M.Sc. Previous Year

Mathematics

MM-04

COMPLEX ANALYSIS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

MADHYA PRADESH BHOJ (OPEN) UNIVERSITY - BHOPAL

COURSE WRITERS

Dr Bikas Chandra Bhui, Prof Dipak Chatterjee
Units (1.0-1.2.1, 1.3-1.3.1, 1.3.2, 1.3.3-1.3.4, 1.3.6, 1.4, 1.5-1.10, 2.3-2.8, 3.0-3.2, 3.3-3.8, 4.5)

K A Lakshminarayanan, K Megalai, P Geetha and D Jayanthi
Units (1.2.2, 1.3.5, 1.4.1-1.4.3, 2.0-2.2)

Published by Registrar, MP Bhoj (Open) University, Bhopal in 2020

VIKAS®

Vikas® is the registered trademark of Vikas® Publishing House Pvt. Ltd.

VIKAS® PUBLISHING HOUSE PVT. LTD.
E-28, Sector-8, Noida - 201301 (UP)
Phone: 0120-4078900 • Fax: 0120-4078999
Regd. Office: A-27, 2nd Floor, Mohan Co-operative Industrial Estate, New Delhi 1100 44
• Website: www.vikaspublishing.com • Email: helpline@vikaspublishing.com

SYLLABI-BOOK MAPPING TABLE

Complex Analysis

Syllabi	Mapping in Book
UNIT - I: Complex Integration, Cauchy-Goursat Theorem, Cauchy's Integral Formula, Higher Order Derivatives, Morera's Theorem, Cauchy's Inequality and Liouville's Theorem, The Fundamental Theorem of Algebra, Taylor's Theorem, Maximum Modules Principle, Schwarz Lemma, Laurent's Series, Isolated Singularities, Meromorphic Functions, The Argument Principle, Rouche's Theorem, Inverse Function Theorem.	**Unit-1:** Complex Integration **(Pages 3–58)**
UNIT - II: Residues, Cauchy's Residue Theorem, Evaluation of Integrals, Branches of Many Valued Functions with Special Reference to arg z, log z, z^{α}.	**Unit-2:** Residues **(Pages 59–96)**
UNIT - III: Bilinear Transformations, Their Properties and Classifications, Definitions and Examples of Conformal Mappings. Space of Analytic Functions, Hurwitz's Theorem, Montel's Theorem, Riemann Mapping Theorem.	**Unit-3:** Bilinear Transformations **(Pages 97–136)**
UNIT - IV: Weierstrass Factorization Theorem, Gamma Function and Its Properties, Riemann Zeta Function, Riemann's Functional Equation, Runge's Theorem, Mittag-Leffler's Theorem, Analytic Continuation, Uniqueness of Direct Analytic Continuation, Uniqueness of Analytic Continuation Along a Curve, Power Series Method of Analytic Continuation, Schwarz Reflection Principle, Monodromy Theorem and Its Consequences, Harmonic Function on a Disk, Harnack's Inequality and Theorem, Dirichlet Problem, Green's Functions.	**Unit-4:** Weierstrass Factorisation Theorem, Analytic Continuation, Inequality Theorem and Functions **(Pages 137–190)**
UNIT - V: Canonical Products, Jensen's Formula, Poisson-Jensen Formula, Hadamard's Three Circle Theorem, Order of an Entire Function, Exponent of Convergence, Borel's Theorem, Hadamard's Factorization Theorem. The Range of an Analytic Function, Bloch's Theorem, The Little Picard Theorem, Schottky's Theorem, Montel Caratheodory and the Great Picard Theorem. Univalent Functions, Bieberbach's Conjecture (Statement Only) and the 1/4 Theorem.	**Unit-5:** Canonical Products, Functions and Theorems **(Pages 191–240)**

CONTENTS

INTRODUCTION

NOTES

Complex analysis, also known as the theory of functions of a complex variable, is the branch of mathematical analysis that studies complex numbers along with their derivatives, manipulation and other properties. A complex number is a number consisting of a real part and an imaginary part. Complex numbers extend the idea of the one-dimensional number line to the two-dimensional complex plane by using the number line for the real part and adding a vertical axis to plot the imaginary part. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that would be impossible with only real numbers. Complex analysis finds applications in number theory, applied mathematics, hydrodynamics, thermodynamics and electrical engineering. Complex analysis is widely applicable to two-dimensional problems in physics because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation.

A complex function is one in which the independent variable and the dependent variable are both complex numbers. More precisely, a complex function is a function whose domain and range are subsets of the complex plane. Complex analysis is mainly concerned with the analytic functions of complex variables. Analytic functions are those which can be locally represented by power series. One of the important facts about the class of analytic functions is that it includes majority of the functions which are encountered in the principal problems of mathematics and its applications to science and technology.

