATION

Differentiation method is specifically used for finding or estimating the **rate of change** when one quantity is compared with another, precisely when the rate of change is not constant. Following are some definitions of differentiation.

Definitions

1. In mathematics, '**Differentiation**' is the process of finding the **derivative**, or **rate of change**, of a **function**.
2. **Differentiation** is the process of finding or evaluating the **rate of change** of **dependent quantity** with respect to **independent quantity**, i.e., $\boldsymbol{y = fx}$ where $\boldsymbol{y}$ is **dependent** with respect to $\boldsymbol{x}$.

As defined above, differentiation method is used for "*finding a function that outputs the rate of change of one variable with respect to another variable*".

In **differential calculus**, the key objects are the **derivative of a function** and interrelated notions, such as the differential and its various applications. Fundamentally, the derivative of a function at a selected input value defines the rate of change of the function that is close or adjacent to that particular input value.

Thus, **differentiation** is the method used to find a **derivative**. Geometrically, the derivative of a function at a specified point is approximated as the slope of the tangent line on the graph of the function at the specified point, as long as the derivative exists and is well-defined at that selected point. **Derivatives** are often used for finding the **maxima** and **minima** of a function and the equations which involve the derivatives are termed as **differential equations**.

Normally, when we consider a real-valued function of a **single real variable**, then the **derivative of a function** at specified or selected point usually defines or determines the **best linear approximation** to the **function** at that specified or selected point. The derivative is used for determining the maximum and minimum values of specific or precise functions, such as profit, loss, cost, etc.

Both the **calculus**, the **differential** and the **integral** are linked or interrelated through the fundamental theorem of calculus, which specifies that the differentiation method is the reverse or opposite to integration method.

Differentiation has its applications in almost all types of quantitative disciplines and helps in solving various types of real-world problems, such as in the field of physics, the displacement derivative of any object in motion with respect to time is termed as the velocity of the moving object, and this derivative of velocity with respect to time is termed as acceleration. Furthermore, in chemistry the derivative can also be approximated for the reaction rate of any chemical reaction while in the operations research, the derivatives are specifically used for efficiently determining the methods for transporting materials or supplies and designing factories.

In various fields of mathematics, the derivatives and their generalizations are frequently used for solving the problems related to functional analysis, complex analysis, differential geometry, abstract algebra and measure theory.

Let us understand the concept of differentiation with the help of a simple example. Assume that we have to track the position of a moving vehicle on a road. To track its position we assign it a variable 'x'. In the meantime the position of the moving vehicle will change as the time changes, i.e., the variable x is dependent on time or in other words $x = f(t)$. Differentiation provides a function dx/dt which characterizes or represents the speed of the moving vehicle, i.e., the rate of change of the position of moving vehicle with respect to time.

Approximating the 'Rate of Change' which is 'Not Constant'

When we throw a round disc straight up in the air, then after some time the speed of disc slows down as the gravity or the gravitational force acts on the disc. As a result the disc direction changes and it starts falling down. During this whole motion, i.e., when the disc was moving straight up and when the disc was falling down the velocity keeps changing continuously. Basically, the motion gradually changes from positive (when the disc is thrown straight up) to negative, i.e., the motion decreases and becomes zero (when the disc is falling downwards). Therefore, when the disc is moving straight up then it has negative acceleration while the acceleration becomes positive when the disc is falling down.

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Figure 2.1 shows the graph of disc height (in metres) against time (in seconds).

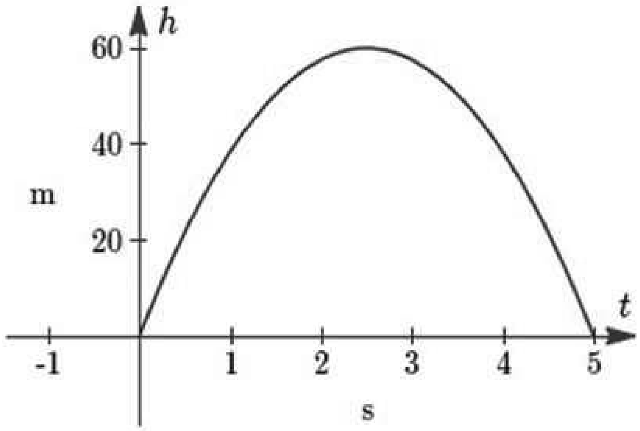

Fig. 2.1 *Graph of Disc Height at Time 't'*

In the Figure 2.1 the slope of the disc height at time *t* graph is continuously changing or varying while in motion. Initially or at the starting, the graph shows a steep positive slope which indicates that the velocity is large or enormous at the time when we throw the disc straight up. Subsequently, the slope decreases gradually or becomes less until it reaches 0, i.e., when the disc is at its maximum point then the velocity becomes zero (Refer Figure 2.2). When the disc starts falling downwards then the slope is referred as negative since it corresponds to the negative velocity as shown in Figure 2.2.

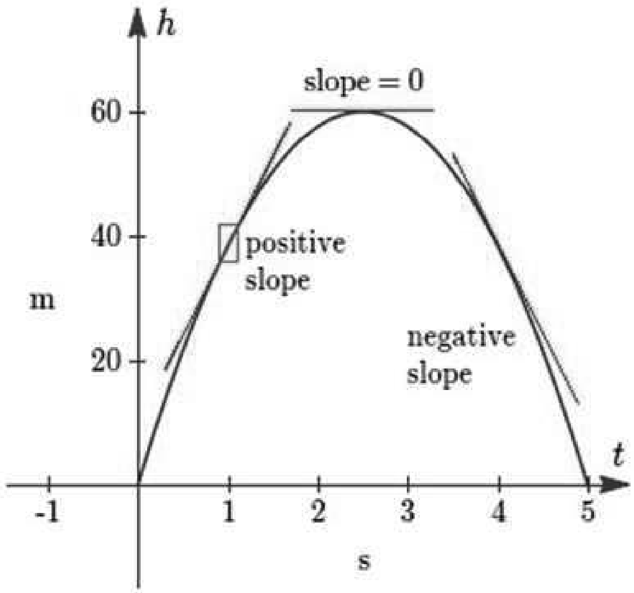

Fig. 2.2 *Graph of Disc Height Showing Slopes*

Now if we zoom in the graph at the section near $t = 1$, i.e., the rectangular section marked in Figure 2.2, then it looks as shown in Figure 2.3, zoomed view. The approximation is calculated on the section between $t = 0.9$ s and $t = 1.1$ s.

NOTES

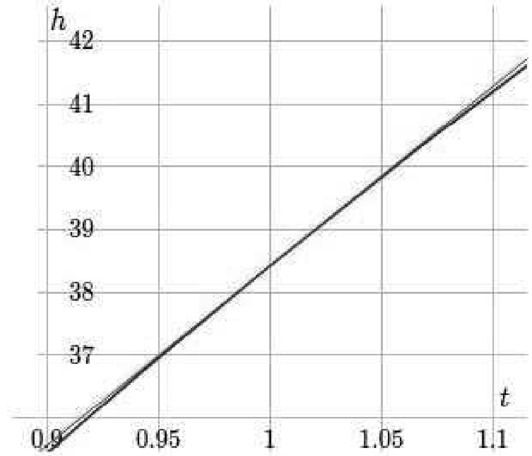

Fig. 2.3 *Zoomed Section of Graph between t = 0.9 s and t = 1.1 s*

When the selected curved section of graph is zoomed in then the curved line appears to be as a straight line. Almost perfect approximation can be made about the slope of the curve at the specified point $t = 1$, which is referred as the slope of the tangent to the curve (see the dim line near the straight line). The approximation is made by identifying or observing all those points through which the curve passes near $t = 1$. Mathematically, a tangent is defined as specific line which touches the curve only at one point.

Perceiving the graph, we comprehend that it passes through the points (0.9, 36.2) and (1.1, 42). Therefore the slope of the tangent at $t = 1$ is approximated as follows:

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{42.0 - 36.2}{1.1 - 0.9}$$

$$= \frac{5.8}{0.2}$$

$$= 29 \text{ m/s}$$

Since this is velocity therefore the unit of measurement is **m/s**. The rate of change is estimated by observing the slope.

Derivative

Assume that for the real numbers x and y, '$\boldsymbol{y}$' is a **function** of '$\boldsymbol{x}$', i.e., for each and every value of x, there will always be a corresponding value of y. We can write this equation of x and y relationship as,

$$\boldsymbol{y = f(x)}$$

If $f(x)$ is considered as the equation, also termed as **linear equation**, for a straight line, then there exist two real numbers m and b such that $y = mx + b$. In the slope intercept method, the 'm' defines the slope and is determined or evaluated using the following formula:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x};$$

NOTES

Here the symbol Δ (Greek letter Delta) is an abbreviation that represents '**change in**'. Therefore $\Delta y = m\,\Delta x$.

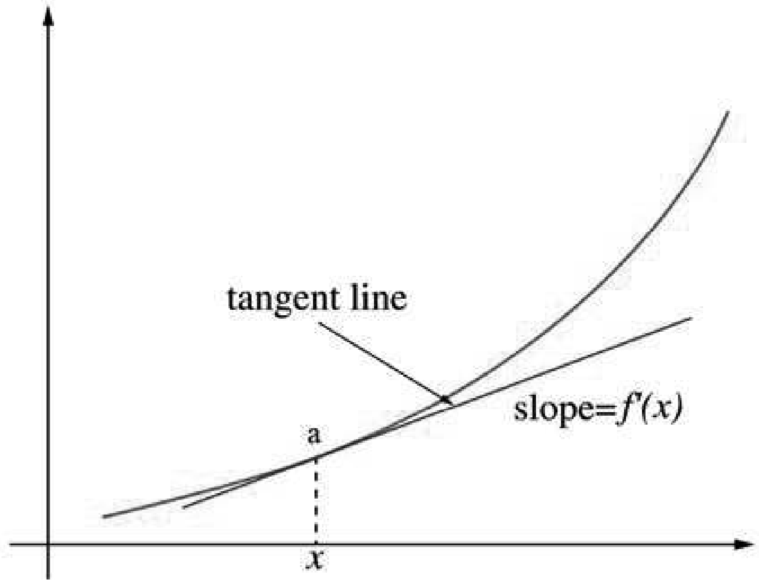

Fig. 2.4 *The Tangent Line at (x, f(x))*

Since the general or common function has no line, hence it has no slope. As per geometric rules, "the **derivative** of ***f*** at the **point *x* = *a*** is the **slope** of the **tangent line** of the **function *f*** at the **specified point *a***" as shown in Figure 2.4.

This is frequently denoted or represented as $f'(a)$ in Lagrange's notation and as in $\frac{dy}{dx}|_{x=a}$ Leibniz's notation. Because the derivative is defined as the slope of the linear approximation towards f at the specified point a, therefore the derivative in conjunction with the value of f at a basically determines or approximates the best 'linear approximation' or the 'linearization' of function f near the specified point 'a'.

In the domain of 'f' there is derivative for each point 'a' which defines a function for sending every specified point 'a' to the derivative of 'f' at 'a', such as when $f(x) = x^2$, then at that point the derivative function is $f'(x) = \frac{dy}{dx} = 2x$.

Another related significant notion is the '***differential of a function***'. For the real variables x and y, we can state that the derivative of f at x is the slope of the tangent line for the required graph of f at x. Additionally, the derivative of 'f' will be real number because both the source and the target of function f are one-dimensional. Now if we take the best linear approximation in a single direction then we can determine a partial derivative denoted as $\partial y/\partial x$. The total derivative can be defined as the linearization of f in all directions simultaneously.

2.2.1 Applications of Differential Calculus Including Maxima and Minima

NOTES

Definition 1

The point $(c, f(c))$ is called a *maximum point* of $y = f(x)$, if (i) $f(c+h) \le f(c)$, and (ii) $f(c-h) \le f(c)$ for small $h \ge 0$. $f(c)$ itself is called a *maximum value* of $f(x)$.

Definition 2

The point $(d, f(d))$ is called a *minimum point* of $y = f(x)$, if

(i) $f(d+h) \ge f(d)$, and

(ii) $f(d-h) \ge f(d)$

for all small $h \ge 0$.

$f(d)$ itself is called a *minimum value* of $f(x)$.

Thus, you observe that points $P[c-h, f(c-h)]$ and $Q[c+h, f(c+h)]$, which are very near to A, have ordinates less than that of A, whereas the points

$R[d-h, f(d-h)]$, and $S[d+h, f(d+h)]$,

Which are very close to B, have ordinates greater than that of B.

Figure 2.5 exhibits these maximum and minima points.

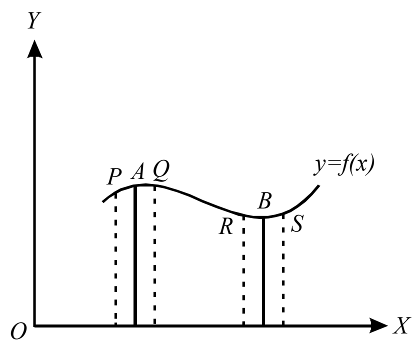

Fig. 2.5 *Maxima and Minima*

We will now prove that at a maximum or minimum point, the first differential coefficient with respect to x must vanish (in other words, tangents at a maximum or minimum point is parallel to x-axis, which is, otherwise, evident from Figure 2.5).

Let $[c, f(c)]$ be a maximum point and let $h \ge 0$ be a small number.

Since $\quad f(c-h) \le f(c)$

We have, $\quad f(c-h) - f(c) \le 0$

$$\Rightarrow \quad \frac{f(c-h) - f(c)}{-h} \ge 0 \qquad \ldots (2.1)$$

Again, $\quad f(c+h) \le f(c) \Rightarrow f(c+h) - f(c) \le 0$

$$\Rightarrow \qquad \frac{f(c+h) - f(c)}{h} \leq 0 \qquad \text{... (2.2)}$$

NOTES

Equation (2.1) implies that $\lim\limits_{k \to 0} \frac{f(c+k) - f(c)}{k} \geq 0,$ [put $k = -h$]

and Equation (2.2) gives that $\lim\limits_{k \to 0} \frac{f(c+h) - f(c)}{k} \leq 0$ [put $k = h$]

Thus, $$0 \leq \lim_{k \to 0} \frac{f(c+k) - f(c)}{k} \leq 0$$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)$ at $x = c$ is equal to zero.

i.e., $f'(c) = 0$

Again, let $[d, f(\mathrm{d})]$ be a minimum point and let $h \geq 0$ be a small number.

Since $f(d-h) \geq f(d)$

we have, $f(d-h) - f(d) \geq 0$

$$\Rightarrow \qquad \frac{f(d-h) - f(d)}{-h} \leq 0 \qquad \text{... (2.3)}$$

Again, $f(d+h) \geq f(d)$

$$\Rightarrow \qquad \frac{f(d+h) - f(d)}{h} \geq 0 \qquad \text{... (2.4)}$$

Equations (2.3) and (2.4) imply $\lim\limits_{k \to 0} \frac{f(d+k) - f(d)}{k} = 0$

i.e., $f'(d) = 0$

Before we proceed to find out the criterion for determining whether a point is maximum or minimum, we will discuss the increasing and decreasing functions of x.

A function $f(x)$ is said to be *increasing* (*decreasing*) if

$$f(x+c) \geq f(x) \geq f(x-c) \; [f(x+c) \leq f(x) \leq f(x-c)] \text{ for all } c \geq 0.$$

Theorem 2.1

If $f'(x) \geq 0$, then $f(x)$ is increasing function of x and if $f'(x) \leq 0$, then $f(x)$ is decreasing function of x.

Proof: $f'(x) \geq 0 \quad \Rightarrow \lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \geq 0$ ... (2.5)

In case $\delta x > 0$, put $c = \delta x$, then Equation (2.5) gives

$$f(x+c) \geq f(x)$$

In case $\delta x < 0$, put $c = -\delta x$, then Equation (2.5) gives

$$\frac{f(x-c) - f(x)}{-c} \geq 0$$

NOTES

$$\Rightarrow \qquad f(x-c) - f(x) \leq 0$$

$$\Rightarrow \qquad f(x) \geq f(x-c)$$

Hence, $\quad f(x+c) \geq f(x) \geq f(x-c)$

In other words, $f(x)$ is increasing function of x.

Suppose that $f'(x) \leq 0$

Then, $$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \leq 0 \qquad \text{... (2.6)}$$

In case $\delta x > 0$, put $c = \delta x$, then Equation (2.6) gives

$$f(x+c) - f(c) \leq 0$$

i.e., $\quad f(x+c) \leq f(x)$

If $\delta x < 0$, put $c = -\delta x$, then Equation (2.6) gives

$$\frac{f(x-c) - f(x)}{-c} \leq 0$$

$$\Rightarrow \qquad f(x-c) - f(x) \geq 0$$

$$\Rightarrow \qquad f(x) \leq f(x-c)$$

So, $\quad f(x+c) \leq f(x) \leq f(x-c)$

This means that $f(x)$ is a decreasing function of x.

Notes:

1. A function $f(x)$ is said to be strictly increasing (strictly decreasing) if

   $f(x+c) > f(x) > f(x-c)$ $[f(x+c) < f(x) < f(x-c)]$ for all $c > 0$.

2. It is seen that $f(x)$ is increasing, if $f(x) > f(y)$, whenever $x > y$, and $f(x)$ is decreasing, if

   $x > y \Rightarrow f(x) < f(y)$ and conversely.

3. It can be proved as above that a function $f(x)$ is strictly increasing or strictly decreasing accordingly, if

   $f'(x) > 0$ or $f'(x) < 0$.

Geometrically, the above theorem means that for an increasing function, tangent at any point makes acute angle with OX whereas for a decreasing function, tangent at any point makes an obtuse angle with x-axis. This is shown in Figure 2.6.

Let A be a maximum point $(c, f(c))$ of a curve $y = f(x)$.

Let $P\,[c-h, f(c-h)]$ and $Q\,[c+h, f(c+h)]$ be two points in the vicinity of A (i.e., h is very small).

NOTES

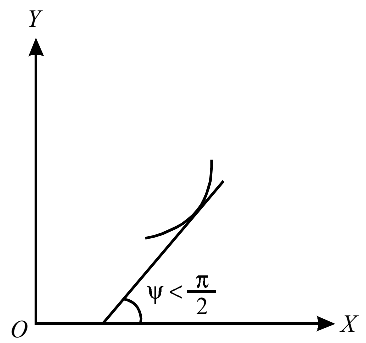

Fig. 2.6 *Tangent for an Increasing Function Making an Acute Angle with X-axis*

Figure 2.7(a) shows the tangent of the function making one obtuse angle with Y-axis. If ψ_1 and ψ_2 are inclinations of tangents at P and Q respectively, it is quite obvious from the Figure 2.7(b), that ψ_1 is acute and ψ_2 is obtuse.

Analytically, it is apparent from the fact that function is increasing from P to A and decreasing from A to Q. So, tanψ decreases as we pass through A (tanψ is +ve when ψ is acute and it is –ve when ψ is obtuse).

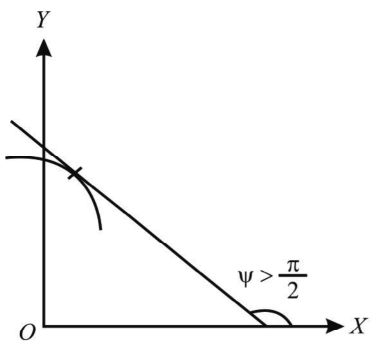

Fig. 2.7(a) *Tangent of the function making an Obtuse Angle with Y-axis*

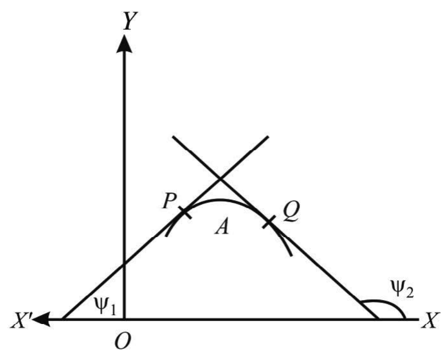

Fig. 2.7(b) *Inclinations of Tangents at P and Q*

Thus, $\frac{dy}{dx} = \tan\psi$ is a decreasing function of x. In other words, $\frac{d^2y}{dx^2} \leq 0$.
Since tanψ is strictly decreasing function of x, ($f(x)$ is not a constant function), so, $\frac{d^2y}{dx^2} < 0$. Consequently, at a maximum point c ($f(c)$),

$$f''(c) < 0.$$

Similarly, it can be easily seen that if $R\,[d-h, f(d-h)]$ and $S\,[d+h, f(d+h)]$ are two points in the neighbourhood of a minimum point $B[d, f(d)]$, slopes of tangents as we pass through B increase. (Here ψ_1 is obtuse, so tan $\psi_1 < 0$ and ψ_2 is acute, so tan $\psi_2 > 0$).

Therefore, for a minimum point $(d, f(d))$, $\frac{d^2y}{dx^2} > 0$, i.e., $f''(d) > 0$.
Figure 2.8 shows the slopes of the tangent passing through B.

NOTES

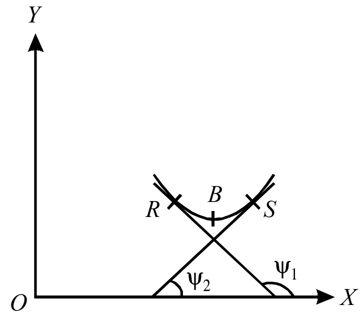

Fig. 2.8 Slopes of the Tangent Passing through B

Notes:

1. A point (α, β), such that $f'(\alpha) = 0, f''(\alpha) \neq 0$ and $f'''(\alpha) \neq 0$ is called a *point of inflexion.*
2. Any point at which $\frac{dy}{dx} = 0$ is called a *stationary point*. Thus, maxima and minima are stationary points. A stationary point need not be a maximum or a minimum point (it could be a point of flexion). Value of $f(x)$ at a stationary point is called *stationary value*.

   We have the following rule for the determination of maxima and minima, if they exist, of a function $y = f(x)$.

Step I. Putting $\frac{dy}{dx} = 0,$ calculate the stationary points.

Step II. Compute $\frac{d^2y}{dx^2}$ at these stationary points.

In case $\frac{d^2y}{dx^2} > 0,$ the stationary point is a minimum point.

In case $\frac{d^2y}{dx^2} < 0,$ the stationary point is a maximum point.

If $\frac{d^2y}{dx^2} = 0,$ then compute $\frac{d^3y}{dx^3}.$

If $\frac{d^3y}{dx^3} \neq 0,$ the stationary point is neither a maximum nor a minmum at that point.

If $\frac{d^3y}{dx^3} = 0,$ find $\frac{d^4y}{dx^4}.$ If the fourth derivative is negative at that point, then there is a maximum and if it is positive then there is a minimum.

Again in case $\frac{d^4y}{dx^4} = 0,$ find the fifth derivative and proceed as above till we get a definite answer.

Example 2.1: Find the maximum and minimum values of the expression

$$x^3 - 3x^2 - 9x + 27.$$

Solution: Let $y = x^3 - 3x^2 - 9x + 27$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

NOTES

For maxima and minima, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 6x - 9 = 0$

$\Rightarrow \quad (x-3)(x+1) = 0$

$\Rightarrow \quad x = -1, 3$

Now, $\frac{d^2y}{dx^2} = 6x - 6$

At $x = -1$, $\frac{d^2y}{dx^2} = -12 < 0,$ so $x = -1$ gives a maximum point of y.

Again, at $x = 3$, $\frac{d^2y}{dx^2} = +12 > 0, x = 3$ gives a minimum point of y.

Hence, maximum value of y is $[(-1)^3 - 3(-1)^2 - 9(-1) + 27]$

$= 36 + 1 - 3 = 34$

While minimum value of y is $3^3 - 3(3)^2 - 9(3) + 27$

$= 54 - 27 - 27 = 0$

Example 2.2: Find the maximum and minimum values of the function

$$8x^5 - 15x^4 + 10x^2.$$

Solution: Let $f(x) = 8x^5 - 15x^4 + 10x^2$

$\Rightarrow \quad f'(x) = 40x^4 - 60x^3 + 20x$

For maxima and minima,

$$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, -\frac{1}{2}$$

So, these are the points where there can be a maximum or a minimum

Now, $f''(x) = 160x^3 - 180x^2 + 20$

Thus, $f''(0) = 20 > 0 \quad \Rightarrow \quad$ There is a minimum at $x = 0$

Again, $f''(-\frac{1}{2}) = 160(-\frac{1}{2})^3 - 180(-\frac{1}{2})^2 + 20 = -45 < 0$

$\Rightarrow \quad$ there is a maximum at $x = -\frac{1}{2}$

Since $f''(1) = 160 - 180 + 20 = 0$, we cannot say anything regarding a maximum or a minimum at $x = 1$ at this stage. So, we find $f'''(x)$.

Now, $f'''(x) = 480x^2 - 360x$

But, $f'''(1) = 480 - 360 \neq 0$

$\Rightarrow \quad$ There is neither a maximum nor a minimum at $x = 1$

Hence, $f(-\frac{1}{2}) = \frac{21}{16}$ is maximum value and $f(0) = 0$ is minimum value.

Example 2.3: Find out maxima and minima of $\sin x + \cos x$, when x lies between 0 and 2π.

Solution: Let $y = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

For maxima and minima, $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}$.

Now, $\frac{d^2y}{dx^2} = -\sin x - \cos x$

At, $\quad x = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

And at, $x = \frac{3\pi}{4}, \frac{d^2y}{dx^2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

So, $x = \frac{\pi}{4}$ gives a maximum and $x = \frac{3\pi}{4}$ gives a minimum point of given function.

Example 2.4: Find maxima and minima of

$\sin x + \cos 2x \quad \text{for } 0 \leq x \leq \pi/2.$

Solution: Let $y = \sin x + \cos 2x$

For maxima and minima $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x - 2\sin 2x = 0$

$\Rightarrow \quad \cos x(1 - 4\sin x) = 0$

$\Rightarrow \quad \cos x = 0 \text{ or } \sin x = \frac{1}{4}$

$\Rightarrow \quad x = \frac{\pi}{2} \text{ or } x = \sin^{-1}\frac{1}{4}$

Now, $\frac{d^2y}{dx^2} = -\sin x - 4\cos 2x$

$= -\sin x - 4(1 - 2\sin^2 x)$

$= 8\sin^2 x - \sin x - 4$

At, $\quad x = \pi/2, \frac{d^2y}{dx^2} = 8 - 1 - 4 = 3 > 0$

So, $\quad x = \frac{\pi}{2}$ gives a minimum point of $\sin x + \cos 2x$

At, $x = \sin^{-1}\frac{1}{4}$, $\frac{d^2y}{dx^2} = 8\left(\frac{1}{16}\right) - \frac{1}{4} - 4$

NOTES

$$= \frac{1}{2} - \frac{1}{4} - 4$$

$$= \frac{1}{2} - \frac{17}{4} = -\frac{15}{4} < 0$$

Consequently, $x = \sin^{-1}\frac{1}{4}$ gives a maximum point of $\sin x + \cos 2x$.

Example 2.5: The sum of two numbers is 24. Find the numbers if the sum of their squares is to be minimum.

Solution: Let x and y be two numbers such that

$$x + y = 24$$

Let, $s = x^2 + y^2 = x^2 + (24 - x)^2$

For maxima and minima, $\frac{ds}{dx} = 0$

$\Rightarrow \quad 2x - 2\,(24 - x) = 0$

$\Rightarrow \quad 2x = 24 \quad \Rightarrow \quad x = 12$

Further, $\frac{d^2s}{dx^2} = 4 > 0$

So, $x = 12$ and $y = 12$ give minimum value.

Hence, the required numbers are 12 and 12.

2.3 RULES OF DIFFERENTIATION

We will now prove the following results for two differentiable functions $f(x)$ and $g(x)$.

(1) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

(2) $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$

(3) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$

(4) $\frac{d}{dx}[cf(x)] = cf'(x)$, where c is a constant

Where, of course, by $f'(x)$ mean $\frac{d}{dx} f(x)$.

NOTES

Proof:

(1) $\frac{d}{dx}[f(x)+g(x)]$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x)+g(x+\delta x)]-[f(x)+g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}+\lim_{\delta x \to 0} \frac{g(x+\delta x)-g(x)}{\delta x}$$

$$= f'(x)+g'(x)$$

Similarly, it can be shown that

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Thus, we have the following rule:

The derivative of the sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.

(2) $\frac{d}{dx}[f(x)\,g(x)]$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)\,g(x+\delta x)-f(x)\,g(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{g(x+\delta x)\big[f(x+\delta x)-f(x)\big]+f(x)\big[g(x+\delta x)-g(x)\big]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[g(x+\delta x).\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]+f(x)\left[\frac{g(x+\delta x)-g(x)}{\delta x}\right]\right]$$

$$= [\lim_{\delta x \to 0} g(x+\delta x)] \lim_{\delta x \to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

$$+\left[\lim_{\delta x \to 0} f(x)\right]\left[\lim_{\delta x \to 0} \frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$= g(x)\,f'(x)+f(x)\,g'(x).$$

Thus, we have the following rule for the derivative of a product of two functions:

The derivative of a product of two functions = (the derivative of first function × second function) + (first function × derivative of second function).

(3) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$

$$= \lim_{\delta x \to 0} \frac{\dfrac{f(x+\delta x)}{g(x+\delta x)}-\dfrac{f(x)}{g(x)}}{\delta x}$$

NOTES

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)\, g(x) - f(x)\, g(x+\delta x)}{\delta x . g(x+\delta x)\, g(x)}$$

$$= \lim_{\delta x \to 0} \frac{g(x)[f(x+\delta x) - f(x)] - f(x)[g(x+\delta x) - g(x)]}{\delta x . g(x+\delta x) g(x)}$$

$$= [\lim_{\delta_x \to 0} \frac{1}{g(x+\delta x)} \cdot \frac{1}{g(x)}] \left[\lim_{\delta x \to 0} \frac{g(x)[f(x+\delta x) - f(x)]}{\delta x} - \lim_{\delta x \to 0} \frac{f(x)[g(x+\delta x) - g(x)]}{\delta x} \right]$$

$$= \frac{1}{[g(x)]^2} \cdot [g(x) f'(x) - f(x) g'(x)]$$

$$= \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}.$$

The corresponding rule is stated as under:

The derivative of quotient of two functions=

$$\frac{(\textit{Derivative of Numerator} \times \textit{Denominator}) - (\textit{Numerator} \times \textit{Derivative of Denominator})}{(\textit{Denominator})^2}$$

(4) $\frac{d}{dx}[cf(x)] = \lim_{\delta x \to 0} \frac{cf(x+\delta x) - cf(x)}{\delta x}$

$$= c \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} \right] = cf'(x).$$

The derivative of a constant function is equal to the constant multiplied by the derivative of the function.

Differential Coefficients of Standard Functions

I. $\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$

Proof: Let, $y = x^n$

Then, $(y + \delta y) = (x + \delta x)^n$

$\Rightarrow$ $\delta y = (x + \delta x)^n - y = (x + \delta x)^n - x^n$

$$= x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1 \right]$$

$$= x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \ldots - 1 \right]$$

$$= nx^{n-1}(\delta x) + \frac{n(n-1)}{2!} x^{n-2} (\delta x)^2 + \ldots$$

$$\frac{\delta y}{\delta x} = nx^{n-1} + \text{terms containing powers of } \delta x$$

$\Rightarrow$ $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}$

Hence, $\frac{dy}{dx} = nx^{n-1}.$

NOTES

II. (*i*) $\frac{d}{dx}(a^x) = a^x \log_e a$

(*ii*) $\frac{d}{dx}(e^x) = e^x$

Proof: (*i*) Let, $y = a^x$

then, $y + \delta y = a^{x+\delta x}$

$\Rightarrow \quad \delta y = a^{x+\delta x} - a^x = a^x(a^{\delta x} - 1)$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{a^x(a^{\delta x}-1)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = a^x \lim_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right)$$

$$= a^x \lim_{\delta x \to 0}\frac{\left[1 + \delta x(\log a) + \frac{(\delta x)^2(\log a)^2}{2} + \ldots - 1\right]}{\delta x}$$

$$= a^x \lim_{\delta x \to 0}\left(\log a + \text{terms containing } \delta x\right)$$

$$= a^x \log a = a^x \log_e a$$

proves the first part.

(ii) Since $\log_e e = 1$, it follows from result (*i*) that $\frac{d}{dx}e^x = e^x$.

III. $\frac{d}{dx}\log_e x = \frac{1}{x}$

Proof: Let, $y = \log x$

$\Rightarrow \quad y + \delta y = \log(x + \delta x)$

$$\Rightarrow \quad \delta y = \log(x + \delta x) - \log x = \log\left(\frac{x+\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\log\left(1 + \frac{\delta x}{x}\right)}{\delta x}$$

$$= \frac{1}{x}\cdot\frac{x}{\delta x}\log\left(1 + \frac{\delta x}{x}\right) = \frac{1}{x}\log\left(1 + \frac{\delta x}{x}\right)^{x/\delta x}$$

$$\Rightarrow \quad \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\lim_{\delta x \to 0}\log_e\left(1 + \frac{\delta x}{x}\right)^{x/\delta x}$$

$$= \frac{1}{x}\log_e e = \frac{1}{x}$$

as $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$ and $\log_e e = 1$

Hence, $\frac{dy}{dx} = \frac{1}{x}$

IV. $\frac{d}{dx}(\sin x) = \cos x$

Proof: Now, $y = \sin x \Rightarrow y + \delta y = \sin(x + \delta x)$

$$\Rightarrow \quad \delta y = \sin(x + \delta x) - \sin x = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

NOTES

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{2\cos\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\delta x} = \cos\left(x+\frac{\delta x}{2}\right)\cdot\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\Rightarrow \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\right]\left[\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right]$$

$$= (\cos x)\left[\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right] = (\cos x)(1) = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x.$$

V. $\boldsymbol{\frac{d}{dx}(\cos x) = -\sin x}$ [The proof is similar to that of (IV).]

Notes:

1. The technique employed in the proofs of (I) to (IV) above is known as '*ab initio*' technique. We have utilized (apart from simple formulas of Algebra and Trigonometry) the definition of differential coefficient only. We have nowhere used the algebra of differentiable functions.
2. In (VI) to (XII) we shall utilize the algebra of differentiable functions.

VI. $\boldsymbol{\frac{d}{dx}(c) = 0}$**, where c is a constant.**

Proof: Let, $\quad y = c = cx^0.$

Then, $$\frac{dy}{dx} = c\left(\frac{dx^0}{dx}\right) = c(0.x^{0-1}) = 0.$$

VII. $\boldsymbol{\frac{d}{dx}(\tan x) = \sec^2 x}$

Proof: Let, $\quad y = \tan x = \frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin x)\cos x - \sin x\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$\text{s} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

VIII. $\boldsymbol{\frac{d}{dx}(\sec x) = \sec x \tan x}$

Proof: Let, $\quad y = \sec x = \frac{1}{\cos x}$

Then, $$\frac{dy}{dx} = \frac{\frac{d}{dx}(1)\cos x - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(0)(\cos x) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.$$

NOTES

Before we proceed further, we will introduce hyperbolic functions.

We define hyperbolic sine of x as $\frac{e^x - e^{-x}}{2}$ and write it as

$\sin h\ x = \frac{e^x - e^{-z}}{2}$.

Hyperbolic cosine of x is defined to be $\frac{e^x + e^{-x}}{2}$ and is denoted by $\cos h\ x$. It can be easily verified that

$$\cos h^2 x - \sin h^2 x = 1$$

Since ($\cos h\ \theta$, $\sin h\ \theta$) satisfies the equation $x^2 - y^2 = 1$ of a hyperbola, these functions are called hyperbolic functions.

In analogy with circular functions (i.e., $\sin x$, $\cos x$, etc.) we define $\tan h\ x$, $\cot h\ x$ $\sec h\ x$ and $\operatorname{cosec} h\ x$.

Thus, by definition, $\tan h\ x = \frac{\sin h\ x}{\cos h\ x}$, $\cot h\ x = \frac{1}{\tan h\ x}$,

$$\sec h\ x = \frac{1}{\cos h\ x} \text{ and } \operatorname{cosec} h\ x = \frac{1}{\sin h\ x}.$$

IX. $\boldsymbol{\frac{d}{dx}(\sin h\ x) = \cos h\ x}$

Proof: Before proving this result, we say that

$$\frac{d}{dx}(e^{-x}) = -e^{-x},$$

Because,

$$e^{-x} = (e^{-1})^x$$

$$\Rightarrow \quad \frac{d(e^{-x})}{dx} = (e^{-1})^x \log_e (e^{-1}) = e^{-x}(-1) = -e^{-x}$$

Now, let $y = \sin h\ x = \frac{1}{2}(e^x - e^{-x})$

Then, $$\frac{dy}{dx} = \frac{1}{2}\left(\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right)$$

$$= \frac{1}{2}[e^x - (-e^{-x})] = \frac{1}{2}(e^x + e^{-x}) = \cos h\ x.$$

X. $\boldsymbol{\frac{d}{dx}(\cos h\ x) = \sin h\ x}$

Proof is similar to that of (IX).

XI. $\boldsymbol{\frac{d}{dx}(\tan h\ x) = \sec h^2 x}$

Proof: Let, $\quad y = \tan h\ x = \frac{\sin h\ x}{\cos h\ x}$

NOTES

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin h\, x)\cos h\, x - \sin h\, x \frac{d}{dx}(\cos h\, x)}{(\cos h\, x)^2}$$

$$= \frac{(\cos h\, x)(\cos h\, x) - (\sin h\, x)(\sin h\, x)}{\cos h^2\, x}$$

$$= \frac{\cos h^2\, x - \sin h^2\, x}{\cos h^2\, x} = \frac{1}{\cos h^2\, x} = \sec h^2\, x.$$

XII. $\frac{d}{dx}(\sec h\, x) = -\sec h\, x \tan h\, x$

Proof: Let,
$$y = \sec h\, x = \frac{1}{\cos h\, x}$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(1)\cos h\, x - (1)\frac{d}{dx}(\cos h\, x)}{\cos h^2\, x}$$

$$= \frac{(0)(\cos h\, x) - \sin h\, x}{\cos h^2\, x}$$

$$= -\left(\frac{\sin h\, x}{\cos h\, x}\right)\left(\frac{1}{\cos h\, x}\right) = -\tan h\, x \sec h\, x.$$

Example 2.6: If $y = x^2 \sin x$, find $\frac{dy}{dx}$.

Solution: This is a problem of the type $\frac{d}{dx}(uv)$.

By applying the formula,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)\sin x + x^2 \frac{d}{dx}(\sin x)$$
$$= 2x \sin x + x^2 \cos x.$$

Example 2.7: If $y = x^2 \operatorname{cosec} x$, find $\frac{dy}{dx}$.

Solution: We can write y as $\frac{x^2}{\sin x}$

Applying the formula,

$$y = \frac{x^2}{\sin x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\frac{d}{dx}(x^2)\sin x - x^2\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{2x \sin x - x^2 \cos x}{\sin^2 x}.$$

Chain Rule of Differentiation

This is the most important and widely used rule for differentiation.

The rule states that:

If y is a differentiable function of z, and z is a differentiable function of x, then y is a differentiable function of x, i.e.,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Proof: Let $y = F(z)$ and $z = f(x)$.

If δx is change in x and corresponding changes in y and z are δy and δz respectively, then $y + \delta y = F(z + \delta z)$ and $z + \delta z = f(x + \delta x)$.

Thus, $\delta y = F(z + \delta z) - F(z)$ and $\delta z = f(x + \delta x) - f(x)$

Now, $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$$

$$\Rightarrow \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta z} \lim_{\delta x \to 0} \frac{\delta z}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\lim_{\delta z \to 0} \frac{\delta y}{\delta z} \right) \frac{dz}{dx},$$

Since $\delta x \to 0$ implies that $\delta z \to 0$

$$= \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Corollary: If y is a differentiable function of x_1, x_1 is a differentiable function of x_2, ... , x_{n-1} is a differentiable function of x, then y is a differentiable function of x.

And $$\frac{dy}{dx} = \frac{dy}{dx_1} \frac{dx_1}{dx_2} \cdots \frac{dx_{n-1}}{dx_n}.$$

Proof: Apply induction on n.

Example 2.8: Find the differential coefficient of sin log x with respect to x.

Solution: Put $z = \log x$, then, $y = \sin z$

Now, $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \cos z \cdot \frac{1}{x} = \frac{1}{x} \cos(\log x).$$

Example 2.9: Find the differential coefficient of (i) $e^{\sin x^2}$ (ii) $\log \sin x^2$ with respect to x.

Solution: (i) Put $x^2 = y$, $\sin x^2 = z$ and $u = e^{\sin x^2}$

Then, $u = e^z$, $z = \sin y$ and $y = x^2$

By chain rule,

$$\frac{du}{dx} = \frac{du}{dz} \frac{dz}{dy} \frac{dy}{dx}$$

$$= e^z \cos y \, 2x = e^{\sin y} \cos y \, 2x = 2xe^{\sin x^2} \cos x^2.$$

(ii) Let, $u = x^2$

$v = \sin x^2 = \sin u$

Then, $y = \log \sin x^2 = \log \sin u = \log v$

So, $$\frac{du}{dx} = 2x, \quad \frac{dv}{du} = \cos u \text{ and } \frac{dy}{dv} = \frac{1}{v}$$

Then, $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{v} \cdot \cos u \cdot 2x$$

$$= \frac{1}{\sin u} \cos u \cdot 2x = 2x \cot u = 2x \cot x^2.$$

Note: After some practice we can use the chain rule, without actually going through the substitutions. For example, If $y = \log(\sin x^2)$, then $\dfrac{dy}{dx} = \dfrac{1}{\sin x^2}\cos x^2 \cdot 2x = 2x\cot x^2$.

NOTES

Note that we have first differentiated log function according to the formula $\dfrac{d}{dt}(\log t) = \dfrac{1}{t}$. Since, here we have log (sin x^2), so the first term on differentiation is $\dfrac{1}{\sin x^2}$.

Now, consider $\sin x^2$ and differentiate it according to the formula $\dfrac{d}{du}(\sin u) = \cos u$. Thus, the second term is $\cos x^2$.

Finally, we differentiated x^2 with respect to x, so, the third term is $2x$.

Then, we multiplied all these three terms to get the answer $2x \cot x^2$. We will illustrate this quick method by few more examples.

Example 2.10: Find $\dfrac{dy}{dx}$, when $y = e^{(2x+3)^3}$.

Solution: Since, $\dfrac{d(e^t)}{dt} = e^t$ and $\dfrac{du^3}{du} = 3u^2$

And $\dfrac{d(2v)}{dv} = 2$

We get, $\dfrac{dy}{dx} = e^{(2x+3)^2} \cdot 3(2x+3)^2 \cdot (2.1+0) = 6(2x+3)^2 e^{(2x+3)^2}$

Example 2.11: Differentiate $y = \log[\sin(3x^2+5)]$ with respect to x.

Solution: $\dfrac{dy}{dx} = \dfrac{1}{\sin(3x^2+5)}\cos(3x^2+5)\,6x = 6x\cot(3x^2+5)$.

Example 2.12: Differentiate $y = \tan^2(\sqrt{x}+3)$.

Solution: $\dfrac{dy}{dx} = 2\tan(\sqrt{x}+3)\cdot\sec^2(\sqrt{x}+3)\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{\tan(\sqrt{x}+3)\sec^2(\sqrt{x}+3)}{\sqrt{x}}$.

2.4 FUNCTION, CONTINUITY AND DIFFERENTIABILITY

In Mathematics, you usually deal with two kinds of quantities, namely constants and variables. A quantity which is liable to vary is called a *variable quantity* or simply a *variable*. Temperature, pressure, distance of a moving train from a station are all variable quantities. On the other hand, a quantity that retains its value through all mathematical operations is termed as a *constant quantity* or a *constant.* Numbers like 4, 5, 2.5, π, etc., are all constants.

If x is a real variable (i.e., x takes up different values that are real numbers). Then in quantities $\log x$, $\sin x$, x^2, etc., log, sin, square are the functions.

Let us write $y = x^2$.

Therefore, if $x = 2$, then $y = 4$, if $x = 3$, then $y = 9$, etc.

Thus, for each value of x, y gets a unique corresponding value and this value is assigned each time by a certain *rule* (namely, square). This rule is what we call a function.

So, in general, by a function of x, we mean a rule that gives us a unique value corresponding to each value of x.

So, if $y = \sin x$, then whenever we give a different value to x, we get a corresponding unique value for y with the assistance of the function sine.

When we are dealing with any function, we simply write

$$y = f(x)$$

and say that y is a function of x although to be very correct we should say that y is the value assigned by the function f corresponding to a value of x. And we call x as the independent variable and y as dependent variable.

Functions play important role in Mathematics, Physics and Social Sciences.

A function which assigns a *fixed* value for every value of x is called a *constant function.* For example, $f(x) = 3$ is a constant function, since for any value of x, $f(x)$ remains equal to 3.

The next important notion is that of the limit of a function. It is quite possible that $f(x)$ may not be defined for all values of x. As an illustration, consider $f(x) = \dfrac{x^2 - 25}{x - 5}$. This is a function of x, provided x takes all real values except 5. If it were defined at $x = 5$, we would have got $f(5) = \dfrac{25 - 25}{5 - 5} = \dfrac{0}{0}$; a meaningless quantity. $\dfrac{0}{0}$ can take any finite value whatsoever, like $\dfrac{0}{0} = 5$. $\dfrac{0}{0} = \pi$, etc., since $0 \times 5 = 0$, $0 \times \pi = 0$. One could perhaps say here that why don't we cancel $x - 5$ first and then put $x = 5$ to get $f(5)$ equal to 10. There is a lapse in this argument as $x - 5$ is zero when $x = 5$ and cancellation of zero factor is not allowed in Mathematics, because you would get very absurd results like $1 = 2$ and $3 = 15$, etc., since $0 \times 1 = 0 \times 2$ and $0 \times 3 = 0 \times 15$. As a consequence you cannot determine $f(5)$, the value of $f(x)$ at $x = 5$. But you should not leave the problem here. Instead you try to evaluate the value of $f(x)$ when x is *very* near to 5 (and this will finally lead you to a value that would *almost* be the value of $f(5)$).

Thus, you can evaluate $f(x)$ at $x = 4.9998$ or $x = 5.00001$. The technique is quite simple. Cancel x–5 first (this step is perfectly legitimate as x is not equal to 5); then substitute the value of x. For example, $f(4.9998) = 4.9998 + 5\ 9.9998$ and $f(5.00001) = 5.00001 + 5 = 10.00001$.

We now write down some of the values given to x and the corresponding values acquired by $f(x)$ in Tables 2.1 and 2.2. In first table values of x are *increasing upto* 5 (being always less than 5) and in second table values of x are *decreasing down* to 5 (being always greater than 5).

NOTES

Table 2.1 *Increasing Value of x*

Value of x	Value of $f(x)$
4	9
4.235	9.235
4.976	9.976
4.99998	9.99998
4.9999999	9.999999
$\downarrow$	$\downarrow$

Table 2.2 *Decreasing Value of x*

Value of x	Value of $f(x)$
7	12
6.31	11.31
5.7984	10.7984
5.2175	10.2175
5.0039	10.0039
5.0000001	10.0000001
$\downarrow$	$\downarrow$

The above is expressed mathematically as: x tends to 5 from the left (in Table 2.1) and x tends to 5 from the right (in Table 2.2). In the first case we write $x \to 5-$ and in the second case we write $x \to 5+$.

Observe the pattern of change in the second column of each table. One could, after slight concentration see that in the first situation $f(x) \to 10-$ while in the second case $f(x) \to 10+$. Thus, you are tempted to assert that $f(x)$ approaches 10 (both from left and from right) as x approaches 5. This number 10, we call limit of $f(x)$ as x approaches 5.

This fact is denoted by, $\operatorname*{Lim}_{x\to 5} f(x) = 10.$

Consider now any small positive number, say, 0.01. When $|x-5| < 0.01$, i.e., $-0.01 < x-5 < 0.01$ or $4.99 < x < 5.01$, then $-0.01 < f(x) - 10 < 0.01$

Or, $\quad |f(x) - 10| < 0.01$

$(4.99 < x < 5.01 \quad \text{and} \quad x \neq 5 \quad \Rightarrow \quad f(x) = x + 5$

This yields in turn $f(x) - 10 = x - 5$

So, $\quad -0.01 < f(x) - 10 < 0.01)$

One can repeat the above experiment by starting with another small positive number say 0.00002 and note that whenever $|x-5| < 0.00002$,

Then, $\quad |f(x) - 10| < 0.00002.$

The expected conclusion will be that, however a small positive number ε we may start with, we shall always be able to find a $\delta > 0$ such that whenever $| x - 5 | < \delta$ then $| f(x) - 10 | < \varepsilon$. In the above illustration $\delta = \varepsilon$ will suffice.

Thus, you are led to the following definition of limit.

You say that $\lim\limits_{x \to a} f(x) = l$ if corresponding to any $\varepsilon > 0$, we can find $\delta > 0$, such that $| f(x) - l | < \varepsilon$ whenever $| x - a | < \delta$.

Let us evaluate some limits using this definition. Later on we'll give other convenient methods too.

Example 2.13: Evaluate $\lim\limits_{x \to a} x$.

Solution: Let $\varepsilon > 0$ be any number.

Take, $\delta = \varepsilon$. Evidently $\delta > 0$.

Now, $| x - a | < \delta \Rightarrow | f(x) - a | < \varepsilon$

Since, $f(x) = x$.

Hence, $\lim\limits_{x \to a} a$.

Example 2.14: Evaluate $\lim\limits_{x \to a} x^2$.

Solution: Let $\varepsilon > 0$ be any number.

Take, $\delta = -2 + \sqrt{4 + \varepsilon}$. Clearly $\delta > 0$

Now, $| x + 2 | = | x - 2 + 4 | \leq | x - 2 | + 4 < \delta + 4$

So, $| x - 2 | < \delta \Rightarrow | x^2 - 4 | = | x + 2 | \; |x - 2| < d^2 + 4\,\delta = \varepsilon$

Hence, $\lim\limits_{x \to 2} x^2 = 4$.

Example 2.15: Determine $\lim\limits_{x \to 3} \dfrac{1}{x}$.

Solution: Let $\varepsilon > 0$.

Put, $\delta = \dfrac{9\varepsilon}{1 + 3\varepsilon}$

Now, $| x - 3 | < \delta \Rightarrow | x - 3 | < \dfrac{9\varepsilon}{1 + 3\varepsilon}$ ...(1)

Also, $| x - 3 | < \delta \Rightarrow -\delta < x - 3$

$\Rightarrow \quad x > 3 - \delta = 3 - \dfrac{9\varepsilon}{1 + 3\varepsilon} = \dfrac{3}{1 + 3\varepsilon}$

$\Rightarrow \quad \dfrac{1}{x} < \dfrac{1 + 3\varepsilon}{3}$ ...(2)

NOTES

Equations (1) and (2) imply

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \left|\frac{x-3}{3x}\right| < \frac{9\varepsilon}{1+3\varepsilon}\;\frac{1+3\varepsilon}{3}\cdot\frac{1}{3} = \varepsilon$$

Consequently, $\lim\limits_{x\to 3}\frac{1}{x} = \frac{1}{3}$.

Example 2.16: Find out the limit of $\frac{x^2-1}{x-1}$ *as* $x \to 1$.

Solution: Let $\varepsilon > 0$ be any number.

Take $\delta = \varepsilon$.

Now $\quad |x-1| < \delta \Rightarrow |x-1| < \varepsilon$

$$\Rightarrow |x+1-2| < \varepsilon$$

$$\Rightarrow \left|\frac{x^2-1}{x-1}-2\right| < \varepsilon$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2-1}{x-1} = 2.$$

h-Method for Determining Limits

You put $a+h$ in place of x and simplify such that h gets cancelled from denominator and numerator. Putting $h = 0$, you get limit of $f(x)$ as $x \to a$.

Example 2.17: Evalute $\lim\limits_{x\to 1}\frac{x^2-1}{x^2-1}$.

Solution: $\lim\limits_{x\to 1}\frac{x^3-1}{x^2-1} = \lim\limits_{h\to 0}\frac{(1+h)^3-1}{(1+h)^2-1}$

$$= \lim_{h\to 0}\frac{3h+3h^2+h^3}{2h+h^2}$$

$$= \lim_{h\to 0}\frac{3+3h+h^2}{2+h}$$

$$= \frac{3}{2}.$$

Expansion Method for Evaluating Limits

This method is applicable to the functions which can be expanded in series. Following expansions are often utilized.

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-1)x^3}{3!} + \ldots$ provided $|x| < 1$

and n is any real number.

Note: If n is a positive integer, the expansion on RHS has finite number of terms only.

NOTES

(2) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$ provided $|x| < 1$.

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$

(4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots \infty$

(5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots \infty$

The method will be illustrated by means of the following examples.

Example 2.18: *Show that* $\operatorname{Lim}_{x \to 0} \frac{\sin x}{x} = 1.$

Solution: $\operatorname{Lim}_{x \to 0} \frac{\sin x}{x} = \operatorname{Lim}_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots}{x}$

$$= \operatorname{Lim}_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \ldots\right) = 1.$$

Note: The result of Example 2.6 shall be frequently used later on.

Example 2.19: Evaluate $\operatorname{Lim}_{x \to 0} \frac{e^x - e^{-x}}{x}$.

Solution: $\operatorname{Lim}_{x \to 0} \frac{e^x - e^{-x}}{x}$

$$= \operatorname{Lim}_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots\right)}{x}$$

$$= \operatorname{Lim}_{x \to 0} \frac{2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right)}{x}$$

$$= \operatorname{Lim}_{x \to 0} 2\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$$

$$= 2.$$

Notes:

1. $\operatorname{Lim}_{x \to 0} f(x) = l$ if and only if

   $\operatorname{Lim}_{x \to a-} f(x) = l = \operatorname{Lim}_{x \to a+} f(x)$.

   If one of the two equalities fails to hold, then we say that limit of $f(x)$ as $x \to a$ does not exist.

   Consider $\operatorname{Lim}_{x \to 0} \frac{1}{x}$. It can be easily seen that, if $x \to 0-$, then $\frac{1}{x} \to -\infty$ while, if $x \to 0+$, then $\frac{1}{x} \to \infty$. So, $\operatorname{Lim}_{x \to 0} \frac{1}{x}$ does not exist.

2. If $\operatorname{Lim}_{x \to a} f(x)$ exists, then it must be unique.

NOTES

Example 2.20: *f(x)* is defined as under

$$f(x) = 0 \qquad \text{for } x \leq 0$$
$$= \frac{1}{2} - x \qquad \text{for } x > 0.$$

Show that $\lim\limits_{x\to 2} f(x)$ does not exist.

Solution: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} f(0-h) = 0.$

Also, $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0+} f(0+h) = \lim\limits_{h\to 0} f\left(\frac{1}{2}-h\right)$

$$= \frac{1}{2}$$

Since, $\lim\limits_{x\to 0-} f(x) \neq \lim\limits_{x\to 0+} f(x)$,

$\lim\limits_{x\to 0} f(x)$ does not exist.

Notes: 1. If $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$, then

(i) $\lim\limits_{x\to a} [f(x) \pm g(x)] = l \pm m$

(ii) $\lim\limits_{x\to a} [f(x)\, g(x)] = lm$

(iii) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$ provided $m \neq 0$

(iv) $\lim\limits_{x\to a} [f(x)]^{g(x)} = l^m$, provided l^m is defined.

2. If $f(x) < g(x)$ for all x, then $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$.

Continuous Functions

We have seen in Example 2.2 that $\lim\limits_{x\to 2} x^2 = 4$, which is same as value of x^2 at $x = 2$.

Whereas, in Example 2.4, $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$, but the function itself is not defined as $x = 1$.

Again consider, $f(x) = \dfrac{x^2-9}{x-3}$, $x \neq 3$

$$= 1, \qquad x = 3$$

In this case, $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = 6$, and $f(3) = 1$.

Thus, a function may possess a limit as $x \to a$ but it may or may not be defined at $x = a$. Even if it is defined at $x = a$, its value may not be equal to its limit. This prompts us to define the following type of functions.

A function $f(x)$ is said to be continuous at $x = a$, if $\lim\limits_{x\to a} f(x) = f(a)$.

In other words, $f(x)$ is said to be continuous at $x = a$, if given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(a)| < \varepsilon$, whenever $|x - a| < \delta$.

Example 2.21: Check for continuity at $x = 0$, the function $(x) = |x|$.

Solution: By definition of absolute value, we can write.

NOTES

$$f(x) = x \text{ for all } x \geq 0$$
$$= -x \text{ for all } x < 0$$

We note that, $\quad f(0) = 0$

Further, $\quad \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} -(-h) = 0$

Also, $\quad \lim_{x \to 0+} f(x) \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} h = 0.$

Thus, $\quad \lim_{h \to 0} f(x) = 0 = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

2.5 MAXWELL DISTRIBUTION

For a closed system involving only pressure-volume work, the four fundamental equations are

$$dE = TdS - PdV$$
$$dH = TdS + PdV$$
$$dA = -PdV - SdT$$
$$dG = VdP - SdT$$

Since dE, dH, dA and dG are exact differentials, the mixed partial second order derivatives of the coefficients of the two terms on the right are equal. Applying this criterion to the equation, Maxwell's equations are obtained.

Now, $\quad dE = TdS - PdV$

$$\therefore \quad \left(\frac{\partial E}{\partial S}\right)_V = T \quad \text{and} \quad \left(\frac{\partial E}{\partial V}\right)_S = -P$$

Taking the second derivative,

$$\left\{\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial S}\right)_V\right\}_S = \left(\frac{\partial T}{\partial V}\right)_S$$

and, $\quad \left\{\frac{\partial}{\partial S}\left(\frac{\partial E}{\partial V}\right)\right\}_V = -\left(\frac{\partial P}{\partial S}\right)_V$

Since the mixed partial derivatives are equal,

$$\left\{\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial S}\right)_V\right\}_S = \left\{\frac{\partial}{\partial S}\left(\frac{\partial E}{\partial V}\right)\right\}_V$$

Hence, $\quad \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \quad$ ...(2.7)

The enthalpy change $dH = TdS + VdP$

$$\therefore \quad \left(\frac{\partial H}{\partial S}\right)_P = T \quad \text{and} \quad \left(\frac{\partial H}{\partial P}\right)_S = V$$

NOTES

Taking second derivatives

$$\left\{\frac{\partial}{\partial P}\left(\frac{\partial H}{\partial S}\right)_P\right\}_S = \left(\frac{\partial T}{\partial P}\right)_S \quad \text{and} \quad \left\{\frac{\partial}{\partial S}\left(\frac{\partial H}{\partial P}\right)_S\right\}_P = \left(\frac{\partial V}{\partial S}\right)_P$$

Since the mixed partial derivatives are equal,

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P \qquad ...(2.8)$$

The Helmholtz free energy change,

$$dA = -PdV - SdT$$

$$\therefore \quad \left(\frac{\partial A}{\partial T}\right)_V = -S \quad \text{and} \quad \left(\frac{\partial A}{\partial V}\right)_T = -P$$

Taking second derivatives,

$$\left\{\frac{\partial}{\partial V}\left(\frac{\partial A}{\partial T}\right)_V\right\}_T = -\left(\frac{\partial S}{\partial V}\right)_V \quad \text{and} \quad \left\{\frac{\partial}{\partial T}\left(\frac{\partial A}{\partial V}\right)_T\right\}_V = -\left(\frac{\partial P}{\partial T}\right)_V$$

Since the mixed partial derivatives are equal,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \qquad ...(2.9)$$

Gibbs free energy change $dG = VdP - SdT$

$$\therefore \quad \left(\frac{\partial G}{\partial T}\right)_P = -S \quad \text{and} \quad \left(\frac{\partial G}{\partial P}\right)_T = V$$

Taking second derivative,

$$\left\{\frac{\partial}{\partial P}\left(\frac{\partial G}{\partial T}\right)_P\right\}_T = -\left(\frac{\partial S}{\partial P}\right)_T \quad \text{and} \quad \left\{\frac{\partial}{\partial T}\left(\frac{\partial G}{\partial P}\right)_T\right\}_P = \left(\frac{\partial V}{\partial T}\right)_P$$

Since the mixed partial derivatives are equal, then

$$-\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial V}{\partial T}\right)_P \qquad ...(2.10)$$

These four relations are known as Maxwell's relations. These relations are quite useful because they help to express any thermodynamic property of a system in terms of easily measured physical quantities.

