M.Sc. Final Year

Mathematics, Paper Option (F)

INTEGRAL TRANSFORMS WITH APPLICATIONS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

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SYLLABI-BOOK MAPPING TABLE

Integral Transforms with Applications

Syllabi	Mapping in Book
UNIT-I The Laplace transforms & its inversions: Definition. Laplace transform of elementary Sectionally continuous and exponential order function including its existence, some important properties of Laplace transforms of derivatives and integrals. Multiplication and division by '*t*' periodic functions. Initial and final value theorems, Laplace transform of some special function. Definition and Uniqueness theorem of inverse Laplace transform. Inversion of some elementary functions, some properties of inverse Laplace transform. Inverse Laplace transform of derivatives and integrals. Multiplication and division by powers of '*s*'. The convolution property. Complex inversion formula, Heaviside expansion formula, Evaluation of integrals.	**Unit-1:** Laplace Transforms and Its Inversions **(Pages 3-103)**
UNIT-II Application of Laplace Transforms. Ordinary differential equations with constant coefficients, ordinary differential equations with variable coefficient. Simultaneous ordinary differential equations. Partial differential equations. Applications to Mechanics, electrical circuits, beams. Application to solution of integral equations - integral equations of convolution type, Abel's integral equation. lntegro - differential equation, difference and differential - difference equations.	**Unit-2:** Application of Laplace Transforms **(Pages 105-163)**
UNIT-III Fourier Series and lntegrals: Fourier series, Odd and Even functions, Half range Fourier sine and cosine series complex form of Fourier series, Parseval's identity for Fourier Cosine and Sine finite Fourier transforms, the Fourier integral/at including its complex form, Fourier transforms, including sine and cosine transforms convolution theorem, Parseval's identity for Fourier integrals. Relations between Fourier and Laplace transforms, Multiple finity Fourier transform, Solution of simple partial differential equations by means of Fourier transforms.	**Unit-3:** Fourier Series and Integrals **(Pages 165-224)**
UNIT-IV Mellin and Hankel Transforms: Elementery properties of the Mellin Transforms, Mellin transforms of derivatives and Integrals Mellin - Inversion Theorem of Some. The Solution convolution Theorem integral equations. The distribution of Potential in a wedge. Application to the summation of series. Elementary properties of Hankel transforms Hankel inversion theorem, Hankel transforms of the derivatives of functions and some elementary function, Relations between Fourier and Hankel Transform, Parseval Relation for Hankel Transforms, The use of Hankel Transforms in the solution of simple partial differential equations.	**Unit-4:** Mellin and Hankel Transfroms **(Pages 225-254)**
UNIT-V Application to Boundary value problems: Boundary value problems ·involving partial differential equations, on dimensional heat conduction equation, one dimensional wave equation, longitudinal and Transverse Vibration of a beam, Solution of boundary value problems by Laplace transform. Simple boundary value problems with applications of Fourier transform.	**Unit-5:** Application to Boundary Value Problems **(Pages 255-291)**

CONTENTS

INTRODUCTION

NOTES

In mathematics, integral transform theory is the study of transforms, which relate a function in one domain to another function in a second domain. The essence of transform theory is that by a suitable choice of basis for a vector space a problem may be simplified or diagonalized as in spectral theory.

In mathematics, the Laplace Transform (LT), named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency). The transform has many applications in science and engineering because it is a tool for solving differential equations. In particular, it transforms differential equations into algebraic equations and convolution into multiplication. In practice, it is typically more convenient to decompose a Laplace transform into known transforms of functions obtained from a table, and construct the inverse by inspection. Laplace's use of generating functions was similar to what is now known as the Z-transform. Laplace also recognised that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space, because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.

The Laplace transform is similar to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Techniques of complex variables can also be used to directly study Laplace transforms.

In mathematics, a Fourier Transform (FT) is a mathematical transform that decomposes functions depending on space or time into functions depending on spatial or temporal frequency, such as the expression of a musical chord in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of space or time.

This book is divided into five units. The topics discussed is designed to be a comprehensive and easily accessible book covering, Laplace transform, inversion of some elementary functions, initial and final value theorems, inverse Laplace transforms, the convolution property, ordinary differential equations with constant coefficients, simultaneous ordinary differential equations, partial differential equations, Abel's integral equation, integro-differential equation, differential-difference equations, Fourier series, odd and even functions, Fourier integral/at including its complex form, Fourier transforms, convolution theorem including sine and cosine transforms, elementary properties of the Mellin transforms, Mellin transforms of derivatives, Hankel inversion theorem, boundary value problems involving partial differential equations, one dimensional heat conduction equation, one dimensional wave equation and longitudinal and transverse vibration of a beam.

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The book follows the Self-Instructional Mode (SIM) wherein each unit begins with an 'Introduction' to the topic. The 'Objectives' are then outlined before going on to the presentation of the detailed content in a simple and structured format. 'Check Your Progress' questions are provided at regular intervals to test the student's understanding of the subject. 'Answers to Check Your Progress Questions', a 'Summary', a list of 'Key Terms', and a set of 'Self-Assessment Questions and Exercises' are provided at the end of each unit for effective recapitulation.

UNIT 1 LAPLACE TRANSFORMS AND ITS INVERSIONS

NOTES

Structure

1.0 INTRODUCTION

The Laplace transform, named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency). The transform has many applications in science and engineering because it is a tool for solving differential equations. In

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particular, it transforms differential equations into algebraic equations and convolution into multiplication. The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform in his work on probability theory. Laplace wrote extensively about the use of generating functions in Essai Philosophique Sur Les Probabilités (1814), and the integral form of the Laplace transform evolved naturally as a result. Laplace's use of generating functions was similar to what is now known as the *z*-transform, and he gave little attention to the continuous variable case which was discussed by Niels Henrik Abel. The theory was further developed in the 19th and early 20th centuries by Mathias Lerch, Oliver Heaviside, and Thomas Bromwich.

In inversion of some elementary functions an integral transform is useful to convent a complicated problem into simpler one. Laplace transform is one of the important type of integral transform that can be used to solve integral and differential equations. As time approaches zero, the initial value theorem is utilised to link frequency domain expressions to time domain behaviour. As time approaches infinity, the Final Value Theorem (FVT) is one of several comparable theorems used to relate frequency domain expressions to time domain behaviour.

The inverse Laplace transform of a function $F(s)$ is the piecewise-continuous and exponentially-restricted real function $f(t)$. The Laplace transform and the inverse Laplace transform together have a number of properties that make them useful for analysing linear dynamical systems. Two integrable functions have the same Laplace transform only if they differ on a set of Lebesgue measure zero. This means that, on the range of the transform, there is an inverse transform. In fact, besides integrable functions, the Laplace transform is a one-to-one mapping from one function space into another in many other function spaces as well, although there is usually no easy characterization of the range. Typical function spaces in which this is true include the spaces of bounded continuous functions, the space $L^{\infty}(0, \infty)$, or more generally tempered distributions on $(0, \infty)$. The Laplace transform is also defined and injective for suitable spaces of tempered distributions. The Laplace transform and the inverse Laplace transform together have a number of properties that make them useful for analysing linear dynamical systems. The Heaviside step function, or the unit step function, usually denoted by H or θ (but sometimes u, 1 or $\mathbb{1}$), is a step function, named after Oliver Heaviside (1850–1925), the value of which is zero for negative arguments and one for positive arguments. It is an example of the general class of step functions, all of which can be represented as linear combinations of translations of this one.

In this unit, you will learn about the Laplace transform, Laplace transform of elementary continuous and exponential order function, some important properties of Laplace transforms of derivatives and integrals, inversion of some elementary functions, initial and final value theorems, multiplication and division by '*t*' periodic functions, inverse Laplace transforms, some elementary inverse Laplace transform, uniqueness theorem of inverse Laplace transform, inverse Laplace transform of derivatives and integrals, multiplication and division by powers of '*s*', the convolution property, complex inversion formula and Heaviside expansion formula, and evaluation of integrals.

1.1 OBJECTIVES

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After going through this unit, you will be able to:

- Explain the Laplace transform
- Analyse the elementary theorems
- Describe the Laplace transform of standard functions
- Explain the inversion of some elementary functions
- Discuss the initial and final value theorem
- Elaborate on the multiplication and division by '*t*' periodic functions
- Explain the inverse Laplace transform and standard formula
- Explain the uniqueness theorem of inverse Laplace transform
- Discuss the basic concept of inverse Laplace transform of derivatives and integrals
- Discuss the multiplication and division by powers of '*s*'
- Explain convolution property, the complex inversion formula and Heaviside expansion formula
- Describe the evaluation of integrals

1.2 LAPLACE TRANSFORM

In mathematics, the Laplace transform is a widely used integral transform and is denoted by $\mathcal{L}\{f(t)\}$. It is a linear operator of a function $f(t)$ including a real argument t ($t \geq 0$) that transforms it to a function $F(s)$ with a complex argument s. As a bijective transformation the respective pairs of $f(t)$ and $F(s)$ are matched in tables. The Laplace transform has the significant property so that various relationships and operations over the originals $f(t)$ correspond to simpler relationships and operations over the images $F(s)$.

The Laplace transform can be related to the Fourier transform. The Fourier transform resolves a function or signal into its modes of vibration and the Laplace transform resolves a function into its moments. The original signal depends on time and therefore Laplace transform is called the *time domain* representation of the signal, whereas the Fourier transform depends on frequency and is called the *frequency domain* representation of the signal. Similar to the Fourier transform, the Laplace transform is also used for solving differential and integral equations. In physics and engineering, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems.

Switching from operations of calculus to *algebraic* operations on transforms is known as **operational calculus** which is an essential area of applied mathematics and with regard to an engineer, the Laplace transform method is basically a very essential operational technique. It is particularly useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function, not merely a sine or cosine.

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Another benefit of the Laplace transform is that it helps in solving the problems in a straightforward manner, initial value problems regardless of initially obtaining a basic solution, and nonhomogeneous differential equation exclusive of initially answering the corresponding homogeneous equation.

In this unit we consider Laplace transforms from a practical approach and exemplify their usage through essential engineering problems wherein many of them are associated with *ordinary* differential equations. Partial differential equations can also be treated by Laplace transforms.

The Laplace transform is named in honor of mathematician and astronomer Pierre-Simon Laplace, who used the transform in his work on probability theory. Leonhard Euler considered integrals of the form,

$$z = \int X(x) e^{ax}\, dx \text{ and } z = \int X(x) x^A\, dx$$

These integrals were the solutions of differential equations but were not used in the long run. Joseph Louis Lagrange was an admirer of Euler and in his work on integrating probability density functions, explored expressions of the form,

$$\int X(x) e^{-ax} a^x\, dx$$

This was interpreted within modern Laplace transform theory. These integrals have attracted Laplace's attention for using the integrals themselves as solutions of equations. He used an integral of the form,

$$\int x^s \phi(s)\, dx$$

This integral was akin to a Mellin transform, to transform the whole of a difference equation in order to look for solutions of the transformed equation.

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\, dt$$

The parameter s is a complex number $s = \sigma + i\omega$ with real numbers σ and ω. The meaning of the integral depends on types of functions of interest. A necessary condition for existence of the integral is that f have to be the neighborhood integrable on $(0, \infty)$. For neighborhood integrable functions that decay at infinity or are of exponential type, the integral can be understood as a (proper) Lebesgue integral. Though, for various applications it is considered as a conditionally convergent improper integral at ∞.

The Laplace transform can be defined of a finite Borel measure μ by the Lebesgue integral of the form,

$$(\mathcal{L}\mu)(s) = \int_{[0,\infty)} e^{-st} d\mu(t)$$

As a special case μ is a probability measure or more specifically the Dirac delta function. In operational calculus, the Laplace transform of a measure is treated as the measure of a distribution function f. In such case the expression is of the form,

$$(\mathcal{L}f)(s) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

Here the lower limit of 0^- is short notation that means, $\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty}$

This limit emphasizes that any point located at 0 is completely acquired by the Laplace transform.

Bilateral Laplace Transform

When the Laplace transform is defined without condition then the unilateral or one-sided transform is normally considered. Alternatively, the Laplace transform can be defined as the *bilateral Laplace transform* or two-sided Laplace transform by extending the limits of integration to be the entire real axis. If that is done the common unilateral transform simply becomes a special case of the bilateral transform where the definition of the function being transformed is multiplied by the Heaviside step function. The bilateral Laplace transform is defined as follows:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

Inverse Laplace Transform

The inverse Laplace transform is also known by various names as the Bromwich integral, the Fourier-Mellin integral and Mellin's inverse formula. It is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

where γ is a real number so that the contour path of integration is in the *region of convergence* of $F(s)$.

Region of Convergence

If f is a locally integrable function, then the Laplace transform $F(s)$ of f converges provided that the following limit exists:

$$\lim_{R \to \infty} \int_0^R f(t) e^{-ts}\, dt$$

The Laplace transform converges absolutely if the following integral exists:

$$\int_0^\infty |f(t)e^{-ts}|\, dt$$

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The Laplace transform is usually understood as conditionally convergent, meaning that it converges in the former instead of the latter sense.

The set of values for which $F(s)$ converges absolutely is either of the form $\text{Re}\{s\} > a$ or else $\text{Re}\{s\} \geq a$, where a is an extended real constant, $-\infty \leq a \leq \infty$. This follows from the dominated convergence theorem. The constant a is known as the abscissa of absolute convergence, and depends on the growth behavior of $f(t)$. Analogously, the two-sided transform converges absolutely in a strip of the form $a < \text{Re}\{s\} < b$ and possibly including the lines $\text{Re}\{s\} = a$ or $\text{Re}\{s\} = b$. The subset of values of s for which the Laplace transform converges absolutely is called the region of absolute convergence or the domain of absolute convergence. In the two-sided case, it is sometimes called the strip of absolute convergence. The Laplace transform is analytic in the region of absolute convergence.

Similarly, the set of values for which $F(s)$ converges (conditionally or absolutely) is known as the region of conditional convergence or simply the region of convergence. If the Laplace transform converges (conditionally) at $s = s_0$, then it automatically converges for all s with $\text{Re}\{s\} > \text{Re}\{s_0\}$. Therefore the region of convergence is a half-plane of the form $\text{Re}\{s\} > a$, possibly including some points of the boundary line $\text{Re}\{s\} = a$. In the region of convergence $\text{Re}\{s\} > \text{Re}\{s_0\}$, the Laplace transform of f can be expressed by integrating by parts as the integral,

$$F(s) = (s - s_0)\int_0^\infty e^{-(s-s_0)t}\beta(t)\, dt, \quad \beta(u) = \int_0^u e^{-s_0 t} f(t)\, dt$$

That is, in the region of convergence $F(s)$ can effectively be expressed as the absolutely convergent Laplace transform of some other function. In particular, it is analytic. A variety of theorems, in the form of Paley–Wiener theorems, exist concerning the relationship between the decay properties of f and the properties of the Laplace transform within the region of convergence.

Differential equations and corresponding initial as well as boundary value problems can be solved through the Laplace transform method. There are three basic steps for the process of solution:

Step 1. Transformation of the provided hard problem is done into a simple equation (subsidiary equation).

Step 2. The use of purely algebraic modifications is done for solving the subsidiary equation.

Step 3. The answer obtained of the subsidiary equation is again transformed for getting the answer of the provided problem.

Through this, Laplace transforms help in decreasing the problem of evaluating a differential equation to an algebraic problem. Tables of functions as well as their transforms have made such process an easy task to perform, whose role is quite equivalent to that of integral tables in calculus. The table is provided at the end of the chapter.

Consider a given function $f(t)$ that is defined for all $t \geq 0$. Multiply $f(t)$ by e^{-st} to integrate t from zero to infinity. If the resultant integral exists with some finite value then it is a function of s, represented as $F(s)$:

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

This function $F(s)$ of the variable s is the Laplace transform of the basic function $f(t)$ and is depicted by $L(f)$. Hence,

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t) dt \qquad (1.1)$$

Here the basic function f is dependent on t and the novel function F which is its transform is dependent on s. The process that provides $F(s)$ from a given $f(t)$ is the Laplace transform.

The basic function $f(t)$ in Equation (1.1) is called the inverse transform or inverse of $F(s)$ and is depicted by $L^{-1}(F)$. It is written as,

$$f(t) = L^{-1}(F)$$

Notation

The basic functions are indicated by **lowercase letters** and the associated transforms by the same letters in **capitals**. Implying $F(s)$ indicates the transform of $f(t)$ and $Y(s)$ indicates the transform of $y(t)$.

Example 1.1: If $f(t) = 1$ for $t \geq 0$ then find $F(s)$.

Solution: From Equation (1.1) using integration we get,

$$L(f) = L(1) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \quad (s > 0)$$

The notation is appropriate. Here the interval of integration in Equation (1.1) is infinite and is termed as an **improper integral**. According to the rule,

$$\int_0^\infty e^{-st} f(t) dt = \min_{T \to \infty} \int_0^T e^{-st} f(t) dt$$

Hence, the notation means,

$$\int_0^\infty e^{-st} dt = \min_{T \to \infty} \left[-\frac{1}{s} e^{-st} \right]_0^T = \min_{T \to \infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0 \right] = \frac{1}{s} \text{ for } (s > 0)$$

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Example 1.2: Let $f(t) = e^{at}$ for $t \geq 0$, where a is a constant. Find $L(f)$ of the exponential function.

Solution: Using Equation (1.1) we get,

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^\infty$$

If $s - a > 0$ then we get,

$$L(e^{at}) = \frac{1}{s-a}$$

Theorem 1.1: Linearity of the Laplace Transform

The Laplace transform is a linear operation; which means, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b,

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.$$

Proof: By the definition,

$$L\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$$

$$= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt.$$

$$= aL\{f(t)\} + bL\{g(t)\}.$$

Example 1.3: Using Theorem 2.1 find $L(f)$ if $f(t) = \cosh at = \frac{1}{2}(e^{at} + e^{-at})$.

Solution: Using Theorem 1.1 and Example 1.2 we have,

$$L(\cosh at) = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

By taking the common denominator while $s > a\ (\geq 0)$ we have,

$$L(\cosh at) = \frac{s}{s^2 - a^2}$$

1.2.1 Transformation Method

As transformation methods provides an effective means for the solution of many problem in engineering stream so that the knowledge of Laplace transform becomes essential for engineers and scientists. The main advantage of Laplace transformation is for the solution of differential equations. As Laplace transformation this is not necessary to find out the general sol. and then application of boundary value conditions. We can find out particular solution for differential equation satisfying the boundary conditions.

Laplace Transformation: Let $f(t)$ be a function of t for $t \geq 0$, then Laplace transformation of $f(t)$ is denoted by $L\{f(t)\}$ or $f(s)$, given by

$$L\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)dt$$

provided the integral exists and s is a parameter which may be a real or complex number.

Laplace Transform of Some Simple Function

1. Let $f(t) = K$, a constant

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty e^{-st} K dt$$

$$= K\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{K}{s},\ s > 0$$

2. Let $f(t) = t^n$

$$L\{f(t)\} = \int_0^\infty e^{-st}, t^n dt$$

$$= \int_0^\infty e^{-\frac{sx}{s}} \left(\frac{x}{s}\right)^n dt \quad \text{Let } st = x,\ t = \frac{x}{s}$$

$$= \frac{1}{s^n}\int_0^\infty e^{-x} x^n dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ provided that } s > 0 \text{ and } (n+1) > 0$$

3. Let $f(t) = e^{at}$

$$L\{f(t)\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[\frac{e^{t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a},\ s > a$$

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4. Let $f(t)$ cosh $at = \frac{1}{2}\left(e^{at}+e^{-at}\right)$

$$L\{f(t)\} = \int_0^\infty e^{-st}\frac{1}{2}\left(e^{at}+e^{-at}\right)dt$$

$$= \frac{1}{2}L\{e^{at}\}+\frac{1}{2}\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a}\right)+\frac{1}{2}L\left(\frac{1}{s+a}\right)$$

$$= \frac{1}{2}\times\frac{2s}{s^2-a^2},\ s>|a|$$

$$= \frac{s}{s^2-a^2}$$

5. Let $f(t) = \sinh at = \frac{1}{2}\left\{e^{at}-e^{-at}\right\}$

$$L\{f(t)\} = \frac{1}{2}L\{e^{at}\}-\frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\frac{1}{s-a}-\frac{1}{2}\frac{s}{s+a}$$

$$= \frac{a}{s^2-a^2} \qquad s>(a)$$

6. $L(1) = \frac{1}{s}$

$$L(t^n) = \frac{\overline{|n+1}}{s^{n+1}} \qquad s>0,\ n>-1$$

$$L(e^{at}) = \frac{1}{s-a} \qquad s>a$$

$$L(\sin at) = \frac{a}{s^2+a^2} \qquad s>0$$

$$L(\cos at) = \frac{s}{s^2+a^2} \qquad s>0$$

$$L(\cosh at) = \frac{s}{s^2-a^2} \qquad s>(a)$$

Example 1.4: Find the Laplace transform of

(i) $\sin 2t \sin 3t$ (ii) $\cos^3 2t$ (iii) $\sin h^3 2t$

Solution (i) $L\{\sin 2t\sin 3t\} = L\left\{\frac{1}{2}(\cos t-\cos 5t)\right\}$

$$= \frac{1}{2}L(\cos t)-\frac{1}{2}L(\cos 5t)$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right]$$

$$= \frac{24s}{(s^2+1)(s^2+25)}$$

$$= \frac{12s}{(s^2+1)(s^2+25)}$$

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Solution (ii) $L\{\cos^3 2t\} = L\left[\frac{1}{4}(3\cos 2t + \cos 6t)\right]$

$$= \frac{3}{4}L(\cos 2t) + \frac{1}{4}L(\cos 6t)$$

$$= \frac{3}{4}\frac{s}{s^2+2^2} + \frac{1}{4}\frac{s}{s^2+6^2}$$

$$= \frac{s(s^2+28)}{(s^2+4)(s^2+36)}$$

Solution (iii) $L\{\sin h^3 2t\} = L\left(e^{2t}\frac{-e^{-2t}}{2}\right)^3$

$$= \frac{1}{8}\left[L(e^{6t}) - 3L(e^{2t}) + 3L(e^{-2t}) - L(e^{-6t})\right]$$

$$= \frac{1}{8}\left[\frac{1}{s-6} - \frac{3}{s-2} + \frac{3}{s+2} - \frac{1}{s+6}\right]$$

$$= \frac{1}{8}\left[\frac{L2}{s^2-36} - \frac{L2}{s^2-4}\right]$$

$$= \frac{48}{(s^2-36)(s^2-4)}$$

Example 1.5: Find the Laplace transform of

(i) $e^{-t}\cos t\cos 2t$ (ii) $e^{-3}t\,(2\cos 5t - 3\sin 5t)$ (iii) $t^2 e^t \sin 4t$

Solution (i) $L[e^{-t}\cos t\cos 2t] = L\left[e^{-t}\frac{1}{2}\{\cos 3t + \cos t\}\right]$

$$= \frac{1}{2}L[e^{-t}\cos 3t] + \frac{1}{2}[e^{-t}\cos t]$$

$$= \frac{1}{2}\left[\frac{s+1}{(s+1)^2+3^2} + \frac{(s+1)}{(s+1)^2+1^2}\right]$$

$$= \frac{(s+1)(s^2+2s+6)}{(s^2+2s+9)(s^2+2s+2)}$$

Solution (ii) $L[e^{-3}t\,(2\cos 5t - 3\sin 5t)]$

$$= 2L[e^{-3t}\cos 5t] - 3L[e^{-3t}\sin 5t]$$

$$= \frac{2(s+3)}{(s+3)^2+s^2} - \frac{3s}{(s+3)^2+s^2}$$

$$= \frac{2s-9}{s^2+6s+34}$$

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Solution (iii) As $L[t^2] = \frac{2}{s^2}$

$$L[t^2 e^{i4t}] = \frac{2}{(5-4i)^3}$$

$$= \frac{2(s+4i)^3}{(s-4i)^3(s+4i)^3}$$

$$= \frac{2(s^3-48s)+8i(3s^2-16)}{(s^2+16)^2}$$

Equating imaginary part on both sides

$$L[t^2 \sin 4t] = \frac{8(3s^2-16)}{(s^2+16)^2}$$

Again applying first shifting theorem

$$L[e^t t^2 \sin 4t] = \frac{8[3(s-1)^2-16]}{[(s-1)^2+16]^3}$$

$$= \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

Example 1.6: Evaluate $L\left[\frac{e^{-at}t^{n-1}}{\lfloor n-1}\right]$

Solution: $L[t^{n-1}] = \frac{\lceil n}{s^n}$

$$L\left[\frac{t^{n-1}}{\lfloor n-1}\right] = \frac{\lfloor n-1}{s^n}\frac{1}{\lfloor n-1}$$

$$= \frac{1}{s^n}$$

$$L\left[e^{-at}\frac{t^{n-1}}{\lfloor n-1}\right] = \frac{1}{(s+a)^n} \qquad \text{(By Ist shifting theorem)}$$

Example 1.7: $L(\cosh at \sin at)$

Solution: As $L(\cosh at) = \frac{s}{s^2-a^2}$

$$\therefore\ L(e^{iat}\cosh at) = \frac{(s-ia)}{(s-ia)^2-a^2} \qquad \text{(By Ist shifting theorem)}$$

$$= \frac{s-ia}{s^2-2a^2-2ias}$$

NOTES

$$= \frac{(s-ia)[(s^2-2a^2)+2ias]}{(s^2-2a^2)^2-4i^2a^2s^2}$$

$$= \frac{s(s^2-2a^2)+2a^2s+ia\{2s^2-(s^2-2a^2)\}}{s^4+4a^4}$$

Equating imaginary parts

$$L[\sin at \cosh at] = \frac{a(2a^2+s^2)}{s^4+4a^4}$$

Example 1.8: Find $L[a(t)]$ where $a(t) = \begin{cases} \sin\left(t-\frac{1}{4}\right), & t > \frac{\pi}{4} \\ 0 & t < \frac{\pi}{4} \end{cases}$

Solution: Hence $f(t) = \sin t$

$$L[f(t)] = L[\sin t] = \frac{1}{s^2+1}, \qquad s > 0$$

$$L[a(t)] = \begin{bmatrix} f\left(t-\frac{\pi}{4}\right) = \sin\left(t\frac{-\pi}{4}\right) & t > \frac{\pi}{4} \\ = 0 & t < \frac{\pi}{4} \end{bmatrix}$$

$$= e^{-\frac{\pi s}{4}} f(s)$$

$$= e^{-\frac{\pi s}{4}} \frac{1}{s^2+1}, \quad s > 0 \text{ (by II shifting theorem)}$$

Example 1.9: If $L[f(t)] = \frac{20-4s}{s^2-4s+20}$, applying the change of scale property evaluate $L[f(3t)]$

Solution: $L[f(3t)] = \frac{1}{3} f\left(\frac{s}{3}\right)$

$$= \frac{1}{3} \frac{20-4\left(\frac{s}{3}\right)}{\left(\frac{s}{3}\right)^2 - 4\left(\frac{s}{3}\right)^2 + 20}$$

$$= \frac{4(15-s)}{s^2-12s+80}$$

Example 1.10: Evaluate $L[f(t)]$ where

Solution: $f(t) = \begin{cases} 0 & 0 < t < 1 \\ t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$

NOTES

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= 0 + \int_1^2 e^{-st} t dt + 0$$

$$= \left[t\frac{e^{-st}}{-s}\right]^2 - \left[\frac{e^{-st}}{s^2}\right]^2$$

$$= \frac{-1}{s}\left[2e^{-2s} - e^{-s}\right]\frac{-1}{s^2}\left(e^{-2s} - e^{-s}\right)$$

$$= \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} - \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-2s}$$

Example 1.11: Obtain $L[f(t)]$ where $f(t) = \cos t$, $0 < t < 2\pi$

$= 0$, $t > 2\pi$

Solution: $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^{2\pi} e^{-st} f(t)dt + \int_{2n}^\infty e^{-st} f(t)dt$$

$$= \int_0^{2\pi} e^{-st} \cos t dt + 0$$

$$= \left\{\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right\}_0^{2\pi}$$

$$= \frac{s(1-e^{-2\pi s})}{1+s^2}$$

Example 1.12: Let $f(t)$ be a periodic function with period T, then

$$L[f(t)] = \frac{1}{1-e^{-st}}\int_0^T e^{-st} f(t)dt$$

Solution: By definition $L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt$

$$= \int_0^\infty e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt$$

Putting $t = v + T$ in second integral and $t = v + 2T$ in third integral and so on.

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(v+T)} f(v+T)dv + \int_0^T e^{-s(v+2T)} f(v+2T)dv$$

As function is periodic with period T

$$f(v) = f(v+T), f(v+2T) \ldots$$

$$\therefore L\{f(t)\} = \int_0^T e^{-st} f(t)dt + e^{-st}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt \ldots$$

$$= (1+e^{-sT}+e^{-2sT}\ldots)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

Example 1.13: Find Laplace transform of the function (Half Wave Rectifier),

$$f(t) - \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{w} \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Solution: $t\{f(t)\} = \dfrac{1}{1-e^{-st}}\displaystyle\int_0^T e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2\frac{\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} \sin \omega t dt + 0$$

$$= \frac{1}{1-e^{-2\frac{\pi s}{\omega}}}\frac{e^{-st}}{s^2 f \omega t}[(-s\sin\omega t - \omega\cos\omega t)]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\frac{\pi s}{\omega}}}\frac{\left(\omega e^{-\frac{\pi s}{w}} + \omega\right)}{\omega^2 + s^2}$$

$$= \frac{1}{\omega^2+s^2}\frac{\omega}{\left(1-e^{-\frac{\pi s}{\omega}}\right)}$$

Existence Theorem: Existence theorem states that the Laplace transform of $f(t)$ exists for all $s > a$, if $f(t)$ is piecewise continous in every finite interval in the domain $t \geq 0$ and is of exponential order a.

[function $f(t)$ is said to of exponential order a if, $\lim\limits_{t\to\infty} e^{-at} f(t) = a$ finite quantity. In this case, there exist real constants M, a, T such that

$|f(t)| < Me^{at}$ for all $t > T$]

That's why Laplace integral exist under certain restriction such as $s > 0$ or $s > a$ etc.

NOTES

1.3 LAPLACE TRANSFORM OF ELEMENTARY CONTINUOUS AND EXPONENTIAL ORDER FUNCTION

The **Laplace transformation** is an important operational method for solving linear differential equations. It is particularly useful in solving *initial value problems* connected with linear differential equations (ordinary and partial). The advantage of Laplace transformation in solving initial value problems lies in the fact that initial conditions are taken care of at the outset and the specific particular solution required is obtained without first obtaining the general solution of the linear differential equation.

1.3.1 Laplace Transforms of Elementary Functions

Using the definitions, we find the Laplace transfomation of some simple functions.

(i) Transform of $f(t) = 1,\ \ t \geq 0$

$$L\{f(t)\} = \int_0^{\infty} e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^{\infty}$$

$$e^{-st} \to 0 \text{ as } t \to \infty, \text{ if } s > 0$$

$$\therefore \qquad L(1) = \frac{1}{s}, s > 0 \qquad (1.2)$$

(ii) Transform of e^{-at}, where a is a constant.

$$L[e^{-at}] = \int_0^{\infty} e^{-st} e^{-at}\,.dt = \int_0^{\infty} e^{-(s+a)t}\,dt = \left[-\frac{e^{-(s+a)t}}{s+a}\right]_0^{\infty}$$

$e^{-(s+a)t}$ tends to zero as, $t \to \infty$, if $(s + a > 0)$

$$\therefore \qquad L[e^{-at}] = \frac{1}{s+a}, s > -a \qquad (1.3)$$

(iii) Transform of e^{-at}, where a is a constant.

$$L[e^{at}] = \frac{1}{s-a}, s > a \qquad (1.4)$$

The result follows from *(ii)* by changing a to $-a$.

(iv) Transform of sin at, where a is a constant.

$$L[\sin at] = \int_0^{\infty} e^{-st} \sin at\,dt = \left[\frac{e^{-st}}{s^2 + a^2}(-s\sin at - a\cos at)\right]_0^{\infty}$$

$$= \frac{-s}{s^2+a^2} \underset{t\to\infty}{\text{Lt}} (e^{-st} \sin at) - \frac{a}{s^2+a^2} \underset{t\to\infty}{\text{Lt}} (e^{-st} \cos at) = \frac{a}{s^2+a^2}$$

(When $s > 0$, both $(e^{-st} \sin at)$ and $(e^{-st} \cos at)$ tend to zero as $t \to \infty$).

NOTES

$$\therefore \quad L[\sin at] = \frac{a}{s^2+a^2}, s > -0 \qquad (1.5)$$

(v) Transform of cos at, where a is a constant.

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at\right]_0^\infty$$

$$= \frac{-s}{s^2+a^2} \underset{t\to\infty}{\text{Lt}} (e^{-st} \cos at) - \frac{a}{s^2+a^2} \underset{t\to\infty}{\text{Lt}} (e^{-st} \cos at)$$

$$= \frac{s}{s^2+a^2}, s > 0 \quad \text{(by the results stated in equation (1.5)}$$

$$\therefore \quad L[\cos at] = \frac{s}{s^2+a^2}, s > 0$$

(vi) Transform of t^n, where n is a positive integer.

$$L[t^n] = \int_0^\infty e^{-st} t^n dt = \left[-\frac{t^n e^{-st}}{s}\right]_0^\infty - \int_0^\infty \left(-\frac{1}{s}\right) e^{-st}.nt^{n-1} dt$$

On integration by parts,

$$= -\frac{1}{s} \cdot \underset{t\to 0}{\text{Lt}} \frac{t^n}{e^{st}} + \frac{n}{s}\int_0^\infty e^{-st}.t^{n-1} dt$$

If $s > 0$, by applying L'hospital's rule successively, it can be shown that as $t \to \infty$,

$$L[t^n] = \frac{n}{s} L[t^{n-1}], s > 0 = \frac{n}{s} \cdot \frac{n-1}{s} L[t^{n-2}], s > 0 \qquad (1.6)$$

By repeated application of equation (1.2)

$$L[t^n] = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L[t^0]$$

But,

$$L[t^0] = L(1) = \frac{1}{s}$$

$$\therefore \quad L[t^n] = \frac{n.(n-1)....2.1}{s^n} \cdot \frac{1}{s}$$

$$\text{Or} \quad L[t^n] = \frac{n!}{s^n+1}, s > 0$$

Note: $L[t] = [1/s^2]$, $L[t^2] = (2/s^3)$

NOTES

1.3.2 Linearity Property of Laplace Transformation

If $f(t)$ and $g(t)$ are functions for which Laplace transforms exist, then

(i) $L[f(t) + g(t)] = L[f(t)] + L[g(t)]$

(ii) $L[k\,f(t)] = kL[f(t)]$

For any constant k.

The result can be proved as follows:

$$L[f(t) + g(t)] = \int_0^\infty e^{-st}[f(t) + g(t)]dt$$

$$= \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} g(t)dt$$

$$\therefore \quad L[f(t) + g(t)] = L[f(t)] + L[g(t)]$$

Also $$L[kf(t)] = \int_0^\infty e^{-st} kf(t)dt$$

$$= k\int_0^\infty e^{-st} f(t)dt$$

$$\therefore \quad L[kf(t)] = kL[f(t)]$$

In view of properties *(i)* and *(ii)*, the Laplace transformation operator L is a *linear operator*.

Note: $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$ for any constants a and b. Also,

$$L\left[\sum_1^n a_i f_i(t)\right] = \sum_{i=1}^n a_i L[f_i(t)]$$

For any constants $a_1, a_2, ... a_n$.

Using this property, the Laplace transforms of functions which can be expressed as linear combination of functions can be written.

1.3.3 Laplace Transforms of sin*h at* and cos*h at*

(i) Transform of sin $h\,at$

$$L[\sin h\,at] = L\left[\frac{1}{2}(e^{at} - e^{-at})\right]$$

$$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right], \; s > a, \text{ and } s > -a$$

$$L[\sin h\,at] = \frac{a}{s^2 - a^2}, \; s > |a|$$

(ii) Transform of cos $h\,at$

$$L[\cos h\, at] = L\left[\frac{1}{2}(e^{at} + e^{-at})\right]$$

$$= \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a} + \frac{1}{s-a}\right],\ s > |\,a\,|$$

$$\therefore \quad L[\cos h\, at] = \frac{s}{s^2 - a^2},\ s > |\,a\,|$$

Note:

$$L[\cos at + i \sin at] = L[e^{iat}] = \frac{1}{s - ia}$$

$$\therefore\ L[\cos at] + iL[\sin at] = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2}$$

Equating the real parts, we get,

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Equating the imaginary parts, we have,

$$L[\sin at] = \frac{a}{s^2 + a^2}$$

Example 1.14: Find $L(5t^3 + 3t^2 - 6t + 3e^{-5t})$

Solution: $L(5t^3 + 3t^2 - 6t + 3e^{-5t}) = 5L(t^3) + 3L(t^2) - 6L(t) + 3L(e^{-5t})$

$$= 5\cdot\frac{3!}{s^4} + 3\cdot\frac{2!}{s^3} - 6\cdot\frac{1}{s^2} + 3\cdot\frac{1}{s+5} = \frac{30}{s^4} + \frac{6}{s^3} - \frac{6}{s^2} + \frac{3}{s+5}$$

Example 1.15: Find $L[\cos^2 3t + \sin 5t \sin 2t]$

Solution: $L[\cos^2 3t + \sin 5t \sin 2t]$

$$= L\left[\frac{1}{2}(1 + \cos 6t) + \frac{1}{2}(\cos 3t - \cos 7t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 36}\right] + \frac{1}{2}\left[\frac{s}{s^2 + 9} - \frac{s}{s^2 + 49}\right]$$

Example 1.16: Find $L(\cos^3 2t)$

Solution: Since, $\cos 6t = 4\cos^3 2t - 3\cos 2t$

$$\cos^3 2t = \frac{1}{4}[\cos 6t + 3\cos 2t]$$

$$\therefore \quad L(\cos^3 2t) = \frac{1}{4}\left[\frac{s}{s^2 + 36} + 3\cdot\frac{s}{s^2 + 4}\right]$$

Example 1.17: Find $L[64 \sin^5 t + \cos(2t + 5)]$

NOTES

Solution: $\sin^5 t = \frac{1}{2^4}[\sin 5t - 5C_1 \sin 3t + 5C_2 \sin t]$

$$= \frac{1}{16}[\sin 5t - 5\sin 3t + 10\sin t]$$

$$\therefore L[64 \sin^5 t + \cos(2t + 5)] = L[4(\sin 5t - 5 \sin 3t + 10 \sin t)] + L[\cos(2t + 5)]$$

$$= 4\left[\frac{5}{s^2+25} - 5\cdot\frac{3}{s^2+9} + 10\cdot\frac{1}{s^2+1}\right]$$

$$+ L[\cos 2t \cos 5 - \sin 2t \sin 5]$$

$$= \frac{20}{s^2+25} - \frac{60}{s^2+9} + \frac{40}{s^2+1}$$

$$+\cos 5\cdot\frac{s}{s^2+4} - \sin 5\cdot\frac{2}{s^2+4}$$

Example 1.18: Find $L[e^{2t} \sin h^2 at]$

Solution: $L[e^{2t} \sin h^2 at] = L\left[e^{2t}\left(\frac{e^{at}-e^{-at}}{2}\right)^2\right] = \frac{1}{4}L\left[e^{2t}\left(e^{2at}+e^{-2at}-2\right)\right]$

$$= \frac{1}{4}L\left[e^{2(a+1)t} + e^{-2(a-1)t} - 2e^{2t}\right]$$

$$\therefore \quad L[e^{2t} \sin h^2 at] = \frac{1}{4}\left\{\frac{1}{s-2a-2} + \frac{1}{s+2a-2} - \frac{2}{s-2}\right\}$$

Note of Notation: In the functions considered above, the independent variable is denoted by t. This is suggestive of the fact that, in many applications, functions involved are functions of time t.

Also, Laplace transforms are functions of the parameter s. In some books, the parameter s is replaced by p. In such cases, the Laplace transform of $f(t)$ is denoted as $F(p)$ or $\bar{f}(p)$ and Laplace transforms of sin at, e^{-at} etc., are $[a/p^2 + a^2]$, $[a/p + a]$, etc.

1.3.4 Transforms of Integrals

The following results show that the Laplace transforms of the derivatives and integrals of a function $f(t)$ can be expressed in terms of the Laplace transform of $f(t)$. These results are important in solving differential equations using the methods of Laplace transformation.

Theorem 1.2: If $f(t)$ is continuous and $f'(t)$ is piecewise continuous in the interval $0 \le t \le T$ for any finite T, and $f(t)$ and $f'(t)$ are of exponential order as $t \to \infty$; then,

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof: Under the conditions stated in the theorem, the Laplace transforms of $f(t)$ and $f'(t)$ exist and,

NOTES

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$$

$$= \int_0^\infty e^{-st} d[f(t)]$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\cdot(-s)e^{-st}dt$$

On integration by parts, we get $L[f'(t)]$ to be,

$$= \underset{t\to\infty}{\text{Lim}}\, e^{-st} f(t) - f(0) + s\int_0^\infty e^{-st} f(t)dt \;= -f(0) + sL[f(t)]$$

Since $\underset{t\to\infty}{\text{Lim}}\, e^{-st} f(t) = 0$, as shown below (see Notes).

$$\therefore \qquad L[f'(t)] = sL[f(t)] - f(0)$$

Notes:

1. As $f(t)$ is of exponential order at $t \to \infty$, there exist constants α and M such that,

$$|f(t)| \le Me^{\alpha t} \text{ for } t \ge t_0$$

$$[|f(t)|/e^{st}] \le Me^{[(s-\alpha)t]} \text{ for } t \ge t_0$$

Now, $\qquad e^{-(s-\alpha)t} \to 0$, as $t \to \infty$, if $s > \alpha$

$\therefore \qquad \underset{t\to\infty}{\text{Lim}} \dfrac{|f(t)|}{e^{st}}$, as also $\underset{t\to\infty}{\text{Lim}} \dfrac{f(t)}{e^{st}}$ vanish if, $s > \infty$

2. Although $f(t)$ is of exponential order, it cannot be said that the derivatives of $f(t)$ will also be of exponential order. However, in most practical cases, the functions considered and their derivatives are all of exponential order.

Theorem 1.3: If $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous in $0 \le t \le T$, for any finite T, and $f(t)$ and $f'(t)$ are of exponential order as $t \to \infty$, then

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f''(0)$$

Proof: By applying the previous theorem to the function $f'(t)$, we have,

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f''(0)$$

Again applying the previous theorem to write $L[f'(t)]$ in terms of $L[f(t)]$, we get,

$$L[f'(t)] = s[sL\{f(t)\} - f(0)] - f'(0)$$

$$L[f'(t)] = s^2L[f(t)] - sf(0) - f'(0)$$

Notes:

1. Conditions stated in the above theorem ensure that the Laplace transforms of $f(t)$, $f'(t)$ and $f''(\text{t})$ exist.

NOTES

2. The result concerning the Laplace transforms of $f^n(t)$ is as follows:

If $f(t), f'(t),..., f^{(n-1)}(t)$ are continuous and $f^n(t)$ is piecewise continuous in the interval $0 \le t \le T$ for any finite T and all these functions are of exponential order as $t \to \infty$, then,

$L[f^{(n)}(t)] = s^n L[f(t)] - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - s^{(n-3)} f''(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$

This result is obtained by successive application of the result,

$$L[f'(t)] = sL[f(t)] - f(0)$$

Example 1.19: Given that $L[t \sin at] = [(2as)/(s^2 + a^2)^2]$, find $L[at \cos at + \sin at]$.

Solution:

$$L[at \cos at + \sin at] = L\left[\frac{d}{dt}(t \sin at)\right]$$

$$= sL[t \sin at] - [t \sin at]_{t=0} = s\frac{2as}{(s^2 + a^2)^2}$$

$$\therefore \quad L[at \cos at + \sin at] = \frac{2as^2}{(s^2 + a^2)^2}$$

Example 1.20: Using the result $L[f'] = sL[f] - f(0)$ and $L[f''] = s^2L[f] - sf(0) - f'(0)$, find $L[e^{at}]$, $L[\sin at]$ and $L[\cos at]$.

Solution:

To find $L(e^{at})$, take $f(t) = e^{at}$ in the result $L(f') = sL(f) - f(0)$.

Then, $L(e^{at}) = sL(e^{at}) - 1$ *i.e.,* $aL(e^{at}) - sL(e^{at}) = -1$

$$\therefore \quad L(e^{at}) = \frac{1}{s-a}$$

Taking $f(t) = \sin at$ in the result $L[f''] = s^2L[f] - sf(0) - f'(0)$, we get,

$$L[-a^2 \sin at] = s^2L[\sin at] - s(0) - a(1)$$

i.e., $\quad -a^2L(\sin at) - s^2L(\sin at) = -a$

$$\therefore \quad L(\sin at) = \frac{a}{s^2 + a^2}$$

Find L(cos *at*) in a similar manner is left as an exercise for the students.

If $L[f(t)] = \bar{f}(s)$, prove that,

$$L[tf'(t)] = -[sf'(s) + f(s)]$$

Theorem 1.4: If $f(t)$ is of exponential order as $t \to \infty$ and piecewise continuous in the interval $0 \le t \le T$ for any finite T, then,

$$L\left[\int_0^t f(u)du\right] = \frac{1}{s}L[f(t)]$$

Proof: This result can be proved using the result,

$$L[g'(t) = sL[g(t)] - g(0) \qquad (1.7)$$

Let $\int_0^t f(u)du$ be denoted as $g(t)$. Then,

$$g'(f) = f(t) \quad \text{and} \quad g(0) = \int_0^0 f(u)du = 0$$

It can be shown that $g(t)$ is continuous in $0 \le t \le T$ and is of exponential order as $t \to \infty$. Therefore, Laplace transforms of both $f(t)$ and $g(t)$ exist and by equation (1.7),

$$L[f(t)] = sL\left[\int_0^t f(u)du\right] - 0$$

$$\therefore \qquad L\left[\int_0^t f(u)du\right] = \frac{1}{s}L[f(t)]$$

Notes:

1. Replacing a dummy variable u by t, the above result is written in form

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

2. Under the conditions stated in the theorem,

$$L\left[\int_0^t\int_0^t\cdots\int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L[f(t)], \text{ for any positive integer } n.$$

3. $\int_a^t f(u)du = \int_a^t f(u)du - \int_a^t f(u)du$

$$\therefore \qquad L\left[\int_a^t f(u)du\right] = L\left[\int_0^t f(u)du\right] - L\left[\int_0^a f(u)du\right]$$

$$= \frac{1}{s}L[f(t)] - \frac{1}{s}\int_0^a f(u)du$$

Since, $\int_0^a f(u)du$ is a constant.

$$\therefore \qquad L\left[\int_a^t f(u)du\right] = \frac{1}{s}L[f(t)] + \frac{1}{s}\int_0^a f(u)du$$

Example 1.21: Find $L\int_0^t [(\sin x)/x]dx$

Solution: $$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t)ds = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}(s)\right]_s^\infty$$

NOTES

$$= \frac{\pi}{2} - \tan^{-1} s \;= \cot^{-1} s$$

$$\therefore \qquad L\left[\int_0^t \frac{\sin x}{x} dx\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right] = \frac{1}{s}\cot^{-1} s$$

1.4 SOME IMPORTANT PROPERTIES OF LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

The Laplace transform is invertible on a large class of functions. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analysing the behaviour of the system, or in synthesizing a new system based on a set of specifications.

The Laplace transform can also be used to solve differential equations. The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The original differential equation can then be solved by applying the inverse Laplace transform. Computation of the Laplace transform of a function's derivative is often used with the differentiation property of the Laplace transform to find the transform of a function's derivative.

1.4.1 Laplace Transformation of Derivatives

1. It $f'(t)$ be continuous for every $t \geq 0$ and be of exponential order s, then

$f'(t) = sf(s) - f(0)$

By definition $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t) dt$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\int_0^\infty e^{-st} f(t) dt$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + sf(s)$$

As $f(t)$ is of order s, $\lim_{t\to\infty} e^{-st} f(t) = 0$

$\therefore L\{f'(t)\} = sf(s) - f(0)$

2. If $f'(t)$ and its first $(n-1)$ derivative be continuous and $f(t)$ be of exponential order s then

$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots f^{n-1}(0)$

By definition

NOTES

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)dt$$

Integrating by parts

$$L\{f^n(t)\} = [e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3} \ldots\infty$$

$$\cdots (-1)^{n-1}(-s)^{n-1} e^{-st} f(t)]_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t)dt$$

Assuming that

$$\lim_{t\to\infty} e^{-st} f^m(t) = 0 \text{ for } m = 0, 1, 2, n-1$$

$$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots f^{n-1}(0)$$

3. **Multiplication by t^n:**

It $L\{f(t)\} = f(s)$

Then $L\{f^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s)$ ...(1.8)

Where $n = 1, 2 \ldots n$.

By definition $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ integrating both sides with respect to s,

$$\frac{d}{ds}\{f(s)\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

By Leibniz's rule for differenciation

$$= \int_0^\infty \frac{d}{2s} e^{-st} f(t)dt$$

$$= \int_0^\infty -te^{-st} f(t)dt$$

$$= (-1)^1 L\{tf(t)\}$$

This proves the theorem for $n = 1$

Now assume that (1) is true for $n = m$ (say)

$$\therefore \quad (-1)^m \frac{d^m}{ds^m} f(s) = \int_0^\infty e^{-st} t^m f(t)dt$$

$$\therefore \quad (-1)^m \frac{d^{m+1}}{ds^{m+1}} f(s) = \frac{d}{ds}\int_0^\infty e^{-st} t^m f(t)dt$$

$$= \int_0^\infty -te^{-st} t^{m+s} f(t)dt$$

$$\therefore \quad (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} f(s) = L\left[t^{m+1} f(t)\right]$$

NOTES

By induction (1) is true.

4. **Laplace Transform of Integrals**

If $L\{f(t)\} = f(s)$

Then $L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s} f(s)$

Let $a(t) = \int_0^t f(t)dt$

$\Rightarrow a(0) = 0$

And $a^1(t) = \frac{d}{dt}\int_0^t f(t)dt = f(t)$

Now $L\{a^1(t)\} = s\,L\{a(t)\} - a(0)$

$L\{f(t)\} = s\,L\{a(t)\} - 0$

$\therefore \quad \frac{1}{s}L\{f(t)\} = L\left[\int_0^t f(t)dt\right]$

5. **Division by *t*:**

If $L\{f(t)\} = f(s)$

Then $T\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(t)dt$ provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists

Let $a(t) = \frac{f(t)}{t}$

Or $f(t) = ta(t)$

$L\{f(t)\} = L\{t, a(t)\}$

$f(s) = \frac{-d}{ds}L\{a(t)\}$

Integrating both sides with respect to s from s to ∞

$$\int_s^\infty f(s)ds = -L\{a(t)\}_s^\infty$$

$$\int_s^\infty f(s)ds = -\lim_{s\to\infty}L\{a(t)\} + L\{a(t)\}$$

$$\int_s^\infty f(s)ds = 0 + L\{a(t)\}\left[as \lim_{s\to\infty}L\{a(t)\} = \lim_{s\to\infty}\int_s^\infty e^{-st}(t)dt = 0\right]$$

$$\therefore \; L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(x)dx$$

Example 1.22: Find $L(t^3 e^{-3t})$

Solution: As $L\{e^{-3t}\} = \dfrac{1}{s+3}$

$$\therefore \quad L\{t^3\ e^{-3t}\} = (t)^3 \frac{d^3}{ds^3}\frac{1}{(s+3)}$$

$$= (-1)^3 \frac{(-1)\lfloor 3}{(s+3)^4}$$

$$= \frac{6}{(s+3)^4}$$

NOTES

Example 1.23: Evaluate $L\{t \sin^2 t\}$

Solution: $L\{t \sin^2 t\} = L\left\{t \dfrac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2}L\{t\} - \frac{1}{2}L\{t\cos 2t\}$$

$$= \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}(-1)\frac{d}{ds}\frac{s}{(s^2+2^2)}$$

$$= \frac{1}{2s^2} - \frac{1}{2}\frac{(s^2-4)}{(s^2+4)^2}$$

$$= \frac{2(3s^2+4)}{s^2(s^2+4)^2}$$

Example 1.24: Find $L\{t^2 \cos at\}$

Solution: $L\{t^2 \cos at\} = (-1)\dfrac{d^2}{ds^2}\dfrac{s}{(s^2+a^2)}$

$$= \frac{d}{ds}\left\{\frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\frac{(a^2 s^2)}{(s^2+a^2)^2}$$

$$= \frac{(s^2+a^2)^2(-2s) - 1a^2 - s^2 \cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

Example 1.25: Evaluate $L\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\}$

Solution: $L\left\{\dfrac{e^{-at}-e^{-bt}}{t}\right\} = L\left\{\dfrac{e^{-at}}{t}\right\} - L\left\{\dfrac{e^{-bt}}{t}\right\}$

NOTES

$$= \int_s^\infty \frac{1}{s+a}\,ds - \int_s^\infty \frac{1}{s+b}\,ds$$

$$= \left[\log(s+a)\right]_s^\infty - \left[\log(s+b)\right]_s^\infty$$

$$= \left[\log\frac{s+a}{s+b}\right]_s^\infty$$

$$= -\log\frac{s+a}{s+b} + \log_{s\to\infty}\frac{1+\frac{a}{s}}{1+\frac{b}{s}}$$

$$= \log\frac{s+b}{s+a}$$

Example 1.26: $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$

Solution: $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = L\left\{\frac{\cos 2t}{t}\right\} - L\left\{\frac{\cos 3t}{t}\right\}$

$$= \int_s^\infty \frac{s}{s^2+4}\,ds - \int_s^\infty \frac{s}{s^2+9}\,ds$$

$$= \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty - \frac{1}{2}\log\left[(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}} - \frac{1}{2}\log\frac{s^2+4}{s^2+9}$$

$$= 0 - \frac{1}{2}\log\frac{s^2+4}{s^2+9}$$

$$= \frac{1}{2}\log\frac{s^2+9}{s^2+4}$$

Example 1.27: Given that $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$

Solution: Prove that $\frac{1}{\sqrt{s}} = L\left\{\frac{1}{\sqrt{\pi t}}\right\}$

Let $f(\mathrm{t}) = 2\sqrt{\frac{t}{\pi}}$ $\quad\quad \therefore f(0) = 2\sqrt{\frac{0}{\pi}} = 0$

NOTES

$$\therefore f'(t) = 2\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$$

Also $L\{f'(t)\} = sf(s) - f(0)$

$$\therefore\ L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s\frac{1}{s^{3/2}} - 0$$

$$= \frac{1}{\sqrt{s}} \text{ (proved)}$$

Example 1.28: Evaluate $L\{\sin\sqrt{t}\}$

As $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7}\cdots$

$$\sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \frac{t^{7/2}}{\lfloor 7}\cdots$$

But $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$$\therefore\qquad L\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{\lfloor 3}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\Gamma(7/2)}{s^{7/2}}\cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} - \frac{1}{\lfloor 3}\frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}\left[1 - \frac{3}{2\lfloor 3 s} + \frac{5.3}{2.2.5.\lfloor 5 s^2}\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[1 - \frac{1}{2^2 s} + \frac{1}{(2^2 s)^2\lfloor 2} - \frac{1}{(2^2 s^2)^3\lfloor 3}\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-\frac{1}{2^2 s}} = \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

Example 1.29: Evaluate $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$

Solution: $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}Lf(t)$ ...(1)

Where $f(t) = \frac{\sin t}{t}$

$$\therefore\qquad L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\}$$

NOTES

$$= \int_s^\infty L(\sin t)ds$$

$$= \int_s^\infty \frac{ds}{s^2+1}$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$(1) \Rightarrow L\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s}\cos^{-1} s$$

1.4.2 Proof of the Laplace Transform of a Function's Derivative

It is often convenient to use the differentiation property of the Laplace transform to find the transform of a function's derivative. This can be derived from the basic expression for a Laplace transform as follows:

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\, dt \quad \text{(by parts)}$$

$$= \left[-\frac{f(0)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},$$

Yielding,

$$\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

In the bilateral case,

$$\mathcal{L}\{f'(t)\} = s \int_{-\infty}^{\infty} e^{-st} f(t)\, dt = s \cdot \mathcal{L}\{f(t)\}.$$

The general result,

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \cdot \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

Where f^n is the nth derivative of f, can then be established with an inductive argument.

Laplace Transform of the Derivative

Consider that the Laplace transform of $y(t)$ is $Y(s)$. Then the Laplace Transform of $y'(t)$ is,

$$L[y'(t)](s) = sY(s) - y(0)$$

For the second derivative we have,

$$L[y''(t)](s) = s^2Y(s) - sy(0) - y'(0)$$

For the n'th derivative we have,

$$L[y^{(n)}(t)](s) = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0)$$

Derivatives of the Laplace Transform

Let $Y(s)$ be the Laplace transform of $y(t)$. Then,

$$L[t^n y(t)](s) = (-1)^n \frac{d^nY}{ds^n}(s)$$

We can compute the Laplace transform of $t \sin(t)$ as follows:

$$L[t \sin t] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2}$$

The Laplace transform is method is used for solving differential equations. The Laplace transform replaces operations of calculus by operations of algebra on transforms. Approximately, differentiation of $f(t)$ is replaced by multiplication of $L(s)$ by s and integration of $f(t)$ is replaced by division of $L(f)$ by s.

Theorem 1.5: Laplace Transform of the Derivative of $f(t)$

Suppose that $f(t)$ is continuous for all $t \geq 0$, satisfies for some k and M, and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > k$, and

$$L(f') = sL(f) - f(0) \quad for \ (s > k)$$

Proof: Consider the situation when $f'(t)$ is continuous for all $t \geq 0$. Then, by the definition and by integration by parts we have,

$$L(f') = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

Since f satisfies the integrated portion on the right is zero at the upper limit when $s > k$ and at the lower limit it contributes $-f(0)$. The last integral $L(f)$ is the existence for $s > k$. This proves that the expression on the right exists when $s > k$ and is equal to $-f(0) + sL(t)$. Consequently, $L(f')$ exists when $s > k$. If the derivative $f'(t)$ is piecewise continuous, then the proof is quite akin. In this case, the range of integration in the original integral which is split into parts such that f' is continuous in each such part. This theorem may be extended to piecewise continuous functions $f(t)$.

Theorem 1.6: Laplace Transform of the Derivative of Any Order n

Let $f(t)$ and its derivatives $f'(t), f''(t), \cdots, f^{(n-1)}(t)$ be continuous functions for all $t \geq 0$, satisfying some k and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of $f^{(n)}(t)$ exists when $s > k$ and is given by,

NOTES

$$L\left(f^{(n)}\right) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Example 1.30: If $f(t) = t^2$ then derive $L(f)$ from $L(1)$.

Solution: Since $f(0) = 0,\ f'(0) = 0,\ f''(t) = 2$ and $L(2) = 2L(1) = 2/s$

We get,

$$L(f'') = L(2) = \frac{2}{s} = s^2 L(f), \quad hence \quad L(f^2) = \frac{2}{s^3}$$

Example 1.31: Derive the Laplace transform of $\cos \omega t$.

Solution: Let $f(t) = \cos \omega t$. Then $f''(t) = -\omega^2 \cos \omega t = -\omega^2 f(t)$. Also $f(0) = 1,\ f'(0) = 0$. Now we take the transform, $L(f'') = -\omega^2 L(f)$.

We get,

$$-\omega^2 L(f) = L(f'') = s^2 L(f) - s,$$

$$hence \quad L(f) = L(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

Example 1.32: If $f(t) = \sin^2 t$ then find $L(f)$.

Solution: Given is, $f(0) = 0,\ f'(t) = 2\sin t \cos t = \sin 2t$

Which gives,

$$L(\sin 2t) = \frac{2}{s^2 + 4} = sL(f) .$$

$$\text{Or} \quad L(\sin^2 t) = \frac{2}{s(s^2 + 4)}$$

Example 1.33: If $f(t) = t \sin \omega t$ then find $L(f)$.

Solution: Given is, $f(0) = 0$ and

$$f'(t) = \sin \omega t + \omega t \cos \omega t ,$$

for $f'(0) = 0$

$$f''(t) = 2\omega \cos \omega t - \omega^2 t \sin \omega t$$

$$= 2\omega \cos \omega t - \omega^2 f(t),$$

Also,

$$L(f'') = 2\omega L(\cos \omega t) - \omega^2 L(f) = s^2 L(f).$$

NOTES

Using the formula for the Laplace transform of $\cos \omega t$, we obtain:

$$\left(s^2+\omega^2\right)L(f)=2\omega L(\cos \omega t)=\frac{2\omega s}{s^2+\omega^2}.$$

The outcome is,

$$L(t\sin \omega t)=\frac{2\omega s}{\left(s^2+\omega^2\right)^2}.$$

1.4.3 Laplace Transform of the Integral of a Function

Differentiation and integration are inverse processes. Consequently, as differentiation of a function corresponds to the multiplication of its transform by *s*, we expect integration of a function to equates to division of its transform by *s*, because division is the inverse operation of multiplication.

Theorem 1.7: Integration of *f*(*t*)

Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piecewise continuous and satisfies an inequality, then

$$L\left\{\int_0^t f(\tau)d\tau\right\}=\frac{1}{s}F(s) \qquad ...(1.9)$$

For $(s > 0,\ s > k)$

Or, if only the inverse transform on both sides of the above equation is taken,

$$\int_0^t f(\tau)d\tau = L^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Proof: Suppose that $f(t)$ is piecewise continuous and satisfies the Equation (1.9) for some *k* and *M*. Clearly, if for Equation (1.9) some negative *k*, it also holds for positive *k* then we may assume that *k* is positive. Then the integral,

$$g(t)=\int_0^t f(\tau)d\tau$$

is continuous and by using Equation (1.9) we obtain for any positive *t*,

$$|\,g(t)\,| \leqq \int_0^t |\,f(\tau)\,|\,d\tau \leqq M\int_0^t e^{k\tau}d\tau=\frac{M}{k}\left(e^{kt}-1\right)\leqq\frac{M}{k}e^{kt} \quad \text{for } (k>0).$$

This shows that $g(t)$ also satisfies an inequality of the form given in Equation (1.9). Also, $g'(t)=f(t)$, except for points at which $f(t)$ is discontinuous. Hence, $g'(t)$ is piecewise continuous on each finite interval and gives,

$$L\{f(t)\}=L\{g'(t)\}=sL\{g(t)\}-g(0) \quad \text{for } (s>k).$$

Here, clearly, $g(0)=0$, so that $L(f) = sL(g)$.

NOTES

Check Your Progress

1. Define the Laplace transform.
2. What is the bilateral Laplace transform?
3. State the inverse Laplace transform.
4. State on the region of convergence.
5. How will you define the elementary theorem?
6. Define the transforms of integrals.
7. What is the linearity of the Laplace transform?
8. State the Laplace transform of the integral of a function.

1.5 INVERSION OF SOME ELEMENTARY FUNCTIONS

An integral transform is useful to convent a complicated problem into simpler one. Laplace transform is one of the important type of integral transform that can be used to solve integral and differential equations. The idea behind a transform is simple, suppose we want to solve a differential equations with unknown function *f*. We first apply the transform to the given differential equation to turn it into an algebraic equation that can be solved very easily in terms of transform F of *f*. One can solve the resultant agebraic equation for F, and finally applying the inversion transform to find *f*. This can be understand diagramatically as follows:

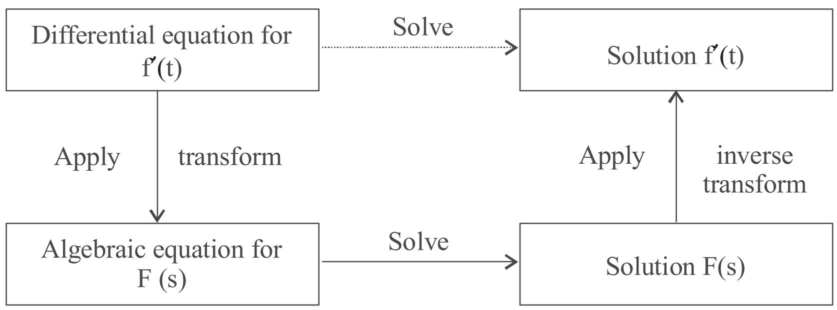

Fig. 1.1 *Working Principle integral transform to solve a differential equation.*

1.5.1 Function of Exponential Order

The function $f(t)$ is said to of exponential order s_0 as $t \to \infty$, if

$$\lim_{t\to\infty} e^{-s_o t} f(t) \text{ is finite}$$

i.e., these exists $M > 0$ and $t_o > 0$ such that

$$\left|e^{-s_o t} f(t)\right| \leq M \qquad \forall\ t \geq t_o$$

or $\quad |f(t)| < Me^{s_o t} \quad$ or $\quad t \geq t_o$

We write $f(t) = 0\,(e^{s_o t})$ as $t \to \infty$

NOTES

Example 1.34: ($f(t)$ 1, t, t^2 are of exponential order so for any $s_o > 0$

Solution: $|f(t)\, e^{-t}| = |e^{-t}| \leq 1$

$\forall\, t \geq 0$

or $\lim_{t\to\infty} \frac{1}{e^t} = 0$

$\Rightarrow$ $f(t) = 1$ is of exponential order 1 as $t \to \infty$

Similarly $\lim_{t\to\infty} t = 0$ for any $s_o > 0$

i.e., $f(t) = t$ is also exponential

order $s_o > 0$

In general we have following remark.