This book is divided into five units which explains complex integration, Cauchy integral formula, Liouville's theorem, Taylor's theorem, Schwarz lemma, Laurent's series, Rouche's theorem, zeros, poles, residues, Cauchy's residue theorem, bilinear transformation, conformal mappings, Hurwitz's theorem, Riemann mapping theorem, Weierstrass and factorization theorem, gamma function, Riemann zeta function, Riemann functional equations, Runge's theorem, Schwarz reflection principle, Dirichlet problem, Green's functions, canonical products, Jensen's formula, Poisson-Jensen formula, Borel's theorem, Hadamard's factorization theorem, range of an entire function, Bloch's theorem, Picard's theorem, Schottky's theorem and univalent functions. The book follows the self-instruction mode or the SIM format where in each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The content is presented in a simple and structured form interspersed with Answers to 'Check Your Progress' for better understanding. A list of 'Summary' along with a 'Key Terms' and a set of 'Self-Assessment Questions and Exercises' is provided at the end of each unit for effective recapitulation.

UNIT 1 COMPLEX INTEGRATION

NOTES

Structure

1.0 INTRODUCTION

In the mathematical field of complex analysis, contour integration is a method of evaluating certain integrals along paths in the complex plane. Contour integration is closely related to the calculus of residues, a method of complex analysis. One use for contour integrals is the evaluation of integrals along the real line that are not readily found by using only real variable methods like direct integration of a complex-valued function along a curve in the complex plane (a contour), application of the Cauchy integral formula and application of the residue theorem. One method can be used, or a combination of these methods, or various limiting processes, for the purpose of finding these integrals or sums.

Complex valued functions are functions which produce complex numbers from complex numbers. Complex function theory is the study of complex analytic functions. It is a simple and powerful method useful in the study of heat flow, fluid dynamics and electrostatics. Two-dimensional potential problem can be solved using analytic functions since the real and imaginary parts of an analytic function are solutions of two-dimensional Laplace's equation.

NOTES

The derivative of a function of a real variable measures the sensitivity to change of the function value (output value) with respect to a change in its argument (input value). Derivatives are a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time advances.

In this unit, you will study about the complex integration, Cauchy-Goursat theorem, Cauchy's integral formula, higher order derivatives, Morera's theorem, Cauchy's inequality, Liouville's theorem, the fundamental theorem of algebra, Taylor's theorem, maximum modulus principle, Schwarz lemma, Laurent's series, isolated singularities, meromorphic functions, the argument principle, Rouche's theorem and inverse function theorem.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand complex integration
- Explain Cauchy-Goursat theorem
- State Cauchy's integral formula
- Understand the higher derivatives in complex integration
- State the Morera's theorem
- Describe Cauchy's inequality
- Analyse the Liouville's theorem
- State the fundamental theorem of algebra
- Explain Taylor's theorem
- State maximum modulus principle and Schwarz lemma
- Explain Laurent's series
- Define the various types of singularities
- Describe meromorphic functions
- Elaborate Rouche's theorem and inverse function theorem
- Discuss the branches of many valued functions

1.2 COMPLEX INTEGRATION

Definition: Let $z = x + iy$ be a point in the Argand's plane where $x = \varphi(t)$, $y = \psi(t)$ are functions of a parameter t. If $\phi(t)$ and $\psi(t)$ are continuous and $t \in [\alpha, \beta] \subset \mathbb{R}$, then z traces out a *continuous arc*. If the curve crosses itself at a point, i.e., if at two or more values of f, z assumes the same value, the corresponding point is called a *multiple point*.

A continuous arc without multiple points is called a Jordan arc.

A Jordan curve is one which is made of a continuous chain of finite number of Jordan arcs.

A contour is a closed Jordan curve, i.e., a Jordan curve whose starting point is the same as its end point.

NOTES

Let A be the starting point of the first arc and B the end point of the last arc, then integral along such a curve is written as

$$\int_{AB} f(z)\, dz$$

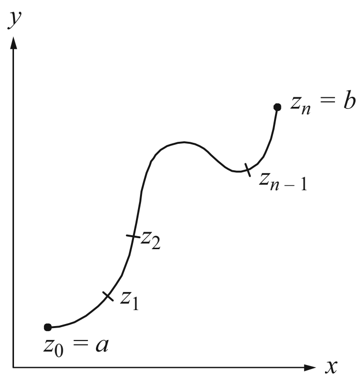
The contour is said to be *closed* if the starting point A of the arc coincides with the end point B of the last arc.