Check Your Progress

1. Define the term differentiation.
2. State the chain rule of differentiation.
3. Define the variable quantity.
4. How Maxwell's equation is obtain?

2.6 BOHR RADIUS

The Bohr radius is a physical constant, approximately equal to the most probable distance between the nucleus and the electron in a hydrogen atom in its ground state. It is named after Niels Bohr, due to its role in the Bohr model of an atom. Its

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value is 5.29177210903×10^{-11} m. In the Bohr model for atomic structure, put forward by Niels Bohr in 1913, electrons orbit a central nucleus under electrostatic attraction. The original derivation posited that electrons have orbital angular momentum in integer multiples of the reduced Planck constant, which successfully matched the observation of discrete energy levels in emission spectra, along with predicting a fixed radius for each of these levels. In the simplest atom, hydrogen, a single electron orbits the nucleus, and its smallest possible orbit, with the lowest energy, has an orbital radius almost equal to the Bohr radius. (It is not exactly the Bohr radius due to the reduced mass effect. They differ by about 0.05%.)

The Bohr model of the atom was superseded by an electron probability cloud obeying the Schrödinger equation as published in 1926. This is further complicated by spin and quantum vacuum effects to produce fine structure and hyperfine structure. However, the Bohr radius formula remains central in atomic physics calculations, due to its simple relationship with fundamental constants.

In Schrödinger's quantum-mechanical theory of the hydrogen atom, the Bohr radius represents the most probable value of the radial coordinate of the electron position, and therefore the most probable distance of the electron from the nucleus.

The Bohr radius is defined as following formula

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = \frac{\varepsilon_0 h^2}{\pi m_e e^2} = \frac{\hbar}{m_e c\alpha},$$

Where as

$\hbar$ is the Reduced Planck Constant

$\in_o$ is the Permittivity of Free Space,

h is the Planck Constant,

$\boldsymbol{m_e}$ is the Mass of Electron,

e is the Elementary Charge,

c is the Speed of Light in Vacuum,

α is the Fine-Structure Constant.

The CODATA (Committee on Data of the International Science Council) value of the Bohr radius (in SI units) is 5.29177210903(80) ×10^{-11} m.

Hydrogen Atom and Similar Systems

The Bohr radius including the effect of reduced mass in the hydrogen atom is given by

$$a_0^* = \frac{m_e}{\mu} a_0,$$

Where $\mu = \frac{m_e m_p}{m_e + m_p}$ is the reduced mass of the electron–proton system (with m_p being the mass of proton). The use of reduced mass is a generalization of the classical two- body problem when we are outside the approximation that the

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mass of the orbiting body is negligible compared to the mass of the body being orbited. Since the reduced mass of the electron–proton system is a little bit smaller than the electron mass, the 'Reduced' Bohr radius is slightly larger than the Bohr radius ($a_0^* \approx 1.00054\, a_0 \approx 5.2946541 \times 10^{-11}$ meter).

This result can be generalized to other systems, such as ***Positronium*** (an electron orbiting a positron) and ***Muonium*** (an electron orbiting an anti-muon) by using the reduced mass of the system and considering the possible change in charge. Typically, Bohr model relations (radius, energy, etc.) can be easily modified for these exotic systems (up to lowest order) by simply replacing the electron mass with the reduced mass for the system (as well as adjusting the charge when appropriate). For example, the radius of Positronium is approximately $2a_0$, since the reduced mass of the Positronium system is half the electron mass ($\mu_{e^-,e^+} = m_e/2$).

A hydrogen-like atom will have a Bohr radius which primarily scales as $r_Z = a_0/Z$ the number of protons in the nucleus. Meanwhile, the reduced mass (μ) only becomes better approximated by m_e in the limit of increasing nuclear mass. These results are summarized in following equation

$$r_{Z,\mu} = \frac{m_e}{\mu}\frac{a_0}{Z}.$$

A Table of Approximate Relationships is given below.

System	Radius
Hydrogen	$1.00054\, a_0$
Positronium	$2a_0$
Muonium	$1.0048\, a_0$
He^+	$a_0/2$
Li^{2+}	$a_0/3$

2.7 EXACT AND INEXACT DIFFERENTIAL IN THERMODYNAMICS AND THEIR APPLICATIONS

In thermodynamics, when **dQ** is exact, the function **Q** is a state function of the system. Generally, neither work nor heat is a state function. An exact differential is sometimes also called a 'Total Differential', or a 'Full Differential', or, in the study of differential geometry, it is termed an exact form.

For a state function, each infinitesimal step that we add together by the process of integration is called an exact differential. When integrated, the sum of exact differentials is a value that is independent of path, depending only on the initial and final states.

State functions depend only on the state of the system. Quantities whose values are independent of path are called state functions, and their differentials are exact **(dP, dV, dG,dT...).** Quantities that depend on the path followed between states are called path functions, and their differentials are inexact **(dw, dq).**

A first-order differential equation (of one variable) is called exact, or an exact differential, if it is the result of a simple differentiation. The equation $\mathbf{P(x, y)\ y^2 + Q(x, y) = 0}$, or in the equivalent alternate notation $\mathbf{P(x, y)\,dy + Q(x, y)\,dx = 0}$, is exact if $\mathbf{P_x(x, y) = Q_y(x, y)}$.

Determine whether the following differential is exact or inexact. If it is exact, determine **z=z(x,y).** If this equality holds, the differential is exact. Therefore, $\mathbf{dz=(2x+y)dx+(x+y)dy}$ is the total differential of $\mathbf{z=x^2+xy+y^2/2+c}$.

In multivariate calculus, a differential is said to be exact or perfect, as contrasted with an inexact differential, if it is of the form ***dQ,*** for some differentiable function ***Q.***

Definition

We work in three dimensions, with similar definitions holding in any other number of dimensions. In three dimensions, a form of the type

$$A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz$$

is called a differential form. This form is called exact on a domain $D \subset \mathbb{R}^3$ in space if there exists some scalar function $Q = Q(x,y,z)$ defined on D such that

$$dQ \equiv \left(\frac{\partial Q}{\partial x}\right)_{y,z} dx + \left(\frac{\partial Q}{\partial y}\right)_{x,z} dy + \left(\frac{\partial Q}{\partial z}\right)_{x,y} dz, \qquad dQ = A\,dx + B\,dy + C\,dz$$

Throughout ***D.*** This is equivalent to saying that the vector field ***(A, B, C)*** is a conservative vector field, with corresponding potential ***Q***.

Note: The subscripts outside the parenthesis indicate which variables are being held constant during differentiation. Due to the definition of the partial derivative, these subscripts are not required, but they are included as a reminder.

One Dimension

In one dimension, a differential form

$$A(x)dx$$

Is exact as long as ***A*** has an antiderivative (but not necessarily one in terms of elementary functions). If ***A*** has an antiderivative, let Q be an antiderivative of ***A*** and this ***Q*** satisfies the condition for exactness. If ***A*** does not have an antiderivative, we cannot write ***dQ=A(x) dx*** and so the differential form is inexact.

Two and Three Dimensions

By symmetry of second derivatives, for any 'Well-Behaved' (non-pathological) function ***Q,*** we have

$$\frac{\partial^2 Q}{\partial x\,\partial y} = \frac{\partial^2 Q}{\partial y\,\partial x}$$

Hence, it follows that in a simply-connected region ***R*** of the ***xy***-plane, a differential

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$$A(x,y)\,dx + B(x,y)\,dy$$

Is an exact differential if and only if the following holds:

$$\left(\frac{\partial A}{\partial y}\right)_x = \left(\frac{\partial B}{\partial x}\right)_y$$

For three dimensions, a differential

$$dQ = A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz$$

Is an exact differential in a simply-connected region ***R*** of the ***xyz***-coordinate system if between the functions A, B and C there exist the relations:

$$\left(\frac{\partial A}{\partial y}\right)_{x,z} = \left(\frac{\partial B}{\partial x}\right)_{y,z} \;;\; \left(\frac{\partial A}{\partial z}\right)_{x,y} = \left(\frac{\partial C}{\partial x}\right)_{y,z} \;;\; \left(\frac{\partial B}{\partial z}\right)_{x,y} = \left(\frac{\partial C}{\partial y}\right)_{x,z}$$

These conditions are equivalent to the following one: If ***G*** is the graph of this vector valued function then for all tangent vectors ***X,Y*** of the surface G then ***s(X, Y) = 0*** with s the symplectic form.

These conditions, which are easy to generalize, arise from the independence of the order of differentiations in the calculation of the second derivatives. So, in order for a differential ***dQ,*** that is a function of four variables to be an exact differential, there are six conditions to satisfy.

In summary, when a differential ***dQ*** is exact:

- the function Q exists;
- $\int_i^f dQ = Q(f) - Q(i),$ independent of the path followed.

In thermodynamics, when ***dQ*** is exact, the function ***Q*** is a state function of the system. The thermodynamic functions ***U, S, H, A*** and ***G*** are state functions. Generally, neither work nor heat is a state function. An exact differential is sometimes also called a '***total Differential'***, or a '***full Differential'***, or, in the study of differential geometry, it is termed an exact form.

Partial Differential Relations

If three variables, ***x, y*** and ***z*** are bound by the condition ***F(x,y,z) =*** Constant for some differentiable function ***F(x,y,z),*** then the following total differentials exist

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy.$$

Substituting the first equation into the second and rearranging, we obtain

$$dz = \left(\frac{\partial z}{\partial x}\right)_y \left[\left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz\right] + \left(\frac{\partial z}{\partial y}\right)_x dy,$$

$$dz = \left[\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial z}{\partial y}\right)_x\right] dy + \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y dz,$$

$$\left[1 - \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y\right] dz = \left[\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial z}{\partial y}\right)_x\right] dy.$$

Since y and z are independent variables, dy and dz may be chosen without restriction. For this last equation to hold in general, the bracketed terms must be equal to zero.

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Reciprocity Relation

Setting the first term in brackets equal to zero yields

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y = 1.$$

A slight rearrangement gives a reciprocity relation,

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}.$$

There are two more permutations of the foregoing derivation that give a total of three reciprocity relations between x, y and z. Reciprocity relations show that the inverse of a partial derivative is equal to its reciprocal.

Cyclic Relation

The cyclic relation is also known as the cyclic rule or the Triple product rule. Setting the second term in brackets equal to zero yields

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -\left(\frac{\partial z}{\partial y}\right)_x.$$

Using a reciprocity relation for $\frac{\partial z}{\partial y}$ is used with subsequent rearrangement, a standard form for implicit differentiation is obtained:

$$\left(\frac{\partial y}{\partial x}\right)_z = -\frac{\left(\frac{\partial z}{\partial x}\right)_y}{\left(\frac{\partial z}{\partial y}\right)_x}.$$

An '**Inexact Differential**' or '**Imperfect Differential**' is a type of differential used in thermodynamics to express changes in path dependent quantities. In contrast, an integral of an exact differential (a differential of a function) is always path independent since the integral acts to invert the differential operator. Consequently, a quantity with an inexact differential cannot be expressed as a function of only the variables within the differential, i.e., its value cannot be inferred just by looking at the initial and final states of a given system. Inexact differentials are primarily used in calculations involving heat and work because they are path functions, not state functions.

Definition

An inexact differential is commonly defined as a differential form δu for which there is no corresponding function f such that: $f = \int \delta u$. More precisely, an

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inexact differential is a differential form that cannot be expressed as the differential of a function. In the language of vector calculus, for a given vector field ***F***, $\delta f = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is an inexact differential if there is no function ***f*** such that

$$\mathbf{F} = \nabla f$$

The fundamental theorem of calculus for line integrals requires path independence in order to express the values of a given vector field in terms of the partial derivatives of another function that is the multivariate analogue of the antiderivative. This is because there can be no unique representation of an antiderivative for inexact differentials since their variation is inconsistent along different paths. This stipulation of path independence is a necessary addendum to the fundamental theorem of calculus because in one-dimensional calculus there is only one path in between two points defined by a function.

First Law of Thermodynamics

Inexact differentials are known especially for their presence in the first law of thermodynamics:

$$\mathrm{d}U = \delta Q + \delta W$$

Instead of the differential symbol ***d***, the symbol δ is used instead, a convention which originated in the 19th century work of German mathematician Carl Gottfried Neumann, indicating that ***Q*** (heat) and ***W*** (work) are path-dependent, while ***U*** (internal energy) is not.

Internal energy ***U*** is a state function, meaning its change can be inferred just by comparing two different states of the system (independently of its transition path), which we can therefore indicate with U_1 **and** U_2. Since we can go from state U_1 to state U_2 either by providing heat $\Delta Q = U_2 - U_1$ or work $\Delta W = U_2 - U_1$, such a change of state does not uniquely identify the amount of work W done to the system or heat **Q** transferred, but only the change in internal energy ΔU.

Heat and Work

A fire requires heat, fuel, and an oxidizing agent. The energy required to overcome the activation energy barrier for combustion is transferred as heat into the system, resulting in changes to the system's internal energy. In a process, the energy input to start a fire may comprise both work and heat, such as when one rubs tinder (work) and experiences friction (heat) to start a fire. The ensuing combustion is highly exothermic, which releases heat. The overall change in internal energy does not reveal the mode of energy transfer and quantifies only the net work and heat. The difference between initial and final states of the system's internal energy does not account for the extent of the energy interactions transpired. Therefore, internal energy is a state function (i.e., exact differential), while heat and work are path functions (i.e., inexact differentials) because integration must account for the path taken.

Integrating Factors

It is sometimes possible to convert an inexact differential into an exact one by means of an integrating factor. The most common example of this in thermodynamics is the definition of entropy:

$$dS = \frac{\delta Q_{\text{rev}}}{T}$$

In this case, δQ is an inexact differential, because its effect on the state of the system can be compensated by δW. However, when divided by the absolute temperature and when the exchange occurs at reversible conditions, it produces an exact differential: the entropy S is also a state function.

2.8 INTEGRAL CALCULUS

Definite integration is a significant and an essential constituent of integral calculus. Both the integrations – **indefinite** and **definite** are **interrelated** processes. Fundamentally, the indefinite integration provides the basis for **definite integral**.

Definitions of Definite Integral

1. The **definite integral** of a **function** has a unique value. A definite integral is denoted by,

   $$\int_a^b f(x)\,dx$$

   In the above notation, **'*a*'** is termed as the **lower limit** of the integral while **'*b*'** is termed as the **upper limit** of the integral.

2. When it is definite or specific that how to start and end the process of integration, i.e., the **upper** and **lower limits** of **integration** (***b*** and ***a***) are defined then the **integral** is termed as **'Definite Integral'**. A definite integral is denoted by,

   $$\int_a^b f(x)\,dx$$

3. Principally, the **definite integral** of the form $\int_a^b f(x)\,dx$ is evaluated either as,

   1. The **limit of the sum**.

      Or as,

   2. The **anti-derivative 'F'** for the **interval [*a*, *b*]**.

   When we consider the case as anti-derivative 'F' for the interval [*a*, *b*], then its evaluated value is defined as the difference between the values of 'F' at the specified end points, to be precise **F(*b*) – F(*a*)**.

2.8.1 Evaluation of Definite Integral as the Limit of a Sum

Assume that '*f*' is a continuous function that is specified on close by interval [*a*, *b*]. Let the function has all non-negative values. In this case the graph of the function will be a curve above the *x*-axis.

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The definite integral certainly will be $\int_a^b f(x)\,dx$, which is the specified area bounded by the curve '$y = f(x)$' with $x = a$, $x = b$ on the x-axis. For estimating the area between the curve, we evaluate the segment PRSQP along with, $x = a$, $x = b$ and the x-axis (Refer Figure 2.9).

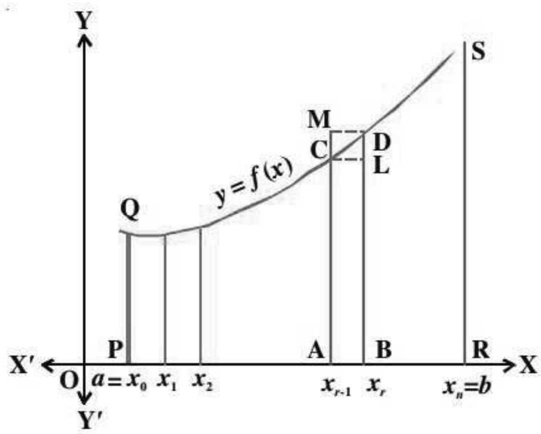

Fig. 2.9 *Area Between the Curve*

The interval $[a, b]$ is further divided into 'n' number of equivalent subintervals which are represented or denoted by,

$$[x_0, x_1], [x_1, x_2], \ldots\ldots\ldots, [x_{r-1}, x_r], \ldots\ldots\ldots, [x_{n-1}, x_n]$$

Here,

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \ \ldots\ldots, \quad x_r = a + rh$$

And,

$$x_n = b = a + nh \qquad \text{or} \qquad n = (b - a)/h$$

As per, $\quad n \to \infty, \ h \to 0.$

The segment PRSQP is defined as the sum of n subsegments and subsegment is defined on the basis of subintervals $[x_{r-1}, x_r]$, where $r = 1, 2, 3, 4, \ldots, n$.

From Figure 2.9, we can state that,

ABLC (Area of the Rectangle) < ABDCA (Area of the Segment) < ABDM (Area of the Rectangle) …(2.11)

Evidently, since,

$$[x_r - x_{r-1}] \to 0, \ \text{specifically as,} \ h \to 0$$

Consequently, with this condition all the three regions that are specified in the Equation (2.11) are considered almost equivalent to each other.

Accordingly, we can define the sum of intervals as follows.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots(2.12)$$

And, $$S_n = h[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots(2.13)$$

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The term 's_n' is used to represent the sum of the regions for all the lower rectangles while the term 'S_n' is used to represent the sum of the regions for all the upper rectangles considered over the subintervals, $[x_{r-1}, x_r]$, where $r = 1, 2, 3, 4, \ldots, n$.

Now, taking into account the Equation (2.13) inequality we can define the following state for an arbitrary or random subinterval $[x_{r-1}, x_r]$,

$$s_n < \text{Area of the Segment PRSQP} < S_n \qquad \ldots(2.14)$$

When '$n \to \infty$' turn out to be narrow then we assume that the limiting values of Equations (2.12) and (2.13) will be similar or equivalent in both the conditions. Hence, under this situation the limiting value which is common will be the required segment under the curve.

Symbolically, the common limiting value can be expressed as,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$$

$$= \text{Area of the Segment PRSQP}$$

$$= \int_a^b f(x)dx \qquad \ldots(2.15)$$

The area specified under the segment PRSQP can also be defined as the limiting value of some other specific area which exists between the rectangles either below the stated curve or above the stated curve. We limit the condition and will consider only those rectangles which are of height equivalent to the stated curve and are at the left side edge of every subinterval.

Therefore, in limit form the Equation (2.15) can be expressed as,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f(a+(n-1)h]$$

Or,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)h] \quad \ldots(2.16)$$

Here, $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$.

Precisely, the **Equation (2.16)** is defined as the definition of the **definite integral** as the **'limit of sum'**.

Definition. The value of the definite integral of a function for some precise interval is specifically determined on the basis of function and the interval, but it will not consider the variable of integration selected for representing the independent variable.

Therefore, if the independent variable is symbolized through 't' or 'u' in place of 'x' then the integral of Equation (2.15),

$$= \int_a^b f(x)dx$$

Will be represented as,

$$\int_a^b f(t)\,dt \text{ or } \int_a^b f(u)\,du$$

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Subsequently, the **variable of integration** will be termed as a **'dummy variable'**.

Example 2.22: Evaluate the given definite integral of the form as the limit of a sum,

$$\int_0^2 (x^2+1)\,dx$$

Solution: Follow the steps given below.

According to the definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)h]$$

For $h = (b-a)/n$.

Hence we can state that,

$a = 0,\quad b = 2,\quad f(x) = x^2 + 1,\qquad h = (2-0)/n = 2/n$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\ (2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}[1+\frac{2}{3}(1-\frac{1}{n})\ (2-\frac{1}{n})]$$

$= 2\ [1 + (4/3)] = 14/3$

2.8.2 Fundamental Theorems of Calculus – Area Function

The fundamental theorems of calculus defines how to evaluate the **area** of the **segment** that is bounded by the **curve** $y = f(x)$.

Area function can be defined with regard to the Equation (2.15),

$$= \int_a^b f(x)dx$$

It states the area of the segment that is bounded through the curve $y = f(x)$, the coordinates $x = a$ and $x = b$ on the x-axis. Consider that 'x' is a specific point in the interval $[a, b]$ as shown in Figure 2.10.

NOTES

Then, $\int_a^x f(x)\ dx$ will specifically represent the area that is on the left-hand side of 'x' denoted as 'A(x)', as shown in Figure 2.10.

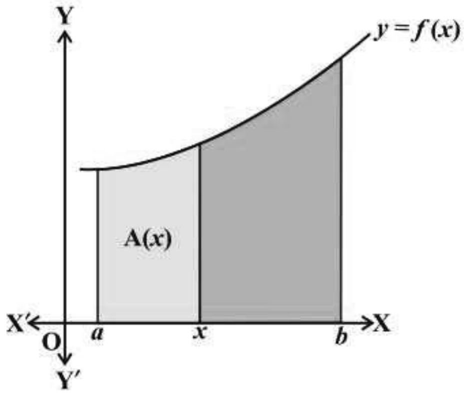


Fig. 2.10 *Area Function*

Consider that $f(x) > 0$ for $x \in [a, b]$. The area of the stated segment is determined on the value of 'x', i.e., the area of the stated segment is a function of 'x' denoted by A(x).

Therefore, the **function of '*x*'** denoted by **A(*x*)** is termed as the **'Area Function'** and is represented as equation,

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \ldots(2.17)$$

On the basis of Equation (2.17) the basic two fundamental theorems of integral calculus can be stated as given below.

Theorem 2.2: Let 'f' be a continuous function on the closed interval $[a, b]$ and let A(x) be the area function. Then,

$$\mathbf{A'(x) = f(x), \text{ for all } x \in [a, b]}$$

Theorem 2.3: Let 'f' be a continuous function defined on the closed interval $[a, b]$ and 'F' be the anti-derivative of 'f'. Then,

$$\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a).$$

Theorem 2.3 is considered as significant theorem of integral calculus as it helps in evaluating the definite integral using anti-derivative.

Example 2.23: Evaluate the given definite integral of the form,

$$\int_2^3 x^2\ dx$$

NOTES

Solution: Follow the steps given below.

Consider that,

$$D = \int_2^3 x^2\, dx$$

Because,

$$\int x^2\, dx = \frac{x^3}{3} = F(x)$$

Hence, applying the Theorem 2 of integral calculus we obtain,

$$\begin{aligned} D &= F(3) - F(2) \\ &= (27/3) - (8/3) \\ &= 19/3 \end{aligned}$$

Evaluation of Definite Integrals by Substitution Method

For evaluating the definite integrals using the substitution method follow the steps given below.

Step 1. To evaluate the definite integral of the form $= \int_a^b f(x)dx$ by the substitution method we first define it to the known indefinite integral form.

Step 2. Reduce the definite integral form to the indefinite integral form by substituting '$y = f(x)$' or '$x = g(y)$', i.e., the integer is without upper and lower limits.

Step 3. Now integrate the new obtained integrand with reference to new variable but no need to mention or use the constant of integration.

Step 4. Again substitute for the new variable to obtain the answer in the form of original expressions of the variable.

Step 5. Use this answer obtained in Step 4, to find the values for the specified limits of the integral, i.e., evaluate the difference of values at the specified upper limit and the lower limit.

Example 2.24: Evaluate the given definite integral of the form using substitution,

$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\, dx$$

Solution: Follow the steps given below.

Let, $t = x^5 + 1$ and then $dt = 5x^4\, dx$

Hence,

$$\int 5x^4\sqrt{x^5+1}\, dx = \int \sqrt{t}\, dt$$

$$= \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Consequently,

$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\, dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

2.8.3 Properties of Definite Integrals

Following are some significant properties of definite integrals using which we can easily evaluate the definite integrals.

Property 1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

To evaluate we substitute as '$x = t$'.

Property 2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Specifically, $\int_a^a f(x)\,dx = 0$

Assume that 'F' be the anti-derivative of 'f'.

As per the Theorem 2 of integral calculus,

$$\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a)$$

$$= -[\mathrm{F}(a) - \mathrm{F}(b)]$$

$$= -\int_b^a f(x)\,dx$$

When $a = b$ then we can state that,

$$\int_a^a f(x)\,dx = 0$$

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Assuming that 'F' be the anti-derivative of 'f'. Then we can state,

$$\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) \qquad \ldots(2.18)$$

$$\int_a^c f(x)\,dx = \mathrm{F}(c) - \mathrm{F}(a) \qquad \ldots(2.19)$$

$$\int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(c) \qquad \ldots(2.20)$$

We add Equations (2.19) and (2.20) to obtain,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)\,dx$$

Hence the Property 3 is proved.

NOTES

Property 4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Consider that $t = a + b - x$. Thus, $dt = -dx$.

While $x = a$, $t = b$ and also when $x = b$, $t = a$.

We can state the definite integral as,

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \qquad \text{.....By Property 2}$$

$$= \int_a^b f(a+b-x)\,dx \qquad \text{.....By Property 1}$$

Property 5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Consider that, when $t = a - x$, then $dt = -dx$.

While $x = 0$, $t = a$ and also when $x = a$, $t = 0$.

The proof is similar to Property 4.

Property 6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Applying Property 3, we obtain,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Assume that,

$t = 2a - x$ for the right-hand side second integral.

Thus, $dt = -dx$.

While $x = a$, $t = a$ and also when $x = 2a$, $t = 0$ and precisely $x = 2a - t$.

Consequently, the right-hand side second integral has the form,

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx$$

Therefore,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Integration by Parts

In calculus and in mathematical analysis, **integration by parts** or **partial integration** method is used for finding the **integral** of a **product of functions** using expressions of the integral, specifically derivative and anti-derivative.

The rule is simply derived by integrating the product rule of differentiation.

Definitions

NOTES

Following are the standrard definitions for the process integration by parts.

1. The **integration by parts** or **partial integration** process is specifically used to find the **integral** of a **product of functions**.

   If $u = u(x)$ and $du = u'(x)\ dx$
   When $v = v(x)$ and $dv = v'(x)\ dx$

   Then integration by parts states that,

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$
$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$$

   Or more precisely,

$$\int u\,dv = uv - \int v\,du.$$

2. Sometimes an **integration** is the **product** of **'2' functions** which can be **integrated** using **Integration by Parts**.

   If u and v are functions of x, then using the product rule for differentiation we obtain the following equation,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

   Reorganizing the equation we have,

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating with reference to 'x' the following formula is obtained for integration by parts:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Precisely, we use the function 'u' for the reason that 'du/dx' can be simple than 'u'. When in an equation the expression contains mix functions that are to be integrated then we can define these mix functions as 'u' to make the integration process easy, for example we can define 'u' as,

$$u = \ln x, \qquad u = x^n, u = e^{nx}, \quad \text{etc.}$$

3. If **'u'** and **'v'** are any **two differentiable functions** of a **single variable 'x'** then through the **product rule of differentiation**,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

When both the sides are integrated then we obtain,

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

NOTES

Alternatively, $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ ...(2.21)

Consider that,

$u = f(x)$ and $dv/dx = g(x)$

Then,

$du/dx = f'(x)$ and $v = +\text{''}g(x)\, dx$

Consequently, we can express Equation (2.23) as follows,

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int [[\int g(x)\, dx]\, f'(x)\, dx$$

Specifically,

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int [\, f'(x) \int g(x)\, dx]\, dx$$

Consider that if first function is 'f' and the second function is 'g' then we can state,

Integral of the Product of Two Functions = [First Function] × [Integral of Second Function] – Integral of [(Differential Coefficient of First Function) × (Integral of Second Function)]

Remember that integration by parts as product of functions can be applied only to specific cases and is not applicable to all.

For example, integration by parts as product of functions will not evaluate the expression of the form $\int \sqrt{x} \sin x\, dx$ because there is not any function which has required derivative as $\sqrt{x}\ \sin x$.

Example 2.25: Evaluate the given integral using integration by parts,

$$\int x \ \sin 2x \ dx$$

Solution: Follow the steps given below.

Consider that $u = x$ and $dv = \sin 2x\, dx$.

Since $u = x$ then $du = dx$.

On integrating $dv = \sin 2x\, dx$ gives,

$$v = \int \sin 2x \ dx$$

$$= \frac{-\cos 2x}{2}$$

Now we substitute these expressions into the formula of integration by parts as follows.

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Substitute the following in the above expression,

$u = x$, $dv = \sin 2x\, dx$, $du = dx$, $v = \frac{-\cos 2x}{2}$

NOTES

We obtain,

$$\int \underbrace{u}\ \underbrace{dv} = \underbrace{u}\ \underbrace{v} - \int \underbrace{v}\ \underbrace{du}$$

$$\int \boxed{x}\ \boxed{\sin 2x\, dx} = \boxed{x}\ \boxed{\frac{-\cos 2x}{2}} - \int \boxed{\frac{-\cos 2x}{2}}\ \boxed{dx}$$

Therefore,

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}\frac{\sin 2x}{2} + K$$

$$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + K$$

Where K is a constant.

2.8.4 Reduction Formulae

In integral calculus, the **integration by reduction formula** is a specific technique or method of integration of the form **recurrence relation**.

Typically, a **Reduction Formula** is used to solve an integral by reducing it to an easier integral form and then further reducing it to the more easier form, and so on.

Definition: A **reduction formula** for a **specified integral** is an **integral** of the form which is **equivalent** in form as the specified integral but of a **lower degree** or **order**.

Characteristically, the reduction formula is used for evaluating those specified integrals which cannot be approximated otherwise. The specified integral is evaluated by repeatedly using the reduction formula.

Definition: For some **specified integrals**, both **definite** and **indefinite**, typically the **function** to be **integrated**, i.e., the **'integrand'** comprises of a product of two functions, where one function contains an **unspecified** or **indefinite integer** **'*n*'**. To evaluate such a specified integral in expressions of similar analogous integral we use the method **integration by parts** where **'*n*'** is replaced by either **$(n-1)$** or at times by **$(n-2)$**. The **correlation** between these **two integrals** is termed as **'reduction formula'**. We can determine the **original integral** in terms of **'*n*'** by repeatedly using this reduction formula.

Definition. Reduction formulae are certain **integrals** which contain some **variable '*n*'** in addition to the standard **variable '*x*'**, generally obtained by means of **integration by parts**. The notation I_n is used for denoting reduction formulae.

Following are some commonly used significant reduction formulae for trigonometric integrals.

NOTES

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

$$\int \csc^n(x)\,dx = -\frac{1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \cot^n(x)\,dx = \frac{-1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx$$

Specifically, we can state that,

Reduction Formula for Sine

$$\int \sin^n x\,dx = -\tfrac{1}{n}\cos x \sin^{n-1} x + \tfrac{n-1}{n}\int \sin^{n-2} x\,dx.$$

Reduction Formula for Cosine

$$\int \cos^n x\,dx = \tfrac{1}{n}\sin x \cos^{n-1} x + \tfrac{n-1}{n}\int \cos^{n-2} x\,dx.$$

Reduction Formula for Natural Logarithm

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Let us understand the concept of reduction formula with the help of following example in which an integral is solved by reducing it to the form of more easy integral.

Let,

$$I_n = \int x^n e^x\,dx$$

This expression is simplified with regard to integration by parts as follows.

$$I_n = x^n e^x - n\int x^{n-1} e^x\,dx$$

$$I_n = x^n e^x - nI_{n-1}$$

Thus, we obtain the above reduction formula.

Similarly, when,

$$I_n = \int \sec^n(\theta)d\theta$$

Again this expression is also simplified with regard to integration by parts as follows.

$$I_n = \sec^{n-2}(\theta)\tan(\theta) - (n-2)\int \sec^{n-2}(\theta)\tan^2(\theta)d\theta$$

Applying the trigonometric identity $\mathbf{\tan^2(\theta) = \sec^2(\theta) - 1}$, we obtain,

$$I_n = \sec^{n-2}(\theta)\tan(\theta) + (n-2)\left(\int \sec^{n-2}(\theta)d\theta - \int \sec^n(\theta)d\theta\right)$$

$$= \sec^{n-2}(\theta)\tan(\theta) + (n-2)(I_{n-2} - I_n)$$

Rearrange or reorganize to obtain,

$$I_n = \frac{\sec^{n-2}(\theta)\tan(\theta)}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

NOTES

When we obtain $n = 1$ or $n = 2$ then we stop integrating, where 'n' is either odd or even, respectively.

Example 2.26: Evaluate the reduction formula for the given integral of the form,

$\int (x^2+1)^n\, dx$ (Where n is a constant)

Solution: Follow the steps given below.

Use the form $\int u dv$.

Select $u = (x^2 + 1)^n$ and $dv = dx$

Then $du = n(x^2 + 1)^{n-1} 2x$ and $v = x$

Therefore,

$$\int (x^2+1)^n\, dx = (x^2+1)^n x - \int x \cdot n (x^2+1)^{n-1} 2x dx$$

$$= x(x^2+1)^n - 2n \int x^2 (x^2+1)^{n-1} dx$$

$$= x(x^2+1)^n - 2n \int \left[(x^2+1)^n - (x^2+1)^{n-1}\right] dx$$

$$= x(x^2+1)^n - 2n \int (x^2+1)^n dx + 2n \int (x^2+1)^{n-1} dx$$

Rearrange the expression to obtain,

$$(2n+1)\int (x^2+1)^n dx = x(x^2+1)^n - 2n \int (x^2+1)^{n-1} dx.$$

Therefore, we obtain the following two recursive formulae.

$$\int (x^2+1)^n dx = \frac{x(x^2+1)^n}{2n+1} - \frac{2n}{2n+1}\int (x^2+1)^{n-1} dx \quad (n \neq -\frac{1}{2})$$

And,

$$\int (x^2+1)^{n-1} dx = \frac{x(x^2+1)^n}{2n} - \frac{2n+1}{2n}\int (x^2+1)^n dx \quad (n \neq 0).$$

Here Formula 1 is for positive 'n' while Formula 2 is for negative 'n'.

Bernoulli's Formula

Bernoulli equation is termed as the **nonlinear differential equations** of the **first order** and is represented as,

$$\boldsymbol{y' + a(x)\, y = b(x)\, y^m}$$

NOTES

Here $a(x)$ and $b(x)$ are termed as the **continuous functions**.

When $m = 0$, then the equation is defined as **linear differential equation** while when $m = 1$ then the equation is defined as **separable**.

Generally, when $m \neq 0, 1$ then the Bernoulli equation is converted to **linear differential equation** using the method **change of variable**,

$$z = y^{1-m}$$

The **function** $z(x)$ will be defined as **differential equation** of the form,

$$z' + (1-m)\,a(x)\,z = (1-m)\,b(x)$$

Example 2.27: Evaluate the general solution of the equation $y' - y = y^2 e^x$.

Solution: Follow the steps given below.

Assume that $m = 2$ for the specified Bernoulli equation and then apply the substitution as follows.

$$z = y^{1-m} = 1/y$$

Differentiating the equations on both sides, considering 'y' in the right-hand side expression as composite function of 'x', we have

$$z' = \left(\frac{1}{y}\right)' = -\frac{1}{y^2}y'.$$

Now the original differential equation is divided both sides by y^2, we have,

$$y' - y = y^2 e^x, \quad \Rightarrow \frac{y'}{y^2} - \frac{1}{y} = e^x.$$

Substituting z and z', we obtain,

$$-z - z = e^x$$

$$\Rightarrow \quad z' + z = -e^x$$

Then obtain the linear equation for the specified function $z(x)$. For solving we apply the integrating factor as follows,

$$u(x) = e^{\int 1dx} = e^x.$$

Consequently, the linear equation has the general solution as follows,

$$z(x) = \frac{\int u(x)\,f(x)\,dx + C}{u(x)} = \frac{\int e^x(-e^x)\,dx + C}{e^x}$$

$$= \frac{-x + C}{e^x}$$

$$= (C - x)\,e^{-x}.$$

When we return to the function $y(x)$, we have the following implicit expression of the form,

$$y = \frac{1}{z} = \frac{1}{(C-x)\,e^{-x}}.$$

This can be expressed as,

$$y\,(C - x) = e^x$$

Therefore,

$$y\,(C - x) = e^x, \quad y = 0$$

2.9 FUNCTIONS OF SEVERAL VARIABLES

NOTES

In mathematics, a **function of several real variables** also sometimes termed as the real multivariate function can be defined as a function having more than one argument in which all the arguments must be real variables. This concept or notion gives the concept or notion of a real variable function to function of several variables. Characteristically, the 'Input' variables include real values, while the 'Output' variable, also termed as the 'Value of the Function', can be either 'Real' or 'Complex'.

Typically, as per the definition, "The domain of a function of n variables is considered as the subset of $\mathbb{R}^n$ specifically for which the function is defined. Generally, the domain of a function of several real variables is assumed to contain a non-empty open subset of $\mathbb{R}^n$."

The term single variable refers to the functions that typically map the real numbers $\mathbb{R}$ to $\mathbb{R}$, occasionally termed as the 'Real Functions of One Variable', implying that for the single real number 'Input' there will be the single real number 'Output'.

Additionally, if the function is considered as the real number which takes the vector form, then it can be perceived as functions $f: \mathbb{R} \to \mathbb{R}^3$ which specifies that for each and every input value we have a precise position in space. Now if the function of several variables is considered, then it specifies that for the 'Several Input Variables', the functions can be perceived as $f: \mathbb{R} \to \mathbb{R}^n$.

Fundamentally, the function of '***n***' **variables** is specifically termed as '**Functions of Several Variables**'.

Essentially, a real valued function of n real variables is referred as a function that typically takes as input n real numbers, generally represented by means of the variables $x_1, x_2, \ldots, x_n$, for generating another real number, in which the value of the function is normally denoted by $f(x_1, x_2, \ldots, x_n)$.

Image of a Function

The **image** of a function $f(x_1, x_2, \ldots, x_n)$ is referred as the set of all values of f when the n-tuple $(x_1, x_2, \ldots, x_n)$ occurs in the whole or full entire domain of f. For a continuous real valued function having a connected domain, the image is defined to be either an interval or a single value. In the latter instance or argument, the function is defined as a **constant function**.

Additionally, the **preimage** of a given real number c is termed as a **level set**. The level set is defined as the set of the solutions of the equation of the form,

$$f(x_1, x_2, \ldots, x_n) = c$$

Domain

The **domain** of a function of several real variables is specifically defined as a subset of $\mathbb{R}^n$ which can be occasionally explicitly defined. Actually, if we restrict or confine that the domain X of a function f as a subset $Y \subset X$, then we formally or properly obtain a different function which typically states the restriction f to Y and is denoted by $f|_Y$.

NOTES

In mathematical analysis, we often identify f and $f|_Y$ and can ignore or exclude the restrictor $|_Y$.

On the contrary, it is also at times possible to naturally enlarge or expand the domain of a given function either by means of continuity or by means of analytic continuation.

Additionally, several functions are typically defined in such a manner that it is sometimes difficult to identify or explicitly specify their domain. For example, for a given function f, it may possibly be difficult to identify or specify the domain of the function $g(x) = 1/f(x)$. If f is a multivariate polynomial having $\mathbb{R}^n$ as a domain, then it is yet difficult to check whether the domain of g is also $\mathbb{R}^n$.

Example 2.28: If D be a subset of $\mathbb{R}^n$, then define the function F of several variables for domain D giving specific notations.

Solution: Given is that D be a subset of $\mathbb{R}^n$.

Consequently, now consider a function F of n variables, also termed as the function F of several variables. Basically, the function F of n variables with domain D is considered as a relation that is typically assigned to each and every ordered n-tuple in domain D is a unique real number in $\mathbb{R}$.

Each specific relation can be denoted by means of the following types of notations:

$$\begin{aligned} F : D &\rightarrow \mathbb{R} \\ x &\mapsto y \\ (x_1, x_2, \ldots\ldots, x_n) &\mapsto y \\ \overline{X} &\mapsto y \\ (x_1, x_2, \ldots\ldots, x_n) &\mapsto y \end{aligned}$$

The range of F refers to the set of all outputs of F.

Therefore, it is typically a subset of $\mathbb{R}$ and not of $\mathbb{R}^n$.

2.9.1 Partial Differentiation

Assume that for a function of two variables, $\boldsymbol{f(x, y)}$, basically the derivative of $\boldsymbol{f}$ is defined simply with respect to $\boldsymbol{x}$ while $\boldsymbol{y}$ is considered as a constant. This derivative is termed as the "**partial derivative of *f* with respect to *x***" and is represented either in terms of $\partial \boldsymbol{f} / \partial \boldsymbol{x}$ or $\boldsymbol{f_x}$. The '∂'symbol is used for denoting partial derivative.

Likewise, we can also define the derivative of $\boldsymbol{f}$ simply with respect to $\boldsymbol{y}$ while $\boldsymbol{x}$ is considered as a constant. This derivative is termed as the "**partial derivative of *f* with respect to *y***" and is represented either in terms of $\partial \boldsymbol{f} / \partial \boldsymbol{y}$ or $\boldsymbol{f_y}$.

Definition 1: The **partial derivative** of a **function *f*(*x*, *y*, ...)** with reference to the variable x is represented using the following specified notations:

$$f'x, \quad fx, \quad \partial_x f, \quad D_x f, \quad D_1 f, \quad \partial f / \partial x, \quad \text{or} \quad \frac{\partial}{\partial x} f$$

NOTES

In addition, for a function $z = f(x, y, \ldots .)$, the partial derivative of z with regard to x is represented by $\partial z / \partial x$.

Generally, the arguments of a partial derivative is same as the original function, however, occasionally its functional dependency is explicitly denoted or represented by the notation as shown below.

$$f_x(x, y, \ldots), \ \frac{\partial f}{\partial x}(x, y, \ldots).$$

Definition 2: Mathematically, a **partial derivative** of a **function** of **several variables** is its derivative with respect to one of those variables, while the remaining others are considered as constant. Partial derivatives has its applications in vector calculus, differential geometry, etc.

Definition 3: The function f can be interpreted or deduced as a family of functions of one variable which is indexed by the other variables as,

$$f(x, y) = f_y(x) = x^2 + xy + y^2$$

Definition 4: A differential equation which expresses one or more measures in terms of partial derivatives is called a partial differential equation.

(1) Partial Derivative, the **Function** of **One Variable (*x*)**:

$$f(x) = x^2$$

Its **derivative** by means of the **Power Rule** is:

$$f'(x) = 2x$$

In calculus, the Power Rule is frequently used to find the derivative.

As per the Power Rule, the **derivative** of $\boldsymbol{x^n}$ is $\boldsymbol{nx^{(n-1)}}$

Example 2.29: Find the derivative of x^2 using the Power Rule.

Solution: To find the derivative of x^2 by means of Power Rule using $n = 2$, follow the given method.

The derivative for,

$$x^2 = 2x^{(2-1)}$$
$$= 2x^1$$
$$= 2x$$

Thus, the derivative of x^2 is $2x$.

(2) Partial Derivative, the **Function** of **Two Variables (*x* and *y*)**:

$$f(x, y) = x^2 + y^3$$

(i) For finding the partial derivative of the above expression of two variables with regard to x, we consider y as constant and evaluate the derivative as follows:

$$f'x = 2x + 0 = 2x$$

Interpretation: Let us understand the process.

(a) The derivative of x^2 with regard to x is defined as $2x$.

(b) Here, since y is considered as a constant, hence y^3 will also be a constant and consequently the derivative of a constant will be 0 (zero).

(ii) For finding the partial derivative of the above expression of two variables with regard to y, we consider x as constant and evaluate the derivative as follows:

$$f'y = 0 + 3y = 3y^2$$

NOTES

Interpretation: Let us understand the process.

(a) The derivative of y^3 with regard to y is defined as $3y^2$.

(b) Here, since x is considered as a constant, hence x^2 will also be a constant and consequently the derivative of a constant will be 0 (zero).

How a Variable Constant is Hold as a Constant in Case of Two Variables

Let us understand the concept with the help of the following example of cylinder where we will consider the variables ***r*** (radius of cylinder) and ***h*** (height of cylinder) as a contant, each at a time (Refer Figure 2.11).

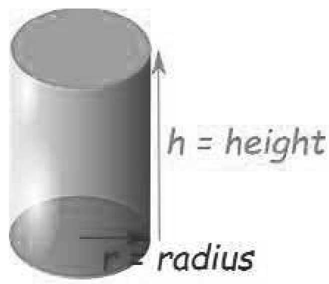


Fig. 2.11 *Radius and Height of a Cylinder*

We know that the volume of a cylinder is evaluated by,

$$\mathbf{V = \pi r^2 h}$$

Now in variable form,

$$V = f(r, h)$$

Hence,

$$f(r, h) = \pi r^2 h$$

For defining the partial derivative with reference to r we consider h as a constant, as variable r changes (Refer Figure 2.12):

$$f'r = \pi (2r) h = 2 \pi r h$$

Fig. 2.12 *Radius 'r' Changes*

NOTES

Therefore, the derivative of r^2 with reference to r is $2r$, where π and h are considered as constants.

Thus it states that, "in this case the radius r only changes by the smallest amount and accordingly the volume of the cylinder changes by $2\ \pi\ r\ h$". The change in the radius is very less as shown in the Figure 2.12 and it seems as if it is a membrane having a circle circumference ($2\ \pi\ r$) and a height h.

For defining the partial derivative with reference to h we consider r as a constant, (Refer Figure 2.13):

$f'h = \pi\ r^2\ (1) = \pi r^2$

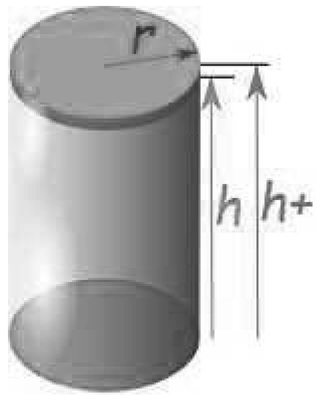


Fig. 2.13 *Height* 'h' *Changes*

Therefore, the derivative of h with reference to h is 1, and where π and r^2 are considered as constants.

Thus it states that, "In this case the height h only changes by the smallest amount and accordingly the volume of the cylinder changes by r^2". The change in the height is very less as shown in the Figure 2.13 and it seems as if a thin disk is added on the top having a circle area of πr^2.

Notations

(i) The partial derivative of 'f' with respect to 'x' is expressed as,

f_x or $\partial f / \partial x$

(ii) The partial derivative of 'f' with respect to 'y' is expressed as,

f_y or $\partial f / \partial y$

(iii) The analogous notation to the most familiarized form is expressed as,

df / dx or f'

(iv) For $z = f(x, y)$.

We can also write the partial derivative $f_x\ (x, y)$ as,

$$\frac{\partial f}{\partial x}(x, y) \text{ or } \frac{\partial z}{\partial x}$$

In the same way, we can write the partial derivative $f_y(x, y)$ as,

$$\frac{\partial f}{\partial y}(x, y) \text{ or } \frac{\partial z}{\partial y}.$$

(v) We can evaluate the partial derivative $f_y(x, y)$ at the point (x_0, y_0) and express as follows:

$$f_x(x_0, y_0), \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$$

Or, $$\frac{\partial f}{\partial x}(x_0, y_0)$$

Try finding the equivalent expression for $f_y(x_0, y_0)$.

We have previously discussed that for a function of one variable $f(x)$, characteristically the derivative $f'(x)$ denotes the rate of change of the function when x changes. This is significant interpretation of derivatives.

Example 2.30: Consider the function $f(x, y) = 2x^2y^3$ and then determine the rate of change by which the function is changing at the specified point (a, b),

(i) When y is constant and x is a varying variable.

(ii) When x is constant and y is a varying variable.

Solution: To determine the rate of change by which the function $f(x, y) = 2x^2y^3$ is changing at the specified point (a, b), let us first consider the case when y is constant and x is a varying.

Hence, we have $\boldsymbol{y = b}$, because $\boldsymbol{y}$ is a constant.

Therefore, then there be a function of the form which will include only 'x' and can be defined as follows:

$$g(x) = f(x, b) = 2x^2b^3$$

At this time, this function has only one variable and for determining the rate of change we take, $g(x)$ at $x = a$.

Alternatively, we evaluate $g'(a)$, the function of one or single variable.

Accordingly the rate of change of the function at (a, b), when x is varying and y is considered as a constant, is evaluated as follows:

$$\boldsymbol{g'(a) = 4ab^3}$$

The notation $\boldsymbol{g'(a)}$ is termed as the **partial derivative** of $\boldsymbol{f(x, y)}$ with reference to $\boldsymbol{x}$ at $\boldsymbol{(a, b)}$ and is represented as follows.

$$\boldsymbol{f_x(a, b) = 4ab^3}$$

Again, let us now consider the case where the variable $\boldsymbol{y}$ is varying and $\boldsymbol{x}$ is a constant. Similarly as we can derive the equation as we have derived in the case when the variable $\boldsymbol{x}$ is varying and $\boldsymbol{y}$ is a constant.

Therefore, when $\boldsymbol{x}$ is a constant then it is constant at $\boldsymbol{x = a}$.

NOTES

Therefore, then there be a function of the form which will include only **'*y*'** and can be defined as follows:

$$h(y) = f(a, y) = 2a^2y^3$$

$$\Rightarrow \qquad h'(b) = 6a^2b^2$$

The notation $h'(b)$ is termed as the **partial derivative** of $f(x, y)$ with reference to y at (a, b) and is represented as follows.

$$f_y(a, b) = 6a^2b^2$$

The **two partial derivatives** that we have derived in this **Example 2.28** are also occasionally termed as the **first order partial derivatives**.

Generally, for partial derivatives the notation (a, b) is not always used, instead for partial derivatives we use the standard notation (x, y). Hence, the partial derivatives of Example 2.28 can be written as follows using the standard notation (x, y):

$$f_x(x, y) = 4xy^3 \quad \text{and} \quad f_y(x, y) = 6x^2y^2$$

Basically, to evaluate $f_x(x, y)$ we consider 'y' as constant for differentiating 'x' and similarly to evaluate $f_y(x, y)$ we consider 'x' as constant for differentiating 'y'.

Using the limit definition we can write the above two partial derivatives in limit form as follows:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Following are some precise and probable alternate notations to evaluate partial derivatives.

For the function $z = f(x, y)$ we have the following equivalent notations:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x, y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x, y)) = z_y = \frac{\partial z}{\partial y} = D_y f$$

The following fractional notations of partial differentiation specify that how the partial derivative is different from the ordinary derivative with reference to single variable calculus.

$$f(x) \quad \Rightarrow \quad f'(x) = \frac{df}{dx}$$

$$f(x, y) \quad \Rightarrow \quad f_x(x, y) = \frac{\partial f}{\partial x} \ \& \ f_y(x, y) = \frac{\partial f}{\partial y}$$

In the following examples, all the remaining variables are considered as a constant value and then we differentiate the derivative taking it as a function of single variable.

NOTES

Example 2.31: For the following given functions, find all the first order partial derivatives.

(i) $f(x,y) = x^4 + 6\sqrt{y} - 10$

(ii) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

Solution: The first order partial derivative is evaluated as follows.

(i) $f(x,y) = x^4 + 6\sqrt{y} - 10$

We first evaluate the partial derivative of the function $f(x,y) = x^4 + 6\sqrt{y} - 10$ with regard to x considering y as a constant. Therefore, the partial derivative with regard to 'x' is,

$$f_x(x, y) = 4x^3$$

In this case, the second and the third terms differentiate to '0' (zero). Since as per the standard rule of differentiation, the term that is considered as a constant differentiates to zero. Now when we differentiate with respect to 'x' and consider 'y' as a constant, then all the terms with 'y' will be considered as constants and therefore will be differentiated to '0' (zero).

Accordingly, when we differentiate with respect to 'y' and consider 'x' as a constant, then the term which includes 'x' will be considered as constants and therefore will be differentiated to '0' (zero). Following partial derivative is evaluated with regard to 'y':

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

Therefore, the partial derivative with regard to x is, $f_x(x, y) = 4x^3$

And the partial derivative with regard to 'y' is, $f_y(x,y) = \dfrac{3}{\sqrt{y}}$

(ii) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

We evaluate the derivatives with reference to 'x' and 'y' of the given function $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$ when 'z' is considered as a constant. On the right-hand side, the 'z' in the denominator is a constant.

The derivatives for x and y are as follows,

$$g_x(x,y,z) = \frac{\sin(y)}{z^2}$$

$$g_y(x,y,z) = \frac{x\cos(y)}{z^2}$$

Additionally, to differentiate 'x' and 'y' with regard to 'z', the derivative is,

Example 2.32: Find dy/dx for $3y^4 + x^7 = 5x$.

Solution: To find dy/dx for $3y^4 + x^7 = 5x$ we first consider y as a function of x, or we can say that $y = y(x)$.

When a term including y is differentiated with regard to x then dy/dx is added to that term as illustrated below.

On either side we differentiate with regard to x as follows:

$$12y^3\frac{dy}{dx} + 7x^6 = 5$$

Solving for dy/dx,

$$\frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}$$

Now we will discuss about the **second partial derivative** of '***f***'. It is represented in the following four notations.

Notations

(i) Differentiate '***f***' two times with regard to '***x***', i.e., we first differentiate 'f' with reference to 'x' and then again differentiate the result with reference to 'x' that is obtained after the first differentiation.

Thus, $\quad \partial^2 \boldsymbol{f} / \partial \boldsymbol{x}^2 \quad$ or $\quad \boldsymbol{f_{xx}}$

(ii) Differentiate '***f***' two times with regard to '***y***', i.e., we first differentiate 'f' with reference to 'y' and then again differentiate the result with reference to 'y' that is obtained after the first differentiation.

Thus, $\quad \partial^2 \boldsymbol{f} / \partial \boldsymbol{y}^2 \quad$ or $\quad \boldsymbol{f_{yy}}$

Mixed Partials

(iii) We first differentiate '***f***' with reference to '***x***' and then again differentiate the result with reference to '***y***' that is obtained after the first differentiation.

Thus, $\quad \partial^2 \boldsymbol{f} / \partial \boldsymbol{y}\, \partial \boldsymbol{x} \quad$ or $\quad \boldsymbol{f_{xy}}$

(iv) We first differentiate '***f***' with reference to '***y***' and then again differentiate the result with reference to '***x***' that is obtained after the first differentiation.

Thus, $\quad \partial^2 \boldsymbol{f} / \partial \boldsymbol{x}\, \partial \boldsymbol{y} \quad$ or $\quad \boldsymbol{f_{yx}}$

Example 2.33: Given is $f(x, y) = 3x^2y + 5x - 2y^2 + 1$, then find the partial derivatives $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ and f_{yx}.

Solution: We evaluate the partial derivatives in the following way.

We first differentiate f with regard to x when y is considered as a constant. This will yield,

$$f_x = 6xy + 5$$

NOTES

Next, we first differentiate f with regard to y when x is considered as a constant. This will yield,

$$f_y = 3x^2 - 4y$$

The f_{xx} is considered as the second partial derivative of f_x, i.e., the partial derivative of f_x with regard to x.

This will yield,

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(6xy + 5) = 6y$$

The f_{yy} is considered as the second partial derivative of f_y, i.e., the partial derivative of f_y with regard to y.

This will yield,

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(3x^2 - 4y) = -4$$

The f_{xy} is considered as the mixed second partial derivative of f_x with regard to y, i.e., the partial derivative of f_x with regard to y.

This will yield,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(6xy + 5) = 6x$$

The f_{yx} is considered as the mixed second partial derivative of f_y with regard to x, i.e., the partial derivative of f_y with regard to x.

This will yield,

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(3x^2 - 4y) = 6x$$

Consequently, see the result in case of the mixed second partial derivatives, f_{xy} and f_{yx}, both are similar as both the result yield '$6x$', i.e.,

$$f_{xy} = f_{yx}$$

Example 2.34: Given is $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the partial derivaties of first order and second order.

Solution: The partial derivaties of first order and second order are evaluated as follows.

$$\frac{\partial z}{\partial x} = 8x - 8y^4$$

$$\frac{\partial z}{\partial y} = -8x(4y^3) + 35y^4 = -32xy^3 + 35y^4$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 8$$

NOTES

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}(-32xy^3 + 35y^4) = -32x(3y^2) + 140y^3$$
$$= -96xy^2 + 140y^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-32xy^3 + 35y^4) = -32y^3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8x - 8y^4) = -32y^3$$

2.10 CURVE SKETCHING

In geometry and mathematical analysis, the term curve sketching or curve tracing are specific techniques used for producing a rough idea of overall shape of a plane curve given its equation, without computing the large numbers of points required for a detailed plot.

Curves are specifically described by means of parametric equations, also termed as the 'Parametric Curves' can range from graphs of the most basic equations to those of the most complex. Parametric equations are typically used for describing all types of curves that can be illustrated or represented and typically exemplified on a plane but are generally used in situations when it is difficult to describe the curves by means of functions on a Cartesian plane, for example when a curve crosses itself. Characteristically, the parametric equations are precisely used in three-dimensional spaces, and also in the spaces with more than three dimensions by applying or implementing additional and essential required parameters.

Mathematically, the curves can be described through the non-parametric curve or parametric curve equations. Non-parametric equations can be either explicit or implicit. For a non-parametric curve, the coordinates y and z of a point on the curve are typically expressed as two distinct and separate functions of the third coordinate x as the independent variable. Parametric curves are used to graph the relationships between two or more quantities and at the same time represent each quantity's directions or orientations.

A parametric equation defines a group of quantities as functions of one or more independent variables termed as parameters. Fundamentally, the parametric equations are generally used to express the coordinates of the points that make up a geometric object, such as a curve or surface and the equations are collectively termed as a parametric representation or parameterization of the object.

Parametric Curves

Parametric curves are used to graph the relationships between two or more quantities and at the same time represent each quantity's directions or orientations.

NOTES

Characteristically, the parametric curve is defined by its corresponding parametric equations of the form, $x = f(t)$ and $y = g(t)$ within a given particular interval. Parametric curves emphasize the orientation of each set of quantities with respect to time.

Parametric Curves and Its Endpoints: We consider the parametric equations of the form,

$$x = f(t) \text{ and } y = g(t)$$

If there are two plane curves that are typically redefined as parametric curves within the interval $[a, b]$, then the parametric curve will have initial points and terminal points at $(x(a), y(a))$ and $(x(b), y(b))$, respectively.

Following are the three key or most significant advantages of using parametric curves:

1. The parametric curve is considered as a remarkable model that emphasizes both the direction and orientation of (x, y) based on 't'. This indicates that the physical representations of quantities can be demonstrated that depend on time.
2. The value of parametric curve can be changed or adjusted for switching of shifting the orientations.
3. The parametric curves, specifically the implicit curves can only be modelled by means of parametric equations, i.e., both the volume of a container and its spillage can be simultaneously modelled with respect to time.

Parametrizing a Plane Curve

The plain curve can be parameterized by precisely redefining the given constraints x and y of a parametric curve as a set of parametric equations characteristically defined by parameter 't'. Considerably, there are numerous significant methods to parametrize a given curve but remember that their definition as a plane curve will remain the same. The simplest and easiest method for parametrizing a plain curve is done by setting $x = t$.

The extremely simple and basic option while parametrizing a plane curve is defined by the form of equation $y = f(x)$ which typically uses the following parametric equations:

$$x = t$$
$$y = f(t)$$

Remember that t should always be within the domains of $f(x)$.

This precise method is explicitly used when a line is to be parametrized. The following form of the parametric equations are used:

$$x = t$$
$$y = mt + b$$

This helps in observing the behaviours and actions of the curve specifically as 't' increases. Even though, this can be best approximated when the right parametric equations are preferred. The standard methodology includes by redefining the equations of circles using parametric equations.