Remark are:

(a) Every bounded function $f(t)$ on $[0, \infty]$ is of exponential order 0

(b) $f(t) = e^{ct}$ has exponential order $s_o = c$

(c) $f(t) = t^n$ for all $t\, \varepsilon\, [0, \infty]$ is of exponential order s_o for any $s_o > 0$.

(d) The functions $f(t) = e^{t^2}, \frac{1}{t}$ are not of exponential order

Solution. (*a*)

Let $f(t)$ is bounded on $[0, \infty]$

$\Rightarrow$ $\exists\, \beta$ such that $|f(t)| \leq \beta\ \forall\, t \in [0, \infty]$

Consider $|f(t)e^{-ot}| \leq \beta\ \forall\, t \geq 0$

$\Rightarrow$ $f(t)$ is of exponential order 0.

(*b*) Let $f(t) = e^{ct}$ then

$\left|f(t)e^{-s_o t}\right| = \left|e^{ct}e^{-s_o t}\right| \leq 1$ for $c = s_o$

that verifics that $f(t) = e^{ct}$ is of exponential order $s_o = c$

(c) For $f(t) = t^n$, we can show that $\lim_{t\to\infty} \frac{t^n}{e^{s_o t}} = M$ (using L' Hospital rule)

thus $t^n = O\left(e^{s_o t}\right)$ for any $s_o > 0$

Hence t^n is for exponential order $s_o > 0$.

(d) $f(t) = e^{t^2}$ is not of exponential order as $\lim_{t\to\infty} \frac{e^{t^2}}{e^{s_o t}} \to \infty$ for any so > 0 .

1.5.2 Laplace Transform of Some Special Function

Let $f(t)$ is any piecewise continuous function in every finite interval and $f(t) = 0$ for all negative values of t, also $f(t)$ is given exponential order as $t \to \infty$ then Laplace transform of f is denoted by F and is defined as

$$F(s) = L(f(t)) = \int_o^\infty e^{-st} f(t)\, dt$$

NOTES

In a systematic manner we can also define as, Let $f : [0, \infty] \to \mathbb{R}$ be a function such that

(i) $f(t)$ is piecewise continuous on $[0, \infty]$

(ii) $f(t)$ in of exponential order as $t \to \infty$ then $L(f) = F(s)$ the Laplace transform of f is defined as the improper integral

$$\int_0^\infty e^{-st} f(f)\, dt$$

or $$L(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

We call e^{-st} as Kernel of Laplace transform.

Note that the conditions (*i*) and (*ii*) given in the definition of 1.1 are sufficient for the existence of Laplace transform of f. But if the integral $\int_0^\infty e^{-st} f(t)\, dt$ exists without obeying (*i*) and (*ii*) then may (*i*) and (*ii*). In that situation we may state as: (by dropping (*i*) and (*ii*) Laplace transform of $f(t)$ is $F(s)$ and is defined as

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt \text{ provided the integral in R.H.S. exists.}$$

This Laplace transform or Laplace transformation is linear. This can be understood with the following (properties of Laplace transform). Before considering the properties of Laplace transform let us first consider some trivial examples.

Example 1.35 (*a*) Let $f(t) = 1$ then $F(s)\frac{1}{s}, s > 0$

$$L(1) = \int_0^\infty 1e^{-st} dt = \lim_{\lambda\to\infty} \int_0 e^{-st} dt$$

$$= \lim_{\lambda\to\infty} \left[\frac{e^{-st}}{-s}\right]_0 \qquad (s > 0)$$

$$= \lim_{\lambda\to\infty} \frac{-1}{s}(e^{-\lambda t} - e^{ot}) \qquad \left(\text{since } \lim_{\lambda\to\infty} e^{-\lambda t} = 0\right)$$

$$= \lim_{\lambda\to\infty} \frac{-1}{s}\left(e^{-\lambda t} - 1\right) = \frac{-1}{s}(0-1) = \frac{1}{s}$$

$$L(1) = \frac{1}{s},\ s > 0$$

(b) If $f(t) = t$ then $L(t) = F(s) = \frac{1}{s^2},\ (s > 0)$

$$F(s) = L(t) = \int_0^\infty e^{-st} t\, dt = \lim_{\lambda \to \infty} \int_0^\lambda te^{-st} dt$$

$$= \lim_{\lambda \to \infty} \left\{ \left[\frac{te^{-st}}{-s} \right]_0^\lambda + \frac{1}{s} \left(\frac{e^{-st}}{-s} \right)_0^\lambda \right\}$$

or $$F(s) = L(t) = \lim_{\lambda \to \infty} \left\{ -\frac{\lambda e^{-\lambda t}}{s} + \frac{1}{s^2} \left(-e^{-\lambda t} + 1 \right) \right\}$$

$$= \frac{1}{s} \lim_{\lambda \to \infty} \frac{\lambda}{e^{\lambda t}} + \frac{1}{s^2} \lim_{\lambda \to \infty} \left(-e^{-\lambda t} + 1 \right)$$

$$= \frac{1}{s^2} \text{ (using } \lim_{\lambda \to \infty} \frac{\lambda}{e^{\lambda t}} = 0 \text{ and } \lim_{\lambda \to \infty} e^{-\lambda t} = 0 \text{)}$$

thus $L(t) = \dfrac{1}{s^2}$

In general $L(t^n) = \dfrac{n!}{s^{n+1}} = \dfrac{\sqrt{n+1}}{s^{n+1}}$ (try yourself and its solution is given is next section using the properties of Laplace transform).

1.5.3 Properties of Laplace Transform

1. **Linearity Property:** Let c_1 and c_2 are constants and Laplace transforms of $f_1(t)$ and $f_2(t)$ are $F_1(s)$ and $F_2(s)$ respectively then laplace transform of $c_1 f_1 + c_2 f_2(f)$ is $c_1 F_1(s) + c_2 F_2(s)$. This property of linearity can be taken as the combination of follwing two properties.

$$L(c_1 f(f)) = c_1 L(f_1(f)) \quad \text{or} \quad L(c_1 f_1(t)) = c_1 F_1(s)$$

and $$L(f_1 + f_2) = F_1 + F_2$$

or $$L(f_1(t) + f_2(t)) = F_1(s) + c_2 F_2(s)$$

Proof:

Proof of property of linearity is easily carried using the fact that integral operator is linear.

$$\mathrm{L}(c_1 f_1(t) + c_2 f_2(t)) = \int_0^\infty \left(c_1 f_1(t) + c_2 f_2(t) \right) e^{-st} dt$$

$$= \int_0^\infty \left(c_1 f_1 e^{-st} + c_2 e^{-st} \right) dt$$

$$= \int_0^\infty c_1 f_1(t) e^{-st} dt + \int_0^\infty c_2 f_2(t) e^{-st} dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$L(c_1 f_1 + c_2 f_2) = c_1 F_1(s) + c_2 F_2(t)^*$$

*This property directly infers that Laplace transform is linear transformation.

NOTES

2. **Change of Scale Property:** If $f(s)$ is the Laplace transform of $f(t)$ then Laplace transform of f(at) is

that is if $L(f(t)) = F(s)$ then

$$L(f(at)) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Proof: $$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(f(at)) = \int_0^\infty e^{-st} f(at)\, dt$$

(taking at $= u$, $dt = \frac{du}{a}, t = \frac{u}{a}$)

$$= \frac{1}{a}\int_0^\infty e^{-\frac{su}{v}} f(u)\, du$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)\, du$$

$$L(f(at)) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

1.5.4 First and Second Shifting Property

Theorem 1.8: If $L(f(t)) = F(s)$ then $L\left(e^{at} f(t)\right) = F(s-a)$

Proof: Let $$L((f)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(e^{at} f(t)) = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$L(e^{at} f(t)) = F(s-a))$$

Second Shifting (Translation) Property (Heaviside Shifting Theorem)

$$L(f(t)) = F(s) \quad \text{and} \quad g(t) = \begin{cases} 0 & 0 < t < a \\ f(t-a) & t > a \end{cases}$$

then $$L(g(t)) = e^{-as} F(s)$$

Proof: $$g(t) = \begin{cases} 0 & 0 < t < a \\ f(t-a) & t > a \end{cases}$$

$$L(g(t) = \int_0^\infty e^{-st} g(t)\, dt \qquad \text{(for } a > 0)$$

NOTES

$$L(g(t)) = \int_0^a e^{-st} \quad 0 \quad dt \quad + \int_a^\infty e^{-st} f(t-a)\, dt$$

Thus $\quad L(g(t)) = \int_a^\infty e^{-st} f(t-a)\, dt$

Taking $\quad t - a = u \Rightarrow \left.\begin{array}{l} t = u + a \\ dt = du \end{array}\right\} \begin{array}{l} t = a \Rightarrow u = 0 \\ t = \infty \Rightarrow u = \infty \end{array}$

then $\quad L(g(t)) = \int_0^\infty e^{-s(uta)} f(u)\, du$

$$= \int_0^\infty e^{-su} e^{-sa} f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

$$L(g(t)) = e^{-as} F(s)$$

Before considersing some other properties, let us see first some example of Laplace transforms.

Example 1.36. $L(e^{at}) = \dfrac{1}{s-a}$

By $L(1) = \dfrac{1}{s} = F(s)$ (say) so by first shift property $L(1.e^{at}) = F(s-a)$

thus $\quad L(e^{at}) = \dfrac{1}{s-a}$

Example 1.37. (a) $L(\cos at) = \dfrac{s}{s^2 + a^2}$

(b) L (sin at) $= L(\sin at) = \dfrac{a}{s^2 + a^2}$

Solution: (a) L(cos at) $= \int_0^\infty e^{-st} \cos at\; dt$

$$= \text{Re}\left(\int_0^\infty e^{-st} e^{+iat} dt\right) = \text{Re}\left(L\left(e^{iat}\right)\right)$$

and L(sin at) $= \text{Im}\left(\int_0^\infty e^{-st} e^{iat} dt\right) = \text{Im}\left(L\left(e^{-iat}\right)\right)$

(b) Consider $L\ (e^{iat}) = \dfrac{1}{s - ia} = \dfrac{s + ia}{s^2 + a^2}$

$$\Rightarrow \quad L(e^{iat}) = \frac{s}{s^2 + a^2} + i\frac{a}{s^2 + a^2}$$

NOTES

thus $L(\cos at) = \frac{s}{s^2 + a^2}$

$L(\sin at) = \frac{a}{s^2 + a^2}$

Example 1.38. Obtain $L(e^{at} \cos wt)$ and $L(e^{at} \sin wt)$

Solution:

Since $L(\cos wt) = \frac{s}{s^2 + w^2} = F(s)$

$\Rightarrow$ $L(e^{at} \cos wt) = F(s - a)$

$= \frac{s - a}{(s - a) + w^2}$

Simlarly $L(e^{at} \sin wt) = \frac{a}{(s - a)^2 + w^2}$

Example 1.39. Obtain Laplace transform of $\cos^2 t$

Solution: Hint: $\cos^2 t = \frac{\cos 2t + 1}{2}$

$$L(\cos^2 t) = L\left(\frac{\cos 2t + 1}{2}\right)$$

$$= \frac{1}{2}\left[L(\cos 2t) + L(1)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + 4} + \frac{1}{s}\right]$$

Theorem 1.9: Multiplication and division by *t* (derivative of Laplace transform.

If $F(s)$ is the laplace transform of $f(t)$

i.e., $L(f(t)) = F(s)$ then

$$L(-f(t)) = \frac{dF}{ds} = F'(s)$$

and in general $L(t^n f(t)) = (-1)^n F^{(n)}(s)$

or $L\left(t^n + (t)\right) = (-1)^n \frac{d^n F(s)}{d.s^n}$

Proof: Let $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$\frac{dF}{ds} = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt$$

NOTES

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\, dt$$

$$= \int_0^\infty (-t) e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st}\left(-t \ \ f(t)\right) dt$$

$$F'(s) = L(-tf(t))$$

or $\quad F'(s) = (-1)' L(1 + f(t))$

Repeat this process again we get

$$(-1)^2 L(t^2 f(t)) = F''(s) = \frac{d^2 f(s)}{ds^2}$$

in general $F^{(n)}(s) = \dfrac{d^n F(s)}{ds^n} = (-1)^n L\left(t^n f(t)\right)$

$$= L((-1)^n t^n f(t))$$

or $\quad L(t^n f(t)) = \left(-1\right)^n \dfrac{d^n F(s)}{ds^n} = (-1)^n F^{(n)}(s)$

Example. 1.40 Obtain the Laplace transform of t^n.

Solution. Since $\quad L(1) = L\left(1\right) = \dfrac{1}{s} = F(s)$ (say)

$$L(t^n) = (-1)^n \frac{d^n F(s)}{ds^n}$$

$$= (-1)^n \left(\frac{1}{s}\right)$$

$$= (-1)^n (-1)^n \left(\frac{n!}{s^{n+1}}\right)$$

$$L(t^n) = \frac{n!}{s^{n+1}} = \frac{\sqrt{n+1}}{s^{n+1}} \qquad \left(\because n! = \sqrt{n+1}\right)$$

Exampl 1.41 Obtain the Laplace transform of $\sqrt{t}$

Solution. $L(t^n) = \dfrac{\sqrt{n+1}}{s^{n+1}}$

$$L\left(\sqrt{t}\right) = \frac{\sqrt{3/2}}{s^{3/2}} = \frac{\sqrt{1/2}+1}{s^{3/2}} \qquad \left(\text{using} \frac{\sqrt{1}}{2} = \sqrt{\pi}\right)$$

$$= \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{3/2}}$$

NOTES

$$L\left(t^{1/2}\right) = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$$

Example 1.42. Find $L\left(t^{-1/2}\right)$

Solution. $L\left(t^{-1/2}\right) = \frac{\sqrt{-1/2+1}}{s^{-1/2+1}} = \frac{\sqrt{1/2}}{s^{1/2}}$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Example 1.43. (a) Obtain Laplace transform of t^2 cosat
(b) Find L(t sinat)

Solution. (a) We know that L(cost) = $\frac{s}{s^2+1} = F(s)$

$$L(\cos at) = \frac{s}{s^2+a^2} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$$

$$= \frac{d}{ds}\left(\frac{(s^2+a^2)1 - s(2s)}{\left(s^2+a^2\right)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-s^2+a^2}{\left(s^2+a^2\right)^2}\right)$$

$$= \frac{\left(s^2+a^2\right)^2(-2s) - \left(-s^2+a^2\right)\left(2\left(s^2+a^2\right)2s\right)}{\left(s^2+a^2\right)^2}$$

$$= \frac{\left(s^2+a^2\right)\left\{\left(s^2+a^2\right)(-2s) - 4s\left(a^2-s^2\right)\right)}{\left(s^2+a^2\right)^4}$$

$$= \frac{2s^2 - 6a^2 s}{\left(s^2+a^2\right)^3}$$

NOTES

$$L(t^2 \cos at) = \frac{2s\left(s^2 - 3a^2\right)}{\left(s^2 + a^2\right)^3}$$

(b) Similarly by use of property 4, one can obtain

$$L(t \sin at) = \frac{2as}{\left(s^2 + a^2\right)^2}.$$

1.6 INITIAL AND FINAL VALUE THEOREMS

Theorem 1.10: Let $f(t)$ be a continously differentiable function and

$\frac{df(t)}{dt} = f'(t)$ be its first order derivative. Suppose L (f(t)) = F(s) then

$$L(f'(t)) = L\left(\frac{df}{at}\right) = s\ L(f(t)) - f(0)$$

or $L\left(\frac{df}{dt}\right) = s\ F(s) - f(0)$

where, $f(0) = f(t = 0)$

Proof : Let L $(F(t))$ = F (s) = $\int_0^\infty \overline{e}^{st}\left(f(t)\right)dt$

Then $L\left(\frac{df}{dt}\right) = \int_0^\infty e^{-st}\left(\frac{df}{dt}\right)dt$

Integrating by parts, we get

$$L\left(\frac{df}{dt}\right) = \overline{e}^{st}.f(t) - \int_0^\infty (-s)\overline{e}^{st}.f(t)\,dt$$

$$= -f(0) + 0 + s\int_0^\infty \overline{e}^{st} f(t)\,dt$$

$$L\left(\frac{df}{dt}\right) = -f(0) + s\ F(s) \qquad \text{(using the fact that } \overline{e}^{st} \to 0 \text{ as } t \to \infty\text{)}$$

thus $L\left(\frac{df}{dt}\right) = sF(s) - f(0)$

Note : Roughly speaking the Laplace transform of corresponds to multiplication of the Laplace transform of $f(t)$ by s.

NOTES

1.6.1 Laplace Transform of Derivative of Order '*n*'

Thorem 1.11: If $f(t)$ is continuously differentiable function and possesses derivatives of all order 1,2,.....,n which all are Laplace transformable, then

$$L\left(\frac{d^n f(t)}{dt^n}\right) = s^n L\left(f(t)\right) - \sum_{i=0}^{n-1} f^{(i)}(0)\, s^{n-i-1}$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f^{22}(0) - \ldots\ldots\ldots f^{(n-1)}(0)$$

Proof : We know that

$$L(f'(t)) = s\,F(s) - f(0)$$

or $$L(f'(t)) = s\,L(f(t)) - f(0) \qquad \ldots(1.9)$$

replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (Equation 1,9) we get

$$L(f''(t)) = s\,L[f'(t)] - f'(0)$$

$$= \text{or } L(f''(t)) = \delta\{\delta L(f(t)) - f(0)\} - f'(0).$$

$$= s^2 L(f(t)) - sf(0) - f'(0)$$

Similarly,

$$L(f'''(t)) = s^3 L(f'(t)) - s^2 f(0) - s f''(0) - f''(0)$$

$$L(f^{(n)}(t)) = s^n L(f(t)) - \sum_{i=0}^{n-1} f^{(i)}(0)\, s^{n-i-1}$$

Theorem 1.12: Initial value theorem:

If $L(f(t)) = F(s)$ them show that $\lim_{s\to\infty} f(t) = \lim_{s\to\infty} s\,F(s)$ provided limits exist.

Proof: We know that

$$L\left(\frac{df}{dt}\right) = L(f^1(t)) = s\,F(s) - f(0)$$

or $$\int_0^\infty e^{-st}\left(\frac{df}{dt}\right)dt = sF(s) - f(0)$$

taking limit $s \to \infty$ we get

$$\lim_{s\to\infty}\int_0^\infty e^{-st}\left(\frac{df}{dt}\right)dt = \lim_{s\to\infty} sF(s) - f(0)$$

$$\int_0^\infty \lim_{s\to\infty} e^{-st}\left(\frac{df}{dt}\right)dt = \lim_{s\to\infty} sF(s) - f(0)$$

(assume $\frac{df}{dt}$ is continuous on $[0, \infty]$

thus $$\lim_{s\to\infty} s\,F(s) - f(0) = 0$$

or $$\lim_{s\to\infty} s\,F(s) = f(0) = \lim_{t\to\infty} f(f)$$ (by Continuous of $f(t)$)

Hence $$\lim_{t\to\infty} f(t) = \lim_{s\to\infty} s\,F(s)$$

Theorem 1.13: Final Value Theorem

Let L($f(t)$) = F(s) then prove that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$, provided limits on both sides exist.

Proof: Again we use L(F[1](t)) = s F(s) – f(0)

or $$\int_0^\infty e^{-st}\left(\frac{df}{dt}\right)dt = s\,F(s) - f(0)$$

taking limit $s \to 0$ on both sides we get

$$\lim_{s\to 0}\int_0^\infty e^{-st}\left(\frac{df}{dt}\right)dt = \lim_{s\to 0} s\,F(s) - f(0)$$

or $$\int_0^\infty \left(\frac{df}{dt}\right)dt = \lim_{s\to 0} s\,F(s) - f(0)$$

or $$f(t)\,|_0^\infty = \lim_{s\to 0}\left(s\,F(s)\right) - F(0)$$

or $$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0}\left(s\,F(s)\right) - f(0)$$

or $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

1.7 MULTIPLICATION AND DIVISION BY 't' PERIODIC FUNCTIONS

Theorem 1.14: If L($f(t)$) = F(s) then

$$L\left(\int_0^t f(t)dt\right) = \frac{F(s)}{s}$$

Proof: Let F(s) = L($f(t)$)

Let $$g(t) = \int_0^t f(t)\,dt$$

$$\Rightarrow \quad g(0) = \int_0^0 f(t)\,dt = 0$$

$$g\,(t) = f(t)$$

$$\Rightarrow \quad L(g'(t)) = s(0)$$

$$Lf(t) = s\,L(g(t)) - g(0) = s\,L(g(t))$$

NOTES

$$\Rightarrow \qquad L\left(g\left(t\right)\right) = \frac{1}{s} L\left(f\left(t\right)\right)$$

$$\Rightarrow \qquad L\left(\int_0^t f(t)dt\right) = \frac{F\left(f\right)}{s}$$

Division by '*t*'

Theorm 1.15: Let L(*f*(*t*)) = F(*s*)

then prove that $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F\left(x\right)dx$, provided integral in R.H.S. exists.

Proof: Let F(*s*) = L(*f*(t))

or $\quad F(s) = \int_s^\infty e^{-st} f(t)dt$

integrating both sides with respect to *s* between *s* to ∞

$$\int_s^\infty F\left(s\right)ds = \int_s^\infty \left(\int_0^\infty e^{-st} f(t)dt\right) \text{ ds}$$

Here *s* and *t* are independent variables, so we can change order of integration in the separated integration in R.H.S.

thus $$\int_s^\infty F(s)ds = \int_0^\infty dt \int_0^\infty e^{-st} f(t)\, ds$$

$$= \int_0^\infty f\left(t\right)dt \int_s^\infty e^{-st}\, ds$$

$$= \int_0^\infty f\left(t\right)dt \left.\frac{e^{-st}}{-t}\right|_s^\infty$$

$$= \int_0^\infty f\left(t\right)dt \left(\frac{0 - e^{-st}}{-t}\right)$$

$$= \int_0^\infty e^{-st} \left(\frac{f\left(t\right)}{t}\right) ds$$

$$\int_s^\infty F\left(s\right)ds = L\left(\frac{f\left(t\right)}{t}\right)$$

or $$L\left(\frac{f\left(t\right)}{t}\right) = \int_s^\infty F\left(x\right)dx$$

Corollary: Prove that $\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(s) ds$

NOTES

Proof:

From the previous theorem

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(x) dx$$

or $$\int_0^\infty e^{-st} \frac{f(t)}{t} dt = \int_s^\infty f(x) dx$$

taking limit s $\to$ 0 on both sides we get

$$\int_0^\infty \frac{f(t)}{t} dt = \int_s^\infty f(x) dx$$

Now we consider same applications of these properties.

Example 1.44 Find the Laplace transform of $t^n e^{at}$

Solution: $$L(t^n e^{at}) = (-1)^n \frac{d^n}{ds^n} F(s)$$

where $$F(s) = L(e^{at}) = \frac{1}{s-a}$$

$\Rightarrow$ $$L(t^n e^{at}) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right)$$

$$= (-1)^n . (-1)^n \frac{n1}{(s-a)^{n+1}}$$

$$L(t^n e^{at}) = \frac{\overline{|n+1}}{(s-a)} n+1 \qquad \left(\text{since } n! = \overline{|n+1}\right)$$

Example 1.45: Laplace transforms of $\int_t^\infty \frac{e^{-x}}{x} dx$

Solution: Let $$f(t) = \int_t^\infty \frac{e^{-x}}{x} dx$$

$\Rightarrow$ $$f'(t) = -\frac{e^{-t}}{t}$$

$\Rightarrow$ $$t' f^1(t) = -e^{-t}$$

NOTES

$\Rightarrow \quad L(t\,t'(t)) = L(-e^{-t}) = -L(e^{-t}) = \dfrac{-1}{s+t}$

or $\quad \dfrac{-d}{ds}\left[L\left(f'(t)\right)\right] = \dfrac{-1}{s+1}$

or $\quad \dfrac{d}{ds}\left[s\,F(s)\right] = \dfrac{1}{s+1}\left(as\,\dfrac{d}{ds}f(0) = 0\right)$

or $\quad \dfrac{d}{ds}\left(s\,F(s)\right) = \dfrac{1}{s+1}$

integrating both side

$$s\,F(s) = \log(s+1) + c$$

by Final value of theorem

$$\lim_{s\to 0} s\,F(s) = \lim_{t\to\infty} f(t)$$

thus $\lim\limits_{s\to 0} s\,F(s) = \lim\limits_{s\to\infty}\left[\log(s+1) + c\right]$

$$= 0 + c = c$$

or $\quad c = \lim\limits_{s\to 0} s\,F(s) = \lim\limits_{t\to\infty} f(t)$

but

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\int_t^\infty \frac{e^{-x}}{x}\,dx = 0$$

so $\quad c = 0$

hence $\quad s\,F(s) = \log(s+1)$

$\Rightarrow \quad F(s) = \dfrac{\log(s+1)}{s}$

or $\quad L\left(\displaystyle\int_t^\infty \frac{e^{-x}}{x}\,dx\right) = \dfrac{\log(s+1)}{s}$

1.8 INVERSE LAPLACE TRANSFORMS

If $f(t)$ be any function of t and

$L\{f(t)\} = f(s)$

then $f(t)$ is known as inverse Laplace transformation and given by

$f(t) = L^{-1}\{f(s)\}$

For example, if $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

NOTES

Then $L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$

The uniqueness of the inverse transform for $t > 0$ is established by Lerch's theorem.

According to Lerch's theorem if $f(t)$ is picewise continuous in every finite interval $O \le t \le a$ and of exponential order for $t > a$, then the inverse Laplace transformation.

$L^{-1}\{f(s)\} = f(t)$ is unique.

Main Laplace inverse are given below:

$$L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sin hat$$

$$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cos hat$$

$$L^{-1}\left[\frac{1}{s^n+1}\right] = \frac{t^n}{\lfloor n} \text{ for } n \text{ positive integer}$$

$$L^{-1}\left[\frac{1}{s^n+1}\right] = \frac{t^n}{\lfloor n+1}, n > -1$$

1.8.1 Properties of Inverse Laplace Transformation

1. Linearly Property: Let $f_1(s)$ and $f_2(s)$ be the Laplace transformation of the functions $f_1(t)$ and $f_2(t)$, respectively, and a and b are the constant, then

   $$L^{-1}\{af_1(s)+bf_2(s)\}=a L^{-1}\{f_1(s)\}+b L^{-1}\{f_2(s)\}$$

2. First Shifting theorem:

   If $L^{-1}\{f(s)\} = f(t)$

   then $L^{-1}\{f(s-a)\} = e^{at} f(t)$

   By definition $f(s) = \int_0^\infty e^{-st} f(t)dt$

   $$\therefore \quad f(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-st} \left\{ e^{at} f(t) \right\} dt$$

$$= L\left\{ e^{at} f(t) \right\}$$

NOTES

$$\therefore \qquad e^{at} f(t) = \text{L}^{-1}\{ f(s-a) \}$$

$\therefore$ There are some important deductions:

1. $\text{L}^{-1}\left\{ \dfrac{1}{(s-a)^2 + b^2} \right\} = \dfrac{1}{b} e^{at} \sin bt$

2. $\text{L}^{-1}\left\{ \dfrac{(s-a)}{(s-a)^2 + b^2} \right\} = e^{at} \cos bt$

3. $\text{L}^{-1}\left\{ \dfrac{1}{(s-a)^2 - b^2} \right\} = \dfrac{1}{b} e^{at} \sin hbt$

4. $\text{L}^{-1}\left\{ \dfrac{(s-a)}{(s-a)^2 - b^2} \right\} = e^{at} \cos hbt$

5. $\text{L}^{-1}\left\{ \dfrac{s}{(s^2 + a^2)^2} \right\} = \dfrac{t}{2a} \sin at$

6. $\text{L}^{-1}\left\{ \dfrac{1}{(s^2 + a^2)^2} \right\} = \dfrac{1}{2a^3} (\sin at - at \cos at)$

3. Second Shifting Property:

If $\text{L}^{-1}\{ f(s) \} = f(t)$

Then $\text{L}^{-1}\left\{ e^{-as} f(s) \right\} = G(t)$

Where $G(t) = \begin{cases} f(t-a), & t > a \\ 0 & t < a \end{cases}$

By definition,

$$\text{L}\{\text{G}(t)\} = \int_0^\infty e^{-st} G(t) dt$$

$$= \int_0^a e^{-st} G(t) dt + \int_a^\infty e^{-st} G(t) dt$$

Let $t - a = v$

$$O + \int_a^\infty e^{-st} f(t-a) dt$$

$$= -\int_0^\infty (e^{-s(a+v)} f(v) dv$$

$$= e^{-as} \int_0^\infty e^{-sv} f(v) dv$$

$$= e^{-as}\mathrm{L}\{f(t)\}$$

$$= e^{-as}f(s)$$

$$\therefore \quad G(t) = L^{-1}\{e^{-as}f(s)\}$$

NOTES

4. Change of Scale Properly:

If $\mathrm{L}^{-1}\{f(s)\} = f(t)$

Then $\mathrm{L}^{-1}\{f(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right),\ a > 0$

By definition $f(s) = \mathrm{L}\{f(t)\}$

$$= \int_0^\infty e^{-st}f(t)dt$$

$$f(as) = \int_0^\infty e^{-ast}f(t)dt$$

Let $at = v$

$$= \int_0^\infty e^{-sv}f\left(\frac{v}{a}\right)\frac{dv}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-sv}f\left(\frac{v}{a}\right)dv$$

$$= \frac{1}{a}\mathrm{L}\left[f\left(\frac{t}{a}\right)\right]$$

$$\therefore \quad \{\mathrm{F}(as)\} = \frac{1}{a}L\left\{f\left(\frac{t}{a}\right)\right\}$$

Example 1.46: Find the Laplace inverse transformation of

(i) $\frac{s+8}{s^2+45+5}$ (ii) $\frac{3s+7}{s^2-2s-3}$ (ii) $\mathrm{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathrm{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-2^2}\right\}$

Solution: (i) $\mathrm{L}^{-1}\left\{\frac{s+8}{s^2+4s+5}\right\} = \mathrm{L}^{-1}\left\{\frac{s+8}{(s+2)^2+1}\right\}$

$$= \mathrm{L}^{-1}\left\{\frac{(s+2)+6}{(s+2)^2+1}\right\}$$

$$= \mathrm{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1^2}\right\} + 6L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

As $\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos t$

NOTES

By first shifting property

$$L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} = e^{-2t}\cos t$$

$$\therefore \quad L^{-1}\left\{\frac{s+8}{s^2+4s+5}\right\} = e^{-2t}\cos t + 6e^{-2t}\sin t$$

(ii) $$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{3s+7}{(s-1)^2-2^2}\right\}$$

$$= L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-2^2}\right\}$$

$$= 3L^{-1}\left\{\frac{(s-1)}{(s-1)^2-2^2}\right\} + 10L^{-1}\left\{\frac{1}{(s-1)^2-2^2}\right\}$$

$$= 3e^t \cosh 2t + 5e^t \sin h\, 2t$$

By first shifting property

Example 1.47: Find the inverse Laplace transform

(i) $\dfrac{e^{-5s}}{(s-2)^4}$

(ii) $\dfrac{se^{-as}}{s^2-\omega^2}$

Solution: (i) $L^{-1}\left\{\dfrac{1}{(s-2)^4}\right\} = e^{2t} L^{-1}\left\{\dfrac{1}{s^4}\right\}$

$$= e^{2t}\frac{t^3}{\lfloor 3} = \frac{e^{2t}t^3}{6}$$

$\therefore$ By second shifting property

$$L^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)}, & t>5 \\ 0 & t<5 \end{cases}$$

(ii) As $L^{-1}\left\{\dfrac{s}{s^2-w^2}\right\} = \cos h\, \omega t$

By second shifting property

$$L^{-1}\left\{\frac{se^{-as}}{s^2-\omega^2}\right\} = \begin{cases} \cos h\omega(t-a) & t>a \\ 0 & t<a \end{cases}$$

Example 1.48: $L^{-1}\left\{\dfrac{s^2-3s+4}{s^3}\right\}$

NOTES

Solution: $L^{-1}\left\{\dfrac{s^2-3s+4}{s^3}\right\}$

$$L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s}\right\}-3L^{-1}\left\{\frac{1}{s^2}\right\}+4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1-3t+\frac{4t^2}{2} = 1-3t+2t^2$$

Example 1.49: $L^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

Solution: $L^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = L^{-1}\left\{\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\}$

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

Solving the partial fraction

$A = \dfrac{1}{2}$, $B = 1$, $C = \dfrac{5}{2}$

$$\therefore\ L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\}-L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$$

Example 1.50: Evaluate $L^{-1}\left\{\dfrac{1+2s}{(s+2)^2(s-1)^2}\right\}$

Solution: As $\dfrac{1+2s}{(s+2)^2(s-1)^2} = \dfrac{A}{s+2}+\dfrac{B}{(s+2)^2}+\dfrac{C}{(s-1)}+\dfrac{D}{(s-1)^2}$

Solving the partial fraction

A=O, B=$\dfrac{-1}{3}$, C=0, D=$\dfrac{1}{3}$

$$\therefore\ L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = \frac{-1}{3}L^{-1}\left\{\frac{1}{(s+2)^2}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= \frac{-1}{3}e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}+\frac{1e^{t}}{3}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{-1}{3}e^{-2t}t+\frac{1}{3}e^{t}t$$

$$= \frac{t}{3}\left(e^{t}-e^{-2t}\right)$$

NOTES

Example 1.51: $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solution: $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{S-1}+\dfrac{Bs+C}{\left(s^2+2s+5\right)}$

Solving the partial fraction

A = 1, B = –1, C = 2

$$\therefore\ L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s\text{-}1}\right\}+L^{-1}\left\{\frac{-s+2}{s^2+2s+5}\right\}$$

$$= e^t - L^{-1}\left\{\frac{s-2}{(s+1)^2+4}\right\}$$

$$= e^t - L^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+2^2}\right\}$$

$$= e^t - L^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\}+3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$$

Example 1.52: Evaluate $L^{-1}\left\{\dfrac{s}{\left(s^4+s^2+1\right)}\right\}$

Solution: As $\dfrac{s}{s^4+s^2+1} = \dfrac{s}{(s^2+1)^2-s^2} = \dfrac{s}{\left(s^2+1+s\right)\left(s^2+1-s\right)}$

$$= \frac{As+B}{s^2+s+1}+\frac{Cs+D}{s^2-s+1}$$

Solving the partial fraction

A = C = 0, $B=\dfrac{-1}{2},\ D=\dfrac{1}{2}$

$$\therefore\quad L^{-1}\frac{s}{s^4+s^2+1} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1}-\frac{1}{s^2+s+1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}t}{2}-\frac{1}{2\frac{\sqrt{3}}{2}}e^{\frac{-t}{2}}\sin\frac{\sqrt{3}t}{2}$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}t}{2}\left(e^{\frac{t}{2}} - e^{\frac{-t}{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \sin h \frac{t}{2}$$

NOTES

Example 1.53: Find the inverse transform of

(i) $\frac{1}{s(s^2+a^2)}$

(ii) $\frac{1}{s(s+1)^3}$

Solution: (i) As $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\, dt$$

$$= \frac{1}{a^2}[-\cos at]_0^t$$

$$= \frac{(1-\cos at)}{a^2}$$

(ii) $$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+1)-1](s+1)^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{(s-1)s^3}\right\}$$

As $$L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s-1)s}\right\} = \int_0^t e^t\, dt$$

$$= e^t - 1$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s-1)s}\right\} = \int_0^t (e^t-1)dt$$

$$= e^t - t - 1$$

and $$L^{-1}\left\{\frac{1}{(s-1)s^3}\right\} = \int_0^t (e^t-1)dt$$

$$= e^t - \frac{t^2}{2} - t - 1$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = e^{-t}\left(e^t - \frac{t^2}{2} - t - 1\right)$$

NOTES

$$= 1 - e^{-t}\left(1 + t + \frac{t^2}{2}\right)$$

Example 1.54: $L^{-1}\left\{\frac{5s-2}{s^2(s+2)(s-1)}\right\}$

Solution: Let $\frac{5s-2}{(s+2)(s-1)} = \frac{A}{S+2} + \frac{B}{S-1}$

Solving the partial fraction

A = 4, B = 1

$$\frac{5s-2}{(s+2)(s-1)} = \frac{4}{(s+2)} + \frac{1}{s-1}$$

$$L^{-1}\left\{\frac{5s-2}{(s+2)(s-1)}\right\} = 4t^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 4e^{-2t} + e^t$$

$$L^{-1}\left\{\frac{5s-2}{s(s+2)(s-1)}\right\} = \int_0^t \left(4e^{-2t} + e^t\right) dt$$

$$= e^t - 2e^{-2t} + 1$$

$$L^{-1}\left\{\frac{5s-2}{s^2(s-1)(s+2)}\right\} = \int_0^t \left(e^t - 2e^{-2t} + t\right) dt$$

$$= e^t + e^{-2t} + t - 2$$

Example 1.55: $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Solution: As $L\{tf(t)\} = -\frac{d}{ds}[\log(s+1) - \log(s-1)]$

$$= -\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

$$= -L\left(e^{-t}\right) + L\left(e^t\right)$$

$$= L\left(e^t - e^{-t}\right)$$

$$= L(2\sin ht)$$

$$\therefore \quad Lf(t) = 2 \sin ht$$

$$f(t) = \frac{\sin ht}{t}$$

NOTES

Example 1.56: $L^{-1}\left[\tan^{-1}\frac{1}{5}\right]$

Solution: $L^{-1}\left[\tan^{-1}\frac{1}{s}\right] = \frac{-1}{t}L^{1}\left[\frac{d}{ds}\tan^{-1}\frac{1}{s}\right]$

$$= \frac{-1}{t}L^{1}\left\{\frac{1}{1+\frac{1}{s^2}}\left(\frac{-1}{s^2}\right)\right\} \quad \left[\begin{array}{l}\text{As } L^{-1}\{f(s)\} \\ = \frac{-1}{t}L^{-1}\frac{d}{ds}f(s)\end{array}\right]$$

$$= \frac{+1}{t}L^{-1}\left\{\frac{1}{1+s^2}\right\}$$

$$= \frac{\sin t}{t}$$

Example 1.57: $L^{-1}\left\{\log\frac{s^2-1}{s^2}\right\}$

Solution: $L^{-1}\left\{\log\frac{s^2-1}{s^2}\right\} = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\log\frac{s^2-1}{s^2}\right\}$

$$= \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left\{\log(s^2-1)-\log s^2\right\}\right\}$$

$$= \frac{-1}{t}L^{-1}\left\{\frac{2s}{s^2-1}-\frac{2}{s}\right\}$$

$$= \frac{1}{t}\{2\cos ht-2\}$$

$$= \frac{2}{t}(1-\cos ht)$$

Example 1.58: $L^{-1}\left\{\cot^{-1}(1+s)\right\}$

Solution: $L^{-1}\left\{\cot^{-1}(1+s)\right\} = \frac{1}{t}L^{-1}\left\{\frac{d}{ds}\cot^{-1}(1+s)\right\}$

$$= \frac{1}{t}L^{-1}\left\{\frac{-1}{1+(1+s)^2}\right\}$$

$$= \frac{1}{t}e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{e^{-t}\sin t}{t}$$

Example 1.59: $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$

NOTES

Solution: $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\log\left(1+\frac{1}{s^2}\right)\right\}$

$$= \frac{-1}{t}L^{-1}\left\{\frac{1}{1+\frac{1}{s^2}}\left(\frac{-2}{s^3}\right)\right\}$$

$$= \frac{2}{t}L^{-1}\left\{\frac{1}{s\left(s^2+1\right)}\right\}$$

$$= \frac{2}{t}\int_0^t \sin t dt$$

$$= \frac{2}{t}(1-\cos t)$$

Example 1.60: $L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\}$

Solution: $= \frac{1}{t}L^{-1}\left\{\frac{d}{ds}\log\left(1+\frac{\omega^2}{s^2}\right)\right\}$

$$= -\frac{-1}{t}L^{-1}\left\{\frac{1}{1+\frac{\omega^2}{s^2}}\left(\frac{-2\omega^2}{s^3}\right)\right\}$$

$$= \frac{2}{t}\omega^2 L^{-1}\left\{\frac{1}{s\left(s^2+\omega^2\right)}\right\}$$

$$= \frac{2\omega^2}{t^2}\frac{1}{\omega}\int_0^t \sin \omega t\, dt$$

$$= \frac{2\omega}{t}\left(-\cos \omega t\right)_0$$

$$= \frac{2\omega}{t}\left(1-\cos \omega t\right)$$

Example 1.61: $L^{-1}\left\{\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}\right\}$

Solution: $L^{-1}\left\{\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}\right\} = \frac{1}{2}L^{-1}\left\{\log\left(s^2+b^2\right)\right\}-\frac{1}{2}L^{-1}\left\{2\log(s-a)\right\}$

$$= \frac{1}{2}L^{-1}\left\{\log\left(s^2+b^2\right)\right\}-L^{-1}\left\{\log(s-a)\right\}$$

$$= \frac{-1}{2t}L^{-1}\left\{\frac{d}{ds}\log\left(s^2+b^2\right)\right\}+\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\log(s-a)\right\}$$

NOTES

$$= \frac{-1}{2t} L^{-1}\left\{\frac{2s}{s^2+b^2}\right\} + \frac{1}{t}\left\{L^{-1}\frac{1}{s+a}\right\}$$

$$= \frac{-1}{2t}\{2\cos bt\} + \frac{1}{t}e^{-at}$$

$$= \frac{e^{-at}\cos bt}{t}$$

Example 1.62: Apply convolution theorem to evaluate

Solution: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

As $= L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$, $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

We have by convolution theorem

$$L^{-}\left\{\frac{s}{s^2+a^2} - \frac{1}{s^2+a^2}\right\} = \int_0^t \cos a(t-u)\frac{\sin au}{a}du$$

$$= \frac{1}{2a}\int_0^t \{\sin at + \sin(2au - at)\}du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au - at)\right]_0^t$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$$

Example 1.63: Apply convolution theorem find $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Solution: As $L^{-1}\left\{\frac{1}{s^2}\right\} = t$, $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sin hat$

We have by convolution theorem

$$L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \int_0^t u\frac{1}{a}\sin ha(t-u)du$$

$$= \frac{1}{a}\int_0^t u\sin h(at-au)du$$

$$= \frac{1}{a}\left[\left\{-\frac{u}{a}\cos h(at-au)\right\}_0^t + \frac{1}{a}\int_0^t \cos h(at-au)du\right]$$

$$= \frac{1}{a^2}\left[-t\cos h(0) + \left\{-\frac{1}{a}\sin h(at-au)\right\}_0^t\right]$$

$$= \frac{1}{a^3}(-at + \sin hat)$$

Example 1.64: Apply convolution to prove that

Solution: $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$, m>0, n>0

Let $\quad F(t) = \int_0^t x^{m-1}(t-x)^{n-1}dx$

And $\quad F_1(x) = x^{m-1}$ and $F_2(x) = x^{n-1}$

Then $\quad F(t) = \int_0^t F_1(x)F_2(t-x)dx$

$$= F_1 * F_2$$

Using covolution, we get

$$L\{F(t)\} = L\{F_1 * F_2\}$$

$$= F_1(S)\,.\,F_2(S)$$

Where $\quad F_1(S) = L\{F_1(t)\}$

$$F_2(S) = L\{F_2(t)\}$$

$$= L\{t^{m-1}\}.L\{t^{n-1}\}$$

$$= \frac{\lfloor m-1}{s^m}\cdot\frac{\lfloor n-1}{s^n}$$

$$= \frac{\Gamma m\,\Gamma n}{s^{m+n}}$$

$$F(t) = L^{-1}\left\{\frac{\Gamma m\,\Gamma n}{s^{m+n}}\right\}$$

$$= \Gamma m\,\Gamma n\,L^{-1}\left\{\frac{1}{s^{m+n}}\right\}$$

$$= \Gamma m\,\Gamma n\,\frac{t^{m+n-1}}{\Gamma(m+n)}$$

$$\int_0^t x^{m-1}(t-x)^{n-1}dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}t^{m+n-1}$$

Putting $t=1$

$$\int_0^t x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

Example 1.65: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}, a \neq b$

Solution: $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$, $L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$

By convolution $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos ax \cos b(t-x)dx$

$$= \frac{1}{2}\int_0^t$$

$$\cos(ax + bt - bx) + \cos(ax + bx - bt)dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)x+bt\}}{a-b} + \frac{\sin\{(a+b)x-bt\}}{a+b}\right]_0^t$$

$$= \frac{\sin at - \sin bt}{2(a-b)} + \frac{\sin at + \sin bt}{2(a+b)}$$

1.8.2 Some Elementary Inverse Laplace Transform

If Laplace transform of $f(t)$ is $F(s)$ then $f(t)$ is called inverse Lalplace transform of $F(s)$ and we write $f(t) = L^{-1}(F(s))$

L^{-1} called inverse Laplace transform operator.

Thus of $L(f(t)) = F(s)$ then $L^{-1}(F(s)) = f(t)$.

Like Laplace transform, inverse Laplace transform is also linear and Possesses some similar properties like change to scale property, shift property (first and second), convolution theorem etc.

In short we are starting these property (without their proof). Proof of these results are very similar to the case considered in case of Laplace transform.

1. Linearity: If $L^{-1}(F_1(s)) = f_1(t)$ and

   $L^{-1}(F_2(s)) = f_2(t)$

   then $L^{-1}(c_1 f_1(s) + c_2 F_2(s)) = c_1 L^{-1}(F_1(s)) + 2\,C_2 L^{-1}(F_2/S)$

   or $L^{-1}(c_1 F_1 + c_2 F_2) = c_1 f_1(t) + c_2 f_2(t)$

2. Change of scale:

   $$L(f(t)) = F(s) \quad \Rightarrow \quad L^{1}(F(s)) = f(t)$$

   then $$L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right)$$

3. First shift/translation property:

   If $L^{-1}(F(s) = f(t)$ then

   $$L^{-1}(F(s+a)) = e^{at} f(t)$$

   and $$L^{-1}(F(s+a)) = e^{-at} f(t)$$

4. Second shift property:

   If $L^{-1}(F(s)) = f(t)$ then

   $$L^{-1}(e^{-as} F(s)) = g(t) \begin{cases} 0 & 0 < t < a \\ f(t-a) & t > a \end{cases}$$

Next we consider some examples to obtain the inverse Laplace transform.

NOTES

Example 1.66. Find the inverse Laplace transform of

(a) $\frac{1}{\sqrt{2s+5}}$ (b) $\frac{1}{s^2-9}$

Solution: We know that

$$L(t^n) = \frac{n!}{s^n+1}$$

$$\Rightarrow \qquad L^1\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!} = \frac{t^2}{\lceil n+1}$$

$$L^{-1}\left(\frac{1}{\sqrt{2s+5}}\right) = L^{-1}\left(\frac{1}{\sqrt{2}\left(s+5/2\right)^{1/2}}\right)$$

$$= \frac{1}{\sqrt{2}} L^{-1}\left(\frac{1}{(s+5)^{1/2}}\right)$$

$$= \frac{1}{\sqrt{2}} e^{-5/2t} L^{-1}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \frac{1}{\sqrt{2}} e^{\frac{-5}{2}t} \frac{t^{-1/2}}{\lceil -1/2+1}$$

$$= \frac{e^{-5/2t}\, t^{-1/2}}{\sqrt{2\pi}} \qquad \left(\because \sqrt{\pi} = \sqrt{1/2}\right)$$

$$\text{(b) } L^{-1}\left(\frac{1}{s^2-9}\right) = L^{-1}\left(\frac{1}{3}.\frac{3}{s^2-3^2}\right) = \frac{1}{3}L^{-1}\left(\frac{3}{s^2-3^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^2-9}\right) = \frac{1}{3} \sin h\, 3t$$

Some Elementary Inverse Laplace Transform

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

3. $L^2\left(\frac{1}{s-a}\right) = e^{at}$

4. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos h \text{ at}$

5. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sin h\, at$

6. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

7. $L^{-1}(F(s-a)) = e^{at} f(t)$

8. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \text{cosat}$

NOTES

9. $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t L^{-1}(F(s))\,ds = \int_0^t f(t)\,dt$

10. $L^{-1}(e^{-as} F(s)) = f(t-a)\, U(f-a)$

1.9 UNIQUENESS THEOREM OF INVERSE LAPLACE TRANSFORM

Suppose $f(t)$ and $g(t)$ are two piecewise continuous functions on $[0, \infty]$ and are of exponential order a and having same Laplace transforms, i.e.

$$L(f(t)) = L(g(t)) \quad \text{for} \quad s > a$$

or $$F(s) = G(s) \quad \text{for} \quad s > a$$

then $$L^{-1} F(s) = L^{-1}(G(s)), \quad i.e., \quad f(t) = g(t) \quad \forall\, t \geq 0$$

Proof: Let $$L(f) = L(g)$$

$$\Rightarrow \quad L(f) - L(g) = 0$$

$$\Rightarrow \quad L(f-g) = 0 \qquad \text{(Since L is a linear transform)}$$

$$\Rightarrow \quad L(f(t) - g(t)) = 0$$

We wish to show $f(t) - g(t) = 0$, *i.e.*, $f = g$ for which it is sufficient to show that $L(f(t)) = 0$ for $s > a$ then $f(t) = 0$. $\forall\, t \geq 0$ or if $F(s) = 0$ for $s > a$ then $f(t) = 0 \ \forall\, t \geq 0$.

Fix $s_o > a$ and make the change of variables in Laplace transform of $e^{-t} = u$. Then for

$$s = s_o + n + 1, \text{ we obtain}$$

$$0 = L(f(t)) = \int_0^\infty f(t)\, e^{-(s_o+n+1)-t}\, dt$$

$$0 = F(s) = \int_0^\infty f(t)\, e^{-nt} e^{-s_o t} e^{t}\, dt$$

$$0 = \int_1^0 f(-\log u)\, u^n u^{s_o} u \left(-\frac{1}{u}\right) du$$

$$0 = \int_0^1 u^n\, u^{s_o} f(-\log u)\, du$$

$$e^{-t} = u$$

NOTES

$$t = -\log u - e^{-t}\,dt = du$$

$$du = -e^{t}\,du$$

$$= -\frac{1}{u}du$$

$$t = 0 \quad \Rightarrow \quad u = 1$$

$$t = \infty \quad \Rightarrow \quad u = 0$$

If we assume

$h(u) = u^{s_o} f(-\log u)$ then h is piecewise continuous on [0, 1]

and $\lim\limits_{u \to 0} h(u) = \lim\limits_{t \to \infty} e^{-s_o t} f(t) = 0$ (as $s_o > a$)

Thus we define $h(0) = 0$ then h is piecewise continuous and satisfies

$$\int_0^1 h(u)\; p(u)\,du = 0 \text{ for every polynomial } p.$$

This implies that if g has or power series expansion which converges uniformly on [0, 1] then

$$\int_0^1 h(u)\,g(u)\,du = 0 \qquad ...(a)$$

If h is not then zero function then replacing h with $-h$ if necessary, we can find a $u_0 \in (0, 1)$ and an interval $J = [u_0 - c, u_0 + c] = c\,[0, 1]$ and on $c_1 > 0$ so that $h \geq c_1$ on J.

Consider the function $g(u) = \frac{1}{d} e^{-\left(\frac{u-u_0}{d}\right)^2}$. If $d > 0$ then g has a power series expansion which converges uniformly on [0, 1] so that (a) holds.

Setting $\quad I_1 = \int_J g(u)du = \int_{u_{0-c}}^{u_0+c} g(u)du$

or $\quad I_1 = \int_{-c/d}^{c/d} e^{-t^2}\,dt \qquad ...\text{(b)}$

and $\quad I_2 = \int_{4_0+c}^{1} g(u)\,du = \int_{c/d}^{(1-u_0)} e^{-t^2}\,dt \qquad ...\text{(c)}$

$\quad I_3 = \int_0^{(u_0-c)} g(u)\,du = \int_{-d_0/d}^{-c/d} et^2\,dt \qquad ...\text{(d)}$

Let $\quad A = \int_{-\infty}^{\infty} e^{-t^2}\,dt$ them $A > 0$

and given $\in > 0 \quad \exists\, \partial > 0$ such that $0 < d \leq \delta$

then $\quad I_1 \geq \frac{A}{2}, \quad 0 \leq I_2 \leq \varepsilon, \quad 0 \leq I_3 \leq \varepsilon$

Because $h \geq c_1 > 0$ and J and $|-h| \leq N$ for some $N < \infty$

$$\int_J h(u)\, g(u)\, du \geq c_1 \frac{A}{2}$$

and

$$\left|\int_{[0,1]\setminus J} h(u)\; g(u)\; du\right| \leq 2N\varepsilon$$

and thus

$$\int_0^1 h(u)\, g(u)\, du \geq c_1 \frac{A}{2} - 2N\,\varepsilon > 0$$

provided $\varepsilon < \dfrac{c_1 A}{4N}$ which Contradicting (a).

This proves that h is a zero function and s_o by definition of f we must have f equal to the zero function.

we $\quad F(s) = 0 \quad$ for $\quad s > 0$

then $\quad f(t) = 0 \quad \forall \quad t \geq 0$

Hence $\quad L(f(t) - g(t)) = 0 \quad$ for $\quad s > a$

$\Rightarrow \quad f(t) - g(t) = 0 \quad \forall \; t \geq 0$

$\Rightarrow \quad f(t) = g(t)$

or $\quad L^{-1}(F(s)) = L^{-1}(G(s)) \quad$ for $\quad$ All $t \geq 0$

i.e., inverse Laplace transform is unique, if their Laplace transforms are given equal for $s > 0$

Convolution Theorem

If $F_1(s) = L(f_1(t))$ and $F_2(s) = L(f_2(t))$ then

$$L\left(\int_0^t f_1(g)\, f_2(t-y)\, dy\right) = L\int_0^t f_2(g) f_1(t-y) dy$$

$$= F_1(s)\, F_2(s)$$

or

$$L^{-1}(F_1(s)F_2(s)) = \int_0^t f_1(g)\, f_2(t-y)\, dy$$

Proof: Left to reader as an exercise.