The integral along such a closed contour is written as $\int_C f(z)\, dz$ and is read as 'Integral $f(z)$ taken over closed contour C'.

Riemann's Definition of Integration: Let a function $f(z)$ be continuous in a domain D and a, b, be two points in the domain, then the integral of $f(z)$ from a to b is defined as below:

Let C be a curve joining a to b and lying entirely in the domain D, so that, $f(z)$ is continuous on C.

Let us consider the partition $P = (a = z_0, z_1, z_2, \ldots z_{r-1}, z_r, \ldots, z_n = b)$ of the curve C.



Consider the sum,

$$S_n = f(\xi_1)(z_1 - z_0) + f(\xi_2)(z_2 - z_1) + \ldots + f(\xi_n)(z_n - z_{n-1})$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the points on the curve chosen between z_i and z_{i+1}, $i = 0, 1, 2, \ldots, n-1$ respectively.

Then clearly $S_n = \sum_{i=0}^{n-1} f(\xi_i)\,\delta z_i$ where $\delta z_i = z_{i+1} - z_i$

If $\lim_{n\to\infty} S_n$ exists finitely when the number of points of division n increases in such a way that the largest of the lengths of δz_n approaches zero, then we say that the line integral $\int_C f(z)\, dz$ exists.

Thus by definition,

$$\int_C f(z)\,dz = \lim_{\substack{x\to\infty \\ 8\to 0}} \sum_{i=0}^{n-1} f(\xi_i)\,\delta z_i \qquad \text{where } \delta = \max_i |\delta z_i|$$

NOTES

Some Elementary Properties of Complex Integrals

(*i*) $\int_C [f(z) \pm g(z)\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$

(*ii*) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

where C_1 and C_2 are the parts of C.

(*iii*) $\int_C k\, f(z)\,dz = k \int_C f(z)\,dz$ where k is a constant.

(*iv*) $\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$

where $-C$ is the same curve as C but have the opposite direction of C.

Example 1.1: Evaluate $\int_C z\,dz$

Solution: From the definition, we get

$$\int_C z\,dz = \lim_{n\to\infty} \left[\sum_{r=1}^{n} (z_r - z_{r-1})\, f(\xi_r)\right]$$

where $z_r \le \xi_r \le z_{r-1}$ and we consider partition $a = z_0, z_1, \ldots z_{r-1}, z_r, \ldots, z_n = b$ of the curve C.

$$\therefore \quad \int_C z\,dz = \lim_{n\to\infty} \left[\sum_{r=1}^{n} \xi_r\,(z_r - z_{r-1})\right]$$

[between z_r and z_{r-1} since $f(z) = z$ where ξ_r lies on C.]

$$\therefore \quad = \lim_{n\to\infty} \left[\sum_{r=1}^{n} z_r\,(z_r - z_{r-1})\right] \text{ taking } \xi_r = z_r \qquad \ldots(1)$$

$$\text{Again} \int_C z\,dz = \lim_{n\to\infty} \left[\sum_{r=1}^{n} z_{r-1}\,(z_r - z_{r-1})\right] \text{taking } \xi_r = z_{r-1} \qquad \ldots(2)$$

By adding Equations (1) and (2), we get

$$2\int_C z\,d\,z = \lim_{n\to\infty} \left[\sum_{r=1}^{n} \{z_r\,(z_r - z_{r-1}) + z_{r-1}(z_r - z_{r-1})\}\right]$$

$$= \lim_{n\to\infty} \left[\sum_{r=1}^{n} (z_r^2 - z_{r-1}^2)\right]$$

$$= \lim_{n\to\infty} (z_1^2 - z_0^2) + (z_2^2 - z_1^2) + (z_3^2 - z_2^2) + \ldots + (z_n^2 - z_{n-1}^2)$$

$$= \lim_{n\to\infty} \left(z_n^2 - z_0^2\right) = b^2 - a^2$$

NOTES

[Since $z_n = b$ and $z_0 = a$ where a, b are the end points of the curve C.]

$$\therefore \quad \int_C z\,dz = \frac{1}{2}(b^2 - a^2)$$

Note: If the curve C is closed, i.e., the end points a and b coincide, then

$$\int_C z\,dz = 0$$

Example 1.2: Find $\int_C dz$.

Solution: From the definition, we get

$$\int_C f(z)\,dz = \lim_{n\to\infty} \left[\sum_{r=1}^{n} (z_r - z_{r-1})\, f(\xi_r)\right]$$

where $z_{r-1} \le \xi_r \le z_r$ and partition $= P = \{a = z_0, z_1, z_2, \ldots, z_n = b\}$.