Parametric Representation of a Circle

NOTES

A circle can be parametrized specifically when the circle is centred typically at the origin. The following form of parametric equations are used:

$$x = r \cos t$$
$$y = r \sin t$$

Circles which are centred at (h, k) can be parametrized using the following form of parametric equations:

$$x = h + r \cos t$$
$$y = k + r \sin t$$

In both the instances, remember that always $0 \le t \le 2\pi$.

These two are considered as the extremely common and popular curves that can be generally parametrized.

Parabola

Mathematically, a parabola is the locus of a fixed point which moves on a plane such a way that its distance from a fixed point in the plane is always equidistant from the fixed point and the fixed straight line.

Consider that S be a fixed point and L, a fixed straight line which is not passing through S on a plane. Figure 2.14 illustrates the basic concept of parabola.

If a point P moves on this plane in such a way that it is always equidistant from the fixed point S and the fixed straight line L then the locus of the point P is termed as a 'Parabola'.

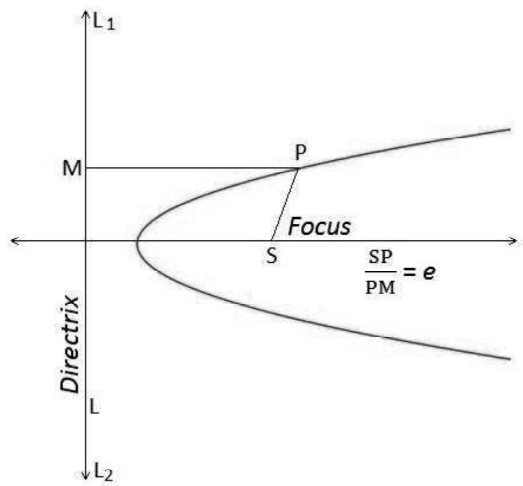


Fig. 2.14 *Basic Concept of Parabola*

Therefore, by definition,

$$SP = PM$$

Where, PM is referred as the length of the perpendicular from P on the directrix ZZ'.

Directrix of the Parabola: The fixed point S is termed as the focus and the fixed straight line L is termed as the 'Directrix' of the parabola.

Axis of the Parabola: The straight line through the focus and perpendicular to the directrix is termed as the 'Axis' of the parabola.

Vertex of the Parabola: The point at which the axis intersects the parabola is termed as the 'Vertex' of the parabola. Alternatively, the point on the axis midway between the focus and directrix is called the vertex of the parabola.

Focal Chord of the Parabola: A chord passing through its focus is termed as the 'Focal Chord' of the parabola.

Double Ordinate of the Parabola: Any chord perpendicular to the axis of the parabola is termed as its 'Double Ordinate'.

Hyperbola

In mathematics, a hyperbola is a type of smooth curve lying in a plane, defined by its geometric properties or by equations for which it is the solution set. A hyperbola has two parts, termed as connected components or branches, that are mirror images of each other and resemble two infinite bows. The hyperbola is one of the three kinds of conic section, formed by the intersection of a plane and a double cone, the other conic sections are the parabola and the ellipse. A circle is a special case of an ellipse. If the plane intersects both halves of the double cone but does not pass through the apex of the cones, then the conic is a hyperbola.

There are two methods to write the parametric form for a hyperbola, namely the east-west opening hyperbola and north-south opening hyperbola, as discussed below.

2.10.1 Curve Tracing in Cartesian Coordinates

The procedure of tracing a curve in Cartesian coordinates is as follows (Refer Figure 2.15 (a) (b) (c) (d) and (e).

1. **Symmetry.** Apply the following rules to ascertain whether the given curve is symmetrical about any line or not:
   (i) **Symmetry about the *x*-axis:** If the equation of a curve remains unaltered when y is changed to $-y$, i.e., if all the powers of y in the given equation are even, the curve is symmetrical about the x-axis. For example, since in the parabola $y^2 = 4ax$, power of y is even, hence it is symmetrical about the x-axis.
   (ii) **Symmetry about the *y*-axis:** If the equation of a curve remains unaltered when x is changed to $-x$, i.e., if all the powers of x are even in the given equation, the curve is symmetrical about the y-axis. For example, since the curve $x^2 = 4ay$ has the even power of x, hence it is symmetrical about the y- axis.
   (iii) **Symmetry about both axes:** If all the powers of both x and y are even in the given equation, the curve is symmetrical about both the axes. For example, since in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, powers of both x and y are even, thus it is symmetrical about both the axes.
   (iv) **Symmetry in opposite quadrants:** If the equation of the curve remains unchanged when x and y are changed to $-x$ and $-y$,

respectively, the curve is symmetrical in opposite quadrants. For example, the hyperbola $xy = c^2$ and the curve $y = x^3$ are symmetrical in opposite quadrants.

(v) **Symmetry about the line $y = x$:** If the equation of the curve is unchanged when x and y are interchanged (i.e., x is changed to y and y to x), the curve is symmetrical about the line $y = x$. For example, the curve $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

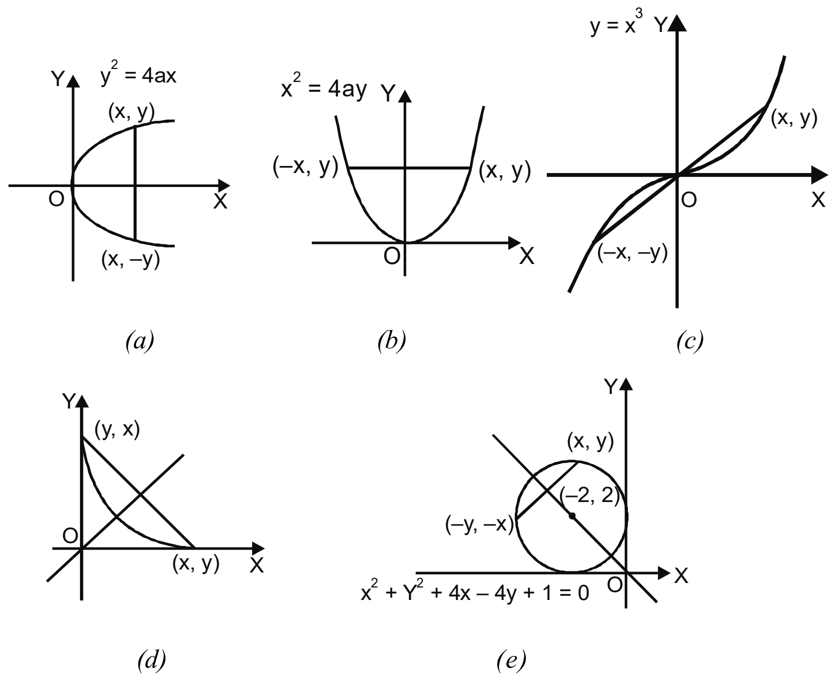


Fig. 2.15 *Tracing a Curve in Cartesian Co-ordinates*

(vi) **Symmetry about the line $y = -x$:** If the equation of the curve is unchanged when x is changed to $-y$ and y is changed to $-x$, the curve is symmetrical about the line $y = -x$. For example, the curve $x^2 + y^2 + 4x - 4y + 1 = 0$ is symmetrical about the line $y = -x$.

Note: The curve which has symmetry about both the axes also has symmetry in opposite quadrants but the converse is not true. For example, in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (symmetrical about both the axes), the equation of the curve remains unchanged when x and y are changed to $-x$ and $-y$, respectively. Thus, it is symmetrical in opposite quadrants also. But the curve $y = x + \frac{1}{x}$ (symmetrical in opposite quadrants) does not have even powers of x or y. Thus, it is not symmetrical about both the axes.

2. **Origin:** Find whether the curve passes through the origin or not.

(i) It will pass through the origin if the equation of the curve does not have any constant term. In this case, find the equation of the tangents at the origin by equating the lowest degree terms (present in the equation of curve) to zero.

NOTES

(ii) If the origin is a double point, then find its nature. The origin is a node if the tangents are real and distinct; cusp if the tangents are real and coincident; and conjugate point if the tangents are imaginary.

3. **Asymptotes:** To find the direction in which the curve extends to infinity, find the following asymptotes:

   (i) **Asymptotes parallel to x-axis or y-axis:** Find the asymptotes parallel to x-axis or y-axis by equating the co-efficients of highest powers of x or y (in the equation of the curve) to zero.

   (ii) **Oblique asymptotes:** Put $x = 1$ and $y = m$ in the highest degree terms of x and y to find $f_n(m)$. Then, equate $f_n(m)$ to zero and solve to get the values of m, which gives the slopes of the asymptotes. Put $x = 1$ and $y = m$ in the next lower degree terms of x and y to find $f_{n-1}(m)$ and similarly find $f_{n-2}(m)$ and obtain the values of m.

   If m has real and distinct values, then find c using $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$ and if m has real and equal values, then find c using $\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$.

   Finally, find the oblique asymptotes by putting the values of m and c in the equation $y = mx + c$.

4. **Point of intersection with the axes**

   (i) Find the points where the curve meets the co-ordinate axes. If the curve meets the x-axis, then on putting $y = 0$, we get the real values of x. Similarly, examine for y-axis where $x = 0$.

   (ii) Shift the origin to all such points and find the tangents at all such points (new origin) by equating to zero the lowest degree terms. If any one of them is a double point, find its nature.

   (iii) Find the points of intersection of the curve with the lines $y = \pm x$ if the curve is symmetrical about these lines.

5. **Region:** Find the region to which curve is bounded in the quadrants using the following steps:

   (i) Solve the equation of the curve for y. Find the corresponding values of x for which y is real. The curve lies between these obtained values of x.

   (ii) Ignore the values of x and y for which L.H.S. and R.H.S. of the equation has opposite sign.

   (iii) Put $x = 0$. Examine how y varies when x increases and tends to ∞, keeping in mind those values of x for which $y = 0$ or $y \to \infty$.

   (iv) Examine how y varies when x decreases from 0 and tends to $-\infty$. If the curve is symmetrical about y-axis, take only the positive values of x and trace the curve for negative values of x by symmetry.

6. **Special points:**

   (i) Find the points on the curve where:

NOTES

(a) $\frac{dy}{dx} = 0$, i.e., the points where the tangents are parallel to the x-axis.

(b) $\frac{dy}{dx} = \infty$, i.e., the points where the tangents are parallel to the y-axis.

(c) $\frac{dy}{dx}$ is positive, i.e., the points where the curve is increasing or

(d) $\frac{dy}{dx}$ is negative, i.e., the points where the curve is decreasing.

(ii) Find the points on the curve where:

(a) $\frac{d^2y}{dx^2}$ is positive, i.e., the points where the curve is concave upward

(b) $\frac{d^2y}{dx^2}$ is negative, i.e., the points where the curve is concave downward

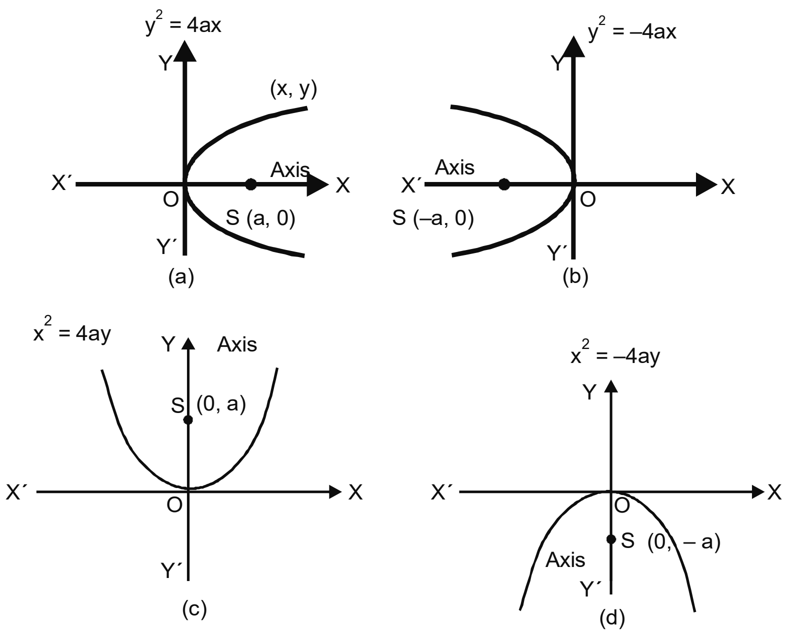
(iii) Find points of inflexion on the curve by equating $\frac{d^2y}{dx^2}$ to zero where $\frac{d^3x}{dy^3} \neq 0$

2.10.2 Some Elementary Curves

Some curves which are useful in drawing the approximate shape of the curves near the origin when either x-axis or y-axis is a tangent at the origin as given in Figure 2.15(a), (b), (c), (d), (e), (f), (g) and (h).

These figure display the cubic parabola.

Figure 2.16(a), (b), (c) and (d) show semi-cubic parabola.



NOTES

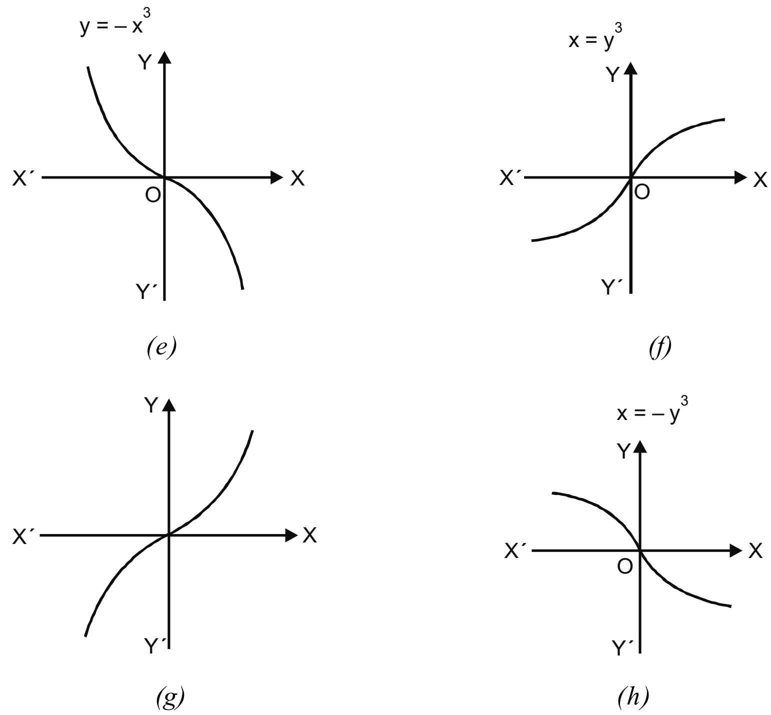


Fig. 2.16 *Cubical Parabola*

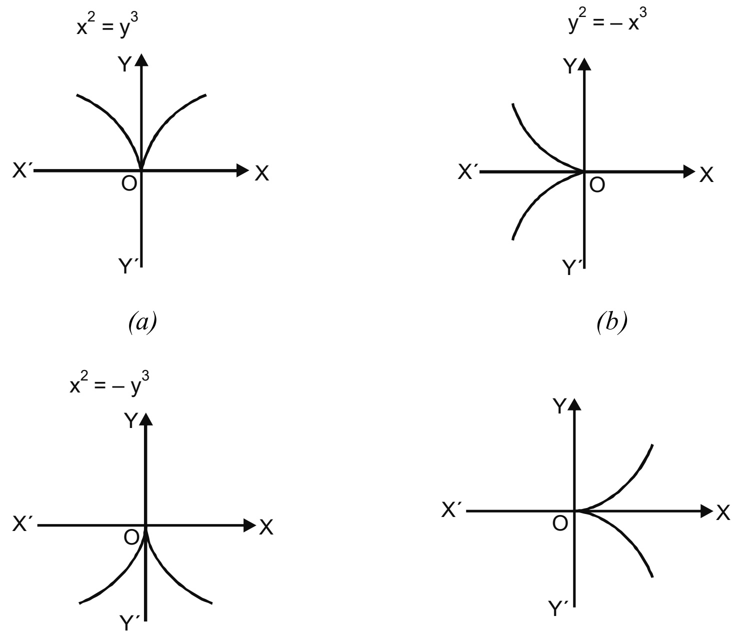


Fig. 2.17 *Semi-Cubical Parabola*

2.10.3 Basic Notation of Curve

The curvature is the amount by which a curve deviates from being a straight line, or a surface deviates from being a plane. For curves, the canonical example is that of a circle, which has a curvature equal to the reciprocal of its radius. Smaller circles bend more sharply, and hence have higher curvature.

The curvature at a point of a differentiable curve is the curvature of its osculating circle, i.e., the circle that best approximates the curve near this point.

The curvature of a straight line is zero. The curvature of a curve at a point is normally a scalar quantity such that it is expressed by a single real number.

The curvature of a differentiable curve was originally defined through osculating circles. A French mathematician, engineer, and physicist Augustin-Louis Cauchy showed that the center of curvature is the intersection point of two infinitely close normal lines to the curve.

NOTES

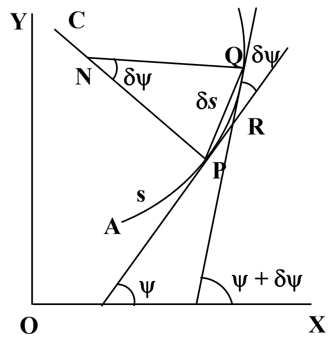


Fig. 2.18 Curvature

In Figure 2.18, Let P, Q be two neighbouring points on a curve AB. Let arc AP = s and arc AQ = s + ds so that the length of the arc PQ = ds, A being the fixed point on the curve, from where arc is measured. Let the tangents at P and Q make angles y and y + dy, respectively with a fixed line say x-axis. Then, angle dy through which the tangent turns as its point of contact travels along the arc PQ is called the total curvature of arc PQ.

The ratio $\frac{\delta\psi}{\delta s}$ represents the average rate of change in the angle y per unit of arc length along the curve. It is called the average curvature of arc PQ.

The limiting value of the average curvature when $Q \to P$ is called the curvature of the curve at the point P.

In general, the ratio $\frac{\delta\psi}{\delta s}$ approaches a limit $\frac{d\psi}{ds}$ as $\delta s \to 0$.

Thus, the curvature at a point P = $\lim\limits_{Q\to P} \frac{\delta\psi}{\delta s} = \lim\limits_{\delta s\to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$.

The curvature of a curve C at a point (x, y) on C is usually denoted by the Greek letter κ (kappa). It is given by the equation $\kappa = \left|\frac{d\psi}{ds}\right|$ where s is the arc length measured along the curve and ψ is the angle made by tangent line to C at (x, y) with positive x-axis.

The reciprocal of the curvature of the curve at P, is called the radius of curvature of the curve at P and is usually denoted by ρ. Thus, $\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$

If PC is normal at P and PC = ρ, then C is called the centre of curvature of the curve at P.

The circle with centre C and radius PC = ρ is called the circle of curvature of the curve at P.

NOTES

The length of the chord drawn through P, intercepted by the circle of curvature at P, is called a chord of curvature.

Note: A straight line does not bend at all (because $\frac{d\psi}{ds}$ is zero as ψ is constant). Therefore, the curvature of a straight line is zero (Refer Figure 2.19).

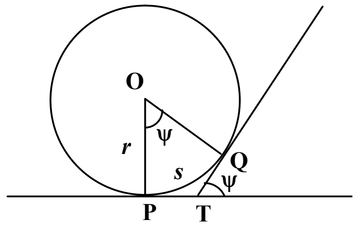


Fig. 2.19 *Chord of Curvature*

Curvature of a Circle

Let O and r be the centre and radius of a circle, respectively. Let P and Q be two points on the circle.

So that, arc PQ = s and the tangent at Q make an angle ψ with the tangent at P. Then, $\angle POQ = \psi$. Therefore, s (= arc PQ) = $r\,\psi$.

Differentiating with respect to ψ, we get $\frac{ds}{d\psi} = r$

$\therefore$ Curvature = $\frac{d\psi}{ds} = \frac{1}{r}$ (constant)

Thus, the curvature at every point of the circle is equal to the reciprocal of its radius and therefore, it is constant.

Note: Circle is the only curve of constant curvature.

Radius of Curvature for Intrinsic Curves

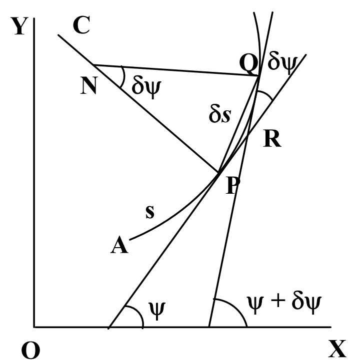


Fig. 2.20 *Radius of Curvature for Intrinsic Curves*

Let P, Q be two neighbouring points on a curve AB. Let the lengths of arc AP = s and arc AQ = $s + \delta s$.

Therefore, the length of the arc $PQ = \delta s$.

Let angles made by the tangents at P and Q with x-axis be ψ be $\psi+\delta s$, respectively. Also let the normals at P and Q intersect at N. Join P and Q.

$\therefore \angle PNQ = \delta\psi$

From the triangle PQN, by sine-rule, we get

$$\frac{\text{PN}}{\sin \text{PQN}} = \frac{\text{chord PQ}}{\sin \delta\psi}$$

$$\Rightarrow \qquad \text{PN} = \frac{\text{chord PQ}}{\sin \delta\psi}.\sin \text{PQN}$$

$$= \frac{\text{chord PQ}}{\delta s}.\frac{\delta s}{\delta\psi}.\frac{\delta\psi}{\sin \delta\psi}.\sin \text{PQN}$$

If ρ be the radius of curvature, then we have

$$\rho = \lim_{Q\to P} \text{PN}$$

$$= \lim_{\delta s\to 0} \frac{\text{chord PQ}}{\delta s}.\frac{\delta s}{\delta\psi}.\frac{\delta\psi}{\sin \delta\psi}.\sin \text{PQN}$$

$$[\because\ \delta s \to 0, \delta\psi \to 0,\ \lim_{\delta s\to 0} \frac{\text{chord PQ}}{\delta s} = 1,\ \angle \text{PQN} \to \frac{\pi}{2} \text{ and } \frac{\delta\psi}{\sin \delta\psi} \to 1]$$

$$\therefore \qquad \rho = \frac{ds}{d\psi}$$

The angle between the tangents to the curve at P and Q, i.e., $\delta\psi$ is called the angle of contingence of the arc PQ. The relation between s and ψ for a curve is called its intrinsic equation.

Thus, we can say that at a point P

$$\rho = \lim_{Q\to P}\left(\frac{\text{arc PQ}}{\text{Angle of Contigence of arc PQ}}\right).$$

Check Your Progress

5. What do you understand by Bohr radius?
6. What is perfect differential in thermodynamics?
7. What is definite integral?
8. Give the formula of integral product of two function.
9. Define the partial derivative.
10. What do you understand by point of inter-section with the axes?

2.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In mathematics, 'Differentiation' is the process of finding the derivative, or rate of change, of a function.
2. If y is a differentiable function of z, and z is a differentiable function of x, then y is a differentiable function of x, i.e.,

NOTES

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

3. In Mathematics, you usually deal with two kinds of quantities, namely constants and variables. A quantity which is liable to vary is called a variable quantity or simply a variable.
4. The mixed partial second order derivatives of the coefficients of the two terms on the right are equal. Applying this criterion to the equation, Maxwell's equations are obtained.
5. The Bohr radius is a physical constant, approximately equal to the most probable distance between the nucleus and the electron in a hydrogen atom in its ground state. It is named after Niels Bohr, due to its role in the Bohr model of an atom. Its value is $5.29177210903 \times 10^{-11}$ m.
6. In thermodynamics, when dQ is exact, the function Q is a state function of the system. Generally, neither work nor heat is a state function. An exact differential is sometimes also called a 'Total Differential', or a 'Full Differential', or, in the study of differential geometry, it is termed an exact form.
7. The definite integral of a function has a unique value. A definite integral is denoted by,

$$\int_a^b f(x)\, dx$$

In the above notation, 'a' is termed as the lower limit of the integral while 'b' is termed as the upper limit of the integral.

8. Integral of the Product of Two Functions = [First Function] × [Integral of Second Function] – Integral of [(Differential Coefficient of First Function) × (Integral of Second Function)]
9. The partial derivative of a function $f(x, y, \ldots.)$ with reference to the variable x is represented using the following specified notations:

$$f'x, \quad fx, \quad \partial_x f, \quad D_x f, \quad D_1 f, \quad \partial f / \partial x, \quad \text{or} \quad \frac{\partial}{\partial x} f$$

In addition, for a function $z = f(x, y, \ldots.)$, the partial derivative of z with regard to $\boldsymbol{x}$ is represented by $\partial z / \partial x$.

10. Point of intersection with the axes
    (i) Find the points where the curve meets the co-ordinate axes. If the curve meets the x-axis, then on putting $y = 0$, we get the real values of x. Similarly, examine for y-axis where $x = 0$.
    (ii) Shift the origin to all such points and find the tangents at all such points (new origin) by equating to zero the lowest degree terms. If any one of them is a double point, find its nature.
    (iii) Find the points of intersection of the curve with the lines $y = \pm x$ if the curve is symmetrical about these lines.

2.12 SUMMARY

NOTES

- Differentiation method is specifically used for finding or estimating the rate of change when one quantity is compared with another, precisely when the rate of change is not constant. Following are some definitions of differentiation.
- In differential calculus, the key objects are the derivative of a function and interrelated notions, such as the differential and its various applications. Fundamentally, the derivative of a function at a selected input value defines the rate of change of the function that is close or adjacent to that particular input value. Thus, differentiation is the method used to find a derivative.
- Both the calculus, the differential and the integral are linked or interrelated through the fundamental theorem of calculus, which specifies that the differentiation method is the reverse or opposite to integration method.
- In various fields of mathematics, the derivatives and their generalizations are frequently used for solving the problems related to functional analysis, complex analysis, differential geometry, abstract algebra and measure theory.
- Geometrically, the above theorem means that for an increasing function, tangent at any point makes acute angle with OX whereas for a decreasing function, tangent at any point makes an obtuse angle with x-axis.
- If y is a differentiable function of x_1, x_1 is a differentiable function of x_2, ... , x_{n-1} is a differentiable function of x, then y is a differentiable function of x.
- In Mathematics, you usually deal with two kinds of quantities, namely constants and variables. A quantity which is liable to vary is called a *variable quantity* or simply a *variable*.
- Temperature, pressure, distance of a moving train from a station are all variable quantities. On the other hand, a quantity that retains its value through all mathematical operations is termed as a *constant quantity* or a *constant.* Numbers like 4, 5, 2.5, π, etc., are all constants.
- The Bohr radius is a physical constant, approximately equal to the most probable distance between the nucleus and the electron in a hydrogen atom in its ground state. It is named after Niels Bohr, due to its role in the Bohr model of an atom.
- In thermodynamics, when dQ is exact, the function Q is a state function of the system. Generally, neither work nor heat is a state function. An exact differential is sometimes also called a 'Total Differential', or a 'Full Differential', or, in the study of differential geometry, it is termed an exact form.
- State functions depend only on the state of the system. Quantities whose values are independent of path are called state functions, and their differentials are exact (dP, dV, dG,dT...). Quantities that depend on the path followed between states are called path functions, and their differentials are inexact (dw, dq).

NOTES

- An 'Inexact Differential' or 'Imperfect Differential' is a type of differential used in thermodynamics to express changes in path dependent quantities. In contrast, an integral of an exact differential (a differential of a function) is always path independent since the integral acts to invert the differential operator.
- Integrating factors is sometimes possible to convert an inexact differential into an exact one by means of an integrating factor.
- Definite integration is a significant and an essential constituent of integral calculus. Both the integrations – indefinite and definite are interrelated processes. Fundamentally, the indefinite integration provides the basis for definite integral.
- The value of the definite integral of a function for some precise interval is specifically determined on the basis of function and the interval, but it will not consider the variable of integration selected for representing the independent variable.
- Integration by parts or partial integration method is used for finding the integral of a product of functions using expressions of the integral, specifically derivative and anti-derivative.
- A reduction formula for a specified integral is an integral of the form which is equivalent in form as the specified integral but of a lower degree or order.
- A function of several real variables also sometimes termed as the real multivariate function can be defined as a function having more than one argument in which all the arguments must be real variables. This concept or notion gives the concept or notion of a real variable function to function of several variables.
- The domain of a function of several real variables is specifically defined as a subset of $\mathbb{R}^n$ which can be occasionally explicitly defined. Actually, if we restrict or confine that the domain X of a function f as a subset $Y \subset X$, then we formally or properly obtain a different function which typically states the restriction f to Y and is denoted by $f|_Y$.
- A function of two variables, $f(x, y)$, basically the derivative of $\boldsymbol{f}$ is defined simply with respect to $\boldsymbol{x}$ while $\boldsymbol{y}$ is considered as a constant. This derivative is termed as the "partial derivative of f with respect to x" and is represented either in terms of $\partial f / \partial x$ or f_x. The '∂'symbol is used for denoting partial derivative.
- In symmetry about the x-axis the equation of a curve remains unaltered when y is changed to $-y$, i.e., if all the powers of y in the given equation are even, the curve is symmetrical about the x-axis. For example, since in the parabola $y^2 = 4ax$, power of y is even, hence it is symmetrical about the x-axis.
- In symmetry about the line $y = -x$ the equation of the curve is unchanged when x is changed to $-y$ and y is changed to $-x$, the curve is symmetrical about the line $y = -x$. For example, the curve $x^2 + y^2 + 4x - 4y + 1 = 0$ is symmetrical about the line $y = -x$.

- Asymptotes parallel to x-axis or y-axis is to the asymptotes parallel to x-axis or y-axis by equating the co-efficients of highest powers of x or y (in the equation of the curve) to zero.

NOTES

2.13 KEY TERMS

- **Differentiation:** In mathematics, 'Differentiation' is the process of finding the derivative, or rate of change, of a function.
- **Derivative:** Derivative, in mathematics, the rate of change of a function with respect to a variable.
- **Variable quantity:** In Mathematics, you usually deal with two kinds of quantities, namely constants and variables. A quantity which is liable to vary is called a variable quantity or simply a variable.
- **Bohr radius:** The Bohr radius is a physical constant, approximately equal to the most probable distance between the nucleus and the electron in a hydrogen atom in its ground state.
- **Reduction formula:** A reduction formula for a specified integral is an integral of the form which is equivalent in form as the specified integral but of a lower degree or order.

2.14 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is differentiation?
2. Define derivative.
3. State the chain rule of differentiation.
4. What do you understand by continuity and differentiability?
5. Determine the Maxwell equation.
6. Calculate the Bohr radius.
7. What is exact differential in chemistry?
8. Why state functions are exact differentials?
9. Define the definite integration.
10. What do you understand by reduction formula?
11. Give the mathematical definition of partial derivative.
12. State the symmetry about line.

Long-Answer Questions

1. Analyse the rules of differentiation with relevant examples.
2. Illustrate the differential calculus including maxima and minima giving examples.

NOTES

3. Describe the function, continuity and differentiability with the help of examples.
4. Analyse the Maxwell distribution.
5. Discuss about the Bohr radius.
6. Briefly explain about the exact and inexact differential in thermodynamics and their applications.
7. Explain in detail about the integral calculus with their relevant examples.
8. Elaborate on the partial differentiation with the help of examples.
9. Illustrate the curve sketching with appropriate examples.

2.15 FURTHER READING

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UNIT 3 ELEMENTARY DIFFERENTIAL EQUATIONS

NOTES

Structure

3.0 INTRODUCTION

A Differential Equation (DE) is an equation that defines a relationship between a function and one or more derivatives of that function. Differential equations play a central role in the mathematical treatment of chemical kinetics.

Secular radioactive equilibrium exists when the parent nucleus has an extremely long half-life. This type of equilibrium is particularly important in nature.

Quantum chemistry, also called molecular quantum mechanics, is a branch of chemistry focused on the application of quantum mechanics to chemical systems. In general, the Differential Equations (DE) of quantum mechanics are special cases of eigenvalue problems. Parallel to the DE eigenvalue problems is an equivalent class of eigenvalue problems expressed in Linear Algebra (LA).

In mathematics, the power series method is used to seek a power series solution to certain differential equations. In general, such a solution assumes a power series with unknown coefficients, then substitutes that solution into the differential equation to find a recurrence relation for the coefficients.

NOTES

The harmonic oscillator model is very important in physics, because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. Harmonic oscillators occur widely in nature and are exploited in many manmade devices, such as clocks and radio circuits. In mathematics and physical science, spherical harmonics are special functions defined on the surface of a sphere. They are often employed in solving partial differential equations in many scientific fields.

In mathematics, a Fourier series is a periodic function composed of harmonically related sinusoids, combined by a weighted summation. The Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution, which is regular at finite points is called a Legendre function of the first kind, while a solution, which is singular at is called a Legendre function of the second kind. A second order ordinary differential equation is an ordinary differential equation in which any derivatives with respect to the independent variable have order no greater than two.

In this unit, you will study about the exact first order differential equation, applications of chemical kinetics, secular equilibria, quantum chemistry, power series method, harmonic oscillator, spherical harmonics, Fourier series, Legendre equation and second order differential equation.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the exact first order differential equation
- Discuss about the applications of chemical kinetics
- Elaborate on the secular equilibria
- Define the quantum chemistry
- Analyse the power series method
- Comprehend the Fourier series
- Interpret the Legendre equation
- Explain about the second order differential equation

3.2 EXACT FIRST ORDER DIFFERENTIAL EQUATIONS

In mathematics, a differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.

Basic Definitions

Equations in which an unknown function, and its derivatives or differentials occur are called *differential equations.*

NOTES

For example,

$$(i)\ x + y\frac{dy}{dx} = 3y \qquad (ii)\ \frac{dy}{dx} + \frac{x-y}{x+y} = 0$$

$$(iii)\ \frac{d^2y}{dx^2} + y = \sin x, \text{ and} \qquad (iv)\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are all differential equations.

If, in a differential equation, the unknown function is a function of one independent variable, then it is known as an *ordinary differential equation.* If the unknown function is a function of two or more independent variables and the equation involves partial derivatives of the unknown function, then it is known as a *partial differential equation.* In the above examples, equations from *(i)* to *(iii)* are ordinary differential equations while the fourth is a partial differential equation.

The order of a differential equation is the order of the highest differential coefficient which occurs in it. In the examples given above, *(i)* and *(ii)* are of first order while *(iii)* and *(iv)* are of second order.

The degree of a differential equation is the degree of the highest order differential coefficient which occurs in it, after the equation has been cleared of radicals and fractions. The above listed equations are all of degree 1. To decide, for example, the degree of the differential equation,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2}, \text{ we rewrite it as}$$

$$\rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 \text{ and observe that it is of degree 2.}$$

If in a differential equation, the derivatives and the dependent variables appear in first power and there are no products of these, and also the coefficients of the various terms are either constants or functions of the independent variable, the equation is said to be a *linear differential equation.*

Solution of Differential Equations

A relation between the dependent and independent variables, which, when substituted in the equation, satisfies it, is known as a *solution* or *a primitive* of the equation. Note that, in the solution, the derivatives of the dependent variable should not be present.

The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the *general solution* or *the complete integral.* By giving particular values to the arbitrary constants appearing in the general solution we obtain *particular solutions* of the equation.

For example, $y = Ae^{2x} + Be^{-2x}$, $y = 3e^{2x} + 2e^{-2x}$ are respectively the general solution and the particular solution of the equation $\frac{d^2y}{dx^2} - 4y = 0$.

Solutions of equations which do not contain any arbitrary constants and which are not derivable from the general solution by giving particular values to one or more of the arbitrary constants, are called *singular solutions.*

NOTES

Formation of Differential Equations

Consider the relation $y = Ae^x + Be^{-x}$ ...(3.1)

Where A and B are constants.

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\frac{d^2y}{dx^2} = y \qquad ...(3.2)$$

Equation (2.2), which is obtained from Equation (3.1) by eliminating the arbitrary constants A and B, is the differential equation whose primitive is Equation (3.1). Further, Equation (3.1) is the general solution of Equation (3.2).

The order of the ordinary differential equation and the number of arbitrary constants appearing in the general solution will be always equal. Also, if there are n arbitrary constants in the relation between the dependent and the independent variables, then by eliminating them, we will arrive at a differential equation of order n.

Example 3.1: Obtain the differential equation associated with the primitive $y = Cx + C - C^3$, where C is an arbitrary constant.

Solution:

$$y = Cx + C - C^3 \qquad \ldots(1)$$

$$\frac{dy}{dx} = C \qquad \ldots(2)$$

Using equations (2) and (1), we get the differential equation,

$$\left(\frac{dy}{dx}\right)^3 - (x+1)\frac{dy}{dx} + y = 0$$

Example 3.2: Form the differential equation satisfied by all circles having their centres on the straight line $y = 10$ and touching the x-axis.

Solution: As the centre lies on the line $y = 10$ and the circle touches the x-axis, the centre of the circle is $(a, 10)$ and its radius will be 10.

Equation of the circle is,

$$(x - a)^2 + (y - 10)^2 = 100 \qquad \ldots(1)$$

Where a is an arbitrary constant. Differentiating equation (1) with respect to x,

$$2(x-a) + 2(y-10)\frac{dy}{dx} = 0$$

$$(x - a) = -(y - 10)\frac{dy}{dx} \qquad \ldots(2)$$

Substituting equation (2) in (1) we get,

$$(y-10)^2\left(\frac{dy}{dx}\right)^2 + (y-10)^2 = 100$$

i.e., $(y-10)^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = 100$

This is the equation for which equation (1) is the primitive.

Example 3.3: Eliminate A and B from $y = Ae^x + Be^{2x}$

Solution:
$$y = Ae^x + Be^{2x} \quad \ldots(1)$$
$$\frac{dy}{dx} = Ae^x + 2Be^{2x} \quad \ldots(2)$$
$$\frac{d^2y}{dx^2} = Ae^x + 4Be^{2x} \quad \ldots(3)$$
$$\frac{dy}{dx} - 2y = -Ae^x \quad \ldots(4)$$
$$\frac{d^2y}{dx^2} - 4y = -3Ae^x \quad \ldots(5)$$

Eliminating A between equations (4) and (5), we get the following differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

NOTES

3.2.1 Solution of Differential Equation of First Order and First Degree

An ordinary differential equation of first order and first degree can be written as

$$\frac{dy}{dx} = f(x, y) \quad ...(3.3)$$

The differential equation can be classified as

(1) Exact equations.

(2) Equations solvable by separation of variables.

(3) Homogeneous equations.

(4) Linear equation of first order.

Exact Equations

An equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact if there exists a function $u(x, y)$ such that

$$du = Mdx + Ndy$$

A necessary and sufficient condition for the exactness of the differential equation $Mdx + Ndy = 0$ is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the differential equation $Mdx + Ndy = 0$ is exact, then the general solution of this differential equation can be obtained by the following rules:

(1) Integrate Mdx taking y as constant.

(2) Integrate Ndy taking x as constant.

(3) Then add the two integrates ignoring the repeated terms if there be any but add a constant of integration to get the general solution.

NOTES

Example 3.4: Solve $(x^2 - y)dx + (y^2 - x)dy = 0$.

Solution: Here $M = x^2 - y$ and $N = y^2 - x$

Now $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = -1$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence the differential equation is exact.

Now integrating M with respect to x keeping y as constant, we get

$$\int M\,dx = \int (x^2 - y)\,dx = \frac{x^3}{3} - yx$$

Integrating N w.r.t. y keeping x as constant, we get

$$\int N\,dy = \int (y^2 - x)dy = \frac{y^3}{3} - xy$$

Then the general solution (or complete primitive) is $\frac{x^3}{3} - xy + \frac{y^3}{3} = c$, where c is an arbitrary constant.

Definition: A function $f(x, y)$ is said to be an *Integrating Factor* (I.F.) of the equation $Mdx + Ndy = 0$ if we can find a function $u(x, y)$ such that $f(x, y)\,(M\,dx + N\,dy) = du$. In other words, an I.F. is a multiplying factor by which the equation can be made exact.

For solving differential equations, the following results will be of much help:

Sl. No.	*Given Expression*	*Integrating Factor*	*Exact Differential*
1.	$x\,dy - y\,dx$	$\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
2.	$x\,dy - ydz$	$\frac{1}{x^2 + y^2}$	$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
3.	$x\,dx + y\,dy$	$\frac{1}{x^2 + y^2}$	$\frac{x\,dx + y\,dy}{x^2 + y^2} = \frac{\frac{1}{2}d(x^2 + y^2)}{x^2 + y^2}$ $= \frac{1}{2}d\,[\log\,(x^2 + y^2)]$
4.	$x\,dy - y\,dx$	$\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left\{\log\left(\frac{y}{x}\right)\right\}$
5.	$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

Example 3.5: Solve $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0.$

Solution: Here $x\,dx + y\,dx + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

or $\quad \dfrac{1}{2}d(x^2 + y^2) + \dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2} = 0$

or $\quad \dfrac{1}{2}d(x^2 + y^2) + d\left\{\tan^{-1}\left(\dfrac{y}{x}\right)\right\} = 0$

By integrating, we get

$\dfrac{1}{2}(x^2 + y^2) + \tan^{-1}\dfrac{y}{x} = c,$ where c is an arbitrary constant.

Example 3.6: Solve $x\ dx + y\ dy = m(x\ dy - y\ dx)$.

Solution: Here $x\ dx + y\ dy = m(x\ dy - y\ dx)$

or $\quad \dfrac{x\,dx + y\,dy}{x^2 + y^2} = m\left(\dfrac{x\,dy - y\,dx}{x^2 + y^2}\right) \qquad \left(\text{Multiplying both sides by } \dfrac{1}{x^2 + y^2}\right)$

or $\quad \dfrac{\dfrac{1}{2}d(x^2 + y^2)}{x^2 + y^2} = m\dfrac{\dfrac{x\,dy - y\,dx}{x^2}}{1 + \dfrac{y^2}{x^2}}$

or $\quad \dfrac{1}{2}d\,\{\log\,(x^2 + y^2)\} = m\ d\left\{\tan^{-1}\dfrac{y}{x}\right\}$

By integrating, we get $\dfrac{1}{2}\log\,(x^2 + y^2) = m\ \tan^{-1}\dfrac{y}{x} + c$, where c is an arbitrary constant.

3.2.2 Variable Separable

Differential equations of the form $f(x)\ dx = g(y)\ dy$ are called equations with variables separated.

Integrating the equation, we get,

$$\int f(x)dx = \int g(y)dy + C$$

Example 3.7: Solve $3e^x \tan y\ dx + (1 - e^x)\ sec^2 y\ dy = 0$

Solution: Rearranging we get,

$$\frac{3e^x}{1-e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

$$-3\int\frac{e^x}{1-e^x}dx = \int\frac{\sec^2 y}{\tan y}dy$$

$$3\log(1-e^x) = \log\tan y + \log C$$

$$(1-e^x)^3 = C\tan y$$

NOTES

Example 3.8: Solve $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

Solution:

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1} y = -\sin^{-1}(x) + C$$

3.2.3 Homogeneous Equations

These equations are of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of x and y of the same degree.

To solve these equations, put $y = vx$. Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting this in the given equation it reduces to the type in which the variables are separable.

Example 3.9: Solve $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$

Solution: Put $y = vx$, then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Using these in the given equation we get,

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3x^3}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{v^4}{1+v^3}$$

$$\frac{1+v^3}{v^4}dv = -\frac{dx}{x}$$

$$\int \frac{1}{v^4}dv + \int \frac{1}{v}dv = -\int \frac{dx}{x}$$

NOTES

$$-\frac{1}{3v^3}+\log v = C-\log x$$

$$\log vx = C+\frac{1}{3v^3}$$

$$\therefore \qquad \log y = C+\frac{x^3}{3y^3} \text{ or } y = Ae^{\frac{x^3}{3y^3}}$$

Homogeneous Equations with Constant Coefficients

An equation of the form,

$$\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+...+...+a_n\frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ...(3.4)$$

Where $a_1, a_2,..., a_n$ are constants and the equation is called a homogeneous linear partial equation of n^{th} order with constant coefficients.

Let, $D^n = \frac{\partial z}{\partial x^n}$; $D'^n = \frac{\partial z}{\partial y^n}$

Substituting above Equation in (3.4) we get,

$(D^n + a_1D^{n-1} D'+ a_2D^{n-2} D'^2+....+ a_nD'^n)z = F(x, y)$

$f(D, D')z = F(x, y)$

The solution will contain two parts:

1. Complementary Function (CF)
2. Particular Integral (PI)

To find complementary function, consider $f(D, D')z = 0$.

Auxiliary equation is,

$$f(m, 1) = 0 \text{ when } (D = m, D' = 1)$$

Now let us find the roots of the equation.

Roots	CF
1. $m_1, m_2, m_3...$	$\phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x) +...$
2. $m_1, m_1, m_1,...$	$\phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) +...$

For example,

1. If the roots are –1, 2, 6

   CF = $\phi_1(y - x) + \phi_2(y + 2x) + \phi_3(y + 6x)$

2. If the roots are –2, –2, –2

   CF = $\phi_1(y - 2x) + x\phi_2(y - 2x) + x^2\phi_3(y - 2x)$

3. If the roots are 1, 1, 2

$$CF = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x)$$

NOTES

Rules to Find PI

Let, $u(x, y)$ be the PI

Then, $$f(D, D')\, u(x, y) = F(x, y)$$

$$\therefore \quad u(x, y) = \frac{1}{f(D, D')} F(x, y)$$

1. $\dfrac{1}{f(D,D')} e^{ax+by} = \dfrac{1}{f(a,b)} e^{ax+by}, f(a,b) \neq 0$

2. (*a*) $\dfrac{1}{f(D,D')}\sin(ax+by)$ (*b*) $\dfrac{1}{f(D,D')}\cos(ax+by)$

   Replace D^2 by $-a^2$, D'^2 by $-b^2$, DD' by $-ab$

3. $\dfrac{1}{f(D,D')} x^r y^s$

   Expand $\dfrac{1}{f(D,D')}$ in ascending powers of $\dfrac{D'}{D}$ up to $\left(\dfrac{D}{D'}\right)^r$ and operate on $x^r y^s$.

   or,

   Expand $\dfrac{1}{f(D,D')}$ in ascending powers of $\dfrac{D}{D'}$ up to $\left(\dfrac{D'}{D}\right)^s$ and operate on $x^r y^s$.

4. $\dfrac{1}{f(D,D')} e^{ax+by} g(x,y) = e^{ax+by} \dfrac{1}{f(D+a, D'+b)} g(x,y)$

5. $\dfrac{1}{D - mD'} f(x,y)$

Change y to $y - mx$ in $f(x, y)$ and integrate with respect to x treating y as constant. In the resulting integral change y to $y + mx$.

Notes: 1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...\infty$; 2. $(1 - x)^{-1} = 1 + x + x^2 + ...\infty$.

Example 3.10: Solve $\dfrac{1}{D^2 + DD' - 6D'^2} y\cos x$

Solution: $\dfrac{1}{(D-2D')(D+3D')} y\cos x$

Replace y by $y + 3x$. Keep y as a constant.

$$= \frac{1}{(D-2D')}\int (y+3x)\cos x \, dx$$

$$= \frac{1}{D-2D'}\left[(y+3x)\sin x - 3(-\cos x)\right]$$

After integration replace y by $y - 3x$

$$= \frac{1}{D-2D'} y\sin x + 3\cos x$$

Replace y by $y - 2x$. While integrating keep y as a constant.

$$= \int\left[(y-2x)\sin x + 3\cos x\right]dx$$

$$= (y - 2x)(-\cos x) - (-2)(-\sin x) + 3\sin x$$

After integration replace y by $y + 2x$

$$= (y + 2x - 2x)(-\cos x) - 2\sin x + 3\sin x$$

$$= -y\cos x + \sin x$$

NOTES

Example 3.11: Solve $\frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$

Solution:

$$= \frac{1}{(D-2D')^2} e^{2x+y} \text{ when, } D = 2, D' = 1, Dr = 0$$

$$= \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$$= \frac{1}{D-2D'}\int e^{2x+y-2x}dx$$

$$= \frac{1}{D-2D'} e^{y} \cdot x$$

$$= \frac{1}{D-2D'} e^{y+2x}.x$$

$$= \int e^{y-2x+2x} \cdot x\ dx = \int e^{y} \cdot x\ dx$$

$$= e^{y}\frac{x^2}{2}$$

$$= e^{y+2x} \cdot \frac{x^2}{2}$$

Example 3.12: Solve $(2D^2 + 5DD' + 2D'^2)z = 0$

Solution: Auxiliary equation is,

$$2m^2 + 5m + 2 = 0$$

$$2m^2 + 4m + m + 2 = 0$$

$$2m(m+2) + 1(m+2) = 0$$

$$(2m+1)(m+2) = 0$$

$$m = \frac{-1}{2}, -2$$

NOTES

$$\therefore \quad z = \phi_1\left(y - \frac{x}{2}\right) + \phi_2(y - 2x)$$

Example 3.13: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

Solution:

Auxiliary equation is,

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m(m-2)^2 = 0$$

$$m = 0, 2, 2$$

$$\therefore \; z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

3.2.4 Non-Homogeneous Equations

The general form of the equation of this type is,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad ...(3.5)$$

Where atleast one c_1 and c_2 is non-zero.

The cases of this type are considered below:

Case (1): When $b_1 = -a_2$

Then the equation becomes,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{-b_1x + b_2y + c_2}$$

Cross-multiplying we get,

$$(-b_1x + b_2y + c_2)dy = (a_1x + b_1y + c_1)dx$$

$$b_2ydy + c_2dy - a_1xdx - c_1dx = b_1(ydx + xdy)$$

i.e., $$(b_2y + c_2)dy - (a_1x + c_1)dx = b_1d(xy)$$

Integrating we get,

$$b_2\frac{y^2}{2} + c_2y - a_1\frac{x^2}{2} - c_1x = b_1xy + K$$

Example 3.14: Solve $\dfrac{dy}{dx} = \dfrac{2x - y + 3}{x - 3y - 1}$

Solution: Cross-multiplying and rearranging the terms, we get,

$$(xdy + ydx) = (2x + 3)dx + (3y + 1)dy$$

Integrating we get,
$$\int d(xy) = \int (2x+3)dx + \int (3y+1)dy$$
$$xy = x^2 + 3x + \frac{3}{2}y^2 + y + K \text{ is the solution.}$$

NOTES

Case (2): When $a_1 = b_2$ and $b_1 = a_2$

Then the equation becomes,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{b_1x + a_1y + c_2}$$

$$\frac{dy}{a_1x + b_1y + c_1} = \frac{dx}{b_1x + a_1y + c_2}$$

$$= \frac{d(x+y)}{(a_1+b_1)(x+y) + c_1 + c_2}$$

$$= \frac{d(x-y)}{(b_1-a_1)(x-y) + c_2 - c_1}$$

$$\int \frac{d(x+y)}{(a_1+b_1)(x+y) + c_1 + c_2} = \int \frac{d(x-y)}{(b_1-a_1)(x-y) + c_2 - c_1}$$

$$\frac{1}{(a_1+b_1)}\log[(a_1+b_1)(x+y) + (c_1+c_2)]$$

$$= \frac{1}{(b_1-a_1)}\log[(b_1-a_1)(x-y) + (c_2-c_1)] + K$$

Example 3.15: Solve $\frac{dy}{dx} = \frac{3x+4y-1}{4x+3y+2}$

Solution:

$$\frac{dy}{3x+4y-1} = \frac{dx}{4x+3y+2} = \frac{d(x+y)}{7(x+y)+1} = \frac{d(x-y)}{(x-y)+3}$$

$$\int \frac{d(x+y)}{7(x+y)+1} = \int \frac{d(x-y)}{(x-y)+3}$$

$$\frac{1}{7}\log[7(x+y)+1] = \log[(x-y)+3] + \log c$$

$$\frac{1}{7}\log[7(x+y)+1] = \log c[(x-y)+3]$$

$$\log[7(x+y)+1] = \log c[(x-y)+3]^7$$
$$7x + 7y + 1 = K(x-y+3)^7$$

Non-Homogeneous Equations with Constant Coefficients

Consider the equation of the form,

$$f(D, D')z = F(x, y) \qquad ...(3.6)$$

If $f(D, D')$ is not homogeneous then Equation (3.6) is called a non-homogeneous linear equation.

PI can be found as in homogeneous linear equations.

NOTES

To find CF, consider $f(D, D')z = 0$ ...(3.7)

Assume, $z = Ce^{hx+ky}$ as a trial solution.

Substituting in Equation (3.7) we get, $f(h, k) = 0$

Find k in terms of h [or h in terms of k]

Let the r values of k be,

$$f_1(h), f_2(h),..., f_r(h)$$

Then, $z = C_n e^{hx+f_n(h)y}$, $n = 1, 2, ... , r$

will be the seperate solution of Equation (3.7).

The general solution of Equation (3.7) is of the form,

$$z = \sum C_1 e^{hx+f_1(h)y} + \sum C_2 e^{hx+f_2(h)y} + ... + \sum C_r e^{hx+f_r(h)y}$$

Example 3.16: Solve $(D^2 - DD' + D' - 1)z = e^x$

Solution:

To find CF, consider,

$(D^2 - DD' + D' - 1)z = 0$

Let, $z = Ce^{hx+ky}$ as a trial solution.

$$f(h, k) = 0$$

$$h^2 - hk + k - 1 = 0$$

$$h = \frac{k \pm \sqrt{k^2 - 4(k-1)}}{2}$$

$$= \frac{k \pm \sqrt{(k-2)^2}}{2}$$

$$= \frac{k \pm (k-2)}{2} = k - 1, 1$$

$$\therefore \qquad \text{CF} = \sum C_1 e^{(k-1)x+ky} + \sum C_2 e^{x+ky}$$

$$= e^{-x} \sum C_1 e^{k(x+y)} + e^x \sum C_2 e^{ky}$$

$$= e^{-x}\phi_1(y + x) + e^x\phi_2(y)$$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1} e^x$$

Put, $D = 1, D' = 0, Dr = 0$

$$= \frac{1}{(D - D' + 1)(D - 1)} e^x$$

$$= \frac{1}{D-1} e^x$$

$$= e^x \frac{1}{D+1-1}(1)$$

$$= xe^x$$

$$\therefore \qquad z = e^{-x}\phi_1(y + x) + e^x\phi_2(y) + xe^x$$

Example 3.17: Solve $(D^2 + DD' + D' - 1)z = \cos(x - y)$

Solution:

To find CF, consider

$$(D^2 + DD' + D' - 1)z = 0$$

NOTES

Assume, $z = Ce^{hx+ky}$ as a trial solution.

$f(h, k) = 0$ becomes, $h^2 + hk + k - 1 = 0$

$$h = \frac{-k \pm \sqrt{k^2 - 4(k-1)}}{2}$$

$$= \frac{-k \pm \sqrt{(k-2)^2}}{2}$$

$$= \frac{-k \pm (k-2)}{2}$$

$$= -1, -k+1$$

$$\text{CF} = \sum C_1 e^{-x+ky} + \sum C_2 e^{(-k+1)x+ky}$$

$$= e^{-x}\sum C_1 e^{ky} + e^{x}\sum C_2 e^{k(y-x)}$$

$$= e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$$

$$\text{PI} = \frac{1}{D^2 + DD' + D' - 1}\cos(x-y)$$

Put,

$$D^2 = -a^2 = -1$$

$$DD' = -ab = 1$$

$$D'^2 = -b^2 = -1$$

$$= \frac{1}{-1+1+D'-1}\cos(x-y)$$

$$= \frac{1}{D'-1}\cos(x-y)$$

$$= \frac{(D'+1)}{D'^2-1}\cos(x-y)$$

$$= -\frac{1}{2}(\sin(x-y) + \cos(x-y))$$

$$\therefore\ z = e^{-x}f_1(y) + e^{x}f_2(y-x) - \frac{1}{2}[\sin(x-y) + \cos(x-y)]$$

Example 3.18: Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{x-y} + x^2y$

Solution: To find CF, consider

$(D^2 + 2DD' + D'^2 - 2D - 2D')z = 0$

$f(h, k) = 0$ becomes,

$$h^2 + 2hk + k^2 - 2h - 2k = 0$$

$$h^2 + 2h(k-1) + k^2 - 2k = 0$$

NOTES

$$h = \frac{-2(k-1) \pm \sqrt{4(k-1)^2 - 4(k^2 - 2k)}}{2}$$

$$= \frac{-2(k-1) \pm \sqrt{4}}{2}$$

$$= 1 - k \pm 1$$

$$= 2 - k, - k$$

$$\text{CF} = \sum C_1 e^{(2-k)x+ky} + \sum C_2 e^{-kx+ky}$$

$$= e^{2x} \sum C_1 e^{k(y-2x)} + \sum C_2 e^{k(y-x)}$$

$$= e^{2x}\, \varphi_1(y - 2x) + \varphi_2(y - x)$$

$$\text{PI}_1 = \frac{1}{(D + D' - 2)(D + D')} e^{x-y}$$

$$= \frac{1}{(1-1-2)(D+D')} e^{x-y}$$

$$= -\frac{1}{2} \int e^{x-(y+x)} dx$$

$$= -\frac{1}{2} e^{-y} x$$

$$= -\frac{1}{2} x e^{x-y}$$

$$\text{PI}_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} x^2 y$$

$$= \frac{1}{-2D\left[1 + \frac{D'}{D} - \frac{D'^2}{2D} - D' - \frac{D}{2}\right]} x^2 y$$

$$= \frac{1}{-2D\left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D}\right)\right]} x^2 y \qquad \left(\because D'^2(x^2 y) = 0\right)$$

$$= \frac{1}{-2D}\left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D}\right)\right]^{-1} x^2 y$$

$$=$$

$$-\frac{1}{2D}\left[1+\frac{D}{2}+D'-\frac{D'}{D}+\frac{D^2}{4}+DD'-D'-\frac{3}{4}DD'+\frac{3}{4}D^2D'\right]x^2y$$

$$= -\frac{1}{2D}\left[1+\frac{D}{2}-\frac{D'}{D}+\frac{1}{4}DD'+\frac{D^2}{4}+\frac{3}{4}D^2D'\right]x^2y$$

$$= -\frac{1}{2}\left[\frac{1}{D}+\frac{1}{2}-\frac{D'}{D^2}+\frac{D'}{4}+\frac{D}{4}+\frac{3}{4}DD'\right]x^2y$$

$$= -\frac{1}{2}\left[\frac{x^3}{3}y+\frac{1}{2}x^2y-\frac{x^4}{12}+\frac{x^2}{4}+\frac{xy}{2}+\frac{3x}{2}\right]$$

$$\therefore\ z = e^{2x}\,\varphi_1(y-2x)+\varphi_2(y-x)-\frac{x}{2}e^{x-y}-\frac{1}{2}\left[\frac{x^3y}{3}+\frac{x^2y}{2}-\frac{x^4}{12}+\frac{x^2}{4}+\frac{xy}{2}+\frac{3x}{2}\right]$$

NOTES

3.2.5 Exact Differential Equations and Integrating Factors

In mathematics, an integrating factor is a function that is used to solve the given equation with the help of differential equations.

Exact Differential Equation

A differential equation is said to be exact if it can be derived directly from its primitive without any further operation of elimination or reduction. Thus the differential equation,

$$M(x, y)dx + N(x, y)dy = 0 \qquad ...(3.8)$$

is exact if it can be derived by equating the differential of some function $U(x, y)$ to zero.

Let, $U(x, y) = C$ be the solution of the Equation (3.8).

Differentiating this we get,

$$\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy = 0 \qquad ...(3.9)$$

Equations (3.8) and (3.9) are identical,

$$M = \frac{\partial U}{\partial x}, \quad N = \frac{\partial U}{\partial y}$$

If we eliminate U between these by means of the equivalence of the relation, then we get,

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial^2 U}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)$$

And, $$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Thus, the condition for $M\,dx + N\,dy = 0$, to be an exact equation is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

NOTES

Rules for Solving $Mdx + Ndy = 0$, when it is Exact

(i) First integrate M with respect to x regarding y as a constant.

(ii) Then integrate with respect to y those terms in N which do not contain x.

(iii) The sum of the expressions obtained in *(i)* and *(ii)*, when equated to an arbitrary constant, will be the solution.