Applications

For example, obtain the inverse Laplace transform of

(a) $\dfrac{1}{(s+a)(s+b)}$ $\qquad$ (b) $\dfrac{s^2}{(s^2+a^2)^2}$

(a) Solution: $L^{-1}\left(\dfrac{1}{(s+a)(s+b)}\right) = L^{-1}\left(F_1(s).F_2(s)\right)$

NOTES

where,

$$F_1(s) = \frac{1}{s+b} \quad \Rightarrow \quad L^{-1}(F_1(s)) = f_1(t) = e^{-at}$$

$$F_2(s) = \frac{1}{s+b} \quad \Rightarrow \quad L^{-1}\left(\frac{1}{s+b}\right) = f_2(t) = e^{-bt}$$

thus by involution theorem

$$L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) = \int_0^f f_1(g) f_2(t-y)\, dy$$

$$= \int_0^t e^{-aty} e^{-(t-y)b} dy$$

$$= \int_0^t e^{-ay} e^{-tb} e^{yb} dy$$

$$= e^{-bt} \int_0^t e^{-(a-b)y}\; dy$$

$$= e^{-bt} \left.\frac{e^{-(a-b)y}}{-(a-b)}\right|_0^t$$

$$= \frac{e^{-bt}}{b-a}\left(e^{-(a-b)t} - 1\right)$$

$$= \frac{e^{-bt}}{b-a}\left(-1 + e^{-(a-b)t}\right)$$

$$L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) = \frac{e^{-bt} - e^{-at}}{a-b}$$

(b) $$L^{-1}\left(\frac{s^2}{\left(s^2+a^2\right)^2}\right) = L^{-1}\left(\frac{s}{s^2+a^2}\right)\frac{s}{\left(s^2+a^2\right)} = L^{-1}\left(F_1(s)F_2(s)\right)$$

$$F_1(s) = L^{-1}\frac{s}{s^2+a^2} = 1\, L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = f_2(t)$$

$$F_2(s) = \frac{s}{s^2+a} = 1 \qquad f_2(t) = \cos at$$

$$L^2\left(\frac{s^2}{\left(s^2+a^2\right)}\right) = L^{-1}\left\{\frac{s}{s^2+a^2}\,\frac{s}{s^2+a^2}\right\}$$

$$= L^{-1}\{F_1(s)\,F_2(s)\}$$

$$F_1(s) = \frac{s}{s^2+a^2} = F_2(s) \quad \Rightarrow \quad L^{-1}(F_1(s)) = f_1(t) = \cos at$$

and f_2 (t) = cosat

thus by Evolution theorem,

$$L^{-1}\frac{s^2}{\left(s^2+a^2\right)^2} = \int_0^t \cos a\,y\;\cos a\,(t-y)\,dy$$

$$= \frac{1}{2}\int_0^t\left[\cos at + \cos\left(2ay - at\right)\right]dy$$

$$= \frac{1}{2}\left[y\cos at + \frac{1}{2a}\sin\left(2ay - at\right)\right]_0^t$$

$$L^2\left(\frac{s^2}{\left(s^2+a^2\right)^2}\right) = \frac{1}{2a}\quad[\text{at}\cos\text{at} + \sin\text{at}]$$

1.10 INVERSE LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

1. Linearity Property of Inverse Laplace Transform

Theorem 16: If $L\{F_1(t)\} = f_1(p)$, $L\{F_2(t)\} = f_2(p)$ and c_1 and c_2 be two constants, then

$$L^{-1}\{c_1 f_1(p) + c_2 f_2(p)\} = c_1F_1(t) + c_2F_2(t)$$

$$i.e., L^{-1}\{c_1f_1(p) + c_2 f_2(p)\} = c_1L^{-1}\{f_1(p)\} + c_2L^{-1}\{f_2(p)\}$$

Proof: Now $L\{F_1(t)\} = f_1(p)$ and $L\{F_2(t)\} = f_2(p)$ ...(1.10)

By linearity property of Laplace transforms, we get

$$L\{c_1F_1(t) + c_2F_2(t)\} = c_1L\{F_1(t)\} + c_2L\{F_2(t)\}$$

$$= c_1 f_1(p) + c_2 f_2(p)$$

By the definition of inverse Laplace transform, we get

$$L^{-1}\{c_1f_1(p) + c_2f_2(p) = c_1F_1(t)\} + c_2F_2(t)\}$$

$$= c_1L^{-1}\{f_1(p)\} + c_2L^{-1}\{f_2(p)\}$$

$$\because \quad L^{-1}\{f_1(p)\} = F_1(t),\; L^{-1}\{f_2(p)\} = F_2(t)$$

Example 1.67: Evaluate $(a)\; L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$

$(b)\; L^{-1}\left\{\frac{2p-5}{9p^2-25}\right\}$

NOTES

Solution: (*a*) Now $L^{-1}\left\{\frac{6}{2p-3}-\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\}$

$$= 6\,L^{-1}\left\{\frac{1}{2p-3}\right\}-3L^{-1}\left\{\frac{1}{9p^2-16}\right\}-4L^{-1}\left\{\frac{p}{9p^2-16}\right\}$$

$$+\,8\,L^{-1}\left\{\frac{1}{16p^2+9}\right\}-6L^{-1}\left\{\frac{p}{16p^2+9}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{p-3/2}\right\}-\frac{1}{3}L^{-1}\left\{\frac{1}{P^2-(4/3)^2}\right\}-\frac{4}{9}\left\{\frac{p}{p^2-\left(\frac{4}{3}\right)^2}\right\}$$

$$+\,\frac{1}{2}L^{-1}\left\{\frac{1}{p^2+\left(\frac{3}{4}\right)^2}\right\}-\frac{3}{8}L^{-1}\left\{\frac{p}{p^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$=3\,e^{3/2t}-\frac{1}{3}\frac{3}{4}\;\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t-\frac{1}{2}.\frac{4}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$$

$$=3\,e^{3/2t}-\frac{1}{4}\;\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{2}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$$

(*b*) Now $L^{-1}\left\{\frac{2p-5}{9p^2-25}\right\}$

$$= L^{-1}\left\{\frac{2p}{9p^2-25}\right\}-5L^{-1}\left\{\frac{1}{9p^2-25}\right\}$$

$$= \frac{2}{9}L^{-1}\left\{\frac{p}{p^2-(5/3)^2}\right\}-\frac{5}{9}L^{-1}\left\{\frac{1}{p^2-(5/3)^2}\right\}$$

$$= 2/9\cosh\;\frac{5}{3}t\;-\;\frac{5}{9}\times\frac{3}{5}\sinh\frac{5}{3}t$$

$$= \frac{2}{9}\cosh\frac{5}{3}t-\frac{1}{3}\sinh\frac{5t}{3}\,.$$

Example 1.68: Evaluate $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

Solution: Here $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

$$=L^{-1}\left\{\frac{1}{2}.\frac{1}{p+1}-\frac{1(p-1)}{2(p^2+1)}\right\}\;\text{[by resolving into partial fractions]}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{p+1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{p}{p^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{1}{2}\left[e^{-t}-\cos t+\sin t\right]$$

NOTES

Example 1.69: Evaluate $L^{-1}\left\{\dfrac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

Solution: Here $L^{-1}\left\{\dfrac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

$$= L^{-1}\left\{\frac{6p^2+22p+18}{(p+1)(p+2)(p+3)}\right\}$$

[Let $\left\{\dfrac{6p^2+22p+18}{(p+1)(p+2)(p+3)}\right\} = \dfrac{A}{p+1}+\dfrac{B}{p+2}+\dfrac{C}{p+3}$

or $6p^2+22p+18 = A(p+2)(p+3)+B(p+1)(p+3)+C(p+1)(p+2)$

Putting $p=-1$, $6-22+18 = A(-1+2)(-1+3) \Rightarrow A = 1$

Putting $p=-2$, $24-44+18 = B(-1)(1) \Rightarrow B = 2$

Putting $p=-3$, $54-66+18 = C(-2)(-1) \Rightarrow C = 3$]

$$= L^{-1}\left\{\frac{1}{p+1}+\frac{2}{p+2}+\frac{3}{p+3}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1}\right\}+2L^{-1}\left\{\frac{1}{p+2}\right\}+L^{-1}\left\{\frac{3}{p+3}\right\}$$

$$= e^{-t}+2e^{-2t}+3e^{-3t}$$

Example 1.70: Evaluate $L^{-1}\left\{\dfrac{5}{p^2}+\left(\dfrac{\sqrt{p}-1}{p}\right)^2-\dfrac{7}{3p+2}\right\}$

Solution: Here $L^{-1}\left\{\dfrac{5}{p^2}+\left(\dfrac{\sqrt{p}-1}{p}\right)^2-\dfrac{7}{3p+2}\right\}$

$$= L^{-1}\left\{\frac{5}{p^2}+\frac{p}{p^2}-\frac{2\sqrt{p}}{p^2}+\frac{1}{p^2}-\frac{7}{3}\frac{1}{p+\frac{2}{3}}\right\}$$

$$= 6L^{-1}\left\{\frac{1}{p^2}\right\}+L^{-1}\left\{\frac{1}{p}\right\}-2L^{-1}\left\{\frac{1}{p^{3/2}}\right\}-\frac{7}{3}L^{-1}\left\{\frac{1}{p+\frac{2}{3}}\right\}$$

$$= 6\frac{t}{\underline{|1}}+1-2\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}-\frac{7}{3}e^{-\frac{2}{3}t} = 6t+1-2\left(\frac{t}{\pi}\right)^{1/2}4-\frac{7}{3}e^{\frac{-2}{3}t}$$

Example 1.71: Show that $L^{-1}\left\{\dfrac{1}{P}\sin\dfrac{1}{p}\right\} = t-\dfrac{t^3}{(\underline{|3})^2}+\dfrac{t^5}{(\underline{|5})^2}-\dfrac{t^7}{(\underline{|7})^2}\ldots$

Solution: Here $L^{-1}\left\{\dfrac{1}{p}\sin\dfrac{1}{p}\right\}$

$$= L^{-1}\left\{\frac{1}{p}\left(\frac{1}{p}-\frac{(1/p)^3}{\underline{|3}}+\frac{(1/p)^5}{\underline{|5}}-\frac{(1/p)^7}{\underline{|7}}+\ldots\right)\right\}$$

NOTES

$$= L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{\underline{|3}}L^{-1}\left\{\frac{1}{p^4}\right\} + \frac{1}{\underline{|5}}L^{-1}\left\{\frac{1}{p^6}\right\} - \frac{1}{\underline{|7}}L^{-1}\left\{\frac{1}{p^8}\right\} + \ldots$$

$$= t - \frac{t^3}{(\underline{|3})^2} + \frac{t^5}{(\underline{|5})^2} - \frac{t^7}{(\underline{|7})^2} + \ldots$$

Similarly $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{P}\right\} = 1 - \frac{t^2}{(\underline{|2})^2} + \frac{t^4}{(\underline{|4})^2} - \frac{t^6}{(\underline{|6})^2} + \ldots$

Example 1.72: Show that $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$

Solution: Let $F(t) = \int_0^\infty e^{-tx^2}dx$ ...(1)

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^\infty e^{-tx^2}dx\right\}dt \text{ by (1)}$$

$$= \int_0^\infty\left[\int_0^\infty e^{-pt}e^{-tx^2}dt\right]dx$$

[changing the order of integration]

$$= \int_0^\infty L\{e^{-tx^2}\}dx = \int_0^\infty \frac{1}{p+x^2}dx$$

$$= \frac{1}{\sqrt{p}}\tan^{-1}\frac{x}{\sqrt{p}}\Big/_0^\infty = \frac{1}{\sqrt{p}}\frac{\pi}{2}$$

$$\therefore \quad F(t) = L^{-1}\left\{\frac{\pi}{2\sqrt{p}}\right\} = \frac{\pi}{2}L^{-1}\left\{\frac{1}{p^{1/2}}\right\} = \frac{\pi}{2}\frac{t^{1/2-1}}{\Gamma(1/2)} = \frac{\pi}{2}.\frac{1}{\sqrt{t\pi}}$$

or $\int_0^\infty e^{-tx^2}dx = \frac{\pi}{2}.\frac{1}{\sqrt{t\pi}}$

Now putting $t = 1$, in the above result, we get

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Example 1.73: For $a > 0$, prove that $L^{-1}\{f(p)\} = F(t)$ implies

$$L^{-1}\{f(ap+b)\} = \left(\frac{1}{a}\right)e^{-\frac{bt}{a}}F\left(\frac{t}{a}\right)$$

Solution: Here $L^{-1}\{f(p)\} = F(t)$

$$\therefore \quad f(p) = L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

$$\therefore \quad f(ap+b) = \int_0^\infty e^{-(ap+b)t}F(t)dt = \int_0^\infty e^{-apt}e^{-bt}F(t)dt$$

$$= \int_0^\infty e^{-px}e^{\frac{-bx}{a}}F\left(\frac{x}{a}\right)\frac{dx}{a} \text{ where } at = x,\ dx = a\,dt$$

$$= \int_0^\infty e^{-px}\left\{\frac{1}{a}e^{\frac{-bx}{a}}F\left(\frac{x}{a}\right)\right\}dx$$

$$\therefore \quad f(ap+b) = L\left\{\frac{1}{a}e^{\frac{-bt}{a}}F\left(\frac{t}{a}\right)\right\}$$

$$\therefore L^{-1}\{f(ap+b)\} = \frac{1}{a}e^{\frac{-bt}{a}}F(t/a)$$

NOTES

Example 1.74: Evaluate $L^{-1}\left\{\frac{p}{(p^2+a^2)(p^2+b^2)}\right\}$

Solution: Now $L^{-1}\left\{\frac{p}{(p^2+a^2)(p^2+b^2)}\right\}$

$$= L^{-1}\left\{\left(\frac{p}{p^2+a^2}-\frac{p}{p^2+b^2}\right)\frac{1}{b^2-a^2}\right\}$$

$$= \frac{1}{b^2-a^2}\left[L^{-1}\left\{\frac{p}{p^2+a^2}\right\}-L^{-1}\left\{\frac{p}{p^2+b^2}\right\}\right]$$

$$= \frac{1}{b^2-a^2}(\cos at - \cos bt)$$

Example 1.75: Evaluate $L^{-1}\left\{\frac{2p^2-4}{(p+1)(p-2)(p-3)}\right\}$

Solution: Let $\frac{2p^2-4}{(p+1)(p-2)(p-3)} = \frac{A}{p+1}+\frac{B}{p-2}+\frac{C}{p-3}$

or $2p^2-4 = A(p-2)(p-3) + B(p+1)(p-3) + C(p+1(p-2)$

Putting $p = -1$, $2-4 = A(-3)(-4) \Rightarrow -2 = 12A \Rightarrow A = -1/6$

Putting $p = 2$, $8-4 = B(3)(-1) \Rightarrow 4 = -3B \Rightarrow B = -4/3$

Putting $c = 3$, $18-4 = C.\,4.1 \Rightarrow 14 = 4C \Rightarrow C = 7/2$

$$\therefore \quad L^{-1}\left\{\frac{2p^2-4}{(p+1)(p-2)(p-3)}\right\}$$

$$= L^{-1}\left\{\frac{-1}{6}.\frac{1}{p+1}-\frac{4}{3(p-2)}+\frac{7}{2}\frac{1}{p-3}\right\}$$

$$= \frac{-1}{6}L^{-1}\left\{\frac{1}{p+1}\right\}-\frac{4}{3}L^{-1}\left\{\frac{1}{p-2}\right\}+\frac{7}{2}L^{-1}\left\{\frac{1}{p-3}\right\}$$

$$= -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$$

2. First Translation (Shifting) Theorem

Theorem 1.17. If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\}$

Proof: Since $L^{-1}\{f(p)\} = F(t)$, so $L\{F(t)\} = f(p)$. Then by the first translation property of Laplace transform,

$$L\{e^{at}F(t)\} = f(p-a)$$

$$\therefore \quad e^{at}F(t) = L^{-1}\{f(p-a)\}$$

or $L^{-1}\{f(p-a)\} = e^{at}L^{-1}\{f(p)\}$

NOTES

Example 1.76 (*a*) $L^{-1}\left\{\dfrac{p+1}{p^2+6p+25}\right\}$ (*b*) $L^{-1}\left\{\dfrac{4p+12}{p^2+8p+16}\right\}$

Solution: (*a*) Here $L^{-1}\left\{\dfrac{p+1}{p^2+6p+25}\right\} = L^{-1}\left\{\dfrac{(p+3)-2}{(p+3)^2+16}\right\}$

$$= e^{-3t}L^{-1}\left\{\frac{p-2}{p^2+16}\right\} \text{ by the first shifting theorem}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{p}{p^2+16}\right\}-2L^{-1}\left\{\frac{1}{p^2+16}\right\}\right] = e^{-3t}\left[\cos 4t - 2\frac{\sin 4t}{4}\right]$$

$$= e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right]$$

(*b*) Now $L^{-1}\left\{\dfrac{4p+12}{p^2+8p+16}\right\} = L^{-1}\left\{\dfrac{4p+12}{(p+4)^2}\right\}$

$$= L^{-1}\left\{\frac{4p+16-4}{(p+4)^2}\right\} = L^{-1}\left\{\frac{4(p+4)-4}{(p+4)^2}\right\}$$

$$= 4e^{-4t}L^{-1}\left\{\frac{p-1}{p^2}\right\}$$

by the first shifting theorem.

$$= 4e^{-4t}\left[L^{-1}\left\{\frac{1}{p}\right\}-L^{-1}\left\{\frac{1}{p^2}\right\}\right] = 4e^{-4t}\ [1-t]$$

Example 1.77: Evaluate $L^{-1}\left\{\dfrac{p}{(p+1)^{5/2}}\right\}$

Solution: Here $L^{-1}\left\{\dfrac{p}{(p+1)^{5/2}}\right\} = L^{-1}\left\{\dfrac{p+1-1}{(p+1)^{5/2}}\right\}$

$$= e^{-t}L^{-1}\left\{\frac{p-1}{p^{5/2}}\right\} \text{ by first shifting theorem}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{1}{p^{3/2}}\right\}-L^{-1}\left\{\frac{1}{p^{5/2}}\right\}\right]$$

$$= e^{-t}\left[\frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)}-\frac{t^{3/2}}{\Gamma(5/2)}\right]$$

$$= e^{-t}\left[\frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}}-\frac{t^{3/2}}{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}\right] \quad \left(\begin{array}{l}\because \Gamma(n+1) = n\Gamma(n)\\ \text{and } \Gamma(1/2) = \sqrt{\pi}\end{array}\right)$$

$$= \frac{2}{3}e^{-t}\left(\frac{t}{\pi}\right)^{1/2}(3-2t)$$

Example 1.78: Prove $L^{-1}\left\{\dfrac{p}{p^4+p^2+1}\right\} = \left(\dfrac{2}{\sqrt{3}}\right)\sinh(t/2)\sin\left(\dfrac{\sqrt{3}t}{2}\right)$

Solution : Now $L^{-1}\left\{\dfrac{p}{p^4+p^2+1}\right\} = L^{-1}\left\{\dfrac{p}{(p^2+1)^2-p^2}\right\}$

$$= L^{-1}\left\{\frac{p}{(p^2+1-p)(p^2+1+p)}\right\} = L^{-1}\left\{\frac{1}{2}\left[\frac{1}{p^2-p+1}-\frac{1}{p^2+p+1}\right]\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)L^{-1}\left\{\frac{1}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}(e^{t/2}-e^{-t/2})\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$= \sinh\left(\frac{t}{2}\right)\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{2}{\sqrt{3}}\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Example 1.79: Evaluate $L^{-1}\left\{\dfrac{1+2p}{(p+2)^2(p-1)^2}\right\}$

Solution: Here $L^{-1}\left\{\dfrac{1+2p}{(p+2)^2(p-1)^2}\right\} = L^{-1}\left\{\dfrac{1}{3}\left[\dfrac{1}{(p-1)^2}-\dfrac{1}{(p+2)^2}\right]\right\}$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(p-1)^2}\right\}-\frac{1}{3}L^{-1}\left\{\frac{1}{(p+2)^2}\right\}$$

$$= \frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{3}e^{-2t}L^{-1}\left\{\frac{1}{p^2}\right\}$$ [by the first shifting theorem]

$$= \frac{1}{3}(e^t-e^{-2t})L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{1}{3}(e^t-e^{-2t})\frac{t}{\lfloor 1}$$

$$= \frac{1}{3}(e^t-e^{-2t})t$$

Example 1.80: Evaluate $L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\}$

NOTES

Solution: Here $\frac{4p+5}{(p-1)^2(p+2)} = \frac{A}{(p-1)} + \frac{B}{(p-1)^2} + \frac{C}{(p+2)}$

or $4p + 5 = A(p-1)(p+2) + B(p+2) + C(p-1)^2$

For $p = 1, \quad 9 = B.3 \Rightarrow B = 3$

For $p = -2, -3 = 9C \Rightarrow C = -\frac{1}{3}$

For $p = 0, \quad 5 = -2A + 2B + C \Rightarrow 2A = 2.3 - \frac{1}{3} - 5 = \frac{2}{3} \Rightarrow A = \frac{1}{3}$

$$\therefore L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\} = L^{-1}\left\{\frac{1}{3}.\frac{1}{p-1} + \frac{3}{(p-1)^2} - \frac{1}{3}\frac{1}{p+2}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{p-1}\right\} + 3L^{-1}\left\{\frac{1}{(p-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$= \frac{1}{3}e^t L^{-1}\left\{\frac{1}{p}\right\} + 3e^t L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{3}e^{-2t} \text{ [by first shifting theorem]}$$

$$= \frac{1}{3}e^t + 3e^t t - \frac{1}{3}e^{-2t}$$

Example 1.81: Evaluate $L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\}$

Solution: Here $L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} = L^{-1}\left\{\frac{p+\frac{1}{2}+\frac{1}{2}}{(p+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$

$$= e^{-\frac{1}{2}t} L^{-1}\left\{\frac{p+\frac{1}{2}}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} \quad \text{[by the first shifting theorem]}$$

$$= e^{-\frac{t}{2}}\left[L^{-1}\left\{\frac{p}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= e^{-\frac{t}{2}}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$= e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

NOTES

Example 1.82: Evaluate $L^{-1}\left\{\frac{p}{(p+1)^5}\right\}$

Solution: Here $L^{-1}\left\{\frac{p}{(p+1)^5}\right\} = L^{-1}\left\{\frac{p+1-1}{(p+1)^5}\right\} = e^{-t}\,L^{-1}\left\{\frac{p-1}{p^5}\right\}$

[by the first shifting theorem]

$$= e^{-t}\,L^{-1}\left\{\frac{1}{p^4}-\frac{1}{p^5}\right\} = e^{-t}\left[L^{-1}\left\{\frac{1}{p^4}\right\}-L^{-1}\left\{\frac{1}{p^5}\right\}\right]$$

$$= e^{-t}\left[\frac{t^3}{\lfloor 3}-\frac{t^4}{\lfloor 4}\right] = e^{-t}\left(\frac{t^3}{6}-\frac{t^4}{24}\right)$$

3. Second Translation (Shifting) Theorem

Theorem 1.18: If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{e^{-ap} f(p)\} = G(t) = \begin{cases} F(t-a), & t > a \\ 0 & , t < a \end{cases}$

Proof: Since $L^{-1}\{f(p)\} = F(t)$, so $L\{F(t)\} = f(p)$.

$\therefore L\{G(t)\} \quad = e^{-ap} f(p)$ [By 2nd shifting theorem of Laplace transform]

$$\therefore\ L^{-1}\{e^{-ap}\,f(p)\} = G(t) = \begin{cases} F(t-a), & t > a \\ 0 & , t < a \end{cases} = F(t-a)\,H(t-a)$$

Another form: Let $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{e^{-ap}\,f(p)\} = F(t-a)\,H(t-a)$$

where $H(t-a)$ is Heaviside's unit function.

Example 1.83: Find (*a*) $L^{-1}\left\{\frac{pe^{-2p\frac{\pi}{3}}}{p^2+9}\right\}$; (*b*) $L^{-1}\left\{\frac{6e^{-3p}}{p^2+9}\right\}$

Solution: (*a*) Here $L^{-1}\left\{\frac{p}{p^2+9}\right\} = \cos 3t$

Then by the second shifting theorem,

$$L^{-1}\left\{\frac{p}{p^2+9}e^{-2p\frac{\pi}{3}}\right\} = \begin{cases} \cos 3\left(t-\frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0 & , t < \frac{2\pi}{3} \end{cases}$$

$$= \cos 3\left(t-\frac{2\pi}{3}\right)H\left(t-\frac{2\pi}{3}\right)$$

(*b*) We have, $L^{-1}\left\{\frac{6}{p^2+9}\right\} = \frac{6}{3}\sin 3t = 2\sin 3t = F(t)$ (say)

Then by the 2nd shifting theorem

$$L^{-1}\left\{\frac{6}{p^2+9}e^{-3p}\right\} = \begin{cases} F(t-3) \text{ for } t > 3 \\ 0 \quad\quad \text{ for } t < 3 \end{cases}$$

$$= \begin{cases} 2\sin 3(t-3) & \text{for } t > 3 \\ 0 & \text{for } t < 3 \end{cases}$$

NOTES

Example 1.84: Find $L^{-1}\left\{\frac{e^{-5p}}{(p-2)^4}\right\}$

Solution: Here $L^{-1}\left\{\frac{1}{(p-2)^4}\right\} = e^{2t} L^{-1}\left\{\frac{1}{p^4}\right\} = e^{2t}\frac{t^3}{\lfloor 3}$

Hence by the second shifting theorem

$$L^{-1}\left\{e^{-5p}\frac{1}{(p-2)^2}\right\} = e^{2(t-5)}\frac{(t-5)^3}{\lfloor 3} H(t-5).$$

Example 1.85: Find $L^{-1}\left\{\frac{e^{-p\pi}}{p^2+1}\right\}$

Solution: Now $L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t$

Hence by the second shifting theorem,

$$L^{-1}\left\{e^{-p\pi}\frac{1}{p^2+1}\right\} = \sin(t-\pi)\, H(t-\pi)$$

$$= -\sin t\, H(t-\pi)$$

Example 1.86: Find $L^{-1}\left\{\frac{3(1+e^{-p\pi})}{p^2+9}\right\}$

Solution: Now $L^{-1}\left\{\frac{3}{p^2+9}\right\} = \sin 3t$

Hence by the second shifting theorem,

$$L^{-1}\left\{\frac{3(1+e^{-p\pi})}{p^2+9}\right\} = L^{-1}\left\{\frac{3}{p^2+9}\right\} + L^{-1}\left\{\frac{3}{p^2+9}e^{-p\pi}\right\}$$

$$= \sin 3t + \sin 3(t-\pi)\, H(t-\pi) = \sin 3t - 3\sin 3t\, H(t-\pi)$$

Theorem 1.19: Change of Scale Property: If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$, where a is constant.

Proof: Since $L^{-1}\{f(p)\} = F(t)$, so $L\{F(t)\} = f(p)$

We have $L\{F(at)\} = \frac{1}{a}f(\frac{p}{a})$ ...(1.11)

Now replacing a by $\frac{1}{a}$ in equation (1.11), we get

$$L\left\{F\left(\frac{t}{a}\right)\right\} = a\, f(pa) \text{ or } L\left\{\frac{1}{a}F\left(\frac{t}{a}\right)\right\} = f(pa)$$

or $$L^{-1}\{f(pa)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

[by the definition of inverse Laplace transform]

Example 1.87: If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$, then find $L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$

Solution: By change of Scale property, we get

$$L^{-1}\left\{\frac{4p}{(4^2p^2+1)^2}\right\} = \frac{1}{4}.\left(\frac{1}{2}.\frac{t}{4}\sin\frac{t}{4}\right)$$

$$\text{or } \frac{1}{8}L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{1}{8}.\frac{t}{4}\sin\frac{t}{4}$$

$$\text{or } L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}$$

Example 1.88: If $L^{-1}\left\{\frac{e^{-\frac{1}{p}}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, then prove that

$$L^{-1}\left\{\frac{e^{-\left(\frac{a}{p}\right)}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Solution: By change of scale property, we get

$$L^{-1}\left\{\frac{e^{-\frac{a}{p}}}{\left(\frac{p}{a}\right)^{1/2}}\right\} = a.\,F(ta) = a.\,\frac{\cos 2\sqrt{ta}}{\sqrt{\pi ta}}$$

$$\text{or} \quad L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\} = \frac{a}{\sqrt{a}}.\frac{1}{\sqrt{a}}\,\frac{\cos(2\sqrt{ta})}{\sqrt{\pi t}} = \frac{\cos(2\sqrt{ta})}{\sqrt{\pi t}}$$

Inverse Laplace Transforms of Derivatives

Theorem 1.20: If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f^n(p)\} = L^{-1}\left\{\frac{d^n f(p)}{dp^n}\right\} = (-1)^n$ $t^n F(t)$, $n = 1, 2, 3 \ldots$

Proof: Since $L^{-1}\{f(p)\} = F(t)$, so $L\{F(t)\} = f(p)$

We have $L\{t^n F(t)\} = (-1)^n \frac{d^n f(p)}{dp^n} = (-1)^n f^n(p)$

$\therefore L^{-1}\{(-1)^n f^n(p)\} = t^n F(t)$

or $L^{-1}\{f^n(p)\} = (-1)^n t^n F(t)$

Example 1.89: Evaluate $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

Solution: Now $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right)\right\}$

NOTES

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right)\right\}$$

$$= \left(-\frac{1}{2}\right)(-1)t\,L^{-1}\left\{\frac{1}{p^2+a^2}\right\}$$

$$= \frac{1}{2}t\frac{\sin at}{a} = \frac{t}{2a}\sin at$$

Example 1.90: Evaluate $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$

Solution: Now $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\} = L^{-1}\left\{\frac{p+1}{[(p+1)^2+1]}\right\}$

$$= e^{-t}L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} \quad \text{[by the first shifting theorem]}$$

$$= e^{-t}L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right\}$$

$$= \left(-\frac{1}{2}\right)e^{-t}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right\}$$

$$= \left(-\frac{1}{2}\right)e^{-t}(-1)tL^{-1}\left\{\frac{1}{p^2+1}\right\} = \frac{1}{2}e^{-t}t\sin t$$

Example 1.91: Evaluate $L^{-1}\left\{\log\frac{p+3}{p+2}\right\}$

Solution: Let $f(p) = \log\frac{p+3}{p+2} = \log(p+3) - \log(p+2)$

$$\therefore \quad f'(p) = \frac{1}{p+3} - \frac{1}{p+2}$$

$$\therefore \quad L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+3}\right\} - L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-3t} - e^{-2t}$$

or $(-1)\,t\,L^{-1}\{f(p)\} = e^{-3t} - e^{-2t}$

or $L^{-1}\{f(p)\} = \frac{e^{-2t}-e^{-3t}}{t}$

or $L^{-1}\left\{\log\frac{p+3}{p+2}\right\} = \frac{e^{-2t}-e^{-3t}}{t}$

Example 1.92: Evaluate $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$

Solution: Let $f(p) = \log\left(\frac{p^2+1}{p^2}\right) = \log(p^2+1) - 2\log p$

$$\therefore \quad f'(p) = \frac{2p}{p^2+1} - \frac{2}{p}$$

$$\therefore \ L^{-1}\{f'(p)\} = 2\,L^{-1}\left\{\frac{p}{p^2+1}\right\} - 2\,L^{-1}\left\{\frac{1}{p}\right\} = 2\cos t - 2$$

NOTES

$$\text{or}(-1)\; t\,L^{-1}\,\{f(p)\} \quad = 2\cos t - 2 \quad \text{or} \quad L^{-1}\{f(p)\} = \frac{2-2\cos t}{t}$$

$$\text{or} \quad L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2(1-\cos t)}{t}$$

1.11 MULTIPLICATION AND DIVISION BY POWERS OF '*s*'

Theorem 1.21: Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order σ as $t \to \infty$ and $L\,\{F\,(t)\,\} = f\,(p)$ exist, then

$$L\{t\,F(t)\} = -\frac{df(p)}{dp} = -f'(p)$$

Proof: From the definition, $f(p) = \int_0^\infty e^{-pt} F(t)dt$ ....(1.12)

Since $F(t)$ is continuous for all $t \geq 0$ and is of exponential order σ as $t \to \infty$, then Leibnitz's rule for differentiation under the sign of integral is justified and so from equation (1.12), we get

$$\frac{d}{dp}\,f(\,p) = \int_0^\infty \frac{\partial}{\partial p}(e^{-pt}F(t))dt \;=\int_0^\infty (-t)e^{-pt}F(t)dt$$

$$= -\int_0^\infty e^{-pt}\{tF(t)\}\,dt \;= -L\{t\,F(t)\}$$

$$\text{or } L\{t\;F(t)\} = -\frac{d}{dp}f(p) = -f'(p)$$

General case: Theorem 1.22: Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order σ as $t \to \infty$ and $L\,\{F\,(t)\} = f(\,p)$ exists, then

$$L\{t^n\;F(t)\} = (-1)^n\,\frac{d^n f(p)}{dp^n} = (-1)^n\,f^n\,(p)$$

Proof: We shall use the principle of mathematical induction to prove it.

For $n = 1$, the theorem reduces to the form:

$$L\{t\,F(t)\} = -\frac{d\,f(p)}{dp} = -f'\,(p)$$

which is true by the above theorem.

Let us assume that the result is true for $n = k$, then

$$L\{t^k\;F(t)\} = (-1)^k\,\frac{d^k\,f(\,p)}{dp^k}$$

$$\text{or} \quad \int_0^\infty e^{-pt}t^kF(t)dt \;=\; (-1)^k\,\frac{d^k\,f(p)}{dp^k}$$

NOTES

Differentiating both sides w.r.t. p and applying the Leibnitz's rule for differentiating under the integral sign, we get

$$\int_0^\infty \frac{\partial}{\partial p}\{e^{-pt}t^k F(t)\}dt = (-1)^k \frac{d^{k+1} f(p)}{dp^{k+1}}$$

or $$\int_0^\infty (-t)e^{-pt}t^k F(t)dt = (-1)^k \frac{d^{k+1} f(p)}{dp^{k+1}}$$

or $$\int_0^\infty e^{-pt}\{t^{k+1}F(t)\}dt = (-1)^{k+1} \frac{d^{k+1} f(p)}{dp^{k+1}}$$

or $$L\{t^{k+1} F(t)\} = (-1)^{k+1} \frac{d^{k+1} f(p)}{dp^{k+1}}$$

Hence by the principle of mathematical induction, the result (1) is true for all positive integers.

Example 1.93: Find $L\{t^2 \sin at\}$

Solution: We know that $L\{\sin at\} = \frac{a}{p^2+a^2} = f(p)$ (say)

$$\therefore \quad L\{t^2 \sin at\} = (-1)^2 \frac{d^2 f(p)}{dp^2} = \frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right)$$

$$= a\frac{d}{dp}\left[\frac{-2p}{(p^2+a^2)^2}\right]$$

$$= (-2a)\frac{(p^2+a^2)^2 - p.2p.2(p^2+a^2)}{(p^2+a^2)^4}$$

$$= (-2a)\frac{p^2+a^2-4p^2}{(p^2+a^2)^3} = \frac{2a(3p^2-a^2)}{(p^2+a^2)^3}$$

Example 1.94: Find $L\{t\,(3 \sin 2t - 2 \cos 2t)\}$

Solution: Now $L\{(3\sin 2t - 2\cos 2t)\} = 3\,L\{\sin 2t\} - 2\,L\{\cos 2t\}$

$$= \frac{3.2}{p^2+4} - \frac{2.p}{p^2+4} = \frac{2(3-p)}{p^2+4} = f(p), \text{ say}$$

$$\therefore L\{t(3\sin 2t - 2\cos 2t)\} = (-1)\frac{d\,f(p)}{dp}$$

$$= (-1)\frac{d}{dp}\left[\frac{2(3-p)}{p^2+4}\right]$$

$$= (-2)\frac{(p^2+4)(-1)-(3-p)2p}{(p^2+4)^2}$$

$$= -2.\frac{-p^2-4-6p+2p^2}{(p^2+4)^2} = \frac{8+12p-2p^2}{(p^2+4)^2}$$

Example 1.95: Find $L\{t^n e^{at}\}$

NOTES

Solution: Now $L\{e^{at}\} = \dfrac{1}{p-a} = f(p)$, say

$$\therefore\ L\{t^n e^{at}\} = (-1)^n \frac{d^n f(p)}{dp^n} = (-1)^n \frac{d^n}{dp^n}\left(\frac{1}{p-a}\right)$$

$$= (-1)^n \frac{(-1)^n \lfloor n}{(p-a)^{n+1}} = \frac{\lfloor n}{(p-a)^{n+1}}$$

Example 1.96: Find $L\{t^3 \cos t\}$

Solution: We know $L\{\cos t\} = \dfrac{p}{p^2+1} = f(p)$, say

$$\therefore\ L\{t^3 \cos t\} = (-1)^3 \frac{d^3 f(p)}{dp^3} = -\frac{d^3}{dp^3}\left(\frac{p}{p^2+1}\right)$$

$$= (-1)\frac{d^2}{dp^2}\left[\frac{p^2+1-p.2p}{(p^2+1)^2}\right] = -\frac{d^2}{dp^2}\left[\frac{-p^2+1}{(p^2+1)^2}\right]$$

$$= \frac{d}{dp}\left[\frac{(p^2+1)^2 2p-(p^2-1)2(p^2+1)2p}{(p^2+1)^4}\right]$$

$$= \frac{d}{dp}\left[\frac{2p^3+2p-4p^3+4p}{(p^2+1)^3}\right] = -\frac{d}{dp}\left[\frac{2p^3-6p}{(p+1)^3}\right]$$

$$= -\left[\frac{(p+1)^3(6p^2-6)-(2p^3-6p)\,3(p^2+1)^2.2p}{(p+1)^6}\right]$$

$$= \frac{6p^4-36p^2+6}{(p^2+1)^4}$$

Example 1.97: Find $L\{(t^2-3t+2)\sin 3t\}$

Solution: We know $L\{\sin 3t\} = \dfrac{3}{p^2+9} = f(p)$, say

$$\therefore\ L\{(t^2-3t+2)\sin 3t\} = (-1)^2 \frac{d^2 f(p)}{dp^2} - 3(-1)\frac{d\,f(p)}{dp} + 2.f(p)$$

$$= \frac{d^2}{dp^2}\left(\frac{3}{p^2+9}\right) + 3\frac{d}{dp}\left(\frac{3}{p^2+9}\right) + 2.\frac{3}{p^2+9}$$

$$= \frac{d}{dp}\left[\frac{-6p}{(p^2+9)^2}\right] - \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}$$

$$= -6\frac{(p^2+9)^2-2.p(p^2+9)2p}{(p^2+9)^4} - \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}$$

$$= -6\frac{p^2+9-4p^2}{(p^2+9)^3} - \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}$$

$$= \frac{18p^2-54}{(p^2+9)^3} - \frac{18p}{(p^2+9)^2} + \frac{6}{p^2+9}$$

NOTES

1.11.1 Effect of Division by 't' on Laplace Transform

Theorem 1.23: Let $L\{F(t)\} = f(p)$, then

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)dp, \text{ provided the integral exists}$$

Proof: From the definition of Laplace transform, we get

$$f(p) = \int_0^\infty e^{-pt} F(t)dt, \qquad ...(1.13)$$

Integrating equation (1.13) w.r.t. 'p' from $p = p$ to $p = \infty$, we get

$$\int_p^\infty f(p)\, dp = \int_p^\infty \left\{\int_0^\infty e^{-pt} F(t)dt\right\} dp$$

Since p and t are independent variables, the order of integration in the double integral of R.H.S. can be interchanged.

$$\therefore \quad \int_0^\infty f(p)dp = \int_0^\infty \left\{\int_p^\infty e^{-pt} dp\right\} F(t)dt$$

$$= \int_0^\infty \left[\frac{e^{-pt}}{-t}\right]_p^\infty F(t)dt$$

$$= \int_0^\infty e^{-pt} \left\{\frac{F(t)}{t}\right\} dt = L\left\{\frac{F(t)}{t}\right\}$$

Example 1.98: Find the Laplace transform of $\frac{\sin at}{t}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Solution: We know $L\{\sin at\} = \frac{a}{p^2 + a^2} = f(p)$, say

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_p^\infty \frac{a}{p^2 + a^2} dp = \frac{a}{a}\tan^{-1}\frac{p}{a}\Big/_p^\infty = \tan^{-1}\infty - \tan^{-1}\frac{p}{a}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{p}{a} = \cot^{-1}\frac{p}{a}$$

$$\text{Now } L\{\cos at\} = \frac{p}{p^2 + a^2} = f(p), \text{ say}$$

$$\therefore L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{p}{p^2 + a^2} dp = \frac{1}{2}\left[\log(p^2 + a^2)\right]_p^\infty$$

$$= \frac{1}{2}\left[\lim_{p\to\infty} \log(p^2 + a^2) - \log(p^2 + a^2)\right]$$

which does not exist since $\log(p^2 + a^2) \to \infty$ as $p \to \infty$.

Example 1.99: Prove that $L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\{(p^2 + 4)/p^2\}$

Solution: We know $L\{\sin^2 t\} = \frac{1}{2} L\{2\sin^2 t\} = \frac{1}{2} L\{1 - \cos 2t\}$

NOTES

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] = \frac{2}{p(p^2+4)}$$

$$\therefore L\left\{\frac{\sin^2 t}{t}\right\} = \int_p^\infty \frac{2}{p(p^2+4)}dp = \int_p^\infty \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]dp$$

$$= \frac{1}{2}\log p - \frac{1}{4}\log(p^2+4) \Big/_p^\infty = \frac{1}{4}\log\frac{p^2}{p^2+4} \Big/_p^\infty$$

$$= \frac{1}{4}\left[\log\frac{1}{1+\frac{4}{\infty}} - \log\frac{p^2}{p^2+4}\right] = \frac{1}{4}\log\frac{p^2+4}{p^2}$$

Example 1.100: Prove that $L\left\{\frac{\sinh t}{t}\right\} = \frac{1}{2}\log\frac{p+1}{p-1}$

Solution: Now $L\ \{\sinh t\} = \frac{1}{p^2-1} = f(p)$, say

$$\therefore L\left\{\frac{\sinh t}{t}\right\} = \int_p^\infty f(p)dp = \int_p^\infty \frac{1}{p^2-1}dp$$

$$= \frac{1}{2}\log\frac{p-1}{p+1} \Big/_p^\infty = \frac{1}{2}\left[\log\left(\frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}\right) - \log\left(\frac{p-1}{p+1}\right)\right]$$

$$= \frac{1}{2}\left[0 + \log\frac{p+1}{p-1}\right] = \frac{1}{2}\log\frac{p+1}{p-1}$$

1.12 CONVOLUTION PROPERTY

Definition 1. Let f and g be two functions defined in $[0, \infty)$. Then the convolution of f and g, denoted by $f * g$, is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \qquad \text{... (a)}$$

Note: it can be shown (easily), that $f * g = g * f$. Hence,

$$(f*g)(t) = \int_0^t g(\tau)f(t-\tau)d\tau \qquad \text{... (b)}$$

We use either (a) or (b) depending on which is easier to evaluate.

Theorem 1.24. (Convolution theorem) The convolution $f * g$ has the Laplace transform property

$$L((f*g)(t)) = \mathrm{F}(s)G(s).$$

Or conversely

$$L^{-1}(\mathrm{F}(s)G(s)) = (f*g)(t)$$

NOTES

Proof: Using definition, we find

$$L\big((f*g)(t)\big)=\int_0^{\infty}(f*g)(t)e^{-st}dt$$

$$=\int_0^{\infty}\left(\int_0^{1}f(\tau)g(t-\tau)d\tau\right)e^{-st}dt$$

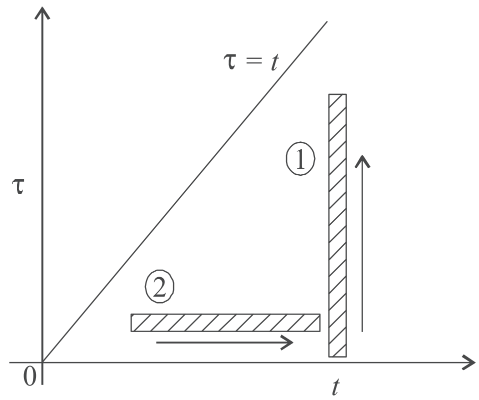

***Fig. 1.2** Effects of unit step function on a function f (t). Here b > a*

The line $\tau = t$. The variable limit of integration is applied on τ which varies from $\tau = 0$ to $\tau = t$.

Let us change the order of integration, thus apply variable limit on t. Then t would vary from $t = \tau$ to $t = \infty$ and τ would vary from $\tau = 0$ to $\tau = \infty$. Hence, we have,

$$L\big((f*g)(t)\big) = \int_0^{\infty}\left(\int_{\tau}^{\infty}e^{-st}g(t-\tau)dt\right)f(\tau)d\tau$$

$$= \int_0^{\infty}\left(\int_0^{\infty}e^{-su}g(u)du\right)f(\tau)e^{-s\tau}d\tau, \quad t-\tau=u$$

$$= \left(\int_0^{\infty}e^{-su}g(u)du\right)\left(\int_0^{\infty}e^{-s\tau}f(\tau)d\tau\right)$$

$$= F(s)G(s)$$

Example 1.101: Consider the same problem as given in Example 1.100 of Lecture. Note, i.e., find inverse Laplace transform of $1/s\,(s+1)^2$.

Solution: We write $H(s) = F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s+1)^2$. Thus $f(t) = 1$ and $g(t) = te^{-t}$. Hence, using convolution theorem, we find

$$h(t) = \int_0^{t}f(t-\tau)g(\tau)d\tau=\int_0^{t}\tau e^{-\tau}d\tau=1-(t+1)e^{-t}$$

Example 1.102: Find the Laplace transform of $1/\,(s^2+\omega^2)^2$.

Solution: Let $H(s) = 1(s)\,G(s)$, where $F(s) = 1/(s^2+\omega^2)$ and $G(s) = 1/(cs^2+\omega^2)$

Thus, $f(t) = \sin(\omega t)/\omega = g(t)$. Hence,

$$h(t) = \frac{1}{w^2}\int_0^{x}\sin(\omega\tau)\sin\big(\omega t-\tau\big)\,d\tau$$

$$= \frac{1}{2\omega^3}\big(\sin(\omega t)-\omega t\cos(\omega t)\big)$$

NOTES

1.13 COMPLEX INVERSION FORMULA AND HEAVISIDE EXPANSION FORMULA

Complex inversion is an useful too for computing the inverse of Laplace transform $f(t) = L^{-1}(F(s))$.

This tool is based on the methods of contour integrations and for this technique it is required that transform parameter s to be a complex variable.

Let us consider a continuous function $f(t)$ possessing Laplace transform. Extend f to $(-\infty, \infty)$ by defining $f(t) = 0$ for $\varepsilon < 0$

Then for $s = a + ib$

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_{-0}^{\infty} e^{-(a+ib)t} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-ibt} (e^{-at} f(t))\,dt \qquad \text{... (1.14)}$$

$$L(f(t)) = F(a, b)$$

In this form $F(a, b)$ represents the fourier transform of the function $e^{-at} f(t)$.

i.e., $$L(f(t)) = f\,(e^{-at} f(t))$$

This may be taken as then relation between Laplace and Fourier transforms. Suppose $f(t)$ is continuous on $[0, \infty]$ and $f(t) = 0$ for $t = 0$ and f is of exponential order d. Also f' (t) is continuous on $[0, \infty]$. Converges absolutely for

$$\text{Real}\,(s) = a > \alpha, \text{ i.e.,}$$

$$\int_0^\infty \left|e^{-st} f(t)\right| dt = \int_{-\infty}^{\infty} e^{-at} \left|f(t)\right| dt < \infty \text{ for } a > a > \alpha.$$

$\Rightarrow$ $$g(t) = e^{-at} f(tt) \text{ is absolutely integrable.}$$

And we may write $g(t)$ (in view of Fourier inversion theorem).

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ibt} F(a, b)\, d\, b \quad t > 0$$

This reads to the representation of $f(t)$ as f(t)

$$= \frac{1}{2n} \int_{-\infty}^{\infty} e^{at} e^{ibt} F\,(a, b)\, db, t > 0 \qquad (1.15)$$

Taking $s = a + ib$ *in* Equation (1.1) since $a > \alpha$

$$dy = \frac{1}{i} ds \text{ and so } f \text{ in given by}$$

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s)\, ds$$

NOTES

or $$= \frac{1}{2\pi i} \lim_{b\to\infty} \int_{a-ib}^{a+ib} e^{tS} \ f(s)\, ds$$

or $$f(t) = \frac{1}{2\pi i} \lim_{b\to\infty} \frac{1}{2\pi i} \int_{a-ib}^{a+ib} e^{fs} f(s)\, ds \qquad ...(1.16)$$

This is called complex inversion formula (or Fourier-Mellin form).

Where the integration is to be performed along a vertical sine at $a > a$, and the vertical line is called Bromwich line.

Heaviside Expansion Formula

Consider $F(s)$ rational function such that $F(s) = \frac{P(s)}{Q(s)}$ where degree of Q(s) is greater than the degree of $P(s)$ (both P(s) and Q(s) are polynomials).

Let $\alpha_1, \alpha_2 .....\alpha_n$ are roots of Q(s) = 0

then $(s) = (s - \alpha_1)(\sigma - \alpha_2) ..... (s - \alpha_n)$

Thus $$F(s) = \frac{P(s)}{(s-\alpha_1)(s-\alpha_2).......(s-\alpha_n)}$$

using partial function we get

$$F(s) = \frac{c_1}{(s-\alpha_1)} + \frac{c_2}{(s-\alpha_2)} + ... + \frac{c_n}{(s-\alpha_n)} \qquad ...(1.14)$$

Any coefficient c_k in (1.14) can be evaluated by multiplying (1.14) by $(s \to \alpha_k)$ and taking the limit s α_a. All terms in the partial partial fraction vanish except $(s - c_k)$.

Thus we get

$$c_k = \lim_{s\to c_k}\left[s - \alpha_k \frac{P(s)}{Q(s)} \right]$$

$$= P(\alpha_k)\left[\frac{(s-\alpha_k)}{Q(s)} \right]_{s=\alpha_k} \qquad (\div \text{ form})$$

So by L. Hospital's Rule we get

$$c_k = P(\alpha_k)\left[\frac{\frac{d}{ds}(s-\alpha_k)}{\frac{d}{ds}Q(s)} \right] \frac{P(\alpha_k)}{\alpha^1(\alpha_k)}$$

and (1.17) gives

NOTES

$$F(s) = \frac{P(s)}{Q(s)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q^2(\alpha_k)} \cdot \frac{1}{(s-\alpha_k)}$$

$$\Rightarrow \qquad f(t) = L^{-1}(F(s)) = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q^1(\alpha_k)} L^{-1}\left(\frac{1}{s-\alpha_k}\right)$$

$$f(t) = \sum_{k=1}^{n} \frac{p(d_k)}{Q'(d_k)} \cdot e^{d_k t}$$

$$= \sum_{k=1}^{n} c_k e^{d_k t}$$

This is called Heaviside expansion formula.

Now we consider some applications of this Heaviside expansion formula.

Example 1.103: Find the inverse Laplace transform of $\frac{s^2+2s-3}{s(s-3)(s+2)}$

Solution: Let $F(s)\ \frac{s^2+2s-3}{s(s-3)(s+2)} = \frac{P(s)}{Q(s)}$ (say)

Here $\qquad Q(s) = s(s-3)(s+2) = s^2 - s^2 - 6s$

roots of Q(s) : $\alpha_1 = 1\alpha_2 = 3,\ \alpha_3 = -2$

$$Q^1(s) = 3s^2 - 2s - 6$$

$$F(s) = \frac{c_1}{s} + \frac{c_2}{s-3} + \frac{c_3}{s+2}$$

$$c_1 = \frac{P(o)}{Q'(o)} = \frac{1}{2}, \qquad c_2 = \frac{P(3)}{Q'(3)} = 4/5$$

$$c_3 = \frac{P(-2)}{Q'(-2)} = \frac{-3}{10}$$

thus

$$f(t) = L^{-1}(F(s)) = \sum_{i=1}^{3} c_i e^{\alpha_i t}$$

$$= \frac{1}{2}e^{ot} + \frac{4}{5}e^{3t} - \frac{3}{10}e^{-2t}$$

$$f(t) = \frac{1}{2} + \frac{4}{5}e^{3t} - \frac{3}{10}e^{-2t}$$

NOTES

Example 1.104: Find the inverse Laplace transform of $\frac{1}{s(s+2)^3}$

Solution. Heve $F(s) = \frac{1}{s(s+2)^3}$ (using partial fractions)

or $F(s) = \frac{c_1}{(s+2)^3} + \frac{c_2}{(s+2)^2} + \frac{c_3}{s+2} + \frac{c_4}{s}$

$$c_4 = \frac{P(0)}{Q'(0)} = \left[\frac{P(s)}{\frac{d}{ds}Q(s)}\right]_{s=0} = \left[\frac{1}{\left\{\frac{d}{ds}s(s+s)^3\right\}}\right]_{s=0} = \frac{1}{8}$$

$$c_1 = \left[\frac{1}{s}\right]_{s=2} = \frac{1}{2}, \quad c_2 = \left[\frac{d}{ds}\left(\frac{1}{s}\right)\right]_{s=2} = \frac{-1}{4}$$

$$c_3 = \left[\frac{d^2}{ds^2}\left(\frac{1}{s}\right)\right]_{s=2} = \frac{1}{4}$$

Thus $F(s) = \frac{1}{2(s+2)^3} - \frac{1}{4(s+2)^2} + \frac{1}{4(s+2)} + \frac{1}{8s}$

$$\Rightarrow \quad L^{-1}(F(s)) = f(t) = \frac{e^{2t}}{4}\left(t^2 - t + 1\right) + \frac{1}{8}$$

1.14 EVALUATION OF INTEGRALS

Wo know that $F(s) = L\left(F(t)\right) = \int_0^\infty e^{-st} f(t)dt$ ...(1.18)

Assuming the integral in R.H.S. of (1.18) is convergent and taking $s \to 0$ we get.

$$\int_0^\infty f(t)dt = \text{F}(0) \qquad ...(1.17)$$

The relations (1.18) and (1.19) can be used to evaluate certain definite integrals.

Example 1.105: Evaluate the following integrals using Laplace transform.

(a) $\int_0^\infty t^2 e^{-t} \text{sint}\ dt$ (b) $\int_0^\infty \cos x^2 dx$

(c) $\int_0^\infty J_0(t)dx$ (d) $\int_0^\infty \frac{e^{-t}\text{sint}}{t}\ dt$

Solution: (a) We know that

NOTES

$$L(\text{sint}) = \frac{1}{s^2+1}$$

$$\Rightarrow \qquad L(t^2 \sin t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{(-1)(2s)}{(s^2+1)^2}\right)$$

$$= (-1)\frac{(-2)(2s)(2s)}{(s^2+1)^3} - \frac{2}{(s^2+1)^2}$$

$$= \frac{8s^2+2(s^2+1)}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3}$$

$$= \frac{-2(1-3s^2)}{(1+s^2)^3}$$

or $$\int_0^\infty e^{st}.t^2\text{sint}\,dt = \frac{-2(1-3s^2)}{(1+s^2)^3} \qquad ...(1)$$

taking $s = 1$ in equation (1) we get

$$\int_0^\infty e^{-1}\,t^2\text{sint}\,dt = \frac{-2(1-3)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2}$$

(b) $$I = \int_0^\infty \cos x^2 dx$$

Let $$F(t) = \int_0^\infty \cos tn^2$$

there $$L(f(t)) = \int_0^\infty e^{-st}\left(\int_0^\infty \cos tx^2 dt\right)dt$$

$$= \int_0^\infty \left\{\int_0^\infty e^{-st} \cos tx^2 dt\right\} dx$$

$$= \int_0^\infty L\left(\text{cost } x^2\right)dx$$

$$L(f(t)) = \int_0^\infty \frac{s}{s^2+x^4}\,dx \;...(1)$$

NOTES

put $x^2 = s \tan\theta \quad \Rightarrow \quad 2x\,dx = s\,sec^2\theta\,d\theta$

or $$dx = \frac{s\sec^2\theta}{\alpha\sqrt{s\tan\theta}}d\theta$$

$$n = 0 \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x^2}{s}\right) = 1$$

$$x = \infty \quad \Rightarrow \quad \theta = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Then (Equation 1.16) becomes

$$L(f(t)) = \int_0^{\pi/2} \frac{s}{s^2 + s^2\tan^2\theta}\frac{s\sec^2\theta}{\sqrt[2]{s\tan\theta}}d\theta$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\alpha/2} \sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$$= \frac{1}{2\sqrt{s}}\frac{\sqrt{\dfrac{-1/2+1}{2}}\sqrt{\dfrac{1/2+1}{2}}}{2\sqrt{\dfrac{(-1/2+1/2+2)}{2}}}$$

$$= \frac{1}{2\sqrt{s}}\frac{\sqrt{1/4}\sqrt{3/4}}{2\sqrt{1}} = \frac{1}{2\sqrt{s}}\frac{\sqrt{1/4}\sqrt{1-1/4}}{2}$$

$$= \frac{1}{4\sqrt{s}}\frac{\pi}{\sin\dfrac{\pi}{4}} = \frac{\pi}{4\sqrt{s}}\cdot\frac{1}{1/\sqrt{2}}$$

$$L(f(t)) = \frac{\pi}{2\sqrt{2s}}$$

$$\Rightarrow \quad f(t) = \int_0^{\infty}\cos t\,x^2 dx = L^{-1}\left(\frac{\pi}{2\sqrt{2s}}\right)$$

or $$f(t) = \frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t\pi}}$$

$$f(t) = \frac{1}{2}\cdot\sqrt{\frac{\pi}{2t}}$$

NOTES

or $\int_0^\infty \cos tx^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$

taking $t \to 1$ we get

$$I = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

(c) $$I = \int_0^\infty J_0(t)\, dt$$

$J_0(t)$ is the Bessel function of order 0.

and we know that $L\left(J_0(t)\right) = \frac{1}{\sqrt{s^2+1}}$

or $\int_0^\infty e^{-st} J_0(t)\, dt = \frac{1}{\sqrt{s^2+1}}$

taking limit $s \to 0$ we get

$$\int_0^\infty J_0(t)\, dt = 1$$

(d) Let $I = \int_0^\infty \frac{e^{-t}\sin t}{t}\, dt$

We know that $L\left(\frac{\sin at}{t}\right) = \cot^{-1}\left(\frac{s}{a}\right)$

or $\int_0^\infty e^{-st} \frac{\sin at}{t} dt = \cot^{-1}\left(\frac{s}{a}\right)$

taking $a = 1, s = 1$ we get

$$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \cot^{-1} \frac{\pi}{4}$$

Check Your Progress

9. How will you define the function of exponential order?
10. Define the inverse Laplace transforms.
11. What is the linearly property of inverse Laplace transform?
12. State the first shifting theorem.
13. Define second shifting property.
14. What do you understand by the change of scale property?
15. Define the term convolution property.
16. Give the Heaviside expansion formula.

1.15 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The Laplace transform is a widely used integral transform and is denoted by $\mathcal{L}\{f(t)\}$. It is a linear operator of a function $f(t)$ including a real argument t (t ³ 0) that transforms it to a function $F(s)$ with a complex argument s.
2. When the Laplace transform is defined without condition then the unilateral or one-sided transform is normally considered. Alternatively, the Laplace transform can be defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis.
3. The inverse Laplace transform is also known by various names as the Bromwich integral, the Fourier-Mellin integral and Mellin's inverse formula. It is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s)\, ds$$

4. If f is a locally integrable function, then the Laplace transform $F(s)$ of f converges provided that the following limit exists:

$$\lim_{R\to\infty} \int_0^R f(t) e^{-ts}\, dt$$

5. The Laplace transform is a linear operation; which means, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b,

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.$$

6. Differentiation and integration are inverse processes. Consequently, as differentiation of a function corresponds to the multiplication of its transform by s, we expect integration of a function to equates to division of its transform by s, because division is the inverse operation of multiplication.
7. The Laplace transformation is an important operational method for solving linear differential equations. It is particularly useful in solving initial value problems connected with linear differential equations (ordinary and partial). The advantage of Laplace transformation in solving initial value problems lies in the fact that initial conditions are taken care of at the outset and the specific particular solution required is obtained without first obtaining the general solution of the linear differential equation.
8. The following results show that the Laplace transforms of the derivatives and integrals of a function $f(t)$ can be expressed in terms of the Laplace transform of $f(t)$. These results are important in solving differential equations using the methods of Laplace transformation.
9. The function $f(t)$ is said to of exponential order s_0 as $t \to \infty$, if

$$\lim_{t\to\infty} e^{-s_o t} f(t) \text{ is finite}$$

i.e., these exists M > 0 and $t_o > 0$ such that

$|e^{-s_o t} f(t)| \leq M \quad \forall\; t \geq t_o$

or $|f(t)| < Me^{s_o t}$ or $t \geq t_o$

We write $f(t) = 0\,(e^{s_o t})$ as $t \to \infty$

10. If $f(t)$ be any function of L and

$L\{f(t)\} = f(s)$

Then $f(t)$ is known as inverse Laplace transformation and given by

$f(t) = L^{-1}\{f(s)\}$

11. Linearly Property: Let $f_1(s)$ and $f_2(s)$ be the Laplace transformation of the functions $f_1(t)$ and $f_2(t)$, respectively, and a and b are the constant, then

$$L^{-1}\{af_1(s) + bf_2(s)\} = a\,L^{-1}\{f_1(s)\} + b\,L^{-1}\{f_2(s)\}$$

12. First Shifting theorem:

If $L^{-1}\{f(s)\} = f(t)$

Then $L^{-1}\{f(s-a)\} = e^{at} f(t)$

By definition $f(s) = \int_0^\infty e^{-st} f(t)dt$

13. If $L^{-1}\{f(s)\} = f(t)$

Then $L^{-1}\{e^{-as} f(s)\} = G(t)$

Where $G(t) = \begin{cases} f(t-a), & t > a \\ 0 & t < a \end{cases}$

By definition,

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)dt$$

$$= \int_0^a e^{-st} G(t)dt + \int_a^\infty e^{-st} G(t)dt$$

14. If $L^{-1}\{f(s)\} = f(t)$

Then $L^{-1}\{f(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right),\ a > 0$

By definition $f(s) = L\{f(t)\}$

$$= \int_0^\infty e^{-st} f(t)dt$$

$$f(as) = \int_0^\infty e^{-ast} f(t)dt$$

15. Let f and g be two functions defined in $[0, \infty)$. Then the convolution of f and g, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

16. $f(t) = \sum_{k=1}^{n} \frac{p(d_k)}{Q'(d_k)} \cdot e^{d_k t}$

NOTES

$$= \sum_{k=1}^{n} c_k e^{d_k t}$$

This is called Heaviside expansion formula.

1.16 SUMMARY

- In mathematics, the Laplace transform is a widely used integral transform and is denoted by $\mathcal{L}\{f(t)\}$. It is a linear operator of a function $f(t)$ including a real argument t (t ³ 0) that transforms it to a function $F(s)$ with a complex argument s.
- The Laplace transform can be related to the Fourier transform. The Fourier transform resolves a function or signal into its modes of vibration and the Laplace transform resolves a function into its moments.
- Switching from operations of calculus to algebraic operations on transforms is known as operational calculus which is an essential area of applied mathematics and with regard to an engineer, the Laplace transform method is basically a very essential operational technique.
- Another benefit of the Laplace transform is that it helps in solving the problems in a straightforward manner, initial value problems regardless of initially obtaining a basic solution, and nonhomogeneous differential equation exclusive of initially answering the corresponding homogeneous equation.
- When the Laplace transform is defined without condition then the unilateral or one-sided transform is normally considered. Alternatively, the Laplace transform can be defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis.
- The inverse Laplace transform is also known by various names as the Bromwich integral, the Fourier-Mellin integral and Mellin's inverse formula. It is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s)\, ds$$

- If f is a locally integrable function, then the Laplace transform $F(s)$ of f converges provided that the following limit exists:

$$\lim_{R\to\infty} \int_0^R f(t) e^{-ts}\, dt$$

- The Laplace transform is a linear operation; which means, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b,

  $$L\{af(t)+bg(t)\}=aL\{f(t)\}+bL\{g(t)\}.$$

- The Laplace transformation is an important operational method for solving linear differential equations. It is particularly useful in solving initial value problems connected with linear differential equations (ordinary and partial).
- The following results show that the Laplace transforms of the derivatives and integrals of a function $f(t)$ can be expressed in terms of the Laplace transform of $f(t)$. These results are important in solving differential equations using the methods of Laplace transformation.
- Differentiation and integration are inverse processes. Consequently, as differentiation of a function corresponds to the multiplication of its transform by s, we expect integration of a function to equates to division of its transform by s, because division is the inverse operation of multiplication.
- If $f(t)$ be any function of L and

  $$\mathrm{L}\{f(t)\} = f(s)$$

  then $f(t)$ is known as inverse Laplace transformation and given by

  $$f(t) = \mathrm{L}^{-1}\{f(s)\}$$

- Linearly Property: Let f_1(s) and f_2 (s) be the Laplace transformation of the functions $f_1(t)$ and $f_2(t)$, respectively, and a and b are the constant, then

  $$\mathrm{L}^{-1}\{af_1(s)+bf_2(s)\}=a\,\mathrm{L}^{-1}\{f_1(s)\}+b\,\mathrm{L}^{-1}\{f_2(s)\}$$

- First Shifting theorem:

  If $\mathrm{L}^{-1}\{f(s)\}=f(t)$

  Then $\mathrm{L}^{-1}\{f(s-a)\} = e^{at} f(t)$

  By definition $f(s) = \int_0^\infty e^{-st} f(t)dt$

- Complex inversion is an useful too for computing the inverse of Laplace transform $f(t) = L^{-1}(F(s))$.

1.17 KEY TERMS

- **Laplace transform:** The Laplace transform is a widely used integral transform and is denoted by $L\{f(t)\}$. It is a linear operator of a function $f(t)$ including a real argument t ($t \geq 0$) that transforms it to a function $F(s)$ with a complex argument s.
- **Bilateral Laplace transform:** The Laplace transform can be defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis.

- **Inverse Laplace transform:** The inverse Laplace transform is also known by various names as the Bromwich integral, the Fourier-Mellin integral and Mellin's inverse formula.
- **Transform of integrals:** The Laplace transform of the derivatives and integrals of a function $f(t)$ can be expressed in terms of the Laplace transform of $f(t)$.
- **Inverse Laplace transform:** Let $f(t)$ be any function of t and $L\{f(t)\} = f(s)$, then $f(t)$ is known as inverse Laplace transformation and given by $f(t) = L^{-1}\{f(s)\}$.
- **Linearly property:** Let $f_1(S)$ and $f_2(s)$ be the Laplace transformation of the functions $f_1(t)$ and $f_2(t)$, respectively, and a and b are the constant, then $L^{-1}\{af_1(s) + bf_2(s)\} = a$ L-1 $\{f_1(s) + bL^{-1}\{f_2(s)\}$.

1.18 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is the Laplace transform?
2. Define the bilateral Laplace transform.
3. What the inverse Laplace transform?
4. Define region of convergence.
5. What do you understand by the elementary theorems?
6. State the transforms of integrals.
7. State the linearity of the Laplace transform.
8. How will you define Laplace transform of the integral of a function?
9. What do you understand by the inversion of some elementary functions?
10. Define the term initial and final value theorems.
11. How will you define the multiplication and division by 't' periodic functions?
12. Define inverse Laplace transforms.
13. Define the linearly property of inverse Laplace transform.
14. How will you define the uniqueness theorem of Laplace transform?
15. State the first shifting theorem.
16. State the second shifting property.
17. What is the change of scale property?
18. What do you mean by the convolution property?
19. Define Heaviside expansion formula.

NOTES

Long-Answer Questions

1. Discuss briefly the Laplace transform.
2. Analyse the bilateral Laplace transform with the help of example.
3. Describe the linearity of the Laplace transform.
4. Explain the Laplace transforms of elementary functions.
5. Discuss the transforms of integrals.
6. Find the Laplace transforms of the following:

   (i) $2t^3 + 3t^2 - 5t + 2$ *(ii)* $\sqrt{e^{3(t+1)}}$

   (iii) $(e^{3t} + e^{-2t})^2$ *(iv)* $\sin at \cos at$

   (v) $\sin^3 bt$ *(vi)* $3t^2 + \cos^3 bt$

   (vii) $\sin at \cos bt$
7. Find the Laplace transforms of the following:

   (i) $t^3 e^5 t$ *(ii)* $e^{-t} \sin(2t+3)$

   (iii) $\cosh at \cos bt$ *(iv)* $\sinh at \sin bt$

   (v) $3t^2e^{-3t} + 5e^{3t} \cos 2t$
8. Find the Laplace transforms of the following:

   (i) $(2t+1)\sin 2t$ *(ii)* $(t+2)\cos 3t$

   (iii) $t^2 \sin at$ *(iv)* $t^2 \cos at$

   (v) $te^{-t} \cos 2t$ *(vi)* $te^{-at} \sin at$
9. Find the Laplace transforms of:

   (i) $\dfrac{e^{-at} - \cos at}{t}$ *(ii)* $\dfrac{\sin^2 t}{t}$

   (iii) $\left[\dfrac{\sin 2t}{\sqrt{t}}\right]^2$ *(iv)* $\left(\dfrac{\sin at}{at}\right)^2$

   (v) $\dfrac{1 - e^{-at}}{t}$
10. Find

    (i) $L\left[\int_0^t \dfrac{\sin at}{t} dt\right]$ *(ii)* Prove that $L\left[\int_0^t \dfrac{f(t)}{t} dt\right] = \dfrac{1}{s} \cdot \int_s^\infty L[f(t)ds$

    (iii) Find $L\left[\int_0^t \dfrac{\sin^2 t}{t} dt\right]$ *(iv)* Find $L\left[\int_0^t \dfrac{t - e^{-at}}{t} dt\right]$

    (v) Find $L[e^{-t} \int_0^t t \cos t \, dt]$
11. Find the Laplace transforms of:

    (i) $L(2t^2 - e^{-t})$ *(ii)* $L(t^2 + 1)^2$

NOTES

(iii) $L(\sin t - \cos t)^2$ *(iv)* $L(\cosh^2 4t)$

(v) $L\{f(t)\}$ if $f(t) = \begin{cases} 0 & \text{when } 0 < t < 2 \\ 4 & \text{when } t > 2 \end{cases}$

(vi) $L\{t^3\ e^{-3t}\}$ *(vii)* $L\{(t+2)^2\ e^t\}$

12. Show that the Laplace transform of $\int_0^\infty te^{-3t} \sin t\ dt = \frac{3}{50}$

13. Solve using $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

(i) Find $L\left[\frac{\sin^2 t}{t}\right]$ *(ii)* Find $L\left[\frac{1-e^t}{t}\right]$

(iii) Evaluate $\int_s^\infty te^{-3t} \cos t\, dt$

14. Evaluate laplace transform of

(i) $f(t) = (\sin t - \cos t)^2$

(ii) $e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t$

(iii) $\cos^3 2t$

(iv) $(t + 2)^2 e^t$

(v) $e^{-t} \sin^2 t$

(vi) $f(t) = \begin{cases} e^t & 0 < + < 1 \\ 0 & t > 1 \end{cases}$

(vii) $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ t-1 & 2 < t < 3 \\ > & t > 3 \end{cases}$

(viii) $f(t) = |t - 1| + |t + 1|,\ t \geq 0$

(ix) $L\{f(9t)\}$ where $f(t) = \begin{cases} \sin\left(t - \frac{2}{3}\pi\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

(x) Prove that $L\{\sin K\,(-\sin hKt)\} = \frac{2K^2 s}{s^4 + 4K^4}$

15. Find the Laplace transformation.

(i) $t \sin 3t \cos 2t$ (ii) $\frac{e^{at} - \cos 6t}{t}$

(iii) $\dfrac{1-\cos t}{t}$ (iv) $\int_0^\infty te^{-3t}\sin t\,dt$

(v) $\int_0^t \dfrac{e^t \sin t}{t}dt$ (vi) Prove that $\int_0^\infty \dfrac{e^{-t}\sin^2 t}{t}dt = \dfrac{1}{2}\log 5$

NOTES

(vii) Prove that $\int_0^\infty \dfrac{e^{-t}-e^{-3t}}{t}dt = \log 3$

(viii) $\int_0^\infty \dfrac{\sin t}{t}dt$ (ix) Prove that $\int_0^\infty \dfrac{e^{-t}\sin}{t}dt = \dfrac{\pi}{4}$

(x) $\int_0^\infty \dfrac{\cos at - \cos bt}{t}dt$ (xi) $t \sin h\, at$

16. Discuss briefly the inverse Laplace transform with help of example.

(i) $\dfrac{a^2}{s(s+a)^2}$

(ii) $\dfrac{s+2}{(s-2)^3}$

(iii) $\dfrac{1}{s(s^2+4)}$

(iv) $\dfrac{2}{(s-1)^2(s^2+1)}$

(v) $\dfrac{s}{s^4+4a^4}$

(vi) $\dfrac{1}{s^3(s^2+1)}$

(vii) $\dfrac{1}{(s^2+2s+5)^2}$

(viii) $\dfrac{s^3+6s^2+14s}{(s+2)^4}$

(ix) $\dfrac{4(s+3)}{(s^2+6s+13)^2}$

(x) $\dfrac{s^4-s^3+2s^2-8}{s^3(s^2+4)}$

(xi) $\dfrac{4s+5}{(s-1)^2(s+2)}$

(xii) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$

(xiii) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

(xiv) $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$

NOTES

(xv) $\frac{1}{s^2(s+2)}$

(xvi) $\frac{1}{s(s+2)^3}$

(xvii) $\frac{1}{s^3(s^2+1)}$

(xviii) $\frac{s^2}{(s+a)^3}$

(xix) $\tan^{-1}\left(\frac{2}{5}\right)$

(xx) $\log\frac{s(s+1)}{s^2+4}$

(xxi) $\log\left(1-\frac{a^2}{s^2}\right)$

Using convolution theorem

(i) $\frac{1}{\left(s^2+a^2\right)^2}$

(ii) $\frac{1}{(s-2)(s+2)^2}$

(iii) $\frac{s}{(s^2+1)(s^2+4)}$

(iv) $L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}$

17. Obtain the Laplace transform of $\frac{\sin 2t}{t}$
18. Discuss convolution property with the help of examples.
19. Explain the complex inversion formula and Heaviside expansion formula. Give appropriate examples.
20. Discuss the applications of Laplace transformation with the help of examples.