Example 3.19: Solve $(\sin x \cos y + e^{2x})dx + (\cos x \sin y + \tan y)dy = 0$

Solution:

Here, $$M = \sin x \cos y + e^{2x}$$
$$N = \cos x \sin y + \tan y$$
$$\frac{\partial M}{\partial y} = -\sin x \sin y;\ \frac{\partial N}{\partial x} = -\sin x \sin y$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Integrating M with respect to x regarding y as a constant, we get, $\left[-\cos x \cos y + \frac{1}{2}e^{2x}\right]$

In N, the term not involving x namely $\tan y$ is integrated with respect to y giving $\log \sec y$.

$\therefore$ The solution is,

$$-\cos x \cos y + \frac{e^{2x}}{2} + \log \sec y = C$$

Example 3.20: Solve $(ye^{xy} - 2y^3)dx + (xe^{xy} - 6xy^2 - 2y)dy = 0$

Solution: $$M = ye^{xy} - 2y^3,\ \frac{\partial M}{\partial y} = e^{xy} + xye^{xy} - 6y^2$$

$$N = xe^{xy} - 6xy^2 - 2y,\ \frac{\partial N}{\partial x} = e^{xy} + xye^{xy} - 6y^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$\therefore \quad \int M dx = \int (ye^{xy} - 2y^3)dx$$

$$= y\frac{e^{xy}}{y} - 2xy^3 = e^{xy} - 2xy^3$$

NOTES

Integrating those terms in N which do not contain x with respect to y, we get,

$\int Ndy = \int -2ydy = -y^2$, omitting terms involving x in N.

$\therefore$ The solution is, $e^{xy} - 2xy^3 - y^2 = C$

Note: Sometimes when the equation is not apparently exact, by suitably regrouping the terms we can find an integrating factor, which when multiplied by the equation will make it an exact equation.

Example 3.21: Solve $y\,(2x^2y + e^x)dx - (e^x + y^3)\,dy = 0$

Solution:

$$M = 2x^2y^2 + ye^x, \qquad \frac{\partial M}{\partial y} = 4x^2y + e^x$$

$$N = -(e^x + y^3), \qquad \frac{\partial N}{\partial x} = -e^x$$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact. However, we can rearrange the equation as,

$$ye^x dx - e^x dy + (2x^2 dx - ydy)y^2 = 0$$

Now dividing by y^2, we have

$$\frac{ye^x dx - e^x dy}{y^2} + 2x^2 dx - ydy = 0$$

i.e., $$d\left(\frac{e^x}{y}\right) + 2x^2 dx - ydy = 0$$

Integrating, we find the solution as, $\frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = C$

Note that, in this example, we used the integrating factor $\frac{1}{y^2}$.

Rules for Finding Integrating Factors

Rule I When $Mx + Ny = 0$, and the equation is a homogeneous one, then $\frac{1}{Mx + Ny}$ is an integrating factor.

Example 3.22: Solve $x^2ydx - (x^3 + y^3)dy = 0$

Solution: The equation is not exact and $Mx + Ny = -y^4 \neq 0$. Hence, $-\frac{1}{y^4}$ can be used as an integrating factor. Then,

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3 + y^3}{y^4}\right)dy = 0$$

NOTES

$$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$$

$$\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$$

Hence, the equation has become exact,

$$\int M dx = -\int \frac{x^2}{y^3} dx = -\frac{x^3}{3y^3}$$

In N, integrating the term not containing x, namely $\frac{1}{y}$ with respect to y we get log y.

$\therefore$ The solution is $-\frac{x^3}{3y^3} + \log y = C$

Rule II If the equation is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor.

Example 3.23: Solve $y(x^2y^2 + xy + 1)dx + x(x^2y^2 - xy + 1)dy = 0$

Solution: The equation is not exact, since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy = 2x^2y^2 \neq 0$$

Using, $\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$ as an integrating factor we get,

$$\left(\frac{x^2y^2 + xy + 1}{2x^2y}\right)dx + \left(\frac{x^2y^2 - xy + 1}{2xy^2}\right)dy = 0$$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0$$

Now, $\frac{\partial M}{\partial y} = 1 - \frac{1}{x^2y^2}$ and $\frac{\partial N}{\partial x} = 1 - \frac{1}{x^2y^2}$

$\therefore$ The equation is exact and the solution is,

$$\int \left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \int -\frac{1}{y} dy = C$$

$$xy + \log x - \frac{1}{xy} - \log y = C$$

NOTES

Rule III

(i) If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor.

(ii) If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone, say $g(y)$, then $e^{\int g(y)dy}$ is an integrating factor.

Example 3.24: Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Solution: The equation is not exact and,

$$\frac{\partial M}{\partial y} = 3xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{y} = g(y)$$

$$e^{\int g(y)dy} = e^{\log y} = y \text{ is an integrating factor.}$$

Multiplying by y we get the equation,

$$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$$

Now, the equation is exact and the solution is,

$$\int (xy^4 + y^2)dx + 2\int y^5\,dy = C$$

i.e., $3y^4x^2 + 6xy^2 + 2y^6 = C$

Rule IV If the equation $Mdx + Ndy = 0$, is of the form $x^ay^b\,(my\,dx + nx\,dy) + x^ry^s(pydx + qxdy) = 0$, where a, b, m, n, r, s, p, q are constants, then x^hy^k, is an integrating factor, where h and k are determined using the condition that after multiplication by x^hy^k, the equation becomes exact.

Example 3.25: Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$

Solution: The equation is not an exact one and it can be rewritten as,

$$y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$$

or, $y^2(ydx + 2xdy) + x^2(-2ydx - xdy) = 0$

So that, this is of the form mentioned in Rule IV above.

Multiplying the equation by x^hy^k we get,

$$\left(x^h y^{3+k} - 2x^{h+2}y^{k+1}\right)dx + \left(2x^{h+1}y^{k+2} - x^{h+3}y^k\right)dy = 0 \qquad \ldots(1)$$

Now, $\frac{\partial M}{\partial y} = (3+k)x^hy^{k+2} - 2(k+1)x^{h+2}y^k$ and,

$$\frac{\partial N}{\partial x} = 2(h+1)x^h y^{k+2} - (h+3)x^{h+2} y^k$$

NOTES

Using the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and equating the coefficients of like powered terms on both sides, we get

$$3 + k = 2(h + 1) \qquad \text{or,} \qquad k - 2h = -1$$

$$2k + 2 = h + 3 \qquad \text{or,} \qquad 2k - h = 1$$

Solving them we get, $k = 1$, $h = 1$, so that the integrating factor is xy.

The equation (1) for these values of h and k becomes,

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0$$

As this equation is exact, the solution is,

$$\int (xy^4 - 2x^3y^2)dx = C$$

$$\frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$$

$$x^2y^4 - x^4y^2 = K$$

3.2.6 Applications of Chemical Kinetics

Chemical kinetics, also known as reaction kinetics, is the branch of physical chemistry, i.e., concerned with understanding the rates of chemical reactions. It is to be contrasted with thermodynamics, which deals with the direction in which a process occurs but in itself tells nothing about its rate. Chemical kinetics includes investigations of how experimental conditions influence the speed of a chemical reaction and yield information about the reaction's mechanism and transition states, as well as the construction of mathematical models that also can describe the characteristics of a chemical reaction. Rate laws are mathematical descriptions of experimentally verifiable data. In other hand rate laws may be written from either of two different but related perspectives. A differential rate law expresses the reaction rate in terms of changes in the concentration of one or more reactants (Δ[R]) over a specific time interval (Δt). Following applications which are related to a differentiations.

- **Ordinary Differential Equations:** Ordinary differential equations applications in real life are used to calculate the movement or flow of electricity, motion of an object to and fro like a pendulum, to explain thermodynamics concepts. Also, in medical terms, they are used to check the growth of diseases in graphical representation.
- **Differential Equation:** In mathematics, differential equation is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. On the other hand the differential method has the advantage

of allowing one to use all of the experimental data. The simplest case would be for a reaction, which is some non-integer order with respect to a reactant. If the reaction is dependent on another reactant, the excess method can be employed to suppress this effect.

NOTES

- **Differential Rate Laws:** It is the express the rate of reaction as a function of a change in the concentration of one or more reactants over a particular period of time; they are used to describe what is happening at the molecular state level during a reaction. These rate laws help us determine the overall mechanism of reaction (or process) by which the reactants turn into products.

  For example: Rate $= –d[A]/dt = [A]^n$

- **Integrated Rate Laws:** integrated rate laws express the reaction rate as a function of the initial concentration and a measured (actual) concentration of one or more reactants after a specific amount of time (t) has passed; they are used to determine the rate constant and the reaction order from experimental data. For example (when $n = 1$):

  $$\mathbf{\ln [A] = – kt + \ln [A]_0}$$

  Where $\mathbf{[A]_0}$ is the initial concentration of the reactant and $\mathbf{[A]}$ is the concentration after a time t has passed.

- **Mathematical Models:** The mathematical models that describe chemical reaction kinetics provide chemists and chemical engineers with tools to better understand and describe chemical processes, such as food decomposition, microorganism growth, stratospheric ozone decomposition, and the chemistry of biological systems. These models can also be used in the design or modification of chemical reactors to optimize product yield, more efficiently separate products, and eliminate environmentally harmful by-products. When performing catalytic cracking of heavy hydrocarbons into gasoline and light gas, for example kinetic models can be used to find the temperature and pressure at which the highest yield of heavy hydrocarbons into gasoline will occur.

- **Numerical Methods:** In some cases, equations are unsolvable analytically, but can be solved using numerical methods if data values are given. There are two different ways to do this, by either using software programmes or mathematical methods, such as the Euler method. Examples of software for chemical kinetics are:

  (i) Tenua, a Java app which simulates chemical reactions numerically and allows comparison of the simulation to real data.

  (ii) Python coding for calculations and estimates.

  (iii) The Kintecus software compiler to model, regress, fit and optimize reactions.

  Numerical integration: for a 1st order reaction A->B

  The differential equation of the reactant A is:

  $$d[A]/dt = –k[A]$$

It can also be expressed as:

$d\,[A]\,/dt = f\,(t,\,[A])$ (Which is, the same as $y' = f(y, x)$)

To solve the differential equations with **Euler and Runge-Kutta methods** we need to have the initial values.

- **Euler method:** Euler method → simple but inaccurate.

  At any point $y' = f(y, x)$ is the same as;

  $$y' = dy/dx$$

  $y' = dy/dx$

  We can approximate the differentials as discrete increases:

  $$y' = dy/dx \simeq \Delta y/\Delta x = [y(x+\Delta x) - y(x)]/\Delta x$$

  The unknown part of the equation is $y(x+\Delta x)$, which can be found if we have the data of initial values.

- **Runge-Kutta Methods:** Runge-Kutta methods is the more accurate than the Euler method. In this method, an initial condition is required: $y = y_0$ at $x = x_0$. The proposal is to find the value of y when $x = x_0 + h$, where h is a given constant. It can be shown analytically that the ordinat at that moment to the curve through (x_0, y_0) is given by the third-order Runge-Kutta formula. In first-order ordinary equations, the Runge-Kutta method uses a mathematical model that represents the relationship between the temperature and the rate of reaction. It is worth it to calculate the rate of reaction at different temperatures for different concentrations.

  The equation obtained is: $dr/dt = R/T + r\,\Delta H^0/RT^2$

- **Stochastic Methods:** Stochastic methods → probabilities of the differential rate laws and the kinetic constants. In an equilibrioum reaction with directed and inverse rate constants, it's easier to transform from A to B rather than B to A. As for probability computations, at each time it choose a random number to be compared with a threshold to know if the reaction runs from A to B or the other way around.

3.2.7 Secular Equilibria

In nuclear physics, ***secular equilibrium*** is a situation in which the quantity of a radioactive isotope remains constant because its production rate (for example due to decay of a parent isotope) is equal to its decay rate.

In Radioactive Decay

Secular equilibrium can occur in a radioactive decay chain only if the half-life of the daughter radionuclide **B** is much shorter than the half-life of the parent radionuclide **A**. In such a case, the decay rate of **A** and hence the production rate of **B** is approximately constant, because the half-life of **A** is very long compared to the time scales considered. The quantity of radionuclide **B** builds up until the number of **B** atoms decaying per unit time becomes equal to the number being produced per unit time. The quantity of radionuclide **B** then reaches a constant, equilibrium value. Assuming the initial concentration of radionuclide **B** is zero, full equilibrium usually takes several half-lives of radionuclide **B** to establish. The quantity of radionuclide **B** when secular equilibrium is reached is determined by the quantity of its parent **A** and the half-lives of the two radionuclide. That can be seen from the time rate of change of the number of atoms of radionuclide **B**:

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B,$$

Where λ_A and λ_B are the decay constants of radionuclide **A** and **B,** related to their half-life $t_{1/2}$ by $\lambda = \ln(2)/t_{1/2}$ and N_A and N_B are the number of atoms of **A** and **B** at a given time.

Secular equilibrium occurs when $dN_B/dt=0$, or

$N_B = \lambda_A / \lambda_B \, N_A$

Over long enough times, comparable to the half-life of radionuclide **A**, the secular equilibrium is only approximate; N_A decays away according to

$N_A(t) = N_A(0)\, e^{\lambda_A t}$

And the 'Equilibrium' quantity of radionuclide **B** declines in turn. For times short compared to the half-life of **A**, $\lambda_A t \ll 1$ and the exponential can be approximated as 1.

3.3 QUANTUM CHEMISTRY

An operator is a rule for transforming a given mathematical function into Another function. Operators will be indicated by their symbols with caret (/\ ) over it.

Example 3.26: The operator A is represented as $\hat{A}$. In general if $\hat{A}$ denotes an operator which transforms the function effects into another function $g(x)$ then mathematically it is written as $\hat{A}.f(x) = k.g(x)$

where, $\hat{A}$ is called operator

$f(x)$ is an operand and

$g(x)$ is an another function and K is constant.

This equation is called as an operator equation. This function which is operated by the operator is called operand.

Example 3.27

Name of the operator	Symbol	Operator on the given function
Differentiate operator	d/dx	$F(x) = x^2$ $d/dx\,(x^2) = 2x$
Integrate operator	?(c) dx	?x^2 $dx = x^3/3$

NOTES

Algebra of An Operator

Addition and Subtraction Operators

If operator $\hat{A}$ and operator $\hat{B}$ the two operators then the new operator obtained by addition and subtract10n of operators are called addition and subtraction operator respectively .It can be represented as:

$$\left(\hat{A}+\hat{B}\right)f\left(x\right)=\hat{A}f\left(x\right)+\hat{B}f\left(x\right)$$

$$\left(\hat{A}-\hat{B}\right)f\left(x\right)=\hat{A}f\left(x\right)-\hat{B}f\left(x\right)$$

Example 3.28: $\hat{A}$ = square root (v)

$\hat{B}$ $=d/dx$

$$(v+d/dx)\ x^2 = \frac{d}{dx}\cdot x^2$$

$$= x + 2x$$

$$= 3x$$

Similarly for subtraction operator

$$(v-d/dx)x^2 = vx^2 - \frac{d}{dx}\cdot x^2$$

$$= x - 2x$$

Linear Non - Linear Operators

Linear Operators

The action of an operator that tunrns the function $f(x)$ into the function $g(x)$ is represented by $\hat{A}F(x) = g(x)$ most common kind of operator encountered are linear operators which satisified the following conditions:

$$\hat{A}(F(x)+g(x)=\hat{A}F(x)+Bg(x)$$

Example 3.29: $\dfrac{d}{dx}\left(4x^2+5x^2\right)=\dfrac{d}{dx}\left(4x^2\right)+\dfrac{d}{dx}\left(5x^2\right)$

$$\frac{d}{dx}\left(9x^2\right)=\frac{d}{dx}\left(4x^2\right)+\frac{d}{dx}\left(5x^2\right)$$

$$18x = 8x + 10x$$

$$18x = 18x$$

L.H.S. = R.H.S

Non - Linear Operators

In operating on the sum of two functions, if an operator does not give the same result as the sum of operations on the two functions separately. Then operator is said to be non linear. Mathematically it can be written as

$$\hat{A}\,(f(x) + g(x)) = \hat{A}\,f(x) + \hat{A}\,g(x)$$

Example 3.30:

Taking the square root function, if $\hat{A}$ is non-linear then

NOTES

$$\sqrt{f(x)+g(x)} \neq \sqrt{f(x)} + \sqrt{g(x)}$$

$$f(x)=4x^2, \;\; g(x)= 16x^2$$

$$\sqrt{4x^2+16x^2} = \sqrt{4x^2} + \sqrt{16x^2}$$

$$\sqrt{20x^2} = \sqrt{4x^2} + \sqrt{16x^2}$$

$$\sqrt{20x} = 2x + 4x$$

$$\sqrt{20x} \neq 6x$$

Del Operator ∇

Operator is a vector quantity or complex quantity, represented in terms of its components

for Example:

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

where i, j, k are the unit vectors along x, y and z axes respectively. Laplacian operator ∇:

Laplacian Operator ∇

The square of the (∇) Del operator ∇^2 is also called Laplacian operator. It is obtained as a dot product of ∇

i.e. $\nabla . \nabla = \nabla^2$

$$\left(k\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)\left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right) = i^2\frac{d^2}{dx^2} + j^2\frac{d^2}{dy^2} + k^2\frac{d^2}{dz^2}$$

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

where $i.i = 1$, $J.J = 1$

$k.k = 1$ and $i.j = 0$

$j.k = 0$ and $k.i = 0$

where x, y and z are cartesian co-ordinates

Position Operator

Position is represented by letter $r(x, y, \mathrm{z})$ in Cartesian co-ordinate system. In tenns of its components

$$\overline{r} = \overline{ix} + \overline{jy} + \overline{kz}$$

NOTES

Momentum Operator

These are two types

1. Linear momentum operator (p)
2. Angular momentum operator (L)

Linear Momentum Operator (P):

Total linear momentum operator is represented by P. In terms of its components

$$P = iP_x + JP_y + KP_z$$

where P_x is component of the linear momentum along the x direction

P_y is component of the linear momentum along the y direction and

P_z is component of linear momentum along z direction.

Properties of Linear Momentum Operator

1. It is a differential operator
2. It is a linear operator
3. It is a hermitian operator
4. Linear momentum operator will not commute with position operator
5. It will commute with Hamiltonian operator .
6. Linear momentum operator is Hermitian

Angular Momentum Operator (L)

Total angular momentum (L) is written in terms of its components as $L = iL_x + jL_y + kL_z$ where L_x, L_y, L_z are the components of the angular momentum about x, y, z axes respectively.

Total angular momentum is obtained by the cross or vector product of the position and the linear momentum vectors. Angular momentum of a particle moving around a fixed point is given by the vector product or cross product of r and p.

Where r is the radius vector and p is the linear momentum vector

$$L = \hat{r} \times \hat{p}$$

$$iL_x + jL_y + kL_z = (ix + jy + kz)\ x\ (ip_x + jp_y + kp_z)$$

$$iL_x + jL_y + kL_z = \begin{bmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{bmatrix}$$

$$= i(yp_z - zp_y) - j(xp_z - zp_x) + k(xp_y - yp_z)$$

$$iL_x + jL_y + kL_z = i(yp_z - zp_y) + j(zp_x - xp_z) + k(xp_y - yp_z)$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Replacing the momentum operators by quantum mechanical operator, we get

$$\hat{L}_x = \frac{h}{2\pi_i}\left(y\cdot\frac{d}{dz} - z\cdot\frac{d}{dy}\right)$$

$$\hat{L}_y = \frac{h}{2\pi_i}\left(z\cdot\frac{d}{dx} - x\cdot\frac{d}{dz}\right)$$

$$\hat{L}_z = \frac{h}{2\pi_i}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)$$

NOTES

$\hat{L}^2$ is the dot product of L with L

$L^2 = L.L.$

$L^2 = (iL_x + jL_y + kL_z).(iL_x + jL_y + kL_z)$

$L^2 = L^2_x + L^2_y + L^2_z$

Hamiltonian Operator, $\hat{H}$

The operator correspondin to the physically observable quantity, energy is known as Hamiltonian Operator.

$E=T+V$

where, T = Kinetic energy

V = Potential energy

Linear momentum

p = mv

$p^2 = m^2v^2$

Divide through out by $2m$

$$\frac{p^2}{2m} = \frac{1}{2}mv^2$$

we know that

$$T = \frac{1}{2}mv^2$$

So $E = \frac{1}{2}mv^2 + V$

$$E = \frac{p^2}{2m} + V$$

If we represent energy (E) by Mamiltonian Operator $\hat{H}$

NOTES

Then,

$$H_x = \frac{p^2 x}{2m} + V$$

$$\frac{1}{2m}\frac{\eta^2}{2m}\frac{\partial^2}{\partial x^2} + V_x \qquad\qquad \Theta\ p = \frac{\eta}{i}\frac{\partial}{\partial x}$$

$$H_x = \frac{\eta^2}{2m}\frac{\partial^2}{\partial x^2} + V_x \qquad\qquad i^2 = -1$$

T he above equation is for x co-ordinate and for x, y, z coordinates

$$H = \frac{\eta^2}{2m}\nabla^2 + V$$

$$H = \frac{\eta^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V$$

Hermitian Operator

The given operator $\hat{A}$ is said to be Hermitian if it obeys "tum over rule" which is defined mathematically as

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi_i^*\left(\hat{A}\psi_j\right) d\tau = \int\int\int_{-\infty}^{+\infty} \psi_i A^* \psi_j^*\ d\tau$$

Where ψ_i and ψ_j are the e1gen functions of the operator $\hat{A}$ and ψ_j^* is the complex conjugate of ψ_i.

Check Your Progress

1. Give the basic definition of differential equation.
2. What is singular solutions?
3. What did you meant by reaction kinetics?
4. State the differential rate laws.
5. What do you understand by secular equilibrium?
6. When operator is said to be non-linear operator?

3.4 POWER SERIES METHOD

The power series method is used to search a power series solution to certain differential equations. Basically, such a solution assumes a power series with unknown coefficients and then substitutes that solution into the differential equation for finding a recurrence relation for the coefficients. The power series method can also be applied to certain non-linear differential equations with less flexibility.

If a homogeneous linear different equation has **constant coefficients,** then it can be solved using algebraic methods and its solutions are elementary functions known from calculus e^x, cos x, etc. However, if such an equation has variable coefficients functions of x, it must be solved by other methods. The standard basic technique used for the purpose of answering linear differential equations containing variable coefficients is known as the **power series method**. The answer is provided in the form of power series which explains the name. The use of these series can be done for the purpose of evaluating the values of solutions, for exploring their properties and for obtaining different kinds of representation of those solutions. The operations on power series include the methods of differentiation, addition, multiplication, etc.)

Power Series

A power series about ***a*** or just power series is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Where a and c_n are numbers. The c_n's are often called the coefficients of the series. The most important thing about a power series is that it is a function of x. We know from calculus that a power series (in powers of $x - x_0$) is an infinite series of the form,

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{3.10}$$

Where $a_0, a_1, a_2, \ldots$ are constants, known as the **coefficients** of the series. Here x_0 is a constant, known as **center** of the series and x is a variable.

If $x_0 = 0$, we obtain a **power series in powers of *x*** as,

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3.11}$$

Assume here that all variables and constants are real. Well known examples of power series are the Maclaurin series.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m + 1 + x + x^2 + \cdots$$

($|x| < 1$, is geometric series),

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

NOTES

The power series method is used for solving differential equations because this method is considered easy and universally used as standard. We first describe the procedure and then exemplify it using simple equations. For a given differential equation,

$$y'' + p(x)y' + q(x)y = 0$$

Now represent $p(x)$ and $q(x)$ as power series in powers of x or of $x - x_0$ if the solutions are required in powers of $x - x_0$. If, $p(x)$ and $q(x)$ are polynomials then this step can be escaped. Let us assume an answer in the form of a power series with anonymous coefficients as,

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (3.12)$$

Insert this series and the series that we have obtained by term wise differentiation into the equation.

$$\text{(a)}\ y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \qquad (3.13)$$

$$\text{(b)}\ y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots \qquad (3.14)$$

Now accumulate like powers of x and evaluate the sum of the coefficient of each power of x to zero that occurs beginning with the constant terms that includes the terms having x, the terms having x^2, etc. This provides relations from which the unknown coefficients can be determined in Equation (3.12) of series successively. This can be explained using some simple equations that can be solved with the help of elementary methods.

Example 3.31 : Solve $y' - y = 0$.

Solution: Step 1, We insert series Equations (3.12) and (3.13) into the equation as follows:

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + \cdots) = 0$$

Step 2, Accumulate the like powers of x to find,

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

Step 3, Evaluating the coefficient of each power of x to zero we get,

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0, \qquad 3a_3 - a_2 = 0, \cdots$$

Step 4, After solving these equations, $a_1, a_2, \cdots$ can be expressed in terms of a_0 which are arbitrary:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots$$

Step 5, Using these coefficients the series in Equation (3.12) becomes,

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots.$$

Step 6, Finally we have obtained the known and common general solution as,

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = a_0 e^x.$$

Example 3.32: Solve $y' = 2xy$.

Solution: Following the same method insert series of Equations (3.12) and (3.13) into the equation:

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

Now do the multiplication by 2*x*. The resulting equation can be easily written as,

$$a_1 + 2a_2 + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$= 2a_0 + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 = 2a_4 x^5 + \cdots$$

Thus, we conclude that:

$$a_1 = 0,\ 2a_2 = 2a_0,\quad 3a_3 = 2a_1,\quad 4a_4 = 2a_2,\quad 5a_5 = 2a_3 \quad \ldots$$

Therefore, $a_3 = 0,\ a_5 = 0, \cdots$ and for the coefficients having even subscripts it becomes,

$$a_2 = a_0, \qquad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \qquad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \cdots;$$

Where a_0 remains arbitrary. Using these coefficients the series in Equation (3.12) provides the solution in the form given below,

$$y = a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2}$$

The answer can be verified using the method of separating variables.

Example 3.33 : Solve $y'' + y = 0$

Solution: Insert series Equations (3.12) and (3.14) into the equation that we have obtained,

$$\left(2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots\right) + \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = 0$$

By accumulating the like powers of *x* we get,

$$\left(2a_2 + a_0\right) + \left(3\cdot 2a_3 + a_1\right)x + \left(4\cdot 3a_4 + a_2\right)x^2 + \cdots = 0$$

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On evaluating the coefficient of each power of x to zero we get,

$$2a_2 + a_0 = 0 \qquad \text{Coefficient of } x^0$$

$$3 \cdot 2a_3 + a_1 = 0 \qquad \text{Coefficient of } x^1$$

$$4 \cdot 3a_4 + a_2 = 0 \qquad \text{Coefficient of } x^2\text{, etc.}$$

After solving the given equations, we observe that $a_2, a_4, \cdots$ can be expressed in terms of a_0 and similarly $a_3, a_5, \cdots$ can be expressed in terms of a_1, where a_0 and a_1 are arbitrary. :

$$a_2 = -\frac{a_0}{2!}, \qquad a_3 = -\frac{a_1}{3!}, \qquad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \cdots;$$

Using these coefficients the series in Equation (3.12) can be written as,

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!} + \frac{a_1}{5!}x^5 + \cdots$$

The acceptable reordering terms for a power series can be written in the form as,

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

The known and common general solution can be distinguished as,

$$y = a_0 \cos x + a_1 \sin x .$$

Probably we finish up using new functions specified by power series. If these equations and its solutions are of realistic or theoretical importance then the names are given to them and are systematically examined. This is how the Legendre's, the Bessel's and the Gauss's hypergeometric equations were established.

Basic Concepts

In calculus, a power series is an infinite series of the form,

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \qquad (3.15)$$

If the variable x, the **center** x_0, and the **coefficients** $a_0, a_1, \cdots$ to be real then the ***n*th partial sum** of series Equation (3.15) is,

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \qquad (3.16)$$

Where $n = 0, 1, \cdots$. If the terms of s_n are omitted from Equation (3.15), then the remaining expression becomes,

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots \qquad (3.17)$$

This expression is termed as the **remainder** of Equation (3.15) subsequent to the term $a_n(x - x_0)^n$.

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For example, consider the following geometric series,

$$1 + x + x^2 + \cdots + x^n + \cdots$$

We can include,

$$s_0 = 1. \qquad R_0 = x + x^2 + x^3 + \cdots,$$

$$s_1 = 1 + x, \qquad R_1 = x^2 + x^3 + x^4 + \cdots,$$

$$s_2 = 1 + x + x^2 \qquad R_2 = x^3 + x^4 + x^5 + \cdots, \text{ etc.}$$

In this manner, Equation (3.15) is associated with the series of the partial sums $s_0(x), s_1(x), s_2(x), \cdots$. If $x = x_1$ series converges as,

$$\lim_{n \to \infty} s_n(x_1) = s(x_1)$$

Then the series in Equation (3.15) is termed **convergent** at $x = x_1$ and the number $s(x_1)$ is termed the **value** or **sum** of Equation (3.15) at x_1. This is written as,

$$s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

For every n we have,

$$s(x_1) = s_n(x_1) + R_n(x_1) \qquad (3.18)$$

If the series diverges at $x = x_1$, then the series in Equation (3.15) is termed **divergent** at $x = x_1$

In convergence, for any positive $\in$ there must be an N which depends on $\in$. Using Equation (3.18) we get,

$$|R_n(x_1)| = |s(x_1) s_n(x_1)| < \in \qquad \text{for all } n > N \qquad (3.19)$$

Mathematically, this signifies that all $s_n(x_1)$ with $n > N$ be positioned between $s(x_1) - \in$ and $s(x_1) + \in$. In fact, it refers that for convergence the sum $s(x_1)$ of Equation (3.19) can be accurately approximated at x_1 by $s_n(x_1)$ considering n too large.

3.4.1 Convergence—Interval and Radius

The convergence of the series may depend upon the value of x that we put into the series. A power series may converge for some values of x and not for other values of x.

We can consider that there is a number R so that the power series will converge for, $|x-a| < R$ and will diverge for $|x-a| > R$. This number is termed as the **radius of convergence** for the series. Remember that the series may or may not converge if $|x-a| = R$. If something happens at these points it will not change the radius of convergence.

Secondly, the interval of all x's, including the end points, for which the power series converges is termed as the **interval of convergence** of the series. These two concepts are quite strongly coupled together. If we know that the radius of convergence of a power series is R then we have the following:

$a - R < x < a + R$	power series converges
$x < a - R$ and $x > a + R$	power series diverges

The interval of convergence must then contain the interval $a - R < x < a + R$ since we know that the power series will converge for these values. We also know that the interval of convergence can not contain x's in the ranges $x < a - R$ and $x > a + R$ since we know the power series diverges for these value of x. Therefore, to completely identify the interval of convergence we have to determine if the power series will converge for $x = a - R$ or $x = a + R$.

If the power series converges for one or both of these values then we must include those in the interval of convergence.

1. If the series of Equation (3.15) converges at $x = x_0$, then all its terms are zero except for the first a_0. In special cases it can be the just x for Equation (3.15) which converges. Such a series is not considered significant.
2. If there are any other values of x for which convergence is done by the series, then such values form an interval, termed as the convergence interval. If this interval is finite, then it contains the midpoint's x_0 of the form,

$$|x - x_0| < R \tag{3.20}$$

The series in Equation (3.20) converges for all x such that $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$. Here the number R is the **radius of convergence** of Equation (3.15). It can be acquired using any one of the following formulas, provided these limits exist and are not zero.

$$\text{(a)}\ R = 1\Big/\lim_{m\to\infty} m\sqrt{|a_m|} \qquad \text{(b)}\ R = 1\Big/\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| \tag{3.21}$$

If they are infinite, then the series in Equation (3.15) converges only at the center x_0.

3. The convergence interval can at times be infinite, i.e, series in Equation (3.15) converges for all x. For example, if the limit in Equation (3.21a) or (3.21b) is zero, then such case takes place and can be written as $R = \infty$. For each x, for which the series in Equation (3.15) converges, has a definite value $s(x)$. We state that series in Equation (3.15) denotes the function $s(x)$ in the convergence interval and can be written as,

$$s(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \qquad (|x - x_0| < R)$$

The following examples will make the concept clear.

Convergence at the Center: Consider the series,

$$\sum_{m=0}^{\infty} m! x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

We include $a_m + m!$ in Equation (3.21b),

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = m + 1 \to \infty \qquad \text{as } m \to \infty$$

This series only converges at the center $x = 0$ and hence it is useless series.

Convergence in a Finite Interval: Consider the geometric series,

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

We include $a_m = 1/m!$. Hence, the Equation (3.21b) becomes,

$$\frac{a_{m+1}}{a_m} = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0 \qquad \text{as } m \to \infty$$

This series converges for all x and is considered significant.

The following hint will help you to solve some specific problems:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + - \cdots$$

This is considered as a series in powers of $t = x^3$ having coefficients $a_m = (-1)^m / 8^m$, so that the series in Equation (3.21b) becomes,

$$\left| \frac{a_{m=1}}{a_m} \right| = \frac{8^m}{8^{m+1}} = \frac{1}{8}$$

The $R = 8$ and therefore the series converges for $|t| < 8$, i.e., $|x| < 2$.

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Example 3.34: Determine the radius of convergence and interval of convergence for the power series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$$

Solution: This power series will converge for $x = -3$ at this point. We have to determine the remainder of the x's for which it convergences by using any one of the appropriate test. Using the test we derive the condition(s) on x which can be used to determine the values of x for which the power series will converge and the values of x for which the power series will diverge. From this we evaluate the radius of convergence and also the interval of convergence. The most suitable test in this case is the ratio or root test. Using the test we get,

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}\frac{4^n}{(-1)^n(n)(x+3)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{-(n+1)(x+3)}{4n}\right|$$

Here x is not dependent on the limit hence it can be factored out of the limit. We must keep the absolute value bars on it so that everything remains positive. The limit then becomes,

$$L = |x+3|\lim_{n\to\infty}\frac{n+1}{4n}$$
$$= \frac{1}{4}|x+3|$$

The ratio test defines that if $L < 1$ the series will converge, if $L > 1$ the series will diverge and if $L = 1$ then any thing may happen. Thus,

$$\frac{1}{4}|x+3| < 1 \quad \Rightarrow \quad |x+3| < 4 \qquad \text{series converges}$$
$$\frac{1}{4}|x+3| > 1 \quad \Rightarrow \quad |x+3| > 4 \qquad \text{series diverges}$$

Now consider the case $L = 1$. Now we have the radius of convergence for this power series which are the required conditions for the radius of convergence. Hence, the radius of convergence for this power series is $R = 4$.

Next we obtain the interval of convergence. For this we obtain most of the interval by solving the first inequality from the above series.

$$-4 < x+3 < 4$$
$$-7 < x < 1$$

Hence, the interval of validity is given by $-7 < x < 1$. Now we determine whether the power series will converge or diverge at the endpoints of this interval. Remember that these values of x will correspond to the value of x to give $L = 1$.

The convergence at these points can be determined by just plugging them into the original power series and observing if the series converges or diverges.

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For $x = -7$: In this case the series is,

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(-1)^n 4^n$$

$$= \sum_{n=1}^{\infty}(-1)^n(-1)^n n \qquad (-1)^n(-1)^n = (-1)^{2n} = 1$$

$$= \sum_{n=1}^{\infty} n$$

This series is divergent since $\lim_{n\to\infty} n = \infty \neq 0$.

For $x = 1$: In this case the series is,

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}(4)^n = \sum_{n=1}^{\infty}(-1)^n n$$

This series is also divergent since $\lim_{n\to\infty}(-1)^n n$ does not exist.

Hence, in this case the power series will not converge for either endpoint. The interval of convergence is $-7 < x < 1$.

Example 3.35: Determine the radius of convergence and interval of convergence for the power series,

$$\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n$$

Solution: Using the ratio test we get,

$$L = \lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{n+1}\ \frac{n}{2^n(4x-8)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n(4x-8)}{n+1}\right|$$

$$= |4x-8|\lim_{n\to\infty}\frac{2n}{n+1}$$

$$= 2|4x-8|$$

Thus the convergence or divergence can be determined as follows,

$$2|4x-8| < 1 \qquad \text{series converges}$$

$$2|4x-8| > 1 \qquad \text{series diverges}$$

For the interval of convergence we need $|x-a| < R$ and $|x-a| > R$, i.e., we have to factorize 4 out of the absolute value bars to obtain the accurate radius of convergence. This gives,

$$8|x-2| < 1 \quad \Rightarrow \quad |x-2| < \frac{1}{8} \qquad \text{series converges}$$

$$8|x-2| > 1 \quad \Rightarrow \quad |x-2| > \frac{1}{8} \qquad \text{series diverges}$$

Thus, the radius of convergence for this power series is $R = \frac{1}{8}$. Next we find the interval of convergence by first solving the inequality that gives convergence.

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$$-\frac{1}{8} < x - 2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

Now verify the end points.

For $x = \frac{15}{8}$**:** The series is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{15}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n} \frac{(-1)^n}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This series is considered as the alternating harmonic series and it converges.

For $x = \frac{17}{8}$**:** The series is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{17}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n} \frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

This series is the harmonic series and it diverges. Hence, the power series converges for one of the end points but not the other. Then, the interval of convergence for this power series is given as,

$$\frac{15}{8} \le x < \frac{17}{8}$$

Example 3.36: Determine the radius of convergence and interval of convergence for the power series,

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution: Using the ratio test we get,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)n!\ (2x+1)}{n!}\right| \\ &= |2x+1|\lim_{n\to\infty}(n+1) \end{aligned}$$

Here, the limit is infinite but we can use the term with the x's in front of the limit. We have $L = \infty -> 1$ provided $x \neq -\frac{1}{2}$. Hence, this power series will only converge if $x = -\frac{1}{2}$.

The basic principle defines that every power series will converge for $x = a$ and here it is $a = -\frac{1}{2}$. We get a from $(x-a)^n$and the coefficient of x must be one. The radius of convergence is $R = 0$ and the interval of convergence is $x = -\frac{1}{2}$.

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Example 3.37: Determine the radius of convergence and interval of convergence for the power series,

$$\sum_{n=1}^{\infty}\frac{(x-6)^n}{n^n}$$

Solution: In this case we use the root test to get,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(x-6)^n}{n^n}\right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\left|\frac{x-6}{n}\right| \\ &= |x-6|\lim_{n\to\infty}\frac{1}{n} \\ &= 0 \end{aligned}$$

Since $L = 0 < 1$, so despite of the value of x this power series will converge for every x. In such conditions we define that the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

Example 3.38: Determine the radius of convergence and interval of convergence for the power series,

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{(-3)^n}$$

Solution: In this case the significant difference is the exponent on the x, which is $2n$ rather than the standard n. We use the root test again to determine the convergence. It gives,

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$$L = \lim_{n\to\infty}\left|\frac{x^{2n}}{(-3)^n}\right|^{\frac{1}{n}}$$
$$= \lim_{n\to\infty}\left|\frac{x^2}{-3}\right|$$
$$= \frac{x^2}{3}$$

We acquire convergence if,

$$\frac{x^2}{3} < 1 \qquad \Rightarrow \qquad x^2 < 3$$

In this case the radius of convergence is NOT 3, however the radius of convergence needs an exponent of 1 on the x. As a result,

$$\sqrt{x^2} < \sqrt{3}$$
$$|x| < \sqrt{3}$$

Here the absolute value bars looks like the radius of convergence and is $R = \sqrt{3}$. The inequality for divergence is the interval for convergence. Next we obtain the interval of convergence. From the inequality we have,

$$-\sqrt{3} < x < \sqrt{3}$$

Now verify the end points.

For $x = -\sqrt{3}$: The power series is,

$$\sum_{n=1}^{\infty}\frac{\left(-\sqrt{3}\right)^{2n}}{(-3)^n} = \sum_{n=1}^{\infty}\frac{\left(\left(-\sqrt{3}\right)^2\right)^n}{(-3)^n}$$
$$= \sum_{n=1}^{\infty}\frac{(3)^n}{(-1)^n(3)^n}$$
$$= \sum_{n=1}^{\infty}(-1)^n$$

This series is divergent since $\lim\limits_{n\to\infty}(-1)^n$ does not exist.

For $x = \sqrt{3}$: Here we square the x so this series will be similar to the preceding step which is divergent.

$$\sum_{n=1}^{\infty}\frac{\left(\sqrt{3}\right)^{2n}}{(-3)^n} = \sum_{n=1}^{\infty}(-1)^n$$

The interval of convergence is $-\sqrt{3} < x < \sqrt{3}$.

3.4.2 Operations on Power Series

In the power series, the acceptable operations are differentiation, integration, addition, subtraction, division and multiplication of power series. A condition regarding the desertion of every coefficient of a power series is listed, which is considered the basic tool for solving power series.

Differentiation

The differentiation of a power series can be done term by term. More specifically, if the series,

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

Converges for $|x - x_0| < R$, where $R > 0$, then the series acquired after differentiating term by term also converges for those x and it also characterizes the derivative y' and y for those x, so that,

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} \qquad (|x - x_0| < R).$$

Also,

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2} \qquad (|x - x_0| < R), \text{ etc.}$$

Addition

Two power series can be added term by term. More specifically, if the series,

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m \ \ and \quad \sum_{m=0}^{\infty} b_m (x - x_0)^m \tag{3.22}$$

Contain positive radii of convergence and the sum is $f(x)$ and $g(x)$, then the series:

$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

Converges and denotes $f(x) + g(x)$ for each x which is in the interior of the convergence interval of each of the given series.

Multiplication

The two given power series can also be multiplied term by term. Assume that the series in Equation (3.22) contains positive radii of convergence and $f(x)$ and $g(x)$ are their sums. Then we obtain the series by multiplying each term of the first series with each term of the second series and accumulating like powers of $x - x_0$, so that,

$$\sum (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x - x_0)^m$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \cdots$$

NOTES

Converges and denotes $f(x)\,g(x)$ for each x in the interior of the convergence interval of every given series.

Vanishing of Coefficients

In case there is a positive radius of convergence of a power series as well as an identically zero sum all through its interval of convergence, then every coefficient of the series will be zero.

Shifting Summation Indices

This can be explained using specific example. Consider the following given series,

$$x_2\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + 2\sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$= x^2\left(2a_2 + 6a_3x + 12a_4x^2 + \cdots\right) + 2\left(a_1 + 2a_2x + 3a_3x^2 + \cdots\right)$$

This series can be written as a single series. For this, we first use x^2 inside the summation, to obtain,

$$\sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=1}^{\infty} 2ma_m x^{m-1}.$$

Assume that we have used s as the summation letter of the series which is to be obtained. We just replace s with m in the first series. This change of notation is possible because a summation letter is only a dummy index and we can use any letter for this dummy index which has not been used before. In the second series shifting happens where you can shift by one unit as $m-1=s$. Then, $m=s+1$. Now the summation starts with $s=0$ because $m=0+1=1$, which is the previous start. Collectively, it becomes:

$$\sum_{s=2}^{\infty} s(s-1)a_s x^2 + \sum_{s=0}^{\infty} 2(s+1)a_{s+1}x^s.$$

In the first series we replace $s=2$ by $s=0$ to obtain,

$$\sum_{s=0}^{\infty}\left[s(s-1)a_s + 2(s+1)a_{s+1}\right]x^2$$

$$= 2a_1 + 4a_2x + \left(2a_2 + 5a_3\right)x^2 + \left(6a_3 + 8a_4\right)x^3 + \cdots$$

3.4.3 Existence of Power Series Solutions and Real Analytic Functions

We have already learned the various properties of power series. We know that an equation has power series solutions. Consider the series where we have the coefficients p and q, and the function r in on the right side of the equation,

$$y'' + p(x)y' + q(x)y = r(x) \tag{3.23}$$

The power series solution has the form as represented in Equation (3.23). This is true if $\tilde{h}$, $\tilde{p}, \tilde{q}$ and $\tilde{r}$ in the series,

$$\tilde{h}(x)y'' = \tilde{p}(x)y = \tilde{r}(x) \tag{3.24}$$

The power series can be represented as shown in Equation (3.24) where $\widetilde{h}(x_0) \neq 0$, here x_0 is the center of the series.

NOTES

Real Analytic Function

A real function $f(x)$ is termed as **analytic** at a point $x = x_0$ if it can be denoted by a power series in powers of $x - x_0$ by radius of convergence $R > 0$. In mathematics, an analytic function is a function that is locally given by a convergent power series. There exist both real analytic functions and complex analytic functions. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not hold generally for real analytic functions. A function is analytic if and only if it is equal to its Taylor series in some neighborhood of every point. This concept can be stated with the help of following basic theorem.

Theorem 3.1: Existence of Power Series Solutions

In Equation (3.23) if p, q and r are analytic at $x = x_0$, then every solution of series in Equation (3.23) is analytic at $x = x_0$ and can be denoted by a power series in powers of $x - x_0$ by radius of convergence $R > 0$. The same is true if $\widetilde{h}, \widetilde{p}, \widetilde{q}$ and $\widetilde{r}$ in Equation (3.24) are analytic at $x = x_0$ and $\widetilde{h}(x_0) \neq 0^3$.

Using this theorem you can prove the existence of power series.

3.5 HARMONIC OSCILLATOR

Bohr's theory based on the quantization of angular momentum and energy of the electron in hydrogen atom was successful in explaining broad features of hydrogen atom and of the spectral lines emitted by it. The concepts used in the theory were new but of fundamental importance and inspired further researches in atomic physics.

Bohr's theory was extended by Arnold Sommerfeld in the year 1915 by introducing elliptical orbits for the electrons in atoms.

In the same year Wilson and Sommerfeld postulated independently a more general statement of quantization rule for systems undergoing periodic motion.

If a periodic system of s degrees of freedom described by generalized coordinates $q_1 \ldots qs$ and generalized momenta $p_1, p_2 \ldots, p$s then phase integrals of the system are defined as

$$J_i = \oint p_i dq_i, \quad i = 1, 2, \ldots, s \qquad \ldots(3.25)$$

the integration being carried over one complete cycle of the variable qi. Wilson and Sommerfeld stated that the stationary states (allowed orbits) for the system are those for which the phase integrals are integral multiples of Planck's constant h, i.e.,

NOTES

$$J_i = \oint p_i dq_i = n_i\, h; \quad n_i = 0,\ 1,\ 2,\ \ldots \qquad \ldots(3.26)$$

In the case of motion of electron in circular orbits, the number of degrees of freedom is only one and the angular momentum $l = mvr$ is a constant of motion so that the quantization rule given by Equation (3.26) reduces t

$$\oint mvr\, d\phi = nh$$

or $mvr\, 2\pi = nh$

so that $$mvr = \frac{nh}{2\pi} \qquad \ldots(3.27)$$

We may note that Equation (3.27) is the quantization rule postulated by Bohr for the electron in hydrogen atom rotating in circular orbits.

The general quantization rule of Wilson and Sommerfeld was used in a number of problems of interest, particularly for finding out the energies that periodic systems could assume. In the following, we present a brief outline of some such systems in the microscopic domain.

The Harmonic Oscillator

Consider a harmonic oscillator of mass m oscillating along the x-axis about the equilibrium position $x = 0$. The displacement of the particle from the equilibrium position at any instant t is given by

$$x = a \sin \omega_0 t \qquad \ldots(3.28)$$

where a is amplitude and w0 is the natural frequency related to the force constant according to

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \ldots(3.29)$$

The potential energy of the oscillator is given by

$$V = \frac{1}{2} kx^2 = \frac{1}{2} m\omega_0^2 a^2 \sin^2(\omega_0 t) \qquad \ldots(3.30)$$

The kinetic energy of the oscillator is

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2} ma^2\omega_0^2 \cos^2(\omega_0 t) \qquad \ldots(3.31)$$

Thus, the total energy of the oscillator becomes

$$E = T + V = \frac{1}{2} m\, \omega_0^2 a^2 \qquad \ldots(3.32)$$

NOTES

3.5.1 Spherical Harmonics

In mathematics and physical science, spherical harmonics are special functions defined on the surface of a sphere. They are often employed in solving partial differential equations in many scientific fields.

Since the spherical harmonics form of a complete set of orthogonal functions and thus an orthonormal basis, each function defined on the surface of a sphere can be written as a sum of these spherical harmonics. This is similar to periodic functions defined on a circle that can be expressed as a sum of circular functions (sines and cosines) via Fourier series. Like the sines and cosines in Fourier series, the spherical harmonics may be organized by (spatial) angular frequency, as seen in the rows of functions in the illustration on the right. Further, spherical harmonics are basis functions for irreducible representations of SO(3), the group of rotations in three dimensions, and thus play a central role in the group theoretic discussion of SO(3).

Spherical harmonics originates from solving Laplace's equation in the spherical domains. Functions that are solutions to Laplace's equation are called harmonics. Despite their name, spherical harmonics take their simplest form in Cartesian coordinates, where they can be defined as homogeneous polynomials of degree l in (x,y,z) that obey Laplace's equation.

The connection with spherical coordinates arises immediately if one uses the homogeneity to extract a factor of radial dependence r^l from the above-mentioned polynomial of degree l the remaining factor can be regarded as a function of the spherical angular coordinates θ and φ only, or equivalently of the orientational unit vector r specified by these angles. In this setting, they may be viewed as the angular portion of a set of solutions to Laplace's equation in three dimensions, and this viewpoint is often taken as an alternative definition.

A specific set of spherical harmonics, denoted $Y_l^m(\theta, \varphi)$ or $Y_l^m(r)$, are known as Laplace's spherical harmonics, as they were first introduced by Pierre Simon de Laplace in 1782. Spherical harmonics are important in many theoretical and practical applications, including the representation of multipole electrostatic and electromagnetic fields, electron configurations, gravitational fields, geoids, the magnetic fields of planetary bodies and stars, and the cosmic microwave background radiation. In 3D computer graphics, spherical harmonics play a role in a wide variety of topics including indirect lighting (ambient occlusion, global illumination, precomputed radiance transfer, etc.) and modelling of 3D shapes.

Spherical harmonics were first investigated in connection with the Newtonian potential of Newton's law of universal gravitation in three dimensions. In 1782, Pierre-Simon de Laplace had, in his Mécanique Céleste, determined that the gravitational potential $\mathbb{R}^3 \rightarrow \mathbb{R}$ at a point x associated with a set of point masses m_i located at points x_i was given by following equation

$$V(\mathbf{x}) = \sum_i \frac{m_i}{|\mathbf{x}_i - \mathbf{x}|}$$

NOTES

Each term in the above summation is an individual Newtonian potential for a point mass. Just prior to that time, Adrien-Marie Legendre had investigated the expansion of the Newtonian potential in powers of $r = |x|$ and $r_1 = |x_1|$. He discovered that if $r \le r_1$ then

$$\frac{1}{|\mathbf{x}_1 - \mathbf{x}|} = P_0(\cos\gamma)\frac{1}{r_1} + P_1(\cos\gamma)\frac{r}{r_1^2} + P_2(\cos\gamma)\frac{r^2}{r_1^3} + \cdots$$

Where γ is the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{x}_1$. The functions $\boldsymbol{P}_i$**:** **[**-1, 1] $\to\mathbb{R}$ are the Legendre polynomials, and they can be derived as a special case of spherical harmonics. Subsequently, in his 1782 memoir, Laplace investigated these coefficients using spherical coordinates to represent the angle γ between $\boldsymbol{x}_1$ and x.

In 1867, William Thomson (Lord Kelvin) and Peter Guthrie Tait introduced the solid spherical harmonics in their Treatise on Natural Philosophy, and also first introduced the name of 'Spherical Harmonics' for these functions. The solid harmonics were homogeneous polynomial solutions $\mathbb{R}^3\to\mathbb{R}$ of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

By examining Laplace's equation in spherical coordinates, Thomson and Tait recovered Laplace's spherical harmonics. The term 'Laplace's Coefficients' was employed by William Whewell to describe the particular system of solutions introduced along these lines, whereas others reserved this designation for the Zonal spherical harmonics that had properly been introduced by Laplace and Legendre.

Whereas the trigonometric functions in a Fourier series represent the fundamental modes of vibration in a string, the spherical harmonics represent the fundamental modes of vibration of a sphere in much the same way. Many aspects of the theory of Fourier series could be generalized by taking expansions in spherical harmonics rather than trigonometric functions. Moreover, analogous to how trigonometric functions can equivalently be written as complex exponentials, spherical harmonics also possessed an equivalent form as complex-valued functions. This was a boon for problems possessing spherical symmetry, such as those of celestial mechanics originally studied by Laplace and Legendre.

The prevalence of spherical harmonics already in physics set the stage for their later importance in the 20th century birth of quantum mechanics. The (complex-valued) spherical harmonics $\boldsymbol{S}^2 \to \mathrm{C}$ are Eigen functions of the square of the orbital angular momentum operator

$$-i\hbar r \times \nabla$$

and therefore they represent the different quantized configurations of atomic orbitals.

Check Your Progress

7. What do you understand by power series method?
8. What is convergence of power series?
9. Give the basic tools for solving power series.
10. State the Bohr's theory based on quantization.
11. Define spherical harmonics.

3.6 FOURIER SERIES

The use of periodic functions is very frequent in engineering problems. But the periodic functions may not be simple all the time. So it is desirable to represent these into simple periodic functions. Trigonometric functions are the simplest periodic functions. A Fourier series decomposes the periodic functions into the sum of a set of simple oscillating functions, namely sines and cosines.

General Properties

Half Range Fourier Series:

1. If a function is defined over half the range, say 0 to L, instead of the full range from -L to L, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a half range Fourier series. An even function can be expanded using half its range from 0 to L or –L to 0 or L to 2L, i.e., the range of integration is L. the Fourier series of the half range even function is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}, \text{ for } n = 1, 2, \ldots..$$

2. An odd function can be expanded using half its range from 0 to L, i.e. the range of integration = L. The Fourier series of the odd function is:
Since $a_o = 0$ and $a_n = 0$, we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}, \text{ for } n = 1, 2, \ldots.$$

Harmonic Analysis:

We can rearrange the Fourier series

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \\ &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \ldots \\ &\quad + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots \end{aligned}$$

And write it as:

$$f(t)\frac{a_0}{2} + (a_1 \cos t + b_1 \sin t) + (a_2 \cos 2t + b_2 \sin 2t) + (a_3 \cos 3t + b_3 \sin 3t) + \ldots$$

The term ($a_1 \cos t + b_1 \sin t$) is known as the fundamental.

The term ($a_2 \cos 2t + b_2 \sin 2t$) is called the second harmonic.

The term ($a_3 \cos 3t + b_3 \sin 3t$) is called the third harmonic, etc.

NOTES

Mean Value Convergence Theorem

If a periodic function $f(x)$ with period $2L$ is piecewise continuous over the interval $[-L, L]$, the Fourier Series of converges to the mean value at point where both the left-hand and right-hand first derivatives of exist.

Fourier Coefficient

A function $f(x)$ defined in the interval $c_1<x<c_2$ can be expressed in the Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a_0, a_n, b_n, (n =1, 2,…) are constants, called the Fourier coefficients of $f(x)$.

3.7 LEGENDRE'S EQUATION

Legendre's Differential Equation

This equation is represented following form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n\,(n+1)y = 0. \qquad ...(3.33)$$

The Legendre's differential equation of such type can be solved in series of ascending or descending powers of x. Suppose, we have to integrate it in a series of descending powers of x. There is no singularity at $x = 0$, so the solution of the equation can be obtained in the form of a series developed about $x = 0$.

Let us assume the solution of the given equation in the form of series.

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}$$

$$\therefore \qquad \frac{dy}{dx} = \sum_{r=0}^{\infty} (k-r)\, a_r x^{k-r-1}$$

And $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (k-r)\,(k-r-1)\, a_r x^{k-r-2}\ .$$

Substituting these values in (3.33), we have

$$\sum_{r=0}^{\infty} [(1-x^2)\,(k-r)\,(k-r-1)x^{k-r-2} - 2x\,(k-r)\,x^{k-r-1} + n(n+1)x^{k-r}]a_r = 0$$

or $$\sum_{r=0}^{\infty} [(k-r)\ (k-r-1)x^{k-r-2} + \{n\,(n+1) - 2(k-r) - (k-r)\,(k-r-1)\}\,x^{k-r}]\,a_r = 0$$

$$\text{or } \sum_{r=0}^{\infty} [(k-r)\ (k-r-1)x^{k-r-2} + \{n(n+1)-(k-r)(k-r+1)\}x^{k-r}]a_r = 0. \quad ...(3.34)$$

The relation Equation (3.34) is an identity and therefore the coefficients of various powers of x can be equated to zero.

Let us first equate the coefficient of x^k the highest power of x {by putting $r = 0$ in (3.34)} to zero; then we get

$$a_0\ \{n\ (n + 1) - k(k + 1) = 0,$$

where a_0 being the coefficient of the first term of the series cannot be zero, *i.e.* $a_0 \neq 0$ and thus

$$n\ (n + 1) - k\ (k + 1) = 0$$

$$\text{or} \quad n^2 + n - k^2 - k = 0$$

$$\text{or} \quad n^2 - k^2 + (n - k) = 0$$

$$\text{or} \quad (n - k)\ (n - k + 1) = 0$$

which gives $\quad k = n$ or $- n - 1 \quad$ ...(3.35)

Again equating the coefficient of x^{k-1} to zero, by putting $r = 1$ in Equation (3.34), we have

$$\{n\ (n + 1) - (k - 1)\ k\}\ a_1 = 0 \quad ...(3.36)$$

From Equation (3.35), $\{n\ (n + 1) - k\ (k - 1)\} \neq 0$

and therefore $\quad a_1 = 0.$

Let us now equate the coefficient of x^{k-r}, the general term in Equation (3.34), to zero,

$$(k - r + 2)\ (k - r + 1)a_{r-2} + \{n\ (n + 1) - (k - r)\ (k - r + 1)\}\ a_r = 0$$

$$\text{Or} \quad a_r = -\frac{(k-r+2)\ (k-r+1)}{n\ (n+1)-(k-r)\ (k-r+1)} a_{r-2}. \quad ...(3.37)$$

Putting $k = n$, the recurrence formula is

$$a_r = -\frac{(n-r+2)\ (n-r+1)}{n^2+n-(n-r)\ (n-r+1)} a_{r-2}$$

$$= -\frac{(n-r+2)\ (n-r+1)}{n^2+n-n^2+nr-n+nr-r^2+r} a_{r-2}$$

$$= -\frac{(n-r+2)\ (n-r+1)}{r\ (2n-r+1)} a_{r-2}\ . \quad ...(3.38)$$

Again putting $k = - n - 1$ in Equation (3.37), we have the recurrence formula as

$$a_r = -\frac{(-n-r+1)\ (-n-r)}{n^2+n-(-n-r-1)\ (-n-r)} a_{r-2}$$

$$= -\frac{(n+r-1)\ (n+r)}{n^2+n-(n+r+1)\ (n+r)} a_{r-2}$$

$$= \frac{(n+r-1)\ (n+r)}{r\ (2n+r+1)} a_{r-2}. \quad ...(3.39)$$

NOTES

Case I. When $k = n$, we have by putting $r = 2, 3, ...$in Equation (3.38),

$$a_2 = -\frac{n(n-1)}{2(2n-1)}a_0$$

$$a_3 = -\frac{(n-1)(n-2)}{3(2n-2)}a_1$$

$$= 0 \text{ since } a_1 = 0.$$

Similarly a_5, a_7, a_9....etc. all the a's having odd suffixes are zero.

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}a_0$$ (by putting value of a_2 etc.)

In general, $a_{2r} = (-1)^r \dfrac{n(n-1)(n-2)(n-3)\cdots(n-2r+1)}{2\cdot 4\cdots 2r\,(2n-1)(2n-3)\cdots(2n-2r+1)}a_0$

(by putting value of a_2, etc.)

Hence the series solution when $k = n$, is

$$y = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}\cdots\right]$$ ...(3.40)

where a_0 is an arbitrary constant and is equal to

$$\frac{1\cdot 3\cdot 5...(2n-1)}{\lfloor n}$$ ...(3.41)

where n is a positive integer.

This solution of Legendre's equation is known as $P_n(x)$, *i.e.*

$$P_n(x) = \frac{1\cdot 3\cdot 5...(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$ ...(3.42)

Case II. When $k = -n-1$, we have by putting $r = 2, 3,...$ in Equation (3.39),

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)}a_0.$$

Now a_3 will contain a_1 and hence is zero. As such a_5, a_7, a_9... all the zero.

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)}a_2$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}a_0,$$ by putting value of a_2 etc.

NOTES

In general, $$a_{2r} = \frac{(n+1)...(n+2r)}{2\cdot 4...2r\,(2n+3)...(2n+2r+1)}a_0.$$

Hence the series solution is

$$y = a_0\left[x^{-n-1} + \frac{(n+1)\,(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2\cdot 4\,(2n+3)\,(2n+5)}x^{-n-5} + \cdots\right] \quad ...(3.43)$$

With $$a_0 = \frac{\lfloor n}{1\cdot 3\cdot 5...(2n+1)} \quad ...(3.44)$$

this solution is known as $Q_n(x)$.

$$\therefore \quad Q_n(x) = \frac{\lfloor n}{1\cdot 3\cdot 5...(2n+1)}\left[x^{-n-1} + \frac{(n+1)\,(n+2)}{2\,(2n+3)}x^{-n-3} + \frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2\cdot 4(2n+3)\,(2n+5)}x^{-n-5} + \cdots\right] \quad ...(3.45)$$

which does not terminate when $n > -1$.

The most general solution of the Legendre's equation is

$$y = AP_n\,(x) + BQ_n\,(x). \quad ...(3.46)$$

where A and B are arbitrary constants.

Thus the series Equations (3.40) or (3.43) will be convergent if $|x| > 1$ *i.e.* the above solutions for Legendre equation are not convergent in the interval $-1 < x < 1$. In order to find the convergent solution of (3.33) we seek for solution in descending powers of x.

Suppose a series solution of (3.33) is

$$y = \sum_{r=0}^{\infty} a_r x^{k+r},\; a_0 \neq 0 \quad ...(3.47)$$

So that $$y' = \sum_{r=0}^{\infty}(k+r)\,a_r x^{k+r-1}$$

$$y'' = \sum_{r=0}^{\infty}(k+r)\;(k+r-1)\;a_r x^{k+r-2}$$

Subsituting these values of y, y' and y'' in (3.33) we get the identity

$$\sum_{r=0}^{\infty}\left[(k+r)\;(k+r-1)\;x^{k+r-2} - (k+r-n)\,(k+r+n+1)x^{k+r}\right]a_r \equiv 0 \quad ...(3.48)$$

Equating to zero the coefficient of x^{k-2} (when $r = 0$) the first term in Equation (3.48) under the assumption $a_0 \neq 0$ yields $k(k-1) = 0$ giving $k = 0, 1$.

Now equating the coefficient of second term *i.e.* x^{k+1} to zero, we have $a_1\,(k+1)k = 0$ giving $a_1 \neq 0$ for $k = -1$ while a_1 may or may not be zero for $k = 0$.

Equating the coefficient of general term *i.e.* x^{k+r} to zero, we find the recursion formula

$$a_{r+2} = \frac{(k+r-n)\,(k+r+n+1)}{(k+r+2)\,(k+r+1)}\,a_r \qquad \text{...(3.49)}$$

NOTES

Putting $k = 0$, Equation (3.49) gives $a_{r+2} = \dfrac{(r-n)\,(r+n+1)}{(r+2)\,(r+1)}\,a_r$ ...(3.50)

And putting $k = 1$, Equation (3.49) gives $a_{r+2} = \dfrac{(1+r-n)\,(2+r+n)}{(3+r)\,(2+r)}\,a_r$ ...(3.51)

Case I. When $k = 0$, we have by putting $r = 0, 1, 2, 3, 4, 5, \ldots$ in Equation (3.50)

$$a_2 = -\frac{n(n+1)}{\underline{|2}}\,a_0 \text{ and } a_3 = -\frac{(n-1)\,(n+2)}{\underline{|3}}\,a_1$$

$$a_4 = \frac{n\,(n-2)\,(n+1)\,(n+3)}{\underline{|4}}\,a_0 \text{ and}$$

$$a_5 = \frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{\underline{|5}}\,a_1$$

and in general $a_{2r} = (-1)^r \dfrac{n(n-2)\ldots(n-2r+1)}{\underline{|2r}}\,a_0$

$$a_{2r+1} = \frac{(-1)^r\,(n-1)\,(n-3)\ldots(n-2r+1)\,(n+2)\ldots(n+2r)}{\underline{|2r+1}}\,a_1$$

Hence the series solution for $k = 0$, is

$$y = a_0\left[1 - \frac{n(n+1)}{\underline{|2}}x^2 + \frac{n\,(n-2)\,(n+1)\,(n+3)}{\underline{|4}}x^4 + \cdots\right]$$

$$+a_1 x\left[1 - \frac{(n-1)\,(n+2)}{\underline{|3}}x^2 + \frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{\underline{|5}}x^4 \cdots\right] \qquad \text{...(3.51}a\text{)}$$

Case II. When $k = 1$, we have from (3.51),

$$a_2 = -\frac{(n-1)\,(n+2)}{\underline{|3}}\,a_0 \text{ and } a_1 = a_3 = a_5 = a_{2\,r+1} = \ldots = 0$$

$$a_4 = \frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{\underline{|5}}\,a_0$$

$$a_{2\,r} = \frac{(-1)^r\,(n-1)\,(n-3)\ldots(n-2r+1)\,(n+2)\ldots(n+2r)}{\underline{|2r+1}}\,a_0$$

Hence the series solution for $k = 1$ is

$$y = a_0 x\left[1 - \frac{(n-1)\,(n+2)}{\underline{|3}}x^2 + \frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{\underline{|5}}x^4 + \cdots\right] \qquad \text{...(3.52)}$$

The solution of Equation (3.52) is included in (3.51a) in the coefficient of a_1 except that a_1 is to be replaced by a_0. Hence setting $a_1 = 0$ for $k = 0$ also, the solution (3.51a) reduces to

$$y = a_0\left[1 - \frac{n\,(n+1)}{\lfloor 2}x^2 + \frac{n(n-2)\,(n+1)\,(n+3)}{\lfloor 4}x^4 + \cdots\right] \quad ...(3.53)$$

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It may be shown by ratio test that the solutions of Equations (3.52) and (3.53) are convergent in the interval $-1 < x < 1$.

Calling the solution of Equations (3.53) as $S_n\,(x)$ and (3.52) as $T_n\,(x)$, the general solution of Legendre equation in ascending powers of x is

$$y = A\,S_n\,(x) + B\,T_n(x). \quad ...(3.54)$$

where A and B are arbitrary constants.

Legendre Polynomials

If we put $n = 2r$ (say) *i.e.* if n be taken as even positive integer then Equation (3.55) gives the Legendre polynomial as

$$y = a_0\left[1 - \frac{n\,(n+1)}{\lfloor 2}x^2 + \cdots + (-1)^{n/2}\,\frac{n(n-2)\cdots 4.2\,(n+1)\,(n+3)\cdots(2n-1)}{\lfloor n}x^n\right] \quad ...(3.56)$$

While Equation (3.40) gives

$$y = a_0 x^n\left[1 - \frac{n\,(n-1)}{2(2n-1)}x^{-2} + \cdots + (-1)^{n/2}\,\frac{\lfloor n}{n(n-2)...2(n+1)\,(n+3)...(2n-1)}x^{-n}\right] \quad ...(3.57)$$

Equations (3.56) and (3.57) will be identical if (3.57) is multiplied by

$$(-1)^{n/2}\,\frac{n(n-2)...4\cdot 2\,(n+1)\,(n+3)...(2n-1)}{\lfloor n}$$

and then the solutions of Equations (3.40), (3.53) will become identical.

Again if we take, n as an odd negative integer then Equation (3.53) is identical with Equation (3.43). Also if n is an odd positive integer then Equation (3.52) reduces to

$$y = a_0\left[x - \frac{(n-1)\,(n+2)}{\lfloor 3}x^3 + \cdots + (-1)^{(n-1)/2}\,\frac{(n-1)\,(n-3)....2(n+2)...(2n-1)}{\lfloor n}x^n\right] \quad ...(3.58)$$

and Equation (3.40) reduces to

$$y = a_0 x^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \cdots + (-1)^{(n-1)/2}\,\frac{\lfloor n}{2\cdot 4....(n-1)\,(n-3)...(2n-1)}x^{-n+1}\right] \quad ...(3.59)$$

which becomes identical when multiplied by the coefficient of last term in Equation (3.59).

Further if n is an even negative integer, Equations (3.52) and (3.43) become identical.

These discussions follow the conclusions:

(*i*) For integral values for n, the solutions Equations (3.40) and (3.43) have great utility of them.

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(*ii*) For positive integral n, Equation (3.40) is a polynomial but Equation (3.43) is an infinite series and the complete integral is a linear combination of them.

(*iii*) For negative integral n, Equation (3.40) is an infinite series and Equation (3.43) is a polynomial.

(*iv*) For positive integral n, in Equations (3.41) or (3.42) we have chosen

$$a_0 = \frac{1\cdot 3\cdot 5...(2n+1)}{\underline{|n}}$$

(*v*) For negative integral n, in Equations (3.43) or (3.44) we have chosen

$$a_0 = \frac{\underline{|n}}{1\cdot 3\cdot 5...(2n+1)}.$$

(*vi*) For positive integral n, the polynomial $P_n(x)$ has the expansion given by Equation (3.44) ending with term from x *i.e.*

$$P_n(x) = \sum_{r=0} (-1)^r \frac{1\cdot 3\cdot 5...(2n-1)}{\underline{|n}} \frac{n(n-1)...(n-2r+1)}{2\cdot 4...2r(2n-1)\,(2n-3)...(2n-2r+1)} x^{n-2r}$$

$$= \sum_{n=0}^{n/2} (-1)^r \frac{1\cdot 3\cdot 5...(2n-2r-1)}{2^r \underline{|r}\underline{|n-2r}} x^{n-2r} \qquad ...(3.60)$$

or more concisely

$$P_n(x) = \sum_{n=0}^{n/2} (-1)^r \frac{\underline{|2n-2r}}{2^n \underline{|r}\underline{|n-r}\underline{|n-2r}} x^{2n-r} \qquad ...(3.61)$$

3.8 DIFFERENTIAL EQUATIONS OF SECOND ORDER

The general form of a linear differential equation of nth order is,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

Where $P_1, P_2 ..., P_n$ and Q are functions of x alone or constants.

The linear differential equation with constant coefficients are of the form,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q \qquad (3.62)$$

Where $P_1, P_2, ..., P_n$ are constants and Q is a function of x.

The equation,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_y y = 0 \qquad (3.63)$$

This is then called the Reduced Equation (R.E.) of the Equation (3.62)

If $y = y_1(x), y = y_2(x), ..., y = y_n(x)$ are n-solutions of this reduced equation,

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then $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of the reduced equation where $c_1, c_2, ..., c_n$ are artbitrary constants.

The solution $y = y_1(x)$, $y = y_2(x)$, $y = y_3(x)$, ..., $y = y_n(x)$ are said to be linearly independent if the **Wronskian** of the functions is not zero where the Wronskian of the functions $y_1, y_2, ..., y_n$, denoted by $W(y_1, y_2, ..., y_n)$, is defined by,

$$W(y_1, y_2, ...y_n) = \begin{vmatrix} y_1 & y_2 & y_3 ... y_n \\ y_1' & y_2' & y_3' ... y_n' \\ y_1'' & y_2'' & y_3'' ... y_n'' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} ... y_n^{(n-1)} \end{vmatrix}$$

Since the general solution of a differential equation of nth order contains n arbitrary constants, $u = c_1y_1 + c_2y_2 + ... + c_ny_n$ is its complete solution.

Let v be any solution of the differential Equation (3.62), then,

$$\frac{d^n v}{dx^n} + P_1 \frac{d^{n-1} v}{dx^{n-1}} + P_2 \frac{d^{n-2} v}{dx^{n-2}} + ... + P_{n-1} \frac{dv}{dx} + P_n v = Q \tag{3.64}$$

Since u is a solution of Equation (3.63), we get,

$$\frac{d^n u}{dx^n} + P_1 \frac{d^{n-1} u}{dx^{n-1}} + P_2 \frac{d^{n-2} u}{dx^{n-2}} + ... + P_{n-1} \frac{du}{dx} + P_n u = 0 \tag{3.65}$$

Now adding Equations (3.64) and (3.65), we get,

$$\frac{d^n (u+v)}{dx^n} + P_1 \frac{d^{n-1}(u+v)}{dx^n} + P_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + ... + P_{n-1} \frac{d(u+v)}{dx} + P_n(u+v) = Q$$

This shows that $y = u + v$ is the complete solution of the Equation (3.62).

Introducing the operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$ etc. The Equation (3.62) can be written in the form,

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + ... + P_{n-1} Dy + y P_n = Q$$

Or $\quad (D^n + P_1 D^{n-1} + P_2 D^{n-2} + ... + P_{n-1} D + P_n) y = Q$

Or $\quad F(D) y = Q$ where $F(D) = D^n + P_1 D^{n-1} P_2 D^{n-2} + ... + P_{n-1} D + P_n$

From the above discussions it is clear that the general solution of $F(D)y = Q$ consists of two parts:

(*i*) The Complementary Function (C.F.) which is the complete primitive of the Reduced Equation (R.E.) and is of the form

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n \text{ containing } n \text{ arbitrary constants.}$$

(*ii*) The Particular Integral (P.I.) which is a solution of $F(D) y = Q$ containing no arbitrary constant.

Rules for Finding The Complementary Function

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Let us consider the 2nd order linear differential equation,

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \qquad (3.66)$$

Let $y = A\, e^{mx}$ be a trial solution of the Equation (3.64); then the auxiliary equation (A.E.) of Equation (3.66) is given by,

$$m^2 + P_1 m + P_2 = 0 \qquad (3.67)$$

The Equation (3.67) has two roots $m = m_1$, $m = m_2$. We discuss the following cases:

(*i*) When $m_1 \neq m_2$, then the complementary function will be,

$y = c_1 e^{m1x} + c_2\, e^{m2x}$ where c_1 and c_2 are arbitrary constants.

(*ii*) When $m_1 = m_2$, then the complementary function will be,

$y = (c_1 + c_2\, x)\, e^{m1x}$ where c_1 and c_2 are arbitrary constants.

(*iii*) When the auxiliary Equation (3.67) has complex roots of the form $\alpha + i\beta$ and $\alpha - i\beta$, then the complementary function will be,

$$y = e^{\alpha x}\, (c_1 \cos \beta\, x + c_2 \sin \beta\, x)$$

Let us consider the equation of order n,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \qquad (3.68)$$

Let $y = A\, e^{mx}$ be a trial solution of Equation (3.68), then the auxiliary equation is,

$$m^n + P_1\, m^{n-1} + P_2\, m^{n-2} + \ldots\ldots + P_{n-1}\, m + P_n = 0 \qquad (3.69)$$

Rule (1): If $m_1, m_2, m_3, \ldots, m_n$ be n distinct real roots of Equation (3.69), then the general solution will be,

$$y = c_1\, e^{m1x} + c_2 e^{\,m2x} + c_3 e^{m3x} + \ldots + c_n e^{mnx}$$

Where $c_1, c_2, c_3 \ldots\ldots c_n$ are arbitrary constants.

Rule (2): If the two roots m_1 and m_2 of the auxiliary equation are equal and each equal to m, the corresponding part of the general solution will be $(c_1 + c_2\, x)\, e^{mx}$ and if the three roots m_3, m_4, m_5 are equal to α the corresponding part of the solution is $(c_3 + c_4 x + c_5 x^2)\, e^{\alpha x}$ and others are distinct, the general solution will be,

$$y = (c_1 + c_2 x)\, e^{mx} + (c_3 + c_4 + c_5 x^2)\, e^{\alpha x} + c_6\, e^{m6x} + \ldots\ldots + c_n e^{mnx}$$

Rule (3): If a pair of imaginary roots $\alpha \pm i\beta$ occur twices, the corresponding part of the general solution will be,

$$e^{\alpha x}\, [(c_1 + c_2 x) \cos\ \beta x + (c_3 + c_4 x) \sin \beta x]$$

And the general solution will be,

$$y = e^{\alpha x}\, [(c_1 + c_2 x) \cos\ \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m5x} + \ldots\ldots + c_n e^{mnx}$$

Where $c_1, c_2 \ldots, c_n$ are arbitrary constants and $m_5, m_6, \ldots., m_n$ are distinct real roots of Equation (3.69).

Rule (4): If the two roots (real) be m and $-m$, the corresponding part of the general solution will be $c_1 e^{mx} + c_2 e^{-mx}$

$$= c_1 (\cosh mx + \sinh mx) + c_2 (\cosh mx - \sinh mx)$$
$$= c'_1 \cosh mx + c'_2 \sinh mx \text{ where } c'_1 = c_1 + c_2,\ c'_2 = c_1 - c_2$$

And general solution will be,

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$$y = c'_1 \cosh mx + c'_2 \sinh mx + c_3 e^{m_3 x} + c_4\, e^{m_4 x} + ...+ c_n e^{mnx}$$

Where $c'_1, c'_2, c_3, ...c_n$ are arbitrary constants and $m_3, m_4 ... m_n$ are distinct real roots of Equation (3.69).

Rules for Finding Particular Integrals

Any particular solution of $F(D)\, y = f(x)$ is known as its Particular Integral (P.I). The P.I. of $F(D)y = f(x)$ is symbolically written as,

$$\text{P.I.} = \frac{1}{F(D)} \{f(x)\} \text{ where } F(D) \text{ is the operator.}$$

The operator $\frac{1}{F(D)}$ is defined as that operator which, when operated on $f(x)$ gives a function $\phi(x)$, such that $F(D)\, \phi(x) = f(x)$

i.e., $\quad \frac{1}{F(D)} \{f(x)\} = \phi(x)$ (= P.I.)

$$\therefore \quad F(D) \left\{\frac{1}{F(D)} f(x)\right\} = f(x) \quad \left[\because \frac{1}{F(D)} f(x) = \phi(x)\right]$$

Obviously, $F(D)$ and $1/F(D)$ are inverse operators.

Case I: Let $F(D) = D$, then $\frac{1}{D} f(x) = \int f(x)\, dx$.

Proof: Let $y = \frac{1}{D}\{f(x)\}$, operating by D, we get $Dy = D \,.\, \frac{1}{D}\{f(x)\}$ or $Dy = f(x)$ or

$\frac{dy}{dx} = f(x)$ or $dy = f(x)\, dx$

Integrating both sides with respect to x, we get,

$y = \int f(x)\, dx$, since particular integrating does not contain any arbitrary constant.

Case II: Let $F(D) = D - m$ where m is a constant, then,

$$\frac{1}{D-m}\{f(x)\} = e^{mx} \int e^{-mx} f(x) dx.$$

Proof: Let $\frac{1}{D-m}\{f(x)\} = y$, then operating by $D - m$, we get,

$$(D - m) \,.\, \frac{1}{D-m}\{f(x)\} = (D - m)\, y$$

Or $\qquad f(x) = \frac{dy}{dx} - my$

Or $\frac{dy}{dx} - my = f(x)$ which is a first order linear differential equation and

$$\text{I.F.} = e^{\int -mdx} = e^{-mx}.$$

Then multiplying above equation by e^{-mx} and integrating with respect to x, we

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get,

$y\,e^{-mx} = \int f(x)e^{-mx}\,dx$, since particular integral does not contain any arbitrary constant,

Or $$y = e^{mx}\int f(x)e^{-mx}\,dx.$$

Note: If $\dfrac{1}{F(D)} = \dfrac{a_1}{D-m_1} + \dfrac{a_2}{D-m_2} + ... + \dfrac{a_n}{D-m_n}$ where a_i and m_i ($i = 1, 2, ..., n$) are constants, then

$$\frac{1}{F(D)}\{f(x)\} = a_1 e^{m^1 x}\int f(x)e^{-m_1 x}dx + a_2 e^{m_2 x}\int f(x)e^{-m_2 x}dx + \dots + a_n e^{m_n x}\int f(x)e^{-m_n x}dx$$

$$= \sum_{i=1}^{n} a_i e^{m_i x}\int f(x)e^{-m_i x}dx$$

We now discuss methods of finding particular integrals for certain specific types of right hand functions

Type 1: $f(D)\,y = e^{mx}$ where m is a constant.

Then $$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}\} = \frac{e^{mx}}{F(m)} \text{ if } F(m) \neq 0$$

If $F(m) = 0$, then we replace D by $D + m$ in $F(D)$,

$$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}\} = e^{mx}\,.\,\frac{1}{F(D+m)}\{1\}$$

Example 3.39: $(D^3 - 2D^2 - 5D + 6)\,y = (e^{2x} + 3)^2 + e^{3x}\cosh x$.

Solution: The reduced equation is,

$$(D^3 - 2D^2 - 5D + 6)\,y = 0 \qquad ...(3.70)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (3.70). Then the auxiliary equation is,

$$m^3 - 2\,m^2 - 5m + 6 = 0 \text{ or } m^3 - m^2 - m^2 + m - 6m + 6 = 0$$

Or $m^2(m-1) - m(m-1) - 6(m-1) = 0$

Or $(m-1)(m^2 - m - 6) = 0$ or $(m-1)(m^2 - 3m + 2m - 6) = 0$

Or $(m-1)(m-3)(m+2) = 0$ or $m = 1, 3, -2$

$\therefore$ The complementary function is,

$y = c_1 e^x + c_2 e^{3x} + c_3\,e^{-2x}$ where c_1, c_2, c_3 are arbitrary constants.

Again $(e^{2x} + 3)^2 + e^{3x}\cosh x = e^{4x} + 6\,e^{2x} + 9 + e^{3x}\left(\dfrac{e^x + e^{-x}}{2}\right)$.

$$= e^{4x} + 6\,e^{2x} + 9e^{0\,.\,x} + \frac{e^{4x}}{2} + \frac{e^{2x}}{2}$$

$$= \frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}$$

$\therefore$ The particular integral is,

$$y = \frac{1}{D^3 - 2D^2 - 5D + 6}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{1}{(D-1)(D-3)(D+2)}\left\{\frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}\right\}$$

$$= \frac{3}{2}\frac{1}{(D-1)(D-3)(D+2)}e^{4x} + \frac{13}{2}\frac{1}{(D-1)(D+2)(D-3)}\{e^{2x}\}$$

$$+ 9\frac{1}{(D-1)(D-3)(D+2)}e^{0.x}$$

$$= \frac{3}{2}\frac{e^{4x}}{(4-1)(4-3)(4+2)} + \frac{13}{2}\frac{e^{2x}}{(2-1))(2+2)(2-3)}$$

$$+ 9\frac{e^{0.x}}{(0-1)(0-3)(0+2)}$$

$$= \frac{3}{2}\frac{e^{4x}}{3.1.6} + \frac{13}{2}\frac{e^{2x}}{1.4.(-1)} + 9\frac{e^{0.x}}{(-1)(-3).2}$$

$$= \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + \frac{3}{2}.$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1e^x + c_2e^{3x} + c_3e^{-2x} + \frac{e^{4x}}{12} - \frac{13}{8}e^{2x} + \frac{3}{2}.$$

Notes: 1. When $F(m) = 0$ and $F'(m) \neq 0$, P.I. $= \dfrac{1}{F(D)}\{e^{mx}\} \quad = x\dfrac{1}{F'(D)}\{e^{mx}\}$

$$= \frac{xe^{mx}}{F'(m)}$$

2. When $F(m)= 0$ $F'(m) = 0$ and $F''(m) \neq 0$, then P.I. $= \dfrac{1}{F(D)}\{e^{mx}\}$

$$= x^2\frac{1}{F''(D)}\{e^{mx}\} = \frac{x^2e^{mx}}{F''(m)}$$

And so on.

Type 2: $f(x) = e^{mx}\,V$ where V is any function of x.

Here the particular integral (P.I.) of $F(D)\,y = f(x)$ is,

$$\text{P.I.} = \frac{1}{F(D)}\{e^{mx}V\} = e^{mx}\frac{1}{F(D+m)}\{V\}.$$

Example 3.40: Solve $(D^2 - 5D + 6)\,y = x^2\,e^{3x}$

Solution: The reduced equation is,

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$$(D^2 - 5D + 6)\, y = 0 \qquad (1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (1) and then auxiliary equation is

$$m^2 - 5m + 6 = 0 \text{ or } m^2 - 3m - 2m + 6 = 0$$

Or $\quad m\,(m-3) - 2\,(m-3) = 0$ or $(m-3)\,(m-2) = 0$

$\therefore \quad m = 2, 3$

$\therefore$ The complementary function is,

$$y = c_1\, e^{2x} + c_2\, e^{3x} \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

The particular integral is,

$$y = \frac{1}{D^2 - 5D + 6}\{x^2 e^{3x}\} = \frac{e^{3x}}{(D+3)^2 - 5(D+3) + 6}\{x^2\}$$

$$= e^{3x}\frac{1}{D^2 + 6D + 9 - 5D - 15 + 6}\{x^2\} = e^{3x}\,\frac{1}{D^2 + D}\{x^2\}$$

$$= e^{3x}\,\frac{1}{D(1+D)}\{x^2\} = e^{3x}\,\frac{1}{D}(1+D)^{-1}\{x^2\}$$

$$= \frac{e^{3x}}{D}(1 - D + D^2 - D^3 + D^4 - \ldots)\{x^2\}$$

$$= \frac{e^{3x}}{D}\{x^2 - 2x + 2\} = e^{3x}\left(\frac{x^3}{3} - x^2 + 2x\right)$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{2x} + c_2 e^{3x} + e^{3x}\left(\frac{x^3}{3} - x^2 + 2x\right).$$

Recall: (*i*) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

(*ii*) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

Type 3: (*a*) $F(D)\, y = \sin ax$ or $\cos ax$ where $F(D) = \phi(D^2)$.

Here $\quad \text{P.I.} = \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{1}{\phi(-a^2)}\sin ax$ (if $\phi(-a^2) \neq 0$)

Or $\quad \text{P.I.} = \dfrac{1}{F(D)}\{\cos ax\} = \dfrac{1}{\phi(-a^2)}\cos ax$ (if $\phi(-a^2) \neq 0$)

[Note D^2 has been replaced by $-a^2$ but D has not been replaced by $-a$.]

(*b*) $F(D)\, y = \sin ax$ or $\cos ax$ and $F(D) = \phi(D^2, D)$

Here $\quad \text{P.I.} = \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{1}{\phi(D^2, D)}\{\sin ax\} = \dfrac{1}{\phi(-a^2, D)}\{\sin ax\}$

if $\quad \phi(-a^2, D) \neq 0$

Or $\quad y = \dfrac{1}{F(D)}\{\cos ax\} = \dfrac{1}{\phi(D^2, D)}\{\cos ax\} = \dfrac{1}{\phi(-a^2, D)}\{\cos ax\}$

if $\quad \phi(-a^2, D) \neq 0$

(c) $F(D)\,y = \sin ax$ or $\cos ax$ and $F(D) = \dfrac{\psi(D)}{\phi(D^2)}$

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Here P.I. $= \dfrac{1}{F(D)}\{\sin ax\} = \dfrac{\psi(D)}{\phi(D^2)}\{\sin ax\} = \dfrac{\psi(D)}{\phi(-a^2)}\{\sin ax\}$ if $\phi(-a^2) \neq 0$

Or $\quad y = \dfrac{1}{F(D)}\{\cos ax\} = \dfrac{\psi(D)}{\phi(D^2)}\{\cos ax\}$

$$= \frac{\psi(D)}{\phi(-a^2)}\{\cos ax\} \text{ if } \phi(-a^2) \neq 0$$

(d) $F(D)\,y = \sin ax$ or $\cos ax$, $F(D) = \phi(D^2)$ but $\phi(-a^2) = 0$.

Here P.I. $= \dfrac{1}{F(D)}\{\sin ax \text{ or } \cos ax\} = x\dfrac{1}{F'(D)}\{\sin ax \text{ or } \cos ax\}$

Alternatively, $\sin ax$ and $\cos ax$ can be written in the form $\sin ax = \dfrac{e^{ixa} - e^{-ixa}}{2i}$

And $\cos ax = \dfrac{e^{aix} + e^{-aix}}{2}$, then find P.I. by the method of Type 1.

Example 3.41: Solve $(D^4 + 2D^2 + 1)\,y = \cos x$.

Solution: The reduced equation is $(D^4 + 2D^2 + 1)\,y = 0$

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equaiton is,

$$m^4 + 2m^2 + 1 = 0 \text{ or } [(m^2 + 1)]^2 = 0 \text{ or } m = \pm i, \pm i$$

$\therefore$ C.F. $= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x$ where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore \quad \text{P.I.} = \frac{1}{D^4 + 2D^2 + 1}\{\cos x\}$$

$$= x\frac{1}{4D^3 + 4D}\{\cos x\}$$

$[\because \phi(D^2) = D^4 + 2D^2 + 1$

$\phi(-1^2) = 1 - 2 + 1 = 0$, then $\dfrac{1}{F(D)}\{f(x)\} = x\dfrac{1}{F'(D)}\{f(x)\}]$

$$= \frac{x}{4}\frac{1}{D^3 + D}\{\cos x\} = \frac{x}{4}.\frac{x}{3D^2 + 1}\{\cos x\}$$

$$= \frac{x^2}{4}\frac{1}{3D^2 + 1}\{\cos x\} = \frac{x^2}{4}.\frac{\cos x}{-3 + 1} = -\frac{x^2}{8}\cos x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{x^2}{8}\cos x.$$

Example 3.42: Solve $(D^2 - 4)y = \sin 2x$.

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Solution: The reduced equation is,

$$(D^2 - 4)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then auxiliary equation is,

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

The complementary function is,

$$y = c_1 e^{2x} + c_2 e^{-2x} \text{ where } c_1, c_2 \text{ are arbitrary constants.}$$

The particular integral is,

$$y = \frac{1}{D^2 - 4}\{\sin 2x\} = \frac{1}{-2^2 - 4}\sin 2x \quad [\text{Replace } D^2 \text{ by} - 2^2]$$

$$= -\frac{1}{8}\sin 2x$$

The general solution is $y = \text{C.F.} + \text{P.I.} = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8}\sin 2x$.

Example 3.43: Solve $(3D^2 + 2D - 8)y = 5 \cos x$.

Solution: The reduced equation is,

$$(3D^2 + 2D - 8)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then the auxiliary equation is,

$$3m^2 + 2m - 8 = 0 \text{ or } 3m^2 + 6m - 4m - 8 = 0$$

Or $\quad 3m\,(m + 2) - 4\,(m + 2) = 0$ or $(m + 2)\,(3m - 4) = 0$

Or $\quad m = -2,\ m = \frac{4}{3}$

$\therefore$ The complementary function is,

$$y = c_1e^{-2x} + c_2e^{\frac{4}{3}x} \text{ when } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

The particular integral is,

$$y = \frac{1}{3D^2 + 2D - 8}\{5\cos x\} = 5\frac{1}{(3D-4)(D+2)}\{\cos x\}$$

$$= 5\frac{(3D+4)(D-2)}{(9D^2-16)(D^2-4)}\{\cos x\} = 5\frac{(3D+4)(D-2)}{[9(-1^2)-16][-1^2-4]}\{\cos x\}$$

$$[D^2 \text{ is replaced by} - 1^2 \text{ in the denominator}] \left[\frac{\psi(D)}{\phi(D^2)}\text{ form}\right]$$

$$= \frac{5}{(-25)\,(-5)}[3D^2 - 6D + 4D - 8]\{\cos x\} = \frac{1}{25}[3D^2 - 2D - 8]\cos x$$

$$= \frac{1}{25}\left(3\frac{d^2}{dx^2}(\cos x) - 2\frac{d}{dx}(\cos x) - 8\cos x\right)$$

$$= \frac{1}{25}[-3\cos + 2\sin x - 8\cos x] = \frac{1}{25}(2\sin x - 11\cos x)$$

NOTES

The general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{-2x} + c_2 e^{4/3x} + \frac{1}{25}(2\sin x - 11\cos x).$$

Type 4: $F(D)\,y = x^n$, n is a positive integer.

Here $\quad \text{P.I.} = \frac{1}{F(D)}\{x^n\} = [F(D)]^{-1}\{x^n\}$

In this case, $[F(D)]^{-1}$ is expanded in a binomial series in ascending powers of D upto D^n and then operate on x^n with each term of the expansion. The terms in the expansion beyond D^n need not be considered, since the result of their operation on x^n will be zero.

Example 3.44: Solve $D^2(D^2 + D + 1)y = x^2$.

Solution: The reduced equation is,

$$D^2(D^2 + D + 1)y = 0 \qquad (1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (1) and then the auxiliary equation is,

$$m^2(m^2 + m + 1) = 0$$

$$\therefore\ m = 0, 0 \text{ and } m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-I \pm \sqrt{3}i}{2}$$

$\therefore$ The complementary function is,

$$y = (c_1 + c_2 x)\, e^{0\,.\,x} + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

$$= c_1 + c_2 x + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

Where c_1, c_2, c_3, c_4 are the arbitrary constant.

The particular integral is,

$$y = \frac{1}{D^2(D^2 + D + 1)}\{x^2\} = \frac{1}{D^2}(1 + D + D^2)^{-1}\{x^2\}$$

$$= \frac{1}{D^2}\{1 - (D + D^2) + (D + D^2)^2 - (D + D^2)^3 + ...\}\{x^2\}$$

$$= \frac{1}{D^2}\{1 - (D + D^2) + (D^2 + 2D^3 + D^4) - (D + D^2)^3 + ...\}\{x^2\}$$

$$= \frac{1}{D^2}\{x^2 - (2x + 2) + (2) + 0\}$$

$$= \frac{1}{D^2}\{x^2 - 2x\} = \frac{1}{D}\left\{\frac{x^3}{3} - x^2\right\} = \frac{x^4}{12} - \frac{x^3}{3}$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 + c_2 x + e^{-x/2}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right) + \frac{x^4}{12} - \frac{x^3}{3}.$$

NOTES

Example 3.45: Solve $(D^2 + 4)y = x \sin^2 x$.

Solution: The reduced equation is,

$$(D^2 + 4)\, y = 0$$

The trial solution $y = A\, e^{mx}$ gives the auxiliary equation as,

$$m^2 + 4 = 0,\; m = \pm\, 2i$$

The complementary function is $y = c_1 \cos 2x + c_2 \sin 2x$

The particular integral is $y = \dfrac{1}{D^2+4}\{x \sin^2 x\}$

$$= \frac{1}{D^2+4}\left\{\frac{x}{2}(1-\cos 2x)\right\} = \frac{1}{D^2+4}\left\{\frac{x}{2} - \frac{x}{2}\cos 2x\right\}$$

$$= \frac{1}{D^2+4}\left\{\frac{x}{2}\right\} - \frac{1}{D^2+4}\left\{\frac{x}{2}\,\frac{(e^{2ix}+e^{-2ix})}{2}\right\}$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}\left\{\frac{x}{2}\right\} - \frac{1}{4}\,\frac{e^{2ix}}{(D+2i)^2+4}\{x\} - \frac{e^{-2ix}}{4(D-2i)^2+4}\{x\}$$

$$= \frac{1}{4}\,\frac{x}{2} - \frac{e^{2ix}}{4}\,\frac{1}{D^2+4Di-4+4}\{x\} - \frac{e^{-2ix}}{4}\,\frac{1}{D^2-4Di-4+4}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}\,\frac{1}{4Di\left(1+\frac{D}{4i}\right)}\{x\} - \frac{e^{-2ix}}{4\,.\,(-4Di)\left(1-\frac{D}{4i}\right)}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}\,.\,\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}\{x\} - \frac{e^{-2xi}}{4(-4Di)}\left(1-\frac{D}{4i}\right)^{-1}\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}\,.\,\frac{1}{4Di}\left(1-\frac{D}{4i}+\frac{D^2}{-16}\ldots\right)\{x\} - \frac{e^{-2xi}}{4(-4Di)}\left(1+\frac{D}{4i}+\ldots\right)\{x\}$$

$$= \frac{x}{8} - \frac{e^{2ix}}{4}\,.\,\frac{1}{4Di}\left(x-\frac{1}{4i}\right) + \frac{e^{-2xi}}{4\,.\,4Di}\left(x+\frac{1}{4i}\right)$$

$$= \frac{x}{8} - \frac{e^{2ix}}{2\,.\,8i}\left(\frac{x^2}{2}-\frac{x}{4i}\right) + \frac{e^{-2xi}}{2\,.\,8i}\left(\frac{x^2}{2}+\frac{x}{4i}\right)$$

$$= \frac{x}{8} - \frac{x^2}{2\,.\,8}\left(\frac{e^{2ix}-e^{-2xi}}{2i}\right) + \frac{x}{2\,.\,16\,.\,i^2}\left(\frac{e^{2ix}+e^{-2xi}}{2}\right)$$

$$= \frac{x}{8} - \frac{x^2}{2.8}\sin 2x - \frac{x}{2.16}\cos 2x$$

$$= \frac{x}{8} - \frac{x^2}{16}\sin 2x - \frac{x}{32}\cos 2x$$

Hence the general solution is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8} - \frac{x^2}{16} \sin 2x - \frac{x}{32}\cos 2x.$$

Example 3.46: Solve $(D^4 + D^3 - 3D^2 - 5D - 2)\, y = 3xe^{-x}$.

Solution: The reduced equation is,

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$$(D^4 + D^3 - 3D^2 - 5D - 2)\, y = 0 \qquad (1)$$

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as,

$m^4 + m^3 - 3m^2 - 5m - 2 = 0$

Or $m^4 + m^3 - 3m^2 - 3m - 2m - 2 = 0$

Or $m^3 (m + 1) - 3m (m + 1) - 2 (m + 1)$

Or $(m + 1)(m^3 - 3m - 2) = 0$ or $(m + 1)\{m^3 + m^2 - m^2 - m - 2m - 2) = 0$

Or $(m + 1)\{m^2 (m + 1) - m (m + 1) - 2 (m + 1)\} = 0$

Or $(m + 1)(m + 1)(m^2 - m - 2) = 0$

Or $(m + 1)^2 (m^2 - 2m + m - 2) = 0$

Or $(m + 1)^2 (m + 1)(m - 2) = 0$

$\therefore\ m = -1, -1, -1, 2$

The complementary function is $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + c_4 e^{2x}$.

The particular integral is,

$$y = \frac{1}{(D+1)^3 (D-2)}\{3e^{-x} x\}$$

$$= 3e^{-x} \frac{1}{(D-1+1)^3 (D-3)}\{x\} = 3e^{-x} \frac{1}{D^3 (-3)(1-D/3)}\{x\}$$

$$= -e^{-x} \frac{1}{D^3}\left(1 - \frac{D}{3}\right)^{-1}\{x\} = -e^{-x} \frac{1}{D^3}\left(1 + \frac{D}{3} + \frac{D^2}{9} + \ldots\right)\{x\}$$

$$= -e^{-x} \frac{1}{D^3}\left(x + \frac{1}{3}\right) = -e^{-x} \frac{1}{D^2}\left(\frac{x^2}{2} + \frac{x}{3}\right) = -e^{-x} \frac{1}{D}\left(\frac{x^3}{6} + \frac{x^2}{6}\right)$$

$$= -e^{-x}\left(\frac{x^4}{24} + \frac{x^3}{18}\right)$$

The general solution is $y = \text{C.F.} + \text{P.I.}$

$$= (c_1 + c_2 x + c_3 x^2) + c_4 e^{2x} - e^{-x}\left(\frac{x^4}{24} + \frac{x^3}{18}\right).$$

Type 5: (*a*) $F(D)\, y = xV$ where V is a function of x.

Here $\quad \text{P.I.} = \dfrac{1}{F(D)}\{xV\} = \left\{x - \dfrac{1}{F(D)} F'(D)\right\}\dfrac{1}{F(D)}\{V\}.$

Example 3.47: Solve $(D^2 + 9)\, y = x \sin x$.

Solution: The reduced equation is $(D^2 + 9)\, y = 0 \qquad (1)$

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as,

$$m^2 + 9 = 0 \text{ or } m = \pm 3i$$

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$\therefore$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$ where c_1 and c_2 are arbitrary constants.

And P.I. $= \dfrac{1}{F(D)}\{x \sin x\}$ where $F(D) = D^2 + 9$

$$= \left\{x - \frac{1}{F(D)}F'(D)\right\}\frac{1}{F(D)}\{\sin x\}$$

$$= \left\{x - \frac{2D}{D^2+9}\right\}\frac{1}{D^2+9}\{\sin x\}$$

$$= \left\{x - \frac{2D}{D^2+9}\right\}\frac{\sin x}{-1+9} = \left\{x - \frac{2D}{D^2+9}\right\}\left\{\frac{\sin x}{8}\right\}$$

$$= \frac{x\sin x}{8} - \frac{1}{4}\frac{1}{-1+9}D\{\sin x\} = \frac{x \sin x}{8} - \frac{1}{32}\cos x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + \frac{x\sin x}{8} - \frac{1}{32}\cos x$$

(b) $F(D)\,y = x^n V$ where V is any function of x.

Here P.I. $= \dfrac{1}{F(D)}\{f(x)\} = \dfrac{1}{F(D)}\{x^n V\} = \left\{x - \dfrac{F'(D)}{F(D)}\right\}^n \dfrac{1}{F(D)}\{V\}$

Example 3.48: Solve $(D^2 - 1)y = x^2 \sin x$

Solution: The reduced equation is $(D^2 - 1)y = 0$ (1)

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equation is,

$$m^2 - 1 = 0 \text{ or } m = \pm 1$$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$ where c_1 and c_2 are arbitrary constants.

$\therefore$ P.I. $= \dfrac{1}{F(D)}\{x^2 \sin x\}$ where $F(D) = D^2 - 1$

$$= \left\{x - \frac{F'(D)}{F(D)}\right\}^2 \frac{1}{F(D)}\{\sin x\} = \left\{x - \frac{1}{D^2-1}2D\right\}^2 \frac{1}{D^2-1}\{\sin x\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{x - \frac{1}{D^2-1}2D\right\}\left\{\frac{1}{-1^2-1}\sin x\right\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{x - \frac{1}{D^2-1}2D\right\}\{-1/2 \sin x\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{-\frac{x}{2}\sin x + \frac{1}{D^2-1}\right\}\{\cos x\}$$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{-\frac{x}{2}\sin x - \frac{1}{2}\cos x\right\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2-1}\{D(x\sin x + \cos x)\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2-1}\{\sin x + x\cos x - \sin x\}$$

$$= -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x + \frac{1}{D^2-1}\{x\cos x\}$$

Again $\frac{1}{D^2-1}\{x\cos x\} = \left\{x - \frac{1}{D^2-1}2D\right\}\frac{1}{D^2-1}\{\cos x\}$

$$= \left\{x - \frac{1}{D^2-1}2D\right\}\left\{\frac{1}{-1-1}\cos x\right\}$$

$$= -\frac{1}{2}x\cos x + \frac{1}{D^2-1}\{-\sin x\}$$

$$= -\frac{1}{2}x\cos x - \frac{\sin x}{-1^2-1} = -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$$

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$$\therefore \text{P.I.} = -\frac{x^2}{2}\sin x - \frac{x}{2}\cos x - \frac{x}{2}\cos x + \frac{1}{2}\sin x$$

$$= -\frac{1}{2}x^2\sin x - x\cos x + \frac{1}{2}\sin x$$

Hence the general solution is,

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{-x} - \frac{1}{2}x^2\sin x - x\cos x + \frac{1}{2}\sin x.$$

Check Your Progress

12. What is half range Fouier series?
13. Give the Legendre's differential equation.
14. Write the general linear differential equation with constant coefficients.

3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Equations in which an unknown function, and its derivatives or differentials occur are called differential equations.
2. Solutions of equations which do not contain any arbitrary constants and which are not derivable from the general solution by giving particular values to one or more of the arbitrary constants, are called singular solutions.
3. Chemical kinetics, also known as reaction kinetics, is the branch of physical chemistry, i.e., concerned with understanding the rates of chemical reactions. It is to be contrasted with thermodynamics, which deals with the direction in which a process occurs but in itself tells nothing about its rate.
4. It is the express the rate of reaction as a function of a change in the concentration of one or more reactants over a particular period of time; they are used to describe what is happening at the molecular state level during a reaction. These rate laws help us determine the overall mechanism of reaction (or process) by which the reactants turn into products.
5. In nuclear physics, secular equilibrium is a situation in which the quantity of a radioactive isotope remains constant because its production rate (for example due to decay of a parent isotope) is equal to its decay rate.

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6. In operating on the sum of two functions, if an operator does not give the same result as the sum of operations on the two functions separately. Then operator is said to be non-linear.
7. The power series method is used to search a power series solution to certain differential equations. Basically, such a solution assumes a power series with unknown coefficients and then substitutes that solution into the differential equation for finding a recurrence relation for the coefficients. The power series method can also be applied to certain nonlinear differential equations with less flexibility.
8. The convergence of the series may depend upon the value of x that we put into the series. A power series may converge for some values of x and not for other values of x.
9. In the power series, the acceptable operations are differentiation, integration, addition, subtraction, division and multiplication of power series. A condition regarding the desertion of every coefficient of a power series is listed, which is considered the basic tool for solving power series.
10. Bohr's theory based on the quantization of angular momentum and energy of the electron in hydrogen atom was successful in explaining broad features of hydrogen atom and of the spectral lines emitted by it. The concepts used in the theory were new but of fundamental importance and inspired further researches in atomic physics.
11. Spherical harmonics form of a complete set of orthogonal functions and thus an orthonormal basis, each function defined on the surface of a sphere can be written as a sum of these spherical harmonics. This is similar to periodic functions defined on a circle that can be expressed as a sum of circular functions (sines and cosines) via Fourier series.
12. If a function is defined over half the range, say 0 to L, instead of the full range from -L to L, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a half range Fourier series.
13. Legendre's Differential Equation

    This equation is represented the following form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\,(n+1)y\ = 0.$$

14. The linear differential equation with constant coefficients are of the form,

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+.....+P_{n-1}\frac{dy}{dx}+P_n y=Q$$

    Where $P_1, P_2, \ldots, P_n$ are constants and Q is a function of x.

3.10 SUMMARY

NOTES

- In mathematics, a differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.
- A relation between the dependent and independent variables, which, when substituted in the equation, satisfies it, is known as a solution or a primitive of the equation. Note that, in the solution, the derivatives of the dependent variable should not be present.
- The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the general solution or the complete integral. By giving particular values to the arbitrary constants appearing in the general solution we obtain particular solutions of the equation.
- A differential equation is said to be exact if it can be derived directly from its primitive without any further operation of elimination or reduction.
- An operator is a rule for transforming a given mathematical function into Another function. Operators will be indicated by their symbols with caret $(\wedge)$ over it.
- Total angular momentum is obtained by the cross or vector product of the position and the linear momentum vectors. Angular momentum of a particle moving around a fixed point is given by the vector product or cross product of r and p.
- The operator correspondin to the physically observable quantity, energy is known as Hamiltonian Operator.
- If a homogeneous linear different equation has constant coefficients, then it can be solved using algebraic methods and its solutions are elementary functions known from calculus e^x, $\cos x$, etc.
- The standard basic technique used for the purpose of answering linear differential equations containing variable coefficients is known as the power series method.
- The power series method is used for solving differential equations because this method is considered easy and universally used as standard. We first describe the procedure and then exemplify it using simple equations.
- A real function $f(x)$ is termed as analytic at a point $x = x_0$ if it can be denoted by a power series in powers of $x - x_0$ by radius of convergence $R > 0$. In mathematics, an analytic function is a function that is locally given by a convergent power series.
- Bohr's theory based on the quantization of angular momentum and energy of the electron in hydrogen atom was successful in explaining broad features of hydrogen atom and of the spectral lines emitted by it. The concepts used in the theory were new but of fundamental importance and inspired further researches in atomic physics.
- The general quantization rule of Wilson and Sommerfeld was used in a

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number of problems of interest, particularly for finding out the energies that periodic systems could assume.

- The use of periodic functions is very frequent in engineering problems. But the periodic functions may not be simple all the time. So it is desirable to represent these into simple periodic functions. Trigonometric functions are the simplest periodic functions.
- A Fourier series decomposes the periodic functions into the sum of a set of simple oscillating functions, namely sines and cosines.
- The Fourier Series of converges to the mean value at point where both the left-hand and right-hand first derivatives of exist.
- The Particular Integral (P.I.) which is a solution of $F(D)\,y = Q$ containing no arbitrary constant.

3.11 KEY TERMS

- **Differential equation:** In mathematics, a differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.
- **Integrated rate laws:** Integrated rate laws express the reaction rate as a function of the initial concentration and a measured (actual) concentration of one or more reactants after a specific amount of time (t) has passed; they are used to determine the rate constant and the reaction order from experimental data.
- **Laplacian operator:** The square of the (∇) Del operator ∇^2 is also called Laplacian operator. It is obtained as a dot product of ∇.
- **Power series method:** The standard basic technique used for the purpose of answering linear differential equations containing variable coefficients is known as the power series method.
- **Spherical harmonics:** Spherical harmonics form of a complete set of orthogonal functions and thus an orthonormal basis, each function defined on the surface of a sphere can be written as a sum of these spherical harmonics.

3.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is differential equations?
2. Write a short note on non-homogeneous equations with constant coefficients.
3. State the integrated rate laws in chemical kinetics.
4. Define the secular equilibrium.

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5. What is quantum chemistry?
6. Define the Laplacian operator.
7. Give the uses of power series method.
8. What is real analytic function?
9. State the Bohr's theory based on quantization.
10. What is harmonic oscillator?
11. Define spherical harmonic.
12. Give the important properties of Fourier series.
13. Define the Fourier series.
14. What is Legendre's differential equation?
15. Give the general form of linear differential equation.

Long-Answer Questions

1. Briefly explain about the variable separable and exact first order differential equation giving appropriate examples.
2. Discuss in detail about the applications of chemical kinetics with the help of examples.
3. Analyse the secular equilibrium.
4. Describe the quantum chemistry and Hermitian operator giving its mathematical relation.
5. What is power series method? Explain their convergence and operations with relevant examples.
6. Discuss the concept of harmonic oscillator giving appropriate examples.
7. Explain in detail about the spherical harmonics.
8. Elaborate on the Fourier formula and Fourier coefficient.
9. What is Legendre's differential equation? Find a convergent solution of Legendre's equation in descending powers of x giving examples.
10. Analyse the differential equation with various types of examples.

3.13 FURTHER READING

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UNIT 4 PERMUTATION AND PROBABILITY

NOTES

Structure

4.0 INTRODUCTION

Permutation and combination are the ways to represent a group of objects by selecting them in a set and forming subsets. When we select the data or objects from a certain group, it is said to be permutations, whereas the order in which they are represented is called combination.

Probability is the branch of mathematics concerning numerical descriptions of how likely an event is to occur, or how likely it is that a proposition is true. Probability theory is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms.

This probability curve is bell shaped, has a peak at mean μ and spread across from entire real line, it use for the probability distributions come in many shapes with different characteristics, as defined by the mean, standard deviation, skewness, and kurtosis. The mean of a probability distribution is the long-run arithmetic average value of a random variable having that distribution. In mathematics and its applications, the Root Mean Square (RMS) is defined as the square root of the mean square. The RMS is also known as the quadratic mean.

NOTES

In statistics, probable error defines the half-range of an interval about a central point for the distribution, such that half of the values from the distribution will lie within the interval and half outside.

The kinetic theory of gases is a simple, historically significant classical model of the thermodynamic behaviour of gases, with which many principal concepts of thermodynamics were established. The model describes a gas as a large number of identical sub microscopic particles (atoms or molecules), all of which are in constant, rapid, random motion.

Curve fitting is the process of constructing a curve, or mathematical function that has the best fit to a series of data points, possibly subject to constraints.

In this unit, you will study about the permutation and combination, probability and probability theorem, probability curve, average, root mean square, most probable error, kinetic theory of gases, curve fitting.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss about the permutation and combination
- Elaborate on the probability and probability theorem
- Know about the probability curve
- Understand the various measure of average
- Analyse the standard error
- Explain the kinetic theory of gases
- Define the curve fitting

4.2 PERMUTATION AND COMBINATION

If there are n objects and they can be placed in any arrangement or order, then any given order of these n objects is called a *permutation of the n objects.*

For instance, assume that there are 4 persons A, B, C and D who can sit on any of the 4 chairs for a group photograph. Since the first person can sit on any one of these chairs, there are 4 ways that person A can be seated. Now, there are 3 chairs left and person B can be seated in 3 ways. Similarly, C can be seated in 2 ways and D must take the last seat left. Hence, by fundamental counting , there are $4 \times 3 \times 2 \times 1 = 24$ possible arrangements of these 4 people occupying 4 seats in any given order. Therefore, the number of permutations of 4 things taken four at a time is 24. This sequential multiplication of $4 \times 3 \times 2 \times 1$ is also known as 4 factorial, symbolized as 4!

In general, the number of different permutations of n distinct items, taken all at a time, is given by,

$$n! = n(n-1)(n-2)..... 1$$

For instance, if there are 6 horses running in a race, so that $n = 6$, then there are 6! number of orders in which these horses can finish as first, second, third, fourth, fifth and sixth. In other words, the total number of such finishing orders is

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Permutation of *x* out of *n* distinct items

Now, let us consider a situation in which we are not interested in taking all n items in a given order, but only some items x in a given order out of the total of n items so that $x \leq n.$ For instance, in the horse race of 6 horses, we may be interested only in the order of first, second and third finish, for which prizes can be awarded. Since there are 6 horses, any one of these horses could finish first. Hence, there are 6 ways to finish in the first place, 5 ways to finish in the second place, since the first place has already been filled by one horse, and there are 4 ways to finish in the third place, and hence, the total number of distinct orders of finish for the first 3 places is,

$$6 \times 5 \times 4 = 120$$

In general, the number of different permutations of x out of n distinct items is given by,

$$n(n-1)(n-2)....(n-x+1) \text{ OR } \frac{n!}{(n-x)!}$$

Combination of *x* items out of *n* distinct items

So far, we have taken either x items out of n distinct items or all the n items in a given order. The order of items has been necessary in the permutation formula. In many cases, however, the order is unimportant. For instance, the probability of any two heads out of three tosses would be different than having two heads and a tail in that order.

Accordingly, the number of combinations of n distinct items taken x at a time without any given order is given by:

$$\frac{n!}{x!(n-x)!}, \text{ where } x \leq n$$

The notation $\frac{n!}{x!(n-x)!}$ is also simply written as ${}^nC_x$ or $\binom{n}{x}$.

(For $x = 0$ or $x = n$, we define $0! = 1$)

Example 4.1: A committee of ten Members of Parliament (MPs) has been selected to investigate the ethical conduct of the ministers. A sub-committee of four MPs is to be selected out of the ten MPs to investigate one Minister. Determine the number of ways in which any four members can be selected out of these ten.

NOTES

Solution: Since the order of such selection is unimportant, the number of ways of choosing the sub-committee is given by:

$$^{n}C_x = {}^{10}C_4 = \binom{10}{4} = \frac{10!}{4!(10-4)!}$$

$$= \frac{10!}{4!6!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1\times 6\times 5\times 4\times 3\times 2\times 1} = \frac{5040}{24} = 210$$

Problems Based on Permutation and Combination

Example 4.2: In the throw of a die, find the probability of obtaining a number greater than 2.

Solution: The numbers on the die are 1, 2, 3, 4, 5, 6 i.e., **six** in number. Out of these, the numbers 3, 4, 5, 6 are greater than 2 and these are **four** in number and the numbers 1, 2 are not greater than 2 and these are **two** in number.

$$\therefore \text{ The required probability } \left(\frac{a}{a+b}\right) = \left(\frac{4}{4+2}\right) = \frac{4}{6} = \frac{2}{3}$$

Example 4.3: A bag contains 5 red, 5 white and 6 green balls. Three balls are drawn at random. What is the probability that a red, a white and a green balls are drawn?

Solution: Three balls can be drawn from (5 + 4 + 6) = 15 balls in $15C_3$ ways ($n(S) = 15C_3$). Similarly, the no. of ways in which one red ball can be drawn out of 5 red balls, one white ball out of 4 white balls and one green ball out of 6 green balls $= 5C_1 \times 4C_1 \times 6C_1 = n(E)$

$$\therefore \text{ The required probability} = P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{5C_1 \times 4C_1 \times 6C_1}{(5+4+6)C_3}$$

$$= \frac{5\times 4\times 6}{15C_3}$$

$$= \left(\frac{5\times 4\times 6}{\frac{15\times 14\times 13}{1\times 2\times 3}}\right)$$

$$= \frac{24}{91}$$

4.3 PROBABILITY AND THEOREM OF PROBABILITY

It seems that the word 'Statistics' was derived from the Latin word '*Status*' which means a political state. The dictionary gives the meaning of statistics as 'Numerical data collected systematically' or 'the science of collecting and interpreting such information'. According to its popular use, by statistics we mean quantitative data affected to a marked extent by multiplicity of causes.

Primarily statistics was supposed to be the science of kings used for the purpose of administration, but later on it was regarded as a branch of economics. Nowadays the meanings of statistics are interpreted in different ways by different classes, for example, for a layman statistics is nothing but a collection of figures, to an economist the field of statistics lies in quantitative analysis and to a physicist statistics is a probability distribution which forms the basis of the theory of errors.

Statistics, which was accepted for some time as necessarily a branch of economics has now become so popular and its application so wide that no branch of human knowledge escapes its approach. Today the naturalists, the biologists, the astronomers, the administrators, the businessmen, the economists, the chemists, the physicists, the photographers, all make frequent use of statistical methods of which probability is the fundamental tool. An astronomer uses statistical methods in making predictions about eclipses, a biologist utilzes them to generalize the laws of variations and heredity, a meteorologist uses them for weather forecasts, regarding temperature pressure and rainfall, etc.

So far as the applications of statistics to social and physical sciences are concerned, the statistical probability can be successfully employed in finding certain inferences in social and physical fields. Dealing with probability, we are able to show that the distribution of r balls in n cells has n^r different ordered samples with replacement of size r and without replacement it has ${}^nP_r = \frac{n!}{(n-r)!}$ different ordered samples of size r. Thus to find the possible configuration of birthday of people in a year, take r the number of people and n the number of days in a year; to clssify accidents according to the weekdays, take r the number of accidents and n the number of days in a week; to classify people according to age or profession, take a group of r people and n will be the number of classes; to observe irradiation in biology when the cells in the retina of the eye are exposed to light r is the number of light particles and n the number of cells; in cosmic ray experiments, r is the number of particles hitting the Geiger counters and n is the number of counters function; in an elevator to find the possible arrangements of discharging the passengers, r is the number of passengers and n is the number of floors; in dice throw r is the number of dice and n is the number of faces, i.e., six; in tossing coins r is the number of coins and $n = 2$; in coupon collecting r is the number of coupons collected and n are the kinds of coupons; in chemistry when a long chain polymer reacts with oxygen, r is the number of the reacting oxygen molecules and n is the number of polymer chains, etc.

Based on classical concepts, we give two definitions of probability.

[A] The Mathematical or 'a Priori' Probability: If there are q number of exhaustive, mutually exclusive and equally likely cases of an event and suppose that p of

them are favourable to the happenings of an event A under the given set of conditions, then the mathematical probability of the event A is defined as

$$P(A) = \frac{p}{q}.$$

We sometimes put this definition in the words 'the odds in favour of A are p to $(q-p)'$ or 'the odds against A are $(q-p)$ to p'. More precisely if we assume that the odds in favour of the event A are m to n (or n to m against A), the probability of happening the event A is defined as

NOTES

$$P(A) = \frac{m}{m+n}.$$

For example, the probability of drawing a white ball from a bag containing 3 white and 4 red balls is $\frac{3}{3+4}$, i.e., $\frac{3}{7}$.

Note 1. *The word 'exhaustive' used in the definition assures the happening of an event either in favour or against and rules out the possibility of happening of neither (in favour or against) in any trial. The word, 'mutually exclusive, is a safeguard against the probability of two simultaneous happenings in a trial, for example, in tossing a coin, the head and tail cannot fall together, but falling of either excludes the other. The word 'equally likely' means equally probable,* i.e., *no happening is biased or partially bound to occur.*

[B] The Statistical or Empirical or 'a Posteriori' Probability: If in a large number of trials performed under the same conditions, the limit of the ratio of the number of happenings of an event A to the total number of trials is unique and finite when the number of trials tends to infinity, then this limit measures the probability of the happening of the event A.