1.19 FURTHER READING

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UNIT 2 APPLICATION OF LAPLACE TRANSFORMS

Structure

2.0 INTRODUCTION

In mathematics, the Laplace transform, named after its inventor Pierre-Simon Laplace a mathematician and astronomer, is an integral transform that converts a function of a real variable 't' to a function of a complex variable 's' (complex frequency). The 'Laplace Transform' has many applications. Two of the most significant are the solution of differential equations and convolution. The Laplace transform operator is used to solve both the first order and second order differential equations with constant coefficients. The differential equations must be IVP's (Initial Value Problem) with the initial condition (s) specified at $x = 0$. Essentially, the Laplace transform is an efficient method for schematically solving the linear differential equations with constant coefficients. Given an IVP, the Laplace transform operator is applied to both sides of the differential equation which will transform the differential equation into an algebraic equation whose unknown function is considered as the Laplace transform of the desired solution.

The Laplace transform is a powerful integral transform used to switch a function from the time domain to the s-domain. The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives. A linear differential equation or a system of linear equations such that the associated homogeneous equations have constant coefficients may be solved by quadrature, which means that the solutions may be expressed in terms of integrals. This is also true for a linear equation of order one, with non-constant coefficients. An equation of order two or higher with non-constant coefficients cannot, in general, be solved by quadrature. For order two, Kovacic's algorithm allows deciding whether there are solutions in terms of integrals, and computing them if any.

NOTES

The Laplace transformation is a mathematical tool which is typically used to solve the differential equations by converting it from one form into another form. It is very useful and effective tool for solving linear differential equations either ordinary or partial. To solve these equations simultaneously, the Laplace transform of each equation is taken for obtaining two simultaneous algebraic equations from which one can determine $X(s)$ and $Y(s)$, the Laplace transforms of $x(t)$ and $y(t)$, respectively. Partial differential equations, which are used to formulate, and thus aid, the solution of problems involving functions of several variables. Differentiate a given equation with respect to x and y and then find the solution for the same by forming partial differential equations.

Convolution is a mathematical operation on two functions (f and g) that produces a third function $f * g$) that expresses how the shape of one is modified by the other. The term convolution refers to both the result function and to the process of computing it. It is defined as the integral of the product of the two functions after one is reversed and shifted. The integral is evaluated for all values of shift, producing the convolution function. The Wronskian of two solutions of a homogeneous second-order linear ordinary differential equation in terms of a coefficient of the original differential equation is expressed by Abel's identity (also known as Abel's formula or Abel's differential equation identity). Retarded, neutral, advanced, and mixed functional differential equations are all terms used to describe differential difference equations. This classification is based on whether the current state of the system's rate of change is influenced by previous values, future values, or both.

In this unit, you will learn about the ordinary differential equations with constant coefficients, ordinary differential equations with variable coefficient, simultaneous ordinary differential equations, partial differential equations, applications to mechanics, electrical circuits, and beams, integral equations of convolution type, Abel's integral equation, integro-differential equation and differential-difference equations.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the significance of Laplace transform
- Solve the ordinary differential equations with constant and variable coefficients using Laplace transform
- Describe the method of solving simultaneous linear equations
- Discuss the types and applications of partial differential equations
- Discuss solution of integral equations of convolution type
- Elaborate on the solution of integro-differential equation
- Explain the differential-difference equations

2.2 ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved. Solving differential equations is performed as per the guidelines given in Figure 2.1 which involve the following three steps:

- Transformation of the differential equation into the mapped space.
- Solving the algebraic equation in the mapped space.
- Back transformation of the solution into the original space.

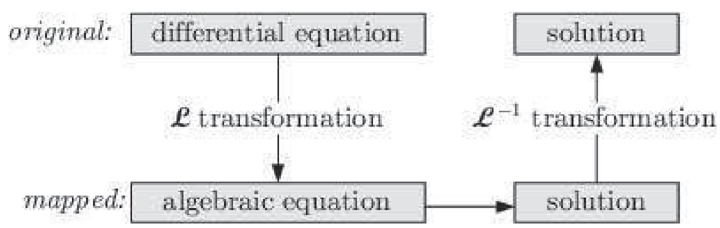


Fig. 2.1 *Schema for Solving Differential Equations Using the Laplace Transformation*

The following examples will make the concept clear.

Example 2.1: Consider the differential equation $\ddot{f}(t) + 3\dot{f}(t) + 2f(t) = e^{-t}$ with the initial conditions $f(0+) = \dot{f}(0+) = 0$.

Solution: We can solve the differential equation using the following steps:

Step 1: $s^2F(s) + 3sF(s) + 2F(s) = \frac{1}{s+1}$

Step 2: $F(s) = \frac{1}{s+1}\frac{1}{s^2+3s+2}$

Step 3: The complex function $F(s)$ must be decomposed into partial fractions in order to get,

$$F(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

From the inverse Laplace transformation the solution of the given differential equation is,

$$f(t) = e^{-2t} - e^{-t} + te^{-t}.$$

Example 2.2: Solve the following system of differential equation using Laplace transform:

$$x_1' = 3x_1 - 3x_2 + 2 \qquad x_1(0) = 1$$
$$x_2' = -6x_1 - t \qquad x_2(0) = -1$$

Solution: Notice that the system is not given in matrix form hence matrix form is not required in solution. The system is nonhomogeneous.

Using Laplace transforms to solve differential equations, we consider the transform of both differential equations as,

$$sX_1(s) - x_1(0) = 3X_1(s) - 3X_2(s) + \frac{2}{s}$$

$$sX_2(s) - x_2(0) = -6X_1(s) - \frac{1}{s^2}$$

Now use the initial condition and simplify to get,

$$(s-3)X_1(s) + 3X_2(s) = \frac{2}{s} + 1 = \frac{2+s}{s}$$

$$6X_1(s) + sX_2(s) = -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2}$$

To solve this for one of the transforms, multiply the top equation by s and the bottom by –3 and then add. We get,

$$(s^2 - 3s - 18)X_1(s) = 2 + s + \frac{3s^2+3}{s^2}$$

Solving for X_1 gives,

$$X_1(s) = \frac{s^3 + 5s^2 + 3}{s^2(s+3)(s-6)}$$

On partial fraction we get,

$$X_1(s) = \frac{1}{108}\left(\frac{133}{s-6} - \frac{28}{s+3} + \frac{3}{s} - \frac{18}{s^2}\right)$$

Taking the inverse transform gives the first solution,

$$x_1(t) = \frac{1}{108}\left(133e^{6t} - 28e^{-3t} + 3 - 18t\right)$$

To find the second solution we can eliminate X_1 to find the transform for X_2. However, in this case notice that the second differential equation is as follows,

$$x_2' = -6x_1 - t \qquad \Rightarrow \qquad x_2 = \int -6x_1 - t\, dt$$

By, endorsing the first solution in and integrating gives,

$$x_2(t) = -\frac{1}{18}\int 133e^{6t} - 28e^{-3t} + 3\, dt$$

$$= -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t\right) + c$$

Reapplying the second initial condition to get the constant of integration gives,

$$-1 = -\frac{1}{108}(133 + 56) + c \qquad \Rightarrow \qquad c = \frac{3}{4}$$

The second solution is,

$$x_2(t) = -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t - 81\right)$$

NOTES

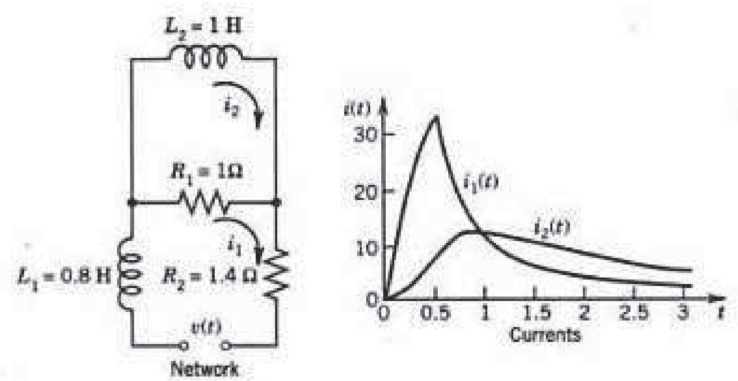
Putting all this together gives the solution to the system as,

$$x_1(t) = \frac{1}{108}\left(133e^{6t} - 28e^{-3t} + 3 - 18t\right)$$

$$x_2(t) = -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t - 81\right)$$

Other systems of differential equations of practical significance can be solved using the Laplace transform method in a related manner, and taking eigenvalues and eigenvectors as shown in Example 2.3 based on electrical network.

Example 2.3: Find the currents $i_1(t)$ and $i_2(t)$ in the network as shown in the following figure with *L* and *R* measured in terms of the usual units, $v(t) = 100$ volts if $0 \le t \le 0.5$ sec and 0 thereafter, and $i(0) = 0,\ i'(0) = 0$.



Solution: The method of the network is obtained using the Kirchhoff's voltage law as,

$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 = 100\left[1 - u\left(t - \frac{1}{2}\right)\right]$$

$$1 \cdot i_2' + l(i_2 - i_1) = 0.$$

Dividing by 0.8 and on ordering we get,

$$i_1' + 3i_1 - 1.25i_2 = 125\left[1 - u\left(t - \frac{1}{2}\right)\right]$$

$$i_2' - i_1 + i_2 = 0$$

NOTES

With $i_1(0)=0, \quad i_2(0)=0$ we acquire the second shifting theorem as the subsidiary equations:

$$(s+3)I_1 - 1.25I_2 = 125\left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right)$$

$$-I_1 + (s+1)I_2 = 0\,.$$

Algebraically solving I_1 and I_2 gives:

$$I_1 = \frac{125(s+1)}{s\left(s+\frac{1}{2}\right)\left(s+\frac{7}{2}\right)}\left(1-e^{s/2}\right)$$

$$I_2 = \frac{125}{s\left(s+\frac{1}{2}\right)\left(s+\frac{7}{2}\right)}\left(1-e^{-s/2}\right),$$

The right hand sides without the factor $1-e^{-s/2}$ contain the partial fraction expansions of the form:

$$\frac{500}{7s} - \frac{125}{3\left(s+\frac{1}{2}\right)} - \frac{625}{21\left(s+\frac{7}{2}\right)},\ \frac{500}{7s} - \frac{250}{3\left(s+\frac{1}{2}\right)} + \frac{250}{21\left(s+\frac{7}{2}\right)},$$

The inverse transform of this equation provides the solution $0 \le t \le \frac{1}{2}$,

$$i_1(t) = -\frac{125}{3}e^{-t/2} - \frac{625}{21}e^{-7t/2} + \frac{500}{7}$$

$$i_2(t) = -\frac{250}{3}e^{-t/2} + \frac{250}{21}e^{-7t/2} + \frac{500}{7} \qquad 0 \le t \le \frac{1}{2}$$

As per the second shifting theorem, the solution for $t > \frac{1}{2}$ is obtained by subtracting from this $i_1\left(t-\frac{1}{2}\right)$ and $i_2\left(t-\frac{1}{2}\right)$, respectively. We get,

$$i_1(t) = -\frac{125}{3}\left(1-e^{1/4}\right)e^{-t/2} - \frac{625}{21}\left(1-e^{7/4}\right)e^{-7t/2}$$

$$i_2(t) = -\frac{250}{3}\left(1-e^{1/4}\right)e^{-t/2} + \frac{250}{21}\left(1-e^{7/4}\right)e^{7t/2} \qquad \left(t > \frac{1}{2}\right)$$

Similarly, the systems of differential equations of higher order can also be solved using the Laplace transform method. The higher order differential equations involve the higher derivatives $x''(t), x'''(t)$, etc. These mathematical models are used to solve physics and engineering problems.

NOTES

2.3 ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENT

Considering $y^{(n)}$ as a constant, then the following equalities hold:

$$\int s^2Y''ds = \frac{1}{3}s^3Y'' \qquad \dots(2.1)$$

$$\int 2sY'ds = s^2Y' \qquad \dots(2.2)$$

In the above given two equations, $y^{(n)}$ is considered as a function of s. However, $y^{(n)}$ can be used as a constant if the difference stands as much as the constant coefficient. For example,

Let $Y = s^2 + 1$. Then

$$\int s^2Y''ds = \int 2s^2ds = \frac{2}{3}s^3$$

And $\frac{1}{3}s^3Y'' = \frac{2}{3}s^3$

Thus the Equation (2.1) holds. New consider the Equation (2.2).

$$\int 2sY'ds = \int 4s^2ds = \frac{4}{3}$$

And

$$s^2Y' = 2s^3.$$

Basically we can state that is the difference is as much as the constant coefficient. This specifies that the left-hand side of the equation has a solution $c_1 + c_2X$ when the right-hand side has a solution X, when both c_1 and c_2 are constant terms. Therefore, these formulas are restricted for the form of s^n only.

Solution of ODEs with Variable Coefficients

Let the solution Y is $c_1 + c_2X$.

Then,

$$\int s^2Y''ds = \int s^2c_2X'' = \frac{1}{3}c_2X''s^3 = \frac{1}{3}s^3c_2X'' = \frac{1}{3}s^3Y''.$$

Similarly,

$$\int 2sY'ds = \int 2sc_2X'ds = c_2X's^2 = s^2c_2X' = s^2Y'.$$

The result is similar when the right-hand side has a solution X.

Theorem 2.1. Let us denote $£(y) = Y = F(s)$, $£(y') = Y'$ and $£(y'') = Y''$. Then Euler-Cauchy equation $t^2y'' + aty' + by = 0$, Legendre's equation $y'' - t^2y'' - 2ty' + n(n+1)y = 0$ and Bessel equation $t^2y'' + ty' + (t^2 - v^2)y = 0$ can be represented by,

$$(\frac{1}{3}s^4 + \frac{4-a}{s}s^2 + b - a + 2)Y = \frac{1}{3}y(0)s^3 + \frac{1}{3}y'(0)s^2 + \frac{4-a}{2}y(0)s,$$

$$\left(\frac{1}{3}s^4+\frac{2}{3}s^2-n(n+1)\right)Y=\frac{1}{3}y(0)s^3+\frac{1}{3}y'(0)s^2+\frac{1}{2}y(0)s-y'(0),$$

NOTES

And $$\left(\frac{1}{3}s^4+\frac{5}{2}s^2+1-v^2\right)Y=\frac{1}{3}y(0)s^3+\frac{1}{3}y'(0)s^2+\frac{5}{2}y(0)s+y'(0),$$

respectively,

Putting

$$Y_1=\frac{d}{ds}Y,$$

We have the solution y as $y=£^{-1}(Y_1)$.

Proof. If we take Laplace transform for above equation, then we have

$$s^2\frac{d^2Y}{ds^2}+(4-a)s\frac{dY}{ds}+(b-a+2)Y=0,$$

$$s^2\frac{d^2Y}{ds^2}+2s\frac{dY}{ds}-(s^2+n(n+1)Y+sy(0)+y'(0)=0,$$

And $$(s^2+1)\frac{d^2Y}{ds^2}+3s\frac{dY}{ds}+(1-v^2)Y=0,$$

respectively. Integrating Euler-Cauchy equation with respect to s, we have,

$$\frac{1}{3}s^3Y''+\frac{4-a}{2}s^2Y'+(b-a+2)Ys=0.$$

Since, $Y'=sY-y(0)$ and $Y''=s^2Y-sy(0)-y'(0)$, we have,

$$\frac{1}{3}s^3(s^2Y-sy(0)-y'(0))+\frac{4-a}{2}s^2(sY-y(0))+(b-a+2)Ys=0.$$

Organizing this equality, we have

$$Y=\frac{\frac{1}{3}y(0)s^3+\frac{1}{3}y'(0)s^2+\frac{4-a}{2}y(0)s}{\frac{1}{3}s^4+\frac{4-a}{2}s^2+b-a+2}$$

Putting,

$$Y_1=\frac{d}{ds}Y,$$

We obtain the solution y as $y=£^{-1}(Y_1)$. For a given number v, we get the above given results using the similar method.

2.3.1 Solution of System of Differential Equations Using the Laplace Transformation

The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved. Solving differential equations is performed as per the guidelines given in Figure 2.2 which involve the following three steps:

- Transformation of the differential equation into the mapped space.
- Solving the algebraic equation in the mapped space.
- Back transformation of the solution into the original space.

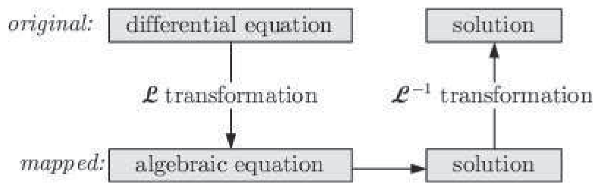


***Fig. 2.2** Schema for Solving Differential Equations Using the Laplace Transformation*

The following examples will make the concept clear.

Example 2.4: Consider the differential equation

$\ddot{f}(t)+3\dot{f}(t)+2f(t)=e^{-t}$ with the initial conditions $f(0+)=\dot{f}(0+)=0$.

Solution: We can solve the differential equation using the following steps:

Step 1: $s^2F(s)+3sF(s)+2F(s)=\dfrac{1}{s+1}$

Step 2: $F(s)=\dfrac{1}{s+1}\dfrac{1}{s^2+3s+2}$

Step 3: The complex function $F(s)$ must be decomposed into partial fractions in order to get,

$$F(s)=\frac{1}{s+2}-\frac{1}{s+1}+\frac{1}{(s+1)^2}.$$

From the inverse Laplace transformation the solution of the given differential equation is,

$$f(t)=e^{-2t}-e^{-t}+te^{-t}.$$

Example 2.5: Solve the following system of differential equation using Laplace transform:

$$x_1'=3x_1-3x_2+2 \qquad x_1(0)=1$$
$$x_2'=-6x_1-t \qquad x_2(0)=-1$$

Solution: Notice that the system is not given in matrix form hence matrix form is not required in solution. The system is nonhomogeneous.

Using Laplace transforms to solve differential equations, we consider the transform of both differential equations as,

NOTES

$$sX_1(s) - x_1(0) = 3X_1(s) - 3X_2(s) + \frac{2}{s}$$

$$sX_2(s) - x_2(0) = -6X_1(s) - \frac{1}{s^2}$$

Now use the initial condition and simplify to get,

$$(s-3)X_1(s) + 3X_2(s) = \frac{2}{s} + 1 = \frac{2+s}{s}$$

$$6X_1(s) + sX_2(s) = -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2}$$

To solve this for one of the transforms, multiply the top equation by s and the bottom by –3 and then add. We get,

$$\left(s^2 - 3s - 18\right)X_1(s) = 2 + s + \frac{3s^2+3}{s^2}$$

Solving for X_1 gives,

$$X_1(s) = \frac{s^3 + 5s^2 + 3}{s^2(s+3)(s-6)}$$

On partial fraction we get,

$$X_1(s) = \frac{1}{108}\left(\frac{133}{s-6} - \frac{28}{s+3} + \frac{3}{s} - \frac{18}{s^2}\right)$$

Taking the inverse transform gives the first solution,

$$x_1(t) = \frac{1}{108}\left(133e^{6t} - 28e^{-3t} + 3 - 18t\right)$$

To find the second solution we can eliminate X_1 to find the transform for X_2. However, in this case notice that the second differential equation is as follows,

$$x_2' = -6x_1 - t \qquad \Rightarrow \qquad x_2 = \int -6x_1 - t\,dt$$

By, endorsing the first solution in and integrating gives,

$$x_2(t) = -\frac{1}{18}\int 133e^{6t} - 28e^{-3t} + 3\,dt$$

$$= -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t\right) + c$$

Reapplying the second initial condition to get the constant of integration gives,

$$-1 = -\frac{1}{108}(133 + 56) + c \qquad \Rightarrow \qquad c = \frac{3}{4}$$

The second solution is,

$$x_2(t) = -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t - 81\right)$$

Putting all this together gives the solution to the system as,

$$x_1(t) = \frac{1}{108}\left(133e^{6t} - 28e^{-3t} + 3 - 18t\right)$$

$$x_2(t) = -\frac{1}{108}\left(133e^{6t} + 56e^{-3t} + 18t - 81\right)$$

NOTES

Other systems of differential equations of practical significance can be solved using the Laplace transform method in a related manner, and taking eigenvalues and eigenvectors.

2.4 SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

Simultaneous ordinary differential equations may be solved by using Laplace transformation

Working Rule

1. Two equations are given for two dependent variables, depending upon a single variable. Apply the Laplace transform on both sides of the equations on both of the equations.
2. Now solving these equations, there will be values of dependent variable in terms of s.
3. Applying Laplace transformation find the value of dependent variables.

Example 2.6: $Dx - y = e^t$, $\quad Dy + x = \sin t$

$x(0) = 1$, $\quad y(0) = 0$

Given $Dx - y = e^t$, $\quad Dy + x = \sin t$

Solution: Apply Laplace transform on both sides

$$[sL(x) - x(0)] - L(y) = \frac{1}{s-1}$$

Or $\quad [sL(x) - 1 - L(y) = \frac{1}{s-1}$

And $\quad [sL(y) - y(0)] + L(x) = \frac{1}{s^2+1}$

Or $\quad sL(y) + L(x) = \frac{1}{s^2+1}$

Solving Equations

$$L(x) = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{s+1}{(s^2+1)}\right] + \frac{1}{(s^2+1)^2} \quad \text{(Resolving into partial fraction)}$$

And $L(y) = \frac{s}{(s^2+1)^2} - \frac{s}{(s^2-1)(s^2+1)}$

NOTES

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{(s-1)}{s^2+1}\right] \qquad \text{(By Partial fraction)}$$

Apply Laplace inverse on both side.

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}(e^t + \cos t + \sin t) + \frac{1}{2}(\sin t - t\cos t)$$

$$= \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

And $\quad y \quad = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s-1} - \frac{(s-1)}{(s^2+1)}\right]$

$$= \frac{1}{2}t\sin t - \frac{1}{2}(e^t - \cos t + \sin t)$$

$$= \frac{1}{2}(\sin t - e^t + \cos t - \sin t)$$

Hence $y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$

$$= \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

Example 2.7: $DX + DY = t,\ D^2X - Y = e^{-t}$

If $X(0) = 3,\ X'(0) = -2,\ Y(0) = 0$

Solution: Taking Laplace transform on both of the eqation

$$[sL\,(x)\,D - X(0)] + [sL(y) - Y(0)] = \frac{1}{s^2}$$

$$\therefore \quad sL(x) + sL(y) = 3 + \frac{1}{s^2} \qquad ...(1)$$

And $\left[s^2L(X)sX(0) - X(0)\right] - [sL(Y) - Y(0)] = \frac{1}{s+1}$

$$\therefore \quad s^2L(X) - L(Y) = -3s = 2 + \frac{1}{s+1} \qquad ...(2)$$

Solving Equations (1) and (2)

$$x = \frac{3s^2+1}{s^3(1+s^2)} + \frac{3s}{1+s^2} - \frac{2}{1+s^2} - \frac{1}{(s+1)(s^2+1)}$$

$$= \frac{1}{s^3}\left(1 + 2s^2 - \frac{2s^4}{1+s^4}\right) + \frac{3s}{1+s^2} - \frac{2}{1+s^2} + \frac{1}{2(s+1)}$$

$$- \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

$$= \frac{2}{s}+\frac{1}{s^3}+\frac{1}{2(s+1)}+\frac{s}{2(1+s^2)}-\frac{3}{2(1+s^2)}$$

And $y = \frac{1}{s(s+1)(s^2+1)}+\frac{2}{(s^2+1)}$

NOTES

$$= \frac{1}{s}-\frac{1}{2(s+1)}-\frac{s}{2(s^2+1)}-\frac{1}{2(s^2+1)}+\frac{2}{s^2+1}$$

Apply Laplace transform on these equation

$$X = 2L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s^3}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]$$

$$+\frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right]-\frac{3}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2+\frac{e^2}{2}+\frac{e^{-t}}{2}+\frac{1}{2}\cos t+3\sin t$$

And $y \quad = L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$

$$= 1-e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

$$\therefore \; x = 2+\frac{t^2}{2}+\frac{e^{-t}}{2}+\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

$$y = 1-e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

Example 2.8: $X'+Y'=t$

$X''-Y=e^{-t}$

Solution: Subject to $X(0)=0$, $X'(0)=-2$, $Y(0)=0$

Apply Laplace transform

$$[sL(X)-X(0)]+[sL(Y)-Y(0)]=1/s^2$$

$$sL(X)+sL(Y)=\frac{1}{s^2} \qquad ...(1)$$

$$[s^2L(X)-sX(0)-X'(0)]-L(Y)=\frac{1}{s+1}$$

$$[s^2L(X)-L(Y)=3s-2+\frac{1}{s^2+1} \qquad ...(2)$$

Solving Equations (1) and (2)

$$X=\frac{1}{(s+1)(s^2+1)}+\frac{3s}{s^2+1}-\frac{2s^2}{s^2+1}+\frac{3s}{s^2+1}+\frac{1}{s(s^2+1)}$$

NOTES

$$Y = \frac{s^2}{(s+1)(s^2+1)} + \frac{3s^3}{s^2+1} - \frac{2s^2}{s^2+1} + \frac{3s}{s^2+1}$$

$$+ \frac{1}{s(s^2+1)} - 3s + 2 - \frac{1}{s+1}$$

Resolving into partial fractions and then applying Laplace inversion.

$$X = 2 + \frac{t^2}{2} + \frac{e^{-t}}{2} - \frac{3}{2}\sin t + \frac{1}{2}\cos t$$

$$Y = 1 - \frac{1}{2}e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t$$

Example 2.9: Solve $(D-2)x - (D+1)y = 6e^{3t}$

$(2D-3)x + (D-3)y = 6e^{3t}$

$x(0) = 3, y(0) = 0$

Solution: Apply Laplace transform on both equation

$$[sL(x) - X(0)] - 2L(X) - [sL(Y) - Y(0) + L(Y)] = \frac{6}{s-3}$$

$$(s-2)L(X) - (s+1)L(Y) = 3 + \frac{6}{s-3} = \frac{3s-3}{s-3} \quad ...(1)$$

$$[2(sL(x) - x(0) - 3L(x)] + [sL(y) - y(0) - 3L(y)] = \frac{e}{s-3}$$

$$(2s-3)L(x) + (s-3)L(y) = \frac{6s-12}{s-3} \quad ...(2)$$

Solving Equations (1) and (2)

$$L(x) = \frac{1}{s-1} + \frac{2(s+1)(s-2)}{(s-3)(s-1)^2}$$

$$= \frac{1}{s-1} + \frac{2}{s-3} + \frac{2}{(s-1)^2}$$

(Resolving into partial fractions)

$$L(y) = -\frac{3}{(s-1)^2} - \frac{y}{(s-1)^2(s-3)}$$

$$= -\frac{3}{(s-1)^2} - \left[\frac{1}{s-3} - \frac{1}{s-1} - \frac{2}{(s-1)^2}\right]$$

$$= \frac{1}{s-1} + \frac{1}{s-3} - \frac{1}{(s-1)^2}$$

$\therefore$ $x = e^t + 2e^{3t} + 2te^t$

$y = e^t + e^{3t} - te^t$

Check Your Progress

1. What is significance of Laplace transform to solve an ordinary differential equations with constant coefficients?
2. Name the three steps for solving an ordinary differential equations with constant coefficients.
3. Define the solution of ordinary differential equations with variable coefficients through the integral and Laplace transform.
4. How Laplace transform is useful for solving the differential equations?
5. Define the working rule for the solution of simultaneous ordinary differential equation.

2.5 PARTIAL DIFFERENTIAL EQUATIONS

Let $z = f(x, y)$ be a function of two independent variables x and y. Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives; $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.

Any equation which contains one or more partial derivatives is called a partial differential equation. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z; \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for partial differential equation (PDE) of first order and second order respectively.

We use the following notations for partial derivatives,

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y},\ t = \frac{\partial^2 z}{\partial y^2}$$

Partial differential equation may be formed by eliminating *(i)* arbitrary constants *(ii)* arbitrary functions.

Example 2.10: Form the parital differential equation by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution: Given, $z = ax + by + a^2 + b^2$ (1)

Here we have two arbitrary constants a and b. Therefore, we need two more equations to eliminate a and b. Differentiating equation (1) partially with respect to x and y respectively we get,

$$\frac{\partial z}{\partial x} = p = a \quad (2)$$

$$\frac{\partial z}{\partial y} = q = b \quad (3)$$

NOTES

From equations (2) and (3), we get,

$$a = p,\ b = q$$

Substituting values of a and b in (1) we get,

$$z = px + qy + p^2 + q^2$$

This is the required partial differential equation.

Example 2.11: Eliminate a and b from $z = (x + a)(y + b)$.

Solution: Differentiating partially with respect to x and y,

$$p = y + b,\ q = x + a$$

Eliminating a and b, we get $z = pq$.

Example 2.12: Form the partial differential equation by eliminating the arbitrary constants in $z = (x - a)^2 + (y - b)^2$.

Solution: Given, $z = (x - a)^2 + (y - b)^2$ (1)

Here we have two arbitrary cosntants a and b. To eliminate these two arbitrary constants we need two more equations connecting a and b. Therefore, differentiating equation (1) partially with respect to x and y, we get,

$$\frac{\partial z}{\partial x} = p = 2(x - a) \quad (2)$$

$$\frac{\partial z}{\partial y} = q = 2(y - b) \quad (3)$$

From equation (2), we get,

$$x - a = \frac{p}{2} \quad (4)$$

From equation (3), we get,

$$y - b = \frac{q}{2} \quad (5)$$

Substituting equations (4) and (5) in (1) we get,

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

Simplifying we get, $4z = p^2 + q^2$

This gives the partial differential equation after elimination of a and b.

Example 2.13: Form the partial differential equation by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Solution: Given, $z = (x^2 + a)(y^2 + b)$ (1)

NOTES

Here we have two arbitrary constants a and b.

Differentiating equation (1) partially with respect to x and y we get,

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \quad (2)$$

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \quad (3)$$

From equation (2) we get, $\frac{p}{2x} = y^2 + b$ (4)

From equation (3) we get, $\frac{q}{2y} = x^2 + a$ (5)

Substituting equations (4) and (5) in (1), we get,

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$pq = 4xyz$$

This gives the required partial differential equation.

Example 2.14: Form the partial differential equation by eliminating *a, b, c* from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution: Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (1)

Differential partially with respect to x and y we get,

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot p = 0 \quad (2)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot q = 0 \quad (3)$$

Differentiating equation (2) partially with respect to y,

$$0 + \frac{2}{c^2}(zs + qp) = 0$$

$$zs + qp = 0$$

Note: More than one partial differential equation is possible in this problem. These partial differential equations are,

$$xzr + xp^2 - zp = 0, \; yzt + yq^2 - zq = 0$$

NOTES

Formation of Partial Differential Equation by Eliminating Arbitrary Functions

The partial differential equations can be formed by eliminating arbitrary junctions. The following examples will make the concept clear.

Example 2.15: Eliminate arbitrary function from,

$$z = f(x^2 + y^2) \qquad (1)$$

Solution: Differentiating partially with respect to x and y, we get,

$$p = f'(x^2 + y^2).2x \qquad (2)$$

$$q = f'(x^2 + y^2).2y \qquad (3)$$

Eliminating $f'(x^2 + y^2)$ from equation (2) and (3), we get, $py = qx$

Example 2.16: Form the partial differential equation by eliminating the arbitrary function ϕ from $xyz = \phi(x^2 + y^2 - z^2)$.

Solution: Given, $xyz = \phi(x^2 + y^2 - z^2)$ (1)

This equation contains only one arbitrary function ϕ and we have to eliminate it.

Differentiating equation (1) partially with respect to x and y we get,

$$yz + xyp = \phi'(x^2 + y^2 - z^2)(2x - 2zp) \qquad (2)$$

$$xz + xyq = \phi'(x^2 + y^2 - z^2)(2y - 2zq) \qquad (3)$$

From equation (2), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{yz + xyp}{2x - 2zp} \qquad (4)$$

From equation (3), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{xz + xyq}{2y - 2zq} \qquad (5)$$

Since, LHS of equations (4) and (5) are equal, we have,

$$\frac{yz + xyp}{2x - 2zp} = \frac{xz + xyq}{2y - 2zq}$$

$$(yz + xyp)(y - zq) = (xz + xyq)(x - zp)$$

i.e., $$y(z + xp)(y - zq) = x(z + yq)(x - zp) \qquad (6)$$

On simplifying equation (6) we get,

$$px(y^2 + z^2) - qy(z^2 + x^2) = z(x^2 - y^2)$$

Which gives the required partial differential equation.

Example 2.17: Eliminate the arbitrary function from $z = (x + y)\, f(x^2 - y^2)$

Solution: Given, $z = (x + y)\, f(x^2 - y^2)$ (1)

Differentiating partially with respect to x and y we get,

$$p = (x + y)f'(x^2 - y^2)2x + f(x^2 - y^2) \cdot 1 \quad (2)$$

$$q = (x + y)f'(x^2 - y^2)(-2y) + f(x^2 - y^2) \cdot 1 \quad (3)$$

Eliminating $f'(x^2 - y^2)$ from equations (2) and (3) we get,

$$\frac{2x(x+y)}{-2y(x+y)} = \frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)}$$

$$2x[q - f(x^2 - y^2)] = -2y[p - f(x^2 - y^2)]$$

$$xq - xf(x^2 - y^2) = -yp + yf(x^2 - y^2)$$

$$xq + yp = (x + y)\, f(x^2 - y^2)$$

$$= (x + y)\frac{z}{(x+y)}$$

$$\therefore \quad z = xq + yp$$

This is a required equation.

Example 2.18: Eliminate the arbitrary function from $z = xy + f(x^2 + y^2)$

Solution: Given, $z = xy + f(x^2 + y^2)$ (1)

Differentiating partially equation (1) with respect to x and y we get,

$$p = y + f'(x^2 + y^2) \cdot 2x \quad (2)$$

$$q = x + f'(x^2 + y^2) \cdot 2y \quad (3)$$

Eliminating $f'(x^2 + y^2)$ from equations (2) and (3) we get,

$$(p - y)y = (q - x)x$$

$$py - y^2 = qx - x^2$$

$$py - qx = y^2 - x^2$$

Which is a required equation.

Example 2.19: Eliminate the arbitrary functions f and ϕ from the relation $z = f(x + ay) + \phi(x - ay)$

Solution: Differentiating partially with respect to x and y we get,

$$p = f'(x + ay) + \phi'(x - ay) \quad (1)$$

$$q = af'(x + ay) - a\phi'(x - ay) \quad (2)$$

Differentiating these again, with respect to x and y we get,

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x + ay) + \phi''(x - ay) \quad (3)$$

$$\frac{\partial^2 z}{\partial y^2} = t = a^2 f''(x + ay) + a^2\phi''(x - ay) \quad (4)$$

NOTES

From equations (3) and (4) we get,

$$t = a^2 r$$

2.5.1 Equations Solvable by Direct Integration

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $\varphi(x, y, z, a, b) = 0$, is the complete integral of a partial differential equation, then the eliminant of a and b from the equations $\frac{\partial\phi}{\partial a} = 0$, $\frac{\partial\phi}{\partial b} = 0$, is called singular integral.

Let us consider four standard types of nonlinear partial differential equations and the procedure for obtaining their complete solution.

Type I: Equations of the form $F(p, q) = 0$. In this type of equations we have only p and q and there is no x, y and z. To solve this type of problems, let us assume that $z = ax + by + c$ be the solution and then proceed as in the following examples.

Example 2.20: .Solve $p^2 + q^2 = 4$

Solution: Given, $p^2 + q^2 = 4$ (1)

Let us assume that $z = ax + by + c$ be a solution of equation (1). (2)

Partially differentiating equation (1) with respect to x and y, we get,

$$\frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b \quad (3)$$

Substituting equation (3) in (1) we get,

$$a^2 + b^2 = 4 \quad (4)$$

To get the complete integral we have to eliminate any one of the arbitrary constants from equation (2).

From equation (4) we get,

$$b = \pm\sqrt{4 - a^2} \quad (5)$$

Substituting equation (5) in (2) we get,

$$z = ax \pm y\sqrt{4 - a^2} + C \quad (6)$$

Which contains only two constants (equal to number of independent variables). Therefore, it gives the complete integral.

NOTES

To check for Singular Integral:

Differentiating equation (6) partially with respect to a and c and equating to zero, we get,

$$\frac{\partial z}{\partial a} = x \pm \frac{1}{2\sqrt{4-a^2}}(-2a) = 0 \qquad (7)$$

and, $$\frac{\partial z}{\partial c} = 1 = 0$$

Here, $1 = 0$ is not possible.

Hence, there is no singular integral.

Example 2.21: Solve $p^2 + q^2 = npq$

Solution. The solution is, $z = ax + by + c$, where $a^2 + b^2 = nab$

Solving, $b = \dfrac{a\left(n \pm \sqrt{n^2-4}\right)}{2}$

The compete integral is,

$$z = ax + \frac{ay}{2}\left(n \pm \sqrt{n^2-4}\right) + c$$

Differentiating partially with respect to c, we see that there is no singular integral, as we get an absurd result.

Example 2.22: Solve $p + q = pq$

Solution: This equation is of the type, $F(p, q) = 0$.

$\therefore$ The complete solution is of the form, $z = ax + by + c$ (1)

Differentiating equation (1) partially with respect to x and y we get,

$$p = a,\ q = b$$

Therefore, the given equation becomes,

$$a + b = ab$$

$$a = b(a-1); \qquad b = \frac{a}{a-1}$$

Therefore, the complete solution is,

$$z = ax + \left(\frac{a}{a-1}\right)y + c$$

This type of equation has no singular solution.

Let, $c = \varphi(a)$

$$z = ax + \left(\frac{a}{a-1}\right)y + \phi(a) \qquad (2)$$

NOTES

Differentiating partially with respect to *a,*

$$0 = x + \left[\frac{(a-1)1-a}{(a-1)^2}\right] y + \phi'(a)$$

$$0 = -\frac{1}{(a-1)^2} y + \phi'(a) \quad (3)$$

The elimination of *a* between equations (2) and (3) gives the general solution.

Type II: Equation of the form $z = px + qy + F(p, q)$ (Clairaut's form). In this type of problems assume that, $z = ax + by + F(a, b)$ be the solution.

Example 2.23: Solve $z = px + qy + ab$

Solution: This equation is of Clairaut's type. Therefore, the complete solution is obtained by replacing *p* by *a* and *q* by *b,* where *a* and *b* are arbitrary constants.

i.e., the complete solution is, $z = ax + by + ab$ (1)

Differentiating equation (1) partially with respect to *a* and *b,* and equating these to zero we get,

$$0 = x + b \quad (2)$$

$$0 = y + a \quad (3)$$

Eliminating *a* and *b* from equations (1), (2) and (3) we get,

$$z = -xy - xy + xy$$

i.e., $z + xy = 0$

This gives the singular solution of the given partial differential equation and to get the general solution.

Put, $b = \phi(a)$ in equation (1)

$\therefore$ $z = ax + \phi(a)y + a\phi(a)$ (4)

Differentiating partially with respect to *a* we get,

$$0 = x + \phi'(a)y + a\phi'(a) + \phi(a) \quad (5)$$

Eliminating *a* from equations (4) and (5) we get the general solution.

Example 2.24: Obtain the complete solution and singular solution of,

$$z = px + qy + p^2 + pq + q^2$$

Solution: This equation is of Clairaut's form. Therefore, the complete solution is,

$$z = ax + by + a^2 + ab + b^2 \quad (1)$$

Where, *a* and *b* are arbitrary constants.

Differentiating equation (1) partially with respect to *a* and *b* we get,

$$0 = x + 2a + b \quad (2)$$

NOTES

$$0 = y + 2b + a$$

$$2x - y = 3a, \text{ and } 2y - x = 3b$$

$$a = \frac{2x - y}{3}, \; b = \frac{2y - x}{3}$$

Substituting this in equation (1) we get,

$$z = \left(\frac{2x-y}{3}\right)x + \left(\frac{2y-x}{3}\right)y + \left(\frac{2x-y}{3}\right)^2$$

$$+ \frac{(2x-y)(2y-x)}{9} + \left(\frac{2y-x}{3}\right)^2$$

Simplifying we get, $3z = xy - x^2 - y^2$. This is the singular solution.

To find singular integral:

Differentiating equation (2) partially with respect to a and b, and then equating to zero, we get,

$$\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}} = 0 \quad (3)$$

$$\frac{\partial z}{\partial b} = y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \quad (4)$$

From equation (3), we get,

$$x^2 = \frac{a^2}{1+a^2+b^2} \quad (5)$$

From equation (4), we get,

$$y^2 = \frac{b^2}{1+a^2+b^2} \quad (6)$$

From equations (5) and (6) we get,

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$\therefore \quad 1 - (x^2 + y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1}{1+a^2+b^2}$$

i.e., $$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$$

$$\therefore \quad \sqrt{1+a^2+b^2} = \frac{1}{1-x^2-y^2} \quad (7)$$

NOTES

Substituting equation (7) in (3) and (4) we get,

$$a = \frac{-x}{\sqrt{1-x^2-y^2}},\ b = \frac{-y}{\sqrt{1-x^2-y^2}} \qquad (8)$$

Substituting equations (7) and (8) in (2) we get,

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$\therefore \quad z = \sqrt{1-x^2-y^2}$ or, $z^2 = 1 - x^2 - y^2$

$\therefore \quad x^2 + y^2 + z^2 = 1$

This is the singular integral.

Type III: Equation of the form, $F(z, p, q) = 0$

Example 2.25: Solve $z = p^2 + q^2$

Solution: Given, $z = p^2 + q^2$ (1)

Assume that, $z = f(x + ay)$ is a solution of equation (1). (2)

Put, $x + ay = u$ in equation (2)

Then, $z = f(u)$ (3)

Partially differentiating equation (3) with respect to x and y we get,

$$p = \frac{dz}{du},\ q = a\frac{dz}{du} \qquad (4)$$

$$\left(\because \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\right)$$

Substituting equation (4) in (1) we get,

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

i.e., $$\left(\frac{dz}{du}\right)^2 (1+a^2) = z$$

i.e., $$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

NOTES

i.e., $$\frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}} \quad (5)$$

Integrating equation (5) we get,

$$\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} \int du$$

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

i.e., $$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + b$$

This gives the complete integral.

Example 2.26: Solve $ap + bq + cz = 0$

Solution: Given, $ap + bq + cz = 0$ (1)

Let us assume that, $z = f(x + ky)$ (2)

By the solution of equation (2).

Put $x + ky = u$ in equation (2)

$\therefore$ $$z = f(u) \quad (3)$$

$$p = \frac{dz}{du};\ q = k\frac{dz}{du} \quad (4)$$

Substituting equation (4) in (5) we get,

$$a \cdot \frac{dz}{du} + b \cdot k\frac{dz}{du} + c \cdot z = 0$$

i.e., $$\frac{dz}{du}(a+bk) = -cz$$

$\therefore$ $$\frac{dz}{du} = -\frac{cz}{a+bk}$$

i.e., $$\frac{dz}{z} = -\frac{c}{a+bk}du \quad (5)$$

Integrating equation (5) we get,

$$\int \frac{dz}{z} = -\frac{c}{a+bk}\int du$$

$$\log z = -\frac{c}{a+bk}(u) + \log b$$

i.e., $$\log z = A[x+ky] + \log b, \quad \text{where } A = -\frac{c}{a+bk}$$

NOTES

i.e., $\log z - \log b = A(x + ky)$

$$\log\left(\frac{z}{b}\right) = A(x + ky)$$

$$\frac{z}{b} = e^{A(x+ky)}$$

$$\therefore \quad z = be^{A(x+ky)}$$

This gives the complete integral.

Type IV: Equation of the form, $F_1(x, p) = F_2(y, q)$

Example 2.27: Solve the equation, $p + q = x + y$

Solution: We can write the equation in the form, $p - x = y - q$

Let, $p - x = a$, then $y - q = a$

Hence, $p = x + a$, $q = y - a$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$= (x + a)dx + (y - a)dy$$

On Integrating,

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b$$

There is no singular integral and the general integral is found as usual.

Example 2.28: Solve $p^2 + q^2 = x + y$

Solution: Given, $p^2 + q^2 = x + y$

$$p^2 - x = y - q^2 = k$$

$$\therefore \quad p^2 - x = k;\ y - q^2 = k$$

$$p = \pm\sqrt{x+k}\ ,\ q = \pm\sqrt{y-k}$$

$$dz = pdx + qdy$$

$$= \pm(\sqrt{x+k})dx \pm (\sqrt{y-k})dy$$

Integrating we get the complete solution.

$$z = \pm\frac{2}{3}(x+k)^{3/2} \pm \frac{2}{3}(y-k)^{3/2} + C$$

$$= \pm\frac{2}{3}\left[(x+k)^{3/2} + (y-k)^{3/2}\right] + C$$

Example 2.29: Solve $p + q = \sin x + \sin y$

NOTES

Solution:

$$p - \sin x = \sin y - q = k$$

$$\therefore \quad p = k + \sin x;\ q = \sin y - k$$

$$dz = pdx + qdy$$

$$= (k + \sin x)dx + (\sin y - k)dy$$

On integrating, we get,

$$z = (kx - \cos x) - (ky + \cos y) + C$$

$$z = k(x - y) - (\cos x + \cos y) + C$$

This is the complete solution.

2.6 APPLICATIONS TO MECHANICS, ELECTRICAL CIRCUITS AND BEAMS

In this section, we will study the applications of Laplace transform to mechanics, electrical circuits and beams.

Electric Circuit

Considers an electric circuit considering of a resistance R, inductance L, a condenser of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit.

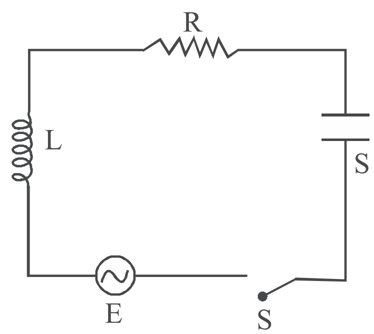


Fig. 2.3 *Electric Circuit*

Here

$$i = \frac{dq}{dt}$$

Voltage developed are

$$\text{Ri}, \ \frac{Ldi}{dt} \text{ and } \frac{q}{c}$$

By Kirchhoff's law,

$$\frac{Ldi}{dt} + \text{Ri} + \frac{q}{c} = \text{E}. \qquad \text{...(2.3)}$$

We can apply/use the Laplace transform to solve the differential equation arises in electronic circuit. Where we first take the Laplace transform of (2.3) and

NOTES

this results (1) will be converted into an algebraic equation in Laplace transform of i and by the use of inverse Laplace transform may retrieve the solution i. This can be better understand by the following adjoined examples.

Example 2.30: A resistance R is a series with inductance L is connected with e.m.f $E(t)$. The current i is give by

$$\frac{Ldi}{dt} + Ri = E$$

If the switch is connected at $t = 0$ and disconnected at $t = a$ then find the current i in terms of t.

Solution:

Given condition are

$$E(t) = \begin{cases} E & 0 < t < a \\ 0 & t > a \end{cases}$$

and current i = a at $t = 0$, i.e., $i(0) = 0$.

Given equation of circuit is

$$\frac{Ldi}{dt} + Ri = E(t) \qquad ...(1)$$

Taking Laplace transform of (1) we get

$$L\left[s\bar{i} - i(0)\right] + R\bar{i} = \int_0^\infty E(t)e^{-st} \text{ at}$$

(Where $\bar{i} = L(i)$ = Laplace transform of i)

or $$Ls\bar{i} + R\bar{i} = \int_0^a e^{-st}.Edt + \int_a^{es} e^{-st}.0dt$$

$$Ls\bar{i} + R\bar{i} = \int_0^a e^{-st}.Edt$$

or $$(Ls + R)\bar{i} = E\left(\frac{e^{-st}}{-s}\right)_0^a$$

or $$(Ls + R)\bar{i} = \frac{E}{s}(1 - e^{-as})$$

$\Rightarrow$ $$\bar{i} = \frac{E(1 - e^{-as})}{s(Ls + R)}$$

or $$\bar{i} = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

NOTES

Taking inverse Laplace transform we have

$$L^{-1}(\bar{i}) = i = L^{-1}\left(\frac{E}{s(Ls+R}\right) - L^{-1}\left(\frac{Ee^{-as}}{s(Ls+R)}\right)$$

or $$i(t) = \frac{E}{R}\left(1 - e^{\frac{-R}{L}t}\right) - \frac{E}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)$$

Where $u(t-a) = \begin{cases} 1 & t > a \\ 0 & 0 < t < a \end{cases}$ $\quad u(t-a)$

or $$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{for } 0 < t < a$$

and for $t > a$

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) - \frac{E}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)$$

$$= \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t}\right]$$

for $t > a$

$$i(t) = \frac{E}{R} e^{-\frac{R}{L}t}\left(e^{\frac{R}{L}a} - 1\right)$$

Example 2.31: Find the current $i(t)$ in the LC-circuit (given in adjoining figure) using Laplace transform. Given $L = 1$ Henry, $C = 1$ ford, zero initial current and charge on the capacitor and

$$V(t)\begin{cases} t & \text{when} \quad 0 < t < 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

Solution:

The differential equation for the LC circuit is given by

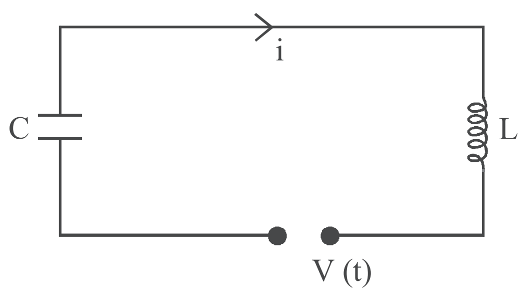


Fig 2.4 *LC Circuit*

NOTES

$$L\frac{d^2q}{dt^2}+\frac{q}{c}=E$$

Given $L = 1$, $C = 1$ and $E = V(t)$ we get

$$\frac{d^2q}{dt^2}+q'=V(t) \qquad ...(1)$$

Taking Laplace transform of (1)

$$L\left(\frac{d^2q}{dt^2}\right)+L(q)=L(V(t))$$

Denote $L(q)=\overline{q}$ then we have

$$s^2\overline{q}-sq(0)-q'(0)+\overline{q}=\int_0^\infty v(t)e^{-st}dt \qquad ...(2)$$

It is given that $q(0) = 0$, $i(0) = q'(0) = 0$

So (2) becomes

$$s^2\overline{q}-\overline{q} = \int_0^\infty v(t)e^{-st}dt$$

$$= \int_0^1 e^{-st}.t\ ds+\int_1^\infty o.e^{-st}dt$$

$$= \int_0^1 te^{-st}dt = \frac{te^{-st}}{-s}\Big|_0^1-\int_0\left(\frac{-e^{-st}}{-s}\right).1.dt$$

$$s^2\overline{q}-\overline{q} = \frac{-1}{s}\left(e^{-s}-0\right)+\frac{1}{s}\int_0^1 e^{-st}dt$$

$$= \frac{1}{s}+\frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^1$$

$$= -\frac{1}{s}e^{-s}\frac{1'}{s^2}\left(-e^{-s}+1\right)$$

Or $(s^2-1)\overline{q} \qquad = \dfrac{-1}{s}e^{-s}+\dfrac{1}{s^2}(1-e^{-s})$

$$\Rightarrow \overline{q}=\frac{1}{s^2+1}\left[\frac{-e^{-s}}{s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right]$$

Or $\overline{q}=\dfrac{-e^{-s}}{s(s^2+1)}-\dfrac{e^{-s}}{s^2(s^2+1)}+\dfrac{1}{s^2(s^2+1)}$

Taking inverse Laplace transform we get

NOTES

$q + L^{-1}(\overline{q})$ and using Linearity property

Or $q = L^{-1}\left(\dfrac{-e^{-s}}{s\left(s^e+1\right)}\right) - L^{-1}\left(\dfrac{e^{-s}}{s^2(s^2+2}\right)$

$$+ L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) \quad ...(3)$$

We know that $L^{-1}(e^{-as}(s)) = f(t-a)\,u(t-a)$

$\Rightarrow$ Since $L^{-1}\left(\dfrac{1}{s(s^2+2)}\right) = \int_0^t \sin t dt$

$$= 1 - \cos t$$

$$L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \int_0^1 (1-\cos t)dt = t - \sin t$$

$$L^{-1}\left(\frac{-e^{-s}}{s(s^2+1)}\right) = [1-\cos(t-1)u(t-1)$$

and $L^{-1}\left(\dfrac{e^{-s}}{s(s^2+1)}\right) = \left[(t-1)-\sin(t-1)u(t-1)\right].$

puting that values of inverse laplace transfrom in (3) we get

$q = [1-\cos(t-1)]u(t-1) - [(t-1)-\sin(t-1)]u(t-1) + t - \sin t$

for $0 < t < 1$ where $u(t-1) = \begin{cases} 1 & t \geq a \\ 0 & 0 < t < a \end{cases}$

$u(t-1) = 0$ (in the unit step function)

then $q(t) = t - \sin t$

and for $t > 1$, $u(t-1) = 1$ in that case

$$q(t) = -[1-\cos(t-1)] - [(t-1) - \sin(t-1)] + t - \sin t$$

thus the current $i(t) = \dfrac{da}{dt}$

for $0 < t < 1$

$i(t) = 1 - \cos t$

and for $t > 1$

$$i(t) = -\sin(t-1) - 1 + \cos t\,(t-1) + 1 - \cos t$$

$$i(t) = \cos(t-1) - \sin(t-1) - \cos t$$

Similarly we can solve the differential equation arising in mechanics and beams.

Now consider application of Laplace transform in beam.

Beam: A bar whose length in much greater than its cross-section and its thickness is called a beam. There are various types of beams. defined so for. We explain few of them.

If a beam may just rest a support like a knife edge then it in called supported beam.

It one or both edges of a beam are firmly fixed then it is called fixed beam.

If one end of a beam in fixed and the other end is loaded, then it is called cantilever.

Bending of Beam: Let a beam be fixed at one end and the other is loaded. The upper surface is elongated and therefore under tension and lower surface is shortened so under compression. Wherever beam is loaded it deflects from its original position. If M is the bending moment of the forces acting on it, then

$$\text{M} = \frac{ER}{R}$$

Where E = Modules of classify of the beam

I = Moment of inertia of the cross-section of beam about neutral areas

R = Radius of curvature of the curved beam

so, $$\text{R} = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{1}{\frac{d^2y}{dx^2}}\left(\text{neglecting } \frac{dy}{dx}\right)$$

Thus $$\text{M} = EI\frac{d^2y}{dx^2}$$

Boundary Condition

(i) **End freely supported:** At the freely supported end 0, there will be no deflction and no bending moment

i.e., $y = 0, \frac{d^2y}{dx^2} = 0$

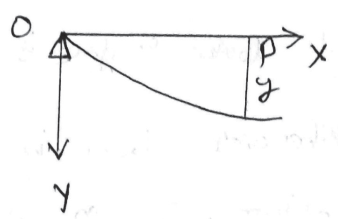


Fig 2.5 *End Freely Supported*

NOTES

(ii) **Fixed end horizontally:** Deflection and slope of the beam are zero

$$y = 0 \text{ and } \frac{dy}{dx} = 0$$

Fig 2.6 *Fixed End Horizontally*

(iii) **Perfectly free end:** At the free end there is no bending moment or shear force, then

$$\frac{d^2y}{dx^2} = 0, \ \frac{d^3y}{dx^3} = 0$$

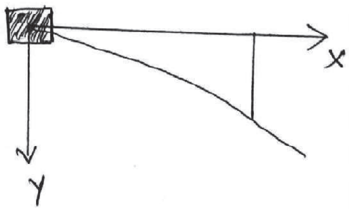


Fig 2.7 *Perfectly Free End*

Example 2.32: The differential equation satisfied by a beam uniformly loaded (Wkg/meter), with one end fixed and the other end tensile force P, is given by

$$\text{E.I} \frac{d^2y}{dx^2} = Py - \frac{1}{2}Wx^2$$

Find the elastic curve for the beam subject to the conditions:

$$y = 0 = \frac{dy}{dx} \text{ at } x = 0$$

Solution:

Given $\quad \text{E.I} \dfrac{d^2y}{dx^2} = Py - \dfrac{1}{2}Wx^2 \qquad ...(1)$

taking Laplace transform both sides (denote $L(y) = \overline{y}$)

$$EI\left(s^2\overline{y} - sy(0) - y^1(0)\right) = p\overline{y} - \frac{1}{2}W.\frac{2!}{s^3}$$

Or $\quad EI\left(s^2\overline{y} - s.0 - 0\right) = p\overline{y} - \dfrac{W}{s^3} \quad$ (using the given conditions)

Or $\quad EIs^2\overline{y} = P\overline{y} - \dfrac{W}{s^3}$

NOTES

Or $$(EI\ s^2 - p)\overline{y} = -\frac{W}{s^3}$$

Or $$\overline{y} = \frac{-W}{s^3(EI\ s^2 - P)} \qquad ...(2)$$

Taking inverse Laplace transform of (2) curve y can be obtained.

i.e., $$y = L^{-1}(\overline{y}) = L^{-1}\left(\frac{W}{s^3(EI\ s^2 - p)}\right)$$

$$= L^{-1}\left[\frac{W}{s^3(EI\ s^2 - p)}\right]$$

Consider

$$\frac{W}{s^3(EI\ s^2 - p)} = \frac{W}{EI\ s^3 - \frac{p}{EI}}$$

$$= \frac{W}{EI}\frac{1}{s^3\left(s - \sqrt{\frac{p}{EI}}\right)\left(s + \sqrt{\frac{p}{EI}}\right)}$$

$$= \frac{W}{Ei}\left\{\frac{c_1}{s^3} + \frac{c_2}{s^2} + \frac{c_3}{s} + \frac{a_1}{\left(s - \sqrt{\frac{p}{EI}}\right)} + \frac{a_2}{\left(s + \sqrt{\frac{p}{EI}}\right)}\right\}$$

$$c_1 = \left[\frac{1}{\left(s^2\ \frac{p}{EI}\right)}\right]_{s=0} = \frac{-1}{\frac{P}{EI}}$$

$$c_2 = \left[\frac{d}{ds}\left(\frac{1}{s^2 - \frac{p}{EI}}\right)\right]_{s=0} = \left[\frac{-(1)(2s)}{\left(s^2 - \frac{p}{EI}\right)^2}\right]_{s=0} = 0$$

$$c_3 = \left[\frac{d^2}{ds^2}\left(\frac{1}{\left(s^2\ \frac{p}{EI}\right)}\right)\right]_{s=0}$$

$$= \left[\frac{d}{ds}\left(\frac{2s}{\left(s^2\ \frac{p}{EI}\right)^2}\right)\right]_{s=0}$$

NOTES

$$c_3 = \left[\frac{2}{\left(s^2 - \frac{p}{EI}\right)^2}\right]_{s=0} + 0 = -\frac{2}{\left(\frac{p}{EI}\right)^2}$$

$$a_1 = \frac{p\left(\sqrt{\frac{p}{EI}}\right)}{q^1\left(\sqrt{\frac{p}{EI}}\right)}$$

Where $p(s) = 1$

$$q(s) = s^3\left(s^2 - \frac{p}{EI}\right)$$

$$q'(s) = 5s^4 - \frac{3s^2 p}{EI}$$

$$= \frac{1}{5\left(\frac{p}{EI}\right)^2 - 3\left(\frac{p}{EI}\right).\frac{p}{EI}}$$

$$= \frac{1}{2\left(\frac{p}{EI}\right)^2}$$

$$a_2 = p\left(-\sqrt{\frac{p}{EI}}\right)\Big/ q'\left(-\sqrt{\frac{p}{EI}}\right)$$

$$a_2 = 1\Big/ 2\left(\frac{p}{EI}\right)^2$$

Thus $$\frac{W}{s^3(EIs^2 - p)} = \frac{W}{EI}\left[-\frac{1}{\left(\frac{p}{EI}\right)}.\frac{1}{s^3} + 0 + \frac{(-2)}{\left(\frac{p}{EI}\right)}.\frac{1}{s} + \right.$$

$$\left.\frac{1}{2\left(\frac{p}{EI}\right)^2}.\frac{1}{\left(s - \sqrt{\frac{p}{EI}}\right)} + \frac{1}{2\left(\frac{p}{EI}\right)^2}.\frac{1}{\left(s + \sqrt{\frac{p}{EI}}\right)}\right]$$

Hence $y = L^{-1}(\overline{y})$

$$= -\frac{W}{EI}\left\{\frac{1}{\left(\frac{p}{EI}\right)}.\frac{x^2}{2!} + \frac{(-2)}{\left(\frac{p}{EI}\right)^2}.1 + \frac{1}{2\left(\frac{p}{EI}\right)^2}\left(e^{+\sqrt{\frac{p}{EI}}x} + e^{-\sqrt{\frac{p}{EI}}x}\right)\right\}$$

NOTES

$$= \frac{-W}{EI}\left\{\frac{-x^2}{2\left(\frac{p}{EI}\right)} + \frac{(-2)}{\left(\frac{p}{EI}\right)^2} + \frac{1}{\left(\frac{p}{EI}\right)^2}\cosh\sqrt{\frac{p_k}{EI}}\right\}$$

Put $\quad n^2 = \frac{p}{EI}\quad$ then $\quad$ (i.e., $n^2\,EI = p$)

$$y = \frac{-W}{p/n^2}\left[\frac{1}{n^4}\cosh nx - \frac{2}{n^4} - \frac{x^2}{2n^2}\right]$$

$$= \frac{Wx^2}{2p} + (2-\cosh nx)\frac{W}{pn^2}$$

$$y(x) = \frac{Wx^2}{2p} + \frac{W}{pn^2}(2-\cosh nx)$$

Application to Mechanices

Example 2.33: Consider the mechanical system in (Figure 2.3) consists of two bodies of mass 1 on these springs and is governed by the differential equations.

$$u'' = ku + R(v - u)$$
$$v'' = -R\,(v - u) - Rv$$

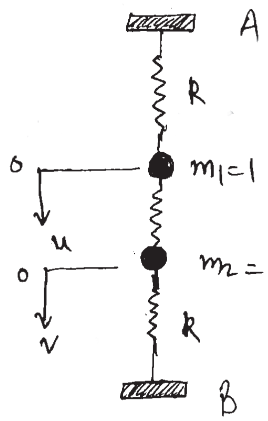


Fig 2.8 *Mechanical system of two bodies*

Where k is the spring constant of each of the three springs and u and v are the displacements of the bodies from their position of static equilibrium, the masses of the springs and damping are neglected.

We shall determine the solution corresponding to the initial conditions $u(0) = 1$, $u'(0) = \sqrt{3R}$ and $v(0) = 1$, $v'(0) = -\sqrt{3k}$

Donote $\bar{u} = L(u)$ and $\bar{v} = L(v)$ and taking the Laplace transform of given system of differential equations, we get,

NOTES

$$\begin{cases} s^2\bar{u} - su(0) - u'(0) = -k\bar{u} + k(\bar{v} - \bar{u}) \\ s^2\bar{v} - su(0) - v'(0) = -k(\bar{v} - \bar{u}) - k\bar{v} \end{cases}$$

Or
$$\begin{cases} s^2\bar{u} - s = -k\bar{u} + k(\bar{v} - \bar{u}) \\ s^2\bar{v} - s = -k(\bar{v} - \bar{u}) - k\bar{v} \end{cases}$$

Solving this system for $\bar{u}$ and $\bar{v}$ we get,

$$\bar{u} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$\bar{v} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}$$

Hence by use of inverse laplace transform, the solutions is given by

$$\begin{cases} u = L^{-1}(\bar{u}) = \cos\sqrt{k}t + \sin\sqrt{k}t \\ v = L^{-1}(\bar{v}) = \cos\sqrt{k}t - \sin\sqrt{k}t \end{cases}$$

From the solution it is obvious that the motion of each mass is harmonic (the system in undamped).