Thus if in a large number of trials performed under the same set of conditions, p is the probability of happening of an event A and q that of its failure, then the probability of its happening in the next trial is $\frac{p}{p+q}$, being assumed to determine the empirical probability that there is no known information relative to the probability of the happening of the event other than the past trials.

In other words, if an event A happens on pN occasions when a large number N is taken out of a series of trials, then the probability $P(A)$ of the event A is p defined as

$$P(A) = \lim_{N\to\infty} \frac{pN}{N} = p.$$

Precisely if m is the number of times in which the event A occurs in a series of n trials, then $P(A) = \lim_{N\to\infty} \frac{m}{n}$.

Note 2. If p is the probability of happening of an event A, i.e., $P(A) = p$ and q that of not happening of that event denoted by $P(\bar{A})$ is given by $P(\bar{A}) = q = 1 - p$, so that

$$P(A) + P(\bar{A}) = 1.$$

Conclusions: (*i*) The probability $P(A)$ of an event A lies between 0 and 1, i.e.,

$$0 \le P(A) \le 1.$$

(*ii*) The probability of an impossible event is zero, i.e.,

$$P(0) = 0.$$

(*iii*) The probability of a certain event E is one, i.e.,

$$P(E) = 1.$$

Example 4.4: From a pack of 52 cards two are drawn at random; find the chance that one is a knave and other a queen.

NOTES

Solution : Total number of ways of drawing 2 cards from 52 cards = ${}^{52}C_2$.

The required cards a knave and a queen appear in different four colours, therefore each card can be drawn in 4 different ways. But the two events happen simultaneously and hence the number of favourable ways = 4 × 4 = 16.

$$\therefore \quad \text{required probability} = \frac{16}{{}^{52}C_2} = \frac{16}{\frac{52.51}{1.2}} = \frac{8}{663}.$$

Example 4.5: What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution: There are 366 days in a leap year. Dividing 366 by 7, the number of days of a week, we conclude that the leap year consists of 52 complete weeks and 2 days more. These two days can be combined in 7 different ways as under :

(*i*) Sunday and Monday, (*ii*) Monday and Tuesday (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, (*vii*) Saturday and Sunday,

Of these seven combinations only (*i*) and (*vii*) are favourable so that the required chance = $\frac{2}{7}$.

Example 4.6: From a bag containing 4 white and 5 black balls a man draws 3 at random; what are the odds against these being all black?

Solution: Total number of ways in which 3 balls can be drawn = ${}^9C_3$.

Number of ways in which 3 black balls can be drawn = ${}^5C_3$.

$$\text{Required chance} = \frac{{}^5C_3}{{}^9C_3} = \frac{5.4.3.}{1.2.3.}\Big/\frac{9.8.7}{1.2.3.} = \frac{5}{42}.$$

Example 4.7:: A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet?

Solution : Number of ways in which a card can be drawn from 52 cards = ${}^{52}C_1$ = 52.

There are four aces so that the number of ways in which a card can be an ace is ${}^4C_1 = 4$.

Now there are 13 spade cards of which one is an ace. Out of the remaining 12 spade cards, a spade card can be drawn in ${}^{12}C_1$, i.e. 12 number of ways.

$\therefore$ The number of ways in which the drawn card may be a spade or an ace = 12 + 4 = 16.

$$\text{Hence the required probability} = \frac{16}{52} = \frac{4}{13} = \frac{4}{9+4},$$

i.e., the odds against the gambler's winning are as 9 to 4.

Example 4.8: The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.

Sobution: Let the chance to the happenings of the first and the second events be p and p' respectively. Then according to the first condition, we have

$$p = p'^2. \quad \text{... (1)}$$

NOTES

According to the second condition, we have

$$\frac{1-p}{p} = \left(\frac{1-p'}{p'}\right)^3. \quad \text{... (2)}$$

Substituting the value of p from Equations (1) to (2), we get

$$\frac{1-p'^2}{p'^2} = \left(\frac{1-p'}{p'}\right)^3$$

or $$\frac{(1-p')(1+p')}{p'^2} = \frac{(1-p')(1-2p'+p'^2)}{p'^2}$$

or $$p'(1+p') = 1 - 2p' + p'^2$$

or $$p' + p'^2 = 1 - 2p' + p'^2$$

or $$3p' = 1, \text{ i.e., } p' = \frac{1}{3}$$

and then from Equation (1) $p = \frac{1}{9}$.

Example 4.9: Three cards are drawn at random from an ordinary pack. Find the chance that they are a king, a queen and a knave.

Solution: Total number of ways in which 3 cards can be drawn from 52 cards $= {}^{52}C_3$.

The pack of cards consists of 4 kings, 4 queens and 4 knaves and therefore each of a king, a queen and a knave can be drawn in ${}^4C_1$, i.e. 4 ways. But all the three events happen together, hence the number of ways in which a king, a queen and a knave can be drawn

$$= 4 \times 4 \times 4 = 64.$$

$\therefore$ required probability$= \frac{64}{{}^{52}C_3} = \frac{64.1.2.3}{52.51.50} = \frac{16}{5525}$.

Example 4.10: Eight letters, to each of which corresponds an envelope, are placed in the envelopes at random. What is the probability that all letters are not placed in their right envelopes.

Solution : Total number of ways in which 8 letters can be placed in 8 envelopes $= 8\,!$.

Also there is only one way in which all the letters are placed in their right envelopes.

Therefore, probability that all the letters are placed in the right envelopes $= \frac{1}{8!}$.

Hence the required probability that all letters are not placed in their right envelopes $= 1 - \frac{1}{8!}$.

Example 4.11: *A* and *B* stand in a ring with 10 other persons. If the arrangement of the persons is at random, find the chance that there are exactly 3 persons between *A* and *B*.

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Solution: In a ring 12 persons can stand in 11 ! ways and 3 persons between *A* and *B* can be selected in $^{10}C_3$ ways. *A* and *B* interchange their positions in 2 ! ways. Also 3 persons between *A* and *B* can stand in 3 ! ways and the other 7 persons in 7 ! ways.

$$\therefore \text{Number of favourable ways} = 2\,!\,3\,!\,7\,!\,{}^{10}C_3$$

$$= 2!3!7!\frac{10!}{7!3!}$$

$$= 2\,!\,10\,!,$$

$$\text{Required probability} = \frac{2!10!}{11!} = \frac{2}{11}.$$

Example 4.12: A number is chosen from each of two sets

(1, 2, 3, 4, 5, 6, 7, 8, 9); (1, 2, 3, 4, 5, 6, 7, 8, 9).

Solution: If p_1 denotes the probability that the sum of the two numbers be 10 and p_2 the probability that their sum is 8, find $p_1 + p_2$.

Each set consists of 9 numbers and hence the total number of ways of choosing one from each = $^9C_1 \times {}^9C_1 = 81$.

A sum of 10 may be found in 9 different ways, like (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1) so that

$$p_1 = \frac{9}{81} = \frac{1}{9}.$$

Similarly a sum of 8 can be found in 7 ways so that $p_2 = \frac{7}{81}$.

$$\therefore \qquad p_1 + p_2 = \frac{1}{9} + \frac{7}{81} = \frac{16}{81}.$$

Simple Properties of Probability

Probability is the measure of the likelihood that an event will occur. Principally, the probability quantifies as a number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty. The higher the probability of an event, the more likely it is that the event will occur. A simple example is the tossing of a fair (unbiased) coin. Since the coin is fair, the two outcomes 'Heads' and 'Tails' are both equally probable, the probability of 'Heads' equals the probability of 'Tails' and since no other outcomes are possible, the probability of either 'Heads' or 'Tails' is ½, which can also be written as 0.5 or 50%. Similar to other theories, the theory of probability and its simple properties can be represented using the rules of mathematics and logic as discussed below.

Events

A collection of all possible outcomes of an experiment is said to be an **event.**

If such a collection contains the outcome of an event, then that event is said to have occurred.

An event is said to be simple or compound according as it cannot or can be decomposed.

For example, if we toss a coin, it will turn up either a head or a tail. Thus there are only two possible outcomes giving a simple event.

If we throw a pair of dice, then to have sum of 5 is a compound event as a sum of 5 can be obtained as (1, 4), (2, 3), (3, 2), (4, 1). Thus this compound event consists of 4 elementary events.

Mutually Exclusive Events: Two or more events are said to be mutually exclusive if the happening or occurrence of any one of them excludes the happening of the others.

For example, if we toss a coin and it falls with head up, then the falling of the head up excludes the simultaneous happening of the tail up, i.e., the two events of the falling head and tail up with a coin cannot happen together, but the happening of one excludes the happening of the other. So the two events are mutually exclusive.

Compound Events (Joint Occurrence): The simultaneous occurrence of two or more events in connection with each other is said to be a compound event.

For example, if we have an urn containing 100 balls of different colours say red and green, and suppose 60 are red and 40 are green. If then it is proposed to draw 10 balls each of red colour, it is a simple event. But if it is proposed to draw first 20 balls of red colour and then 10 balls of green colour, then it is a compound event.

Dependent and Independent Events: Two or more events are said to be *dependent* or *independent* according as the occurrence of one does or does not affect the occurrence of the other or others. The dependent events are sometimes known as *contingent*.

For example, if from an urn containing 10 balls, it is proposed to draw 2 balls, then if a ball is drawn and it is not replaced unless the second ball is drawn, the event of the drawing of the second ball is dependent on that of the first. But if the first ball is replaced and then the second ball is drawn, the event of drawing the second ball is independent.

Random Variable

In probability and statistics, a random variable or a random quantity is a variable whose possible values are outcomes of a random phenomenon. More specifically, a random variable is defined as a function that maps the outcomes of an unpredictable process to numerical quantities, typically real numbers. It is referred as a variable, precisely a dependent variable, in the sense that it depends on the outcome of an underlying process providing the input to this function, and it is random in the sense that the underlying process is assumed to be random.

The possible values of a random variable may possibly represent the possible or probable outcomes of an experiment that is yet to be performed or the possible outcomes of a past experiment whose already-existing value is uncertain, i.e., not certain. The potential or probable values assigned to a random variable helps in the interpretation of probability. As a function, a random variable is required to be measurable, which allows for probabilities to be assigned to sets of its potential or probable values. It is common that the outcomes depend on some physical variables that are not predictable. For example, when tossing a fair coin, the final outcome of heads or tails depends on the uncertain physical conditions. Which outcome will be observed is not certain, i.e., there are only two possible outcomes, namely heads or tails. Therefore, the domain of a random variable is the set of possible outcomes.

A random variable, usually written X, is a variable whose possible values are numerical outcomes of a random phenomenon. There are two types of random variables, discrete and continuous.

Discrete Random Variables: A discrete random variable is one which may take on only a countable number of distinct values, such as 0,1,2,3,4,........ Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children in a family, the employee attendance at an organization, the number of children present in the class/school on the specific day, the number of defective light bulbs in a box of ten, etc. The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the probability mass function.

Continuous Random Variables: A continuous random variable is a random variable whose cumulative distribution function is continuous everywhere. There are no 'Gaps', which would correspond to numbers which have a finite probability of occurring, for example a continuous random variable can be based on a spinner that can select a horizontal direction. Then the values taken by the random variable are directions and these directions can be represented by North, West, East, South, North-West, South-East, etc. However, it is generally more convenient to record the sample space to a random variable which takes values which are real numbers. Essentially, the continuous random variables typically state Probability Density Functions (PDF).

Characteristically, a random variable takes on different values as a result of the outcomes of a random experiment. In other words, a function which assigns numerical values to each element of the set of events that may occur (i.e., every element in the sample space) is termed a random variable. The value of a random variable is the general outcome of the random experiment. One should always make a distinction between the random variable and the values that it can take on.

Theorem of Total Probability

If there are n mutually exclusive events $A_1, A_2, A_3, ..., A_n$ whose probabilities are $P(A_1), P(A_2), P(A_3),..., P(A_n)$ respectively, then the probability that one of them will happen is the sum of their separate probabilities, i.e.,

$P(A_1 + A_2 + A_3 +...+ A_n) = P(A_1) + P(A_2) + P(A_3) +...+ P(A_n)$, *where* $P(A_1 + A_2 +...+ A_n)$ *denotes the probability of occurrence of at least one of the n events* $A_1, A_2, A_3, ..., A_n$.

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Suppose there are N total number of exhaustive, mutually exclusive and equally likely cases of which $m_1, m_2,...m_n$ are favourable to the events A_1, $A_2,...A_n$, respectively. Then the total number of cases favourable to either A_1 or A_2 or A_3 or ... or A_n is $m_1 + m_2 + m_3 +...+ m_n$ so that the probability of happening of at least one of these events is

$$P(A_1 + A_2 + A_3 +...+ A_n) = \frac{m_1 + m_2 + m_3 + ... + m_n}{N}$$

$$= \frac{m_1}{N} + \frac{m_2}{N} + ... + \frac{m_n}{N}$$

$$= P(A_1) + P(A_2) + P(A_3) +...+ P(A_n).$$

Conclusions: (*i*) In case an event A is comprised by n mutually exclusive forms $A_1, A_2,....A_n$, i.e.,

$$A = A_1 + A_2 +...+ A_n,$$

then probability A, i.e., $P(A)$ is the sum of the probabilities of $A_1, A_2, ..., A_n$ separately, i.e.,

$$P(A) = P(A_1) + P(A_2) +...+ P(A_n).$$

(*i*) In case the n mutually exclusive events are exhaustive also, so that there is certainty of happening of at least one, i.e., $P(A_1 + A_2 +...+ A_n) = 1$, we have

$$P(A_1) + P(A_2) +...+ P(A_n) = 1.$$

Note. We generally use the following notations:

$P(A)$ denotes the probability for an event A to happen,

$P(\bar{A})$ denotes the probability for an event A not to happen,

$P(A + B)$ denotes the probability of the occurrence of at least one of the events A and B

$P(AB)$ denotes the probability of the occurrence of both the events A and B,

$P(A\bar{B})$ denotes the probability of the hpening of A and not of B,

$P(\bar{A}B)$ denotes the probability of the happening of B and not of A,

$P(\overline{AB})$ denotes the probability of the happening of neither of A and B.

If A and B are two events such that AB and $A\bar{B}$ are two exhaustive and mutually exclusive forms in which A can occur, then we have

$$P(A) = P(AB) + P(A\bar{B}) \text{ from conclusion } (ii)$$

Similarly, $$P(B) = P(BA) + P(B\bar{A})$$

$$= P(AB) + P(\bar{A}B).$$

so that $$P(A) + P(B) = P(AB) + \{P(A\bar{B}) + P(\bar{A}B) + P(AB)\}.$$

But from the above theorem of total probability, we can write

$$P(A + B) = P(A\bar{B}) + P(\bar{A}B) + P(AB),$$

i.e., probability that at least one of A and B happens = the sum of probabilities that A happens B not, B happens A not, and A, B both happen.

Thus, $$P(A) + P(B) = P(AB) + P(A + B)$$

or $$P(A + B) = P(A) + P(B) - P(AB). \quad ... (4.1)$$

Generalization of this Result: *To prove that the formula for the probability of occurrence of at least one of the n given events A_1, A_2,...A_n is*

$P(A_1 + A_2 + ... + A_n) = S_1 - S_2 + S_3 - ... + (-1)^{n-1} S_n$

where S stands for the sum of probabilities of simultaneous occurrence of exactly r of n events, the summation extending over all possible combinations.

NOTES

For two mutually exclusive events A_1 and A_2 the Result Equation (4.1) gives $P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1A_2)$.

Similarly,

$$\begin{aligned} P(A_1 + A_2 + A_3) &= P(A_1) + P(A_2 + A_3) - P\{A_1(A_2 + A_3)\} \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_2A_3) - P(A_1A_2 + A_1A_3) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_2A_3) \\ &\qquad - P(A_1A_2) - P(A_1A_3) + P(A_1A_2A_3) \\ &= \sum_{i=1}^{3} P(A_i) - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} P(A_iA_j) + P(A_1A_2A_3). \qquad ... (4.2) \end{aligned}$$

Thus it follows from induction method that if there are n events A_1, A_2, A_3,...A_n, then the generalization of the Result Equation (4.2) gives

$$\begin{aligned} P(A_1 + A_2 + ... + A_n) &= \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} P(A_iA_j) + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{3} P(A_iA_jA_k) \\ &\qquad - ... + (-1)^{n-1} P(A_1A_2...A_n) \qquad ... (4.3) \end{aligned}$$

Denoting by S_1, S_2, ..., S_n the sum of the probabilities of simultaneous occurrence of exactly 1, 2,..., n of the n events, the summation extending over all possible combinations, we have

$$P(A_1 + A_2 + ... + A_n) = S_1 - S_2 + S_3 - ... + (-1)^{n-1} S_n.$$

COROLLARY. In case the events A_1, A_2,..., A_n are mutually exclusive, then

$$P(A_iA_j) = 0,$$
$$P(A_iA_jA_k) = 0,$$
....
$$P(A_1A_2...A_n) = 0,$$

so that Result Equation (4.2) reduces to the theorem of total probability, i.e.,

$$P(A_1 + A_2 + ... + A_n) = \sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + ... + P(A_n).$$

Example 4.13: If the probability of a horse A winning a race is $\frac{1}{5}$ and the probability of a horse B winning the same race is $\frac{1}{6}$, what is the probability that one of the horses wins.

Solution: Let p_1 and p_2 be the probabilities of A and B respectively; then

$p_1 = \frac{1}{5}$ and $p_2 = \frac{1}{6}$.

NOTES

The two events being mutually exclusive, the probability that one of them wins

$$= p_1 + p_2$$

$$= \frac{1}{5} + \frac{1}{6} = \frac{11}{30}.$$

Example 4.14: Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?

Solution: Total number of ways of drawing 6 cards out of 52 = ${}^{52}C_6$.

There are 26 red and 26 black cards; therefore the number of ways in which 3 red and 3 black cards can be drawn

$$= {}^{26}C_3 \times {}^{26}C_3.$$

$$\therefore \text{Required probability} = \frac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6}$$

$$= \frac{\frac{26.25.24}{1.2.3} \times \frac{26.25.24}{1.2.3}}{\frac{52.51.50.49.48.47}{1.2.3.4.5.6}} = \frac{13000}{39151}$$

Example 4.15: The first twelve letters of the alphabet are written down at random. What is the probability that there are four letters between the letters *A* and *B*?

Solution: Denoting the positions of letters as

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,

when *A* is kept at 1, *B* should be placed at 6 to have four letters in between

”	2	”	7	”
”	3	”	8	”
”	4	”	9	”
”	5	”	10	”
”	6	”	11	”
”	7	”	12	”

Thus *A* and *B* can be placed in 7 ways so as to have four letters in between. Also *A* and *B* can interchange their positions in 2! ways. The four letters between *A* and *B* can be chosen in ${}^{10}C_4$ ways out of remaining 10 letters when *A* and *B* have already been placed. Moreover, these four letters can be arranged in 4! ways and the remaining 6 letters in 6! ways.

$\therefore$ Number of favourable ways = $7.2!.\ {}^{10}C_4\ 4!\ 6!$

and total number of arrangements in which 12 letters can be put = 12!

$$\text{Hence the required probabilities} = \frac{7.2!.\ {}^{10}C_4\ 4!\,6!}{12!}$$

$$= \frac{7.2!.10!\cdot 4!\cdot 6!}{4!\cdot 6!\cdot 12!} = \frac{14}{12.11} = \frac{7}{6.6}$$

Example 4.16: One of two events must happen; given that the chance of the one is two-thirds that of the other, find the odds in favour of the other.

NOTES

Solution: Let p_1, p_2 be the probabilities of the two events and suppose that x is the chance of happening of either, say

$$p_1 = x \text{ so that } p_2 = \frac{2}{3}x.$$

But we have $p_1 + p_2 = 1$ for a sure event,

i.e., $$x + \frac{2}{3}x = 1 \text{ or } \frac{5}{3}x = 1 \text{ giving } x = \frac{3}{5}.$$

$$p_1 = \frac{3}{5} \text{ and as such } p_2 = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}.$$

Thus odds in favour of the other $= \frac{2}{5} : \frac{3}{5}$, *i.e.*, 2 : 3.

Theorem of Compound Probability or Multiplicative Law of Probability

If there are two events A and B, probabilities of their happening being P(A) and P(B) respectively, then the probability P(AB) of the simultaneous occurrence of the events A and B is equal to the probability of A multiplied by the conditional probability of B (i.e., *the probability of B when A has occurred*) *or the probability of B multiplied by the conditional probability of A,* i.e.,

$$\begin{aligned} P(AB) &= P(A)\,P(B/A) \\ &= P(B)\,P(A/B), \end{aligned}$$

where P(B/A) denotes conditional probability of B and P(A/B) that of A and that if the two events are independent, then the theorem of compound probability is

$$P(AB) = P(A)\,P(B).$$

Suppose there are N total number of mutually exclusive and equally likely cases of which m are favourable to A. Let m_1 be the number of cases favourable to A and B both, while m_1 included in m. Thus

$$P(B/A) = \frac{m_1}{m} \text{ and } P(A) = \frac{m}{N}.$$

Now $P(AB)$ = the probability of happening of A and B both

$$\begin{aligned} &= \frac{m_1}{N} = \frac{m_1}{m} \cdot \frac{m}{N} \\ &= P(A)\,P(B/A). \qquad \text{... (4.4)} \end{aligned}$$

The interchange of A and B will yield a similar result

$$P(AB) = P(BA) = P(B)\,(A/B) \qquad \text{... (4.5)}$$

In case the two events A and B are independent, i.e., the occurrence of one does not affect the other, $P(B/A)$ is the same as $P(B)$ and $P(A/B)$ is the same as $P(A)$, so that the Results Equations (4.4) and (4.5) both become $P(AB) = P(A)$ $P(B)$.

Generalization. The result Equation (4.4) may be generalized as, if there are A_1, A_2,...A_n, n mutually independent events, then the compound probability is given by

$$P(A_1, A_2, A_3 ... A_n) = P(A_1)\,P(A_2) ... P(A_n) \qquad \text{... (4.6)}$$

In case of n mutually exclusive events $A_1, A_2, ... A_n$, the Result Equation (4.4) may be generalized as

$$P(A_1A_2...A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1A_2)...P(A_n/A_1A_3...A_{n-1}).$$

NOTES

Conclusions: (i) If p be the chance that an event will happen in one trial, the chance that it will happen in any assigned succession of r trials is p^r; for in this case

$$P(A_1) = P(A_2) = ... = P(A_r) = p$$

$$\therefore \text{ required probability} = P(A_1)\,P(A_2)...P(A_r)$$

$$= p.p...r \text{ times} = p^r.$$

(ii) If $p_1, p_2, p_3, ... p_n$ are the probabilities that n events happen, then probability that all the events fail is

$$(1 - p_1)\,(1 - p_2)\,(1 - p_3)...(1 - p_n)$$

Hence the chance that at least one of these events happens

$$= 1 - (1 - p_1)\,(1 - p_2)...(1 - p_n)$$

Example 4.17: In shuffling a pack of cards three are accidentally dropped; find the chance that the missing cards should be from different suits.

Solution: The pack consists of 52 cards.

The chance of dropping a card $= \dfrac{{}^{52}C_1}{{}^{52}C_1} = 1.$

When one card is dropped, there remain 51 cards of which 39 cards are of suits different from that of dropped one. Thus the chance of dropping a card of different suit in second draw

$$= \frac{{}^{39}C_1}{{}^{51}C_1} = \frac{39}{51}.$$

When two cards are dropped, there remain 50 cards of which 26 cards are of suits different from that of dropped cards. Thus the chance of dropping a card of different suit in third draw.

$$\frac{{}^{26}C_1}{{}^{50}C_1} = \frac{26}{50}.$$

The events being dependent, the required chance

$$= 1 \times \frac{39}{51} \times \frac{26}{50} = \frac{169}{425}.$$

Example 4.18: The face cards (three from each suit) are removed from a full pack. Out of the 40 remaining cards, 4 are drawn at random:

(a) What is the probability that they belong to different suit?

(b) What is the probability that the 4 cards drawn belong to different suits and different denominations.

Solution: Having removed 12 face cards, the remaining 40 consist of 10 cards of each suit :

(a) Chance of drawing a card in first draw $= \dfrac{{}^{40}C_1}{{}^{40}C_1} = 1.$

NOTES

Having drawn 1 card, there remain 39 cards of which 30 are of suits different from the drawn one.

$\therefore$ Chance of drawing a card of different suit in second draw

$$= \frac{{}^{30}C_1}{{}^{39}C_1} = \frac{30}{39}.$$

Having drawn two cards, there remain 38 cards of which 20 are of suits different from the drawn cards.

$\therefore$ Chance of drawing a card in third draw $= \frac{{}^{20}C_1}{{}^{38}C_1} = \frac{20}{38}$

Having drawn three cards, there remain 37 cards of which 10 are of suits different from drawn cards.

$\therefore$ Chance of drawing a card in fourth draw $= \frac{{}^{10}C_1}{{}^{37}C_1} = \frac{10}{37}.$

All the events being dependent, the required probability

$$= 1 \times \frac{30}{39} \times \frac{20}{38} + \frac{10}{37} = \frac{1000}{9139}.$$

(*b*) Chance of drawing a card in first draw $= \frac{{}^{40}C_1}{{}^{40}C_1} = 1.$

Having drawn one card, there remain 39 cards of which 9 cards are of the same suit and 3 of the same denomination (value) so that 27 cards out of 39 are such that they are of different colours and different denominations from the drawn one.

$\therefore$ Chance of drawing a card in second draw $= \frac{{}^{27}C_1}{{}^{39}C_1} = \frac{27}{39}.$

Similarly chance of drawing a card in third draw $= \frac{{}^{16}C_1}{{}^{38}C_1} = \frac{16}{38}$

and chance of drawing a card in fourth draw $= \frac{{}^{7}C_1}{{}^{37}C_1} = \frac{7}{37}.$

Example 4.19: From a pack of cards two cards are drawn, the first being replaced before the second is drawn. Find the probability that the first is a diamond and the second is a king.

Solution: Let A denote the event of drawing a diamond and B denote the event of drawing a king in the second draw, when the first card has been replaced. Then

$$P(A) = \text{Probability of drawing a diamond}$$

$$= \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{1}{4}.$$

$$P(B) = \text{Probability of drawing a king}$$

$$= \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{1}{13}.$$

The two events being independent, we have

$$P(AB) = P(A)\,P(B)$$
$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}.$$

NOTES

Example 4.20: Find the chance of throwing a 6 at least once in two throws of a single die.

Let A denote the event of throwing a six in the first throw and B that in the second throw. Then probability of throwing a 6 at least once in two throws may be represented by $P(A + B)$.

Now $\quad P(A) =$ Prob. of throwing a 6 in first throw.

$$= \frac{1}{6}$$

and $\quad P(B) =$ Prob. of throwing a 6 in second throw

$$= \frac{1}{6}.$$

But we have

$$P(A + B) = P(A) + P(B) - P(AB)$$

where $P(AB) = P(A)\,P(B)$, A, B being independent events

$$\therefore \quad P(A + B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{6} \cdot \frac{1}{6}$$
$$= \frac{1}{3} - \frac{1}{36} = \frac{11}{36}.$$

Example 4.21: A coin is tossed three times. Find the probability of getting head and tail alternately.

Solution: Let $P(A)$ and $P(B)$ represent the probability of getting head and tail respectively. Then

$$P(A) = \frac{1}{2} = P(B).$$

The alternate occurring of head and tail may happen in two ways:

(*i*) starting with head,

(*ii*) starting with tail.

In case of first, the probability of the event

$$= P(A)\,P(B)\,P(A)$$
$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

In second case the probability of the event

$$= P(B)\,P(A)\,P(B)$$
$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

But the two events being mutually exclusive, the total probability of happening of any one of them $= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$

Probability: A New Standpoint

[A] Sample Space or Outcome Space: *A set consisting of the elementary events as its elements is said to be a sample space. It is generally denoted by S.*

NOTES

A sample space provides a mathematical model of an ideal experiment in the sense that every conceivable outcome of the experiment is completely described by one and only one sample point.

In fact an element in S is known as *sample point* or *sample* and an event, for example, event A is a subset of the sample space S. The event $A = \{a\}$ consisting of a single sample point $a \in S$ is said to be an *elementary event.* The null sets ϕ and S itself are events. The null set ϕ is said to be an *impossible event* while S is said to be a *sure* or *certain event*.

[B] Correspondence between Sets and Events: Let there be two events A and B. Then

(*i*) $A \cup B$ denotes an event which occurs iff (if and only if) A occurs or B occurs (or both occur).

(*ii*) $A \cap B$ denotes an event which occurs iff A occurs and B occurs.

(*iii*) The complementary event of A denoted by A' or $\overline{A}$ is an event which occurs iff A does not occur.

$B - A$ will be the event consisting of all points not contained in the event A but contained in B, i.e.,

$$B - A = B \cap A'$$

$$\therefore \quad A - A' = \phi = A \cap A'.$$

$$\text{Also} \quad A \cup A' = S.$$

A complementary event A' is always *mutually exclusive* and *exhaustive*.

(*iv*) Two events A and B are called *mutually exclusive* or *disjoint* if $A \cap B = \phi$.

Example 4.22: A coin is tossed and it is observed whether a head or tail is up. Describe the suitable sample space of the experiment.

Solution: Denoting by H the event in which the coin turns up head and by T, the event in which the coin turns up tail, the sample space of the experiment consists of only two elements, i.e., $S = \{H, T\}$.

Example 4.23: An urn contains three red and two white balls. Two balls are drawn and their colour is noted. Set up the sample space of this experiment.

Solution: Let the red balls be numbered as R_1, R_2, R_3 and white balls as W_1, W_2.

Denoting the event of drawing ball R_1 in first draw and R_2 in second draw by $R_1 R_2$, the sample space consists of 20 outcomes as given below:

$S = \{R_1R_2, R_1R_3, R_1W_1, R_1W_2, R_2R_1, R_2R_3, R_2W_1, R_2W_2, R_3R_1, R_3R_2, R_3W_1, R_3W_2,$
$W_1R_1, W_1R_2, W_1R_3, W_1W_2, W_2R_1, W_2R_2, W_2W_3, W_2W_1\}$.

NOTES

[C] The Modern Concept of Probability: It is a well known fact that the concept of probability developed from evil habits of games of chance or gambling used in France in the 17th century. In this connection the French nobleman and gambler Chevalier de méré consulted the well known mathematician Blaise Pascal (1623-1662) who began to think over the problem how and to what degree of accuracy a gambler can be assured of his chance of success. Pascal solved the problem of de méré and had a correspondence with Pierre de Fermat (1601-1665) who becam interested in this and other similar problems.

The phenomena occurring in nature of any secrets can be either *Deterministic* or *Probabilistic,* for example, if a train moves at the rate of 20 km./hr., it is deterministic that it will travel 100 km. in 5 hours, but if a coin is tossed, then it is probabilistic to say that the chance of each either 'head up' or 'tail up' is equal.

Actually the theory of probability deals with the things likely to occur and their chance or probable values. It is a measure of degree of uncertainty rather than accuracy.

The concepts of probability are connected with the events or occurrences and the repeated trials.

[D] Probability of an Event: If in a series of n trials all made under the same conditions an event A is observed m times, then m is said to be the frequency of success and the ratio $\frac{m}{n}$ is said to be the *relative frequency* of success.

The *probability* of the event A is defined to be the limit p of the ratio $\frac{m}{n}$ when n tends to infinity (if it exists), i.e.,

$$P(A) = p = \lim_{n\to\infty} \frac{m}{n}.$$

This is sometimes known as the *frequency definition of probability*. In other words, if the event A consists of r clear events out of n, then

$$P(A) = \frac{r}{n} = \frac{\text{Number of elementary events in } A}{\text{Number of elementary events in sample space } S} = \frac{N(A)}{N(S)}.$$

In terms of sample space S associated with Σ the class of events if P be a real-valued function defined on Σ, then P is said to be the *probability function* and $P(A)$ the *probability* of the event A which satisfies the following conditions:

(*i*) For every event A, $0 \le P(A) \le 1$.

(*ii*) $P(S) = 1$, i.e., $\sum_{i=1}^{n} P(A_i) = 1$

where $$S = \{A_1\} \cup \{A_2\} \cup ... \cup \{A_n\}.$$

If $S = \phi$ (a null set), then $\sum_{i=1}^{n} P(A_i) = 0$, *i.e.*, $P(\phi) = 0$,

(*iii*) If A and B are two mutually exclusive (i.e., disjoint) events, i.e., $A \cap B = \phi$, then

$$P(A \cup B) = P(A) + P(B).$$

NOTES

(iv) If $A_1, A_2, A_3, ..., A_n$ are n mutually exclusive events then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n).$$

4.3.1 Baye's Theorem

In order to prove the thorem given by Thomas Baye, let us first introduce a Lemma, required for its proof.

Lemma. *Let* $\{A_1, A_2, ..., A_n\}$ *be a partition of the sample space S and suppose that each of the events* $A_1, A_2, ..., A_n$ *has non-zero probability,* i.e., $P(A_j) > 0$ *for* $j = 1, 2, ...n$. *Then for any event A, we have*

$$P(A) = P(A_1).P(A \mid A_1) + P(A_2).P(A \mid A_2) + ... + P(A_n).\ P(A \mid A_n)$$

$$= \sum_{j=1}^{n} P(A_j)\ P(A \mid A_j).$$

Its proof. As $A_1, A_2, ..., A_n$ are partitions of S, therefore

$\{A \cap A_1, A \cap A_2, ..., A \cap A_n\}$

will represent the partition of A.

Thus $$A = (A \cap A_1) \cup (A \cap A_2) \cup ... \cap (A \cap A_n).$$

so that $$P(A) = P(A \cap A_1) + P(A \cap A_2) + ... + P(A \cap A_n)$$

$$= \sum_{j=1}^{n} P(A \cap A_j)$$

Applying the result of the conditional probability, i.e.,

$$P(A \mid A_j) = \frac{P(A \cap A_j)}{P(A_j)} \text{ giving } P(A \cap A_j)$$

$$= P(A_j).\ P(A \mid A_j)$$

we have $$P(A) = \sum_{j=1}^{n} P(A_j).\ P(A \mid A_j), \qquad ... (4.7)$$

which proves the lemma.

Statement of Bayes' Theorem: *If an event A can occur only if one of the mutually exclusive events* $A_1, A_2, ..., A_n$, i.e., $A \subset \bigcup_{k=1}^{k} A_k,\ A_k \cap A_j = \phi$ *when* $k \neq j$ *and suppose we are given the probabilities* $P(A_k)$, $k = 1, 2, ..., n$ *and the conditional probabilities* $P(A \mid A_k)$, *then we are required to find the probability of* A_k *when it is given that A has already occurred and* $P(A) > 0$ *for each integer k* $(1 \leq k \leq n)$ *then Baye's formula is*

$$P(A_k \mid A) = \frac{P(A_k)\ P(A \mid A_k)}{\sum_{j=1}^{n} P(A_j)\ P(A \mid A_j)}.$$

Proof. By the definition of conditional probability, we have

$$P(A_k \mid A) = \frac{P(A \cap A_k)}{P(A)}$$

NOTES

i.e.,
$$P(A \cap A_k) = P(A)\, P(A_k) \mid A)$$
$$= P(A_k)\, P(A \mid A_k)$$

and by Equation (4.7).
$$P(A) = \sum_{j=1}^{n} P(A_j).P(A \mid A_j).$$

$$\therefore \quad P(A_k/A) = \frac{P(A_k).P(A/A_k)}{\sum_{j=1}^{n} P(A_j).P(A/A_j)}.$$

Example 4.24: There are three coins, identical in appearance, one of which is ideal and the other two biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice. If a head appears both the times, what is the probability that the ideal coin was chosen.

Solution: Let A_1, A_2, A_3 denote the events of choosing the 1st (ideal), 2nd and 3rd coins respectively. Then

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}.$$

Let A be the event of obtaining 2 heads in two tosses of the selected coin.

Probability of getting a head in a toss $= \frac{1}{2}$.

$$\therefore \quad P(A/A_1) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Probability of turning head up with 2nd coin is $\frac{1}{3}$ and that with 3rd coin is $\frac{2}{3}$; therefore in two tosses,

$$P(A/A_2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \text{ and } P(A/A_3) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Using Bayes' formula,

$$P(A_1/A) = \frac{P(A_1)\, P(A/A_1)}{\sum_{j=1}^{3} P(A_j).P(A/A_j)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{4}{9}}$$

$$= \frac{\frac{1}{12}}{\frac{1}{12}+\frac{1}{27}+\frac{4}{27}} = \frac{\frac{1}{12}}{\frac{29}{108}} = \frac{1}{12}\times\frac{108}{29} = \frac{9}{29}.$$

Example 4.25: In a bolt factory, machines A, B, C manufacture respectively 25, 35 and 40% of the total. Of their output, 5, 4 and 2% are defective bolts. A bolt is drawn random from the procedure and is found defective. What are the probabilities that it was manufactured by machine A, B or C?

Solution: Here $P(A) = \frac{25}{100}$, $P(B) = \frac{35}{100}$, $P(C) = \frac{40}{100}$.

NOTES

Let E denote the event of drawing a defective bolt.

$$P(E/A) = \frac{5}{100},\ P(E/B) = \frac{4}{100},\ P(E/C) = \frac{2}{100}.$$

Using Bayes' formula

$$P(A/E) = \frac{P(A)\ P(E \mid A)}{P(A)\ P(E \mid A) + P(B)\ P(E \mid B) + P(C)\ P(E \mid C)}$$

$$= \frac{\frac{25}{100} \times \frac{5}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}}$$

$$= \frac{125}{125 + 140 + 80} = \frac{125}{345}.$$

Similarly $P(B \mid E) = \frac{140}{345}$ and $P(C \mid E) = \frac{80}{345}$.

[E] Independent Events: *The two events A and B are said to be stochastically independent if and only if* $P(A \mid B) = P(A)$, i.e., *the happening of the event B does not affect the happening of A.*

We have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

The two events are independent of each other when

$$P(A \mid B) = P(A),\ P(A) > 0,$$

$$P(B \mid A) = P(B),\ P(B) > 0.$$

Thus the two events A and B are stochastically independent if and only if

$$P(A \cap B) = P(A)\ P(B).$$

In general, $m (m > 2)$ events $A_1, A_2 ..., A_m$ are independent if

$$P(A_1 \cap A_2 \cap ... \cap A_m) = P(A_1).P(A_2)...P(A_m).$$

Example 4.26: If A and B are two independent events in a sample space S, then prove that

(*i*) $\bar{A}$ and $\bar{B}$ are independent,

(*ii*) $\bar{A}$ and B are independent,

(*iii*) A and $\bar{B}$ are independent,

(*iv*) $P(A \cup B) = 1 - P(\bar{A})\ \ P(\bar{B})$.

Solution: (*i*) We have $P(\bar{A} \cup \bar{B}) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A).P(B)$$

as A and B are independent

$$= [1 - P(A)]\ [1 - P(B)]$$

$$= P(\bar{A}).P(\bar{B})$$

showing that $\bar{A}$ and $\bar{B}$ are independent.

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(ii) We have $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$$= P(B) - P(A).P(B)$$

$$= P(B)\,[1 - P(A)]$$

$$= P(B).\ P(\overline{A})$$

showing that $\overline{A}$ and B are independent.

(iii) We have $\quad A = A \cap \overline{B} \cup A \cap B.$

$$\therefore \quad P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

$$= P(A \cap \overline{B}) + P(A).P(B)$$

or $\quad P(A \cap \overline{B}) = P(A)\,[1 - P(B)]$

$$= P(A) \cdot (\overline{B})$$

showing that A and $\overline{B}$ are independent.

(iv) We have $(A \cup B) \cup (\overline{A} \cup \overline{B}) = S.$

$$\therefore \quad P(A \cup B) + P(\overline{A} \cap \overline{B}) = \text{P}(S) = 1$$

or $\quad P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$

$= 1 - P(\overline{A}).P(\overline{B})$ as $\overline{A}$ and $\overline{B}$ independent by (i).

Example 4.27: From a pack of cards if the event of drawing a spade card is denoted by A and that of drawing an honour card by B, then show that A and B are independent events.

Solution: We have $P(A) = \dfrac{{}^{13}C_1}{{}^{52}C_1} = \dfrac{1}{4},$

$$P(B) = \frac{{}^{12}C_1}{{}^{52}C_1} = \frac{12}{52} = \frac{3}{13},$$

$$P(A \cap B) = \frac{{}^{3}C_1}{{}^{52}C_1} = \frac{3}{52}.$$

Then $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{3}{52}}{\frac{3}{13}}$

$$= \frac{1}{4} = P(A).$$

Since $P(A \mid B) = P(A)$, the events A and B are independent.

4.4 PROBABILITY CURVE

Having started with certain general hypotheses, if it is possible to deduce mathematically the frequency distributions of certain populations or universes, then such distributions are said to be theoretical distributions, which constitute three types of distributions.

(i) **Binomial Distributions**: It was discovered by James Bernoulli (1654 – 1705) in the year 1700 and was first published in 1713, eight years after the death of Bernoulli.

(*ii*) **Poisson Distribution**: It was discovered by the French mathematician and physicist Siméon Denis Poisson (1781-1840) who published it in 1837.

(*iii*) **Normal Distribution**: Though it was first discovered by De Moivre as early as 1733, but associated with the names of the distinguished French mathematician Pierre Simon, Marquis de Laplace (1749-1827) and the German mathematician and physicist Carl Friedrich Gauss (1777-1855) who discussed it independently at the close of 18th and the beginning of 19th centuries.

NOTES

Here below we shall discuss these theree distributions taking one by one.

4.4.1 The Binomial Distribution

If we toss a coin which is a uniform, homogeneous circular disc, then nothing is biased to make it to tend to fall more often on the one side than on the other. It is therefore expected that in a series of throws the coin will fall heads-up and tails-up an approximately equal number of times and so the chance of throwing heads or tails with a coin is $\frac{1}{2}$. Similarly the chance of throwing an ace with a fair die is $\frac{1}{6}$. Instead of considering the particular instances we generally use to say that the chance of success of an event is p and chance of its failure is q such that $p + q = 1$. Assuming the events in a number of trials to be independent, the chances p and q may be supposed to remain constant throughout.

If we take a number of sets of n trials and count the number of successes in each set, then there will be some sets with no success, some with one success, some with two successes, some with three successes and so on. the classification of the sets according to the number of successes which they contain, will give us a frequency distribution.

Suppose there are N sets of n trials in which the chances of the success and failure are respectively p and q. We have to find the frequencies of 0, 1, 2, 3, ... successes in cases of one event, two events, three events and so on.

When $n = 1$, i.e. in case of single event, out of N sets of 1 trial each we expect Np successes and Nq failures.

When $n = 2$, i.e., there are N sets of two events or N sets of two trials each, the event which has taken place once is repeated again. We have Nq failure of the first event or trial and the events being independent among these Nq there will be $Nq \times q$ failures and $Nq \times p$ sucesses of the second event on the average. Similarly among the Np successes of the first event there are $Np \times p$ successes and $Np \times q$ failures of the second event on the average. Hence there are in total Nq^2 failures of both events. $2Nqp$ cases of the two events with one success and one failure and Np^2 successes of both events. Thus the frequencies of 0, 1, 2 successes are respectively.

$$Nq^2,\ 2Nqp,\ Np^2$$

When $n = 3$, i.e., there are N sets of three events, we see that of the Nq^2 cases in which the first two events were failures, we have $Nq^2 \times q$ a third failure and $Nq^2 \times p$ one success; of the $2Nqp$ cases, $2Npq^2$ will give two failures and one success; and $2Np^2q$ one failure and two successes; of the Np^2 cases, Np^2q will give one failure and two successes and Np^3 a third success. Hence the frequencies of 0, 1, 2, 3 successes are respectively

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$$Nq^3,\ 3Nq^2p,\ 3Nqp^2,\ Np^3.$$

From the foregoing discussions we conclude that the frequencies 0, 1, 2, ... successes are given:

for one event by the binomial expansions of $N(q + p)$

for two events by the binomial expansion of $N(q + p)^2$

for three events by the binomial expansion of $N(q + p)^3$.

In general, for N sets of n-events (trials) the frequencies of 0, 1, 2, 3,... successes are given by the successive terms in the binomial expansion of $N(q + p)^n$, i.e.,

$$N[q^n + nq^{n-1} + \frac{n(n-1)}{1.2}\ q^{n-2}p^2 + \frac{n(n-1)(n-2)}{1.2.3}\ q^{n-3}\ q^3 + ...]$$

This is called the *Binomial Frequency Distribution* or simply *Binomial Distribution* and the quantities n, p(or q) are said to be parameters of Binomial distribution.

Characterisitcs of Binomial Distribution

(1) Its general form depends on parameters p, q and n.

(2) The probability that there are r successes in n independent trials is given by ${}^nC_r p^r q^{n-r}$ and hence in all the N sets, this is given by $N.{}^nC_r p^r q^{n-r}$.

(3) The numerical coefficients of the binomial expansions can be found by Pascal's triangle

Exponent power of $(p + q)$ i.e. n	*Coefficients of terms*											*Sum of coefficients*
1	1	1										2
2	1	2	1									4
3	1	3	3	1								8
4	1	4	6	4	1							16
5	1	5	10	10	5	1						32
...	...	...	...	...	...	...	...	...	...	...	...	...
10	1	10	45	120	210	252	210	120	45	10	1	1024
11	1	11	55	...	...	...	...	...	...	...	...	
etc.												

(4) It is chiefly applied when the population being sampled is infinite so that 'p' remains unchanged by sampling.

(5) It can be applied to finite poplations also if they are not too small.

(6) It is used under the conditions:

(*i*) The variable is discrete.

(*ii*) A dichotomy (i.e., process of classification of collected individuals into two classes according to whether they do or do not possess a particular attribute) exists.

(*iii*) Statistical independence is assumed.

(*iv*) The exponent power n is finite and small.

(*v*) For symmetrical distribution $p = q$ and for asymmetrical $p \neq q$.

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Constants of the Binomial Distribution

Let us take an arbitrary origin at 0 (zero) successes so that the successive deviations are 0, 1, 2, 3,...n.

1. The Mean

We have the mean $= \mu_1'$ (about the origin)

$$= \sum_{r=0}^{n} {}^nC_r . p^r q^{n-r} . r$$

$$= (q.0) + ({}^nC_1 q^{n-1} p.1) + ({}^nC_2 q^{n-2} p^2.2) + ... + (p^n.n)$$

$$= p[nq^{n-1} + n(n-1)\, q^{n-2}\, p + ... + np^{n-1}]$$

$$= np[q^{n-1} + (n-1)\, q^{n-2}\, p + ... + n^{n-1}]$$

$$= np(q + p)^{n-1}$$

$$= np \text{ since } q + p = 1$$

2. The Variance and Standard Deviation

We have, μ_2' (about the origin)

$$= \sum_{0}^{n} {}^nC_r p^r q^{n-r} . r^2$$

$$= (q^n.0) + ({}^nC_1 q^{n-1} p.1^2) + ({}^nC_2 q^{n-2} p^2.2^2) + ... + (p^n.n^2)$$

or $$\mu_2' = np\left[q^{n-1} + 2(n-1)q^{n-2}p + \frac{3(n-1)(n-2)}{2}q^{n-3}p^2 + ... + np^{n-1}\right]$$

$$= np\,[(n-1)\,p + 1]$$

since the bracketed expression is the first moment of $(q + p)^{n-1}$ about origin -1 and hence is equal to $(n-1)\,p + 1$.

As an *alternative*,

$$\mu'_2 = \sum_{0}^{n} {}^nC_r p^r q^{n-r} . r^2 = \sum_{0}^{n} {}^nC_r p^r q^{n-r}\ [r\,(r-1) + r]$$

$$= n(n-1)\,p^2\,\Sigma\,{}^{n-2}C_{r-2}\,p^{r-2}\,q^{n-r} + np\,\Sigma\,{}^{n-1}C_{r-1}\,p^{r-1}\,q^{n-r}$$

$$= n(n-1)\,p^2\,(p + q)^{n-2} + np\,(p + q)^{n-1}$$

$$= n(n-1)\,p^2 + np \text{ as } p + q = 1$$

$$= np\,[(n-1)\,p + 1].$$

$\therefore$ The variance $= \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2$

$$= np\,[(n-1)p + 1] - (np)^2$$

$$= np\,(1 - p)$$

$$= nqp,$$

so that standard deviation $= \sigma = \sqrt{\mu_2} = \sqrt{(nqp)}$.

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3. Third Moments about the Origin and about the Mean

$$\mu_3 = \sum_0^n {}^nC_r p^r q^{n-r} . r^3$$

$$= \sum_0^n {}^nC_r p^r q^{n-r} \ \{r(r-1)(r-2) + 3r(r-1) + r\}$$

$$= n(n-1)(n-2).p^3 \ \Sigma\ {}^{n-3}C_{r-3} p^{r-3} q^{n-r} + 3n(n-1)p^2 \ \Sigma\ {}^{n-2}C_{r-2} p^{r-2} q^{r-r}$$

$$+ np \ \Sigma\ {}^{n-1}C_{r-1} + p^{r-1} q^{n-r}$$

$$= n(n-1)(n-2)p^3 (p+q)^{n-3} + 3n(n-1)p^2 (p+q)^{n-2} + np(p+q)^{n-1}$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np \text{ as } p + q = 1$$

$$= np\{(n-1)(n-2)p^2 + 3(n-1)p + 1\}$$

and $\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$

$$= npq(q-p).$$

4. Fourth Moments about the Origin and about the Mean

$$\mu_4' = \sum_0^n {}^nC_r p^r q^{n-r} . r^4$$

$$= \sum_0^n {}^nC_r p^r q^{n-r} . \{r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 7r(r-1) + r\}$$

$$= np\ \{(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 + 7(n-1)p + 1\}$$

as above

and

$$\mu_4 = \mu_4' - 4\mu'_3\mu_1' + 6\mu_2' \mu_1'^2 - 3\mu_1'^3$$

$$= 3p^2q^2n^2 + pqn(1 - 6qp).$$

Example 4.28: A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of a 5 or a 6 is called a success. In what proportion of the sets would you expect 3 successes.

Solution: Number of faces in a die = 6.

a '5' + a '6' = 2 successes.

$$\therefore\ p = \frac{2}{6} = \frac{1}{3}, \text{ so that } q = 1 - p = \frac{2}{3} \text{ and } n = 8.$$

The binomial distribution is therefore $N\left(\frac{2}{3}+\frac{1}{3}\right)^8$.

Probability of 3 successes in one set of 8 = ${}^8C_3\, p^3q^5$

$$= \frac{8.7.6}{1.2.3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5 = \frac{8\times7\times32}{81\times81}.$$

$$\therefore \text{ Probability of 3 successes in 100 sets} = \frac{8\times7\times32}{81\times81} \times 100$$

$$= \frac{179200}{6561} = 27.31\%.$$

Example 4.29: An irregular six-faced die is thrown and the expectation that in 10 throws it will give 5 even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even numbers?

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Solution: If p is the expectation of getting an even number then the probability of 5 even numbers in 10 throws of a die is twice the probability of 4 even numbers in 10 throws of the die, i.e.,

$$^{10}C_5\, p^5 q^5 \;=\; 2 \times {}^{10}C_4\, p^4 q^6,$$

$$\frac{2}{5}p = q = 1 - p \text{ giving } p = \frac{5}{8} \text{ and } q = \frac{3}{8}.$$

Therefore, frequency for no even number in 10,000 sets of 10 throws

$$= \; 10{,}000.q^{10} = 10{,}000 \left(\frac{3}{8}\right)^{10} = 1 \text{ nearly.}$$

Example 4.30: Show the results of throwing 12 dice, 4096 times, a throw of 4,5 or 6 being called a success.

Find the expected frequencies and compare the actual mean with those of the expected distribution. Calculate the standard deviation.

Solution: Probability of success of falling 4, 5 or 6 = $\frac{3}{6} = \frac{1}{2} = p$ (say).

$$\therefore\; q = 1 - p = \frac{1}{2}.$$

Success	0	1	2	3	4	5	6	7	8	9	10	11	12	
Frequency	–	7	60	198	430	731	948	847	536	257	71	11	–	Total 4,096

Binomial distribution is $4096\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$.

Frequency of 0 successes = $4096\left(\frac{1}{2}\right)^{12} = 1$,

Frequency of 1 success $= 4096\ {}^{12}C_1 q^{11}.p = 12 \times \frac{4096}{2^{12}} = 12.$

Frequency of 2 successes = $4096\ {}^{12}C_2\, q^{10}p^2 = 66$

Frequency of 3 successes = $4096\ {}^{12}C_3 q^9 p^3 = 220$

Frequency of 4 successes = $4096\ {}^{12}C_4 \cdot q^8 p^4 = 495$...etc.

Now, expected mean = $np = 12.\frac{1}{2} = 6$,

$$\sigma \;=\; nqp = 12 \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

$\therefore$ Standard deviation $\sigma \;=\; \sqrt{3} \;=\; 1.732.$

Example 4.31: If a coin is tossed N times, where N is a very large even number, show that the probability of getting exactly $\frac{1}{2}N - p$ heads and $\frac{1}{2}N + p$ tails is approximately

$$\left(\frac{2}{\pi N}\right)^{12} e^{-2p^2/N}$$

NOTES

Solution: Since the coin is unbiased $p = q = \dfrac{1}{2}$.

$$\therefore \text{ Required probability} = {}^{N}C_{\frac{1}{2}\text{N}-p}\; p^{\frac{n}{2}-p}\; q^{\frac{n}{2}+p}$$

$$= \frac{N!}{\left(\frac{1}{2}N-p\right)!\left(\frac{1}{2}N+p\right)!} \cdot \frac{1}{2^N} \text{ as } p = q = \frac{1}{2}$$

Applying *Stirling's theorem*, i.e.,

$$n! = \sqrt{(2\pi n)}\,.n^n e^{-n} \text{ approx.}$$

$$= e^{-n}.\, n^{n+\frac{1}{2}} \sqrt{(2\pi)}\text{ , } n \text{ being large,}$$

We have the required probability

$$= \frac{\sqrt{(2\pi)}.N^{N+\frac{1}{2}}e^{-N}}{\left[\sqrt{(2\pi)}\right]^2\left(\frac{1}{2}N\ -\ p\right)^{\frac{1}{2}N-p+\frac{1}{2}} e^{-\left(\frac{1}{2}N-p\right)}\left(\frac{1}{2}N+p\right)^{\frac{1}{2}N+p+\frac{1}{2}} e^{-\left(\frac{1}{2}N+p\right)}}\cdot\frac{1}{2^N}$$

$$= \frac{1}{\sqrt{(2\pi)}} \cdot \frac{N^{N+\frac{1}{2}}}{\left(\frac{1}{2}N\right)^{N+1}\left(1-\frac{2p}{N}\right)^{\frac{1}{2}N-p+\frac{1}{2}}\left(1+\frac{2p}{N}\right)^{\frac{1}{2}N+p+\frac{1}{2}}}\cdot\frac{1}{2^N}$$

$$= \frac{2}{\sqrt{(2\pi)}N^{\frac{1}{2}}}\left(1+\frac{2p}{N}\right)^{\frac{1}{2}N-p+\frac{1}{2}}\left(1-\frac{2p}{N}\right)^{\frac{1}{2}N+p+1}$$

$$= \left(\frac{2}{\pi N}\right)^{\frac{1}{2}}\left(1-\frac{4p^2}{N^2}\right)^{N/2} \text{ neglecting other terms}$$

$$= \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} e^{-2p^2/N} \text{ approx. when } N \text{ is large, for then}$$

$$\lim_{N\to\infty}\left(1-\frac{4\pi^2}{N^2}\right)^{N/2} = \lim_{N\to\infty}\left(1-\frac{2p^2/N}{\frac{1}{2}N}\right) - e^{-2p^2/\text{N}}$$

Limiting form of the Binomial Distribution when *n* is Large

We know that for large values of n, each term of the binomial becomes small, but we consider the sum of the terms lying within certain ranges. It is observed that as n increases and becomes infinitely large, the curve representing the binomial distribution becomes smooth and continuous and approaches the *normal curve* which is a symmmetrical curve such that the maximum frequencies are clustered around the mean and the deviations below the mean are equal in number and magnitude to the deviations above the mean. Thus normal curve is a standardized and special case of the binomial and the distribution giving this particular curve is said to be the *normal distribution*. We consider it in two cases:

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Case I. *Normal distribution as a limiting case of binomial distribution where $p = q$.*

When $p = q = \frac{1}{2}$, the terms of the series $N\left(\frac{1}{2}+\frac{1}{2}\right)^n$ are

$$N\left(\frac{1}{2}\right)^n \left[1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+...\right]$$

The frequency of m successes is

$$N\left(\frac{1}{2}\right)^n . {}^nC_m, \quad \text{i.e.,} \quad N\left(\frac{1}{2}\right)^n \frac{n!}{m!(n-m!)}$$

and the frequency of $(m + 1)$ success in $N\left(\frac{1}{2}\right)^n \cdot {}^nC_{m+1}$

$$= N\left(\frac{1}{2}\right)^n \frac{n!}{(n+1)!(n-m-1)!}$$

$$= \left\{N\left(\frac{1}{2}\right)^n \frac{n!}{m!(n-m!)}\right\}\frac{n-m}{m+1}$$

$$= \left(\frac{n-m}{m+1}\right) \text{th times the frequency of } m \text{ successes.}$$

So that the frequency of $(m + 1)$ successes is greater than that of m successes if

$$\frac{n-m}{m+1} > 1$$

or if $\qquad n - m > m + 1$

or if $\qquad m < \frac{n-1}{2}$

For the sake of convenience let us suppose that $n = 2k$; then the frequency of k successes say $y_0 = N\left(\frac{1}{2}\right)^{2k} \frac{(2k)!}{k!k!}$ and the frequency of $(k + x)$ successes

$$y_x = N\left(\frac{1}{2}\right)^{2k} \frac{(2k)!}{\lfloor(k+x)\lfloor(k-x)}$$

$$\therefore \quad \frac{y_x}{y_0} = \frac{k!k!}{(k+x)!(k-x)!} = \frac{k!k(k-1)(k-2)...(k-x+1)(k-x)!}{k!(k+1)(k+2)...(k+x).(k-x)!}$$

$$= \frac{k^x\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)...\left(1-\frac{x-1}{k}\right)}{k^x\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)...\left(1+\frac{x-1}{k}\right)\left(1+\frac{x}{k}\right)}$$

or $\quad \log\frac{y_x}{y_0} = \log\left(1-\frac{1}{k}\right) + \log\left(1-\frac{2}{k}\right) + ... \log\left(1-\frac{x-1}{k}\right)$

$$-\log\left(1+\frac{1}{k}\right) - \log\left(1+\frac{2}{k}\right) - ...- \log\left(1+\frac{x-1}{k}\right) - \log\left(1-\frac{x}{k}\right).$$

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Supposing k to be very large as compared with x so that $\left(\frac{x}{k}\right)^2$ may be neglected, expanding the logarithmic functions as $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3}$ ... and $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3}\frac{z^2}{2} - \frac{z^3}{3}$ ... and then neglecting the squares of $\frac{x}{k}$, *i.e.*, $\left(\frac{x}{k}\right)^2$, we find

$$\log\frac{y_x}{y_0} = -\frac{1}{k} - \frac{2}{k} - \frac{3}{k} ... - \frac{x-1}{k} - \frac{1}{k} - \frac{2}{k} - ... - \frac{x-1}{k} - \frac{x}{k}$$

$$= -\frac{2}{k}(1+2+3+...+\overline{x-1}) - \frac{x}{k}$$

$$= -\frac{2}{k}\cdot\frac{x(x-1)}{2} - \frac{x}{k}$$

$$= -\frac{x^2}{k}$$

or $$\frac{y_x}{y_0} = e^{-x^2/k}$$

i.e., $$y_x = y_0\, e^{-x^2/k}$$

But for a binomial distribution with $p = q = \frac{1}{2}$, $n = 2k$, we have

$$\sigma^2 = npq = 2k.\ \frac{1}{2}.\ \frac{1}{2} = \frac{1}{2}k, \quad \text{i.e.,} \quad k = 2\sigma^2$$

Hence $$y_x = y_0 e^{-\left(x^2/2\sigma^2\right)}$$

which gives the normal distribution and the equation $y = y_0 e^{-x^2/2\sigma^2}$ *represents normal curve.*

Case II. *Normal distribution as a limiting case of binomial distribution $p \neq q$ but $p \approx q$.*

The frequency of m successes is

$$N.{}^nC_m q^{n-m}p^m = N.\frac{n!}{m!(n-m)!}\, q^{n-m}p^m$$

and the frequency of $(m+1)$ successes is $N.{}^NC_{m+1}\, q^{n-m-1}\, p^{m+1}$

$$= \frac{n!}{(m+1)!\,(n-m-1)!}\, q^{n-m-1}p^{m+1}$$

$$= \left\{N.\frac{n!}{m!(n-m)!}q^{n-m}p^m\right\}.\ \frac{n-m}{m+1}\cdot\frac{p}{q}$$

$$= \left(\frac{n-m}{m+1}.\frac{p}{q}\right) \text{th times the frequency of } m \text{ successes.}$$

so that the frequency of $(m + 1)$ successes is greater than that of m successes if

NOTES

$\frac{n-m}{m+1}.\frac{p}{q} > 1$ or if $(n - m)\, p > (m + 1)q$, i.e., if $m < n\,p - q$.