2.7 INTEGRAL EQUATIONS OF CONVOLUTION TYPE

Consider the integral equations of Volterra type (of fifth kind).

$$f(x) = \int_0^x k(x-t)y(t)dt \qquad ...(2.3)$$

Taking Laplace transform of (2.3) both sides

$$L(f(x) = L\left\{\int_0^x k(z-t)y(t)dt\right\}$$

Or
$$f(s) = k(s)\, y(s) \quad \Rightarrow \quad y(s) = \frac{F(s)}{K(s)} \qquad ...(2.4)$$

By inverse Laplace transform one can

Where

$$\begin{cases} L(f(u)) = f(s) & \text{find by (Equation 2.4)} \\ L(k(x)) = k(s) & y(x) = L^{-1}(y(s) \\ L(y(x)) = 1/(s) & \end{cases}$$

Similarly the Laplace transform method is applicable to the Volterra integral equation of 2^{nd} kind with a convolution type kernel.

NOTES

Consider a non-homogeneous integral equation of 2[nd] kind as

$$y(x) = f(x) + \int_0^x k(x-t)y(t)dt \quad ...(2.5)$$

Taking Laplace transform of back sides to (2.5) we get,

$$L(y(x)) = L(f(x)) + L\left(\int_0^x k(x-t).y(t)dt\right)$$

Using convolution theorem we get,

$$y(s) = F(s) + K(s)y(s)$$

Or $$y(s)\ (1 - k(s)) = f(x)$$

$$\Rightarrow \quad y(s) = \frac{f(s)}{1-k(s)} \quad ...(2.6)$$

And again by use of inverse Laplace transfrom we get,

$$y(x) = L^{-1}\left(\frac{f(s)}{1-k(s)}\right) \quad ...(2.7)$$

The resolvant kernel of the non-homogeneous integral equation can also be determined by the method of Laplace transfrom.

Let $k(x, t)$ is defined as difference kernel and so is the resolvent kernel.

Since the resolvent karnel $k(x, t)$ is the sum of the iterated kernel and they all depend on the difference $(x - t)$ then

$$y(t) = k_2(x,t) = \int_t^x k(x-z)\, k(z-t)dz$$

Let $$z - t = u \Rightarrow z = u + t$$

$$\Rightarrow dz = du$$

thus $$k^2(x,\ t) = \int_0^{x-t} k(x-t-u)k(u)du \quad ...(2.8)$$

The other integrals can be determined similarly. Thus we can obtain the resolvent kernel.

$$k(x,\ t,\ \lambda) = \sum_{\nu=1}^{\infty} \lambda^{\nu} k_{\nu+1}(x,t)$$

With the help of resolvent kernel we can find the solution of integral equation (2.5) as

$$y(x) = f(x) + \int_0^x k(x-t) f(t)\, dt$$

Taking the Laplace transform of (2.9) both sides

$$L(y_{(x)}) = L\big(f(x)\big) + L\left(\int_0^x k\big(x-t\big) f(t)\, dt\right)$$

NOTES

$$\text{or} \qquad Y(s) = F(s) + \bar{k}(s)F(s)$$

$$\left(where\ \bar{k}(s) = L(k(x-t)) \right)$$

$$\text{or} \qquad \frac{F(s)}{1-k(s)} = F(s) + \bar{k}(s)\,F(s)$$

$$\text{or} \qquad \left(1+\bar{k}(s)\right)F(s) = \frac{F(s)}{1-K(s)}$$

$$\text{or} \qquad \bar{k}(s) = \frac{1}{1-k(s)} - 1$$

$$\text{or} \qquad \bar{k}(s) = \frac{K(s)}{1-K(s)} \qquad \text{...(2.10)}$$

and thus the resolvent kernel $k(x,t) = L^{-1}\left(\bar{k}(s)\right)$, i.e., from (2.9) we can recover the resolvent kernel.

Now we consider some example.

Example 2.34

Solve the integral equation.

$$x = \int_0^x e^{x-t}\, y(t)\, dt$$

Solution: The given integral equation can be writen as

$$x = y(x) * e^x \qquad \text{...(1)}$$

where * denotes the convolution product. Taking Laplace transform of (1) we get.

$$\frac{1}{s^2} = L(y(x)).L(e^x)$$

$$\frac{1}{s^2} = Y(s) = \frac{1}{s-1}$$

$$\text{or} \qquad Y(s) = \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}$$

$$\text{thus} \qquad Y(s) = \frac{1}{s} - \frac{1}{s^2}$$

Taking inverse Laplace transform one can get solution of given integral equation as

$$y(x) = L^{-1}(y(s)) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^2}\right)$$

$$y(x) = 1 - x$$

NOTES

Example 2.35 Find the solution of the integral equation.

$$\sin x = \int_0^x J_0(x-t)y(t)\,dt$$

Solution: Taking Laplace transform of given integral equation both sides.

$$L(\sin x) = L\left(\int_0^x J_0(x-t)y(t)\,dt\right)$$

and using convolution theorem we get.

$$\frac{l}{s^2+1} = Y(s).\frac{1}{\sqrt{s^2+1}} \quad \left(\because L\left(J_0(x)\right) = \frac{1}{\sqrt{s^2+1}}\right)$$

or $$Y(s) = \frac{1}{\sqrt{s^2+1}}$$

taking inverse Laplace we get solution $y(x)$

$$y(x) = L^{-1}(Y(s)) = J_0(x).$$

Example 2.36 Solve the integral equation of convolution $\int_0^x y(t)\,y(x-t)\,dt = 16\sin 4x$

Solution: Taking the Laplace transform of given integral, in view of convolution theorem, we have

$$Y(s)\,Y(s) = 16\frac{4}{s^2+16}$$

(where $$L(y(x)) = Y(s))$$

or $$Y(s) = \pm\sqrt{\frac{64}{s^2+16}}$$

or $$Y(s) = \pm\frac{8}{\sqrt{s^2+4}}$$

by inverse Laplace transform the desired solution of given integral equation is given by

$$y(x) = L^{-1}\left(\pm\frac{8}{\sqrt{s^2+4}}\right)$$

or $$y(x) = \pm\,8\,J_0(4x)$$

Example 2.37: Solve the following inhomogeneous integral equations.

$$\text{(a)}\,y(x) = 1-\int_0^x (x-t)y(t)\,dt$$

$$\text{(b)}\,y(x) = 1+\int_0^x \sin(x-t)y(t)\,dt$$

$$(c)y(x) = \frac{s \sin \pi d}{\pi}.L\left(x^{\alpha-1} * f(t)\right)$$

$$(d)y(x) = e^{-x} - 2\int_0^1 \cos(x-t)y(t)\,dt$$

NOTES

Solution: (a) Given integral equation is (a) can be written (in term of convolution product *)

$$y(x) = 1 - y(x) * x \qquad ...(1)$$

taking Laplace transform of (1) bron side.

we get,

$$Y(s) = \frac{1}{s} - y(s)\frac{1}{s^2}$$

or $$\left(1+\frac{1}{s^2}\right)Y(s) = \frac{1}{s}$$

or $$Y(s) = \frac{s}{s^2+1}$$

(b) By taking inverse Laplace transform we get the solution as

$$y(x) = L^{-1}(Y(s)) = L^{-1}\left(\frac{s}{s^2+1}\right)$$

or $$y(x) = \cos x$$

(c) Given equation is given as

$$y(x) = x + 2y(x) * \cos x \qquad ...(2)$$

taking Laplace transform of (2) we get

$$Y(s) = \frac{1}{s^2} + 2Y(s)\frac{s}{s^2+1}$$

or $$\left(1-\frac{2s}{s^2+1}\right)Y(s) = \frac{1}{s^2}$$

or $$Y(s) = \frac{s^2+1}{s^2(s)}$$

or $$Y(s) = \frac{1}{(s-1)^2} + \frac{1}{s^2(s-1)^2}$$

(d) By Taking inverse Laplace transform we get.

$$y(x) = xe^x + (x-2)e^x + x + 2$$

or $$y(x) = 2e^x(x-1) + x + 2$$

Parts (*b*) and (*d*) can be solved is the same way. Let *f* as an exercise to the reader.

2.8 ABEL'S INTEGRAL EQUATION

NOTES

The following Abel integral equation.

$$f(x) = \int_0^x \frac{y(t)}{(x-t)^{\alpha}}\,dt \qquad 0 < \alpha < 1 \quad ...(2.11)$$

This (2.11) can be rewritten as

$$f(x) = y(x) * x^{-\alpha} \quad ...(2.12)$$

taking Laplace transform of (2.12) we get

$$F(s) = Y(s)\frac{\sqrt{1-\alpha}}{s^{1-\alpha}}$$

or $$Y(s) = \frac{F(s^{1-\alpha})}{\sqrt{1-\alpha}} = \frac{F(s)s}{\sqrt{a}\sqrt{1-\alpha}}\left(\sqrt{\alpha}s^{-\alpha}\right)$$

or $$Y(s) = \frac{s}{\sqrt{\alpha}\sqrt{1-\alpha}}\sqrt{\alpha}s^{-\alpha}F(s)$$

or $$Y(s) = \frac{s}{(\pi/\sin\pi\,\alpha)}L\left(x^{\alpha-1} * f(t)\right)$$

or $$Y(s) = \frac{s\sin\pi d}{\pi}.L\left(x^{\alpha-1} * f(t)\right)$$

or $$Y(s) = \frac{\alpha\sin\pi\,\alpha}{\alpha}.\text{ and. }L\left(\int_0^x (x-t)^{\alpha-1}F(t)\,dt\right)$$

denote $$g(x) = \int_0^x (x-t)^{\alpha-1} f(t)\,dt$$

then $$g(0) = 0$$

thus $$Y(s) = \frac{\sin\pi\,\alpha}{\alpha}sL(g(x)) \quad ...(2.13)$$

We know that

$$L(g'(x)) = sL\left(g(x)\right) - g'(0)$$

$$L(g'(x)) = s\,L(g(x))$$

or $$L(g(x)) = \frac{1}{s}L\left(g'(u)\right)$$

From (2.13) we get

$$Y(s) = \frac{\sin\pi\alpha}{\alpha}L\left(g'(x)\right)$$

NOTES

or $$L(y(x)) = \frac{\sin \pi\alpha}{\alpha} L\left(\frac{d}{dx}(g(x))\right)$$

or $$y(x) = \frac{\sin \pi\alpha}{\alpha}\left(\frac{d}{dx} g(x)\right)$$

or $$y(x) = \frac{\sin \pi\alpha}{\alpha}\frac{d}{dx}\left(\int_0^x (x-t)^{a-1} F(t)\, dt\right)$$

is the desired solution.

But if the Abel integral equation is given the following form:

$$f(x) = \int_0^x \frac{y(t)\, dt}{\sqrt{x-t}} \qquad ...(2.14)$$

Then we may proceed as

Here $k(x) = x^{-1/2}$ and the given equation is $f(x) = y(x) \times k(x)$

Here $k(x) = x^{-1/2}$ is not piecewise continuous but it does have a Laplace transform as

$$K(x) = \int_0^\infty x^{-1/2} e^{-sx}\, dx = \sqrt{\frac{1}{2}} s^{-1/2}$$

or $$K(s) = \sqrt{\pi} s^{-1/2}$$

Taking Laplace transform of (1.14) we get

$$K(s) = Y(s) . K(s)$$

$$Y(s) = L(y(x)) = \frac{F(s)}{K(s)}$$

or $$y(s) = \frac{F(s)}{\sqrt{\pi} s^{-1/2}}$$

or $$y(s) = \frac{s^{1/2}}{\sqrt{\pi}} F(s)$$

$$= \frac{s}{\pi}\left(\sqrt{\pi} s^{-1/2} F(s)\right)$$

or $$Y(s) = \frac{s}{\pi}(k(s)F(s))$$

these fore (by use of inverse Laplace transform)

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt \text{ is the required solution of (2.13)}$$

NOTES

Example 2.38 Solve the following integral equations

(a) $\int_0^x \frac{y(t)}{(x-t)^{1/3}}\,dt = x(1+x)$

(b) $\int_0^x \frac{y(t)}{(x-t)^{1/2}}\,dt = 1+x+x^2$

Solution: (a) The given integral equation is given as $y(x) * x^{-1/3} = x(1+x)$ ...(2.15)

*denotes the convolution product taking the Laplace transform we get.

$$L(y(x))L\left(x^{-1/3}\right) = L(x + x^2)$$

$$Y(s)\frac{\sqrt{2/3}}{s^{2/3}} = \frac{1}{s^2}+\frac{2}{s^3}$$

$$Y(s) = \frac{1}{\sqrt{2/3}}\left\{\frac{1}{s^{4/3}}+\frac{1}{s^{7/3}}\right\}$$

$$\Rightarrow \quad Y(x) = L^{-1}(Y(s)) = \frac{1}{\sqrt{2/3}}\left\{L^{-1}\left(\frac{1}{s^{4/3}}\right)+L^{-1}\left(\frac{1}{s^{4/3}}\right)\right\}$$

$$\text{or} \quad Y(x) = \frac{1}{\sqrt{2/3}}\left\{\frac{x^{1/3}}{\frac{1}{3}\sqrt{1/3}}+2\frac{x^{4/3}}{4/3 \cdot 1/3\sqrt{1/3}}\right\}$$

$$\text{or} \quad y(x) = \frac{2x^{1/3}}{\sqrt{2/3}\sqrt{1/3}}\left(1+\frac{3}{2}x\right)$$

$$\text{or} \quad y(x) = \frac{2x^{1/3}}{(\pi/\sin \pi/3)}\cdot\left(\frac{1+3}{2}x\right)$$

$$\text{or} \quad y(x) = \frac{3\sqrt{3}}{4\pi}x^{1/3}(2+3x)$$

Past (b) is similar and it is left as an exercise to the readers.

2.9 INTEGRO-DIFFERENTIAL EQUATION

Laplace transforms can be used to solve Integro-Differential Equation (IDE) consisting the kernel of convolution type, i.e., the kernel $k(x, t)$ depends on $(x-t)$ like $K(x, t) = e^{x-t}$, cos $(x - t)$, $(x - t)^2$ ect. This is also called difference kernel. Consider an Integro-difference equation.

$$\frac{d^n y(x)}{dx^n} = f(x) + \lambda \int_0^x k(x-t) y(t)\, dt \qquad ...(2.16)$$

we can rewrite (1) as

$$y^{(n)}(x) = f(x) + \lambda\, k(x) * y(x) \qquad ...(2.17)$$

where * denotes the consolution.

Taking Laplace transform of (2.17) both sides and using Convolution theorem we get

$$s^n Y(s) - \sum_{i=1}^{n-1} s^{n-i-1} y^{(i)}(0) = F(s) + K(s).Y(s)$$

Solution (2.18) for $Y(s)$ and by the use of inverse Laplace transform one can find $y(x)$ as

Example 2.39 Solve the following Volterra integro differential equation.

$$\frac{dy}{dx} = 1 + \int_0^x y(t)\, dt \text{ subject to } y(0) = 1$$

taking Laplace transform both sides we get.

$$(y(x)) - y(0) = L(1) + L\left(\int_0^x 1.y(t)\, dt\right)$$

or

$$sY(s) - 1 = \frac{1}{s} + L(1).L\big(y(x)\big)$$

$$sY(s) - 1 = \frac{1}{s} + \frac{1}{s}.Y(s)$$

$$\left(s - \frac{1}{s}\right)Y(s) = \frac{1}{s} + 1 = \frac{1+s}{s}$$

$$Y(s) = \frac{\frac{1+s}{s}}{\frac{s^2-1}{s}}$$

or

$$Y(s) = \frac{1}{s-1}$$

$$\Rightarrow \quad y(x) = L^{-1}\left(y(s)\right) = L^{-1}\left(\frac{1}{s-1}\right)$$

$$\Rightarrow \quad y(x) = e^x$$

NOTES

Example 2.40 Solve the following integro differential equation of order *z*.

$$y''(x) = -1 + x + \int_0^x (x-t)\, y(t)\, dt$$

Subject so the conditions

$$y(0) = 1, \quad y'(0) = 1$$

Solution: The given IDF can be written as y" $Y(x) = -1 - x + (x) * y(x)$

Taking Laplace transform both sides.

$$y^2Y(s) - sy_{(0)} - y'_{(0)} = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}Y(s)$$

$$\text{or} \quad s^2Y(s) - s\,.\,1 - 1 = \frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s^2}.Y(s)$$

$$\text{or} \quad \left(s^2 - \frac{1}{s^2}\right)Y(s) = \frac{-1}{s} - \frac{1}{s^2} + s + 1$$

$$\text{or} \quad \left(\frac{s^4-1}{s^2}\right)Y(s) = \frac{-s-1+s+s^2}{s^2}$$

$$\text{or} \quad Y(s) = \frac{s^3+s^2-s-1}{s^4-1}$$

$$\text{or} \quad Y(s) = \frac{s^3+s+s^2-1}{s^4-1}$$

$$\text{or} \quad Y(s) = \frac{(s^2-1)(s+1)}{(s^2-1)(s^2+1)}$$

$$\text{or} \quad Y(s) = \frac{s+1}{s^2+1}$$

$$\text{or} \quad Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow \quad L^{-1}(y(s)) = y(x) = \cos x + \sin x$$

2.10 DIFFERENTIAL-DIFFERENCE EQUATIONS

Like differential equations, differential-difference equations also used to model variety of problems in mechanics, electrical and electronic systems. These equations also arise frequently in economics, business and particularly in problems concerning interest, annuities, amortization, loan and mortgages. Here we see the use of Laplace transfer to find the solution of simple diffrential-difference equations.

Suppose $\{y_i\}_{i=1}^{\infty}$ is a given sequence. We first see the definitions of difference operators. $\Delta, \Delta^2, \ldots \Delta^n$ defined as

$$\Delta y_i = y_{i+1} - y_i \qquad \text{...(2.19)}$$

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(\Delta y_{i+1} - y_i)$$

or $$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i \qquad \text{...(2.20)}$$

Similarly $$\Delta^2 y_i = \Delta^2(\Delta y_i) = \Delta^2(y_{i+1} - y_i)$$

or $$\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i \qquad \text{...(2.21)}$$

In general

$$\Delta^n y_i = \Delta^{n-1}(y_{i+1} - y_i)$$

or $$\Delta^n y_i = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, y_{i+n-k} \qquad \text{...(2.22)}$$

These operaters $\Delta, \Delta^2, \Delta^3, \ldots \Delta^n$ respectively called as first, second, third and nth order finite differences. Any equation consisting the finite differences is called difference equation.

The highest order finite difference involved in the equation is referred to as its order. A difference equation containing the derivatives of the unknown function is called the differential-difference equation. Thus the differential-differnce equation has two distinct orders – one is realted to the highest order finite difference and the other is associated with highest order derivatives.

The equations

$$\Delta y_i = y_i = 0$$

$$\Delta^2 y_i = 2\Delta y_i = 0$$

are the example of difference equations of the first and second order respectively. The general n^{th} order differnce equation has the form:

$$a = \Delta^n y_i + a_1 \Delta^{n-1} y_i + \ldots + a_{n-1} \Delta y_i + a_n y_i = f(n) \qquad \text{...(2.23)}$$

where $a_0, a_1, \ldots a_n$, and $f(n)$ are either constans or functions of non negative integer n.

Like in ordinary defferential, Equation (2.23) is called a homogenous or inhomogeneous according to $f(n) = 0$ or $\neq 0$.

NOTES

The following equations

$y^1\ (t) - y\ (t-1) = 0$ ...(2.24)

$y^1\ (t) - a\ y\ (t-1) = f\ (t)$...(2.25)

are the examples of differential-difference equations where $f(t)$ is a given function of t. The study of above such equations can be carried by introducing the function.

$S_n\ (t) = H\ (t-n) - H\ (t-n-1),\ n \le t \le n+1$

where H (t) is the Heaviside unit function. The Laplace transform of p_n(t) is given by

$$\overline{S}n(s) = \mathcal{L}\ (S_n\ (t)) = \int_n^{\infty} e^{-st}\ \{H\ (t-n) - H\ (t-n-1)\}\,dt$$

$$= \int_n^{n+1} e^{-st}\ dt = \frac{1}{s}\left(1-e^{-s}\right)e^{-ns}$$

$$= \overline{s}_0(s)e^{-ns}$$

where $\overline{S}_0(s) = \frac{1}{s}\left(1-e^{-s}\right)$

We, next define the function y(t) by a series

$$y(t) = \sum_{n=0}^{\infty} y_n S_n(t) \qquad ...(2.26)$$

where $\{y_n\}_{n=0}^{\infty}$ is a given sequence.

Thus it follows that

$y(t) = y_n$ for $n \le t \le n+1$, and

represents a staircase function.

Further

$$y(t+1) = \sum_{n=0}^{\infty} y_n S_n(t+1)$$

or

$$y\ (t+1) = \sum_{n=0}^{\infty} y_n\left[\mu\left(t+1-n\right)-\mu\left(t-n\right)\right]$$

$$= \sum_{n=1}^{\infty} y_n S_{n-1}(t)$$

or

$$y(t+1) = \sum_{n=0}^{\infty} y_{n+1} S_n(t) \qquad ...(2.27)$$

NOTES

Similarly,

$$y(t+2) = \sum_{n=0}^{\infty} y_{n+2} S_n(t)$$

In general

$$y(t+k) = \sum_{n=0}^{\infty} y_{n+k} S_n(t) \quad ...(2.28)$$

The Laplace transform of $y(t)$ is given by

$$\bar{y}(s) = \mathcal{L}(y(t)) = \int_0^{\infty} e^{-st} y(t) dt$$

or $$(s) = \frac{1}{s}\left(1-e^{-s}\right)\sum_{n=0}^{\infty} y_n e^{-ns}$$

Thus $$\bar{y}(s) = \frac{1}{s}\left(1-e^{-s}\right)\mathcal{L}(S) = S_0(s)\mathcal{L}(S)$$

Thus $$\mathcal{G}(s) = \frac{1}{s}\left(1-e^{-s}\right) \mathcal{G}(s) = \bar{S}_0(s)\, \mathcal{G}(s)$$

where $\mathcal{G}$(s) represents the Dirichlet function defined by

$$\mathcal{G}(s) = \sum_{n=0}^{\infty} y_n \ell xp\left(-ns\right)$$

Thus we can reduce that

$$y(t) = \mathcal{L}^{-1}(\bar{y}(s))$$

or $$y(t) = \mathcal{L}^{-1}(\bar{S}_0(\text{s})\, \mathcal{G}(s)) \quad ...(2.29)$$

In particular if $y_n = a^n$ is a geometric sequence then

$$\mathcal{G}(s) = \sum_{n=0}^{\infty}\left(ae^{-s}\right)^n$$

$$= \frac{1}{\left(1-ae^{-s}\right)} = \frac{e^s}{e^s - a} \quad ...(2.30)$$

or $$\mathcal{L}{-1}\left(\bar{S}_0(s)\frac{e^s}{e^s-a}\right) = a^n \quad ...(2.31)$$

From the identity

$$\sum_{n=0}^{\infty}\left(n+1\right)\left(ae^{-s}\right)^n = (1-ae^{-s})^{-2}$$

it follows that

NOTES

$$\mathcal{L}((n+1)a^n) = \overline{S}_0(s)(1 - ae^{-s})^{-2}$$

$$= \frac{e^{2s}\overline{S}_0(s)}{(e^s - a)^2}$$

Thus $$\mathcal{L}^{-1}\left(\frac{e^{2s}\overline{S}_0(s)}{(e^s - a)^2}\right) = (n+1)a^n$$

or $$\sum_{n=0}^{\infty} na^n e - ns = \frac{ae^s}{(1 - ae^{-s})^2}$$

Hence $$\mathcal{L}((na^n) = \overline{S}_0(\text{s}) \frac{ae^s}{(e^s - a)^2}$$

$$\mathcal{L}^{-1}\left(\frac{a\overline{S}_0(s)e^s}{(e^s - a)^2}\right) = na^n$$

Theorem 2.2. If $l(y(t) = \mathcal{L}(y(t)) = \overline{y}(s)$ then

$$\mathcal{L}(y(t+1)) = e^s(\overline{y}(s) - y_0\overline{S}_0(s)), \quad y_0 = y(0)$$

Proof: Consider

$$\mathcal{L}(y(t+1)) = \int_0^{\infty} e^{-st} y(t+1)dt \text{ (Put t + 1 = } \tau)$$

$$= \int_0^{\infty} e^s e^{-s\tau} y(\tau)d\tau$$

$$= e^s\left[\overline{y}(s) - \int_0^1 e^{-s\tau} y(\tau)d\tau\right]$$

$$= e^s\left[\overline{y}(s) - y(0)\int_0^1 e^{-s\tau} d\tau\right]$$

$$\mathcal{L}(y(t+1)) = e^s[\overline{y}(s) - y_0\overline{S}_0(s)]$$

Similarly we derive

$$\mathcal{L}(y(t+2)) = e^s[\mathcal{L}(y(t+1)) - y(1) - (s)]$$

$$= e^s [e^s((s) - y_0(s)] - y(1(s))$$

$$\mathcal{L}(y(t+2)) = e^{2s} [(s) - (y_0 + y_1 e^{-s})(s)]$$

$$\text{where} \quad y_0 = y(0), y_1 = y(1)$$

Similarly

$$\mathcal{L}(y(t+3)) = e^{3s} [(s) - (y_0 + y_1 e^{-s} + y_2 e^{-2s})(s)]$$

In general

$$\mathcal{L}(y(t+k)) = e^{ks}\left(\overline{y}(s) - \overline{S}_0(s)\sum_{r=0}^{k-1} yi\ e^{-rs}\right)$$

Example 2.41 Solve the following differential-difference equation

$y'(t) = y(t-1), y(0) = 1$

Solution: Taking the Laplace transform of given equation we get

$$s(s) - y(0) = e^{-s}((s) - y(0)(s))$$

$$\text{or} \qquad (s)(s - e^{-s}) = 1 + \frac{e^{-s}}{s}\left(e^{-s} - 1\right)$$

$$\text{or} \qquad (s) = \left\{\frac{1}{s - e^{-s}} + \frac{e^{-s}}{s\left(s - e^{-s}\right)}\right\} + \frac{e^{-2s}}{s\left(s - e^{-s}\right)}$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1}$$

$$\overline{y}(s) = \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^3} + \frac{e^{-4s}}{s^4} + \ldots + \frac{e^{-ns}}{s^n} + \ldots \qquad \ldots(1)$$

Since we know that $\mathcal{L}{-1}\left(\frac{e^{-as}}{s^n}\right) = \frac{\left(t-a\right)^{n-1}}{\lceil n} H\left(t-a\right)$

Thus we get (from (1))

$$\text{y(t)} = \mathcal{L}^{-1}(\overline{y}\text{(s)})$$

$$= 1 + \frac{\left(t-2\right)}{1!} + \frac{\left(t-3\right)^2}{2!} + \ldots + \frac{\left(t-n\right)^{n-1}}{\left(n-1\right)!} t > n.$$

NOTES

Example 2.42 Solve the following differntial-difference equation.

y' $(t) - au\,(t-1) = b, y(0) = 0$

Solution: Taking the Laplace transform of given equation

$$s\,\bar{y}\,(s) - y(0) - a\,e^{-s}\,[(s) - y(0)\,(s)] = \frac{b}{s}\ .$$

or $$\bar{y}\,(\mathrm{s}) = b\left[\frac{1}{s^2} + \frac{ae^{-s}}{s^3} + \frac{a^2e^{-2s}}{s^4} + \frac{a^n e^{-ns}}{s^{n+2}} + \ldots + \frac{a^n e^{-ns}}{s^{n+2}} + \ldots\right]$$

Taking inverse Laplace transform we get

$$y(t) = b\left[1 + \frac{a(t-1)^2}{\lceil 3} + \frac{a^2(t-2)^3}{\lceil 4} + \ldots + \frac{a^n(t-n)^{n+1}}{\lceil n+2}\right] t > n.$$

Example 2.43 Solve the following difference equation.

$$y_{n+2} - 2\,\lambda y_{n+1} + \lambda^2 y_n = 0$$

Subject to the conditions $y_0 = 0$ and $y_1 = 1$.

Solution: Taking the Laplace transform of given difference equation we get

$$e^{2s}\left[\bar{y}(s) - e^{-s}\bar{S}_0(s)\right] - 2\lambda\bar{y}(s)e^s + \lambda^2\bar{y}(s) = 0 \text{ (using } y_0 = 0 \text{ and } y_1 = 1)$$

or $$\bar{y}(s) = \frac{e^s\bar{S}_0(s)}{(e^s - \lambda)^2}$$

or $$y_n = \mathcal{L}^{-1}\,(y\,(s)) = \mathcal{L}^{-1}\left(\frac{e^s\bar{S}_0(s)}{(e^s - \lambda)^2}\right)$$

or $$y_n = \frac{1}{\lambda}.n\lambda^n = n\lambda^{n-1}$$

Check Your Progress

6. What are partial differential equations?
7. How is a partial differential equation formed?
8. What do you mean by the terms 'Complete integral' and 'Particular integral'?
9. What is the electric circuit?
10. Define the term beam.
11. What do you understand by the bending of beam?
12. State the Abel integral equation.
13. Where are differential-difference equations used?

2.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved.
2. These are three steps for solving an ordinary differential equations with constant coefficients:
   - Transformation of the differential equation into the mapped space.
   - Solving the algebraic equation in the mapped space.
   - Back transformation of the solution into the original space.
3. Considering $y^{(n)}$ as a constant, then the following equalities hold:

   $$\int s^2 Y'' ds = \frac{1}{3} s^3 Y''$$

   $$\int 2sY' ds = s^2 Y'$$

   In the above given two equations, $y^{(n)}$ is considered as a function of s. However, $y^{(n)}$ can be used as a constant if the difference stands as much as the constant coefficient.
4. The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved.
5. Working Rule:
   - Two equations are given for two dependent variables, depending upon a single variable. Apply the Laplace transform on both sides of the equations on both of the equations.
   - Now solving these equations, there will be values of dependent variable in terms of s.

   Applying Laplace transformation find the value of dependent variables.
6. Any equation which contains one or more partial derivatives is called a partial differential equation. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$; $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for partial differential equation of first order and second order respectively.
7. Partial differential equation may be formed by eliminating *(i)* Arbitrary constants *(ii)* Brbitrary functions.
8. A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

NOTES

In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $\varphi(x, y, z, a, b) = 0$, is the complete integral of a partial differential equation, then the eliminant of a and b from the equations

$\frac{\partial \phi}{\partial a} = 0$, $\frac{\partial \phi}{\partial b} = 0$, is called singular integral.

9. An electric circuit considering of a resistance R, inductance L, a condenser of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit.
10. A bar whose length in much greater than its cross-section and its thickness is called a beam. There are various types of beams. defined so for. We explain few of them.
11. A beam be fixed at one end and the other is loaded. The upper surface is elongated and therefore under tension and lower surface is shortened so under compression. Wherever beam is loaded it deflects from its original position. If M is the bending moment of the forces acting on it, then

$$M = \frac{ER}{R}$$

12. The following Abel integral equation.

$$f(x) = \int_0^x \frac{y(t)}{(x-t)^\alpha}\, dt$$

13 Laplace transforms can be used to solve Integro-Differential Equation (IDE) consisting the kernel of convolution type, i.e., the kernel $k(x, t)$ depends on $(x - t)$ like $K(x, t) = e^{x-t}$, $\cos(x - t)$, $(x - t)^2$ ect. This is also called difference kernel. Consider an Integro-difference equation.

14. Differential-difference equations are used to model variety of problems in mechanics, electrical and electronic systems. These equations also arise frequently in economics, business and particularly in problems concerning interest, annuities, amortization, loan and mortgages.

2.12 SUMMARY

- The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved.
- Considering $y(n)$ as a constant, then the following equalities hold:

$$\int s^2 Y'' ds = \frac{1}{3} s^3 Y''$$

$$\int 2sY' ds = s^2 Y'$$

NOTES

In the above given two equations, $y^{(n)}$ is considered as a function of s. However, $y^{(n)}$ can be used as a constant if the difference stands as much as the constant coefficient.

- The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficients. As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved.
- Two equations are given for two dependent variables, depending upon a single variable. Apply the Laplace transform on both sides of the equations on both of the equations.

In this unit, you have learned about partial differential equations and the formation of partial differential equations by eliminating arbitrary constants and arbitrary functions from ordinary equations.

- Considers an electric circuit considering of a resistance R, inductance L, a condenser of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit.
- A bar whose length in much greater than its cross-section and its thickness is called a beam. There are various types of beams. defined so for. We explain few of them.
- If a beam may just rest a support like a knife edge then it in called supported beam.
- It one or both edges of a beam are firmly fixed then it is called fixed beam.
- If one end of a beam in fixed and the other end is loaded, then it is called cantilever.
- A beam be fixed at one end and the other is loaded. The upper surface is elongated and therefore under tension and lower surface is shortened so under compression. Wherever beam is loaded it deflects from its original position. If M is the bending moment of the forces acting on it, then

$$M = \frac{ER}{R}$$

- Laplace transforms can be used to solve Integro-Differential Equation (IDE) consisting the kernel of convolution type, i.e., the kernel $k(x, t)$ depends on $(x - t)$ like $K(x, t) = e^{x-t}$, cos $(x - t)$, $(x - t)^2$, ect. This is also called difference kernel. Consider an Integro-difference equation.
- Differential-difference equations are used to model variety of problems in mechanics, electrical and electronic systems. These equations also arise frequently in economics, business and particularly in problems concerning interest, annuities, amortization, loan and mortgages.

2.13 KEY TERMS

NOTES

- **Solving ODE with constant coefficient:** As an alternative of solving the differential equation with the initial conditions directly in the original domain, a mapping into the frequency domain is taken where only an algebraic equation has to be solved.
- **ODEs with variable coefficients:** Considering $y(n)$ as a constant, then the following equalities hold:

$$\int s^2 Y''ds = \frac{1}{3}s^3 Y''$$

$$\int 2sY'ds = s^2 Y'$$

- **Solution of ODE using Laplace transform:** The Laplace transform is an elegant way for fast and schematic solving of linear differential equations with constant coefficient.
- **Solution of simultaneous ordinary differential equation:** Simultaneous ordinary differential equations may be solved by using Laplace transformation.
- **First order partial derivatives:** $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives.
- **Second order partial derivatives:** $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.
- **Complete integral:** A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.
- **Partial differential equation:** Any equation which contains one or more partial derivatives is called a partial differential equation. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$; $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for partial differential equation of first order and second order respectively.
- **Beam:** A bar whose length in much greater than its cross-section and its thickness is called beam.

2.14 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the significance of Laplace transform to solve an ordinary differential equations with constant coefficients.

2. What is the three steps for solving an ordinary differential equations with constant coefficients?
3. Define the solution of ordinary differential equations with variable coefficients through the integral and Laplace transform.
4. Why Laplace transform is useful for solving the differential equations?
5. Give the solution of simultaneous ordinary differential equations.
6. Define partial differential equation.
7. How will you identify the order of a partial differential equation? Give an example.
8. What do you understand by the beam?
9. State solution of integral equations of convolution type.
10. What is Abel integral equation?
11. Define integro-differential equation.
12. Write a short note on differential- difference equations.

Long-Answer Questions

1. Discuss briefly the significance of Laplace transform to solve an ordinary differential equations with constant coefficients. Give appropriate example.
2. Describe the three steps for solving an ordinary differential equations with constant coefficients.
3. Discuss the solution of ordinary differential equations with variable coefficients through the integral and Laplace transform.
4. Explain why Laplace transform is useful for solving the differential equations.
5. Discuss the solution of simultaneous ordinary differential equations with the help of example.
6. Explain the working rule for the solution of simultaneous ordinary differential equations.
7. Obtain a partial differential equation by eliminating the arbitrary constants of the following:

   (i) $z = ax + by + \sqrt{a^2 + b^2}$ *(ii)* $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$

   (iii) $z = xy + y\sqrt{x^2 - a^2} + b$ *(iv)* $z = ax^3 + by^3$

   (v) $(x-a)^2 + (y-b)^2 + z^2 = a^2 + b^2$ *(vi)* $2z = (ax+y)^2 + b$

8. Eliminate the arbitrary function from the following:

   (i) $z = e^y f(x+y)$ *(ii)* $z = f(my - lx)$

   (iii) $z = f(x^2 + y^2 + z^2)$ *(iv)* $z = x + y + f(xy)$

   (v) $z = f(x) + e^y g(x)$ *(vi)* $z = f(x + 4y) + g(x - 4y)$

   (vii) $z = f(2x + 3y) + y\, g(2x + 3y)$ *(viii)* $z = f(x + y) \cdot \phi(x - y)$

NOTES

9. Solve the following differential equations:

   (i) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ *(ii)* $y^2zp + x^2zq = y^2x$

   (iii) $x^2p - y^2q = (x - y)z$ *(iv)* $xp + yq = 2z$

   (v) $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

10. Eliminate the arbitrary function(s) from the following and form the partial differential equations:

    (i) $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ *(ii)* $z = f(x^2 + y^2 + z^2)$

    (iii) $u = e^y f(x - y)$ *(iv)* $z = f(\sin x + \cos y)$

    (v) $\phi(x + y + z, x^2 + y^2 - z^2) = 0$ *(vi)* $z = f(2x + 3y) + \phi(y + 2x)$

    (vii) $u = f(x^2 + y) + g(x^2 - y)$ *(viii)* $u = x f(ax + by) + g(ax + by)$

11. Discuss the applications of mechanics, electrical circuits and beams with the help of examples.
12. Explain the Abel integral equation with the help of examples.
13. Explain the integro-differential equation. Give appropriate examples.
14. Solve the following IDE $\frac{dy}{dt} + 2y + 5\int_0^x y(t)dt = f(t)\ y(0) = 0$

    Hint: Take differente Kernel $k(x - t) = 1$.
15. Discuss the differential-difference equation with the help of relevant examples.
16. Find the solution of

    $\Delta yb - y_n = 0, y_0 = 1$ using Laplace transforms.

2.15 FURTHER READING

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UNIT 3 FOURIER SERIES AND INTEGRALS

NOTES

Structure

3.0 INTRODUCTION

A Fourier series is a periodic function composed of harmonically related sinusoids, combined by a weighted summation. With appropriate weights, one cycle (or period) of the summation can be made to approximate an arbitrary function in that interval (or the entire function if it too is periodic). As such, the summation is a synthesis of another function. The discrete-time Fourier transform is an example of Fourier series. The process of deriving weights that describe a given function is a form of Fourier analysis. For functions on unbounded intervals, the analysis and synthesis analogies are Fourier transform and inverse transform.

Even functions and odd functions are functions which satisfy particular symmetry relations, with respect to taking additive inverses. They are important in many areas of mathematical analysis, especially the theory of power series and Fourier series. They are named for the parity of the powers of the power functions which satisfy each condition: the function $f(x) = x^n$ is an even function if n is an even integer, and it is an odd function if n is an odd integer.

$f(x), x \in [0, L]$ A half range Fourier series is a Fourier series defined on an interval $[0, L]$ instead of the more common $[-L, L]$, with the implication that the analysed function should be extended to $[-L, 0]$ as either an even ($f(-x)=f(x)$) or odd function ($f(-x)=-f(x)$). This allows the expansion of the function in a series solely of sines (odd) or cosines (even). The choice between odd and even is typically motivated by boundary conditions associated with a differential equation satisfied by $f(x)$.

NOTES

The Parseval's identity, named after Marc-Antoine Parseval, is a fundamental result on the summability of the Fourier series of a function. Geometrically, it is a generalised Pythagorean theorem for inner-product spaces (which can have an uncountable infinity of basis vectors).

The Fourier transform can be formally defined as an improper Riemann integral, making it an integral transform, although this definition is not suitable for many applications requiring a more sophisticated integration theory. For example, many relatively simple applications use the Dirac delta function, which can be treated formally as if it were a function, but the justification requires a mathematically more sophisticated viewpoint. The Fourier transform can also be generalized to functions of several variables on Euclidean space, sending a function of 3-dimensional 'Position Space' to a function of 3-dimensional momentum (or a function of space and time to a function of 4-momentum). A Fourier Transform (FT) is a mathematical transform that decomposes functions depending on space or time into functions depending on spatial or temporal frequency, such as the expression of a musical chord in terms of the volumes and frequencies of its constituent notes. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of space or time.

The convolution theorem states that under suitable conditions the Fourier transform of a convolution of two functions (or signals) is the pointwise product of their Fourier transforms. More generally, convolution in one domain (e.g., time domain) equals point-wise multiplication in the other domain (e.g., frequency domain). Other versions of the convolution theorem are applicable to various Fourier-related transforms.

In this unit, you will learn about the Fourier series, odd and even functions, half range Fourier sine and cosine series, complex form of Fourier series, Parseval's identities for Fourier cosine and sine transform, Fourier integral, Fourier transforms, convolution theorem including sine and cosine transforms, relation between Fourier and Laplace transform, multiple finite Fourier transform and solution of simple partial differential equations by means of Fourier transform.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the Fourier series
- Elaborate on the even and odd functions in Fourier series
- Discuss the half range Fourier series
- Explain complex Fourier transform
- Discuss the Parsival's identity in Fourier transforms
- Describe Fourier's integral formula and Fourier transformation
- Explain the convolution theorem including sine and cosine transform
- Discuss the multiple finite Fourier transform

3.2 FOURIER SERIES

NOTES

Fourier series is used as infinite series representation of periodic function and it uses trigonometric sine and cosine functions for expansion. Its main application is to solve ordinary and partial differential equations. It is a powerful tool to solve differential equations specially with periodic functions appearing as non-homogeneous terms. It has wider applications as it is valid for periodic functions as well as continuous functions and for functions, which are discontinuous.

Function $f(x)$ is said to be periodic if $f(x + T) = f(x)$, and real x for some positive number T is period of $f(x)$. Smallest positive period of $f(x)$ is called primitive or fundamental period of $f(x)$.

For example, cosec$x$, sin$x$, sec$x$, cosec$x$ are periodic function with period 2π and cot$x$, tan$x$, are periodic with period π. In general, it can be defined as

If $\quad f(x + nT) = f(x), \quad n \neq 0$

Then T is period of $f(x)$ and nT is also period of f for any integer n.

If f(x) is a periodic function of period T, then $f(ax)$ with $a \neq 0$, is a periodic function of period $\frac{T}{a}$.

For example cos$2x$ has period $\frac{2\pi}{3} = \pi$ and sin $3x$ has period $\frac{2\pi}{3}$.

If $f(x)$ and $g(x)$ have period T, then $f(x) = af(x) + bg(x)$ has period T, where a and b are constants.

A constant function is periodic for any positive period T. The period of a sum of a number of periodic functions is the least common multiple of the periods.

If a function is periodic with period 2π, then the trigonometric series of $f(x)$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $\quad a_0 = \frac{1}{\pi}\int_{a_0}^{2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\, dx$$

And $\quad b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\, dx$

Where $\quad n = 1, 2, 3$ ....

a_0, a_n and b_n are called Fourier coefficients as

1. $\int_{0}^{2\pi} \cos mx . \cos nx \, dx = 0 \qquad m \neq n, \quad m \neq 0, \quad n \neq 0$

NOTES

2. $\int_{0}^{2\pi} \cos mx = 0$

3. $\int_{0}^{2\pi} \sin mx = 0$

4. $\int_{0}^{2\pi} \sin mx \sin nx \, dx = 0 \qquad m \neq n, \quad m \neq 0, \quad n \neq 0$

5. $\int_{0}^{2\pi} \sin mx \cos nx \, dx = 0 \qquad m \neq n, \quad m \neq 0, \quad n \neq 0$

6. $\int_{0}^{2\pi} \sin^2 mx \, dx = \pi$

7. $\int_{0}^{2\pi} \cos^2 mx \, dx = \pi$

8. $\int_{0}^{2\pi} \cos mx \sin mx \, dx = 0$

Example 3.1: Obtain Fourier series for $f(x) = e^{2x}$ in $(0, 2\pi)$

Solution: We know that Fourier expansion is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin ax \right)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{ax} dx$$

$$= \left[\frac{e^{ax}}{\pi} \right]_0^{2\pi}$$

$$\boxed{a_0 = \frac{1}{\pi}\left(e^{2a\pi} - 1\right)}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

NOTES

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}\left[a\cos nx + n\sin nx\right]}{a^2+n^2}\right]_0^{2\pi}$$

$$\boxed{a_n = \frac{1}{\pi}\left(\frac{ae^{2a\pi}-1}{a^2+n^2}\right)}$$

And $\quad b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax} \sin nx \, dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}\left[a\sin nx + n\cos nx\right]}{a^2+n^2}\right]_0^{2\pi}$$

$$= \boxed{b_n = \frac{n}{\pi\left(a^2+n^2\right)}\left(1-e^{2a\pi}\right)}$$

Therefore $\quad f(x) = \frac{\left(e^{2a\pi}-1\right)}{\pi}\left[\frac{1}{2}+\sum_{a=1}^{\infty}\frac{-n\sin nx}{a^2+n^2}\right]+\frac{\left(ae^{2a\pi}-1\right)}{\pi}\sum_{a=1}^{\infty}\frac{\cos nx}{a^2+n^2}$

Example 3.2: Find the Fourier series of $f(x) = x$ for $(0, 2\pi)$ and sketch the graph of $f(x)$ from -6π to 6π.

Solution: We know that

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Where $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(4\pi^2-0\right) = 2\pi$$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\left(x\right)\cos nx\, dx$$

NOTES

$$= \frac{1}{\pi}\int_0^{2\pi} x.\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{(1-1)}{n^2}\right]$$

$$\boxed{a_n = 0}$$

And $$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

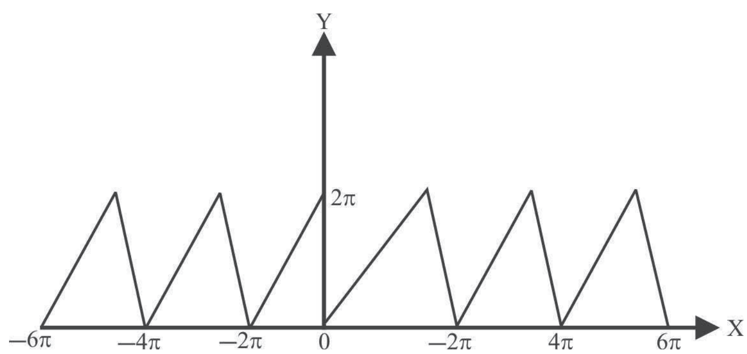
$$= \frac{1}{\pi}\left(\frac{-2\pi}{n}\right)$$

$$b_n = -\frac{2}{n}$$

Therefore, Fourier series is

$$f(x) = \frac{2\pi}{2} + 0 - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$= \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$



Example 3.3: Obtain Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{-(\pi - x)^3}{3}\right]_0^{2\pi}$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^2 \sin nx}{n} - \frac{2[-(\pi - x)](\cos nx)}{n^2} + \left(\frac{-2x \sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$\boxed{a_n = \frac{1}{n^2}}$$

And $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) + 2(\pi - x)\frac{\sin nx}{n^2} - \frac{2}{n^3}(-\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}(0 + 0)$$

$$\boxed{b_n = 0}$$

Therefore, Fourier series,

$$\left(\frac{\pi - x}{2}\right)^2 = \left(\frac{\pi - x}{2}\right)^2 = \frac{1}{2}\frac{\pi}{6} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} + 0$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

NOTES

Example 3.4: Find Fourier series of $f(x)$, where

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

NOTES

And hence prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ----= \frac{\pi^2}{8}$

Solution: We know that $f(x) = \frac{a_0}{0} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{\pi}^{0} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)dx + \int_{\pi}^{0} xdx\right]$$

$$\boxed{a_n = \frac{-\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x).\cos nx\,dx + \int_{0}^{\pi} f(x).\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{-\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{0} + \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= 0 + \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]}$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x).\sin x\,dx + \int_{0}^{\pi} f(x).\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

NOTES

$$= \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin x}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{n}\left[1-(-1)^n\right] - \frac{(-1)}{n}$$

$$\boxed{b_n = \frac{1}{n}\left[1-2(-1)^n\right]}$$

Therefore

$$f(x) = \frac{1}{2}\left(\frac{-\pi}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1^n-1)}{n^2}\right\}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left\{1-2(-1)^n\right\}\sin nx$$

Now $x = 0$ is a point of discontinuity of $f(x)$

As $$f(x) = \frac{1}{2}\left[f(0-0)+f(0+0)\right]$$

$$= \frac{1}{2}\left[-\pi+0)\right]$$

$$= \frac{-\pi}{2}$$

Putting $x = 0$ in Fourier expansion, we have

$$\frac{-\pi}{2} = f(0) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{-2}{(2n-1)^2}$$

Or $$\frac{\pi}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$= \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}$$

Example 3.5: Expand $f(x)$ as Fourier series if

$$f(x) = \begin{cases} -c & \text{for} \quad -\pi < x < 0 \\ c & \text{for} \quad 0 < x < \pi \end{cases}$$

And hence prove that $1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - -- = \frac{\pi}{4}$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n nx$$

Where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

NOTES

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -c\,dx + \int_{0}^{\pi} c\,dx\right]$$

$$= \frac{1}{\pi}(-c)[x]_{-\pi}^{0} + \frac{c}{\pi}[x]_{0}^{\pi}$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x).\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x).\cos nx\,dx + \int_{0}^{\pi} f(x).\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -c.\cos nx\,dx + \int_{0}^{\pi} c.\cos nx\,dx\right]$$

$$= \frac{c}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{c}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x).\sin nx\,dx + \int_{0}^{\pi} f(x).\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-c).\sin nx\,dx + \int_{0}^{\pi} c.\sin nx\,dx\right]$$

$$= \frac{-c}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{c}{\pi}\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$\boxed{b_n = \frac{2c}{n\pi}\left[1-(-1)^n\right]}$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} \frac{2c}{n\pi}\left[1-(-1)^n\right].\sin nx$$

$$= \frac{4c}{\pi}.\sin nx = 0 \quad n \text{ is odd}, n \text{ is even}$$

Now putting $x = \frac{\pi}{2}$

$$c = f\left(\frac{\pi}{2}\right)$$

$$= \frac{4c}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-)\frac{\pi}{2}}{2n-1}$$

Or $$\frac{4}{\pi} = \sum_{n=1}^{\infty}\frac{\sin(2n-)\frac{\pi}{2}}{2n-1}$$

$$\frac{4}{\pi} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}----$$

NOTES

Example 3.6: Expand $f(x) = x\sin x$ for the interval $(0, 2\pi)$

Solution: We know that

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \sin nx + b_n \sin nx)$$

Where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx$$

$$= \frac{1}{\pi}[-x\cos + \sin x]_0^{2\pi}$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\sin(n+1)x - \sin(n-1)x\}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-x\cos(n+1)x}{n+1}+\frac{\cos(n-1)}{n-1}\right]_0^{2\pi} - \frac{1}{2\pi}\left[\frac{-\sin(n+1)x}{n+1}+\frac{-\sin(n-1)}{n-1}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{\frac{-\cos(n+1)2\pi}{n+1}+\frac{\cos(n-1)2\pi}{n-1}\right\}\right],\ n \neq 1$$

$$a_n = \frac{1}{n-1}-\frac{1}{n+1}$$

$$\boxed{a_n = \frac{2}{n^2-1}} \qquad \text{for } n \neq 1$$

At $n = 1$, $$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx$$

NOTES

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$\boxed{a_1 = \frac{-1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,.\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - \left\{\frac{-\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right],\ n \neq 1$$

$$\boxed{b_n = 0} \text{ for } n \neq 1$$

At n = 1,
$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{x - \sin 2x}{2}\right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi - 0) - \left(2\pi^2 + \frac{1}{4}\right) + \frac{1}{4}\right]$$

$$= \pi$$

Therefore

$$f(x) = x\sin x = \frac{1}{2}(-2) - \frac{1}{2}\cos + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx + \pi\sin x + 0$$

$$= -1\frac{-\cos x}{2} + \pi\sin x + 2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1}$$

3.3 ODD AND EVEN FUNCTIONS

Functions can be defined as even and odd functions.

When $f(-x) = f(x),\ \forall x$, then function $f(x)$ is said to be even. For even functions, graph is symetric about y-axis. It uses the property of Integration,

NOTES

$$\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a} f(x)\,dx$$

Even functions contains only even powers of x and in trignometric terms, they contain only cos$x$ and sec$x$.

For example, $-x^2$, $x^4 + 2$, $x^6 + \cos x$, $3x^8 + \cos 2x$ etc.

When $f(x)$ and $g(x)$ are even functions, then sum of two even functions is even, i.e., $f(x) = f_1(x) + g(x)$ is also even. Product of two even function is even, i.e., $f_2(x)$ - $f(x)$. $g(x)$

For even functions all $b_n{}'s$ will be zero as integer and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ becomes an odd function. Fourier series for even function is defined

As $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

And $$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x).\cos nx\,dx$$

This is also called Fourier cosine series.

When $f(-x) = -f(x)$, $\forall x$, then function f(x) is said to be odd function. Graph of odd function is symmetric about origin and it uses the property of integration,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Odd functions contain only odd powers of x and in trignometric function, it contains only sin$x$ and cosec$x$.

For example $-x^3$, $3\sin x + x$, $4\sin 2x + x^2$ etc.

It $f(x)$ and $g(x)$ are odd functions, then sum of odd functions is odd, for example,

$f_1(x) = f(x) + g(x)$ is odd

$f_2(x) = f_1(x)$. $g(x)$ is even.

For odd function, all a_n and a_0 are zero as integrand become an odd function. Therefore, Fourier series for odd function is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

When $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\, dx$$

It is called Fourier sine series.

NOTES

To work out the Fourier series, first step is to identify whether the given function is even or odd.

If f is odd then only $b_n's$ are calculated and subsituted in the formula. If f is even function, there a_0 and $a_n's$ are calculated.

Example 3.7: Expand $f(x) = x^3$ in $(-\pi, \pi)$ as Fourier series.

Solution: As $f(x) = x^3$ and

$f(-x) = -f(x)$, therefore $f(x)$ is odd function.

Therefore $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$b_n = \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin nx\, dx$$

$$= \int_{0}^{\pi} x^3 \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \sin nx}{n} + \frac{3x^2 \sin nx}{n^2} + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{6\pi}{n^3}\cos n\pi\right]$$

$$= 2(-1)^n\left[\frac{x^2}{n} + \frac{6}{n^3}\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\left(\frac{6}{n^3} + \frac{\pi^2}{n}\right)(-1)^n \sin x$$

Example 3.8: Expand $f(x) = \sin x$ in $(-\pi, \pi)$ as Fourier series.

Solution: As $f(-x) = -f(x)$, $f(x)$ is odd function.

Therefore $$f(x) = \sum_{n=1}^{\infty} f(x).\sin x$$

Where $$b_n = \frac{2}{\pi}\int_{0}^{\pi} \sin x.\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$$

$$\boxed{b_n = 0}, \quad n \neq 1$$

NOTES

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x . \sin x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{(1-\cos 2x)\,dx}{2}$$

$$= \frac{1}{\pi}\left[\frac{x-\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{\pi}$$

$$\boxed{b_1 = 1}$$

$$f(x) = \sin x$$

$$= b_1 \sin x$$

$$= \sin x$$

Example 3.9: Expand $f(x)$ as Fourier series in $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}, (-\pi\ \pi)$

Solution: As $f(x) = f(x)$, $f(x)$ is even function.

Therefore $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[\frac{\pi^2}{12}x - \frac{x^3}{12}\right]_0^{\pi}$$

$$\boxed{a_0 = 0}$$

And $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\frac{\sin nx}{n} - \left(\frac{2x}{4}\right)\frac{\cos nx}{n^2} + \frac{1}{2}\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-1}{n^2}\right)\frac{2}{\pi}.\cos n\pi$$

$$\boxed{a_n = \frac{(-1)^{n+1}}{n^2}}$$

NOTES

Therefore $\quad f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$

Example 3.10: Expand $f(x) = |x|$ in $(-\pi, \pi)$ as Fourier series.

Solution: We know that $f(x) = |x|$

And
$$\begin{aligned} f(-x) &= |-x| \\ &= |x| \\ &= f(x) \end{aligned}$$

Therefore, $f(x)$ is an even function and

$$f(x) = \frac{a_n}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where
$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} |x|\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\, dx \\ &= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \end{aligned}$$

$$\boxed{a_0 = \pi}$$

And
$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} |\, x\,|.\cos nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx \\ &= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \end{aligned}$$

$$\boxed{\begin{aligned} a_n &= \frac{-4}{\pi n^2}, && n \text{ is odd} \\ &= 0, && n \text{ is even} \end{aligned}}$$

Therefore $$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

Example 3.11: Expand $f(x) = x \sin x$ for $(-\pi, \pi)$ and hence prove that

$$\frac{\pi-2}{4} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + ----$$

Solution: We know that

$$\begin{aligned} f(-x) &= (-x)\sin(-x) \\ &= -x[-\sin x] \\ &= x \sin x \\ &= f(x) \end{aligned}$$

Therefore, $f(x)$ is even function and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}(-\pi\cos\pi)$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right\} - \left\{\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{\cos n\nu}{n+1} + \frac{\cos n\pi}{1-n}$$

$$\boxed{a_n = \frac{(-1)^{n+1}.2}{n^2-1}, \quad n \neq 1}$$

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And
$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \frac{x\sin 2x}{2}\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$
$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2}+\frac{\sin 2x}{4}\right]_0^{\pi}$$
$$\boxed{a_1 = \frac{-1}{2}}$$

Therefore $(x\ \sin x) = \frac{2}{2}-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos x.$
$$= 1-\frac{\cos x}{2}+2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

Putting $x = \frac{\pi}{2}$
$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1-0+2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$
$$\frac{\pi}{2}-1 = 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$
$$\frac{\pi-2}{4} = \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}----$$

3.4 HALF RANGE FOURIER SINE AND COSINE SERIES

Fourier expansion has been defined for function, which is periodic with period $2l$. Now suppose we are given a function $f(x)$, which is non-periodic and is defined in half interval $(0, l)$ of length l. These types of expansions are known as half range Fourier series. In this case, $f(x)$ is neither even nor odd nor periodic. Only information is to obain Fourier cosine series for $f(x)$ in the interval $(0, l)$. In this negard, let as define a new function $f_1(x)$ such that

1. $f_1(x) \equiv f(x)$ in interval $(0, l)$ and
2. $f_1(x)$ is even function in $(-l, l)$ and is periodic with period $2l$.

This, $f_1(x)$ is called 'even period extension of $f(x)$,' and can be expressed as follows:

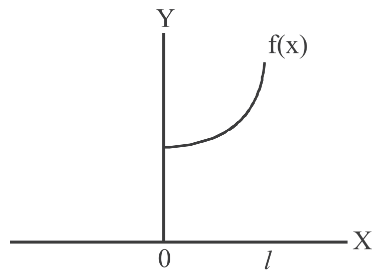

Fig. 3.1

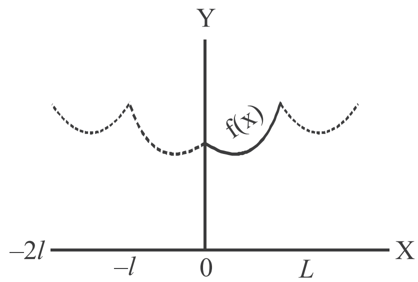

Fig. 3.2

Figure 3.1 represents function $f(x)$, whereas Figure 3.2 represents the extension of $f(x)$, i.e., $f_1(x)$.

Since by construction $f(x)$ and $f_1(x)$ are equal in (0, l), the half range Fourier cosine series for $f(x)$ is given as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{l}$$

Where $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

And $$a_n = \frac{2}{l}\int_0^l f(x)\frac{\cos n\pi x}{l}\,dx$$

It can be conderstand as $f_1(x) = f(x)$ for (0, l) = $f(-x)$ for (–l, 0) and this series expansion of $f(x)$ is valid only for the interval (0, l) but not outside of this interval.

Now suppose we are interested in finding half range Fourier sine series for $f(x)$ in $(0, l)$ let $f_2(x)$ is a function such that

1. $f_2(x) = f(x)$ in (0, l)

2. $f_2(x)$ is odd function in $(-l, l)$ periodic with period $2l$.

Then $f_2(x)$ is called 'odd periodic continuation of $f(x)$' and can be expressed as,

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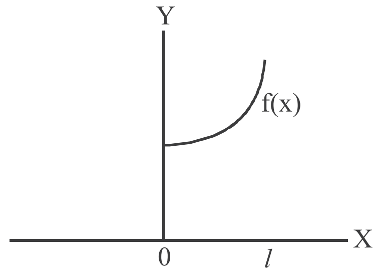

Fig. 3.3

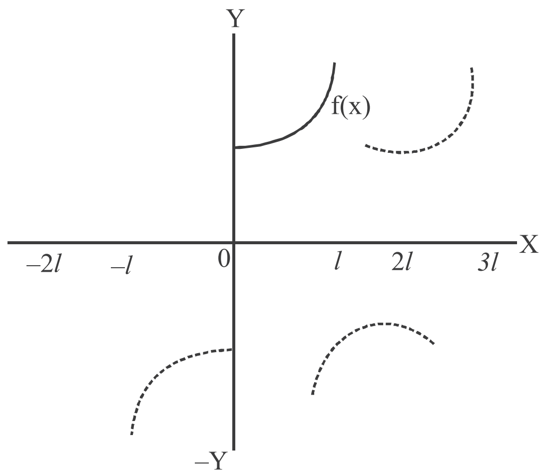

Fig. 3.4

Figure 3.3 represents function $f(x)$, whereas Figure 3.4 represents the extension of $f(x)$, i.e., $f_2(x)$.

Since by construction $f(x)$ and $f_2(x)$ are equal in (0, l), the required half range Fourier sine series expansion of $f(x)$ in interval $(0, l)$ is given as,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l} dx$$

Example 3.12: Find Fourier series expansion of

$$f(x) = \frac{\pi - \pi}{2} \text{ in } 0 < x < 4.$$

Solution: Given that interval = 4 – 0

$$2l = 0$$

$$l = 0$$

Now $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

Where $$a_0 = \frac{1}{2}\int_0^4 f(x)\, dx$$

NOTES

$$= \frac{1}{2}\int_0^4 \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{4}\left[\frac{\pi x - x^2}{2}\right]_0^4$$

$$= \frac{1}{4}[4\pi - 8]$$

$$\boxed{a_0 = (\pi - 2)}$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2} dx$$

$$= \frac{1}{2}\int_0^4 \left(\frac{\pi - \pi}{2}\right)\cos\frac{n\pi x}{2} dx$$

$$= \frac{1}{4}\left[\left\{(\pi - x)\left(\sin\frac{n\pi x}{2}\right)\frac{2}{n\pi}\right\} - \int_0^4 (-1)\sin\frac{n\pi x}{2}.\frac{2}{n\pi}\right]$$

$$= \frac{1}{4}\left\{0 + \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{2}.\frac{2\pi}{n\pi}\right]_0^4\right\}$$

$$= -\frac{1}{n^2\pi^2}(\cos 2n\pi - \cos 0)$$

$$= -\frac{1}{n^2\pi^2}(1-1)$$

$$\boxed{a_n = 0}$$

And $$b_n = \frac{1}{2}\int_0^4 f(x).\sin\frac{n\pi x}{2} dx$$

$$= \frac{1}{2}\int_0^4 \frac{\pi - x}{2}.\sin\frac{n\pi x}{2} dx$$

$$= \frac{1}{4}\left\{\left[(\pi - x)\left(-\cos\frac{n\pi x}{2}\right).\frac{2}{n\pi}\right]_0^4 - \int_0^4 (-1)\left(-\cos\frac{n\pi x}{2}\right).\frac{2}{n\pi} dx\right\}$$

$$= \frac{-2}{4n\pi}\left[(\pi - 4)\cos 2n\pi - \pi\cos 0\right] - \frac{1}{2n\pi}\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_0^4$$

$$= \frac{-2}{4n\pi}(\pi - 4 - \pi)$$

$$\boxed{b_n = \frac{2}{n\pi}}$$

NOTES

Therefore

$$f(x) = \frac{\pi - x}{2} = \frac{\pi - 2}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi x}{2}$$

Example 3.13: Find Fourier series expansion of $f(x)$.

When $$f(x) = \begin{matrix} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{matrix}$$

Solution: Given that interval = 5 – (–5)

$$2l = 10$$

$$l = 5$$

And $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{5} + b_n \sin\frac{n\pi x}{5}\right)$$

When $$a_0 = \frac{1}{5}\int_{-5}^{5} f(x)\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x)\,dx + \int_{0}^{5} f(x)\,dx\right]$$

$$= \frac{1}{5}\left[0 + \int_{0}^{5} 3dx\right]$$

$$= \frac{3}{5}[x]_0^5$$

$$= \frac{3}{5}[5 - 0]$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x)\cos\frac{n\pi x}{5}\,dx + \int_{0}^{5} f(x)\cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[0 + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{3}{5}\left[\sin\frac{n\pi x}{5} . \frac{5}{n\pi}\right]_0^5$$

$$= \frac{3}{n\pi}\left[\sin 5\pi - \sin 0\right]$$

$$\boxed{a_n = 0}$$

NOTES

And $$b_n = \frac{1}{5}\int_{-5}^{5} f(x).\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x).\frac{\sin n\pi x}{5}\,dx + \int_{0}^{5} f(x).\frac{\sin n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[0 + \int_{0}^{5} 3.\frac{\sin n\pi x}{5}\,dx\right]$$

$$= \frac{3}{5}\left[\frac{-\cos n\pi x}{5}.\frac{5}{n\pi}\right]_0^5$$

$$= \frac{-3}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-3}{n\pi}\left[(-1)^n - 1\right]$$

$$= \boxed{b_n = \frac{3}{n\pi}\left[1-(-1)^n\right]}$$

Therefore

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\left[-1(-1)^n\right]\frac{\sin n\pi x}{5}$$

Example 3.14: Find Fourier series for $f(x) = x^2$ in $(-1, 1)$

Solution: We are given that interval $= 1-(-1)$

$$2l = 2$$

$$l = 1$$

As $$f(-x) = (-x)^2$$
$$= x^2$$
$$= f(x)$$

$f(x)$ is even function, therefore $b_n = 0$ and Fourier series is given as,

$$f(x) = \frac{a_n}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{1}$$

Where $$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx$$

NOTES

$$= 2\int_0^1 x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$\boxed{a_0 = \frac{2}{3}}$$

$$a_n = \frac{2}{1}\int_0^1 f(x).\frac{\cos n\pi x}{1}\,dx$$

$$= 2\int_0^1 x^2 \cos n\pi x\,dx$$

$$= \left[x^2\frac{1}{n\pi}\sin n\pi x + \frac{2x}{n^2\pi^2}\cos n\pi + \frac{2}{n^3\pi^3}\sin n\pi\right]_0^1$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}(-1)^n}$$

Therefore

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n \cos n\pi x$$

Example 3.15: Find half range sine series for $f(x) = x(\pi - x)$ 0 for $0 < x < \pi$.

Solution: Half range sine series is give as,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x(\pi - x)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right).\left(\frac{-2}{n^3}\cos nx\right)\right]_0^{\pi}$$

$$\boxed{b_n = \frac{4}{\pi n^3}\left[1-(-1)^n\right]}$$

There $$f(x) = \sum_{n=1}^{\infty}\frac{4}{\pi n^3}\left[-1(-1)^n\right]\sin nx$$

Example 3.16: Find half range series (sine and cosine) in $0 \le x \le \pi$.

$f(x) = x$. Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - -$

NOTES

Solution: Half rang sine series is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$\boxed{b_n = \frac{-2}{n}(-1)^n}$$

Therefore,

$$f(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Fourier half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\boxed{a_0 = \pi}$$

And
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x.\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

NOTES

$$\boxed{a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$$

Therefore,

$$f(x) = x = \frac{\pi}{2} + \frac{4}{\pi}\sum \frac{\left[(-1)^n - 1\right]}{n^2}\cos nx$$

Putting $x = 0$

$$0 = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

Or

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^3} + \frac{1}{5^2} - - - -$$

Example 3.17: Find Fourier sine and cosine series for $f(x) = x^2$ in $(0, \pi)$.

Solution: Half range sine series ifs

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^2}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

Therefore,

$$f(x) = x^2 = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\pi^2(-1)^{n+1}}{n} + \frac{2\left[(-1)^n - 1\right]}{n^3}\right\}\sin nx$$

Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_0^{n}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n + 0\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2} + 0}$$

Therefore

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

3.4.1 Half Range Series

The given finction is defined in the interval $(0, x)$.

To get the series of cosines only assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \ \ \& \ b_n = 0$$

So when $f(x)$ is defined as sine half range series then

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx, \ \ \& \ a_n = 0$$

NOTES

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Example 3.18: Represent the following function by a half range fourier series.

$$f(t) = \begin{bmatrix} t & 0 < t \le \pi/2 \\ \pi/2 & \dfrac{\pi}{2} < t \le \pi \end{bmatrix}$$

Solution: For half sine series $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt dt$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} t\sin nt dt + \int_{\pi/2}^{\pi/2} \frac{\pi}{2}\sin nt dt\right]$$

$$= \frac{2}{\pi}\left\{\left[t\left(\frac{-\cos nt}{n}\right) - \left(\frac{-\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{\pi}{2}\left[\frac{-\cos nt}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[\frac{-\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[\frac{-\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2}\right] + \left[-\cos\pi + \frac{\cos\pi}{2}\right]$$

$$= \frac{2}{\pi}[0+1] + (1) = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[\frac{-\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[\frac{-\cos 2\pi}{2} + \frac{\cos\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2}\frac{(-1)}{2} + 0\right] + \left[\frac{-1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4}\right) - 1 = \frac{1}{2} - 1 = \frac{-1}{2}$$

$$b_3 = \frac{-2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t \frac{-1}{2}\sin 2t + \left(\frac{-2}{9\pi} + \frac{1}{3}\right)\sin 3t$$

Change of Interval and Functions Having Arbitrary Period

Now suppose period for the function is $2c$ instead of 2π, then independent variable x is also to be changed proportionally.

Let $f(x)$ is defined for the interval $(-c, c)$

Now $\quad 2c$ is the interval for x

1 is the interval for $\dfrac{x}{2c}$

2π is the interval for $x \times \frac{2x}{2c} = \frac{\pi x}{c} = z\,(\text{said})$ or $x = \frac{cz}{\pi}$

Now $\quad f(c)$ is the function of period 2π

NOTES

$\therefore$ $F(z)$ may be expanded by Fourier series i.e.

$$\text{F(z)} = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \Sigma a_n \cos cz + \Sigma b_n \sin cz \qquad ...(3.1)$$

Where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)dz$$

$$= \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) d\left(\frac{\pi x}{c}\right)$$

$$\boxed{a_0 = \frac{1}{c}\int_0^{2\pi} f(x)dx}$$

Similarly $a_n = \frac{1}{c}\int_0^{2\pi} f(x)\frac{\cos n\pi x}{c}dx$

$$bn = \frac{1}{c}\int_0^{2\pi} f(x)\frac{\sin n\pi x}{c}dx$$

By Equation 3.1

$$f(x) = \frac{a_0}{2} + \Sigma a_n \frac{\cos n\pi x}{c} + \Sigma b_n \frac{\sin n\pi x}{c}$$

Where
$$a_0 = \frac{1}{c}\int_0^{2c} f(x)dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\frac{\cos n\pi x}{c}dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\frac{\sin n\pi x}{c}dx \qquad \text{(for interval 0 to } c\text{)}$$

For Half Range Sine Series

$a_n = 0$

$$b_n = \frac{2}{c}\int_0^{c} f(x)\frac{\sin n\pi x}{c}dx$$

For Half Range Cosine Series

$b_n = 0$

$$a_n = 0\ \frac{2}{c}\int_0^{c} f(x)\frac{\cos n\pi x}{c}dx$$

$$a_0 = \frac{2}{c}\int_0^{c} f(x)dx$$

Example 3.19: Obtain Fourier series for

$$f(x) = \pi x \qquad 0 \le x \le 1$$
$$= \pi(2 - x) \qquad 1 \le x \le 2$$

$$2c = 2 - 0 = 2$$

$$\therefore\ c = 1$$

NOTES

Solution: Let $f(x) = \dfrac{a_0}{2} + \Sigma a_n \dfrac{\cos n\pi x}{1} + \Sigma b_n \dfrac{\sin n\pi x}{1}$

Where $a_0 = \dfrac{1}{1}\displaystyle\int_0^2 f(x)dx$

$$= \int_0^1 \pi x dx + \int_1^2 \pi(2-x)dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x\frac{-x^2}{2}\right]_1^2$$

$$= \pi$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\frac{\cos n\pi x}{1}dx$$

$$= \pi\int_0^1 x\cos n\pi x dx + \pi\int_1^2 (2-x)\cos nx dx$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} - \pi\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+\pi\left[(2-x)\frac{\sin n\pi x}{n\pi} - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] + \pi\left[\frac{-\cos 2n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$= 0$ even

$= \dfrac{-4}{n^2\pi}$ odd

$$b_n = \frac{1}{1}\int_0^2 f(x)\frac{\sin n\pi x}{1}dx$$

$$= \pi\int_0^1 x\sin n\pi x dx + \pi\int_1^2 x\sin n\pi x dx$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) + \left(\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2}\right)$$

Eample 3.20: Expand $f(x)$ in Fourier series

(–2, 2) where

$f(x) = 0 \qquad -2 < x < 0$

$\quad = 1 \qquad 0 < x < 2$

$2c = 2(-2) = 4$

$\therefore\ c = 2$

NOTES

Solution: Let $f(x) = \frac{a_0}{2} + \Sigma a_n \frac{\cos n\pi x}{2} + \Sigma b_n \frac{\sin n\pi x}{2}$

Where $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0ax + \int_{0}^{2} 1dx\right]$$

$$= \frac{1}{2}[x]_0^2 = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\frac{\cos n\pi x}{2}dx$$

$$= \frac{1}{2}\int_{0}^{2} 1\frac{\cos n\pi x}{2}dx$$

$$= \frac{1}{2}\left[\frac{\sin n\pi x}{\frac{n\pi}{2}}\right]_0^2 = 0$$

And $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\frac{\sin n\pi x}{2}dx$

$$= \frac{1}{2}\int_{0}^{2} 1\frac{\sin n\pi x}{2}dx$$

$$= \frac{1}{2}\left[\frac{\frac{-\cos n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{1}{n\pi}(1 - \cos n\pi)$$

$= 0$ (n is even)

$= \frac{2}{n\pi}$ odd

$$\therefore \qquad f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin \pi x}{2} + \frac{1}{3}\frac{\sin 3n\pi x}{2} + \frac{1}{5}\frac{\sin 5\pi x}{2}\cdots\right)$$

Example 3.21: Expand $f(x) = x - x^2$ as a Fourier series in the interval (–1, 1)

Interval $2c = 1 - (-1)2$ ....

NOTES

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{1} + \Sigma b_n \frac{\sin n\pi x}{1}$$

Where

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{1}(x - x^2)dx$$

$$= \int_{-1}^{1} xdx - \int_{-1}^{1} x^2 dx$$

$$= 0 - 2\int_{0}^{1} x^2 dx$$

$$= -2\left[\frac{x^3}{3}\right]_0^1$$

$$= -\frac{2}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos n\pi x dx$$

$$= \int_{-1}^{1} x\cos n\pi x dx - \int_{-1}^{1} x^2 \cos n\pi x dx$$

$$= 0 - 2\int_{0}^{1} x^2 \cos n\pi x dx$$

$$= -2\left[x^2 \frac{\sin n\pi x}{n\pi} - 2x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^2\pi^3}\right)\right]_0^1$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right]$$

$$= -\frac{4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

And

$$b_n = \int_{-1}^{1}(x - x^2)\sin n\pi x dx$$

$$= 2\int_{0}^{1}(x - x^2)\sin n\pi x dx$$

$$= 2\left[x\left(\frac{-\cos n\pi x}{\pi x}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{-2\cos n\pi}{n\pi}$$

$$= \frac{-2(-1)^n}{n\pi} = \frac{2(-1)n+1}{n\pi}$$

$$\therefore \qquad x - x^2 = \frac{-1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} \cdots\right)$$

$$+ \frac{2}{\pi}\left(\frac{\sin \pi x - 1}{2} \frac{\sin 2\pi x + 1}{3} \sin 3\pi x \cdots\right)$$

NOTES

Example 3.22: Expand $f(x) = e^x$ in a cosine series over (0, 1)

$f(x) = e^x$ and $c = 1$

Solution:

$$a_0 = \frac{2}{c}\int_0^c f(x)dx$$

$$= \frac{2}{1}\int_0^1 e^x dx$$

$$= 2(e - 1)$$

$$a_n = \frac{2}{1}\int_0^1 e^x \frac{\cos n\pi x}{1} dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\{\cos n\pi x + n\pi \sin n\pi x\}\right]_0^1$$

$$= 2\left[\frac{1}{1+n^2\pi^2}(-1)^n e - 1\right]$$

$$b_n = 0$$

$$\therefore \qquad f(x) = e - 1 + 2\left[\frac{-e-1}{\pi^2+1}\cos \pi x + \frac{(e-1)}{4\pi^2+1}\cos 2\pi x \frac{-(e+1)}{9\pi^2+1}\cos 3\pi x\right]$$

Example 3.23: Obtain Fourier cosins expansion of periodic function defined by

$$f(t) = \sin\left(\frac{\pi t}{r}\right); 0 < t < l$$

For Fourier cosins transformation $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)dt$$

$$= \frac{2}{l}\left(-\frac{l}{\pi}\cos\frac{\pi t}{l}\right)_0^l$$

$$= -\frac{2}{l}(\cos \pi - \cos 0) = \frac{4}{\pi}$$

$$an = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)\cos\frac{n\pi t}{l} dt$$

$$= \frac{1}{l}\int_0^l \sin\left(\frac{\pi t + n\pi t}{l}\right) - \sin\left(\frac{n\pi t - \pi t}{l}\right)dt$$

$$= \frac{1}{l}\left[\frac{-\cos(n+1)\pi t}{l}\frac{1}{(n+1)\pi}\right]_0^l - \left[\frac{1}{(n-1)\pi}\frac{\cos(n-1)\pi t}{l}\right]_0^l$$

NOTES

$$= \frac{-1}{(n+1)\pi}[\cos(n+1)\pi - \cos 0]$$

$$\frac{1}{(n-1)\pi}[\cos(n-1)\pi - \cos 0]$$

$$= \frac{1}{(n+1)\pi}[(-1)^{n+1} - 1] + \frac{1}{(n-1)\pi}[(-1)^{n+1} - 1]$$

$$= (-1)^{n+1}\left[\frac{-1}{(n+1)\pi} + \frac{1}{(n-1)\pi}\right] + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi}$$

$$= (-1)^{n+1}\left[\frac{2}{(n^2-1)\pi} - \frac{2}{(n^2-1)\pi}\right]$$

$$= \frac{2}{(n^2-1)\pi}[(-1)^{n+1} - 1]$$

$$= -\frac{4}{(n^2-1)\pi}; \qquad (n \text{ even provided } n+1)$$

$$= 0; \qquad n \text{ odd}$$

At $n = 1$, $a_1 = \frac{2}{l}\int_0^l \frac{\sin \pi t}{l} \frac{\cos \pi t}{l} dt$

$$= \frac{1}{l}\int_0^l \frac{\sin 2\pi t}{l} dt$$

$$= \frac{1}{l}\left(-\frac{l}{2\pi}\frac{\cos 2\pi t}{l}\right)_0^l$$

$$= 0$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\frac{\cos 2\pi t}{l} + \frac{1}{15}\frac{\cos 4\pi t}{l} + \frac{1}{35}\frac{\cos 6\pi t}{l}\cdots\right]$$

Example 3.24: Find the Fourier series expansion of the periodic function of period 1

$$f(x) = \frac{1}{2} + x; \qquad \frac{-1}{2} < x \le 0$$

$$= \frac{1}{2} - x; \qquad 0 < x < \frac{1}{2}$$

Interval $2c = 1$ or $c = \frac{1}{2}$

Solution: Let $f(x)$ $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1/2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi l}{1/2}$

Where $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx$

NOTES

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx+\frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)dx$$

$$= 2\left[\frac{x}{2}+\frac{x^2}{2}\right]_{-1/2}^{0}+2\left[\frac{x}{2}-\frac{x^2}{2}\right]_{0}^{1/2}$$

$$= 2\left[\frac{1}{4}-\frac{1}{8}\right]+2\left[\frac{1}{4}-\frac{1}{8}\right]=\frac{1}{2}$$

$$a_n = \frac{1}{c}\int_{-c/2}^{c}\left(\frac{1}{2}+x\right)(\cos 2n\pi x dx) + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)(\cos 2n\pi x dx)$$

$$= 2\left[\left(\frac{1}{2}+x\right)\sin\frac{2n\pi x}{2n\pi}-\left(-\cos\frac{2n\pi x}{2n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+2\left[\left(\frac{1}{2}-x\right)\sin\frac{2n\pi x}{2n\pi}-(-1)\left(-\cos\frac{2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{1/2}$$

$$= 2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right]+2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4\pi^2 n^2}\right]$$

$$= \frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}; \qquad n \text{ odd}$$

$$= 0 \qquad n \text{ even}$$

And $$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi dx+2\int_{-1/2}^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right)-\left(\frac{-\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+2\left[\left(\frac{1}{2}-x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right)-(-1)\left(\frac{-\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{-1/2}$$

$$= 2\left(\frac{-1}{2n\pi}\right)+2\left(\frac{1}{4n\pi}\right)=0$$

$$f(x) = \frac{1}{4}+\frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2}+\frac{\cos 6\pi x}{3^2}+\frac{\cos 10\pi x}{5^2}\cdots\right]$$

3.4.2 Complex Form of Fourier Series

Let $f(x)$ be a piecewise continuous in each finite partial interval of $(-\infty, \infty)$ in which $f(x)$ is defined and absolutely integrable, then.

NOTES

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)dx \qquad (3.2)$$

It is called Fourier transform of $f(x)$ and is denoted by $f\{f(x)\}$. Fourier transform of $f(x)$ is a function of P and is also denoted as $\tilde{f}(P)$.

Check Your Progress

1. What do you understand by the Fourier series?
2. When a function is called periodic?
3. Define the even functions in Fourier series.
4. Learn about the odd functions in Fourier series.
5. What is the half range Fourier series?
6. Define the complex Fourier transform.

3.5 PARSEVAL'S IDENTITIES FOR FOURIER COSINE AND SINE TRANSFORM

In mathematical analysis, Parseval's identity, named after Marc-Antoine Parseval, is a fundamental result on the summability of the Fourier series of a function. Geometrically, it is generalised Pythagorean theorem for inner-product spaces (which can have an uncountable infinity of basis vectors).

Informally, the identity asserts that the sum of the squares of the Fourier coefficients of a function is equal to the integral of the square of the function,

$$\| f \|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} | f(x)|^2 \, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

Where the fourier coefficients c_n of f are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

More formally, the result holds as stated provided f is square-integrable or, more generally, in $L^2[-\pi, \pi]$. A similar result is the plancherel theorem, which asserts that the integral of the square of the fourier transform of a function is equal to the integral of the square of the function itself. In one-dimension, for $f \in L^2$ (R),

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} | f(x)|^2 \, dx$$

3.5.1 The Generalization of Parseval's Theorem

NOTES

The result is

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{-\infty} \overline{f}(\omega)\overline{g}(\omega)^* d\omega \qquad ...(3.3)$$

This has many names but is often called Plancherel's formula.

The key step in the proof of this is the use of the integral representation of the δ-function

$$\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\tau\omega} d\omega \text{ or } \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\tau\omega} d\tau \qquad ...(3.4)$$

We firstly invoke the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega) e^{\pm i\omega t} d\omega \qquad ...(3.5)$$

And then use this to re-write the LHS of Equation (3.3) as

$$\int_{-\infty}^{\infty} f(t)g(t)^* dt \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{i\omega t} d\omega\right)$$

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega')...d\omega'\right) dt \qquad ...(3.6)$$

Re-arranging the order of integration we obtain

$$\int_{-\infty}^{\infty} f(t)f(t)^* dt = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(\omega)\overline{g}(\omega')\underbrace{\left(\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt\right)}_{\text{Use delta }-\int_n} d\omega' d\omega \qquad ...(3.7)$$

The version of the integral representation of the δ-function we use in Equation (3.4) above is

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\omega-\omega')} dt \qquad ...(3.8)$$

Using this in Equation (3.7), we obtain

$$\int_{-\infty}^{\infty} f(t)g(t)^* dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\left(\int_{-\infty}^{\infty} \overline{g}(\omega')^* \delta(\omega-\omega')d\omega'\right) d\omega \qquad ...(3.9)$$

Equation (3.9) comes about because of the $f(\omega)$ general δ function property

$$\int_{-\infty}^{\infty} F(\omega')\delta(\omega-\omega')d\omega'$$

Taking $g = f$ in we immediately obtain

$$\int_{-\infty}^{\infty} [f(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega \qquad ...(3.10)$$

3.5.2 The Convolution Theorem and the Auto-Correlation Function

NOTES

The statement of the Convolution theorem is this: for two function $f(t)$ and $g(t)$ with Fourier transforms $F\,|\,f(t)\,|= \overline{f}(\omega)$ and $F\,|\,g(t)\,|= \overline{g}(\omega)$, with convolution integral defined by[1],

$$f * g = \int_{-\infty}^{\infty} f(u)g(t-u)du, \qquad ...(3.11)$$

Then the Fourier transform of this convolution is given by

$$F(f * g) = \overline{f}(\omega)\overline{g}(\omega). \qquad ...(3.12)$$

To prove Equation (3.12) we write it as

$$F(f * g) = \int_{-\infty}^{\infty} e^{-i\omega t}\left(\int_{-\infty}^{\infty} f(u)g(t-u)du\right)dt \qquad ...(3.13)$$

Now define $\tau = t - u$ and divide the order of integration to find

$$F(f * g) = \int_{-\infty}^{\infty} e^{-i\omega t} f(u)du \int_{-\infty}^{\infty} e^{-i\omega t} g(\tau)d\tau = \overline{f}(\omega) - \overline{g}(\omega) \qquad ...(3.14)$$

This step is allowable because the region of integration in the $\tau - u$ plane is infinite. As we shall later, with Laplace transforms this is not the case and requires more case.

The normalised auto-correlation function is related to this and is given by

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f(u) f^*(t-u)du}{\int_{-\infty}^{\infty} |\,f(u)\,|^2\, du}$$

Practical Harmonic Analysis: If function is not given by a formula, but by a graph or by a table of corresponding values, then process of finding the Fourier series for the function is known as Harmonic analysis.

As Mean $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{2}{2\pi - 0}\int_0^{2\pi} f(x)dx$$

$= 2$ |Mean of $f(x)$ in $(0, 2\pi)$]

Similarly $a_n = 2$[Mean of $f(x)$ cos nx in $(0, 2\pi)$]

$b_n = 2$[Mean of $f(x)$ sin nx in $(0, 2\pi)$]

Example 3.25: The turning moment T units of the crank shaft of a steam engine for a series of values of the crank-angle θ in degrees:

θ:	0°	30°	60°	90°	120°	150°	180°
T:	0	5224	8097	7850	5499	2626	0

Find the first 4 terms in a series of sines to represent T also calculate T when $\theta = 75°$

Solution: $T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta$

$b_n = 2[\text{Mean of } f(x) \sin nx \text{ in } (0, 2\pi))]$

NOTES

0°	T	sin θ	sin 2θ	sin 3θ	sin 4θ	T sin θ	T sin 2θ	T sin 3θ	T sin 4θ
0°	0	0	0	0	0	–0	0	0	0
30°	5224	0.5	0.866	1	0.866	–2612	4523.984	5224	4523.984
60°	8097	0.866	0.866	0	–0.866	–7012.002	7012.002	0	–7012.002
90°	7850	1	0	–1	0	7850	0	–7850	0
120°	5499	0.866	–0.8660	0	0.866	4767.0831	4767.0831	0	4767.0831
150°	2626	0.5	–0.866	1	–0.866	1313	–2274.116	–2626	–2274.116
						23554.08591	14028.9531	0	0

$$b_1 = 2\,[\text{Mean of } T \sin\theta \text{ in } (0, 2\pi)]$$

$$= 2\left[\frac{23554.0851}{6}\right]$$

$$= 7851.36$$

$$b_2 = 2\,[\text{Mean of } T \sin 2\theta \text{ in } (0, 2\pi)]$$

$$= 2\left[\frac{14028.9531}{6}\right]$$

$$= 1358.7725$$

$$b_3 = 2\,[\text{Mean of } T \sin 3\theta \text{ in } (0, 2\pi)]$$

$$= 0$$

$$b_4 = 2\,[\text{Mean of } T \sin 4\theta \text{ in } (0, 2\pi)]$$

$$= 0$$

$$T = (7851.36) \sin\theta + (1558.7725) \sin 2\theta$$

At $\theta = 75°$

$$T = 7851.56 \sin 75° + (1558.7725) \sin 150°$$

$$= (7851.56)\,(.9659) + (1558.7725)\,(0.5)$$

$$= (7583.8218) + (799.38625)$$

Example 3.26: Find the Fourier series as far as the second harmonic to represent the function given by table below:

x :	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$:	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solution:

x°	sin x	sin 2x	cos x	cos 2x	f(x)	f(x) sin x	f(x) sin 2x	f(x) cos x	f(x) cosx
0°	0	0	1	1	2.34	0	0	2.340	2.340
30°	0.50	0.87	0.87	0.50	3.01	1.505	2.619	2.619	1.505
60°	0.87	0.87	0.30	–0.50	3.69	3.210	3.210	1.845	1.845

NOTES

90°	1.00	0	0	–1.00	4.15	4.150	0	0	–4.150
120°	0.87	–0.87	–0.50	–0.50	3.69	3.210	–3.210	–1.845	–1.845
150°	0.50	–0.87	–0.87	0.50	2.20	1.100	–1.914	–1.914	1.100
180°	0	0	–1	1.00	0.83	0	0	–0.830	0.830
210°	–0.50	0.87	–0.87	0.50	0.51	–0.255	0.444	–0.444	0.255
240°	–0.87	0.87	–0.50	–0.50	0.88	–0.766	0.766	–0.440	–0.440
270°	–1.00	0	0	–1.00	1.09	–1.090	0	0	–1.090
300°	–0.87	–0.87	0.50	–0.50	1.19	–1.035	–1.035	0.595	–0.595
330°	–0.50	–0.87	0.87	0.50	1.64	–0.820	–1.427	1.427	0.820
					25.22	9.209	–0.547	3.353	–3.115

$$b_1 = 2[\text{Mean of } f(x) \sin x]$$

$$= 2\left(\frac{9.209}{12}\right) = 1.535$$

$$b_2 = 2[\text{Mean of } f(x) \sin 2x]$$

$$= 2\left(\frac{-0.547}{12}\right) = -0.091$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

$$= 2.1018 + 0.557 \cos x - 0.519 \cos 2x + 11.535 \sin x - 0.091 \sin 2x$$

Example 3.27: The following values of y give the displacement of a certain machine part for the rotation x of the flywheel.

x :	0°	60°	120°	180°	240°	300°	360°
y :	1.98	2.15	2.77	–0.22	–0.31	1.43	1.93

Express y is Fourier series upto the IIIrd harmonic.

Solution: Let $y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$

$$+(a_3 \cos 3x + b_3 \sin 3x)$$

x	y	cos$x$	sin$x$	cos 2x	sin 2x	cos 3x	sin 3x	y cos$x$	y sin$x$	y cos 2x	y sin 2x	y cos 3x	y sin 2x
0°	1.98	1.0	0	1.0	0	1.0	0	1.98	0	1.98	0	1.98	0
60°	2.15	0.5	0.866	–0.5	0.866	–1.0	0	1.075	1.8619	–1.075	1.8619	–2.15	0
120°	2.77	–0.5	0.866	–0.5	–0.866	1.0	0	–1.385	2.3988	–1.385	–2.3988	2.77	0
180°	–0.22	7.0	0	1.0	0	–1.0	0	0.22	0	0	0	.22	0
240°	–0.31	–0.5	–0.866	–0.5	0.866	1.0	0	0.155	0.2685	0.2685	–0.2685	–.31	0
360°	–1.43	0.5	–0.866	–0.5	–0.866	–1.0	0	0.715	–1.2383	–.715	–1.2383	–1.43	0
	7.8							2.76	3.2909	–1.4635	–2.0437	1.08	0

$$b_3 = 0$$

$$\therefore \quad y = 1.3 + (0.92 \cos x + 1.0969 \sin x)$$

$$+ (-0.4878 \cos 2x - 0.6812 \sin 2x) + (0.36 \cos 3x)$$

Example 3.28: The following table gives the variations of a periodic current over a period

t (sec) :	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A (amp) :	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

Show that there is a direct current part of 0.75 amp. in the variable current, and obtain the amplitude of the first harmonic.

NOTES

Solution: Let $A = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} \cdots$

t	A	$\cos\left(\frac{2\pi t}{T}\right)$	$\sin\left(\frac{2\pi t}{T}\right)$	$A\cos\frac{2\pi t}{T}$	$A\sin\frac{2\pi t}{T}$
.0	1.98	1	0	1.98	1.1258
T/6	1.3	0.5	0.866	0.65	0
T/3	1.05	–0.5	0.866	–0.525	0.9093
T/2	1.3	–1	0	–1.3	0
2T/3	–0.88	–0.5	–0.866	.44	0.76208
5T/6	–0.25	0.5	–0.866	–0.125	0.2165
	4.5			1.12	3.01348

a_0, a_1 and a_2

$$\therefore\ A = 0.75 + 0.373 \cos\left(\frac{2\pi t}{T}\right) + 1.005 \sin\left(\frac{2\pi t}{T}\right)$$

$\therefore$ A has a direct current part of 0.75 amp.

The amplitude of first harmonic is given by

$$= \sqrt{(.373)^2 + (1.005)^2}$$

$$= \sqrt{1.1491}$$

$$= 1.072$$

1. Analyse the current i(amp) given in the table below upto III harmonic.

$\theta°$:	0	30	60	90	120	150	180	210	240	270	300	330
i:	.0	24	33.5	27.5	18.2	13	0	–24	–33.5	–27.5	–18.2	–13

2. Find the Fourier series upto III harmonic.

(i)

x :	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y :	0.8	0.6	0.4	0.7	0.9	1.1	0.8

(ii)

x :	0	30	60	90	120	150	180	210	240	270	300	330
y :	25	40	50.5	57.5	61.5	63.3	68.2	59.2	52.2	44.2	35.8	28.7

3. The turning moment T on the crank-shaft of a steam engine for the crank angle θ (degrees)

θ :	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
T:	0	2.7	5.7	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto second harmonic.

3.6 FOURIER INTEGRAL AT INCLUDING ITS COMPLEX FORM

NOTES

Let $f(x)$ be a function defined in $(-\ell, \ell)$ that satisfies Dirichlet's condition. Suppose at every point of discontinuity, $f(x)$ is defined as, $\frac{1}{2}\left[f(x+0)+f(x-0)\right]$ and $f(x)$ is absolutely integrable in $-\infty < x < \infty$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} \cos v(x-u)\,du\right\}$$

Or $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\int_{-\infty}^{\infty} f(u)\cos v(x-u)\,du \quad ....(3.15)$$

This is called Fourier Integral formula.

3.6.1 Fourier Transforms

Let $f(x)$ be a function on the interval (a, b), then it is said to satisfy Dirichlet's conditions: (i) $f(x)$ is defined and single valued except at a finite number of points in the interval (a, b), and (ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a, b).

It $f(x)$ is a periodic function with period $2l$, that means if $f(x) = f(x) + 2l$ and stisfies Dirichlet's conditions in the interval $(-l, l)$ then at every point of continuity Fourier series is defined as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right]$$

Where, $$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

And $$b_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

a_n and b_n are called Fourier coefficients corresponding to $f(x)$.

If function $f(x)$ is even function in the interval $(-\ell, \ell)$

Or $f(-x) = f(x)$, then,

$$a_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = 0$$

In this case, it is called Fourier cosine series.

If function $f(x)$ is odd function in the interval $(-\ell, \ell)$

Or $f(-x) = -f(x)$, then

$$a_n = 0$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

NOTES

In this case, it is called Fourier sine series.

3.7 CONVOLUTION THEOREM INCLUDING SINE AND COSINE TRANSFORMS

Therom 3.1. Let $f_1(t)$ and $f_2(t)$ are two functions having Fourier transforms $\hat{f}_1(w)$ and $\hat{f}_2(w)$ respectively then

$$F(f_1 * f_2) = \hat{f}_1(w).\hat{f}_2(w)$$

*denote the convolution product and is defined as

$$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(x) f_2 \; dx$$

The Convolution theorem states that the Fourier transform of convolution product of two functions is equal to the $\sqrt{2\pi}$ times their product of Fourior transforms.

Proof: $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x)\,dx$

$$\Rightarrow \quad F(f_1 * f_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_1(x) f_2(t-x)dx\right)e^{-iwt}dt\,dx$$

Let $t - x = u \Rightarrow t = u + x$

$dt = du$

$$F(f_1 * f_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x)\, f_2(u)\, e^{-iw(w+x)} du\,dx$$

The double integral in R.H.S. can be written as product of two integrals as

$$F(f_1 * f_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)\, e^{-iwx}\, dx \int_{-\infty}^{\infty} f_2(u) e^{-iwn}\, du$$

$$= \frac{1}{\sqrt{2\pi}}.\sqrt{2\pi}F(f_1).\sqrt{2\pi}F(f_2)$$

thus $F(f_1 * f_2) = \sqrt{2\pi} f(f_1).f(f_2)$

or $(f_1 * f_2)(w) = \sqrt{2\pi}\hat{f}_1(w).\hat{f}_2(w)$

or $f(f_1 * f_2) = \sqrt{2\pi}\hat{f}_1(w)\hat{f}_2(w)$

NOTES

Similar convolution theorem for Fourier sine and Fourier cosine transforms can also be written

i.e., $$F_c(f_1 * f_2) = \hat{f}_{c'}(w).\hat{f}_{c^2}(\omega)$$

$$F_s(f_1 * f_2) = \hat{f}_{c},(w).\hat{f}_{c^2}(\omega)$$

But proof of these theorems are beyond the scope of the book.

Example 3.29: Let $f_1(t) = f_1(t) = \begin{cases} 1-t & 0 \le t,1 \\ 0 & \text{otherwise} \end{cases}$

$$f_2(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Solution:

$$\text{F}(f_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t)\, e^{-iwt} dt = \frac{1}{\sqrt{2\pi}} \int_0^1 (1-t) e^{-iwt} dx$$

$$\hat{f}_1(\omega) = F_1(f_1) = \frac{1}{\sqrt{2\pi}} \left[\int_0^1 e^{-iwt} dt - \int_0^1 te^{-iwt} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwt}}{-iw} \right]_0^1 - \left\{ \left. \frac{te^{-wt}}{-iw} \right|_0^1 + \frac{1}{i\omega} \int_0^1 1e^{-iwt} dt \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{iw} \left(e^{-i\omega} - 1 \right) + \frac{1}{iw} e^{-iw} - \frac{1}{i^2 w^2} \left[e^{-i\omega fl} \right]_0^1 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\left(1 - e^{-i\omega}\right) + e^{-iw}}{iw} + \frac{1}{w^2} \left(e^{-iw} - 1 \right) \right]$$

$$\hat{f}_1(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{iw} + \frac{1}{w^2} (-1 + e^{-iw}) \right]$$

Let $$f_2(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$F(f_2) = \hat{f}_2(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega t} e^{-t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(i\omega+1)t} dt = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(i\omega+1)+t}}{-(iw+1)} \right]_0^\infty$$

$$\hat{f}_2(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{(iw+1)}$$

NOTES

$$(f_1 \times f_2)(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

$$(f_1 * f_2)(t) = \begin{cases} 0 & t < 0 \\ 2 - t - 2e^{-t} & 0 \le t \\ (e-2)e^{-t} & t \ge 1 \end{cases}$$

The Fourier transform of $(f_1 * f_2)(t)$ will be obtained using convolution theorem of $(f_1 \times f_2) = \sqrt{2\pi}\, \hat{f}_1(w) . \hat{f}_2(w)$

$$= \sqrt{2\pi} . \frac{1}{\sqrt{2\pi}} \left[\frac{1}{iw} + \frac{1}{w^2}\left(-1 + e^{-iw}\right) \right] = \frac{1}{\sqrt{2\pi}} . \frac{1}{(iw+1)}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{iw(iw+1)} + \frac{1}{w^2(iw+1)} \left(-1 + e^{-iw}\right) \right]$$

$$F(f_1 * f_2) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(-w^2 + iw)} + \frac{1}{(iw^3 + w^2)} \left(-1 + e^{-iw}\right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{w} \left[\frac{1}{(-w^2 + 1)} + \frac{1}{(iw^2 + w)} \left(-1 + e^{-iw}\right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{w} \left[\frac{(-w^2 + i)}{(w^2 + 1)} + \frac{(w - iw^2)}{(w^2 + w^4)} \left(-1 + e^{-iw}\right) \right]$$

$$f(f_1 * f_2) = \frac{1}{\sqrt{2\pi}} \frac{1}{w} \left[\frac{(-w^2 + i)}{(1 - w^2)} + \frac{(1 - iw^2)}{w(1 + w^2)} \left(-1 + e^{-iw}\right) \right]$$

3.7.1 Relations Between Fourier and Laplace Transforms

We know that the Laplace transform of $f(t)$ is

$$F(s) = \int_0^{\infty} f(t)\, e^{-st} dt \qquad ...(3.16)$$

If we take $s = iw$ along the imaginary axis then (3.16) becomes

$$F(iw) = \int_0^{\infty} f(t)\, e^{-iwt} ds$$

Assume $\quad f(t) =$ for $t < 0$ then

$$F(iw) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt} \qquad ...(3.17)$$

or $$\text{F}(iw) = \sqrt{2\pi}\, F(f)$$

NOTES

This is the between Laplace and Fourier transform. In nutshell we can say. Laplace of $f(t)$ with $s = iw$ converted into Fourier transform.

3.8 MULTIPLE FINITE FOURIER TRANSFORM

In many physical phenomena we frequently see the function

$$\hat{f}(w) = \int_a^b f(t)\, k(\omega, t)\, dt$$

the function $\hat{f}(w)$ is called the integral transform of $f(t)$ by the kernel $k(w, t)$. by we choose the kernel e^{-iwt} for $t\ \varepsilon\ (-\infty, \infty)$ then we get

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$ and is called as Fourier transform

of $f(t)$ and in respect of $\hat{f}(\omega)$, $f(t)$ is the inverse Fourier transform of $f(t)$ and in respect of $\hat{f}(\omega)$, $f(t)$ is the inverse Fourier Transform.

Fourier transform : Fourier transform of $f(t)$ is as the function $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$

we say $$F(f(t)) = \hat{f}(\text{w})$$

and $$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwt} \hat{f}(w)\, dw$$ is called

inverse fourier transform.

We write $$f^{-1}\left(\hat{f}(w)\right) = f(t)$$

Fourier sine and Fourier cosine transform :

$$\hat{f}_0(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt\, dt$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t)\ \sin wt\, dt$$

are respectively called as Fourier cosine and Fourier sine transform.

and their respective inverses are

$$f_c(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w) \text{ con wt } dt$$

NOTES

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w) \text{ sin wt } dt$$

3.8.1 Solution of Simple Partial Differential Equations by Means of Fourier Transforms

An extension of the Fourier Series represent action to an infinite domain, several additional notions such as 'Fourier integral representation' and 'Fourier transform' of a function are presented.

Example 3.30: Solve the following boundary value problem $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2}(x,t),\ x > 0,\ t > 0$

Subject of the boundary conditions:

$$u(0, t) = 0 \quad u(x,0)\begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

and $u(n, t)$ is bounded.

Solution. The given differential equation is $\frac{2\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 1$.

Taking Fourior sine transform of (1) both side.

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t} \sin wx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin wx\, dx$$

or

$$\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial t}\int_0^\infty u \sin wx\, dx = \sqrt{\frac{2}{\pi}}\left[\left(\frac{\partial u}{\partial x}\sin wx\right)_0^\infty - w\int_0^\infty \frac{\partial u}{\partial x}\cos wx\, dx\right] \quad ...(2)$$

or but $\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin wn\, dx = \hat{u}_s(w,t)$

and $\frac{\partial u}{\partial x} \to 0 \quad$ as $x \to \infty$

so (2) becomes

$$\frac{\partial}{\partial x}\hat{u}_s(w,t) = \sqrt{\frac{2}{\pi}}\left[0 - w\left\{(u \cos wx)_0^\infty\right.\right.$$

NOTES

$$+\int_0^\infty w \sin wx\ u\ dx$$

$$\frac{\partial}{\partial t}\hat{u}_s(w,t) = \sqrt{\frac{2}{\pi}}w\left[0 - u(0_1 t) - w\int_0^\infty u \sin wx\ dx\right]$$

$$+\sqrt{\frac{2}{\pi}}w^2\left[\int_0^\infty u \sin wx\ dx\right]$$

using $u(0, t) = 0; \quad u \to 0 \quad$ as $x \to \infty$

or $$\frac{\partial \hat{u}_s}{\partial t} = -w^2\hat{u}_s(w,t)$$

or $$\frac{\partial \hat{u}_s}{\partial t} + w^2\hat{u}_s(w,t) = 0$$

or $$\int \frac{\partial \hat{u}_s}{\hat{u}_s} = -\int w^2 \partial t + \log c_1$$

or $$\log \hat{u}_s = -w^2 t + \log c_1$$

or $$\hat{u}_s(w,t) = c_1 e^{-w^2 t} \qquad ...(3)$$

using the BC : at $t = 0$

$$\hat{u}_s(w,0) = c_1$$

or $$c_1 = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0) \sin wx\, dx$$

or $$c_1 = \sqrt{\frac{2}{\pi}}\left[\int_0^1 1 \sin wx\, dx + \int_1^\infty 0 \sin wx\, dx\right]$$

$$= \left[\sqrt{\frac{2}{x}}\frac{\cos wx}{w}\right]_0^1$$

or $$c_1 = -\sqrt{\frac{2}{\pi}}\frac{1-\cos w}{w}$$

thus $$\hat{u}_s = \sqrt{\frac{2}{\pi}}\frac{1-\cos w}{w}e^{-w^2 t}$$

Applying inverse Fourier sine transform we get the desired solution of given BVP as

$$u(x,t) = F_s^{-1}(\hat{u}_s)$$

NOTES

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_s \sin wx \, dx$$

$$u(x, t) = \left(\frac{2}{\pi}\right) \int_0^\infty \left(\frac{1-\cos w}{w}\right) \sin wx \, dx$$

is the desired solution

Example 3.31: Using the method of Fourier transform, determine the displacement $u(x, t)$ of an infinite string, given that the string is initially at rest and the initial displacement is $g(x) - \infty < x < \infty$. Show that the solution can be put in the form

$$u(x,t) = \frac{1}{2}\left[g(x+kt) + (x-kt)\right]$$

Solution: We know that the displacement of a string is governed by one - dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

where $u(x, t)$ is the displacement of string at any time t and $-\infty < x < \infty$,

$$k^2 = \frac{T}{\delta}$$

Subject to the boundary conditions (i) $u(x, 0) = f(x)$

and (ii) $\frac{\partial u}{\partial t} = 0$ at $t = 0$ (since string is initially at rest)

Taking fourier stransform of (1) both sides we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{-iwx} dx = k^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-iwx} dx$$

or
$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-iwx} dx\right) = k^2 \hat{U}(w, t)(iw)^2$$

$\left(\text{wheve } \hat{U}(w, t) = F\left(u(x_1 t)\right)\right)$

or
$$\frac{\partial^2}{\partial t^2} \hat{U}(w, t) + k^2 w^2 \hat{U}(w,t) \quad ...(2)$$

The solution of (2) is given by

$$\hat{U}(w,t) = c_1 \cos k\, wt + c_2\, sin\, kwt \quad ...(3)$$

Using the boundary condition

$$\frac{\partial u}{\partial t} = 0 \quad \text{at } t = 0 \text{ gives}$$

$$F\left(\frac{\partial u}{\partial t}\right) = 0$$

NOTES

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2u}{\partial t}e^{-iwx}dx = 0$$

or $$\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ve^{-iwx}dx = 0$$

or $$\frac{\partial \hat{U}(w,t)}{\partial t} = 0 \quad \text{at } t = 0 \qquad ...(4)$$

Using (4) for (3) (i.e., differentiating partially (3) with respect to t and putting $t = 0$) we get $c_2 \Rightarrow \hat{U}\left(w,t\right) = c_1 \cos kwt$ ...(5)

Again it is given that

$$u(x, 0) = f(x)$$

$$\Rightarrow \quad F(u(x, 0)) = F(f(x))$$

$$\hat{U}\left(w,0\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$$

$$\hat{U}(w, 0) = \hat{f}(w) \qquad ...(6)$$

From (5) and (6) we can estimaten the value of c_1 as

$$\hat{U}(w,0) = c_1 = \hat{f}(w)$$

Hence $$\hat{U}(w,\ t) = \hat{f}(w)\ \cos kwt$$

by taking inverse Fourier transform, we have

$$u(x,\ t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{U}(w,t)e^{+iwx}dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\ \cos kwt\ e^{iwx}dw$$

$$u(x,\ t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\left(\frac{e^{ikwt}+e^{-ikwt}}{2}\right)e^{iw}dx$$

or $$u(x,\ t) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\ e^{iw(x+kt)}dw + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\ e^{+iw(x-kt)}dw\right]$$

$$u(x,\ t) = \frac{1}{2}\left[f\left(x+kt\right)+\left(x-kt\right)\right]$$

in view of definition of inverse Fourier transform.

NOTES

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}\hat{f}(w)\,e^{iwx}dx$$

Next we consider some examples of ordinary differential equations.

Example 3.32: Find the solution of $-y^{11}+y=f(t)$ given the condition $\lim_{1\to\infty} u(t)=0$.

Solution: Taking the Fourier transform of given ODE we have

$$w^2\hat{y}(w)+\hat{y}(w) = \hat{f}(w)$$

or $$\hat{y}(w) = \frac{\hat{f}(w)}{1+w^2} \qquad ...(1)$$

From (1) we can recover the solution of given ODE by of inverse of Fourier transform.

$$y(t) = F^{-1}\left(t\,\hat{y}(w)\right)$$

$$= \left(F^{-1}(\hat{f}w)\frac{1}{1+w^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}f(t)*F^{-1}C\left(\frac{1}{1+w^2}\right)$$

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-|1t-y|}f(y)\,dy$$

Example 3.33. Obtain the solution of Airy equation

$$y'' - ty = 0$$

Solution. Subject to field condition

$$\lim_{|t|\to\infty} y(t) = 0$$

$$y'' - ty = 0 \quad \text{we get}$$

$$-w^2\hat{y}(w)-\frac{id\hat{y}\,(w)}{dw} = 0$$

or $$\frac{d\hat{y}(w)}{dw} = iw^2\hat{y}(w)$$

whose solution is

$$\hat{y}(w) = c\,e^{iw^3/3}$$

NOTES

taking inverse Fourier we get

$$y(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{i\left(wx+\frac{w^2}{3}\right)}\, dw$$

for the choice $c = 1$, we get

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(wx+\frac{w^3}{3}\right)}\, dw$$

which is called Airy function and is denoted by $A_i(t)$

Example 3.34 Find the solution of Laplace equation is upper half plane

$$\text{i.e., solve} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty$$

$$y > 0, \quad u(x, 0) = f(x), \ \lim_{y\to\infty} u(x, y) = 0$$

Solution. We denote the Fourier transform in the x variable as

$$\text{F}(u(x, y)) = \hat{u}(w, y)$$

$$= \int_{-\infty}^{\infty} e^{-iwx} u(x, y)\, dx \qquad ...(1)$$

we note that the y-derivative commutes with the Fourier integral in x, so that the transform of u_{yy} is simply denoted by $\hat{U}_{yy}$. Taking the Fourier transform of given equation with boundary conditions we get

$$-w^2\hat{u} + \hat{U}_{yy} = 0, \quad \hat{U}(w, 0) = \hat{f}(w)$$

$$\lim_{y\to\infty} \hat{U}(w, y) = 0$$

$$\hat{U}_{yy} = w^2\hat{U}$$

whose solution is

$$\hat{U}(w, y) = c, e^{|w|y} + c_2 e^{-|w|y}$$

taking $y \to \infty$, $\hat{U}(w, y) = 0$ gives

$$\hat{U}(w, y) = c_2\, e^{-|w|y}$$

using

$$\hat{U}\left(w, 0\right) = \hat{f}(w) \text{ we get}$$

$$c_2 = \hat{f}(w)$$

Thus $$\hat{U}(w, y) = \hat{f}(w)e^{-|w|y}$$

Taking inverse Fourier transform we get

$$u(x, y) = f(x) * F^{-1}\left(e^{-|w|y}\right)$$

$$= f(x) * \left(\frac{y}{\pi(x^2 + y^2)}\right)$$

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)}{(x-t)^2 + y^2}\, dt$$

(in view of convolution theorem)

is the desired solution of given BVP.

NOTES

Check Your Progress

7. Define the Parseval's identity in Fourier series.
8. What is the generalisation of Parseval's theorem?
9. State the convolution theorem.
10. What is the auto-correlation function?
11. Define the practical harmonic analysis.
12. What do you understand by the Fourier's integral formula?
13. Define Fourier transformation.

3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Fourier series is used as infinite series representation of periodic function and it uses trigonometric sine and cosine functions for expansion. Its main application is to solve ordinary and partial differential equations. It is a powerful tool to solve differential equations specially with periodic functions appearing as non-homogeneous terms. It has wider applications as it is valid for periodic functions as well as continuous functions and for functions, which are discontinuous.
2. Function $f(x)$ is said to be periodic if $f(x + T) = f(x)$, and real x for some positive number T is period of $f(x)$. Smallest positive period of $f(x)$ is called primitive or fundamental period of $f(x)$.
3. When $f(-x) = f(x)$, $\forall x$, then function $f(x)$ is said to be even. For even functions, graph is symetric about y-axis. It uses the property of Integration,

   $$\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a} f(x)\,dx$$

   Even functions contains only even powers of x and in trignometric terms, they contain only cos$x$ and sec$x$.
4. When $f(-x) = -f(x)$, $\forall x$, then function f(x) is said to be odd function. Graph of odd function is symmetric about origin and it uses the property of integration,

NOTES

$$\int_{-a}^{a} f(x)\, dx = 0$$

Odd functions contain only odd powers of x and in trignometric function, it contains only sin$x$ and cosec$x$.

5. Fourier expansion has been defined for function, which is periodic with period $2l$. Now suppose we are given a function $f(x)$, which is non-periodic and is defined in half interval $(0, l)$ of length l. These types of expansions are known as half range Fourier series. In this case, $f(x)$ is neither even nor odd nor periodic. Only information is to obain Fourier cosine series for $f(x)$ in the interval $(0, l)$.

6. Let $f(x)$ be a piecewise continuous in each finite partial interval of $(-\infty, \infty)$ in which $f(x)$ is defined and absolutely integrable, then.

   $$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)dx$$

   It is called Fourier transform of $f(x)$ and is denoted by $f\{f(x)\}$. Fourier transform of $f(x)$ is a function of P and is also denoted as $\tilde{f}(P)$.

7. In mathematical analysis, Parseval's identity, named after Marc-Antoine Parseval, is a fundamental result on the summability of the Fourier series of a function. Geometrically, it is generalised Pythagorean theorem for inner-product spaces (which can have an uncountable infinity of basis vectors).

8. $$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{-\infty} \bar{f}(\omega)\bar{g}(\omega)^* d\omega$$

   This has many names but is often called Plancherel's formula.

9. The statement of the Convolution theorem is this: for two function $f(t)$ and $g(t)$ with Fourier transforms $F\,|\,f(t)\,| = \bar{f}(\omega)$ and $F\,|\,g(t)\,| = \bar{g}(\omega)$, with convolution integral defined by

   $$f * g = \int_{-\infty}^{\infty} f(u)g(t-u)du,$$

   Then the Fourier transform of this convolution is given by

   $$F(f * g) = \bar{f}(\omega)\bar{g}(\omega).$$

10. The normalised auto-correlation function is related to this and is given by

    $$\gamma(t) = \frac{\int_{-\infty}^{\infty} f(u) f^*(t-u)du}{\int_{-\infty}^{\infty} |\,f(u)\,|^2\, du}$$

11. Practical Harmonic Analysis: If function is not given by a formula, but by a graph or by a table of corresponding values, then process of finding the Fourier series for the function is known as Harmonic analysis.

    $$\text{As Mean} = \frac{1}{b-a}\int_a^b f(x)dx$$

    $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

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12. Let $f(x)$ be a function defined in $(-\ell, \ell)$ that satisfies Dirichlet's condition. Suppose at every point of discontinuity, $f(x)$ is defined as, $\frac{1}{2}[f(x+0)+f(x-0)]$ and $f(x)$ is absolutely integrable in $-\infty < x < \infty$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} \cos v(x-u)du\right\}$$

13. Let $f(x)$ be a function on the interval (a, b), then it is said to satisfy Dirichlet's conditions: (i) $f(x)$ is defined and single valued except at a finite number of points in the interval (a, b), and (ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a, b).

3.10 SUMMARY

- Fourier series is used as infinite series representation of periodic function and it uses trigonometric sine and cosine functions for expansion. Its main application is to solve ordinary and partial differential equations. It is a powerful tool to solve differential equations specially with periodic functions appearing as non-homogeneous terms. It has wider applications as it is valid for periodic functions as well as continuous functions and for functions, which are discontinuous.
- Function $f(x)$ is said to be periodic if $f(x + T) = f(x)$, and real x for some positive number T is period of $f(x)$. Smallest positive period of $f(x)$ is called primitive or fundamental period of $f(x)$.
- When $f(-x) = f(x)$, $\forall x$, then function $f(x)$ is said to be even. For even functions, graph is symetric about y-axis. It uses the property of Integration,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a} f(x)\,dx$$

  Even functions contains only even powers of x and in trignometric terms, they contain only cos$x$ and sec$x$.
- When $f(-x) = -f(x)$, $\forall x$, then function f(x) is said to be odd function. Graph of odd function is symmetric about origin and it uses the property of integration,

$$\int_{-a}^{a} f(x)\,dx = 0$$

  Odd functions contain only odd powers of x and in trignometric function, it contains only sin$x$ and cosec$x$.
- Fourier expansion has been defined for function, which is periodic with period $2l$. Now suppose we are given a function $f(x)$, which is non-periodic and is defined in half interval $(0, l)$ of length l. These types of expansions are known as half range Fourier series. In this case, $f(x)$ is neither even nor odd nor

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periodic. Only information is to obain Fourier cosine series for $f(x)$ in the interval $(0, l)$.

- Let $f(x)$ be a piecewise continuous in each finite partial interval of $(-\infty, \infty)$ in which $f(x)$ is defined and absolutely integrable, then.

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-ipx} f(x)dx$$

It is called Fourier transform of $f(x)$ and is denoted by $f\{f(x)\}$. Fourier transform of $f(x)$ is a function of P and is also denoted as $\tilde{f}(P)$.

- In mathematical analysis, Parseval's identity, named after Marc-Antoine Parseval, is a fundamental result on the summability of the Fourier series of a function. Geometrically, it is generalised Pythagorean theorem for inner-product spaces (which can have an uncountable infinity of basis vectors).

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{-\infty} \overline{f}(\omega)\overline{g}(\omega)^{*} d\omega$$

This has many names but is often called Plancherel's formula.

- The statement of the Convolution theorem is this: for two function $f(t)$ and $g(t)$ with Fourier transforms $F \mid f(t) \mid = \overline{f}(\omega)$ and $F \mid g(t) \mid = \overline{g}(\omega)$, with convolution integral defined by[1].

$$f * g = \int_{-\infty}^{\infty} f(u)g(t-u)du,$$

Then the Fourier transform of this convolution is given by

$$F(f * g) = \overline{f}(\omega)\overline{g}(\omega).$$

- The normalised auto-correlation function is related to this and is given by

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f(u)f^{*}(t-u)du}{\int_{-\infty}^{\infty} \mid f(u) \mid^{2} du}$$

- Practical Harmonic Analysis: If function is not given by a formula, but by a graph or by a table of corresponding values, then process of finding the Fourier series for the function is known as Harmonic analysis.

$$\text{As Mean} = \frac{1}{b-a}\int_{a}^{b} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx$$

- In Fourier transformation, Let $f(x)$ be a function on the interval (a, b), then it is said to satisfy Dirichlet's conditions: (i) $f(x)$ is defined and single valued except at a finite number of points in the interval (a, b), and (ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a, b).
- The Convolution theorem states that the Fourier transform of convolution product of two functions is equal to the times their product of Fourior transforms.

3.11 KEY TERMS

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- **Fourier series:** Fourier series is used as infinite series representation of periodic function and it uses trigonometric sine and cosine functions for expansion.
- **Even functions in Fourier series:** Functions can be defined as even and odd functions. When $f(-x) = f(x)$, $\forall x$, then function $f(x)$ is said to be even. For even functions, graph is symmetric about y-axis. Even functions contains only even powers of x and in trigonometric terms, they contain only cos x and sec x.
- **Odd functions in Fourier series:** When $f(-x) = -f(x)$, $\forall x$, then function $f(x)$ is said to be odd function. Graph of odd function is symmetric about origin. Odd functions contain only odd powers of x and in trigonometric function, it contains only sin x and cosec x.
- **Half range Fourier series:** Fourier expansion has been defined for function, which is periodic with period $2l$. Now, suppose we are given a function $f(x)$, which is non-periodic and is defined in half interval $(0, l)$ of length l. These types of expansions are known as half range Fourier series.
- **Parseval's identity:** The Parseval's identity, named after Marc-Antoine Parseval, is a fundamental result on the summability of the Fourier series of a function. Geometrically, it is a generalised Pythagorean theorem for inner-product spaces (which can have an uncountable infinity of basis vectors).
- **Practical harmonic analysis:** If function is not given by a formula, but by a graph or by a table of corresponding values, then process of finding the Fourier series for the function is known as harmonic analysis.
- **Fourier transformation:** Let $f(x)$ be a function on the interval (a,b), then it is said to satisfy Drichlet's conditions: (i) $f(x)$ is defined and single valued except at a finite number of points in the interval (a,b), and (ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a,b).

3.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the Fourier series.
2. What is the even functions in Fourier series?
3. How will you define the odd functions in Fourier series?
4. What do you understand by the the half range Fourier series?
5. State the complex Fourier transform.
6. Define the Parseval's identity in Fourier series.
7. State the generalisation of Parseval's theorem.

8. State the convolution theorem.
9. What is the auto-correlation function?
10. Define the practical harmonic analysis.

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11. Write the Fourier's integral formula.
12. Define the Fourier transformation.
13. State convolution theorem.
14. What is multiple finite Fourier transform?

Long-Answer Questions

1. Discuss the Fourier series with appropriate example.
2. Explain the even functions in Fourier series. Give example.
3. Explain the odd functions in Fourier series.
4. Analyse the half range Fourier series.
5. Find a series of cosine for $f(x)$ in $(0, \pi)$

   Where $f(x) = 0$ for $0 < x < \frac{\pi}{2}$

   $= \frac{\pi}{2}$ $\quad \frac{\pi}{2} < x < \pi$

   Deduce $1 \frac{-1}{3} + \frac{1}{5} - \frac{1}{7} \quad \infty = \frac{\pi}{9}$
6. Find sine and cosine series

   $f(x) = x + 1$ for $0 < x < \pi$
7. Find half range cosine & sine series for

   $f(x) = e^{\pi}$ for $0 < x < \pi$
8. Find half range cosine series for

   $f(x) = x^2$ for $0 < x < \pi$
9. Find cosine series for $f(x) = \pi - x$, $0 < x < \pi$
10. If $f(x) = x = \pi - x$ for $0 < x < \pi/2$ $\quad \pi/2 < x < \pi$

    Prove that $f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x\right)$

    $= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x\right)$
11. Find Fourier series corresponding to

    $f(x) = 2$ is $-2 \leq x \leq 0$

    $= x$ $\quad 0 < x < 2$
12. Prove that $\frac{1}{2} - x = \frac{1}{\pi}\sum_{0}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{1}, 0 < x < 1$

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13. $f(x) = -a \qquad -c < x < 0$

$\qquad a \qquad 0 < x < c$

14. $f(t) = 2t, \qquad 0 < t < 1$

$\qquad = 2(2 - t), \quad 1 < t < 2$

Find Fourier half range cosine series

15. Find Fourier series of

$f(x) = |x|, \qquad -2 < x < 2$

16. Let $f(x) = 5x, \qquad 0 \le x \le \frac{l}{2}$

$\qquad = 5(l - x), \qquad \frac{l}{2} \le x \le l$

P.T. $f(x) = \frac{45l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{l}$

hence prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots \frac{\pi^2}{8}$

17. Express $x(\pi - x)$ in a half range sine series is the interval $(0, x)$

18. Find the Fourier series of

$$f(x) = \begin{bmatrix} -2 & \text{for} & -4 < x < -2 \\ x & & -2 < x < 2 \\ 2 & & 2 < x < 4 \end{bmatrix}$$

19. Discuss the Parseval's identity in Fourier series.
20. Describe the generalisation of Parseval's theorem.
21. Analyse the convolution theorem.
22. Explain the convolution theorem and the auto-correlation function.
23. Describe the Fourier integral theorem with the help of giving examples.
24. Discuss the Fourier transformation. Give an example.
25. Explain briefly about the convolution theorem including sine and cosine transform with the help of giving examples.
26. What do you understand by the multiple finite Fourier transform? Give appropriate examples.

3.13 FURTHER READING

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UNIT 4 MELLIN AND HANKEL TRANSFORMS

NOTES

Structure

4.0 INTRODUCTION

In mathematics, the Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions; it is closely related to the Laplace transform and the Fourier transform, and the theory of the gamma function and allied special functions.

The Hankel transform expresses any given function $f(r)$ as the weighted sum of an infinite number of Bessel functions of the first kind $Jv(kr)$. The Bessel functions in the sum are all of the same order *í*, but differ in a scaling factor k along the r axis. The necessary coefficient Fv of each Bessel function in the sum, as a function of the scaling factor k constitutes the transformed function. The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval.

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In this unit, you will study about the Mellin and Hankel transforms, elementary properties of Mellin transforms, Mellin transforms of derivatives, inversion theorem of Mellin integrals, distribution of potential in a wedge, elementary properties of Hankel transforms, Hankel transforms of the derivatives of functions and some elementary function, relations between Fourier and Hankel transform, parseval relation for Hankel transform and use of Hankel transforms in the solution of simple partial differential equations.

4.1 OBJECTIVES

After going through this unit you will be able to:

- Describe the basics of Mellin and Hankel transforms
- Explain the Mellin transforms of derivatives
- Discuss the inversion theorem of Mellin integrals
- Elaborate on the distribution of potential in a wedge
- Describe the various elementary properties of Hankel transforms
- Explain the basics of Hankel transforms of the derivatives of functions and some elementary functions
- Explain the relations between Fourier and Hankel transform
- State the parseval relation for Hankel transform
- Discuss the use of Hankel transforms in the solution of simple partial differential equations

4.2 ELEMENTARY PROPERTIES OF THE MELLIN TRANSFORMS

Denote $\quad M(f(t)) = F(s) = \int_0^\infty f(t)t^{s-1}dt$

or $\quad \mathrm{M}(f i\, s) = F(s)$

1. Scaling: Let $\quad M(f i\, s) = F(s)$

then $\quad M(f(rt)_i s) = r^{-s} F(s),\ \mathrm{r} > 0$

Proof:

$$M(f(rt i\, s) = \int_0^\infty f(rt)t^{s-1}dt$$

Put $\quad rt = u \quad \Rightarrow \quad t = \frac{u}{r}$ in R.H.S

$$\mathrm{dt} = \frac{du}{r}$$

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Then $M(f(rt)i\,s) = \int_0^\infty f(u)\left(\frac{u}{r}\right)^{s-1} \frac{du}{r}$

or $M(f(rt)i\,s) = r^{-s}\int_0^\infty f(u)\,u^{s-1}\,du$

or $M(f(rt)i\,s) = r^{-s} F(s)$

2. Raising of the original variable t to a real power by say t^n

$$M(f(t)is) = F(s) \text{ than}$$

$$M(f(t^p)i,s) = \frac{1}{p}F\left(\frac{s}{p}\right);\ \frac{s}{p} \in s,\quad p \neq 0 \text{ (real)}$$

where $s = \{s: \sigma_1 < \text{Re}\,(s) < \sigma_2\}$ is the strip of homomorphy.