Assuming that np is a whole number, since there is no loss of generality as n ultimately tends to infinity, the frequency of np successes may be taken as the maximum frequency.

The frequency of np successes (say)

$$y_0 = N.\frac{n!}{np!(n-np)!}\,.\,q^{n-np}\,p^{np}$$

$$= N.\ \frac{n!}{np!nq!}\ q^{nq}\,p^{np} \text{ as } 1 - p = q$$

and the frequency of $(np + x)$ succcesses (say)

$$y_x = N.\frac{n!}{(np+x)!(np-x)!}\,q^{nq-x}\,p^{np+x}$$

$$= N.\frac{n!}{np!\,nq!}\,q^{nq}\,p^{np} \text{ as } 1 - p = q$$

$$\therefore \qquad \frac{y_x}{y_0} = \frac{(np)!\,(nq)!}{(np+x)!(nq-x)!}\ q^{-x}p^{x}.$$

Applying the Stirling's theorem that when n is large

$$n! = \sqrt{(2\pi n)}\ n^n\,e^{-n} = e^{-n}n^{n+1/2}\sqrt{(2\pi)} \text{ approx.}$$

we have

$$\frac{y_x}{y_0} = \frac{e^{-np}\,(np)^{np+1/2}\sqrt{(2\pi)}\times e^{-nq}\,(nq)^{nq+1/2}\sqrt{(2\pi)}}{e^{-(np+x)}\,(np+x)^{np+x+1/2}\sqrt{(2\pi)}\times e^{-(nq-x)}\,(nq-x)^{nq-x+1/2}\sqrt{(2\pi)}}.\left(\frac{np}{nq}\right)^x$$

$$= \frac{(np)^{np+1/2}\,(nq)^{nq+1/2}}{(np)^{np+x+1/2}\left\{1+\frac{x}{np}\right\}^{np+x+1/2}\times\left\{1-\frac{x}{nq}\right\}^{nq-x+1/2}}\left(\frac{np}{nq}\right)^x$$

$$= \frac{1}{\left(1+\frac{x}{np}\right)^{np+x+1/2}\left(1-\frac{x}{nq}\right)^{nq-x+1/2}}$$

$$\text{or } \log\frac{y_x}{y_0} = -(np + x + \tfrac{1}{2})\log\left(1+\frac{x}{np}\right) - (nq - x + \tfrac{1}{2})\log\left(1-\frac{x}{nq}\right)$$

$$= \left(np+x+\frac{1}{2}\right)\left[\frac{x}{np}-\frac{x^2}{2n^2p^2}+\ldots\right] + \left(nq-x+\frac{1}{2}\right)\left[\frac{x}{nq}+\frac{x^2}{2n^2q^2}+\ldots\right]$$

$$= -x+\frac{x^2}{2np}-\frac{x^2}{np}-\frac{x}{2np}+x+\frac{x}{2nq}-\frac{x^2}{nq}+\frac{x}{2nq}$$

neglecting the terms containing $\frac{1}{n^2}$

NOTES

$$= \frac{x^2}{n}\left\{\frac{1}{2p}-\frac{1}{p}+\frac{1}{2q}-\frac{1}{q}\right\}-\frac{x}{2n}\left\{\frac{1}{p}-\frac{1}{q}\right\}$$

$$= \frac{x^2(p+q)}{2npq}-\frac{x(q-p)}{2npq} = -\frac{x^2}{2npq}-\frac{(q-p)}{2npq}x \quad \text{as} \quad p+q=1$$

Since q and p both are less than 1 and very nearly equal, therefore $\frac{q-p}{2npq}$ can be neglected and then we are left with

$$\log\frac{y_x}{y_0} = -\frac{x^2}{2npq} = -\frac{x^2}{2\sigma^2} \quad \text{as } \sigma^2 = npq$$

or $$\frac{y_x}{y_0} = e^{-x^2/2\sigma^2}$$

i.e., $$y_x = y_0\, e^{-x^2/2\sigma^2}$$

Note: The curve given by $y = y_0\, e^{-x^2/2\sigma^2}$ is said to be the normal curve.

A **normal curve** *is symmetrical about the point x = 0 where the ordinate has its maximum value. In a normal curve, the mean, the median and mode coincide.*

4.4.2 The Normal Distribution

The normal distribution is a probability distribution or specifically a continuous probability distribution. It is also termed as Gaussian or Gauss distribution because it was discovered by Carl Friedrich Gauss. The standard normal distribution, also known as the Z distribution, is the normal distribution and is often called the 'Bell Curve' because the shape of the graph of its probability density looks like a bell.

Normal distributions are a family of distributions of the same general form. These distributions differ in their location and scale parameters, the mean 'average' of the distribution defines its location while the standard deviation 'variability' defines the scale. Typically, the normal distributions are important in statistics and are often used to represent real-valued random variables whose distributions are not known. A random variable with a Gaussian distribution is said to be normally distributed and is called a normal deviate.

We have just introduced that the equation to the normal curve is

$$y = y_0\, e^{-x^2/2\sigma^2} \qquad \text{... (4.8)}$$

Its area $$= \int_{-\infty}^{\infty} y_0\, e^{-x^2/2\sigma^2}\, dx$$

$$= 2y_0 \int_0^{\infty} e^{-x^2/2\sigma^2}\, dx, \text{ being symmetrical}$$

$$= 2y_0\cdot\frac{\sqrt{\pi}}{2\cdot\frac{1}{\sqrt{2}\sigma}} \quad \text{as} \int_0^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a} \text{ if } a > 0$$

$$= \sigma y_0\,\sqrt{(2\pi)} \qquad \text{... (4.9)}$$

$$= 2.506627\, y_0\sigma \quad \text{by putting the value of } \sqrt{(2\pi)}.$$

NOTES

In order to make Equation (4.8), the normal probability curve, the value of y_0 be so determined that the total frequency is one, i.e., the area of the normal curve is unity and this will be so if from Equation (4.9).

$$\sigma y_0 \sqrt{(2\pi)} = 1$$

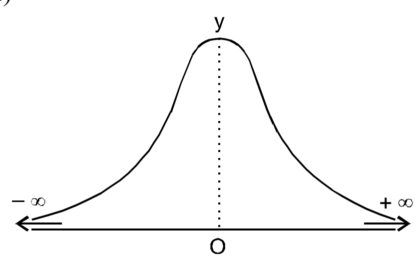


Fig. (b)

i.e.,
$$y_0 = \frac{1}{\sigma\sqrt{(2\pi)}} \quad \text{... (4.10)}$$

Substituting this value of y_0 in Equation (4.8), the standard form of the normal curve becomes

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/2\sigma^2} \quad \text{... (4.11)}$$

which is the normal distribution.

In deriving the Form Equation (4.8) of normal curve, we have taken the mean at the origin, but if however we take another point as the origin such that the excess of the mean over the arbitrary origins is m, the form of the normal curve is

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-m)^2/2\sigma^2} \quad \text{... (4.12)}$$

which represents the standard form of the normal curve with origin at $(m, 0)$.

Physical Conditions Leading a Normal Curve

(*i*) The casual forces that affect individual events are mutually independent.

(*ii*) The casual forces of equal magnitude are very large in number.

(*iii*) The casual forces operate in such a way that the maximum frequencies are clustered around the mean value thereby giving a symmetrical curve. Conclusively the deviations below the mean are equal in number and magnitude to those above the mean.

(*iv*) The normal distribution can be used as an error distribution by inquiring what law of distribution errors of observation should obey in order to make the mean of a set of measurements the most probable value of the 'true' magnitude.

Hence according to Gauss, if we call the '**Precision**' h, such that $h^2 = \frac{1}{2\sigma^2}$, the form Result (4.10) becomes

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2} \quad \text{... (4.13)}$$

It is clear form Result (4.13) that as h increases, the normal curve would become narrower and as such h is a measure of closeness of the mass of observations to the true value.

NOTES

Definition of a Normal Distribution: *A normal distribution is a continuous distribution given by*

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}\left[(x-m)/\sigma\right]^2}$$

where x is a continuous normal variate distributed with probability density function $f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}\left[(x-m)/\sigma\right]^2}$, *with mean m and standard deviation* σ.

Properties of a Normal Distribution

(1) When $p = q$ or $p \approx q$ (i.e., p is very nearly equal to q), the distribution fitted is symmetrical. But a normal curve is distinguished from the other symmetrical curves in a markable point that a normal curve is symmetrical not only with regard to skewness as are all symmetrical curves; but it is also symmetrical with regard to peakedness (i.e., kurtosis).

(2) The normal curve is a mathematical abstraction, not found in practical work, but used to describe the form of distribution that would be obtained by some continuous data in very large numbers.

(3) The normal curves are based upon regular variation and uniformity conjoined so that no single force playing on each item of the distribution is dominating.

(4) In a normal distribution, items differing from the mean (or median or mode which coincide in a symmetrical distribution) by the same amount in either direction occur with the same frequency. As such above and below the mean at equal distances, there are same number of measurements.

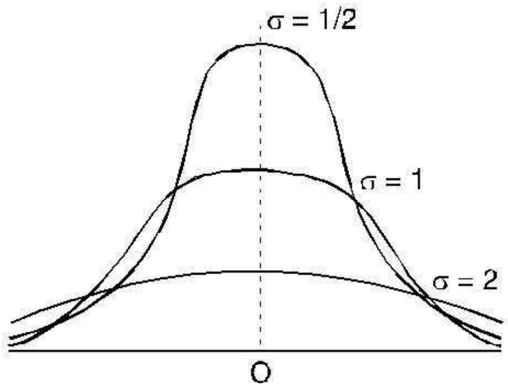


Fig. 4.1 *Normal Distribution Graph*

(5) Normal curve is a single peaked.

(6) Normal curve is asymptotic to the horizontal base as y decreases rapidly when x increases numerically.

(7) The mean, median and mode coincide and lower and upper quartiles are equidistant from the median.

(8) The curve can be completely specified by mean (i.e. origin of x) and the standard deviation along with the value of y_0 found as in Equation (4.13).

(9) The points of inflexion of the normal curve are obtained by putting $\frac{d^2y}{dx^2} = 0$ provided $\frac{d^3y}{dx^3} \neq 0$ for these points.

These are found to be

$$x = \pm \sigma,\ y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-1/2}.$$

NOTES

Note. The points of inflexion are the points where the curvature changes its direction.

Constants of Normal Distribution

1. Mean

Mean = μ_1' (about the origin)

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} x.e^{-\left[(x-m)^2/2\sigma^2\right]} dx \quad \text{Put } \frac{x-m}{\sqrt{2}\sigma} = z,\ i.e.,\ dx = \sqrt{2}\sigma\ dz$$

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left(m + \sqrt{2}.\sigma z\right) e^{-z^2} dz.\ \sqrt{2}\sigma$$

$$= \frac{2m}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\ dz,$$ the second integral vanishes, being an odd function of z

$$= \frac{2m}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = m.$$

2. Standard Deviation and Variance

We have μ_2 (about the origin)

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} x^2.e^{-[(x-m)^2/2\sigma^2]} dx$$

or $\mu_2' = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left(m + \sqrt{2}\sigma z\right)^2 e^{-z^2} \sqrt{2}\sigma\, dz$ Put $\frac{x-m}{\sqrt{2}\sigma} = z$, i.e., dx $= \sqrt{2}\sigma\, dz$

$$= \frac{m^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz + \frac{2\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} z.e^{-z^2} dz + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2.e^{-z^2} dz$$

$$= \frac{2m^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz + 0 + \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z^2 e^{-z^2} dz$$

the second integral vanishes by the property of definite integrals

$$= \frac{2m^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z^2 e^{-z^2} dz$$

$$= m^2 - \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z(-2z) e^{-z^2} dz$$

$$= m^2 - \frac{2\sigma^2}{\sqrt{\pi}} \left[\left(ze^{-z^2}\right)_0^{\infty} - \int_0^{\infty} e^{-z^2} dz\right]$$ integrating by parts

$$= m^2 + \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = m^2 + \sigma^2.$$

$\therefore$ The variance $\mu_2 = \mu_2' - \mu_1'^2$

$$= m^2 + \sigma^2 - m^2 = \sigma^2$$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sigma$.

3. Mean Deviation from the Mean

The Mean deviation about the mean

NOTES

$$m = \int_{-\infty}^{\infty} |x-m| \cdot \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx$$

$$= \frac{\sigma\sqrt{2}}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |\sqrt{2}\sigma \cdot z| \cdot e^{-z^2} dz \qquad \text{Put } \frac{x-m}{\sqrt{2}\sigma} = z$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^{0} -ze^{-z^2} dz + \int_{0}^{\infty} ze^{-z^2} dz\right], \quad i.e., \quad dx = \sqrt{2}\sigma\, dz$$

$$= \sigma\sqrt{\frac{2}{\pi}} \left\{\left[\frac{e^{-z^2}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-z^2}}{-2}\right]_{0}^{\infty}\right\}$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left\{\frac{1}{2}+\frac{1}{2}\right\} = \sigma\sqrt{\frac{2}{\pi}} = 0.7979\sigma = \frac{4}{5}\sigma \text{ approx.}$$

4. Moments about the Mean

Let us first consider the odd moments about the mean

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-m)^{2n+1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{(-x-m)^2 2\sigma^2}\, dx.$$

$$\text{Put} \quad \frac{x-m}{\sqrt{2}\sigma} = z, \text{ i.e., } dx = \sqrt{2}\sigma\ dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2\sigma}\, z)^{2n+1} e^{-z^2} dz$$

= 0 being an odd function of z, by the properties of definite integral.

$\therefore \quad \mu_3 = \mu_5 = \mu_7 = ... = 0$

i.e., all odd moments about the mean are zero.

Let us now consider the even moments about the mean

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^{2n} \cdot e^{-(x-m)^2/\sqrt{2}\sigma} dx \qquad \text{Put } \frac{x-m}{\sqrt{2}\sigma} = z,\ dx = \sqrt{2}\sigma\, dz$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^{2n}.e^{-z^2} dz$$

$$= \frac{2^{n+1}.\sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2 z^{2n-1}}\ z\, dz \qquad \text{Put } z^2 = t,\ 2z\, dz = dt$$

$$= \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{(2n-1)/2} dt = \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{(n+1/2)-1} dt$$

$$= \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right) \qquad \text{by the definition of Gamma integral.}$$

In particular, $\mu_k = \dfrac{2^{k/2}.\sigma^k}{\sqrt{\pi}}\Gamma\left(\dfrac{k+1}{2}\right)$ when $k = 2n$

NOTES

Now we have $\mu_2 = \frac{2\sigma^2}{\sqrt{\pi}}\Gamma\frac{3}{2} = \frac{2\sigma^2}{\sqrt{\pi}}.\frac{1}{2}\sqrt{\pi} = \sigma^2$

$$\mu_4 = \frac{2^2\sigma^4}{\sqrt{\pi}}\Gamma\frac{5}{2} = -\frac{4\sigma^4}{\sqrt{\pi}}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = 3\sigma^4.$$

5. The Normal Probability Integral or Error Functions

It has been shown that the total area of the normal curve being unity, it is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

or putting $\frac{1}{2\sigma^2} = h^2$, this becomes

$$y = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}$$

where h is known as *percision*.

Thus the probability that a deviation lies between x and $-x$ is

$$P = \frac{h}{\sqrt{\pi}}\int_{-x}^{x} e^{-h^2x^2}dx$$

$$= \frac{2h}{\sqrt{\pi}}\int_{0}^{x} e^{-h^2x^2}dx$$

$$= \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-h^2x^2}h\,dx$$

or $$\phi(hx) = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-h^2x^2}h\,dx$$

so that $$\phi(y) = \frac{2}{\sqrt{\pi}}\int_{0}^{y} e^{-y^2}\,dy$$

which is known as *error function* or the *probability integral*.

Note. The Probable error λ or quartile deviation is defined to be the error such that the chance of an error lying within the limits $m - \lambda$ and $m + \lambda$ is exactly the same as the chance of an error lying outside these limits, i.e.,

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{m-\lambda}^{m+\lambda} e^{-\frac{1}{2}\left[(x-m)/\sigma\right]^2}dx = \frac{1}{2}.\text{ Put } \frac{x-m}{\sigma} = z$$

or $$\frac{1}{\sqrt{2\pi}}\int_{0}^{\lambda/\sigma} e^{-z^2/2}dz = \frac{1}{4}$$

whence from table $\frac{\lambda}{\sigma} = 0.6745$, i.e., $\lambda = 0.6745\sigma = \frac{2}{3}$approx.

$$\therefore \quad Q_1 = m - \frac{2}{3}\sigma \quad \text{and} \quad Q_2 = m + \frac{2}{3}\sigma.$$

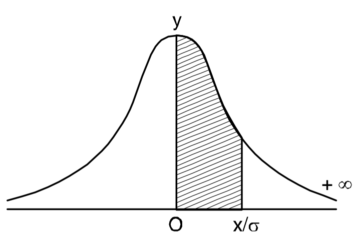


Fig. 4.2 *Normal Probability Interal or Error Function Graph*

NOTES

Table for Ordinates: Normal curve is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

So the tables are to be prepared to give the values of $\frac{1}{\sqrt{(2\pi)}} e^{-x^3/2\sigma^2}$, the origin of x being taken at the origin whence division of these values by σ will yield the ordinates y as required by normal curve equation.

Table 4.1 *Area Under the Normal Curve*

x/σ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	0.319	0.359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1960	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2966	.2956	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4329	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4626	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4836	.4850	.4854	.5857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.3932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.49865	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.49903	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.499313	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.499517	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.499663	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.499767	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.499841	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.499892	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.499928	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.499952	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4.0	.499968	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

NOTES

Table 4.2 *Ordinates of the Standard Normal Curve*

$z = x/\sigma$	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	.3989	.3989	.3989	.3988	.3986	.3984	.3982	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	.3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3772	.3555	.3538
0.5	.3521	.3503	.3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	.2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	.2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1965
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	.1758	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1354	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0669
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	.0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	.0396	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.1054	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.0088	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	.0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	.0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0011	.0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

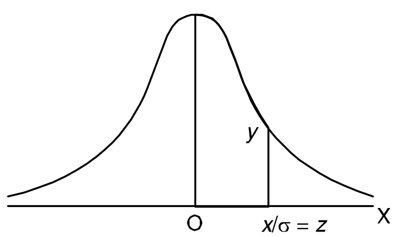


Fig. 4.3 *Standard Normal Curve*

NOTES

Proportions of Items Included Within ± σ, ± 2σ, ± 3σ of the Mean in Normal Curve: The total area of a normal curve being treated as unity, the probability corresponding to any interval in the range of the variate is measured by the area under the curve within that interval given by Table 4.1. Hence if *m* is the mean of the normal distribution, then *P*, the probability from *m* to any value *x* of the variate is given by

$$P = \frac{1}{\sigma\sqrt{(2\pi)}} \int_m^x e^{-(x-m)^2/2\sigma^2}\, dx \quad \text{Put } \frac{x-m}{\sigma} = z, \text{ i.e., } dx = \sigma\, dz$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_0^x e^{-z^2/2} dz$$

This value *P* is known as the *Probability integral* or *Error function*. Thus

$$P[m - \sigma < x < m + \sigma] = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{m-\sigma}^{m+\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx$$

$$\text{Put } \frac{x-m}{\sigma} = z,\ dx = \sigma\, dz$$

$$\therefore \quad P[-1 < z < 1] = \frac{1}{\sqrt{(2\pi)}} \int_{-1}^{1} e^{-z^2/2} dz$$

$$= \frac{2}{\sqrt{(2\pi)}} \int_0^1 e^{-z^2/2} dz$$

$$= 2 \times .3413 \qquad \text{since from table 1, for } z = 1,$$

$$\frac{1}{\sqrt{(2\pi)}} \int_0^1 e^{-z^2/2} dz = .3413$$

$$= .6826$$

which follows that 68.26% of the items in the normal distribution fall between the range ± σ of the mean.

Now

$$P[m - 2\sigma < x < m + 2\sigma]$$

$$= \frac{1}{\sigma\sqrt{(2\pi)}} \int_{m-2\sigma}^{m+2\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2} \quad \text{Put } \frac{x-m}{\sigma} = z, \text{ i.e., } dx = \sigma\, dz$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-2}^{2} e^{-z^2/2} dz$$

$$= \frac{2}{\sqrt{(2\pi)}} \int_0^2 e^{-z^2/2} dz$$

$$= 2 \times .4772 \text{ from Table 4.1 for } z = 2$$

$$= .9544.$$

So that 95.44% of the items in the normal distribution fall within the range ± 2σ of the mean.

Again

$$P[m - 3\sigma < x < m + 3\sigma] = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{m-3\sigma}^{m+3\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx$$

NOTES

Put $\frac{x-m}{\sigma} = z, \quad \therefore\ dx = \sigma\, dz$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-3}^{3} e^{-z^2/2}dx$$

$$= \frac{2}{\sqrt{(2\pi)}}\int_{0}^{3} e^{-z^2/2}dz$$

$$= 2 \times .49865$$

$$= .99720,$$

i.e., 99.72% of the items in the normal distribution fall within the range $\pm\, 3\sigma$ of the mean.

Conclusively, a normal curve can be used to find:

(*i*) The number of cases at any given distance from the mean

(*ii*) The number of cases lying within certain range of values in the distribution;

(*iii*) The probability that a case selected at random lies above or below a given point.

Example 4.32: If n is large and neither of p and q is too close to zero, then show that the binomial distribution can be closely approximated by a normal distribution with standarized variable given by $z = \frac{x-np}{\sqrt{(npq)}}$, where x is the binomial variate with mean np and standard deviation $\sqrt{(npq)}$.

Solution: From the given properties of x, we conclude that z is a variate with mean zero and variance unity. Also as x goes from 0 to n, z takes values $\frac{-np}{\sqrt{(npq)}}$ to $\frac{nq}{\sqrt{(npq)}}$ so that the total range of z is $\frac{nq}{\sqrt{(npq)}} - \left(\frac{-nq}{\sqrt{(npq)}}\right) = \left(\frac{n(p+q)}{\sqrt{(npq)}}\right) = \frac{n}{\sqrt{(npq)}}$, *i.e.* z jumps $\frac{1}{\sqrt{(npq)}}$ at each stage.

Now $\quad \frac{1}{\sqrt{(npq)}} \to 0$ as $\to \infty$

$$\frac{nq}{\sqrt{(npq)}} \to +\infty \text{ as } n \to \infty,\ p,\ q \neq 0$$

$$\frac{-nq}{\sqrt{(npq)}} \to -\infty \text{ as } n \to \infty,\ p,\ q \neq 0$$

As such the distribution of z is a continuous distribution from $-\infty$ to $+\infty$, with mean zero and variance unity.

If $P(n, x)$ denotes the probability for the variate taking the value x, then

$$P(n,x) = {}^nC_x p^x q^{n-x}$$

$$= \frac{n!}{x!(n-x)!}p^x q^{n-x}$$

NOTES

Applying Stirling's theorem, i.e., $n! = \sqrt{(2\pi)}.e^{-n}n^{n+1/2}$, we get

$$P_n = \lim_{n\to\infty} P(n,\ x) = \lim_{n\to\infty} \frac{\sqrt{(2\pi)}.e^{-n}.n^{n+1/2}.p^x\,p^{n-x}}{\sqrt{(2\pi)}.e^{-x}.x^{x+1/2}.\sqrt{(2\pi)}.e^{-(n-x)}\,(n-x)^{n-x+1/2}}$$

$$= \lim_{n\to\infty}\left(\frac{1}{B}.\frac{1}{\sqrt{(2npq)}}\right)$$

where $$B = \left(\frac{x}{np}\right)^{x+1/2}\left(\frac{n-x}{nq}\right)^{n-x+1/2} \qquad (1)$$

Now given substitution is $z = \dfrac{x-np}{\sqrt{(npq)}}$

or $$x = np + z\sqrt{(npq)}$$

i.e., $$\left.\begin{aligned} \frac{x}{np} = 1 + z\sqrt{\left(\frac{q}{np}\right)} \quad \text{and} \quad \frac{n-x}{nq} &= \frac{1}{nq}[n - np - z\sqrt{(npq)}] \\ &= \frac{1}{np}\left[nq - z\sqrt{(npq)}\right] \\ &= 1 - z\sqrt{\left(\frac{p}{nq}\right)} \end{aligned}\right\} \qquad (2)$$

Taking logarithms of both sides of Equation (1), we have

$$\log B = \left(x+\frac{1}{2}\right)\log\frac{x}{np} + \left(n-x+\frac{1}{2}\right)\log\frac{n-x}{nq}.$$

Making substitution from Equation (2), this becomes

$$\log B = \left[np + z\sqrt{(npq)}\right]\log\left[1+z\sqrt{\left(\frac{q}{np}\right)}\right] + \left[nq - z\sqrt{(npq)}\right]\log\left[1-z\sqrt{\left(\frac{p}{nq}\right)}\right]$$

$$= \left[np + z\sqrt{(npq)}\right]\left\{z\sqrt{\left(\frac{q}{np}\right)} - \frac{1}{2}\frac{z^2q}{np} + \ldots\right\} + \left[nq - z\sqrt{(npq)}\right]\left\{-z\sqrt{\left(\frac{p}{nq}\right)} - \frac{1}{2}\frac{z^2p}{np} + \ldots\right\}$$

$$= \frac{z}{2\sqrt{n}}\left(\sqrt{\frac{q}{p}} - \sqrt{\frac{p}{q}}\right) + \frac{z^2}{2} - \frac{z^2}{4n}\left(\frac{q}{p}+\frac{p}{q}\right) + \text{terms containing higher power of } (1/n)$$

when $n \to \infty$, $\log B \to \dfrac{z^2}{2}$, i.e., $B \to e^{+z^2/2}$

Now as x takes integral values, z jumps through $\dfrac{1}{\sqrt{(npq)}}$, i.e., $(npq)^{-1/2}$ so that increment in z, i.e., $dz = (npq)^{-1/2}$ when $n \to \infty$. Thus, if dP represents the probability for the variate z to lie within the range $z - \dfrac{1}{2}\,dz$ to $z + \dfrac{1}{2}\,dz$, then

$$dP = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} dz, \; -\infty \le z \le \infty$$

which is the required normal distribution for z.

Hence $$f(z) = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} \text{ as } dP = f(z)\, dz.$$

NOTES

If m be the mean and σ the standard deviation of the normal distribution, then we can replace z by $\frac{x-m}{\sigma}$ and dz by $\frac{1}{\sigma}\, dx$, whence we have

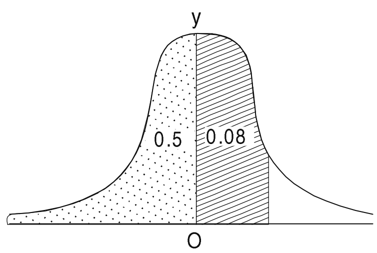
$$dP = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx$$

giving $$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

so that $y = f(x)$, i.e., $y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$

gives the normal probability curve.

Example 4.33: If skulls are classified *A, B, C* according as the length, breadth idex as under 75, between 75 and 80 or over 80 find approximately (assuming that the distribution is normal) the mean and standard deviation of series in which *A* are 58 per cent, *B* are 38 per cent and *C* are 4 per cent being given that if



$$f(t) = \frac{1}{\sqrt{(2\pi)}} \int_0^t e^{-(x^2/2)} dx,$$

then $f(0.20) = 0.08$ and $f(1.75) = 0.46$.

Solution: Let m be the mean and σ the standard deviation for the given distribution.

As given the area between $t = 0$ and $t = 0.20$ is 0.08, so that the area to the left of this ordinate is $0.5 + 0.08 = 0.58$ which corresponds to $x = 75$.

$$\therefore \quad \frac{75-m}{\sigma} = 0.20$$

Also the area to the right of the ordinate at $x = 80$ is 0.04, so that the area to the left of this ordinate is $1 - 0.04 = 0.96$, i.e., the area from 0 to $t = \frac{80-m}{\sigma}$ is $0.96 - 0.5 = 0.46$, which corresponds to $t = 1.75$.

$$\therefore \quad \frac{80-m}{\sigma} = 1.75.$$

are $0.20\sigma = 75 - m$, $1.75\sigma = 80 - m$.

Subtracting $1.55\sigma = 5$, i.e. $\sigma = \frac{5}{1.55.} = 3.2$ approx.

and then $$m = 75 - 0.20 \times 3.2$$
$$= 75 - 0.64 = 74.4 \text{ approx.}$$

NOTES

Example 4.34: In a normal distribution 31 per cent of the items are under 45 and 48 per cent are over 64. Find the mean and standard deviation of the distribution.

Solution: Let *m* be the mean and s the standard deviation of the distribution. Since 31 per cent of the items are under 45; therefore 19 per cent of the items lie between 45 and *m*.

Thus if $$\phi(t) = \frac{1}{\sqrt{(2\pi)}} e^{-t^2/2} \qquad \text{where } t = \frac{x-m}{\sigma},$$

then $$\int_{(45-m)\sigma}^{0} \phi(t)\,dt = .19, \text{ i.e.} \int_{0}^{(m-45)/\sigma} \phi(t)\,dt = .19.$$

From Table I, we have $\int_0^{0.5} \phi(t)\,dt = .1915$

So $$\frac{m-45}{\sigma} = .5 \text{ approx} \qquad \text{... (1)}$$

Again since 8 per cent of the items are over 64, therefore 42 per cent lie between *m* and 64, so that $\int_0^{(64-m)\sigma} \phi(t)\,dt = .42 = \int_0^{1.4} \phi(t)\,dt$ nearly from Table I.

$$\therefore \qquad \frac{64-m}{\sigma} = 1.4. \qquad \text{... (2)}$$

Solving Equations (1) and (2), we get $m = 50$, $\sigma = 10$.

Example 4.35: If *x* is a normal variate with mean *x* and *S.D.* σ*,* find the mean and variance of variate *y* defined by

$$y = \frac{1}{2}\left(\frac{x-\bar{x}}{2}\right)^2.$$

Solution: For the normal variate *x*, we have

$$f(x)\ dx = \frac{1}{\sigma\sqrt{(2\pi)}}\ e^{-\frac{1}{2}\{(x-\bar{x})/\sigma\}^2}\,dx. \qquad \text{... (1)}$$

If $y = \frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2$, i.e. $dy = \frac{x-\bar{x}}{\sigma}.\ \frac{dx}{\sigma}$ or $dx = \frac{\sigma}{\sqrt{(2y)}}\ dy,$

then Equation (1) becomes

$$f(y)\ dy = \frac{1}{\sigma\sqrt{(2\pi)}}\ e^{-y}.\ \frac{\sigma}{\sqrt{(2y)}}\ dy = \frac{1}{2\sqrt{\pi}}\ y^{-1/2}\ e^{-y}\,dy,$$

so that

mean = μ_1' (about the origin), for a normal distribution

$$= 2\int_0^{\infty} y f(y)\,dy = \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{1/2} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} \cdot \Gamma\frac{3}{2} \text{ (by gamma integrals)} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi} = \frac{1}{2}$$

NOTES

And $$\mu'_2 = 2\int_0^\infty y^2 f(y)\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty y^{3/2} e^{-y}\,dy = \frac{1}{\sqrt{\pi}} \Gamma\frac{5}{2} = \frac{1}{\sqrt{\pi}} \cdot \frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{3}{4}.$$

$\therefore$ $$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \text{variance.}$$

Example 4.36: If a normal distribution is grouped in intervals of total frequency N and S is the sum of squares of the frequencies, an estimate of the standard deviation σ is given by $\frac{N^2}{2S\sqrt{\pi}}$.

Solution: The normal frequency distribution with mean m and standard deviation σ is given by

$$dF = f(x)\ dx = y\ dx = \frac{N}{\sigma\sqrt{(2\pi)}}\ e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

where the ordinate of the normal curve is

$$y = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

$\therefore$ Square of the ordinate is

$$y^2 = \frac{N^2}{\sigma^2 s^2 .2\pi} e^{-[(x-m)/\sigma]^2},$$

As such the distribution function for y^2 is

$$dP = y^2 dx = \frac{N^2}{2\pi\sigma^2} .e^{-[(x-m)/\sigma]^2}\ dx$$

$$= \frac{N^2}{2\sigma\sqrt{\pi}} \cdot \frac{1}{\left(\frac{\sigma}{\sqrt{2}}\right)\sqrt{(2\pi)}} \cdot e^{-\frac{1}{2}(x-m)^2/(\sigma/\sqrt{2})^2}\ dx,$$

which gives a normal distribution with S.D. $\sigma/\sqrt{2}$ and total frequency $N^2/(2\sigma/\sqrt{\pi})$.

Hence taking S as equivalent to this value, we have

$$S = \frac{N^2}{2\sigma\sqrt{\pi}}, \text{ i.e., } \sigma = \frac{N^2}{2S\sqrt{\pi}}.$$

Example 4.37: The local authorities in a certain city instal 200 lamps in the streets of the city. If the lamps have an average life of 1000 burning hours with a standard deviation of 200 hours, (i) what number of lamps might be expected to fail in first 700 burning hours, and (ii) after what period of burning hours would we expect that 10 per cent of the lamps would have failed? Assume that the lives of the lamps are normally distributed. You are given that

$$F(1.50) = .933,\ F(1.28) = .900 \text{ where } F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} dz$$

NOTES

(*i*) We have the average life of the lamp, i.e., $m = 1000$ hours and standard deviation $\sigma = 200$ hours.

Solution: (*i*) The normal distribution being symmetrical, the area of the normal curve to the left of $x = 700$ is equal to that to the right of $x = (2000 - 700) = 1300$.

$$\therefore \qquad z = \frac{x-m}{\sigma} = \frac{1300-1000}{200} = 1.50$$

and area to the left of $x = 1300$ is $= \dfrac{1}{\sqrt{(2\pi)}} \displaystyle\int_{-\infty}^{z} e^{-z^2/2} dz = F(z)$

where $z = 1.50$ and also $F(1.50) = 0.933$ which gives the probability of a value of x lying to the left of $x = 1300$ or right of $x = 700$.

$\therefore$ probability of its failure = 1 – .933 = 067.

So the number of lamps expected to fail in first 700 hours

$$= 2000 \times .067 = 134.$$

(*ii*) The failure of 10 per cent of lamps gives that a value of x is to be so found that the area of the standard normal curve to the left of it is $\frac{10}{100}$, i.e., 0.1

But the area to the left of $z = 1.28$ is .900, hence by symmetry in the distribution of a normal curve it follows that the area to the left of $z = -1.28$ is equals to the area to the right of $z = 1.28$, i.e., 0.1.

Solution: (*ii*)

$$\text{Hence} \qquad z = -1.28 = \frac{x-m}{\sigma} = \frac{x-1000}{200}$$

giving $x = 1000 - 256 = 744$.

Hence after 744 hours, it is expected that 10 per cent of the lamps fail.

4.4.3 The Poisson Distribution

We have shown in the preceding sections that the normal curve is the limit to the binomial, whether p is or is not equal to q provided that n becomes sufficiently large in order to make $(q - p)$ negligibly small as compared to $\sqrt{(npq)}$. Now we have to consider the limit to the same series if p (or q) becomes sufficiently small and n is increased sufficiently to keep the product np (or nq) a finite number say m, i.e., $np = m$.

The probability of r successes in the binomial distribution is given by the $(r + 1)$th term in the binomial expansion $(q + p)^n$, i.e.

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)} p^r q^{n-r}$$

$$= \frac{n!}{r!(n-r)} \left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r}$$

$$\left(\because\ np = m, \text{ i.e., } p = \frac{m}{n} \text{ and so } q = 1 - p = 1 - \frac{m}{n} \text{ as } p + q = 1\right)$$

NOTES

$$= \frac{m^r}{r!}\left(1-\frac{m}{n}\right) \cdot \frac{n}{(n-r)!\cdot n^n\left(1-\frac{m}{n}\right)^r}$$

Here $\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty}\left\{\left(1-\frac{m}{n}\right)^{-n/m}\right\}^{-m} = e^{-m}$

and applying Stirling's approximation, i.e., $n! = n^n e^{-n}\sqrt{(2\pi n)}$, when n is large,

$$P(r) = \frac{m^r e^{-m}}{r!} \cdot \frac{n^n e^{-n}\sqrt{(2\pi n)}}{(n-r)^{n-r}e^{-(n-r)}\sqrt{\{2\pi(n-r)\}\times n^r\left(1-\frac{m}{n}\right)^r}}$$

$$= \frac{m^r e^{-m} e^{-r}}{r!}\left(1-\frac{r}{n}\right)^{-n}\left(1-\frac{r}{n}\right)^{r-\frac{1}{2}}\left(1-\frac{m}{n}\right)^{-r}$$

$$= \frac{m^r e^{-m} e^{-r}}{r!} \because \lim_{n\to\infty}\left(1-\frac{r}{n}\right)^{-n} = \lim_{n\to\infty} = \left\{\left(1-\frac{r}{n}\right)^{-n/r}\right\}^r = e^r;$$

$$\lim_{n\to\infty}\left(1-\frac{r}{n}\right)^{r-\frac{1}{2}} = 1 \text{ and } \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-r} = 1$$

$$= \frac{m^r e^{-m}}{r}$$

Hence successive terms in the expansion of $(q+p)^n$ are

$e^{-m}, e^{-m}, m, e^{-m}\frac{m^2}{2!}, e^{-m}.\frac{m^3}{3!}$, etc.

and the limiting value of $(q+p)^n$ is

$$e^{-m}\left\{1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+......\right\}$$

This expression is known as *Poisson's distribution* or *Poisson's Exponential limit*.

Note 1: The limiting form of $N(q+p)^n$ is

$$N.\, e^{-m}\left\{1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+......\right\}$$

Note 2: The quantity m introduced in *Poisson distribution* is said to be the parameter of *Poisson distribution*.

Note 3: Characteristics of the Poisson distribution.

(*i*) This is the limiting form of binomial distribution when n is large and p (or q) is small.

(*ii*) Here p or q is very close to zero or unity, but if p is close to zero, the distribution is J-shaped or unimodal.

(*iii*) Since it consists of a single parameter m, the entire distribution can therefore be obtained by knowing the mean only.

Note 4: Some examples of *Poisson distribution* are:

(*i*) The number of defective screws per box of 100 screws.

(*ii*) The numebr of typographical errors per page in typed material.

(*iii*) The number of cars passing through a certain street in time t.

Constants of the Poisson Distribution

Assume that origin is located at the first term of the distribution, so that the values of the deviation from the assumed origin are 0, 1, 2....

NOTES

1. The Mean

$$\text{Mean} = \mu_1' \text{ (about the origin) } \sum_0^{\infty} e^{-m}\frac{m^r}{r!}.r$$

$$= e^{-m}\left[0+m+\left(\frac{m^2}{2!}\times 2\right)+\left(\frac{m^3}{3!}\times 3\right)+...\right]$$

$$= me^{-m}\left(1+\frac{m}{1!}+\frac{m^2}{2!}+...\right) = me^{-m}.e^m = m$$

2. The Variance

We have μ_2' (about the origin)

$$= \sum_0^{\infty} e^{-m}\frac{m^r}{2!}.r^2$$

$$= e^{-m}\left[0+m+\left(\frac{m^2}{2!}\times 2^2\right)+\left(\frac{m^3}{3!}\times 3^2\right)+...\right]$$

$$= me^{-m}\left[1+\left(\frac{m}{1!}\times 2\right)+\left(\frac{m^2}{2!}\times 3\right)+...\right]$$

$$= me^{-m}\left[1+\frac{m}{1!}(1+1)+\frac{m^2}{2!}(2+1)+...\right]$$

$$= me^{-m}\left[\left(1+\frac{m}{1!}+\frac{m^2}{2!}+...\right)+\left(m+\frac{m^2}{1!}+\frac{m^3}{2!}+...\right)\right]$$

$$= me^{-m}\left[e^m+m\left(1+\frac{m}{1!}+\frac{m^2}{2!}+...\right)\right]$$

$$= me^{-m}\left[e^m + me^m\right] = m\,(m+1)$$

$\therefore$ variance $= \mu_2 = \sigma^2 = \mu_2' - \mu_1'^2 = m^2 + m - m^2 = m$

So that standard deviation, $\sigma = \sqrt{m}$.

3. The Moments

$$\mu_3' \text{ (about the origin)} = \Sigma\, e^{-m}\frac{m^r}{r!}.\, r^3$$

$$= e^{-m}\,\Sigma m^r.\left\{\frac{r(r-1)(r-2)+3r(r-1)+r}{r!}\right\}$$

$$= e^{-m}\,m^3\,\Sigma\frac{m^{r-3}}{(r-3)!} + e^{-m}.3m^2\,\Sigma\frac{m^{r-2}}{(r-2)!} + e^{-m}.$$

$$m\,\Sigma\frac{m^{r-1}}{(r-1)!}$$

or $\quad \mu_3' = m^3 + 3m^2 + m$

so that $\mu_3 = \mu_3' - 3\mu_2'.\,\mu_1' + 2\mu_1'^3$

$$= m^3 + 3m^2 + m - 3m^2 - m^3 + 2m^3 = m$$

NOTES

and $\mu_4' = \Sigma e^{-m}. \frac{m^r}{r!}. r^4$

$= e^{-m} \Sigma m^r. \left\{\frac{r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+7(r-1)r+r}{r!}\right\}$

$= e^{-m}\ m^4.\ \Sigma \frac{m^{r-4}}{(r-4)!} + e^{-m}.\ 6m^3\ \Sigma \frac{m^{r-3}}{(r-3)!} + e^{-m}.\ 7m\ \Sigma \frac{m^{r-2}}{(r-2)!} + e^{-m}.$

$m\Sigma \frac{m^{r-1}}{(r-1)!}$

$= m^4 + 6m^3 + 7m^2 + m$

so that $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= m^4 + 6m^3 + 7m^2 - 4m\ (m^3 + 3m^2 + m) + 6m^2(m^2 + m) - 3m^4$

$= 3m^2 + m.$

Example 4.38: Define a Poisson random variable and give some physical situations illustrating it. Find out its mean and variance.

If X and Y are independently distributed as Poisson random variates with parameters λ and μ respectively find the probability distribution of $X + Y$.

Solution: We know that a random variable is a function defined as a sample space. A random variable x assuming values 0, 1, 2,...r,... with probabilities

$$e^{-m}, me^{-m}, \frac{m^2}{2!}\ e^{-m},..., \frac{m^r e^{-m}}{r!},...$$

i.e.,
$$P(x = r) = \frac{m^r e^{-m}}{r!}, r = 0, 1, 2,...$$

is said to be a *Poisson random variable*.

In order to illustrate it with the help of a physical situation, let us consider a random variate X denoting the number of calls during time t at a telephone switch board. Let us assume that the calls are independent and the probability of a call in time dt is λdt.

Denoting by $P_x(t)$ and $P_x\ (t + \delta t)$, the chance of x calls in times t and $t + \delta t$ respectively we have two mutually exclusive possibilities, (*i*) there are x-calls in t and no call in time dt, (*ii*) there are $(x - 1)$ calls in t and one call in dt, neglecting the possibility of more than one call in dt, for it would be of order 2 (i.e., dt^2) and higher.

Thus, $P_x\ (t + \delta t) = P_x(t).\ \{1 - \lambda\ \delta t\} + P_{x-1}\ (t)\ \lambda\ \delta t$

$= P_x(t) - P_x(t).\ \lambda\ \delta t + P_{x-1}\ (t)\ .\ \lambda\delta t$

or $P_x\ (t + \delta t) - P_x(t) = \{P_{x-1}\ (t) - P_x(t)\}.\ \lambda\ \delta t$

or $\frac{P_x(1+\delta t) - P_x(t)}{\delta t} = \lambda\ [P_{x-1}(t) - P_x(t)]$

Proceeding to the limit as $\delta t \to 0$, we have

$$\lim_{\delta t \to 0} \frac{P_x(t+\delta t) - P_x(t)}{\delta t} = \lim_{\delta t \to 0} \lambda\ \left[P_{x-1}(t) - P_x(t)\right]$$

i.e.,
$$\frac{dP_x(t)}{dt} = \lambda[P_{x-1}(t) - Px(t)]$$

NOTES

Putting $$P_x(t) = \frac{(\lambda t)^x}{x!} f(t)$$

So that $$P_0(t) = f(t) \text{ and } P_0(0) = F(0) = 1 \qquad \text{... (1)}$$

We have $$\frac{d}{dt}\left[\frac{(\lambda t)^x}{x!} f(t)\right] = \lambda\left[\frac{(\lambda t)^{x-1}}{(x-1)!} f(t) - \frac{(\lambda t)^x}{x!} f(t)\right]$$

or $$\frac{(\lambda t)^x}{x!} f'(t) + \frac{\lambda^x}{x!} f(t).\ xt^{x-1} = \frac{\lambda^x t^{x-1}}{(x-1)!} f(t) - \frac{\lambda t}{x}.\ \frac{\lambda^x t^{x-1}}{(x-1)!} f(t)$$

Dividing throughout by $\frac{\lambda^x t^{x-1}}{(x-1)!} f(t)$, this reduces to

$$\frac{\lambda t}{x}.\frac{f'(t)}{f(t)} + 1 = 1 - \frac{\lambda t}{x} \text{ or } \frac{f'(t)}{f(t)} = -\lambda.$$

Integrating with regard to t, this yields

$$\log f(t) = -\lambda t + A, \ A \text{ being constant of integration.}$$

Initially when $t = 0, f(0) = 1$, from Relations (1), giving $A = 0$

$$\therefore \qquad \log f(t) = -\lambda t \text{ or } f(t) = e^{-\lambda t}$$

or $$\frac{(\lambda t)^x}{x!} f(t) = P_z(t) = e^{-\lambda t}.\frac{(\lambda t)^x}{x!}$$

where $$\sum_{x=0}^{\infty} P_x(t) = e^{-\lambda t} \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} = e^{-\lambda t}, e^{\lambda t} = 1.$$

If follows that the number of calls in a fixed time t is a Poisson variate with parameter λt.

For the second part, we have

$$P(X = x) = e^{-\lambda}, \frac{\lambda^x}{x!}, P(Y = y) = e^{-\mu}\frac{\mu^y}{y!}$$

so that $P(X = x,\ Y = y) = P\ (X = x).\ P(Y = y)$

$$= e^{-\lambda}\frac{\lambda^x}{x!}.e^{-\mu}\frac{\mu^y}{y!} = e^{-(\lambda+\mu)}\frac{\lambda^x \mu^y}{\lfloor x \lfloor y}$$

Since the variables take values 0, 1, 2, 3,..., let us find the Probability that their sum, i.e., $X + Y$ takes values r so that $y = r - x$. Summing for all values of x from 0 to r, we have

$$P(x + y = r) = e^{-(\lambda+\mu)} \sum_{x=0}^{r} \frac{\lambda^x \mu^{r-x}}{\lfloor x \lfloor r-x} = e^{-(\lambda+\mu)}.\frac{\mu^x}{\lfloor r} \sum_{x=0}^{r} \frac{\lfloor r}{\lfloor x \lfloor r-x}\left(\frac{\lambda}{\mu}\right)^x$$

$$= e^{-(\lambda+\mu)} \frac{\mu^r}{\lfloor r} \sum_{x=0}^{r} {}^rC_x \left(\frac{\lambda}{\mu}\right)^x$$

$$= e^{-(\lambda+\mu)} \frac{\mu^r}{\lfloor r}\left[{}^rC_0 + {}^rC_1\frac{\lambda}{\mu} + {}^rC_2\left(\frac{\lambda}{\mu}\right)^2 + ... + {}^rC_r\left(\frac{\lambda}{\mu}\right)^r\right]$$

$$= e^{-(\lambda+\mu)} \frac{\mu^r}{\lfloor r}\left(1+\frac{\lambda}{\mu}\right)^r = e^{-(\lambda+\mu)}.\frac{(\lambda+\mu)^r}{\lfloor x+y} \quad \because\ r = x + y$$

which is a Poisson distribution with mean $(\lambda + \mu)$.

Example 4.39: Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 per cent of such fuses are defective.

NOTES

Solution: Here$n$ = 200, $q = \frac{2}{100}$ = .02, so that $m = nq = 4$.

$$\text{Hence} P(x \le 5) = \sum_{x=0}^{5} e^{-4}.\frac{4^x}{x!}$$

$$= e^{-4}\left[1+4+\frac{4^2}{2!}+\frac{4^3}{3!}+\frac{4^4}{4!}+\frac{4^5}{5!}\right]$$

$$= e^{-4}\left[1+4+8+\frac{32}{3}+\frac{32}{3}+\frac{128}{15}\right]$$

$$= e^{-4}.\ \frac{643}{.15} = 0.0183 \times \frac{643}{15} = 0.785$$

approx. [$\because\ e^{-4} = 0.0183$]

Example 4.40: Fit a Poissons's distribution to the set of observations:

Deaths	0	1	2	3	4
Frequency	122	60	15	2	1

and calculate theoretical frequencies.

Solution: We have, $\text{Mean} = \frac{0\times122+1\times60+2\times15+3\times2+4\times1}{122+60+15+2+1}$

$$= \frac{100}{200} = 0.5$$

$$\therefore \qquad e^{-0.5} = 1-(0.5)+\frac{1}{2}(0.5)^2-\frac{1}{6}(0.5)^3+...$$

$$= 1 - 0.5 + 0.125 - 0.0208 + ... = 0.61 \text{ approx}$$

The theoretical frequency of r deaths is given by

$$Ne^{-m}.\frac{m^r}{r!} = 200.\ e^{-0.5}\ \frac{(0.5)^r}{r!} = 122.\ \frac{(0.5)^r}{r!}$$

which gives for r = 0, 1, 2, 3, 4 the theoretical frequencies as 122, 61, 15, 2 and 0 respectively.

Example 4.41: In a certain factory turning out razor blades, there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective respectively in a consignment of 10,000 packets, given that $e^{-.02} = 0.9802$.

Solution: Here N = 10,000, $m = np = 10 \times \frac{1}{500} = 0.02$ and $e^{-.02} = .9802$.

$\therefore$ Required frequencies are given by Ne^{-m}, $Ne^{-m}.m$, $Ne^{-m}.\ \frac{m^2}{2!}$

i.e., $10{,}000 \times .9802;\ 10{,}000 \times .9802 \times 0.02;\ 10{,}000 \times .9802 \times \frac{(.02)^2}{2}$

i.e., 9802; 196, 2 packets

Example 4.42: If x is a Poissonian variate with mean m,

(a) What would be the expectation of e^{-kx}. kx where k is a constant.

(b) Show that, expectation of $(e^{-kx}) = e^{-m(1-e^{-k})}$.

NOTES

Solution: (a) We have, $E(kxe^{-kx}) = \sum_{x=0}^{\infty} kx.e^{-kx} . \frac{e^{-m}m^x}{x!}$

$$= mke^{-(m+k)} \sum_{x=0}^{\infty} e^{-k(x-1)} \frac{m^{x-1}}{(x-1)!}$$

$$= mke^{-(m+k)}. \sum_{x=0}^{\infty} \frac{(me^{-k})^{x-1}}{(x-1)!}$$

$$= mke^{-(m+k)}e^{me^{-k}} = mke^{m(e^{-k}-1)-k}$$

and $$E(e^{-kx}) = \sum_{x=0}^{\infty} e^{-kx} \frac{e^{-m}m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^{-k})^x}{x!}$$

$$= e^{-m}.e^{me^{-k}} = e^{-m(1-e^{-k})}$$

4.4.4 Central Limit Theorem

In probability theory, the Central Limit Theorem (CLT) establishes that, in some situations when independent random variables are added then their appropriately normalized sum tends toward a normal distribution, usually a 'Bell Curve' even if the original variables themselves are not normally distributed. The theorem is a key concept in probability theory because it implies that probabilistic and statistical methods that work for normal distributions can be applicable to many problems involving other types of distributions.

Definition: The Central Limit Theorem (CLT) is a statistical theory which states that given a sufficiently large sample size from a population with a finite level of variance, the mean of all samples from the same population will be approximately equal to the mean of the population.

For example, suppose that a sample is obtained containing a large number of observations, each observation being randomly generated in a way that does not depend on the values of the other observations, and that the arithmetic mean of the observed values is computed. If this procedure is performed many times, the central limit theorem says that the distribution of the average will be closely approximated by a normal distribution. A simple example of this is that if one flips a coin many times the probability of getting a given number of heads in a series of flips will approach a normal curve, with mean equal to half the total number of flips in each series.

Thus, the Central Limit Theorem (CLT) is considered significant in the probability theory and statistics.

Example 4.43: Under certain conditions, the sum of a large number of random variables is approximately normal. Consider the CLT that is applied to Independent and Identically Distributed (I.I.D.) random variables or alternatively random variables.

Solution: Suppose that $X_1, X_2, \ldots, X_n$ are I.I.D. random variables with,

Expected values $= EX_i = \mu < \infty$

And, Variance $= \mathrm{Var}(X_i) = \sigma^2 < \infty$.

NOTES

Then, the sample mean, $\overline{X} = \frac{X_1+X_2+...+X_n}{n}$ has mean $E\overline{X} = \mu$ and variance $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

Thus, the normalized random variable,

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma}$$

has mean $EZ_n = 0$ and variance Var $(Z_n) = 1$. The central limit theorem states that the CDF (Cumulative Distribution Function) of (Z_n) converges to the standard normal CDF.

Check Your Progress

1. What is permutation of *n* objects?
2. Define the event.
3. What is random variable?
4. Define the normal distribution.
5. State the central limit theorem.

4.5 VARIOUS MEASURES OF AVERAGE

In statistics, the term central tendency specifies the method through which the quantitative data have a tendency to cluster approximately about some value. A measure of central tendency is any precise method of specifying this 'Central Value'. In the simplest form, the measure of central tendency is an average of a set of measurements, where the word *average* refers to as mean, median, mode or other measures of location. Typically the most commonly used measures are arithmetic mean, mode and median. These values are very useful not only in presenting the overall picture of the entire data but also for the purpose of making comparisons among two or more sets of data. As an example, questions like 'How hot is the month of June in Delhi?' can be answered, generally by a single figure of the average for that month. Similarly, suppose we want to find out if boys and girls at age 10 years differ in height for the purpose of making comparisons. Then, by taking the average height of boys of that age and average height of girls of the same age, we can compare and record the differences.

While arithmetic mean is the most commonly used measure of central location, mode and median are more suitable measures under certain set of conditions and for certain types of data. However, each measure of central tendency should meet the following requisites.

1. It should be easy to calculate and understand.
2. It should be rigidly defined. It should have only one interpretation so that the personal prejudice or bias of the investigator does not affect its usefulness.

NOTES

3. It should be representative of the data. If it is calculated from a sample, then the sample should be random enough to be accurately representing the population.
4. It should have sampling stability. It should not be affected by sampling fluctuations. This means that if we pick 10 different groups of college students at random and compute the average of each group, then we should expect to get approximately the same value from each of these groups.
5. It should not be affected much by extreme values. If few very small or very large items are present in the data, they will unduly influence the value of the average by shifting it to one side or other, so that the average would not be really typical of the entire series. Hence, the average chosen should be such that it is not unduly affected by such extreme values.

All these measures of central tendency are discussed in this section.

4.5.1 Mean

There are several commonly used measures, such as arithmetic mean, mode and median. These values are very useful not only in presenting the overall picture of the entire data, but also for the purpose of making comparisons among two or more sets of data.

As an example, questions like 'How hot is the month of June in Delhi?' can be answered generally by a single figure of the average for that month. Similarly, suppose we want to find out if boys and girls of age 10 years differ in height for the purpose of making comparisons. Then, by taking the average height of boys of that age and the average height of girls of the same age, we can compare and record the differences.

While arithmetic mean is the most commonly used measure of central tendency, mode and median are more suitable measures under certain set of conditions and for certain types of data. However, each measure of central tendency should meet the following requisites:

(*i*) It should be easy to calculate and understand.

(*ii*) It should be rigidly defined. It should have only one interpretation so that the personal prejudice or the bias of the investigator does not affect its usefulness.

(*iii*) It should be representative of the data. If it is calculated from a sample, the sample should be random enough to be accurately representing the population.

(*iv*) It should have a sampling stability. It should not be affected by sampling fluctuations. This means that if we pick ten different groups of college students at random and compute the average of each group, then we should expect to get approximately the same value from each of these groups.

(*v*) It should not be affected much by extreme values. If few, very small or very large items are present in the data, they will unduly influence the value of the average by shifting it to one side or other, so that the average would not be really typical of the entire series. Hence, the average chosen should be such that it is not unduly affected by such extreme values.

Arithmetic mean is also commonly known as the mean. Even though average, in general, means measure of central tendency, when we use the word average in our daily routine, we always mean the arithmetic average. The term is widely used by almost everyone in daily communication. We speak of an individual being an average student or of average intelligence. We always talk about average family size or average family income or Grade Point Average (GPA) for students, and so on.

For discussion purposes, let us assume a variable X which stands for some value such as the ages of students. Let the ages of 5 students be 19, 20, 22, 22 and 17 years. Then variable X would represent these ages as,

$$X: 19, 20, 22, 22, 17$$

Placing the Greek symbol Σ(Sigma) before X would indicate a command that all values of X are to be added together. Thus,

$$\Sigma X = 19 + 20 + 22 + 22 + 17$$

The mean is computed by adding all the data values and dividing it by the number of such values. The symbol used for sample average is $\overline{X}$, so that,

$$\overline{X} = \frac{19+20+22+22+17}{5}$$

In general, if there are n values in the sample, then,

$$\overline{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

In other words,

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \qquad i = 1,\ 2\ ...\ n$$

According to this formula, mean can be obtained by adding all values of X_i, where the value of i starts at 1 and ends at n with unit increments so that $i = 1, 2, 3, ... n$.

If instead of taking a sample, we take the entire population in our calculations of the mean, then the symbol for the mean of the population is μ(mu) and the size of the population is N, so that,

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}, \qquad i = 1,\ 2\ ...N$$

If we have the data in grouped discrete form with frequencies, then the sample mean is given by,

$$\overline{X} = \frac{\Sigma f(X)}{\Sigma f}$$

Here, Σf = Summation of all frequencies

$= n$

$\Sigma f(X)$ = Summation of each value of X multiplied by its corresponding frequency (f)

NOTES

Example 4.44: Let us take the ages of 10 students as follows:

19, 20, 22, 22, 17, 22, 20, 23, 17, 18

Solution: This data can be arranged in a frequency distribution as follows:

(X)	(f)	$f(X)$
17	2	34
18	1	18
19	1	19
20	2	40
22	3	66
23	1	23
	Total = 10	200

In this case, we have $\Sigma f = 10$ and $\Sigma f(X) = 200$, so that,

$$\overline{X} = \frac{\Sigma f(X)}{\Sigma f}$$

$$= 200/10 = 20$$

Characteristics of the Mean

The arithmetic mean has some interesting properties. These are as follows:

(*i*) The sum of the deviations of individual values of X from the mean will always add up to zero. This means that if we subtract all the individual values from their mean, then some values will be negative and some will be positive, but if all these differences are added together then the sum will be zero. In other words, the positive deviations must balance the negative deviations. Or symbolically,

$$\sum_{i=1}^{n}(X_i - \overline{X}) = 0, i = 1, 2, \ldots n$$

(*ii*) The second important characteristic of mean is that it is very sensitive to extreme values. Since the computation of mean is based upon inclusion of all values in the data, an extreme value in the data would shift the mean towards it, thus making the mean unrepresentative of the data.