Proof:

$$M(f(t^p)i,\,s) = \int_0^\infty f(t^p)t^{s-1}dt$$

Put $t^p = u \quad \Rightarrow \quad t = u^{1/p}$

$$dt = \frac{1}{p}u^{1/p-1}du \quad \text{in R.H.S.}$$

$$M(f(t^p)i,\,s) = \int_0^\infty f(u)\,.u^{\frac{s-1}{p}}\,.\frac{1}{p}u^{1/p-1}du$$

$$= \frac{1}{p}\int_0^\infty f(u)\quad u^{\frac{s}{p}-\frac{1}{p}+\frac{1}{p}-1}du$$

$$M(f(t^p)i,\,s) = \frac{1}{p}\int_0^\infty f(u)\,u^{\frac{s}{p}-1}du$$

$$= \frac{1}{p}F\left(\frac{s}{p}\right)$$

3. Multiplication by t^a

If $M(f(t)i,\,s) = F(s)$ then

on $\left(f(t)t^a i,s\right) = F(s+a)$

Proof: $M\left(f(t)t^a i,s\right) = \int_0^\infty f(t)t^a_{.}t^{s-a}dt$

$$= \int_0^\infty f(t)\,t^{s+a-1}dt$$

$$M(f(t).\,t^a i,\,s) = F(s+a)$$

4. Multiplication by $\log_e t$ and in general $(\log_e t)^n$

If $M(f(t)) = F(s)$ then

$$M(f(t) \log ti, s) = \frac{d}{ds} F(s)$$

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Proof: $$M(f(t) \log ti, s) = \int_0^\infty f(t) \log t \,.\, t^{s-1} dt$$

and $$\frac{d}{ds} F(s) = \frac{d}{ds}\left(\int_0^\infty f(t)\, t^{s-1} dt\right)$$

$$= \int_0^\infty f(t) \frac{d}{ds}(t^{s-1})\, dt$$

or $$\frac{d}{ds} F(s) = \int_0^\infty f(t)\, t^{s-1} \log^t\, dt$$

$$= M(f(t) \log t; s)$$

Similarly

$$\frac{d^n}{ds^n} F(s) = \frac{d^n}{ds^n} \int_0^\infty f(t)\, t^{s-1} dt$$

$$= \int_0^\infty f(t) \left(\frac{d^n}{ds^n} t^{s-1}\right) dt$$

$$= \int_0^\infty f(t)\, t^{s-1} (\log_e t)^n\, dt$$

$$= \int_0^\infty (\log t)^n f(t)\, t^{s-1} dt$$

$$\frac{d^n}{ds} F(s) = M\left((\log t)^n f(t) i, s\right)$$

Using the fact $\frac{d^n}{ds^n} t^{s-1} = (\log t)^n\, t^{s-1}$

5. Inverse of independent variable

If $M(f(t)i, s) = F(s)$ then

$$M\left(\frac{1}{t} f\left(t^{-1}\right) i, s\right) = F(1-s)$$

Proof: $$M\left(t^{-1} f\left(t^{-1}\right) i, s\right) = \int_0^\infty \frac{1}{t} f\left(\frac{1}{t}\right) t^{s-1}\, dt$$

$$= \int_0^\infty f\left(\frac{1}{t}\right) t^{s-2}\, dt$$

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or $$M\left(t^{-1}f\left(t^{-1}\right)i,s\right) = \int_{\infty}^{0} f(u).\left(\frac{1}{u}\right)^{s-2}\left(\frac{-1}{u^2}\right)du$$

Put $$\frac{1}{t} = u$$

$$\frac{-1}{t^2}dt = du$$

$$dt = -t^2\,du$$

$$dt = \frac{1}{u^2}du$$

or $$= \int_0^{\infty} f(u)\,u^{2-s}.\frac{1}{u^2}\,du \qquad t = 0 \Rightarrow u = \infty$$

$$t = \infty \Rightarrow u = 0$$

$$= \int_0^{\infty} f(u)\,u^{(1-s)-1}\,du$$

$$M(t^{-1}f(t^{-1})i,s) = F(1-s)$$

6. Derivative of $f(t)$

Let $M(f(t)i, s) = F(s)$ then

$$M\left(\frac{d^n}{dt^n}f(t)i,s\right) = (-1)^n\,(s-n)_n\;F(s-n) \qquad k \in \mathbb{Z}^+$$

Where $$(s-n)_n = (s-n)\,(s-n+1)\,(s-n+2)...(s-1)$$

Proof: $$M\left(\frac{d^n}{dt^n}f(t)i,s\right) = \int_0^{\infty}\frac{d^n}{dt^n}f(t).\,t^{s-1}\,dt$$

for $n = 1$

$$M\left(\frac{d}{dt}f(t)i,s\right) = \int_0^{\infty} t^{s-1}f'(t)\;dt$$

integrating by parts we get

$$M\left(\frac{d}{dt}f(t)i,s\right) = t^{s-1}f(t)\Big]_0^{\infty} - (s-1)\int_0^{\infty} t^{p-2}f(t)\;dt$$

$$= 0-(s-1)\;F(s-1)$$

$$= (-1)^1\,(s-1)\;F(s-1)$$

provided $t^{s-1}\,f(t) \to 0$ as $x \to \infty$ and $x \to 0$

Similarly for 2nd order derivative we again

assume $t^{s-1}\,f(t) \to 0$ and $t^{s-2}\,f'(t) \to 0$ as $x \to \infty$ and $x \to 0$.

and the $M(f''(t)i,s) = (-1)^2\,(s-1)\,(s-2)\,F(s-2)$

Continue in this way we may derive in general

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$$M(f^n(t) i\, s) = (-1)^n (s-1)(s-2) \ldots (s-n) F(s-n)$$

provider $t^{s-i-1} f^{(i)}(t) = 0$ as $x \to 0$

for $i = 0, 1, 2 \ldots (n-1)$

or we may rewrite

$$M\left(\frac{d^n f}{dt^n} is\right) = (-1)^n (s-n)_n F(s-n)$$

where $(s-n)_n = (s-n)(s-n+1) \ldots (s-2)(s-1)$

Similarly one can show the property of derivative where derivative is multiplied by independent variable:

$$M\left(t^n \frac{d^n}{dt^n} f(t); s\right) = (-1)^n (s)_n F(s)$$

$$= (-1)^n \frac{\Gamma s+n}{\Gamma s} F(s)$$

where

$$(s)_n = s(n+1) \ldots (s+n-1).$$

4.3 MELLIN TRANSFORMS OF DERIVATIVES

Unlike Fourier and Laplace transform that were introduced to solve physical problems, Mellin transform arose in a mathematical content Mellin transform was occurred during the study of famous zeta function.

Mellin transform has wide area of applicability in mathematics as well as physics and engineering. Its most famous application is found in computation of solution of potential problem. Another domain of applicability of Mellin transformation is in the resolution of linear differential equation arising of in electrical circuit/engineering by a procedure analogous to Laplace transform.

Definition 4.1: Let $f(t)$ be a function defined for $0 < t < \infty$ use for positive real axis. The Mellin transform M is the operation mapping function f into the function F defined on the complex plane by the relation

$$M[f,s] = F(s) = \int_0^\infty f(t) t^{s-1} dt \qquad \ldots(4.1)$$

The function $F(s)$ in called Mellin transform of $f(t)$. In genral the integral does exist only for $s = a + ib$ (a complex number) such that $a < a_1 < a_2$ where a_1 and a_2 depend on the function $f(t)$ to transform.

Thus, $M(f(t)) = F(s) = \int_0^\infty t^{s-1} f(t)\, dt$

Where $s \in c$, $s = a + ib$

and $\begin{cases} a = \text{Re}(s) \\ b = \text{Im}(s) \end{cases}$

and $a < a_1 < a_2$ this introduces a strip of definition of the Mellin transform and is denoted by $S(a_1, a_2)$. In some cases this strip may be extended to a half plane ($a_1 = -\infty$ or $a_2 = \infty$) or to the whole complex s-plane ($a_1 = -\infty$ and $a_2 = \infty$).

Example 4.1: Consider the function

$$f(t) = H(t - t_0)\, t^z$$

where H is Heaviside function, $t_o > 0$ and z is complex. Obtain the Mellin transform of $f(t)$.

Solution: $M(f(t)) = F(s) = \int_0^{\infty} f(t) t^{s-1}\, dt$

or $M(f) = \int_0^{t_0} 0 \cdot t^{s-1} dt + \int_{t_0}^{\infty} 1 \cdot t^z t^{s-1} dt$

$$F(s) = \int_{t_0}^{\infty} t^{z+s-1} dt = \left. \frac{t^{z+s}}{z+s} \right|_{t_0}^{\infty}$$

provided s in such that $\text{Re}(s) < -\text{Re}(z)$.

Example 4.2: Obtain the Mellin transform of

$$f(t) = e^{-pt} \quad p > 0$$

Solution:

$$M(f(t)) = F(s) = \int_0^{\infty} e^{-pt} t^{s-1} dt$$

or $F(s) = \int_0^{\infty} e^{-pt} t^{s-1} dt$

$$= \frac{1}{p^s}\, Fs.$$

$$F(s) = \frac{\sqrt{s}}{p^s} \qquad R(s) > 0.$$

Example: 4.3: Find the Mellin transform of

$$f(t) = (1 + t)^{-1}$$

Solution:

$$M(f) = F(s) = \int_0^{\infty} x^{s-1} (1 - x)^{-s} dx$$

or $M(f) = F(s) = \beta(s_1, 1 - s)$ (using definition of Beta function)

$$= \sqrt{s}\ \sqrt{1-s}$$

or $$F(s) = \frac{\pi}{s_n \pi s}$$

NOTES

4.3.1 Inverse Mellin Transform

Let $F(s)$be the Mellin transform of $f(t)$ then $f(t)$ is called inverse Mellin transform of $F(s)$ and we write

$$f(t) = M^{-1}\,(F(s))$$

4.4 INVERSION THEOREM OF MELLIN INTEGRALS

Since $$M(f(t)) = F(s) = \int_0^{\infty} f(t) t^{s-1}\, dt \qquad ...(4.2)$$

changing the variable in (1) by

$$t = e^{-x} \quad \Rightarrow dt = -e^{-x}\, dx\ \&\ t^{s-1} = e^{-(s-1)}$$

$$\Rightarrow \quad x = -\log t$$

Limits are also changed as

$$t = 0 \quad \Rightarrow \quad x = \infty$$

$$t = \infty \quad \Rightarrow \quad x = -\infty$$

thus $$F(s) = \int_{\infty}^{-\infty} f(e^{-x})\, e^{-(s-1)s} (-e^{-x})\, dx$$

or $$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\, e^{-sx}\, dx$$

or $$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\, e^{-sx}\, dx \qquad ...(4.3)$$

let $$g(x) = f(e^{-x})$$

then (2) becomes

$$F(s) = \int_{-\infty}^{\infty} g(x)\, e^{-sx}\, dx \qquad ...(4.4)$$

$$F(s) = \int_{-\infty}^{0} g(x)\, e^{-sx}\, dx + \int_0^{\infty} g(x) e^{-sx} dx$$

We called the R.H.S of (4.4) as two sided Laplace transform and in symbols we write

$$M(f(t)) = \mathcal{T}f(e^{-x})) \qquad ...(4.5)$$

NOTES

$\mathcal{J}$ denote 2-side Laplace transform operator in particular if we take s complex and say $s = a + 2\pi ib$ the (4.3) can be written as

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\, e^{-ax}\, e^{-i2\pi b} dx \qquad ...(4.6)$$

or $$M(f(t)), s = a + i2\pi b) = \mathcal{J}\,[f(e^{-x}).\, e^{-ax},\, b] \qquad ...(4.7)$$

Where $$\mathcal{J}(f) = \hat{f}(b) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi b} dt$$

(4.5) and (4.7) repectively represents the selation of Mellin transform with Laplace and Fourier transform.

With the help of these relations we may redefine the inversion formulas as.

1. Inversion Formula

We know that inverse Fourier transform of $\hat{f}(b)$ as

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(b) e^{i2\pi b} db \qquad ...(4.8)$$

Using this formula to (5) with $s = a + i2\pi b$ we have

$$F(e^{-x}).e^{-ax} = \int_{-\infty}^{\infty} F(s) e^{i2\pi b} db \qquad ...(4.9)$$

back substituting $t = e^{-x}$ we get

$$f(t)\ t^{a} = \int_{-\infty}^{\infty} F(s)\, t^{-i2\pi b} db$$

or $$f(t) = t^{-a} \int_{-\infty}^{\infty} F(s)\, t^{-i2\pi b} db$$

or $$f(t) = \int_{-\infty}^{\infty} F(s)\, t^{-(a + i2\pi b)} db$$

or $$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)\, t^{-s} ds \qquad ...(4.10)$$

$$\begin{cases} a + i2\pi b = s & \Rightarrow \quad db = \dfrac{1}{2\pi i} ds \\ b = \infty & \Rightarrow \quad s = a + i\,\infty \\ b = -\infty & \Rightarrow \quad s = a - i\infty \end{cases}$$

thus (9) can be written as

$$M^{-1}(F(s)) = f(t) = \lim_{b \to \infty} \left(\frac{1}{2\pi i} \int_{a-ib}^{a+ib} F(s) t^{-s} ds \right) \qquad ...(4.11)$$

where integration is made along the vertical line through $\text{Re}(s) = a$ (10) is the Mellin inversion formula.

4.5 DISTRIBUTION OF POTENTIAL IN A WEDGE

NOTES

The problem is to solve Laplace's equation in an infinite two-dimensional wedge with Dirichlet boundary conditions. Polar coordinates with origin at the apex of the wedge are used and the sides are located at $\theta = \pm\,\alpha$. The unknown function $u(r,\theta)$ is supposed to verify:

$$\Delta u = 0, 0 < r < \infty, -\alpha < \theta < \alpha \qquad ...(4.12)$$

with the following boundary conditions:

1. On the sides of the wedge, if R is a given positive number:

$$u\,(r, \pm\,\alpha) \quad = \begin{cases} 1 \; if \;\; 0 < r < R \\ 0 \; if \;\; r > R \end{cases} \qquad ...(4.13)$$

   or, equivalently:

$$u\,(r, \pm\,\alpha) \quad = \quad H(R-r) \qquad ...(4.14)$$

2. When r is finite, $u\,(r, \theta)$ is bounded.
3. When r tends to infinity, $u\,(r, \theta) \sim r^{-\beta}$, $\beta > 0$.

   In polar coordinates, Equation (4.12) multiplied by r^2 yields:

$$r^2 \frac{\partial^2 U}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(4.15)$$

The above conditions on $u(r, \theta)$ ensure that its Mellin transform $U(s, \theta)$ with respect to r exists as a holomorphic function in some region $0 < Re\,(s) < \beta$. The equation satisfied by U is obtained from (4.15) by using property in this book 4.16 of the Mellin transformation and reads:

$$\frac{\partial^2 U}{\partial \theta^2}(s,\theta) + s^2 U(s,\theta) = 0 \qquad ...(4.16)$$

The general solution of this equation can be written as:

$$U\left(s,\theta\right) = A(s)e^{js\beta} + B(s)e^{-js\theta} \qquad ...(4.17)$$

Functions A, B are to be determined by the boundary condition (4.14) which leads to the following requirement on U:

$U\,(s, \pm\,\alpha) = R^s\, s^{-1}$ for $Re(s) > 0$

Explicitly, this is written as:

$A\,(s)\, e^{js\alpha} + B\,(s)\, e^{-js\alpha} = a^s\, s^{-1}$

$A(s)\,e^{-js\alpha} + B(s)\,e^{js\alpha} = a^s s^{-1}$

and leads to the solution:

$$A(s) = B(s) = \frac{R^s}{2s\cos(s\alpha)}$$

The solution of the form is given by:

$$U(s,\theta) = \frac{R^s \cos(s\theta)}{s\cos(s\alpha)}$$

NOTES

4.6 ELEMENTARY PROPERTIES OF HANKEL TRANSFORMS

1. Scaling:

$$\mathcal{H}_n(f(r)) = \hat{f}(s) \text{ then } \left(\hat{f}(s) = \int_0^\infty rJ_n(sr)f(r)\,dr\right)$$

$$\mathcal{H}_n(f(ar)) = \frac{1}{a^2}\hat{f}\left(\frac{8}{a}\right)$$

Proof: Since $\mathcal{H}_n = \left(f(ar)\right) = \int_0^\infty \pi J_n(rs)f(ar)\,dr$

$$\left\{\begin{aligned} ar &= u \\ r &= \frac{u}{a} \\ dr &= \frac{dr}{a} \end{aligned}\right\} \begin{aligned} &= \int_0^\infty \frac{u}{a}J_n\left(s\frac{u}{a}\right)f(u)\frac{du}{a} \\ &= \frac{1}{a^2}\int_0^\infty u\,J_n\left(\frac{s}{a}u\right)f(u)\,dr \\ &= \frac{1}{a^2}\hat{f}\left(\frac{s}{a}\right) \end{aligned}$$

2. Parseval's Relation

$$y\;\hat{f}(s) = \mathcal{H}_n(f(n)) \text{ and } \hat{g}(s) = \mathcal{H}_n(g(r))$$

then

$$\int_0^\infty r\,f(r)g(r)\,dr = \int_0^\infty s\,\hat{f}(s)\hat{g}(s)\,ds$$

Proof: Consider

NOTES

$$\int_0^\infty s\,\hat{f}(s)\hat{g}(s)\,g(r)ds$$

$$= \int_0^\infty s\,\hat{f}(s)ds\int_0^\infty r\,J_n(srg(r))dr$$

interchanging the order of integration we get

$$\int_0^\infty s\,\hat{f}(s)\hat{g}(s)ds = \int_0^\infty r\,g(r)dr\int_0^\infty sJ_n(sr)\hat{f}(s)ds$$

3. Hankel transform of deriatives:

$$\text{y} : \mathcal{H}_n(f(s)) = \hat{f}_n(s) \text{ then}$$

(i) $$\mathcal{H}_n(f'(r)) = \frac{s}{2n}\left[(n-1)\hat{f}_{n+1}(s)-(n+1)\hat{f}_{n+1}(s)\right]n\geq,1$$

(ii) $$\mathcal{H}_n(f'(r)) = -s\hat{f}_0(s)$$

provided $rf(r)\to 0$ as $r\to 0$ and $r\to\infty$.

Proof: By definition

$$\mathcal{H}_n(f'(r)) = \int_0^\infty \lambda\,J_n(sr)f'(r)dr \qquad ...(4.18)$$

Integrating by parts we get

$$\mathcal{H}_n(f'(r)) = rf(s)\,J_n(sr)\int_0^\infty - \int_0^\infty f(r)\frac{d}{dr}\left(rJ_n(sr)\right)dr$$

...(4.19)

We know that the properties of Bessel function that,

$$\frac{d}{d^r}\left(rJ_n(sr)\right) = J_n(sr)+rsJ_n^1(sr)$$

$$= J_n(sr) + rs\,J_{n-1}(sr) - nJ_n(sr)$$

$$= (1-n)\,J_n(sr) + rs\,J_{n-1}(sr) \qquad ...(4.20)$$

in view of (4.20), (4.19) may yield (assuming $r\,f(r)\to 0$ as $r\to 0$ and $n\to\infty$)

$$\mathcal{H}_n(f'(r)) = (n-1)\int_0^\infty f(r)J_n(sr)dr$$

$$s\int_0^\infty rJ_{n-1}(sr)f(r)dr$$

$$= (n-1)\int_0^\infty f(r)\,Jn\,(sr)\,dr - s\hat{f}_{n-1}(s) \qquad ...(4.21)$$

Again use the recurrence relation of Bessel fucntion

$$J_n(rs) = \frac{rs}{2n}\left[J_{n-1}(sr) + J_{n+1}(sr)\right] \qquad ...(4.22)$$

NOTES

Thus, (4.21) can have (in view of (4.22))

$$\mathcal{H}_n(f'(r)) = -s\, f_{n-1}(s) + s\left(\frac{n-1}{2n}\right)\cdot\left[\hat{f}_{n-1}(s) + \hat{f}_{n+1}(s)\right]$$

or $$\mathcal{H}_n(f'(r)) = \frac{s}{2n}\left[(n-1)\hat{f}_{n+1}(s) - (n+1)\hat{f}_{n-1}(s)\right]$$

i.e. (4.18) established

In particular take $n = 1$ we get

$$\mathcal{H}_1(f'(r)) = \frac{s}{2}\left[-(2)\hat{f}_0(s)\right]$$

$$\mathcal{H}_1(f'(r)) = -s\ \hat{f}_0(s)$$

i.e., (ii) follows

By the repeated use of (i) we may generalize this to 2nd derivative

and $$\mathcal{H}_n(f''(r)) = \frac{s}{2n}\ [(n-1)\ \mathcal{H}_{n+1}(f'(r)) - (n+1)\ \mathcal{H}_{n-1}$$
$(f'(r))]$

or $$\mathcal{H}_n(f''(r)) = \frac{s^2}{4}\left[\left(\frac{n+1}{n-1}\right)\hat{f}_{n-2}(s) - 2\left(\frac{n^2-3}{n^2-1}\right)\hat{f}_n(s) + \frac{(n-1)}{(n+1)}\hat{f}_{n+2}(s)\right]$$

4.7 HANKEL INVERSION THEOREM

Let $F(s)$ and $G(s)$ are the Mellin transform of $f(t)$ and $g(t)$ respectively then

$$M\left(\int_0^\infty f(xt)\,g(t)\,dt\right) = F(s)\ G\ (1\text{–}8)$$

Proof:

$$M\left(\int_0^\infty f(xt)\,g(t)\,dt\right) = \int_0^\infty x^{s-1}\left(\int_0^\infty f(xt)g(t)dt\right)dx$$

NOTES

$$=\int_0^{\infty}{}^{s-1}\left(\int_0^{\infty} f(u)\,g(t)\,dt\right)\frac{du}{t}$$

$$=\int_0^{\infty}\left(\frac{u}{t}\right)^{s-1}\left(\int_0^{\infty} f(u)\,g(t)\,dt\right)\frac{du}{t}$$

$$\left[\begin{array}{l} putting\quad xt = u \\ \quad du = t\,dx \\ \quad dx = \frac{1}{t}du \end{array}\right.$$

$$=\int_0 u^{5-1} f(u)\,du \int_0^{\infty} t^{(1-s)-1} g(t)\,dt$$

$$=F(s)\,.\,G(1-s)$$

Hence

$$M\left(\int_0^{\infty} f(nt)g(t)\ dt\right) = f(s)\ G(1-s)$$

Example 4.4: Find the Mallin Transform of

$$f(t)=\frac{1}{(e-1)}$$

Solution. $M\left(\frac{1}{e-1}\right) = f(t) = \int_0^{\infty} t^{s-1}\frac{1}{e^{t-1}}\,dt$ ...(1)

using the result $\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{1-e^{-t}}$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^{t}-1}$$

So (1) becomes

$$M\left(\frac{1}{e^{t}-1}\right) = \sum_{n=1}^{\infty}\int_0^{\infty} t^{s-1}e^{-nt}\ dt$$

$$= \sum_{n=1}^{\infty}\frac{\Gamma s}{n^{s}} = \Gamma s\sum_{n=1}^{\infty}\frac{1}{n^{s}}$$

$$= M\left(\frac{1}{e^{+}-1}\right) = \Gamma s\ G(s) \quad \left(Since\ M\left(e^{-pt}\right) = \frac{\Gamma s}{p^{s}}\right)$$

where $G(s) = \sum_{n=1}^{\infty}\frac{1}{n^{s}}$ $(Re\,(s) > 1)$ is

Riemann Zota Function.

Example 4.5 : Obtain the Mellin transform of

$$f(t) = \frac{2}{e^{2t}-1}$$

Solution:

$$M\left(\frac{2}{e^{2t}-1}\right) = F(s)$$

$$= 2\int_0^\infty t^{s-1}\frac{dt}{e^{2t-1}}$$

$$= 2\sum_{n=1}^\infty \int_0^\infty t^{s-1}.\, e^{-2nt}dt$$

$$= 2\sum_{n=1}^\infty \frac{\overline{|s}}{(2n)^s}$$

$$= 2^{1-s}\overline{|s}\sum_{n=1}^\infty \frac{1}{n}s$$

$$M\left(\frac{2}{e^{2t}-1}\right) = 2^{1-s}\overline{|s}\; G(s)$$

Example 4.6: Show that

$$M\left(\frac{1}{e+1}\right) = \left(1-2^{1-s}\right)\overline{|s}\; G(s)$$

Solution : Left as an exercise to the readers.

Example 4.7: Find the Mellin transform of

cos *nt* and sin *nt*

We that

$$M\left(\overline{e}^{\text{int}}i, s\right) = \frac{\overline{|s}}{(in)^s} = \frac{\overline{|s}}{s^n} = \overline{e}^{\frac{i\pi s}{2}}$$

Equating real and imaginary parts we get

$$M(cos\ nt;i\ s) = (n)^{-s}\ \overline{|s}\ cos\frac{s\pi}{2}$$

$$M(sin\ nti\ s) = n^{-s}\ \overline{|s}\ sin\frac{s\pi}{2}$$

Remark:

Thus $\quad M(cos\ nti\ s) = (n)^{-s}\overline{|s}\,cos\frac{s\pi}{2}$

or $\quad \int_0^\infty t^{s-1}cos\,nt\,dt = \frac{\overline{|s}}{n^s}\ cos\frac{s\pi}{2}$

NOTES

or $$\mathcal{J}_c\left(\sqrt{\frac{\pi}{2}}t^{s-1}\right) = \frac{\lceil s}{n^s}\ cos\ \frac{s\pi}{2}$$

or $$\mathcal{F}_c\,(t^{s-1}) = \sqrt{\frac{2}{\pi}}\,\frac{\lceil s}{n^s}\ \ cos\ \frac{s\pi}{2}$$

that gives the fourier cosine transforming t^{s-1}.

Similarly $$\mathcal{J}_s\,(t^{s-1}) = \sqrt{\frac{2}{\pi}}\,\frac{\lceil s}{n^s}\ \ sin\ \frac{s\pi}{2}$$

gives the Fourier sine transform of t^{s-1}.

4.7.1 Mellin Transform Integrals

Theorem:

If, $$M(f(t)) = F(s) \text{ then}$$

$$M\left(\int_0^x f(t)\ dt\right) = -\frac{1}{s}F\left(s+1\right)$$

Proof : Denote $$'(t) = \int_0^t f(x)\,dx$$

$\Rightarrow$ $$(t) = f(t) \text{ and } f(\text{o}) = 0$$

$$M(f(t)) = '(t)\ i\ s) = -(s-1).\ M\left\{\int_0^t f(x)dx, s-1\right\}$$

$$\left\{\begin{array}{c}\text{using the property}\\ \text{M}\left(f\,'(t)_i\,s\right) = -(s-1)\ .\ F(s-1)\\ \text{or M}\left(f\,'(t)_i\,s\right) = -(s-1)\int_0^\infty t^{(s-2)}\ f(t)\ dt\end{array}\right\}$$

Thus $$M(f(t) = i(t);\ \text{s}) = -(s-1)\ M\left\{\int_0^t f(x)dx_i s-1\right\}$$

Replacing s by $s+1$ we get

$$\text{M}\left\{\int_0^t f(x)dxi\ s\right\} = -\frac{1}{s}M\left(f(t)\right).$$

$$-\frac{1}{s}F\left(s+1\right)$$

4.8 HANKEL TRANSFORMS OF THE DERIVATIVES OF FUNCTIONS AND SOME ELEMENTARY FUNCTION

Theorem: If: $H_n(f(s)) = \hat{f}_n(s)$ then

(i) $H_n(f'(s) = \dfrac{s}{2n}\left[(n-1)\hat{f}_{n+1}(s) - (n+1)\hat{f}_{n+1}(s)\right] n \geq 1$

(ii) $H_1(f(s) = -s\hat{f}_0(s)$

provided $rfs) \to 0$ as $r \to \infty$ and $r \to \infty$.

Proof: By definiton

$$H_n(\hat{f}(s) = \int_0^\infty \pi J_n(sr)\, f'(s)\, dr \qquad \text{... (a)}$$

Integrating by parts we get

$$H_n(f'(s) = r\, f(s)\, J_n(sr)\Big|_0^\infty - \int_0^\infty f(r) \frac{d}{dr}(r J_n(sr)\, dr. \qquad \text{... (4.23)}$$

We know that the propertie of Bessel function that:

$$\frac{d}{dr}(r J_n(sr)) = J_n(sr) + rsJ_n^1(sr)$$

$$= J_n(sr) + rs\, J_{n-1}(sr) - n\, J_n(sr)$$

$$= (1-n)\, J_n(sr) + rs\, J_{n-1})(sr) \qquad \text{... (4.24)}$$

In view of (4.24), (4.23) may yeild (assuming $r\, f(r) \to 0$ as $r \to 0$ and $n \to \infty$)

$$H_n(\hat{f}(r)) = (n-1)\int_0^\infty f(r)\ \text{Jn (sr) dr} -$$

$$s\int_0^\infty r\, J_{n-1}(sr)\, f(r)\, dr$$

$$= (n-1)\int_0^\infty f(r)\, J_n(sr)\, dr - s\,\hat{f}(s)\, n-1 \qquad \text{... (4.25)}$$

Again use the recurrence relatation of Bessel function

$$J_n(rs) = \frac{rs}{2n}\left[J_{n-1}(sr) + J_{n+1}(sr)\right] \qquad \text{... (4.26)}$$

Thus (4.25) can have (in view of (4.26))

$$H_n(f'(r) = -s\,\hat{f}_{n-1}(s) + s\left(\frac{n-1}{2n}\right)$$

$$\left[\int_{\infty}^{\infty} rf(r)\left\{J_{n-1}(sr)+J_{n+1}(sr)\right\} dr\right]$$

NOTES

or $$= -s\,\hat{f}_{n-1}(s) + s\,\frac{(n-1)}{2n}\left[\hat{f}_{n-1}(s)+\hat{f}_{n+1}(s)\right]$$

$$\text{or } H_n(f'(r)) = \frac{s}{2n}\left[(n-1)\,\hat{f}_{n+1}(s)-(n+1)\,\hat{f}_{n-1}(s)\right]$$

i.e. Equation (a) established

In particular take $n = 1$ we get

$$H_1(f'(r)) = \frac{s}{2}\left[-(2)\,\hat{f}_0(s)\right]$$

$$H_1(f'(r)) = -s\,\hat{f}_0(s)$$

i.e. (ii) follows.

By the repeated use of (i) we may generalize this to 2nd order derivative.

and $$H_n(f''(r)) = \frac{s}{2n}\left[(n-1)\,H_{n+1}(f'(r)-(n+1)\,H_{n-1}(f'(r))\right]$$

$$\text{or } H_n(f''(r)) = \frac{s^2}{4}\left[\left(\frac{n+1}{n-1}\right)\hat{f}_{n-2}(s)-2\left(\frac{n^2-3}{n^2-1}\right)\hat{f}_n(s)+\frac{(n-1)}{(n+1)}\,\hat{f}_{n+2}(s)\right]$$

4.8.1 Hankel Transform of Some Elementary Function

Example 4.8: Find the nth ($n > -1$) order Hankel transform of

(a) $f(r) = r^n\, \mathrm{H}(a-r)$ (b) $f(r) = r^n e^{-ar^2}$

Solution: (a)

$$\hat{f}(s) = H_n(r^n H(a-r)) = \int_0^{\infty} r\,.r^n\; H(a-r).J_n(rs)\,dr$$

$$= \int_0^{a} r^{n+1} J_n(sr)\,dr + \int_a^{\infty} r^{n+1}.0.J_n(rs)\;\; dr$$

$$\int_0^{a} r^{n+1} J_n(rs)\,dr$$

$$\hat{f}(s) = \frac{a^{n+1}}{s} J_{n+1}(as)$$

(b) $$H_n\left(r^n e^{-ar^2}\right) = \int_0^{\infty} r\,.r^n e^{-ar^2}\;\; J_n(rs)\,dr$$

$$= \int_0^{\infty} r^{n+1} J_n(rs)\,e^{-ar^2}\,dr$$

or $\hat{f}(s) = \dfrac{s^n}{(2a)^{n+1}} e^{-\frac{s^2}{4a}}$

NOTES

Example 4.9. Find the first order Hankel transform of following functions

(a) $f(r) = \dfrac{\sin r}{r}$ (b) $f(r) = \dfrac{e^{-2r}}{r}$

Solution:

$$\text{(a)} \quad \mathcal{H}_1\left(\frac{\sin r}{r}\right) = \int_0^\infty r \frac{\sin r}{r} . J_1(rs)\, dr$$

$$= \int_0^\infty \sin r J_1(rs)\, dr$$

$$= \frac{s}{(1+s^2)^{3/2}}$$

$$\text{(b)} \quad \mathcal{H}_1\left(\frac{e^{-2r}}{r}\right) = \int_0^\infty e^{-2r} J_1(rs)\, ds$$

$$= \frac{1}{s}\left[1 - 2\left(s^2 + 2^2\right)^{-1/2}\right]$$

$$\mathcal{H}_1 = \left(\frac{e^{-2r}}{r}\right) = \frac{1}{s}\left(1 - \frac{2}{\sqrt{s^2+4}}\right)$$

4.9 RELATIONS BETWEEN FOURIER AND HANKEL TRANSFORM

We know that the double (two-dimensional) Fourior transform and its inverse is given by

$$\mathcal{J}(f(x, y)) = f(a, b) = \frac{1}{2\pi} \int_{+\infty}^\infty \int_{-\infty}^\infty e^{-i(\underline{u}.\underline{v})} f(x, y)\, dx\, dy$$

$$\mathcal{J}^{-1}(f(a, b)) = f(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i(\underline{u}.\underline{v})} f(a, b)\, da\, db$$

where

$$\underline{u} = (x, y)$$

$$\underline{v} = (a, b)$$

Let $(x, y) = u\,(\cos\theta, \sin\phi)$ and

$(a, b) = v\,(\cos\phi, \sin\phi)$ and the polar representation of $\underline{u}$ and $\underline{v}$ respectively.

Then $\underline{u}.\underline{v} = uv\cos(\theta-\phi)$ and then

$$\mathcal{F}\,(v_1\phi) = \frac{1}{2\pi}\int_0^{\infty} udu \int_0^{2\pi} e^{-iuv\cos\,(\theta-\phi)} f(u,\theta)\,d\theta \qquad \text{...(4.27)}$$

assuming $f(u, \theta) = e^{-in\theta} f(u)$

Then by making a change of variable by setting $\theta - \phi = \alpha - \frac{\pi}{2}$ we get

$$\mathcal{F}\,(v_1\phi) = \frac{1}{2\pi}\int_0^{\infty} uf(u)\,du \times \int_{\phi_0}^{\phi_0+2\pi} e^{in\left(\phi-\frac{\pi}{2}\right)} \times e^{i(n\alpha-uv\sin\alpha)} d\alpha$$

where $\phi_0 = \phi_0 = \frac{\pi}{2} - \phi$

In view of integral representation of Bessel function of order n

$$Jn(uv) = \frac{1}{2\pi}\int_{\phi_0}^{\phi_0+2\pi} e^{i(n\alpha-uv\sin\alpha)} d\alpha$$

(4.27) becomes

$$F\,(v_1\,\phi) = e^{in\left(\phi-\frac{\pi}{2}\right)} \int_0^{\infty} u\,Jn\,(uv)\,f\,(u)\,du$$

$$\mathcal{F}\,(v_1\phi) = e^{in\left(\phi-\frac{\pi}{2}\right)} \hat{f}_n(v) \qquad \text{...(4.28)}$$

where $\hat{f}_n(v)$ is the Hankel transform of $f(u)$ and is defined as

$$\mathcal{H}_n\,(f(u)) = \hat{f}_n(v) = \int_0^{\infty} uJ_n(uv)\,f(u)du$$

and λ(4.28) is the desired relation between fourior and Hankel tranform.

Then

$$\mathcal{F}(\mathrm{v}, \phi) = e^{in\left(\phi-\frac{\pi}{2}\right)} \hat{f}_n(v)$$

or

$$\mathcal{F}\,(\mathrm{f}(\mathrm{u}, \phi)) = e^{in\left(\phi-\frac{\pi}{2}\right)} \mathcal{H}_{\mathrm{n}}\,(f(u))$$

4.10 PARSEVAL RELATION FOR HANKEL TRANSFORM

We introduce the definition of Hankel transform from the two-dimensional fourier transform and its inverse is given by

$$\mathcal{J}(f\,(x, y)) = F\,(a, b) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\mathrm{r.s})} f(x, y)\,dxdy$$

$$\mathcal{F}^{1}(F(a, b)) = f(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(rs)} f(a,b)\, da\, db$$

where $r = (x, y)$ and $s = (a, b)$.

Let $(x, y) = (cos\,\theta, sin\,\theta)$ and

$(a, b) = s\,(\cos\phi, \sin\phi)$ be the polar representation of $\underline{r}$ and $\underline{s}$ respectively.

Then $\underline{r}\,.\,\underline{s} = rs\cos(\theta - \phi)$ and then

$$F(s, \phi) = \frac{1}{2\pi}\int_{0}^{\infty} r dr \int_{0}^{2\pi} e^{(-irs\cos(\theta-\phi))} . f(r,\theta)\, d\theta .$$

Assume $f(r, \theta) = e^{in\theta} f(r)$.

and by make a change of variable by $\theta - \phi = \alpha - \pi/2$

we get $F(s, \phi) = \frac{1}{2\pi}\int_{0}^{\infty} rf(r)dr\ x$

$$\int_{\phi_0}^{\phi_0+2\pi} \exp(in\,(\phi - \frac{\pi}{2}) + i(n\alpha - \pi s\sin\alpha)dr \qquad ...(4.29)$$

where $\phi_0 = \frac{\pi}{2} - \phi$ ...(1)

in view of integral representation of Bessed function of order n

$$\mathcal{J}_{n}(sr) = \frac{1}{2\pi}\int^{\phi_0+2\pi} \exp\,(i\,(n\alpha - rs\sin\alpha))\, d\alpha$$

(4.29) becomes

$$F(s_1\, \phi) = exp\left(in\left(\phi - \frac{\pi}{2}\right)\right)\int_{0}^{\infty} r\ J_n(rs)\ f(r)\ dr$$

$$= exp\left(in\left(\phi - \frac{\pi}{2}\right)\right)\hat{f}_n(s)$$

where $\hat{f}_n(s)$ is called the Hankel transform of $f(r)$ and in defined formally as

$$\mathcal{H}n\,(f(r)) = \hat{f}_n(s)\int_{0}^{\infty} r\ J_n(rs)\ f(r)\ dr \qquad ...(4.30)$$

and the inverse Hankel transform of

$\mathcal{H}_{n}(f(r)) = \hat{f}_n(s)$ is designed as

$$\mathcal{H}_n^{-1}\left(\hat{f}_n(s)\right) = f(r) = \int_0^\infty s\ J_n(rs)\,\hat{f}_n(s)\ ds \qquad (4.31)$$

NOTES

Provided the integrals in (4.30) and (4.31) exist Alliterative Hankel integral formula

$$f(\pi) = \int_0^\infty s\ J_n(rs)\ ds$$

$$\int_0^\infty s\ Jn(sp)\,f(p)\ dp$$

can be used to design the Hankel transform (4.30) and its inverse (4.31). In particular the Hankel transforms of the zero order ($n = 0$) and of order one ($n = 1$) are often useful for the solution of problems involving Laplace's equation in an axisymmetric geometry.

The zero order Hankel transform

$$\mathcal{H}_0(f(r)) = \int_0^\infty r\ J_\alpha(rs)\ f(r)\ dr \qquad (4.32)$$

Example 4.10: Obtain zero-order Hankel transform of

(a) $\dfrac{\delta(r)}{r}$

(b) $H(a-r)$

(c) $\dfrac{\delta^{-ar}}{r}$

where $H(r)$ is the Heaviside unit step function.

Solution:

(a) $$\underset{n}{\hat{f}}(s) = \mathcal{H}_0\left(\frac{\delta(r)}{r}\right) = \int_0^\infty r\frac{s(r)}{\pi}\ J_0(rs)\,dr \qquad ...(1)$$

$$\underset{0}{\hat{f}}(s) = \int_0^\infty S(r)\ .J_0(rs)\ dr$$

$$= J_0(0) = 1$$

or $$\hat{f}_0(s) = 1$$

(b) $$\hat{f}_0(s) = \mathcal{H}_0(H(a-r))$$

$$= \int_0^\infty rH(a-r)J_0(rs)\ dr$$

$$= \int_0^a rJ_0(rs)dr + \int_0^\infty o\ rJ_0(rs)\ dr$$

$$= \frac{1}{s^2}\int_0^{as} p\ J_0(p)\,dp$$

NOTES

where $\qquad p = rs$

or $\qquad \mathcal{H}_0(H(a–r)) = \frac{1}{s^2}\left(p\ J_1(p)^{\ as}\right)_0$

$$\hat{f}_0(s) = \frac{a}{s}J_1(as)$$

(c) $\qquad \mathcal{H}_0\left(\frac{1}{r}e^{-ar}\right) = \int_0^\infty r.\frac{1}{r}e^{-ar}\ J_0(rs)\,dr$

$$= \int_0^\infty e^{-ar}\ J_0(rs)\,dr$$

$$\mathcal{H}_0\left(\frac{e^{-ar}}{r}\right) = \frac{1}{\sqrt{s^2+a^2}}$$

This can be shown using relation

$$\mathcal{H}_n = \left(\frac{e^{-ar}}{r^{1-n}} is\right) = \frac{s^n 2^n \Gamma n + ½}{\sqrt{\pi}\left(a^2+s^2\right)^{½+n}}, n > -½$$

and setting $n = 0$, i.e.,

$$\mathcal{H}_0\left(\frac{e^{-ar}}{r} is\right) = \frac{1}{\left(a^2+s^2\right)^{½}},\ a > 0 \qquad \left(\because \Gamma ½ = \sqrt{\pi}\right)$$

and if we take $a = 0$ and $n = 0$

$$\mathcal{H}_0\left(\frac{1}{r} is\right) = \frac{1}{s}$$

We now consider the general example as

Example 4.11 Obtain the Hankel transform of orden n for $\left(\frac{e^{-ar}}{r^{1-n}}\right)$

or show that

$$\mathcal{H}_n\left(\frac{e^{-ar}}{r} is\right) = \frac{2^n s^n \Gamma n+\frac{1}{2}}{\sqrt{\pi}\left(a^2+s^2\right)^{n+\frac{1}{2}}} i\ n > -\frac{1}{2}$$

Consider $\qquad \mathcal{H}_n\left(\frac{e^{-ar}}{r^{1-n}} is\right) = \int_0^\infty \frac{re^{-ar}}{r^{1-n}}\ J_n(sr)\,dr$

$$= \int_0^\infty r^n e^{-ar}\ J_n(rs)\,dr$$

$$= \int_0^\infty \frac{1}{s^{n+1}} t^n J_n(t)^{\frac{-a}{s}} + dt \qquad \text{(where } t = rs\text{)}$$

NOTES

$$= \frac{1}{s^{n+1}} \int_0^\infty t^n . J_n(t), e^{-\frac{a}{s}}\, dt$$

$$\mathcal{H}_n \left(\frac{e^{-ar}}{r^{1-n}} is \right) = \frac{1}{s^{n+1}} \mathcal{L} \left(t^n J_n(t) ip = \frac{a}{s} \right) \qquad ...(1)$$

when $t = rs$ and $\mathcal{J}$ denots the Laplace transform operator.

Consider $t^n J_n(t) = \sum_{v=o}^{\infty} \frac{(-1)^v t^{2n+2v}}{v! \sqrt{n+v+1} + 2^{2v+n}}$

$$\Rightarrow \qquad \mathcal{L}(t^n J_n(t) i\, p) = \sum_{v=o}^{\infty} \frac{(-1)^v}{v! \sqrt{n+v+1}\, 2^{2v+1}} \mathcal{L}(t^{2n+2v} i\, p)$$

$$\text{or} \qquad \mathcal{L}(t^n J_n(t)\, \mathrm{i}\, p) = \sum_{v=o}^{\infty} \frac{(-1)^v \lceil 2v + 2n + 1}{v! \sqrt{n+v+1}\;\; 2^{2v+1}\;\; p^{2n+2v+1}}$$

By Legendre Duplication formula, we have the relation

$$\frac{\lceil 2n + 2v + 1}{\sqrt{n+v}+1} = \frac{1}{\sqrt{\pi}}\; 2^{2n+2v} \sqrt{n+v} + \frac{1}{2}$$

Thus

$$\mathcal{L}(t^n J_n(t) i\, p) = \frac{2^n}{\sqrt{\pi}\, p^{2n+1}} \sum_{v=0}^{\infty} \frac{(-1)^v \sqrt{n+v+\frac{1}{2}}}{v!(p^2)^v}$$

$$\text{or} \qquad \mathcal{L}(t^n J_n(t) i\, p) = \frac{2^n}{\sqrt{\pi}\, p^{2n+1}} \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n+v+\frac{1}{2}}}{v!} \left(\frac{1}{p^2} \right)^v$$

$$\text{but} \qquad (1+x)^{-v} = \sum_{n=0}^{\infty} {}^{-v}C_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{= (-1)^n \sqrt{n+v}}{n! \lceil v} x^n \quad | x | < 1$$

$$\text{and} \qquad {}^{-v}C_n = \frac{(-1)^n \sqrt{n+v}}{n! \lceil v}$$

In view of above relation, we get

$$\mathcal{L}(t^n J_n(t) ip) = \frac{2^n \lceil n + \frac{1}{2}}{\sqrt{\pi}(p^2+1)^{n+\frac{1}{2}}}$$

$$\text{or} \qquad \mathcal{L}(t^n J_n(t) : p) = \frac{2^n \lceil n + \frac{1}{2}}{\sqrt{\pi} \left(\left(\frac{a}{s} \right)^2 + 1 \right)^{n+\frac{1}{2}}}, \ \mathrm{Re}(p) > 1$$

NOTES

Hence, $$\mathcal{H}_n\left(\frac{e^{-ar}}{r^{1-n}} is\right) = \frac{1}{s^{n+1}} \frac{2^n \sqrt{n+1/2}}{\sqrt{\pi}\left(\left(\frac{a}{s}\right)^2+1\right)^{n+\frac{1}{2}}}$$

or $$\mathcal{H}_n\left(\frac{e^{-ar}}{r^{1-n}} is\right) = \frac{s^n 2^n \sqrt{n+\frac{1}{2}}}{\sqrt{\pi}\left(a^2+s^2\right)^{n+\frac{1}{2}}}, \quad n > \frac{-1}{2}$$

for $n = 0$

$$\mathcal{H}_0\left(\frac{e^{-ar}}{r} is\right) = \frac{1}{\sqrt{a^2+s^2}} \quad n > -\frac{1}{2}$$

and for $a = 0, n = 0$ (both)

$$\mathcal{H}_0\left(\frac{1}{r}, s\right) = \frac{1}{s}$$

4.11 USE OF HANKEL TRANSFORMS IN THE SOLUTION OF SIMPLE PARTIAL DIFFERENTIAL EQUATIONS

Hankel transforms are extremely useful in solving variety of partial differrential equations in cylindrical coordinator. We consider following examples that illustrate the application of Hankel transforms.

Example 4.12 : (Free vibration of a large circular membrane) : If the free vibration of a large circular elastic membrane is governed by the initial value problem:

$$c^2\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{2u}{dr}\right) = \frac{d^2u}{dt^2}, \qquad 0 < r < \infty, \ t > 0$$

$$u(r_1 0) = f(r), \ u_t(x_1 0) = g(r) \qquad \text{for } 0 \le r < \infty,$$

where $c^2 = \frac{T}{\delta}$ = constant and T is the tension in the membrane and in δ the surface density of the membrane.

Solution: Using zero order Hankel transform with respect to *r* we get

$$\mathcal{H}_0(u(r, t)i, s) = \bar{u}(s, t) = \int_0^\infty rJ_0(sr)u(r,t)\,dr \qquad ...(1)$$

Taking the Hankel transform of given partial differential equation, i.e.,

$$\mathcal{H}_0\left(\frac{d^2u}{dt^2}\right) = c^2\mathcal{H}_0\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right)$$

$$\frac{d^2\overline{u}}{dt^2}(st) = c^2\left(\int_0^\infty rJ_0(sr)\frac{d^2u}{dr^2}dr + \int_0^\infty rJ_0(sr)\frac{1}{r}\frac{dy}{dn}dr = -c^2\overline{u}(s,t)\right)$$

NOTES

or $$\frac{d^{2\overline{u}}}{dt^2}(s,t) = -c^2\overline{u}(s,t)$$

or $$\frac{d^{2\overline{u}}}{dt^2} + c^2\overline{u}(s,t) = 0 \qquad ...(3)$$

from the initial conditions

$$\overline{u}(s,0) = \overline{f}(s) \text{ and } \overline{u}_t(s,0) = \overline{g}(s) \qquad ...(4)$$

where $$\overline{f}(s) = \mathcal{H}_0(f(r)\ is) \text{ and } \overline{g}(s) = \mathcal{H}_0(g\ (r)is)$$

General solution of (2) is

$$\overline{u}(s,t) = A\cos(\text{cst}) + B\sin(\text{cst}) \qquad ...(4.35)$$

Taking the help of initial conditions (3) we can compute the values A and B as

$$\overline{u}(s,0) = \overline{f}(s) = A$$

and $$\frac{d\overline{u}}{dt} = -A\sin(\text{cst}).\ \text{cs} + B\ \text{as}\ (\text{cst}).\ \text{cs}$$

$$\frac{d\overline{u}}{dt}(s,0) = \overline{g}(s) = B\ cs$$

$\Rightarrow$ $$\text{B} = \frac{1}{cs}\overline{g}(s)$$

hence the solution (3) can be written as

$$\overline{u}(s_1t) = \overline{f}(s)\cos(cst) + \left(\frac{1}{cs}\right)\overline{g}(s)\sin(cst)$$

then the solution of given p.d.e is obtained by taking inverse Hankel transform. And the solution is given by

$$\overline{u}(r_1t) = \mathcal{H}_0^{-1}\left(\overline{u}(s_1t)ir\right)$$

$$= \int_0^\infty s\ \overline{f}(s)\cos(cst)J_0(rs)ds + \frac{1}{c}\int_0^\infty \overline{g}(s)\sin(cst)J_0(sr)ds \qquad ...(4)$$

In particular if the conditions are given as

$$u(r_1 0) = f(r)$$

$$= ka\left(r^2 + a^2\right)^{-1/2}$$

and $$u_t(r_1 0) = g(0) = 0$$

then solution is obtained by substituting

$$\vec{f}(s) = ka\int_0^\infty r\left(a^2 + r^2\right)^{1/2} J_0(sr)dr$$

$$\vec{f}(s) = \frac{ka}{s} e -^{as} \text{ and } \overline{g}(s) = 0$$

thus $$u(r, t) = ka \int_0^{\infty} e^{-as} J_0(sr) \cos\left(cst\right) ds$$

$$u(r, t) = Ka \left(r^2 + \left(a + ict\right)^2\right)^{1/2}$$

NOTES

Example 4.13: Find the solution of Laplace equation in polar coordinates

$$\nabla^2 u = \frac{d^2 u}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \frac{d^2 u}{dt^2} = 0$$

subject to the boundary conditions $0 \le r < 1$

$$u\left(r_1 0\right) = u_0, \frac{du}{dr}\left(r_1 0\right) = 0, \; r > 1$$

Solution:

Let $\overline{u}\left(s,t\right) = A\,\mathcal{H}_0(u(r, t))$

Taking the Hankel transform of given *pd* we have

$$\mathcal{H}_0\left(\nabla^2 u\right) = -s^2 \overline{u}\left(s_1 t\right) + \frac{d^2 \overline{u}}{dt^2}(r,t) = 0$$

or $$\frac{d^2 \overline{u}}{dt^2}(s,t) - s^2 \overline{u}(g,t) = 0$$

whose solution is

$$\overline{u}\left(s_1 t\right) = A\,(s)\,\mathrm{e}^{-st} + B\,(s)\,e^{+st} \qquad ...(1)$$

Since potential vanishes as $t \to \infty$

so $$B(s) = 0$$

thus $$\overline{u}\left(s_1 t\right) = \mathrm{A}\,|\,\mathrm{s}\,|\mathrm{e}^{-\mathrm{st}} \qquad ...\,(2)$$

by inverse Hankel transform we get the solution as $u(r, t)$

$$= \int_0^{\infty} s\,A(s) e^{-st} J_0(sr) ds \qquad ...(3)$$

From the bounding conditions

$$u\,(\mathrm{r}_1\,0) = \mathrm{u}0\,;\, 0 \le \mathrm{r} < 1$$

$$\frac{du}{dt}(r_1 0) = 0, \qquad r > 1$$

$$\begin{cases} u\left(r_1 0\right)\right) = \int_0^{\infty} s\,A(s) J_0(sr) ds = u_0\, 0 \le r < 1 \\ \text{and } \dfrac{du}{dt}\left(r_1 0\right) = \int_0^{\infty} s^2\, A(s) J_0(sr) ds = 0 \;\; r > 0 \end{cases}$$

solving we get $A(s) = \dfrac{\sin s}{s^2}$

NOTES

Hence the desired solution of given Laplace equation (i.e., the potential is given by)

$$u(r, t) = \frac{240}{\pi}\int_0^{\infty}\frac{\sin s}{s}e^{-st}J_o(sr)\,ds$$

Check Your Progress

1. What is Mellin transform?
2. Define the term inverse Mellin transform.
3. Give the definition of Hankel transform.
4. Write the use of Hankel transform.
5. State the potential problem in a wedge.

4.12 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Mellin transform has wide area of applicability in mathematics as well as physics and engineering. It's most famous application is found in computation of solution of potential problem.
2. $F(s)$ be the Mellin transform of $f(t)$ then $f(t)$ is called inverse Mellin transform of $F(s)$.
3. We introduce the definition of Hankel transform from the two-dimensional Fourier transform and its inverse is called Hankel transform.
4. Hankel transform are extremely useful in solving variety of partial differential equations in cylindrical coordinator.
5. The problem is to solve Laplace's equation in an infinite two-dimensional wedge with Dirichlet boundary conditions. Polar coordinates with origin at the apex of the wedge are used and the sides are located at $q = \pm a$.

4.13 SUMMARY

- Fourier and Laplace transform that were introduced to solve physical problems, Mellin transform arose in a mathematical content Mellin transform was occurred during the study of famous zeta function.
- Mellin transform has wide area of applicability in mathematics as well as physics and engineering. It's most famous application is found in computation of solution of potential problem.
- Another domain of applicability of Mellin transformation is in the resolution of linear differential equation arising of in electrical circuit/engineering by a procedure analogous to Laplace transform.
- $F(s)$ be the Mellin transform of $f(t)$ then $f(t)$ is called inverse Mellin transform of $F(s)$.
- We introduce the definition of Hankel transform from the two-dimensional Fourier transform and its inverse is called Hankel transform.

- Hankel transform are extremely useful in solving variety of partial differential equations in cylindrical coordinator.
- The problem is to solve Laplace's equation in an infinite two-dimensional wedge with Dirichlet boundary conditions. Polar coordinates with origin at the apex of the wedge are used and the sides are located at $q = \pm a$.

4.14 KEY TERMS

- **Mellin transform:** Mellin transform has wide area of applicability in mathematics as well as physics and engineering. It's most famous application is found in computation of solution of potential problem.
- **Inverse Mellin transform:** $F(s)$ be the Mellin transform of $f(t)$ then $f(t)$ is called inverse Mellin transform of $F(s)$.
- **Hankel transform**: Hankel transform are extremely useful in solving variety of partial differential equations in cylindrical coordinator.
- **Potential problem:** The problem is to solve Laplace's equation in an infinite two-dimensional wedge with Dirichlet boundary conditions. Polar coordinates with origin at the apex of the wedge are used and the sides are located at $q = \pm a$.

4.15 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Give the definition of Mellin transform.
2. Write the inversion formula of Fourier transform.
3. What is Mellin transform integrals?
4. Write the formula of Hankel integral.
5. State about the scaling in Hankel transform.
6. Differentiate between Fourier and Hankel transforms.
7. State the elementary function of Hankel transform.

Long-Answer Questions

1. Explain the Mellin transform derivatives with the help of appropriate examples.
2. Discuss the relation of Mellin with Fourier and Laplace transforms.
3. Describe the properties of the Mellin transforms.
4. Explain the convolution theorem for Mellin transform.
5. Discuss Hankel transform with the help of appropriate example.
6. Describe the properties of the Hankel transform by giving appropriate examples.

7. Explain the applications of Hankel transforms to partial differential equations with the help of example.
8. Discuss the distribution of potential in a wedge.

NOTES

4.16 FURTHER READING

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UNIT 5 APPLICATION TO BOUNDARY VALUE PROBLEMS

Structure

5.0 INTRODUCTION

In mathematics, the boundary value problems involving partial differential equations is the integral equations can be found in the theories of ordinary differential equations. The boundary conditions or initial conditions must be explicitly included in the search for the representation formula for the solution of a linear differential equation in this manner. A boundary value problem can be converted to an equivalent Fredholm integral equation. But this method is complicated and so is rarely used.

The heat equation simulates heat transfer in a rod that is insulated all the way around except at the ends. This equation has two variables: one spatial variable (position along the rod) and one temporal variable. The term 'One-Dimensional' in the differential equation's description alludes to the fact that we are only examining one spatial dimension.

A one dimensional wave equation is a first order partial differential equation used in scientific fields such as geophysics, whose solution is - in contrast to the well-known 2nd order two-way wave equations with a solution consisting of two waves travelling in opposite directions - a single propagating wave travelling in a pre-defined direction (the direction in 1D is defined by the sign of the wave velocity). In the one-dimensional case, the one dimensional wave equation allows wave propagation to be calculated without the mathematical complication of solving a 2nd order differential equation. Due to the fact that in the last decades no 3D one dimensional wave equation could be found numerous approximation methods based on the 1D one dimensional wave equation are used for 3D seismic and other geophysical calculations.

Boundary Value Problems (BVP) arise in a number of different applications that include for example, deflection of beam, mechanical and electronics etc. Laplace transforms can be used to solve these BVPs. The key strategy to find the solve the solution of BVP, is to first convert the given BVP into the algebraic equation of the Laplace transform of the solution say L(y) and after solving one can find L(y). The desired solution of the given BVP can be obtained by taking inverse Laplace transform.

A longitudinal wave is one in which the medium or channel flows in the same direction as the wave. The particles in this example move from left to right, causing other particles to vibrate. The medium or channel flows perpendicular to the wave's direction in a transverse wave.

In this unit, you will learn about the boundary value problems involving partial differential equations, one dimensional heat conduction equation, one dimensional wave equation, longitudinal and transverse vibration of a beam, solution of boundary value problems by Laplace transform and simple boundary value problems with applications of Fourier transform.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Solve the boundary value problems involving partial differential equations
- Explain one dimensional heat conduction equation and one dimensional wave equation
- State the solution of the boundary value problems by the Laplace transform
- Analyse the simple boundary value problems with applications of Fourier transform
- Elaborate on the longitudinal and transverse vibration of a beam

5.2 BOUNDARY VALUE PROBLEMS INVOLVING PARTIAL DIFFERENTIAL EQUATIONS

A boundary value problem can be converted to an equivalent Fredholm integral equation. But this method is complicated and so is rarely used. This method is demonstrated with the help of following illustration:

Consider the differential equation

$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$ with boundary conditions

$$x = a: \quad y(a) = \alpha$$

$$y = b: \quad y(b) = \beta$$

NOTES

Where α and β are given constants. Make the transformation,

$$y''(x) = u(x)$$

Integrating both sides from a to x, we get

$$y'(x) = y'(a) + \int_a^x u(t)dt$$

Integrating with respect to x from a to x and applying the given boundary condition at $x = a$, we get

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt$$

$$= \alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt$$

And using the boundary condition at $x = b$ gives,

$$y(b) = \beta = \alpha + (b-a)y'(a) + \int_a^b \int_a^b u(t)dtdt,$$

And the unknown constant $y'(a)$ is determined as

$$y'(a) = \frac{\beta-\alpha}{b-a} - \frac{1}{b-a}\int_a^b \int_a^b u(t)dtdt.$$

Hence the solution can be rewritten as,

$$y(x) = \alpha + (x-a)\left\{\frac{\beta-\alpha}{b-a} - \frac{1}{b-a}\int_a^b \int_a^b u(t)dtdt\right\}$$

$$+ \int_a^x \int_a^x u(t)dtdt$$

Therefore,

$$u(x) = f(x) - P(x)\left\{y'(a) + \int_a^x u(t)dt\right\}$$

$$- Q(x)\left\{\alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt\right\}$$

where $u(x) = y''(x)$ and so $y(x)$ can be determined. It is a complicated procedure to determine the solution of a Boundary Value Problem (BVP) by equivalent Fredholm equation.

NOTES

If $a = 0$ and $b = 1$, i.e., $0 \le x \le 1$, then

$$y(x) = \alpha + xy'(0) + \int_0^x \int_0^x u(t)\,dt\,dt$$

$$= \alpha + xy'(a) + \int_0^x (x - t)u(t)\,dt$$

And hence the unknown constant $y'(0)$ can be determined as

$$y'(0) = (\beta - \alpha) - \int_0^1 (1 - t)u(t)\,dt$$

$$= (\beta - \alpha) - \int_0^x (1 - t)u(t)\,dt - \int_x^1 (1 - t)u(t)\,dt$$

Thus,

$$u(x) = f(x) - P(x)\left\{y'(0) + \int_0^x u(t)\,dt\right\}$$

$$- Q(x)\left\{\alpha + xy'(0) + \int_0^x (x - t)u(t)\,dt\right\}$$

$$u(x) = f(x) - (\beta - \alpha)(P(x) + xQ(x)) - \alpha Q(x) + \int_0^1 K(x,t)u(t)\,dt$$

Where the kernel $K(x,t)$ is given by,

$$K(x,t) = \begin{cases} (P(x) + tQ(x))(1 - x) & 0 \le t \le x \\ (P(x) + xQ(x))(1 - t) & x \le t \le 1 \end{cases}$$

It can be easily verified that $K(x,t) = K(t,x)$ confirming that the kernel is symmetric. The Fredholm integral equation is given by $u(x)$.

NOTES

Example 5.1: Consider the boundary value problem,

$$y''(x) = f(x, y(x)), \quad 0 \le x \le 1$$
$$y(0) = y_0, \quad y(1) = y_1$$

Solution: Integrating the equation with respect to x from 0 to x two times yields

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x f(t, y(t))dtdt$$
$$= y_0 + xy'(0) + \int_0^x (x - t)f(t, y(t))dt$$

To determine the unknown constant $y'(0)$, we use the condition at $x = 1$, i.e., $y(1) = y_1$. Hence,

$$y(1) = y_1 = y_0 + y'(0) + \int_0^1 (1 - t)f(t, y(t))dt,$$

And

$$y'(0) = (y_1 - y_0) - \int_0^1 (1 - t)f(t, y(t))dt.$$

Therefore,

$$y(x) = y_0 + x(y_1 - y_0) - \int_0^1 K(x, t)f(t, y(t))dt, \quad 0 \le x \le 1$$

Where the kernel is given by,

$$K(x, t) = \begin{cases} t(1 - t) & 0 \le t \le x \\ x(1 - t) & x \le t \le 1. \end{cases}$$

If we specialize our problem with simple linear BVP $y''(x) = -\lambda y(x)$, $0 < x < 1$ with the boundary conditions $y(0) = y_0$, $y(1) = y_1$, then $y(x)$ reduces to the second kind Fredholm integral equation,

$$y(x) = F(x) + \lambda \int_0^1 K(x, t)y(t)dt, \quad 0 \le x \le 1$$

where $F(x) = y_0 + x(y_1 - y_0)$. It can be easily verified that $K(x, t) = K(t, x)$ confirming that the kernel is symmetric.

5.3 ONE DIMENSIONAL HEAT CONDUCTION EQUATION

NOTES

One dimensional Heat conduction equation is:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = h^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} = k \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$$

[A] Independent derivation of $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = k \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

Consider one dimensional flow of electricity in a long insulated cable and specify the current i and voltage E at any time in the cable by x-coordinate and time-variable t.

The potential drop E in a line-element δx of length at any point x is given by

$$-\delta E = iR\,\delta x + L\,\delta x \frac{\partial i}{\partial t} \quad \text{... (5.1)}$$

where R and L are respectively resistance and induction per unit length.

If C and G are respectively capacitance to earth and conductance per unit length, then we have

$$-\delta i = GE\,\delta x + C\,\delta x \frac{\partial E}{\partial t} \quad \text{... (5.2)}$$

Rewriting (5.1) and (5.2), $\frac{\partial E}{\partial x} + Ri + L\frac{\partial i}{\partial t} = 0$ ... (5.3)

and

$$\frac{\partial i}{\partial x} + GE + C\frac{\partial E}{\partial t} = 0 \quad \text{... (5.4)}$$

Differentiating (5.3) with respect to x and (5.4) with respect to t, we have

$$\frac{\partial^2 E}{\partial x^2} + R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x\,\partial t} = 0 \quad \text{... (5.5)}$$

and

$$\frac{\partial^2 i}{\partial x\,\partial t} + G\frac{\partial E}{\partial t} + C\frac{\partial^2 E}{\partial t^2} = 0 \quad \text{... (5.6)}$$

Eliminating $\frac{\partial^2 i}{\partial x\,\partial t}$ from (5.5) and (5.6), we get

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + LG\frac{\partial E}{\partial t} - R\frac{\partial i}{\partial x} \quad \text{... (5.7)}$$

Again eliminating $\frac{\partial i}{\partial x}$ from (5.4) and (5.7), we find

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + (CR + GL)\frac{\partial E}{\partial t} + RGE \quad \text{... (5.8)}$$

Differentiation of (5.3) with respect to 't' and (5.4) with respect to 'x' yields

$$\frac{\partial^2 E}{\partial x \partial t} + R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} = 0 \qquad \text{... (5.9)}$$

and $$\frac{\partial^2 i}{\partial x^2} + G\frac{\partial E}{\partial x} + C\frac{\partial^2 E}{\partial x \partial t} = 0 \qquad \text{... (5.10)}$$

Elimination of $\frac{\partial E}{\partial x}$ and $\frac{\partial^2 E}{\partial x \partial t}$ from (5.3), (5.9) and (5.10) gives

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + (CR + GL)\frac{\partial i}{\partial t} + RGi \qquad \text{... (5.11)}$$

(5.7) and (5.11) follow that E and i satisfy a second order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = CL\frac{\partial^2 u}{\partial t^2} + (CR + GL)\frac{\partial u}{\partial t} + RGu \qquad \text{... (5.12)}$$

which is known as *telegraphy equation.*

If the leakage to the ground is small then $G = 0 = L$ and hence (5.12) reduces to $\frac{\partial^2 u}{\partial x^2} = CR\frac{\partial u}{\partial t} = \frac{1}{k}\frac{\partial u}{\partial t}$ where $k = \frac{1}{CR}$.

which is one dimensional diffusion equation.

[B] Solution of $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{h}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$ or $\mathbf{u}_t = \mathbf{h}^2 \mathbf{u}_{xx}$

The solution of this equation by the method of separation of variables has already been discussed in this book. Here below we discuss the solution in different conditions.

[b_1] (Both the ends of a bar at temperature zero)

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function F(x) in the bar, then find the temperature at a subsequent time t.

One dimensional heat equation is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$ ... (5.13)

we have to find a function $u(x, t)$ satisfying (1) with the boundary conditions $u(0, t) = u(l, t) = 0$, $t \geq 0$, l being the length of bar ... (5.14)

and $u(x, 0) = F(x), 0 < x < l$ ... (5.15)

In order to apply the method of seperation of variables, let us assume that

$u(x, t) = X(x)\, T(t)$, X and T being respectively the function of x and t alone.

NOTES

So that $\frac{\partial u}{\partial t} = X\frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$.

Their substitution in (5.13) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt} \qquad \text{... (5.16)}$$

The L.H.S. and R.H.S. of (5.16) are constants because of variables being separated and hence we can write $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2$ (constant of separation).

Here $\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$ *i.e.*, $\frac{d^2X}{dx^2} + \lambda^2 X = 0$ gives $X = A\cos\lambda x + B\sin\lambda x$ ...(5.17)

and $\frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2$ *i.e.*, $\frac{dT}{dt} + h^2\lambda^2 T = 0$ gives $T = Ce^{-\lambda^2 h^2 t}$ ... (5.18)

In view of condition (5.14), i.e., $u = 0$ at $x = 0$ or (5,17) gives $A = 0$ and λ be chosen such that $\sin\lambda l = 0$, i.e., $\lambda = \frac{n\pi}{l}$, n being an integer.

Hence the solution (5.13), i.e., $u = XT$ takes the form

$$u = B\sin\frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2h^2}{l}t} \qquad \text{... (5.19)}$$

Summing over for all values of n, this becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2h^2}{l}t} \qquad \text{... (5.20)}$$

Applying condition (5.15) i.e., $u(x, 0) = F(x)$ at $t = 0$, we have

$$F(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x \quad \text{for } 0 < x < l \qquad \text{... (5.21)}$$

So that
$$B_n = \frac{2}{l}\int_0^t F(x)\sin\frac{n\pi x}{l}\,dx \qquad \text{... (5.22)}$$

which is obtained by multiplying (5.21) by $\sin\frac{n\pi x}{l}$ and then integrating from $x = 0$ to $x = l$.

Hence the required solution is

$$u(x, t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2h^2}{l}t}\sin\frac{n\pi}{l}x\int_0^t F(u)\sin\frac{n\pi}{l}u\,du \qquad \text{... (5.23)}$$

NOTES

Deduction: (Insulated faces)

If instead of the ends of a bar of length l having kept at temperature zero, they are impervious to heat and the initial temperature is the prescribed function F(x) in the bar, then to find the temperature at a subsequent time t, we have the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ or } l \text{ for all } t \quad \dots (5.24)$$

$$u(x, 0) = F(x),\ 0 < x < l \quad \dots (5.25)$$

Then the solution follows from (5.17) as

$$u = A \cos \lambda x + B \sin \lambda x \quad \dots (5.26)$$

which in view of (5.26) requires $B = 0$ and $\sin \lambda x = 0$ i.e., $\lambda = \frac{n\pi}{l}$, $n = 0, 1, 2, 3...$

So that the general solution of the one dimensional diffusion equation will be of the form

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2h^2}{l}t} \cos \frac{n\pi x}{l} \quad \dots (5.27)$$

where B_0 corresponds to $n = 0$.

By (5.25), this yields, $F(x) = u(x, 0) = B_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}$ ... (5.28)

from which we can easily find the coefficients

$$A_n = \frac{2}{l} \int_0^l F(x) \cos \frac{n\pi x}{l}\, dx \quad \dots (5.29)$$

and $$B_0 = \tfrac{1}{2} A_0. \quad \dots (5.30)$$

Note 1. The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0$, $x = \pi$ thermally insulated is given by

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n^2h^2t} \cos nx. \quad \dots (5.31)$$

where $$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \quad \dots (5.32)$$

and $$B_0 = \frac{1}{2} A_0 = \frac{1}{\pi} \int_0^{\pi} F(x)\, dx. \quad \dots (5.33)$$

Note 2. The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0$, $x = l$ thermally insulated is given by

$$u(x, t) = \frac{1}{l} \int_0^l F(x)\, dx + \frac{2}{l} \sum_{n=1}^{\infty} e^{\frac{-n^2\pi^2h^2}{l}t} \cos \frac{n\pi x}{l} \int_0^l F(x) \frac{n\pi x}{l}\, dx \quad \dots (5.34)$$

NOTES

[b_2] (**One end of a bar at temperature u_0 and other at zero temperature**).

If a bar of length l is at a temperature v_0 such that one of its ends $x = 0$ is kept at zero temperature and the other end $x = l$ is kept at temperature u_0, then find the temperature at any point x of the bar at an instant of time $t > 0$.

or

A rod of length l and thermal conductivity h^2 is maintained at a uniform temperature v_0. At $t = 0$ the end $x = 0$ is suddenly cooled to 0°C by application of ice and the end $x = l$ is heated to the temperature u_0 by applying steam, the rod being insulated along its length so that no heat can transfer from the sides. Find the temperature of the rod at any point at any time.

The equation is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < l, t > 0$ ... (5.35)

With boundary conditions $u(0, t) = 0$, $u(l, t) = u_0$ for all t ... (5.36)

and $u(x, 0) = v_0$ ... (5.37)

Let the solution of (5.35) be $u(x, t) = X(x)\, T(t)$ ... (5.38)

where X is a function of x alone and T is a function of t alone.

Substituting from (5.38) $\frac{\partial u}{\partial t} = X \frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$, in (5.35) we get $\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{h^2 T}\frac{dT}{dt}$ where variables are separated and hence terms on either side are constants.

Now there arise these possibilities:

[1] $\frac{d^2 X}{dx^2} = 0, \frac{dT}{dt} = 0$ whence the solution is $X = Ax + B$, $T = C$ ... (5.39)

[2] $\frac{d^2 X}{dx^2} = \lambda^2 x, \frac{dT}{dt} = h^2\lambda^2 T$, the solution being $X = Ae^{\lambda x} + Be^{-\lambda x}$, $T = Ce^{h^2\lambda^2 t}$

... (5.40)

[3] $\frac{d^2 X}{dx^2} = -\lambda^2 x, \frac{dT}{dt} = -h^2\lambda^2 T$, the solution being $X = A \cos \lambda x + B \sin \lambda x$, $T = Ce^{-h^2\lambda^2 t}$. ... (5.41)

The combined solution in any of the three cases is $u = XT$. But $u = XT$ increases indefinitely with time t so possibility [2] is ruled out since then $u \to 0$ as $t \to \infty$. Conclusively the possibilities [1] and [3] determine the solution of (5.35) in the form

$$u(x, t) = u_S(x) + u_T(x, t) \quad ... (5.42)$$

where $u_S(x)$ is the temperature distribution after a long interval of time when there exists steady state of temperature and $u_T(x, t)$ is the transient effects which die down when the time passes. Consequently there exists uniform temperature after one and $x = 0$ being kept at zero temperature and the end $x = l$ at $u = u_0$ so that

$$u_S(x) = \frac{u_0}{l}x, \text{ whence (5.42) yields } u(x, t) = \frac{u_0}{l}x + u_T(x, t) \quad \text{... (5.43)}$$

with boundary conditions $u_T(0, t) = 0 = u_T(l, t)$ by (2) ... (5.44)

and $$u_T(x, 0) = v_0 - \frac{u_0}{l}x \quad \text{[by (5.37)]} \quad \text{... (5.45)}$$

Hence the possibility [3] i.e., the solution (5.41) reduces to

$$u_T(x, t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 h^2 t} \quad \text{... (5.46)}$$

whence in view of (5.36), this requires $A = 0$ and $\sin\lambda l = 0$ i.e., $\lambda = \frac{n\pi}{l}$, being an integer.

We thus obtain a solution

$$u_T(x, t) = \sum_{n=1}^{\infty} B_n\, e^{-\frac{n^2\pi^2h^2}{l^2}t} \sin\frac{n\pi}{l}x \quad \text{... (5.47)}$$

In view of (5.45), this gives $u_T(x, 0) = v_0 - \frac{u_0}{l}x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x.$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \left(v_0 - \frac{u_0}{l}x\right)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{n\pi}[v_0 - (-1)^n(v_0 - u_0)] \quad \text{(on integrating by parts)}$$

Hence the general solution of (5.35) with the help of (5.45) and (5.47) is

$$u(x, t) = \frac{u_0}{l}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[v_0 - (-1)^n(v_0 - u_0)]\, e^{-\frac{n^2\pi^2h^2}{l^2}t}\sin\frac{n\pi}{l}x \quad \text{... (5.48)}$$

which gives temperature at any point x of the bar at any time $t > 0$.

Note. If we set $v_0 = 0$, then (5.48) takes the form

$$u(x, t) = \frac{u_0}{l}\left[x + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\, e^{-\frac{n^2\pi^2h^2}{l^2}t}\sin\frac{n\pi}{l}x\right] \quad \text{... (5.49)}$$

[b_3] (Temperature in an infinite bar)

If an infinite bar of small cross-section is insulated such that there is no transfer of heat at the surface and the temperature of the bar at t = 0 is given by an arbitrary function F(x) of x (taking the bar along x-axis), then find the temperature of the rod at any point of the bar at any time t.

NOTES

The boundary value example is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$ ... (5.50)

With initial condition, $u(x, 0) = F(x), -\infty < x < \infty$ ... (5.51)

Let the solution be $u(x, t) = X(x)\, T(t)$ ... (5.52)

whence (5.50) gives $\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{h^2 T}\frac{dT}{dt} = -\lambda^2$ (say) ... (5.53)

Then the solution of (5.50) is

$$u(x, t) = XT = (A \cos \lambda x + B \sin \lambda x)\, e^{-h^2\lambda^2 t} \quad \text{... (5.54)}$$

Here the arbitrary constants A and B being periodic may be taken as $A = A(\lambda)$, $B = B(\lambda)$ and due to the linearity and homogeneity of the heat equation we may write

$$u(x, t) = \int_0^\infty u(x, t, \lambda)\, d\lambda = \int_0^\infty e^{-h^2\lambda^2 t}[A(\lambda)\cos \lambda x + B(\lambda)\sin(\lambda x)]\, d\lambda \quad \text{... (5.55)}$$

The condition (5.51) claims that

$$u(x, 0) = F(x) = \int_0^\infty [A(\lambda)\cos \lambda x + B(\lambda)\sin \lambda x]\, d\lambda$$

In view of Fourier's integrals we have

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\mu)\cos(\mu\lambda)\, d\mu \text{ and } B(\lambda) = \frac{1}{\pi}\int_0^\infty F(\mu)\sin(\lambda\mu)\, d\mu$$

so that $u(x, 0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} F(\mu)\cos \lambda(x-\mu)\, d\mu\right] d\lambda$

As such (5.55) takes the form

$$u(x, t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} F(\mu)\cos \lambda(x-\mu)\, e^{-h^2\lambda^2 t}\, d\mu\right] d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} F(\mu)\left[\int_0^\infty e^{-h^2\lambda^2 t}\cos \lambda(x-\mu)\, d\lambda\right] d\mu \quad \text{... (5.56)}$$

But we know that $\int_0^\infty e^{-x^2}\cos 2bx\, dx = \sqrt{\pi}\, e^{-b^2/2}$

So that $\int_0^\infty e^{-h^2\lambda^2 t}\cos \lambda(x-\mu)\, d\lambda = \frac{\sqrt{\pi}}{2h\sqrt{t}}\, e^{-\left(\frac{x-\mu}{2h\sqrt{t}}\right)^2}$

$$= \frac{\sqrt{\pi}}{2h\sqrt{t}}\, e^{-\frac{(x-\mu)^2}{4h^2 t}}$$

Hence (5.56) gives

$$u(x, t) = \frac{1}{2h\sqrt{\pi t}}\int_{-\infty}^{\infty} F(\mu)\, e^{-\frac{(x-\mu)^2}{4h^2 t}}\, d\mu \quad \text{... (5.57)}$$

which gives the required temperature at any point at any time.

NOTES

Example 5.2: *Solve* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ *where u = 0 for t = ∞ and x = 0 or l.*

Solution. Taking $u(x, l) = X(x)\, T(t)$, the solution of the given equation is

$$X = A \cos \lambda x + B \sin \lambda x,\ T = Ce^{-\lambda^2 t}$$

with boundary conditions, $u(0, t) = 0$ and $u(x, \infty) = 0$.

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{l^2}t}$$

Example 5.3: *Solve* $\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$ *under the boundary conditions*

$$\theta(0, t) = \theta(l, t) = 0,\ t > 0 \quad \text{... (1)}$$

and $$\theta(x, 0) = x,\ 0 < x < l, \quad \text{... (2)}$$

l being the length of the bar.

Solution: $$\theta(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-n^2\pi^2 h^2/l^2 t}$$

where, $$B_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l}\int_0^l x \sin \frac{n\pi}{l} x\, dx$$

$$= -\frac{2l}{n\pi} \cos n\pi$$

$$= \begin{cases} \frac{2l}{n\pi} \text{ when } n \text{ is odd} \\ \frac{-2l}{n\pi} \text{ when } n \text{ is even} \end{cases}$$

Hence,

$$\theta(x, t) = \frac{2l}{\pi}\left[e^{-\pi^2 h^2 t/l^2} \sin \frac{\pi}{l} - \frac{1}{2} e^{-2^2\pi^2 h^2 t/l^2} \sin \frac{2\pi}{l} x + \frac{1}{3} e^{-3^2\pi^2 h^2 t/l^2} \sin \frac{3\pi}{l} x ...\right]$$

Example 5.4: *Find the temperature u (x, t) in a bar of length l, perfectly insulated, and whose ends are kept at temperature zero while the initial temperature is given by*

$$F(x) = \begin{cases} x,\ 0 < x < l/2 \\ l - x,\ \frac{l}{2} < x < l. \end{cases}$$

Solution. The boundary value example is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$.

With conditions $u(0, t) = u(l, t) = 0$ and $u(x, 0) =$

$$F(x) = \begin{cases} x, 0 < x < l/2 \\ l - x, \dfrac{l}{2} < x < l. \end{cases}$$

NOTES

$$u(x, t) = \sum_{n=1}^{\infty} B_n\ F(x) \sin \frac{n\pi x}{l}\, dx$$

where $B_n = \dfrac{2}{l}\int_0^l F(x) \sin \dfrac{n\pi x}{l}\, dx = \dfrac{2}{l}\int_0^{l/2} x \sin \dfrac{n\pi}{l}\, x\, dx + \int_{l/2}^{l} (l - x) \sin \dfrac{n\pi x}{l}\, dx$

$$= \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} = \begin{cases} \dfrac{4l}{n^2\pi^2} & \text{for } n = 1, 5, 9, \ldots \\ 0 & \text{for } n = 2, 4, 6, \ldots \\ -\dfrac{4l}{n^2\pi^2} & \text{for } n = 3, 7, 11, \ldots \end{cases}$$

Hence the solution is

$$u(x, t) = \frac{4l}{\pi^2}\left[-\frac{1}{1^2} \sin \frac{\pi x}{l}\, e^{-h^2\pi^2 t/l^2} - \frac{1}{3^2} \sin \frac{3\pi x}{l}\, e^{-3^2 h^2 \pi^2 t/l^2} + \ldots \right]$$

Note. Had we considered the case of slab with its ends $x = 0$ and $n = l$ maintained at temperature zero and initial temperature being

$$F(x) = \begin{cases} T_0, & 0 < x < l/2 \\ 0, & l/2 < x < l \end{cases}$$

Then we should have

$$B_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l}\, dx = \frac{2T_0}{l}\int_0^l \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{4T_0}{n\pi} \sin^2 \frac{n\pi}{4}$$

and the solution would be

$$u(x, t) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4}\, e^{-n^2\pi^2 t/l^2} \sin \frac{n\pi x}{l}.$$

Example 5.5: *Solve* $\dfrac{\partial u}{\partial t} = h^2 \dfrac{\partial^2 u}{\partial x^2}$, *$0 < x < \pi$, $t > 0$, under the boundary conditions $u_x(0, t) = 0 = u_x(\pi, t)$ and $u_x(x, 0) = \sin x$.*

Solution. We have,

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}\, e^{-n^2\pi^2 h^2 t/l^2} \quad \text{where } B_0 = \frac{A_0}{2} \text{ and } l = \pi$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx\, e^{-n^2 h^2 \cdot t}$$

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where $$A_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \begin{cases} 0, \text{ when } n \text{ is odd} \\ -\dfrac{4}{\pi\left(4m^2-1\right)}, \text{ when } n = 2m \end{cases}$$

and $$A_n = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{4}{\pi}.$$

Hence the required solution is

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}(4m^2-1)^{-1} e^{-4m^2h^2t} \cos 2mx.$$

Example 5.6. *The face x = 0 of a slab is maintained at temperature zero and heat is supplied at constant rate at the face x = π, so that* $\frac{\partial u}{\partial x} = \mu$ *when x = π. If the initial temperature is zero, show that*

$$u(x, t) = \mu x + \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j-\dfrac{1}{2}\right)^2} \sin\left(j-\tfrac{1}{2}\right) x e^{-(j-\frac{1}{2})^2 t}$$

where the unit of time is so chosen that k = 1.

Solution. Taking $u(x, t)$ as the temperature of the slab, the boundary value example is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi, t > 0$. ... (1)

with condition $u(0, t) = 0$ ... (2)

$u(x, 0) = 0$ ... (3)

and $\frac{\partial}{\partial x} u(\pi, t) = \mu$ ... (4)

Applying the method of separation of variables, the solutions of the given equation are

(*i*) $u = (Ae^{\lambda x} + Be^{-\lambda x})\, e^{\lambda^2 t}$

(*ii*) $u = A_1 + B_1 x$

(*iii*) $u = (A_2 \cos \pi x + B_2 \sin \pi x)\, e^{-\lambda^2 t}$

according as the constant of variation is λ^2 or $-\lambda^2$.

Here (*i*) is inadmissible as $u \to \infty$ when $t \to \infty$.

(*ii*) alone is inadequate to give the complete solution and hence the complete solution is given by (*ii*) and (*iii*) jointly

i.e., $u(x, t) = u_S(x) + u_T(x, t)$ ... (5)

where $u_S(x)$ is the temperature distribution after a long period of time when the slab has reached the steady state of the temperature distribution and $u_T(x, t)$

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denotes the transient effects which die down with the passage of time.

From (*ii*) $u_S(x) = A_1 + B_1 x$ ... (6)

and from (*iii*) $u_T(x, t) = (A_2 \cos \lambda x + B_2 \sin \lambda x)\ e^{-\lambda^2 t}$ ... (7)

Applying (2), (6) gives $A_1 = 0$ and by (4), (6) gives $\mu = B_1$

so that $u_S = \mu x$ ... (8)

Thus with the help of (7) and (8), (5) reduces to

$$u(x, t) = \mu x + (A_2 \cos \lambda x + B_2 \sin \lambda x)\ e^{-\lambda^2 t} \quad \text{... (9)}$$

Applying (2), i.e., $u(0, t) = 0$, we get $A_2 = 0$ ... (10)

Applying (4) i.e., $\mu = \dfrac{\partial}{\partial x} u(\pi, t)$,

we have $(\mu + \lambda B_2 \cos \lambda\pi)\ e^{-\lambda^2 t} = \mu$

i.e., $\cos \lambda\pi = 0$ giving $\lambda\pi = (2j - 1)\dfrac{\pi}{2}$ *i.e.,* $\lambda = j - \dfrac{1}{2}$ ... (11)

As such $u_T(x, t) = B_j \sin\left(j - \dfrac{1}{2}\right) x \cdot e^{\left(j - \frac{1}{2}\right)^2 t}$ where we have set $B_j = B_2$.

Summing over all j, the general solution is

$$u_T(x, t) = \sum_{j=1}^{\infty} B_j \sin\left(j - \frac{1}{2}\right) x\, e^{-(j - \frac{1}{2})^2 t} \quad \text{... (12)}$$

Hence from (5)

$$u(x, t) = \mu x + \sum_{j=1}^{\infty} B_j \sin\left(j - \frac{1}{2}\right) x\, e^{-(j - \frac{1}{2})^2 t} \quad \text{... (13)}$$

Applying the condition (3), $0 = \mu x + \displaystyle\sum_{j=1}^{\infty} B_j \sin\left(j - \frac{1}{2}\right) x$

i.e., $-\mu x = \displaystyle\sum_{j=1}^{\infty} B_j \sin\left(j - \frac{1}{2}\right) x$ so that $B_j = \dfrac{1}{\pi}\displaystyle\int_j^{\pi} (-\mu x) \sin\left(j - \frac{1}{2}\right) x\, dx$

$$= \frac{2\mu}{\pi} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2}$$

Hence (13) reduces to

$$u(x, t) = \mu x + \frac{2\mu}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2} \sin\left(j - \frac{1}{2}\right) x\, e^{-(j - \frac{1}{2})^2 t}$$

which is the required relation.

Check Your Progress

1. State the resolvent kernel as a ratio of two series.
2. Give the one dimensional heat conduction equations.

NOTES

5.4 ONE DIMENSIONAL WAVE EQUATION

[A] Derivation of One Dimensional Wave Equation

Consider a flexible string of length l tightly stretched between two points x = 0 and x = l on x-axis, with its ends at these ends. If the string is set into small transverse vibration, the displacement say u (x, t) from the x-axis of any point x of the string at any time t is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$, T *being tension and* ρ *the linear density.*

The equation $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{... (5.58)}$$

is known as one dimensional wave equation.

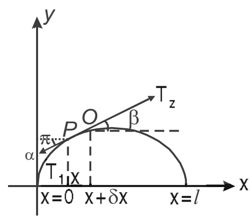

Fig. 5.1 *One Dimensional Wave Equation*

Let the string (assumed to be perfectly flexible) of length l tightly stretched between the points $x = 0$ and $x = l$ on x-axis be distorted and then at a certain instant of time say $t = 0$, it is released and allowed to vibrate. To determine its deflection (displacement from x-axis) at any point x at any time t, let us take the following assumptions:

(*i*) The string is uniform, i.e., its mass m per unit length is constant.

(*ii*) The string is perfectly elastic and so offers no resistance to any bending.

(*iii*) The tension T is so large that the action of gravitational force on the string is negligible.

(*iv*) The motion of the string is a small transverse vibration in a vertical plane, i.e., each particle of the string moves strictly in the vertical plane so that the deflection and slope (gradient) at any point of the string are very small in absolute value.

Cosider the motion of an element PQ of length δs of the string. The string being perfectly elastic the tension T_1 at P and T_2 at Q are tangential to the curve of the string. Let T_1 and T_2 make angle α and β respectively with the horizontal.

There being no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (say)} = \text{Constant} \quad \ldots (5.59)$$

Mass of the element PQ is $\rho\, \delta\, s$. By Newton's second law of motion we therefore have

$$T_2 \sin \beta - T_1 \sin \alpha = (\rho\, \delta\, s)\,.\, \frac{\partial^2 u}{\partial t^2}, \quad \ldots (5.60)$$

$\frac{\partial^2 u}{\partial t^2}$ being upward acceleration of PQ

Using (5.58), (5.59) yields $\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \frac{\rho\, \delta s}{T} \cdot \frac{\partial^2 u}{\partial t^2}$

i.e., $$\tan \beta - \tan \alpha = \frac{\rho\, \delta s}{T} \frac{\partial^2 u}{\partial t^2} \quad \ldots (5.61)$$

Replacing δs by δx since the gradient of the curve is very small, (5.61) gives

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho\, \delta s}{T} \frac{\partial^2 u}{\partial t^2} \quad \ldots (5.62)$$

since $\tan \alpha$ and $\tan \beta$ are slopes at x and $x + \delta x$ respectively.

or $$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

i.e., $$\frac{u_x (x + \delta x, t) - u_x (x, t)}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

Proceeding to the limit as $\delta x \to 0$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \text{ where } \frac{1}{c^2} = \frac{\rho}{T}.$$

Note 1. $c^2 = \frac{T}{\rho}$ reveals that the constant $\frac{T}{\rho}$ is positive.

Note 2. Since u is dependent of x and t both, therefore we have used the partial derivative $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial t^2}$.

Note 3. If a force $F(x, t)$ per unit of mass acts in the u-direction along the string, in addition to the tension of the string, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F.$$

[B] Derivative of Two Dimensional Wave Equation

In case of a rectangular membrane, the two dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \ldots (5.63)$$

Consider the motion of a stretched membrane supposed to be stretched and fixed along its entire boundary in the *x-y* plane. Let us take the following assumptions:

(*i*) The membrane is homogeneous, i.e., mass (say) ρ per unit area is constant.

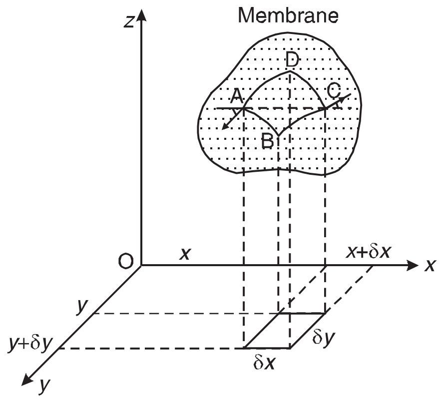

Fig. 5.2 *Membrane in Homogeneous*

(*ii*) The membrane is perfectly flexible and so thin that it offers no resistance to any bending.

(*iii*) The tension *T* per unit length caused by the stretching of the membrane is invariant during the motion and retains the same value at each of its points and in all the directions.

(*vi*) The deflection $u(x, y, t)$ of the membrane during the motion is negligible as compared to the size of the membrane. Also all the angles of inclination are small.

Consider the motion of an element *ABCD* of the membrane. Let its area be $\delta x\, \delta y$. *T* being the tension per unit length, the force acting on the edges are $T\, \delta x$ and $T\, \delta y$ approximately. Also the membrane being perfectly flexible, the tensions $T\, \delta x$ and $T\, \delta y$ are tangential to the membrane. Let α, β be the inclinations of these tensions with the horizontal. Then the horizontal components of the forces at one pair of opposite edges are $T\, \delta y \cos \alpha$ and $T\, \delta y \cos \beta$. When α and β are very small $\cos \alpha \to 1$ and $\cos \beta \to 1$ so that $T\, \delta y \cos \alpha \to T\, \delta y$ and $T\, \delta y \cos \beta \to T\, \delta y$, i.e., the horizontal components of the forces at opposite edges are nearly equal and hence the motion of the particles of the membrane in horizontal direction is negligibly small. As such we assume that every particle of the membrane moves vertically.

The resultant vertical force $= T\, \delta y \sin \beta - T\, \delta y \sin \alpha$

$$= T\, \delta y (\tan \beta - \tan \alpha)$$

$$(\because \alpha, \beta \text{ being small } \sin \alpha = \alpha = \tan \alpha \text{ and } \sin \beta = \beta = \tan \beta)$$

$$= T\, \delta y\, [u_x (x + \delta x, y_1) - u_x (x, y_2)] \quad \text{... (5.64)}$$

where u_x denotes the partial derivative of *u* with respect to. *x* and y_1, y_2 are the values of *y* between *y* and $y + \delta y$.

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Similarly, the resultant vertical force acting on the other two edges

$$= T\,\delta x\,[u_y\,(x_1, y + \delta y) - u_y\,(x_2, y)] \quad \text{... (5.65)}$$

where u_y denotes the partial derivative of u with respect to. y and x_1, x_2 are the values of x between x and $x + \delta x$.

By Newton's second law of motion, we have

Total vertical force on the element $= \rho\,\delta x\,\delta y\,\dfrac{\partial^2 u}{\partial t^2}$

i.e., $T\delta y[u_x(x+\delta x, y_1) - u_x(x, y_2)] + T\,\delta x[u_y(x_1, y+\delta y) - u_y(x_2, y)]$

$$= \rho\,\delta x\,\delta y\,\frac{\partial^2 u}{\partial t^2}$$

where $\dfrac{\partial^2 u}{\partial t^2}$ is the acceleration of the element.

Thus $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{T}{\rho}\left[\dfrac{u_x\,(x+\delta x, y_1) - u_x(x, y_2)}{\delta x}\right] + \dfrac{T}{\rho}\left[\dfrac{u_y\,(x_1, y+\delta y) - u(x_2, y)}{\delta y}\right]$

Proceeding to the limit as $\delta x \to 0$ and $\delta y \to 0$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[u_{xx} + u_{yy}\right] = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\,\nabla^2 \text{ where } \nabla^2 \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{... (5.66)}$$

Note 1. If $u = v\,(x, y)\,e^{iwt}$, (9) yields $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + k^2 v = 0$ where $k^2 = \left(\dfrac{w}{c}\right)^2$ ... (5.67)

Note 2. The three dimensional wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = c^2\nabla^2 u \quad \text{... (5.68)}$$

where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

5.4.1 Green's Functions for the Wave Equation

The wave equation is

$$\nabla^2\psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \quad \text{... (5.69)}$$

Also written as,

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^3}\right)\psi = 0 \quad \text{... (5.70)}$$

If its solution is of the form

$$\psi\,(x, y, z, t) = \Psi\,(x, y, z)\, e^{\pm ic\lambda t} \quad \text{... (5.71)}$$

then (5.69) gives, $\nabla^2\Psi + \lambda^2\Psi = 0$ ... (5.72)

which is known as *Space form of the wave equation* or *Helmholtz's equation.*

Taking $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ as the position vector of a point (x, y, z) and $\mathbf{r}' = x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k}$ as the position vector of an isolated point (x', y', z'), the *Green's function G(***r, r′***)* is defined as

$$G(\mathbf{r}, \mathbf{r}') = H\,(\mathbf{r}, \mathbf{r}') + \frac{1}{|\mathbf{r}' - \mathbf{r}|} \quad \text{... (5.73)}$$

where $H\,(\mathbf{r}, \mathbf{r}')$ satisfies $\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) H\,(\mathbf{r}\,\mathbf{r}') = 0$ ... (5.74)

Using *Green's formula,* i.e., $\psi\,(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{ \frac{1}{|\mathbf{r}' - \mathbf{r}|} \frac{\partial\,\psi(\mathbf{r}')}{\partial n} \right.$

$$\left. -\psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} \right\} dS' \quad \text{.. (5.75)}$$

it may be shown that

$$\psi\,(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{ G(\mathbf{r} - \mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}')\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \quad \text{... (5.76)}$$

where $\mathbf{n}$ is the unit outward drawn normal to the surface S.

Now we claim that the solution of space form of the wave equation under certain boundary conditions can be made to depend on the determination of the appropriate Green's function. Let us assume that $G\,(\mathbf{r}, \mathbf{r}')$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) G(\mathbf{r}, \mathbf{r}') + \lambda^2 G(\mathbf{r}, \mathbf{r}') = 0 \quad \text{... (5.77)}$$

under the assumption that $G\,(\mathbf{r}, \mathbf{r}')$ is finite and continuous with respect to. either their variables x, y, z or x', y', z' for the points $\mathbf{r}, \mathbf{r}'$ belonging to a region V bounded by a closed surface S except in $\mathbf{r}$-neighbourhood where there is a singularity of the type

$$\frac{e^{-i\lambda|\mathrm{r} - \mathrm{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \text{ as } |\mathbf{r}'| \to |\mathbf{r}| \quad \text{... (5.78)}$$

Now $\Psi\,(\mathbf{r})$ being the solution of (5.72) and its partial derivatives of the first and second orders being continuous within the volume V on the closed surface S we have

$$\frac{1}{4\pi}\int_S \frac{e^{-i\lambda|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial\Psi\,(\mathbf{r}')}{\partial n} - \Psi(\mathbf{r}')\frac{\partial}{\partial n}e^{i\lambda|\mathbf{r} - \mathbf{r}'|}\cdot\frac{1}{|\mathbf{r} - \mathbf{r}'|}\int dS'$$

$$= \begin{cases} \Psi(\mathbf{r}), \text{ if } \mathbf{r} \text{ lies inside } V \\ 0, \quad \text{ if } \mathbf{r}' \text{ does not lie inside } V \end{cases} \quad \text{... (5.79)}$$

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Using (5.78), we therefore have

$$\Psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{ G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r}')}{\partial n} - \Psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \quad \text{... (5.80)}$$

Taking $G(\mathbf{r}, \mathbf{r}')$ such that it satisfies the boundary condition

$$G_1(\mathbf{r}, \mathbf{r}') = 0 \quad \text{... (5.81)}$$

whereas the point $\mathbf{r}'$ lies on the surface S, then (5.80) reduces to

$$\Psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \Psi(\mathbf{r}') \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} dS' \quad \text{... (5.82)}$$

which gives Ψ at any point $\mathbf{r}$ within S.

Again if $G_2(\mathbf{r}, \mathbf{r}')$ is such a function satisfying $\frac{G_2(\mathbf{r}, \mathbf{r}')}{\partial n} = 0$

for $\mathbf{r}'$ lying inside S we have ... (5.83)

$$\Psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \frac{\partial \Psi(\mathbf{r}')}{\partial n} G_2(\mathbf{r}, \mathbf{r}') dS' \quad \text{... (5.84)}$$

which gives Ψ at any point within S provided $\frac{\partial \Psi}{\partial n}$ is known at every point of S

Corollary. *Green's function for Diffusion equation*:

The diffusion equation is $\frac{\partial u}{\partial t} = h^2 \nabla^2 u$ ... (5.85)

Let $u(\mathbf{r}, t)$ be a solution of it. Then for a volume V enclosed by a surface S, the boundary condition is

$$u(\mathbf{r}\, t) = \phi(\mathbf{r}, t) \quad \text{... (5.86)}$$

when $\mathbf{r}$ lies inside S.

The Initial condition is

$$u(\mathbf{r}, o) = f(\mathbf{r}) \quad \text{... (5.87)}$$

when $\mathbf{r}$ lies inside V.

If we define Green's function $G(\mathbf{r}, \mathbf{r}', t - t'), t > t'$

such that $$\frac{\partial G}{\partial t} = h^2 \nabla^2 G \quad \text{... (5.88)}$$

With boundary condition

$$G(\mathbf{r}, \mathbf{r}', t - t') = 0 \quad \text{... (5.89)}$$

when $\mathbf{r}'$ lies inside S and initial condition

$$\lim_{t \to t'} G \to 0 \qquad \text{... (5.90)}$$

at the points of V except at the point $\mathbf{r}$ where G takes the form

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$$\frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{4h^2(t-t')}}}{8\left[\pi h^2(t-t')\right]^{3/2}} \qquad \text{... (5.91)}$$

Now G being a function of t and hence of $(t-t')$ only, (20) is equivalent to

$$\frac{\partial G}{\partial t'} + h^2\nabla^2 G = 0 \qquad \text{... (5.92)}$$

Physically interpreted $G(\mathbf{r}, \mathbf{r}', t-t')$ is the temperature at any point $\mathbf{r}'$ at time t due to an instantaneous point source of unit strength generated at time t' of the point $\mathbf{r}$. Initially, the temperature of the solid is zero and the surface is kept at zero temperature.

Equations (5.85) and (5.86) being valid for $t' < t$, can be rewritten as

$$\frac{\partial u}{\partial t'} = h^2\nabla^2 u, t' < t \qquad \text{... (5.93)}$$

and $\qquad u(\mathbf{r}', t) = \phi(\mathbf{r}', t)$ when $\mathbf{r}'$ lies inside S $\qquad$ ... (5.94)

Equations (5.92) and (5.93) yield,

$$\frac{\partial}{\partial t'}(uG) = u\frac{\partial G}{\partial t'} + G\frac{\partial u}{\partial t'} = h^2\left[G\,\nabla^2 u - u\nabla^2 G\right]$$

so that for an arbitrary small $\varepsilon > 0$, we find

$$\int_0^{t-\varepsilon}\left\{\int_V \frac{\partial}{\partial t'}(uG)\,dv'\right\}dt' = h^2\int_0^{t-\varepsilon}\left\{\int_V\left[G\nabla^2 u - u\nabla^2 G\right]dv'\right\}dt' \qquad \text{... (5.95)}$$

or, changing the order of integration,

$$\int_V (u\,G)_{t'=t-\varepsilon}\,dv' - \int_V (u\,G)_{t'=0}\,dv'$$

$$= u\,(\mathbf{r}, t)\int_V [G(\mathbf{r}, \mathbf{r}'), t-t']_{t'=t-\varepsilon}\,dv' - \int_V G(\mathbf{r}, \mathbf{r}', t)\; f(\mathbf{r}')\,dv'$$

By Eqn. (5.91), for $G(\mathbf{r}, \mathbf{r}', t-t')$ we have $\int_V G(\mathbf{r}, \mathbf{r}', t-t']_{t'=t-0}\,dv' = 1$

so that when $\varepsilon \to 0$, L.H.S. of (5.95)

$$= u\,(\mathbf{r}', t) - \int_V f(\mathbf{r}')\,G\,(\mathbf{r}, \mathbf{r}', t)\,dv'$$

Hence applying Green's theorem to the R.H.S. of (5.95) and using (5.86) and (5.89) we may find

$-h^2\int_0^t dt'\int_S \phi(\mathbf{r}', t)\frac{\partial G}{\partial n}\,dS'$ in limit when $\varepsilon \to 0$ and $\frac{\partial G}{\partial n}$ denoting the derivative of G along outward drawn normal to the surface S.

We shall ultimately find,

$$u(\mathbf{r}, t) = \int_V f(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}', t)\, dv' - h^2 \int_0^t dt' \int_s^t \phi(\mathbf{r}', t)\, \frac{\partial G}{\partial n}\, dS' \qquad \text{... (5.96)}$$

which gives the solution of (5.85) with boundary conditions (5.86) and (5.87).

5.4.2 Homogeneous and Inhomogeneous Wave Equations

The following is the Maxwell's electromagnetic field equations in the form,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t} \qquad \text{... (5.97)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{... (5.98)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{... (5.99)}$$

and $\nabla \cdot \mathbf{D} = \rho$ ... (5.100)

In addition to these equations, we have few more relations in a homogeneous isotropic medium.

$$\mathbf{D} = k\mathbf{E} \qquad \text{... (5.101)}$$

$$\mathbf{B} = \mu\mathbf{H} \qquad \text{... (5.102)}$$

and $\mathbf{J} = \sigma\mathbf{E}$ ... (5.103)

The method of integration to be used here for electrodynamical equations actually leads us to homogeneous wave equation as shown below. For the purpose of their integration, we introduce a vector **A** known as *magnetic vector potential* such that

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{... (5.104)}$$

(5.97) and (5.104) yield $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$ ... (5.105)

(on changing the order of time and space derivatives).

We can write (5.105) as $\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$ ... (5.106)

Which follows that $\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)$ is an irrotational vector and hence it is expressible as the gradient of a scalar point function such that,

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi, \ \phi \text{ being a scalar potential} \qquad \text{... (5.107)}$$

or $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$ ... (5.108)

Multiplying by μ and using (5.102), we have

NOTES

$$\nabla \times \mathbf{B} = \mu\mathbf{J} + \mu\,\frac{\partial \mathbf{D}}{\partial t} \qquad \text{... (5.109)}$$

But $\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla\,(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ ... (5.110)

$\therefore$ (5.109) gives $\nabla\,(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} + \mu\,\frac{\partial \mathbf{D}}{\partial t}$ ... (5.111)

Differentiation of (5.108) with respect to 't' yields

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\,\frac{\partial \phi}{\partial t} \qquad \text{... (5.112)}$$

Elimination of $\frac{\partial \mathbf{E}}{\partial t}$ from (5.111) and (5.112) with the help of (5.101) gives,

$$\nabla\,(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} + \mu\,\kappa\left(\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\,\frac{\partial \phi}{\partial t}\right) \qquad \text{... (5.113)}$$

or $\quad -\nabla^2\mathbf{A} = \mu\,\mathbf{J} - \mu\,\kappa\,\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A}\mu\kappa\,\frac{\partial \phi}{\partial t}\right)$ ... (5.114)

It follows from (5.114) that curl of **A** is specified by its divergence but div **A** is not specified. But to find **A** uniquely, curl **A** and div **A** both should be specified and hence let us assume that

$$\nabla \cdot \mathbf{A} = -\mu\kappa\,\frac{\partial \phi}{\partial t} \qquad \text{... (5.115)}$$

So that (5.114) yields $\nabla^2\mathbf{A} - \mu\kappa\,\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$ ... (5.116)

Also (5.100) with the help of (5.101) gives $\nabla \cdot \mathbf{E} = \frac{\rho}{\kappa}$ ... (5.117)

which with the help of (5.108) becomes $\nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla\,\phi\right) = \frac{\rho}{\kappa}$ ... (5.118)

or

$$-\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) - \nabla^2\phi = \frac{\rho}{\kappa} \qquad \text{... (5.119)}$$

Elimination of $\nabla\mathbf{A}$ from (5.115) and (5.119), yields

$$\nabla^2\phi - \mu\kappa\,\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\kappa} \qquad \text{... (5.120)}$$

If we put $c = \frac{1}{\sqrt{\mu\kappa}}$, (5.116) and (5.120) reduce to

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \qquad \text{... (5.121)}$$

NOTES

and $$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\kappa} \qquad \text{... (5.122)}$$

which have got the same form and known as *Inhomogeneous wave equations* or *Lorentz's equations* and they lead to the conclusion that magnetic vector potential **A** and scalar potential ϕ are propagated in accordance with the equation of the form

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -f(x, y, z, t) \qquad \text{... (5.123)}$$

which is claimed to solve with initial conditions

$$u = 0 \text{ and } \frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \qquad \text{... (5.124)}$$

In order to use the method of Laplace transform, assume that

$$L\{u(x, y, z, t)\} = U(x, y, z, s) \text{ and } L\{f(x, y, z, t)\} = F(x, y, z, s) \qquad \text{... (5.125)}$$

Taking Laplace transform of (5.123), we get

$$\nabla^2 U - \frac{s^2}{c^2} = -F \qquad \text{... (5.126)}$$

It we put $\kappa^2 = -\frac{s^2}{c^2}\phi, \kappa = \frac{s}{c}i, i = \sqrt{-1},$ then it becomes

$$\nabla^2 U + \kappa^2 U + F = 0 \qquad \text{... (5.127)}$$

which is *Helmholtz's equation.*

In particular case (5.127) can be taken as

$$\nabla^2 U_0 + \kappa^2 U_0 = 0 \qquad \text{... (5.128)}$$

which is the standard form of Helmholtz's equation and its particular solution is

$$U_0 = \frac{e^{\pm i\kappa r}}{r} \qquad \text{... (5.129)}$$

where r is the distance from a point and U_0 is determined at another point.

Using this particular solution, we can find the general solution of (5.127) as

$$U(x, y, z, s) = \frac{1}{4\pi}\iiint F(x_1, y_1, z_1)\, e^{\pm i\kappa r} \cdot \frac{1}{r}\, dv \qquad \text{... (5.130)}$$

where $r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$ and $dv = dx\, dy\, dz$ ... (5.131)

It may be verified that (5.130) satisfies (5.127).

Now substituting $\kappa = \frac{s}{c}i$ (5.131) becomes

$$U(x, y, z, s) = \frac{1}{4\pi}\iiint \frac{F(x_1, y_1, z_1)}{r} e^{\pm s/cr}\, dv \qquad \text{... (5.132)}$$

Taking inverse Laplace transform of (5.133) we find the solution of inhomogeneous wave equation (5.123) as

$$u\,(x, y, z, s) = \frac{1}{4\pi}\iiint \frac{f\left(x, y, z, t - \frac{r}{c}\right)}{r}\, dv \qquad \text{... (5.133)}$$

Since we define the Laplace transform of $F(t)$ as $f(s) = L\{F(t)\}$

$= \int_0^{\infty} e^{-st}\, F(t)\, dt$. Under the condition that definite integral of $F(t)$ exists

and

$F(t) = 0$ for $t < 0$. Also we define the inverse transform $L^{-1}\{f(s)\} = F(t)$

and $L\left\{\frac{d^2F}{dt^2}\right\} = s^2 f - sF'(0) - s^2 F(0)$ where

$F'(0)$ is $\frac{dF}{dt}$ evaluated at $t = 0$ and

$$L^{-1}\left\{e^{as}\, f(s)\right\} = \left[\begin{matrix} o, t < a \\ F(t-a), t > a. \end{matrix} \text{ Also } L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U\right]$$

The equation (5.133) shows that the effects in variation of $F(x_1, y_1, z_1, t)$ do not approach the point (x, y, z) unless the time t is retarded by r/c.

As such we can write the solutions of (5.121) and (5.122) as

$$A = \frac{\mu}{4\pi}\iiint \frac{\left(J(x_1, y_1, z_1, t) - \frac{r}{c}\right)}{r}\, dv \qquad \text{... (5.134)}$$

and $$\phi = \frac{1}{4\pi\kappa}\iiint \frac{\rho\left(x_1, y_1, z_1, t - \frac{r}{c}\right)}{r}\, dv \qquad \text{... (5.135)}$$

These give *retarded potentials of* electro dynamics.

5.5 SOLUTION OF BOUNDARY VALUE PROBLEMS BY LAPLACE TRANSFORM

Boundary Value Problems (BVP) arise in a number of different applications that include for example, deflection of beam, mechanical and electronics, etc. Laplace transforms can be used to solve these BVPs. The key strategy to find the solution of BVP, is to first convert the given BVP into the algebraic equation of the Laplace transform of the solution say $L(y)$ and after solving one can find $L(y)$. The desired solution of the given BVP can be obtained by taking inverse Laplace transform.

Example 5.7: Consider the second order boundary value problem

$$y''(t) + 2y'(t) + y = t$$

subject to the $y(0) = 0$ and $y(1) = -1$ boundary conditions

NOTES

Solution: Taking the Laplace transform of

$$y''(t) + 2y'(t) + y = t$$

$$s^2 y(s) - sy(0) - y'(0) + 2s\, y(s) - y(0) + y(s) = \frac{1}{s^2}$$

where $\gamma(s) = L(y(t)^-)$

given that $y(0) = 0$ and assume $y'(0) = a$ the we have

$$s^2 \gamma(s) - a + 2s\, y(s) + y(s) = \frac{1}{s^2}$$

or $$y(s) = \frac{a}{(s^2 + 2s + 1)} + \frac{1}{s^2(s^2 + 2s + 1)}$$

or $$y(s) = \frac{a}{(s^2 + 1)} + \frac{1}{s^2(s+1)^2}$$

by partial fraction we may so write

$$y(s) = \frac{1}{s^2} - \frac{2}{s} + \frac{2}{s+1} + \frac{a+1}{(s+1)^2}$$

Taking inverse Laplace transform we get

$$y(t) = t - 2 + 2e^{-t} + (a + 1)\, te^{-t} \quad ...(1)$$

Again we use the other boundary condition

$y(1) = -1$ to compute a

$$-1 = 1 - 2 + 2e^{-1} + (a + 1)\, e^{-1}$$

$$= -1 + e^{-1}\,(a + 1 + 2)$$

$$= -1 + e^{-1} + ae^{-1}$$

or $$e^{-1}(3 + a) = 0 \quad \Rightarrow \quad a = -3$$

Hence the required solution of given BVP is

$$y(t) = t - 2 + 2e^{-t} - 2t\, e^{-t}$$

Example 5.8: Solve the following BVP using Laplace transform

$$y''(t) + 4y(t) = -8t^2$$

$$y(0) = 3, \quad y\left(\frac{\pi}{t}\right) = 0$$

Solution: Taking Laplace transform of given BVP we get (assume $y'(0) = c$)

$$s^2 y(s) - sy(0) - y'(0) + 4y(s) = \frac{8 \times 21}{s^3}$$

$$s^2 y(s) - 3s - c + 4\, y(s) = \frac{-16}{s^3}$$

NOTES

$$(s^2+4)\,y(s) = 3s + c - \frac{16}{s^3}$$

$$y(s) = \frac{3s}{s^2+4} + \frac{c}{s^2+4} - \frac{16}{s^3(s^2+4)}$$

$$y(t) = L^{-1}\left(\frac{3s}{s^2+4}\right) + L^{-1}\left(\frac{c}{s^2+4}\right) - L^{-1}\left(\frac{16}{s^3\left(s^2+4\right)}\right)$$

$$L^{-1}\left(\frac{3s}{s^3+4}\right) = 3\cos 2t,\quad L^{-1}\left(\frac{c}{s^2+4}\right) = \frac{c}{2}\sin 2t$$

$$L^{-1}\left(\frac{16}{s^3\left(s^2+4\right)}\right) = 16\int_0^t \frac{(t-x)^2}{2!}\cdot\frac{1}{2}\sin 2x\ dx$$

since $L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2!}$ and $L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$

(and using convolution theorem)

Thus $L^{-1}\left(\frac{16}{s^3\left(s^2+4\right)}\right) = -1 + 2t^2 + \cos 2t$

Hence,

$$L^{-1}(y(s)) = y(t) = 3\cos 2t + \frac{c}{2}\sin 2t + 1 - 2t^2 - \cos 2t$$

To evaluate the value of c we use the 2nd boundary condition (given)

$$y(\pi/4) = 0$$

$$0 = 3\cos\pi/2 + \frac{c}{2}\sin\frac{\pi}{2} + 1\,2\frac{\pi^2}{16}\quad\cos\frac{\pi}{2}$$

$$= \frac{c}{2} + 1\quad 2\frac{\pi^2}{16}$$

or $$\frac{c}{2} = -1 + \ 2\frac{\pi^2}{16}$$

or $$c = \ 2\left(\frac{\pi^2}{8} - 1\right)$$

Hence, $$y(t) = 2\cos 2t + \left(\frac{\pi^2}{8} - 1\right)\sin 2t + 1 - 2t^2$$

Next we consider an example governing partial differential equation.

Example 5.9: A string is stretched between two fixed points (0, 0), and (1, 0) and is released from position $y = A\sin\left(\frac{\pi x}{L}\right)$ then show that the expression for its subsequent displacement $u(x, t)$ is given by

NOTES

$$u(x,\ t) = \text{A} \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{h\pi t}{L}\right)$$

where h^2 is diffusivity

Solution: Equation of vibration of string is given

$$\frac{\partial^2 u}{\partial t^2} = h^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

The boundary condition are

$$u(0,\ t) = 0 \text{ and } u(L,\ t) = 0 \qquad ...(2)$$

and the initial conditions are

$$\left\{\begin{aligned} u(x,0) &= A\sin\left(\frac{\pi x}{L}\right) \\ \left(\frac{\partial u}{dt}\right)_{t=o} &= u_t(x,0) = 0 \end{aligned}\right\} \qquad ...(3)$$

Taking Laplace transform of (1) we get

$$s^2\,\overline{u}(x,s) - s\,u(x,0) - u_t(x,0) = h^2 \frac{\partial^3 \overline{u}(x,s)}{\partial x^2}$$

where $\quad \overline{u} = u(x,\ s) = L(u(x,\ t))$

usint (3) we have

$$\frac{\partial^2 \overline{u}}{\partial x^2} - \frac{s^2}{h^2}\overline{u} = -\frac{s}{s^2}\sin\left(\frac{\pi x}{L}\right) \quad ...(4)$$

From (2) (taking Laplace transform)

$$\overline{u}(o,s) = 0, \quad \overline{u}(L,s) = 0 \qquad ...(5)$$

So the solution of (4) is

$$\overline{u}(x,s) = c_1\,e^{sx/h} + c_2\,e^{-sx/h} + \frac{A\,s\sin\left(\frac{\pi x}{L}\right)}{s^2 + \frac{\pi^2 h^2}{L^2}}$$

from (5)

$$\begin{aligned} \overline{u}(0,s) = 0 &\quad \Rightarrow \\ \overline{u}(L,s) = 0 &\quad \Rightarrow \end{aligned} \left\{\begin{aligned} & c_1 + c_2 = 0 \\ & c_1\,e^{sx/h} + c_2\,e^{-sx/h} = 0 \end{aligned}\right.$$

Solving we get $\quad c_1 = c_2 = 0$

Hence $\quad \overline{u}(x,s) = \dfrac{A\,s\sin\left(\frac{\pi x}{L}\right)}{s^2 + \frac{\pi^2 h^2}{L^2}}$

by use of inverse Laplace transform, the solution $u(x, t)$ is

$$u(x, t) = L^{-1}\left(\bar{u}(x, s)\right)$$

$$= A\sin\left(\frac{\pi x}{L}\right) \quad L^{-1}\left(\frac{s}{s^2 + \frac{\pi^2 h^2}{L^2}}\right)$$

$$u(x, t) = A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi h t}{L}\right)$$

5.5.1 Simple Boundary Value Problems with Applications of Fourier Transform

Example 5.10: Solve the transport equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad ..(1)$$

$-\infty < x < \infty, t > 0$

subject to $u(x, 0) = f(x)$

Solution: Let $F((u(x, t)) = \hat{u}(w, t)$ be the Fourier transform of $u(x, t)$ in x-variable

and $F(u_t) = \hat{u}_t$

thus the Fourier transform of in given by

$$\hat{u}_t + iwc\,\hat{u}(w, t) = 0$$

or, $$\frac{\partial \hat{u}}{\partial t} + iwc\,\hat{u} = 0 \text{ subject to}$$

$$\hat{u}(w, 0) = \hat{f}(w)$$

which is a first order differential equation in $\hat{u}$.

Solution of this equation is given by

$$\hat{u}(w, t) = e^{-icwt}\hat{f}(w)$$

By inverse Fourier transform, we can find the solution of given transport equation as

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}e^{-icwt}\hat{f}(w)\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iw(x-ct)}\hat{f}(w)\,dw$$

$$u(x, t) = f(x - c\,t)$$

which is a traveling wave solution describing a pulse shape $f(x)$ moving uniformly at speed c.

5.6 LONGITUDINAL AND TRANSVERSE VIBRATION OF A BEAM

NOTES

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by application of external force and then released, they execute a vibratory motion.

There are three types of vibratory motion:

1. Free or Natural Vibration: When no external force acts on the body, after giving it an initial displacement, then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called free or natural frequency.

2. Forced Vibrations: When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The vibrations have the same frequency as the applied force.

3. Damped vibrations: When there is reduction in amplitude in every cycle of vibrations, the motion is said to be damped vibration. This is due to the fact that a certain of energy possessed by the vibrating system is always dissipated in overcoming functional resistance to the motion.

Types of Free vibration

There are 3 types of free vibrations:

1. Longitudinal vibrations
2. Transverse vibrations
3. Torsional vibrations

But here we consider only longitudinal and transverse vibrations.

1. Longitudinal Vibrations: When a particle of the shaft or disc moves parallel to the axis of the shaft then the vibrations are known as longitudinal vibrations. In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stress are induced alternately in the shaft.

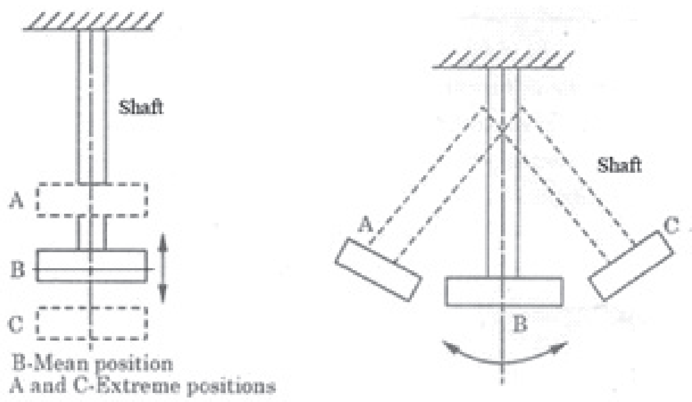

Fig. 5.3. *Free Vibrations*

2. Transverse Vibrations: When a particles of the shaft or disc move approximately perpendicular to the axis of the shaft then vibrations are called as transverse vibrations. In this case, the shaft is straight and bent alternately and bending stress are induced in the shaft.

3. Torsional Vibrations: Torsional vibration is the angular vibration of an object-typically a shaft-along its rotational axis. Torsional vibration is a common source of failure in power transmission systems with rotating shafts or couplings if it is not regulated. In passenger cars, torsional vibrations have a second consequence. At certain speeds, torsional vibrations might cause seat vibrations or noise. Both decrease the level of comfort.

Check Your Progress

3. State one dimensional wave equation.
4. Define the solution of boundary value problems by Laplace transform method.
5. Give the three dimensional wave equation.
6. What is the wave equation for Green's function?
7. How will you define diffusion equation?
8. Define vibratory motion.
9. How many types of vibratory motion?

5.7 ANSWERS TO 'CHECK YOUR PROGRESS'

1. An integral equation is an equation in which an unknown function appears under an integral sign.
2. $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = h^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} = k \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$
3. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

   is known as one dimensional wave equation.
4. Boundary Value Problems (BVP) arise in a number of different applications that include for example, deflection of beam, mechanical and electronics, etc. Laplace transforms can be used to solve these BVPs. The key strategy to find the solve the solution of BVP, is to first convert the given BVP into the algebraic equation of the Laplace transform of the solution say $L(y)$ and after solving one can find $L(y)$. The desired solution of the given BVP can be obtained by taking inverse Laplace transform.
5. The properties of the Dirac delta function $\delta(x)$ are,

$$\int_{-\infty}^{\infty} \delta(x - x')f(x')dx' = f(x)$$

$$\int_{-\infty}^{\infty} \delta(x')dx' = 1$$

NOTES

6. The Green's function depends upon the distance between the source and field points.

7. Let $G(x;\xi)$ be a function which

   (a) Considered as a function of x, satisfies the differential equation,

$$\left[\frac{d}{dx}p(x)\frac{d}{dx}+\gamma(x)\right]G(x;\xi)\equiv LG(x;\xi)=0$$

   in (a,b) except at the point $x=\xi$,

   (b) Satisfies the given homogeneous boundary conditions,

   (c) For fixed ξ is continuous, even at $x=\xi$,

   (d) has continuous 1st and 2nd derivatives everywhere in (a,b), except at $x=\xi$, where it has a jump discontinuity given by,

$$\left.\frac{d}{dx}G(x;\xi)\right|_{\xi^-}^{\xi^+}=\frac{-1}{p(\xi)}\quad .$$

   Then,

$$u(x)=\int_a^b G(x;\xi)\ f(\xi)\ d\xi \iff \begin{array}{l} u(x) \text{ satisfies} \\ (i) \text{ the given boundary conditions} \\ (ii)\quad Lu=-f(x), \\ \qquad \text{where } f \text{ is piecewise} \\ \qquad \text{continuous in } (a,b) \end{array}$$

8. When elastic bodies such as spring, a beam and a shaft are displaced from the equilibrium position by application of external force and then released, they execute a vibratory motion.

9. There are three types of vibratory motion:
   - Free or natural vibrations
   - Forced vibrations
   - Damped vibrations

5.8 SUMMARY

- A boundary value problem can be converted to an equivalent Fredholm integral equation. But this method is complicated and so is rarely used.

NOTES

- $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$ If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $F(x)$ in the bar, then the temperature at a subsequent time t is .
- If a bar of length l is at a temperature v_0 such that one of its ends $x = 0$ is kept at zero temperature and the other end $x = l$ is kept at temperature u_0, then find the temperature at any point x of the bar at an instant of time $t > 0$.
- From Green's theorem it follows that if the normal derivative of a harmonic function is zero on a closed surface within which there is no singularity the function is constant.
- In case of a rectangular membrane, the two dimensional wave equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}\ \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

- The wave equation is $\nabla^2 y = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$.

  It is also written as $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^3}\right)\psi = 0$.
- Boundary Value Problems (BVP) arise in a number of different applications that include for example, deflection of beam, mechanical and electronics etc. Laplace transforms can be used to solve these BVPs. The key strategy to find the solve the solution of BVP, is to first convert the given BVP into the algebraic equation of the Laplace transform of the solution say L(y) and after solving one can find L(y). The desired solution of the given BVP can be obtained by taking inverse Laplace transform.
- When elastic bodies such as spring, a beam and a shaft are displaced from the equilibrium position by application of external force and then released, they execute a vibratory motion.
- When no external forces acts on the body after giving it an initial displacement then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called free or natural frequency.
- When the body vibrates under the influence of external forces then the body is said to be under forced vibrations. The vibrations have the same frequency as the applied force.
- When there is reduction in amplitude over every cycle of vibrations. The motion is said to be damped vibrations, this is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming functional resistance to the motion.
- When a particle of the shaft or disc moves parallel to the axis of the shaft then the vibrations are known as longitudinal vibrations.
- When a particle of the shaft or disc moves approximately perpendicular to the axis of the shaft then vibrations are called as transverse vibrations.

NOTES

5.9 KEY TERMS

- **Boundary value problems:** A boundary value problem is made up of a differential equation and a set of extra limitations known as boundary conditions. A boundary value problem solution is a differential equation solution that also meets the boundary criteria.
- **Wave equation:** The wave equation is a second-order linear partial differential equation for the description of waves as they occur in classical physics such as, mechanical waves (e.g., water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics.
- **Vibratory motion:** When elastic bodies such as spring, a beam and a shaft are displaced from the equilibrium position by application of external force and then released, they execute a vibratory motion.
- **Free or natural frequency:** When no external forces acts on the body after giving it an initial displacement then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called free or natural frequency.
- **Longitudinal vibrations:** When a particle of the shaft or disc moves parallel to the axis of the shaft then the vibrations are known as longitudinal vibrations.
- **Transverse vibrations:** When a particle of the shaft or disc moves approximately perpendicular to the axis of the shaft then vibrations are called as transverse vibrations.

5.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is the boundary value problems involving differential equation?
2. State the one dimensional heat conduction equation.
3. How will you define the one dimensional wave equation?
4. What do you understand by the solution of boundary value problems by Laplace transform method?
5. Define longitudinal and transverse vibration of a beam.
6. What is vibratory motion?
7. How will you define longitudinal vibrations and transverse vibrations?

Long-Answer Questions

1. Discuss the boundary value problems involving differential equation with the help of examples.
2. Explain one dimensional heat conduction equation. Give appropriate examples.
3. Discuss one dimensional wave equation with the help of examples.

4. Describe the solution of boundary value problems by Laplace transform method. Give appropriate examples.
5. Explain the longitudinal and transverse vibration of a beam with the help of examples.
6. Explain the types of vibratory motion with the help of relevant examples.
7. Differentiate between the longitudinal vibration and transverse vibration. Give appropriate examples.

5.11 FURTHER READING

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