(*iii*) The third property of mean is that the sum of squares of the deviations about the mean is minimum. This means that if we take the differences between individual values and the mean and square these differences individually and then add these squared differences, then the final figure will be less than the sum of the squared deviations around any other number other than the mean. Symbolically, it means that

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \text{Minimum}, i = 1, 2, \ldots n$$

(*iv*) The product of the arithmetic mean and the number of values on which the mean is based is equal to the sum of all given values. In other words, if we replace each item in series by the mean, then the sum of these substitutions will equal the sum of individual items. Thus, if we take random figures as an example like 3, 5, 7, 9, and if we substitute the mean for each item 6, 6, 6, 6 then the total is 24, both in the original series and in the substitution series.

This can be shown like,

NOTES

Since, $$\bar{X} = \frac{\Sigma X}{N}$$

$\therefore$ $$N\bar{X} = \Sigma X$$

For example, if we have a series of values 3, 5, 7, 9, the mean is 6. The squared deviations will be:

X	$X - \bar{X} = X'$	X'^2
3	$3 - 6 = -3$	9
5	$5 - 6 = -1$	1
7	$7 - 6 = 1$	1
9	$9 - 6 = 3$	9
		$\Sigma X'^2 = 20$

This property provides a test to check if the computed value is the correct arithmetic mean.

Example 4.45: The mean age of a group of 100 persons (grouped in intervals 10–, 12–,..., etc.) was found to be 32.02. Later, it was discovered that age 57 was misread as 27. Find the corrected mean.

Solution: Let the mean be denoted by X. So, putting the given values in the formula of arithmetic mean, we have,

$$32.02 = \frac{\sum X}{100}, \text{ i.e., } \sum X = 3202$$

$$\text{Correct } \sum X = 3202 - 27 + 57 = 3232$$

$$\therefore \quad \text{Correct AM} = \frac{3232}{100} = 32.32$$

Example 4.46: The mean monthly salary paid to all employees in a company is ₹ 500. The monthly salaries paid to male and female employees average ₹ 520 and ₹ 420, respectively. Determine the percentage of males and females employed by the company.

Solution: Let N_1 be the number of males and N_2 be the number of females employed by the company. Also, let x_1 and x_2 be the monthly average salaries paid to male and female employees and $\bar{x}$ be the mean monthly salary paid to all the employees.

$$\bar{x} = \frac{N_1x_1 + N_2x_2}{N_1 + N_2}$$

Or $$500 = \frac{520N_1 + 420N_2}{N_1 + N_2} \quad \text{or} \quad 20N_1 = 80N_2$$

Or $$\frac{N_1}{N_2} = \frac{80}{20} = \frac{4}{1}$$

Hence, the males and females are in the ratio of 4 : 1 or 80 per cent are males and 20 per cent are females in those employed by the company.

Example 4.47: The ages of twenty husbands and wives are given in the following table. Form frequency tables showing the relationship between the ages of husbands and wives with class intervals 20 – 24; 25 – 29; etc.

NOTES

Calculate the arithmetic mean of the two groups after the classification.

S.No.	*Age of Husband*	*Age of Wife*
1	28	23
2	37	30
3	42	40
4	25	26
5	29	25
6	47	41
7	37	35
8	35	25
9	23	21
10	41	38
11	27	24
12	39	34
13	23	20
14	33	31
15	36	29
16	32	35
17	22	23
18	29	27
19	38	34
20	48	47

Solution:

Calculation of Arithmetic Mean of Husbands' Age

Class Intervals	*Midvalues* m	*Husband Frequency* (f_1)	$x_1' = \frac{m-37}{5}$	f_1x_1'
20–24	22	3	–3	–9
25–29	27	5	–2	–10
30–34	32	2	–1	–2
				– 21
35–39	37	6	0	0
40–44	42	2	1	2
45–49	47	2	2	4
				6
		$\Sigma f_1 = 20$		$\Sigma f_1 x_1' = -15$

Arithmetic mean of husbands age,

$$\overline{x} = \frac{\Sigma f_1 x_1'}{N} \times i + A = \frac{-15}{20} \times 5 + 37 = 33.25$$

Calculation of Arithmetic Mean of Wives' Age

Class Intervals	*Midvalues* m	*Wife Frequency* (f_2)	$x_2' = \frac{m-37}{5}$	f_2x_2'
20–24	22	5	–3	–15
25–29	27	5	–2	–10
30–34	32	4	–1	–4
35–39	37	3	0	0
40–44	42	2	1	2
45–49	47	1	2	2
		$\Sigma f_2 = 20$		$\Sigma f_2 x_2' = -25$

Arithmetic mean of wife's age,

$$\bar{x} = \frac{\sum f_2 x_2'}{N} \times i + A = \frac{-25}{20} \times 5 + 37 = 30.75$$

NOTES

Advantages of Mean

(*i*) Its concept is familiar to most people and is intuitively clear.

(*ii*) Every data set has a mean, which is unique and describes the entire data to some degree. For example, when we say that the average salary of a professor is ₹ 25,000 per month, it gives us a reasonable idea about the salaries of professors.

(*iii*) It is a measure that can be easily calculated.

(*iv*) It includes all values of the data set in its calculation.

(*v*) Its value varies very little from sample to sample taken from the same population.

(*vi*) It is useful for performing statistical procedures, such as computing and comparing the means of several data sets.

Disadvantages of Mean

(*i*) It is affected by extreme values, and hence are not very reliable when the data set has extreme values especially when these extreme values are on one side of the ordered data. Thus, a mean of such data is not truly a representative of such data. For example, the average age of three persons of ages 4, 6 and 80 years gives us an average of 30.

(*ii*) It is tedious to compute for a large data set as every point in the data set is to be used in computations.

(*iii*) We are unable to compute the mean for a data set that has open-ended classes either at the high or at the low-end of the scale.

(*iv*) The mean cannot be calculated for qualitative characteristics, such as beauty or intelligence, unless these can be converted into quantitative figures such as intelligence into IQs.

4.5.2 Median

The second measure of central tendency that has a wide usage in statistical works is the median. Median is that value of a variable which divides the series in such a manner that the number of items below it is equal to the number of items above it. Half the total number of observations lie below the median, and half above it. The median is thus a positional average.

The median of ungrouped data is found easily if the items are first arranged in order of the magnitude. The median may then be located simply by counting, and its value can be obtained by reading the value of the middle observations. If we have five observations whose values are 8, 10, 1, 3 and 5, the values are first arrayed: 1, 3, 5, 8 and 10. It is now apparent that the value of the median is 5, since two observations are below that value and two observations are above it. When there is an even number of cases, there is no actual middle item and the median is taken to be the average of the values of the items lying on either side of $(N + 1)/2$, where N is the total number of items. Thus, if the values of six items of a series are 1, 2, 3, 5, 8 and 10, then the median is the value of item number $(6 + 1)/2 = 3.5$, which is approximated as the average of the third and the fourth items, i.e., $(3+5)/2 = 4$.

NOTES

Thus, the steps required for obtaining median are as follows:

(*i*) Arrange the data as an array of increasing magnitude.

(*ii*) Obtain the value of the $(N+ 1)/2$th item.

Even in the case of grouped data, the procedure for obtaining median is straightforward as long as the variable is discrete or non-continuous as is clear from Example 4.48.

Example 4.48: Obtain the median size of shoes sold from the following data:

Number of Shoes Sold by Size in One Year

Size	*Number of Pairs*	*Cumulative Total*
5	30	30
$5\frac{1}{2}$	40	70
6	50	120
$6\frac{1}{2}$	150	270
7	300	570
$7\frac{1}{2}$	600	1170
8	950	2120
$8\frac{1}{2}$	820	2940
9	750	3690
$9\frac{1}{2}$	440	4130
10	250	4380
$10\frac{1}{2}$	150	4530
11	40	4570
$11\frac{1}{2}$	39	4609
	Total	4609

Solution: Median is the value of $\frac{(N+1)}{2}\text{th} = \frac{4609+1}{2}\text{th} = 2305$th item. Since the items are already arranged in ascending order (size-wise), the size of 2305th item is easily determined by constructing the cumulative frequency. Thus, the median size of shoes sold is 8½, the size of 2305th item.

In the case of grouped data with continuous variable, the determination of median is a bit more involved. Consider the following table where the data relating to the distribution of male workers by average monthly earnings is given. Clearly the median of 6291 is the earnings of $(6291 + 1)/2 = 3146$th worker arranged in ascending order of earnings.

From the cumulative frequency, it is clear that this worker has his income in the class interval 67.5 – 72.5. However, it is impossible to determine his exact income. We therefore, resort to approximation by assuming that the 795 workers of this class are distributed uniformly across the interval 67.5 – 72.5. The median worker is (3146–2713) = 433rd of these 795, and hence, the value corresponding to him can be approximated as,

$$67.5 + \frac{433}{795} \times (72.5 - 67.5) = 67.5 + 2.73 = 70.23$$

NOTES

Distribution of Male Workers by Average Monthly Earnings

Group No.	*Monthly Earnings (₹)*	*No. of Workers*	*Cumulative No. of Workers*
1	27.5–32.5	120	120
2	32.5–37.5	152	272
3	37.5–42.5	170	442
4	42.5–47.5	214	656
5	47.5–52.5	410	1066
6	52.5–57.5	429	1495
7	57.5–62.5	568	2063
8	62.5–67.5	650	2713
9	67.5–72.5	795	3508
10	72.5–77.5	915	4423
11	77.5–82.5	745	5168
12	82.5–87.5	530	5698
13	87.5–92.5	259	5957
14	92.5–97.5	152	6109
15	97.5–102.5	107	6216
16	102.5–107.5	50	6266
17	107.5–112.5	25	6291
		Total	6291

The value of the median can thus be put in the form of the formula,

$$Me = l + \frac{\frac{N+1}{2} - C}{f} \times i$$

Where l is the lower limit of the median class, i its width, f its frequency, C the cumulative frequency upto (but not including) the median class, and N is the total number of cases.

Advantages of Median

(*i*) Median is a positional average and hence the extreme values in the data set do not affect it as much as they do to the mean.

(*ii*) Median is easy to understand and can be calculated from any kind of data, even from grouped data with open-ended classes.

(*iii*) We can find the median even when our data set is qualitative and can be arranged in the ascending or the descending order, such as average beauty or average intelligence.

(*iv*) Similar to mean, median is also unique, meaning that, there is only one median in a given set of data.

(*v*) Median can be located visually when the data is in the form of ordered data.

NOTES

(*vi*) The sum of absolute differences of all values in the data set from the median value is minimum. This means that, it is less than any other value of central tendency in the data set, which makes it more central in certain situations.

Disadvantages of Median

(*i*) The data must be arranged in order to find the median. This can be very time consuming for a large number of elements in the data set.

(*ii*) The value of the median is affected more by sampling variations. Different samples from the same population may give significantly different values of the median.

(*iii*) The calculation of median in case of grouped data is based on the assumption that the values of observations are evenly spaced over the entire class interval and this is usually not so.

(*iv*) Median is comparatively less stable than mean, particularly for small samples, due to fluctuations in sampling.

(*v*) Median is not suitable for further mathematical treatment. For example, we cannot compute the median of the combined group from the median values of different groups.

4.5.3 Mode

Mode is that value of the variable which occurs or repeats itself the greatest number of times. The mode is the most 'Fashionable' size in the sense that it is the most common and typical, and is defined by Zizek as 'the value occurring most frequently in a series (or group of items) and around which the other items are distributed most densely'.

The mode of a distribution is the value at the point around which the items tend to be most heavily concentrated. It is the most frequent or the most common value, provided that a sufficiently large number of items are available, to give a smooth distribution. It will correspond to the value of the maximum point (ordinate), of a frequency distribution if it is an 'Ideal' or smooth distribution. It may be regarded as the most typical of a series of values. The modal wage, for example, is the wage received by more individuals than any other wage. The modal 'hat' size is that, which is worn by more persons than any other single size.

It may be noted that the occurrence of one or a few extremely high or low values has no effect upon the mode. If a series of data are unclassified, not have been either arrayed or put into a frequency distribution, the mode cannot be readily located.

Taking first an extremely simple example, if seven men receive daily wages of ₹ 5, 6, 7, 7, 7, 8 and 10, it is clear that the modal wage is ₹ 7 per day. If we have a series such as 2, 3, 5, 6, 7, 10 and 11, it is apparent that there is no mode.

There are several methods of estimating the value of the mode. However, it is seldom that the different methods of ascertaining the mode give us identical results. Consequently, it becomes necessary to decide as to which method would be most suitable for the purpose in hand. In order that a choice of the method may be made, we should understand each of the methods and the differences that exist among them.

The four important methods of estimating mode of a series are: (i) Locating the most frequently repeated value in the array; (ii) Estimating the mode by interpolation; (iii) Locating the mode by graphic method; and (iv) Estimating the mode from the mean and the median. Only the last three methods are discussed in this unit.

Estimating the Mode by Interpolation: In the case of continuous frequency distributions, the problem of determining the value of the mode is not so simple as it might have appeared from the foregoing description. Having located the modal class of the data, the next problem in the case of continuous series is to interpolate the value of the mode within this 'modal' class.

The interpolation is made by the use of any one of the following formulae:

$$(i)\ Mo = l_1 + \frac{f_2}{f_0 + f_2} \times i; \quad (ii)\ Mo = l_2 - \frac{f_0}{f_0 + f_2} \times i$$

$$(iii)\ Mo = l_1 + \frac{f_1 - f_0}{(f_1 - f_0) + (f_1 - f_2)} \times i$$

Where l_1 is the lower limit of the modal class, l_2 is the upper limit of the modal class, f_0 equals the frequency of the preceding class in value, f_1 equals the frequency of the modal class in value, f_2 equals the frequency of the following class (class next to modal class) in value, and i equals the interval of the modal class. Example 4.63 explains the method of estimating mode.

Example 4.49: Determine the mode for the data given in the following table:

Wage Group	*Frequency (f)*
14 — 18	6
18 — 22	18
22 — 26	19
26 — 30	12
30 — 34	5
34 — 38	4
38 — 42	3
42 — 46	2
46 — 50	1
50 — 54	0
54 — 58	1

Solution:

In the given data, 22 – 26 is the modal class since it has the largest frequency. The lower limit of the modal class is 22, its upper limit is 26, its frequency is 19, the frequency of the preceding class is 18, and of the following class is 12. The class interval is 4. Using the various methods of determining mode, we have,

NOTES

$$(i)\ Mo = 22 + \frac{12}{18+12} \times 4 \qquad (ii)\ Mo = 26 - \frac{18}{18+12} \times 4$$

$$= 22 + \frac{8}{5} \qquad\qquad = 26 - \frac{12}{5}$$

$$= 23.6 \qquad\qquad = 23.6$$

$$(iii)\ Mo = 22 + \frac{19-18}{(19-18)+(19-12)} \times 4 = 22 + \frac{4}{8} = 22.5$$

In formulae (i) and (ii), the frequency of the classes adjoining the modal class is used to pull the estimate of the mode away from the midpoint towards either the upper or lower class limit. In this particular case, the frequency of the class preceding the modal class is more than the frequency of the class following and therefore, the estimated mode is less than the midvalue of the modal class. This seems quite logical. If the frequencies are more on one side of the modal class than on the other it can be reasonably concluded that the items in the modal class are concentrated more towards the class limit of the adjoining class with the larger frequency.

Formula (iii) is also based on a logic similar to that of (i) and (ii). In this case, to interpolate the value of the mode within the modal class, the differences between the frequency of the modal class, and the respective frequencies of the classes adjoining it are used. This formula usually gives results better than the values obtained by the other and exactly equals results obtained by graphic method. The formulae (i) and (ii) give values which are different from the value obtained by formula (iii) and are more close to the central point of modal class. If the frequencies of the class adjoining the modal are equal, the mode is expected to be located at the midvalue of the modal class, but if the frequency on one of the sides is greater, the mode will be pulled away from the central point. It will be pulled more and more if the difference between the frequencies of the classes adjoining the modal class is higher and higher.

Advantages of Mode

(*i*) Similar to median, the mode is not affected by extreme values in the data.

(*ii*) Its value can be obtained in open-ended distributions without ascertaining the class limits.

(*iii*) It can be easily used to describe qualitative phenomenon. For example, if most people prefer a certain brand of tea, then this will become the modal point.

(*iv*) Mode is easy to calculate and understand. In some cases, it can be located simply by observation or inspection.

Disadvantages of Mode

(*i*) Quite often, there is no modal value.

(*ii*) It can be bi-modal or multi-modal, or it can have all modal values making its significance more difficult to measure.

(*iii*) If there is more than one modal value, the data is difficult to interpret.

(*iv*) A mode is not suitable for algebraic manipulations.

(*v*) Since the mode is the value of maximum frequency in the data set, it cannot be rigidly defined if such frequency occurs at the beginning or at the end of the distribution.

(*vi*) It does not include all observations in the data set, and hence, less reliable in most of the situations.

NOTES

4.6 STANDARD ERROR

By far the most universally used and the most useful measure of dispersion is the standard deviation or root mean square deviation about the mean. We have seen that all the methods of measuring dispersion so far discussed are not universally adopted for want of adequacy and accuracy. The range is not satisfactory as its magnitude is determined by most extreme cases in the entire group. Further, the range is notable because it is dependent on the item whose size is largely matter of chance. Mean deviation method is also an unsatisfactory measure of scatter, as it ignores the algebraic signs of deviation. We desire a measure of scatter which is free from these shortcomings. To some extent standard deviation is one such measure.

The calculation of standard deviation differs in the following respects from that of mean deviation. First, in calculating standard deviation, the deviations are squared. This is done so as to get rid of negative signs without committing algebraic violence. Further, the squaring of deviations provides added weight to the extreme items, a desirable feature for certain types of series.

Secondly, the deviations are always recorded from the arithmetic mean, because although the sum of deviations is the minimum from the median, the sum of squares of deviations is minimum when deviations are measured from the arithmetic average. The deviation from $\bar{x}$ is represented by d.

Thus, standard deviation, σ (sigma) is defined as the square root of the mean of the squares of the deviations of individual items from their arithmetic mean.

$$\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}} \qquad ...(4.14)$$

For grouped data (discrete variables)

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} \qquad ...(4.15)$$

and, for grouped data (continuous variables)

$$\sigma = \sqrt{\frac{\Sigma f(M-\bar{x})}{\Sigma f}} \qquad ...(4.16)$$

where M is the mid-value of the group.

The use of these formulae is illustrated by the following examples.

NOTES

Example 4.50: Compute the standard deviation for the following data:

11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Solution: Here formula (7) is appropriate. We first calculate the mean as $\bar{x} = \sum x/N = 176/11 = 16$, and then calculate the deviation as follows:

x	$(x - \bar{x})$	$(x - \bar{x})^2$
11	–5	25
12	–4	16
13	–3	9
14	–2	4
15	–1	1
16	0	0
17	+1	1
18	+2	4
19	+3	9
20	+4	16
21	+5	25
176		10

Thus by formula Equation (4.21).

$$\sigma = \sqrt{\frac{110}{11}} = \sqrt{10} = 3.16$$

Example 4.51: Find the standard deviation of the data in the following distributions:

x	12	13	14	15	16	17	18	20
f	4	11	32	21	15	8	6	4

Solution: For this discrete variable grouped data, we use Equation 4.22. Since for calculation of $\bar{x}$, we need $\sum fx$ and then for s we need $\sum f(x - \bar{x})^2$, the calculations are conveniently made in the following format.

x	f	fx	$d = x - \bar{x}$	d^2	fd^2
12	4	48	–3	9	36
13	11	143	–2	4	44
14	32	448	–1	1	32
15	21	315	0	0	0
16	15	240	1	1	15
17	8	136	2	4	32
18	5	90	3	9	45
20	4	80	5	25	100
	100	1500			304

Here $\bar{x} = \sum fx / \sum f = 1500/100 = 15$

And
$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$$

$$= \sqrt{\frac{304}{100}} = \sqrt{3.04} = 1.74$$

Example 4.52: Calculate the standard deviation of the following data.

Class	1–3	3–5	5–7	7–19	9–11	11–13	13–15
frequency	1	9	25	35	17	10	3

NOTES

Solution: This is an example of continuous frequency series and formula 9 seems appropriate.

Class	*Mid-point* x	*Frequency* f	*fx*	*Deviation of mid-point x from mean* (8)	*Squared deviation* d^2	*Squared deviation times frequency* d^2
1–3	2	1	2	–6	36	36
3–5	4	9	36	–4	16	144
5–7	6	25	150	–2	4	100
7–9	8	35	280	0	0	0
9–11	10	17	170	2	4	68
11–13	12	10	120	4	16	160
13–15	14	3	42	6	36	108
		100	800			616

First the mean is calculated as

$$\bar{x} = \sum fx/\sum x = 800/100 = 8.0$$

Then the deviations are obtained from 8.0. The standard deviation

$$\sigma = \sqrt{\frac{\sum f(M-\bar{x})^2}{\sum f}}$$

$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f}} = \sqrt{\frac{616}{100}}$$

$$= 2.48$$

4.6.1 Calculation of Standard Deviation by Short-cut Method

The three examples worked out above have one common simplifying feature, namely $\bar{x}$ in each, turned out to be an integer, thus, simplifying calculations. In most cases, it is very unlikely that it will turn out to be so. In such cases, the calculation of d and d^2 becomes quite time-consuming. Short-cut methods have consequently been developed. These are on the same lines as those for calculation of mean itself.

In the short-cut method, we calculate deviations x' from an assumed mean A. Then,

for ungrouped data

$$\sigma = \sqrt{\frac{\sum x'^2}{N} - \left(\frac{\sum x'}{N}\right)^2} \quad ...(4.17)$$

and for grouped data

$$\sigma = \sqrt{\frac{\sum fx'^2}{\sum f} - \left(\frac{fx'}{\sum f}\right)^2} \quad ...(4.18)$$

This formula is valid for both discrete and continuous variables. In case of continuous variables, x in the equation $x' = x - A$ stands for the mid-value of the class in question.

NOTES

Note that the second term in each of the formulae is a correction term because of the difference in the values of A and $\bar{x}$. When A is taken as $\bar{x}$ itself, this correction is automatically reduced to zero. Examples 4.7 to 4.11 explain the use of these formulae.

Example 4.53: Compute the standard deviation by the short-cut method for the following data:

11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21

Solution: Let us assume that $A = 15$.

	$x' = (x - 15)$	x'^2
11	–4	16
12	–3	9
13	–2	4
14	–1	1
15	0	0
16	1	1
17	2	4
18	3	9
19	4	16
20	5	25
21	6	36
$N = 11$	$\sum x' = 11$	$\sum x'^2 = 121$

$$\sigma = \sqrt{\frac{\sum x'^2}{N} - \left(\frac{\sum x'}{N}\right)^2}$$

$$= \sqrt{\frac{121}{11} - \left(\frac{11}{11}\right)^2}$$

$$= \sqrt{11 - 1}$$

$$= \sqrt{10}$$

$$= 3.16.$$

Another method

If we assumed A as zero, then the deviation of each item from the assumed mean is the same as the value of item itself. Thus, 11 deviates from the assumed mean of zero by 11, 12 deviates by 12, and so on. As such, we work with deviations without having to compute them, and the formula takes the following shape:

x	x^2
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
176	2,926

$$\sigma = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$$

$$= \sqrt{\frac{2926}{11} - \left(\frac{176}{11}\right)^2} = \sqrt{266 - 256} = 3.16$$

NOTES

Example 4.54: Calculate the standard deviation of the following data by short method.

Person	1	2	3	4	5	6	7
Monthly income (Rupees)	300	400	420	440	460	480	580

Solution: In this data, the values of the variable are very large making calculations cumbersome. It is advantageous to take a common factor out. Thus, we use $x' = \frac{x - A}{20}$. The standard deviation is calculated using x' and then the true value of σ is obtained by multiplying back by 20. The effective formula then is

$$\sigma = C \times \sqrt{\frac{\sum x'^2}{N} - \left(\frac{\sum x'}{N}\right)^2}$$

where C represents the common factor.

Using $x' = (x - 420)/20$.

x	*Deviation from Assumed mean $x' = (x - 420)$*	x'	x'^2
300	–120	–6	36
400	–20	–1	1
420	0	0	0
		–7	
440	20	1	1
460	40	2	4
480	60	3	9
580	160	8	64
		+ 14	
$N = 7$		7	115

$$\sigma = 20 \times \sqrt{\frac{\sum x'^2}{N} - \left(\frac{\sum x'}{N}\right)^2}$$

$$= 20\sqrt{\frac{115}{7} - \left(\frac{7}{7}\right)^2}$$

$$= 78.56$$

Example 4.55: Calculate the standard deviation from the following data:

Size	6	9	12	15	18
Frequency	7	12	19	10	2

NOTES

Solution:

x	Frequency f	Deviation from assumed mean 12	Deviation divided by common factor 3 x'	x' times frequency fx'	x'^2 times frequency fx'^2
6	7	–6	–2	–14	28
9	12	–3	–1	–12	12
12	19	0	0	0	0
15	10	3	1	10	10
18	2	6	2	4	8
	$N = 50$			$\sum fx' = -12$	$\sum fx'^2 = 58$

Since deviations have been divided by a common factor, we use

$$\sigma = C\sqrt{\frac{\sum fx'^2}{N} - \left(\frac{\sum fx'}{N}\right)^2}$$

$$= 3\sqrt{\frac{58}{50} - \left(\frac{-12}{50}\right)^2}$$

$$= 3\sqrt{1.1600 - .0576} = 3 \times 1.05 = 3.15.$$

Example 4.56: Obtain the mean and standard deviation of the first N natural numbers, i.e., of 1, 2, 3, ..., N – 1, N.

Solution: Let x denote the variable which assumes the values of the first N natural numbers.

Then

$$\bar{x} = \frac{\sum_1^N x}{N} = \frac{\frac{N(N+1)}{2}}{N} = \frac{N+1}{2}$$

because $\sum_1^N x = 1 + 2 + 3 + ... + (N-1) + N$

$$= \frac{N(N+1)}{2}$$

To calculate the standard deviation σ, we use 0 as the assumed mean A. Then

$$\sigma = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$$

But $\sum x^2 = 1^2 + 2^2 + 3^2 + ... (N-1)^2 + N^2 = \frac{N(N+1)(2N+1)}{6}$

Therefore

$$\sigma = \sqrt{\frac{N(N+1)(2N+1)}{6N} - \frac{N^2(N+1)^2}{4N^2}}$$

$$= \sqrt{\frac{(N+1)}{2}\left[\frac{2N+1}{3} - \frac{N+1}{2}\right]} = \sqrt{\frac{(N+1)(N-1)}{12}}$$

Thus for first 11 natural numbers

$$\bar{x} = \frac{11+1}{2} = 6$$

and $$\sigma = \sqrt{\frac{(11+1)(11-1)}{12}} = \sqrt{10} = 3.16$$

NOTES

Example 4.57:

	Mid-point *x*	*Frequency* *f*	*Deviation from class of assumed mean* *x′*	*Deviation time frequency* *fx′*	*Squared deviation times frequency* *fx′²*
0–10	5	18	–2	–36	72
10–20	15	16	–1	–16	16
				–52	
20–30	25	15	0	0	0
30–40	35	12	1	12	12
40–50	45	10	2	20	40
50–60	55	5	3	15	45
60–70	65	2	4	8	32
70–80	75	1	5	5	25
				– 60	
		79		60	242
				–52	
				$\sum fx' = 8$	

Solution: Since the deviations are from assumed mean and expressed in terms of class-interval units,

$$\sigma = i \times \sqrt{\frac{\sum x'^2}{N} - \left(\frac{\sum fx'}{N}\right)^2}$$

$$= 10 \times \sqrt{\frac{242}{79} - \left(\frac{8}{79}\right)^2}$$

$$= 10 \times 1.75 = 17.5.$$

4.6.2 Combining Standard Deviations of Two Distributions

If we were given two sets of data of N_1 and N_2 items with means $\bar{x}_1$ and $\bar{x}_2$ and standard deviations σ_1 and σ_2 respectively, we can obtain the mean and standard deviation $\bar{x}$ and σ of the combined distribution by the following Equation:

$$\bar{x} = \frac{N_1\bar{x}_1 + N_2\bar{x}_2}{N_1 + N_2} \qquad ...(4.19)$$

and $$\sigma = \sqrt{\frac{N_1\sigma_1^2 + N_2\sigma_2^2 + N_1(\bar{x}-\bar{x}_1)^2 + N_2(\bar{x}-\bar{x}_2)^2}{N_1 + N_2}} \qquad ...(4.20)$$

Example 4.58: The mean and standard deviations of two distributions of 100 and 150 items are 50, 5 and 40, 6 respectively. Find the standard deviation of all taken together.

NOTES

Solution: Combined mean

$$\bar{x} = \frac{N_1\bar{x}_1 + N_2\bar{x}_2}{N_1 + N_2} = \frac{100 \times 50 + 150 \times 40}{100 + 150}$$

$$= 44$$

Combined standard deviation

$$\sigma = \sqrt{\frac{N_1\sigma_1^2 + N_2\sigma_2^2 + N_1(\bar{x} - \bar{x}_1)^2 + N_2(\bar{x} - \bar{x}_2)^2}{N_1 + N_2}}$$

$$= \sqrt{\frac{100 \times (5)^2 + 150\,(6)^2 + 100\,(44 - 50)^2 + 150\,(44 - 40)^2}{100 + 150}}$$

$$= 7.46.$$

Example 4.59: A distribution consists of three components with 200, 250, 300 items having mean 25, 10 and 15 and standard deviation 3, 4 and 5, respectively. Find the standard deviation of the combined distribution.

Solution: In the usual notations, we are given here

$N_1 = 200, N_2 = 250, N_3 = 300$

$\bar{x}_1 = 25, \bar{x}_2 = 10, \bar{x}_3 = 15$

The Equations (4.19) and (4.20) can easily be extended for combination of three series as

$$\bar{x} = \frac{N_1\bar{x}_1 + N_2\bar{x}_2 + N_3\bar{x}_3}{N_1 + N_2 + N_3}$$

$$= \frac{200 \times 25 + 250 \times 10 + 300 \times 15}{200 + 250 + 300}$$

$$= \frac{12000}{750} = 16$$

And

$$\sigma = \sqrt{\frac{N_1\sigma_1^2 + N_2\sigma_2^2 + N_3\sigma_3^2 + N_1(\bar{x} - \bar{x}_1)^2 + N_2(\bar{x} - \bar{x}_2)^2 + N_3(\bar{x} - \bar{x}_3)^2}{N_1 + N_2 + N_3}}$$

$$= \sqrt{\frac{200 \times 9 + 250 \times 16 + 300 \times 25 + 200 \times 81 + 250 \times 36 + 300 \times 1}{200 + 250 + 300}}$$

$$= \sqrt{51.73} = 7.19.$$

4.6.3 Comparison of Various Measures of Dispersion

The range is the easiest to calculate the measure of dispersion, but since it depends on extreme values, it is extremely sensitive to the size of the sample, and to the sample variability. In fact, as the sample size increases the range increases dramatically, because the more the items one considers, the more likely it is that some item will turn up which is larger than the previous maximum or smaller than the previous minimum. So, it is, in general, impossible to interpret properly the significance of a given range unless the sample size is constant. It is for this reason that there appears to be only one valid application of the range, namely in statistical quality control where the same sample size is repeatedly used, so that comparison of ranges are not distorted by differences in sample size.

NOTES

The quartile deviations and other such positional measures of dispersions are also easy to calculate but suffer from the disadvantage that they are not amenable to algebraic treatment. Similarly, the mean deviation is not suitable because we cannot obtain the mean deviation of a combined series from the deviations of component series. However, it is easy to interpret and easier to calculate than the standard deviation.

The standard deviation of a set of data, on the other hand, is one of the most important statistics describing it. It lends itself to rigorous algebraic treatment, is rigidly defined and is based on all observations. It is, therefore, quite insensitive to sample size (provided the size is 'large enough') and is least affected by sampling variations.

It is used extensively in testing of hypothesis about population parameters based on sampling statistics.

In fact, the standard deviations has such stable mathematical properties that it is used as a standard scale for measuring deviations from the mean. If we are told that the performance of an individual is 10 points better than the mean, it really does not tell us enough, for 10 points may or may not be a large enough difference to be of significance. But if we know that the s for the score is only 4 points, so that on this scale, the performance is $2.5s$ better than the mean, the statement becomes meaningful. This indicates an extremely good performance. This sigma scale is a very commonly used scale for measuring and specifying deviations which immediately suggest the significance of the deviation.

The only disadvantages of the standard deviation lies in the amount of work involved in its calculation, and the large weight it attaches to extreme values because of the process of squaring involved in its calculations.

Check Your Progress

6. What did you meant by measure of central tendency?
7. Define the median.
8. How will you calculate the standard deviation?

4.7 EXAMPLES OF KINETIC THEORY OF GASES BASED ON PROBABILITY

The kinetic theory of gases is a simple, historically significant classical model of the thermodynamic behaviour of gases, with which many principal concepts of thermodynamics were established. The model describes a gas as a large number of identical sub microscopic particles (atoms or molecules), all of which are in constant, rapid, random motion. Their size is assumed to be much smaller than the average distance between the particles. The particles undergo random elastic collisions between themselves and with the enclosing walls of the container. The basic version of the model describes the ideal gas, and considers no other interactions between the particles.

NOTES

The kinetic theory of gases explains the macroscopic properties of gases, such as volume, pressure, and temperature, as well as transport properties, such as viscosity, thermal conductivity and mass diffusivity. The model also accounts for related phenomena, such as Brownian motion. Historically, the kinetic theory of gases was the first explicit exercise of the ideas of statistical mechanics. The kinetic theory of gases explains the macroscopic properties of gases, such as volume, pressure, and temperature, as well as transport properties such as viscosity, thermal conductivity and mass diffusivity. The model also accounts for related phenomena, such as Brownian motion. Kinetic Theory describes the random motion of atoms.

There are Four Assumptions of the Theory

1. A large number of molecules are present, but the space they occupy is also large and keeps the individual molecules far apart.
2. The molecules move randomly.
3. The collisions between molecules are elastic and therefore exert no net forces.
4. The molecules obey Newtonian mechanics.

1. **The Kinetic Molecular Theory of Matter States:** Matter is made up of particles that are constantly moving. All particles have energy, but the energy varies depending on the temperature the sample of matter is in. This in turn determines whether the substance exists in the solid, liquid, or gaseous state.
2. **Diffusion:** The kinetic theory of matter is also shown by the process of diffusion. Diffusion is the movement of particles from a high concentration to a low concentration. It can be seen as a spreading-out of particles resulting in their even distribution. Placing a drop of food colouring in water provides a visual representation of this process – the colour slowly spreads out through the water. If matter were not made of particles, then we would simply see a clump of colour, since there would be no smaller units that could move about and mix in with the water.

The Application of Kinetic Theory to Ideal Gases Makes the Following Assumptions:

1. The gas consists of very small particles. This smallness of their size is such that the sum of the volume of the individual gas molecules is negligible compared to the volume of the container of the gas. This is equivalent to stating that the average distance separating the gas particles is large compared to their size, and that the elapsed time of a collision between particles and the container's wall is negligible when compared to the time between successive collisions.
2. The number of particles is so large that a statistical treatment of the problem is well justified. This assumption is sometimes referred to as the thermodynamic limit.

3. The rapidly moving particles constantly collide among themselves and with the walls of the container. All these collisions are perfectly elastic, which means the molecules are perfect hard spheres.
4. Except during collisions, the interactions among molecules are negligible. They exert no other forces on one another.

Thus, the dynamics of particle motion can be treated classically, and the equations of motion are time-reversible.

As a simplifying assumption, the particles are usually assumed to have the same mass as one another; however, the theory can be generalized to a mass distribution, with each mass type contributing to the gas properties independently of one another in arrangement with Dalton's Law of partial pressures. Many of the model's predictions are the same whether or not collisions between particles are included, so they are often neglected as a simplifying assumption in derivations.

More modern developments relax these assumptions and are based on the Boltzmann equation. These can accurately describe the properties of dense gases, because they include the volume of the particles as well as contributions from intermolecular and intramolecular forces as well as quantized molecular rotations, quantum rotational-vibrational symmetry effects, and electronic excitation.

4.8 CURVE FITTING

In this section, we consider the problem of approximating an unknown function whose values, at a set of points, are generally known only empirically and are, thus subject to inherent errors, which may sometimes be appreciably larger in many engineering and scientific problems. In these cases, it is required to derive a functional relationship using certain experimentally observed data. Here the observed data may have inherent or round-off errors, which are serious, making polynomial interpolation for approximating the function inappropriate. In polynomial interpolation the truncation error in the approximation is considered to be important. But when the data contains round-off errors or inherent errors, interpolation is not appropriate.

The subject of this section is curve fitting by least square approximation. Here we consider a technique by which noisy function values are used to generate a smooth approximation. This smooth approximation can then be used to approximate the derivative more accurately than with exact polynomial interpolation.

There are situations where interpolation for approximating function may not be efficacious procedure. Errors will arise when the function values $f(x_i)$, $i = 1, 2, \ldots, n$ are observed data and not exact. In this case, if we use the polynomial interpolation, then it would reproduce all the errors of observation. In such situations one may take a large number of observed data, so that statistical laws in effect cancel the errors introduced by inaccuracies in the measuring equipment. The approximating function is then derived, such that the sum of the squared deviation between the observed values and the estimated values are made as small as possible.

Mathematically, the problem of curve fitting or function approximation may be stated as follows:

NOTES

To find a functional relationship $y = g(x)$, that relates the set of observed data values $P_i(x_i, y_i)$, $i = 1, 2, ..., n$ as closely as possible, so that the graph of $y = g(x)$ goes near the data points P_i's though not necessarily through all of them.

The first task in curve fitting is to select a proper form of an approximating function $g(x)$, containing some parameters, which are then determined by minimizing the total squared deviation.

For example, $g(x)$ may be a polynomial of some degree or an exponential or logarithmic function. Thus $g(x)$ may be any of the following:

(*i*) $g(x) = \alpha + \beta x$ (*ii*) $g(x) = \alpha + \beta x + \gamma x^2$

(*iii*) $g(x) = \alpha e^{\beta x}$ (*iv*) $g(x) = \alpha e^{-\beta x}$

(*v*) $g(x) = \alpha \log(\beta x)$

Here α, β, γ are parameters which are to be evaluated so that the curve $y = g(x)$, fits the data well. A measure of how well the curve fits is called the goodness of fit.

In the case of least square fit, the parameters are evaluated by solving a system of normal equations, derived from the conditions to be satisfied so that the sum of the squared deviations of the estimated values from the observed values, is minimum.

4.8.1 Method of Least Squares

Let (x_1, f_1), (x_2, f_2),..., (x_n, f_n) be a set of observed values and $g(x)$ be the approximating function. We form the sums of the squares of the deviations of the observed values f_i from the estimated values $g(x_i)$,

i.e.,
$$S = \sum_{i=1}^{n} \{f_i - g(x_i)\}^2 \tag{4.21}$$

The function $g(x)$ may have some parameters, α, β, γ. In order to determine these parameters we have to form the necessary conditions for S to be minimum, which are

$$\frac{\partial S}{\partial \alpha} = 0, \frac{\partial S}{\partial \beta} = 0, \frac{\partial S}{\partial \gamma} = 0 \tag{4.22}$$

These equations are called normal equations, solving which we get the parameters for the best approximate function $g(x)$.

Curve Fitting by a Straight Line: Let $g(x) = \alpha + \beta x$, be the straight line which fits a set of observed data points (x_i, y_i), $i = 1, 2, ..., n$.

Let S be the sum of the squares of the deviations $g(x_i) - y_i$, $i = 1, 2,...,n$, given by,

$$S = \sum_{i=1}^{n} \left(\alpha + \beta x_i - y_i\right)^2 \tag{4.23}$$

NOTES

We now employ the method of least squares to determine α and β so that S will be minimum. The normal equations are,

$$\frac{\partial S}{\partial \alpha} = 0, \text{ i.e., } \sum_{i=1}^{n} (\alpha + \beta\, x_i - y_i) = 0 \qquad (4.24)$$

And,
$$\frac{\partial S}{\partial \beta} = 0, \text{ i.e., } \sum_{i=1}^{n} x_i (\alpha + \beta\, x_i - y_i) = 0 \qquad (4.25)$$

These conditions give,

$$n\alpha + S_1\beta - S_{01} = 0$$
$$S_1\alpha + S_2\beta - S_{11} = 0$$

where,
$$S_1 = \sum_{i=1}^{n} x_i, \quad S_{01} = \sum_{i=1}^{n} y_i, \quad S_2 = \sum_{i=1}^{n} x_i^2, \quad S_{11} = \sum_{i=1}^{n} x_i y_i$$

Solving,
$$\frac{\alpha}{-S_1S_{11} + S_1S_2} = \frac{\beta}{nS_{11} - S_1S_{01}} = \frac{1}{nS_2 - S_1^2}. \quad \text{Also} \quad \alpha = \frac{S_{01}}{n} - \beta\frac{S_1}{n}.$$

Algorithm: Fitting a straight line $y = a + bx$.

Step 1: Read n [n being the number of data points]

Step 2: Initialize : sum $x = 0$, sum $x^2 = 0$, sum $y = 0$, sum $xy = 0$

Step 3: For $j = 1$ to n compute

Begin

Read data x_j, y_j

Compute sum x = sum $x + x_j$

Compute sum $x^2 + x_j \times x_j$

Compute sum y = sum $y + y_i \times y_j$

Compute sum xy = sum $xy + x_j \times y_j$

End

Step 4: Compute $b = (n \times \text{sum } xy - \text{sum } x \times \text{sum } y)/ (n \times \text{sum } x^2 - (\text{sum } x)^2)$

Step 5: Compute x bar = sum x / n

Step 6: Compute y bar = sum y / n

Step 8: Compute $a = y$ bar $- b \times x$ bar

Step 9: Write a, b

Step 10: For $j = 1$ to n

Begin

Compute y estimate $= a + b \times x$

write x_j, y_j, y estimate

End

Step 11: Stop

Curve Fitting by a Quadratic (A Parabola): Let $g(x) = a + bx + cx^2$, be the approximating quadratic to fit a set of data (x_i, y_i), $i = 1, 2, ..., n$. Here the parameters are to be determined by the method of least squares, i.e., by minimizing the sum of the squares of the deviations given by,

$$S = \sum_{i=1}^{n} (a + bx_i + cx_i^2 - y_i)^2 \tag{4.26}$$

NOTES

Thus the normal equations, $\frac{\partial S}{\partial a} = 0, \quad \frac{\partial S}{\partial b} = 0, \quad \frac{\partial S}{\partial c} = 0$, are as follows: (4.27)

$$\begin{aligned} &\sum_{i=1}^{n} (a + bx_i + cx_i^2 - y_i) = 0 \\ &\sum_{i=1}^{n} x_i (a + bx_i + cx_i^2 - y_i) = 0 \\ &\sum_{i=1}^{n} x_i^2 (a + bx_i + cx_i^2 - y_i) = 0. \end{aligned} \tag{4.28}$$

These equations can be rewritten as,

$$\begin{aligned} &na + s_1 b + s_2 c - s_{01} = 0 \\ &s_1 a + s_2 b + s_3 c - s_{11} = 0 \\ &s_2 a + s_3 b + s_4 c - s_{21} = 0 \end{aligned} \tag{4.29}$$

where $s_1 = \sum_{i=1}^{n} x_i, \quad s_2 = \sum_{i=1}^{n} x_i^2, \quad s_3 = \sum_{i=1}^{n} x_i^3, \quad s_4 = \sum_{i=1}^{n} x_i^4$

$$s_{01} = \sum_{i=1}^{n} y_i, \quad s_{11} = \sum_{i=1}^{n} x_i \quad y_i, \quad s_{21} = \sum_{i=1}^{n} x_i^2 y_i \tag{4.30}$$

It is clear that the normal equations form a system of linear equations in the unknown parameters a, b, c. The computation of the coefficients of the normal equations can be made in a tabular form for desk computations as shown below.

	x	x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
	1	x_1	y_1	x_1^2	x_1^3	x_1^4	$x_1 y_1$	$x_1^2 y_1$
	2	x_2	y_2	x_2^2	x_2^3	x_2^4	$x_2 y_2$	$x_2^2 y_2$
	...	...	...	...	...	...	...	...
	n	x_n	y_n	x_n^2	x_n^3	x_n^4	$x_n y_n$	$x_n^2 y_n$
Sum		s_1	s_{01}	s_2	s_3	s_4	s_{11}	s_{21}

The system of linear equations can be solved by Gaussian elimination method. It may be noted that number of normal equations is equal to the number of unknown parameters.

Example 4.60: Find the straight line fitting the following data:

x_i	4	6	8	10	12
y_1	13.72	12.90	12.01	11.14	10.31

Solution: Let $y = a + bx$, be the straight line which fits the data. We have the normal equations $\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$ for determining a and b, where

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$S = \sum_{i=1}^{5}(y_i - a - bx_i)^2$.

Thus, $\sum_{i=1}^{5} y_i - na - b\sum_{i=1}^{5} x_i = 0$

and , $\sum_{i=1}^{5} x_i y_i - a\sum_{I=1}^{5} x_i - b\sum_{i=1}^{5} x_i^2 = 0$

The coefficients are computed in the table below.

	x_i	y_i	x_i^2	$x_i y_i$
	4	13.72	16	54.88
	6	12.90	36	77.40
	8	12.01	64	96.08
	10	11.14	100	111.40
	12	10.31	144	123.72
Sum	40	60.08	360	463.48

Thus the normal equations are,

$$5a + 40b - 60.08 = 0$$
$$40a + 360b - 463.48 = 0$$

Solving these two equations we obtain,

$$a = 15.448, b = 0.429$$

Thus, $y = g(x) = 15.448 - 0.429\,x$, is the straight line fitting the data.

Example 4.61: Use the method of least square approximation to fit a straight line to the following observed data:

x_i	60	61	62	63	64
y_i	40	40	48	52	55

Solution: Let the straight line fitting the data be $y = a + bx$. The data values being large, we can use a change in variable by substituting $u = x - 62$ and $v = y - 48$.

Let $v = A + B\,u$, be a straight line fitting the transformed data, where the normal equations for A and B are,

$$\sum_{i=1}^{5} v_i = 5\,A + B\sum_{i=1}^{5} u_i$$
$$\sum_{i=1}^{5} u_i v_i = A\sum_{i=1}^{5} u_i + B\sum_{i=1}^{5} u_i^2$$

NOTES

The computation of the various sums are given in the table below,

	x_i	y_i	u_i	v_i	$u_i v_i$	u_i^2
	60	40	–2	–8	16	4
	61	42	–1	–6	6	1
	62	48	0	0	0	0
	63	52	1	4	4	1
	64	55	2	7	14	4
Sum			0	–3	40	10

Thus the normal equations are,

$$-3 = 5A \quad \text{and} \quad 40 = 10B$$

$$\therefore A = -\frac{3}{5} \quad \text{and} \quad B = 4$$

This gives the line, $v = -3/5 + 4u$

or, $20u - 5v - 3 = 0.$

Transforming we get the line,

$$20\,(x - 62) - 5\,(y - 48) - 3 = 0$$

or, $$20\,x - 5y - 1003 = 0$$

Curve Fitting with an Exponential Curve: We consider a two parameter exponential curve as,

$$y = ae^{-bx} \tag{4.31}$$

For determining the parameters, we can apply the principle of least squares by first using a transformation,

$z = \log y$, so that Equation (4.31) is rewritten as, (4.32)

$$z = \log a - bx \tag{4.33}$$

Thus, we have to fit a linear curve of the form $z = \alpha + \beta \quad x$, with $z - x$ variables and then get the parameters a and b as,

$$a = e^{\alpha}, \quad b = -\beta \tag{4.34}$$

Thus proceeding as in linear curve fitting,

$$\beta = \frac{n \sum_{i=1}^{n} x_i \log y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \log y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \tag{4.35}$$

and, $\alpha = \bar{z} - p\bar{x}$, where $\bar{x} = \left(\sum_{i=1}^{n} x_i\right)\Big/ n$, $\bar{z} = \left(\sum_{i=1}^{n} \log y_i\right)\Big/ n$ (4.43)

After computing α and β, we can determine a and b given by Equation (4.33). Finally, the exponential curve fitting data set is given by Equation (4.31).

Algorithm: To fit a straight line for a given set of data points by least square error method.

Step 1: Read the number of data points, i.e., n

Step 2: Read values of data-points, i.e., Read (x_i, y_i) for $i = 1, 2, ..., n$

Step 3: Initialize the sums to be computed for the normal equations,

i.e., $sx = 0$, $sx^2 = 0$, $sy = 0$, $syx = 0$

Step 4: Compute the sums, i.e., For $i = 1$ to n do

Begin

$$sx = sx + x_i$$
$$sx^2 = sx^2 + x_i^2$$
$$sy = sy + y_i$$
$$syx = syx + x_i y_i$$

End

Step 5: Solve the normal equations, i.e., solve for a and b of the line $y = a + bx$

Compute $d = n * sx^2 - sx * sx$

$$b = (n * sxy - sy * sx) / d$$
$$x\text{bar} = sx / n$$
$$y\text{bar} = sy / n$$
$$a = y\text{bar} - b * x\,\text{bar}$$

Step 6: Print values of a and b

Step 7: Print a table of values of $x_i,\ \ y_i,\ \ y_{pi} = a + bx_i$ for $i = 1,\ 2,\ ...,\ n$

Step 8: Stop

Algorithm: To fit a parabola $y = a + bx + cx^2$, for a given set of data points by least square error method.

Step 1: Read n, the number of data points

Step 2: Read (x_i, y_i) for $i = 1, 2, ..., n,$ the values of data points

Step 3: Initialize the sum to be computed for the normal equations,

i.e., $sx = 0$, $sx^2 = 0$, $sx^3 = 0$, $sx^4 = 0$, $sy = 0$, $sxy = 0$.

Step 4: Compute the sums, i.e., For $i = 1$ to n do

Begin

$$sx = sx + x_i$$
$$x^2 = x_i * x_i$$
$$sx^2 = sx^2 + x^2$$
$$sx^3 = sx^3 + x_i * x^2$$
$$sx^4 = sx^4 + x^2 * x^2$$
$$sy = sy + y_i$$
$$sxy = sxy + x_i * y_i$$
$$sx^2 y = sx^2 y + x^2 * y_i$$

End

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Step 5: Form the coefficients $\{a_{ij}\}$ matrix of the normal equations, i.e.,

$$a_{11} = n, \quad a_{21} = sx, \quad a_{31} = sx^2$$
$$a_{12} = sx, \quad a_{22} = sx^2, \quad a_{32} = sx^3$$
$$a_{13} = sx^2, \quad a_{23} = sx^3, \quad a_{33} = sx^4$$

Step 6: Form the constant vector of the normal equations.

$$b_1 = sy, b_2 = sxy, b_3 = sx^2y$$

Step 7: Solve the normal equation by Gauss-Jordan method

$$a_{12} = a_{12} / a_{11}, a_{13} = a_{13} / a_{11}, b_1 = b_1 / a_{11}$$
$$a_{22} = a_{22} - a_{21}a_{12}, a_{23} = a_{23} - a_{21}a_{13}$$
$$b_2 = b_2 - b_1a_{21}$$
$$a_{32} = a_{32} - a_{31}a_{12}$$
$$a_{33} = a_{33} - a_{31}a_{13}$$
$$b_3 = b_3 - b_1a_{31}$$
$$a_{23} = a_{23} / a_{22}$$
$$b_2 = b_2 / a_{22}$$
$$a_{33} = a_{33} - a_{23}\, a_{32}$$
$$b_3 = b_3 - a_{32}\, b_2$$
$$c = b_3 / a_{33}$$
$$b = b_2 - c\, a_{23}$$
$$a = b_1 - b\, a_{12} - c\, a_{13}$$

Step 8: Print values of a, b, c (the coefficients of the parabola)

Step 9: Print the table of values of x_k, y_k and y_{pk} where $y_{pk} = a + bx_k + cx^2k$,

i.e., print x_k, y_k, y_{pk} for $k = 1, 2, \ldots, n.$

Step 10: Stop.

Check Your Progress

9. What do you understand by kinetic theory of gases?
10. What is curve fitting by least square approximation?
11. State the function approximation.

4.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. If there are n objects and they can be placed in any arrangement or order, then any given order of these n objects is called a permutation of the n objects.
2. A collection of all possible outcomes of an experiment is said to be an event.
3. In probability and statistics, a random variable or a random quantity is a variable whose possible values are outcomes of a random phenomenon. More specifically, a random variable is defined as a function that maps the outcomes of an unpredictable process to numerical quantities, typically real numbers.

NOTES

4. Definition of a Normal Distribution: A normal distribution is a continuous distribution given by

$$y = \frac{1}{\sigma\sqrt{(2\pi)}}\, e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

where x is a continuous normal variate distributed with probability density function $f(x) = \dfrac{1}{\sigma\sqrt{(2\pi)}}\, e^{-\frac{1}{2}[(x-m)/\sigma]^2}$, with mean m and standard deviation σ.

5. The Central Limit Theorem (CLT) is a statistical theory which states that given a sufficiently large sample size from a population with a finite level of variance, the mean of all samples from the same population will be approximately equal to the mean of the population.

6. A measure of central tendency is any precise method of specifying this 'Central Value'. In the simplest form, the measure of central tendency is an average of a set of measurements, where the word *average* refers to as mean, median, mode or other measures of location.

7. The second measure of central tendency that has a wide usage in statistical works is the median. Median is that value of a variable which divides the series in such a manner that the number of items below it is equal to the number of items above it. Half the total number of observations lie below the median, and half above it. The median is thus a positional average.

8. The calculation of standard deviation differs in the following respects from that of mean deviation. First, in calculating standard deviation, the deviations are squared. This is done so as to get rid of negative signs without committing algebraic violence. Further, the squaring of deviations provides added weight to the extreme items, a desirable feature for certain types of series.

9. The kinetic theory of gases is a simple, historically significant classical model of the thermodynamic behaviour of gases, with which many principal concepts of thermodynamics were established. It is explains the macroscopic properties of gases, such as volume, pressure, and temperature, as well as transport properties, such as viscosity, thermal conductivity and mass diffusivity.

10. The subject of this section is curve fitting by least square approximation. Here we consider a technique by which noisy function values are used to generate a smooth approximation. This smooth approximation can then be used to approximate the derivative more accurately than with exact polynomial interpolation.

11. To find a functional relationship $y = g(x)$, that relates the set of observed data values $P_i(x_i, y_i)$, $i = 1, 2,..., n$ as closely as possible, so that the graph of $y = g(x)$ goes near the data points P_i's though not necessarily through all of them.

4.10 SUMMARY

- The probability of any two heads out of three tosses would be different than having two heads and a tail in that order.

NOTES

- Primarily statistics was supposed to be the science of kings used for the purpose of administration, but later on it was regarded as a branch of economics.
- Statistics, which was accepted for some time as necessarily a branch of economics has now become so popular and its application so wide that no branch of human knowledge escapes its approach.
- An astronomer uses statistical methods in making predictions about eclipses, a biologist utilzes them to generalize the laws of variations and heredity, a meteorologist uses them for weather forecasts, regarding temperature pressure and rainfall, etc.
- Probability is the measure of the likelihood that an event will occur. Principally, the probability quantifies as a number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty. The higher the probability of an event, the more likely it is that the event will occur.
- Two or more events are said to be mutually exclusive if the happening or occurrence of any one of them excludes the happening of the others.
- The simultaneous occurrence of two or more events in connection with each other is said to be a compound event.
- Two or more events are said to be dependent or independent according as the occurrence of one does or does not affect the occurrence of the other or others. The dependent events are sometimes known as contingent.
- A continuous random variable is a random variable whose cumulative distribution function is continuous everywhere.
- A normal curve is symmetrical about the point $x = 0$ where the ordinate has its maximum value. In a normal curve, the mean, the median and mode coincide.
- The total area of a normal curve being treated as unity, the probability corresponding to any interval in the range of the variate is measured by the area under the curve within that interval.
- In probability theory, the Central Limit Theorem (CLT) establishes that, in some situations when independent random variables are added then their appropriately normalized sum tends toward a normal distribution, usually a 'Bell Curve' even if the original variables themselves are not normally distributed.
- Typically the most commonly used measures are arithmetic mean, mode and median. These values are very useful not only in presenting the overall picture of the entire data but also for the purpose of making comparisons among two or more sets of data.
- There are several commonly used measures, such as arithmetic mean, mode and median. These values are very useful not only in presenting the overall picture of the entire data, but also for the purpose of making comparisons among two or more sets of data.
- The median of ungrouped data is found easily if the items are first arranged in order of the magnitude. The median may then be located simply by counting, and its value can be obtained by reading the value of the middle observations.

NOTES

- By far the most universally used and the most useful measure of dispersion is the standard deviation or root mean square deviation about the mean.
- The range is the easiest to calculate the measure of dispersion, but since it depends on extreme values, it is extremely sensitive to the size of the sample, and to the sample variability.
- In polynomial interpolation the truncation error in the approximation is considered to be important. But when the data contains round-off errors or inherent errors, interpolation is not appropriate.
- In the case of least square fit, the parameters are evaluated by solving a system of normal equations, derived from the conditions to be satisfied so that the sum of the squared deviations of the estimated values from the observed values, is minimum.
- The system of linear equations can be solved by Gaussian elimination method. It may be noted that number of normal equations is equal to the number of unknown parameters.

4.11 KEY TERMS

- **Joint occurrence:** The simultaneous occurrence of two or more events in connection with each other is said to be a compound event.
- **Random variable:** In probability and statistics, a random variable or a random quantity is a variable whose possible values are outcomes of a random phenomenon.
- **Continuous random variables:** A continuous random variable is a random variable whose cumulative distribution function is continuous everywhere.
- **Central Limit Theorem (CLT):** The Central Limit Theorem (CLT) is a statistical theory which states that given a sufficiently large sample size from a population with a finite level of variance, the mean of all samples from the same population will be approximately equal to the mean of the population.
- **Kinetic theory of gases:** The kinetic theory of gases explains the macroscopic properties of gases, such as volume, pressure, and temperature, as well as transport properties, such as viscosity, thermal conductivity and mass diffusivity. The model also accounts for related phenomena, such as Brownian motion.

4.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the combination.
2. What is mutually exclusive events?
3. State the theorem of total probability.
4. State the multiplication low of probability.

NOTES

5. Give the statement of Bayes' theorem.
6. Give the characteristics of binomial distribution.
7. What do you understand by poission distribution?
8. Define the median.
9. What is standard error?
10. What do you understand by measure of dispersion?
11. State the kinetic molecular theory of matter.
12. Define the curve fitting.
13. State the least square method.

Long-Answer Questions

1. Discuss about the permutation and combination giving examples.
2. What is probability? Explain about the simple properties of probability giving appropriate examples.
3. Describe the theorem of probability with the help of examples.
4. Explain in detail about the normal, binomial and poission distribution with the help of examples.
5. Explain in detail about the mean, median and mode giving appropriate examples.
6. Elaborate on the standard error with various types of examples.
7. Briefly explain about the kinetic theory of gases with the help of examples.
8. Analyse the curve fitting with appropriate examples.

4.13 FURTHER READING

Dass, HK. 2008. *Mathematical Physics*. New Delhi: S. Chand & Company.

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Datta, K. B. 2002. *Matrix and Linear Algebra.* New Delhi: Prentice Hall of India Pvt. Ltd.

Shanti Narayan, P.K. Mittal.1987. *A Textbook of Vector Calculus*. New Delhi: S. Chand & Company.