Here $\quad f(z) = 1$

$$\therefore \quad \int_C dz = \lim_{n\to\infty} \left[\sum_{r=1}^{n} (z_r - z_{r-1})\, 1\right]$$

$$= \lim_{n\to\infty} [(z_1 - z_0) + (z_2 - z_1) + \ldots + (z_n - z_{n-1})]$$

$$= \lim_{n\to\infty} [z_n - z_0]$$

$$= b - a$$

where $z_n = b$ and $z_0 = a$ are the end points of the curve.

Example 1.3: Find the value of the integral,

$$\int_0^{1+i} (x - y + ix^2)\,dz$$

(*a*) Along the straight line from $z = 0$ to $z = 1 + i$

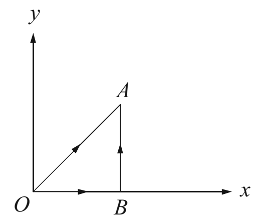
(*b*) Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

Solution: Let A be the point corresponding to $z = 1 + i$ and B be the point corresponding to $z = 1$.

(*a*) Let OA be the line from $z = 0$ to $z = 1 + i$.

Along $OA \qquad y = x,\ z = x + ix$

$\therefore \qquad dz = (1 + i)\,dx$



NOTES

$$\therefore \int_{OA} (x - y + ix^2)\, dz = \int_0^1 ix^2 (1 + i)\, dx = (i^2 + i) \int_0^1 x^2\, dx$$

$$= (i - 1)\left[\frac{x^3}{3}\right]_0^1 = \frac{i-1}{3}$$

(*b*) The real axis from $z = 0$ to $z = 1$ is the line OB and then from $z = 1$ to $z = 1 + i$ a line parallel to imaginary axis is the line BA.

So, the contour of integration consists of the lines OB and BA.

Now, along OB, $y = 0$ and $z = x + iy = x$, $dz = dx$

$$\therefore \int_{OB} (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx = \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 = \frac{1}{2} + \frac{i}{3}$$

Along the line BA, $x = 1$; then $z = (1 + iy)$ and $dz = idy$

$$\therefore \int_{BA} (x - y + ix^2)\, dz = \int_0^1 (1 - y + i)\, i\, dy$$

$$= i\left[(1+i)\, y - \frac{y^2}{2}\right]_0^1 = i\left[(1+i) - \frac{1}{2}\right]$$

$$= i\left[\frac{1}{2} + i\right] = \frac{i}{2} + i^2 = \frac{i}{2} - 1$$

Hence $\int_0^{1+i} (x - y + ix^2)\, dz$ along the contour OBA

= Integral along OB + Integral along BA

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1 = \frac{5i}{6} - \frac{1}{2}$$

Example 1.4: Evaluate the integral $\int_0^{1+i} z^2\, dz$

Solution: Since $f(z) = z^2$ is analytic for all finite values of z, then its integral along a curve joining two fixed points will be independent of the path. Here we now integrate z^2 between two fixed points (0, 0) and (1, 1).

Let the path of integration joining these points be along the curve made up of,

(1) Part of the real axis from (0, 0) to the point (1, 0). On this line $z = x$, $dz = dx$ and x varies from 0 to 1.

(2) Followed by the line parallel to the axis of imaginaries from the point (1, 0) to the point (1, 1). On this line $z = 1 + iy$, $dz = idy$ and y varies from 0 to 1.

NOTES

$$\therefore \quad \int_{0}^{1+i} z^2\, dz = \int_0^1 x^2 dx + \int_0^1 (1+iy)^2\, dy$$

$$= \left[\frac{1}{3}x^3\right]_0^1 + \left[\frac{(1+iy)^3}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3}\left[(1+i)^3 - 1\right] = \frac{1}{3}(1+i)^3$$

Example 1.5: Using the definition of the integral of $f(z)$ on a given path, find

$$\int_{-2+i}^{5+3i} z^3\, dz$$

Solution: Since $f(z) = z^3$ is analytic for all finite values of z, then its integration along a curve joining two fixed points will be independent of the path. Here we have to integrate z^3 between two points $(-2, 1)$ and $(5, 3)$. Let the path of integration joining these two points be along the curve made up of,

(1) A line parallel to the real axis from the point $(-2, 1)$ to the point $(5,1)$. On this line, $z = x + i$, $dz = dx$ and x varies from -2 to 5.

(2) Followed by a line parallel to the imaginary axis from $(5, 1)$ to $(5, 3)$. On this line $z = 5 + i\,y$, $dz = i\,dy$ and y varies from 1 to 3.

$$\therefore \int_{-2+i}^{5+3i} z^3\, dz = \int_{-2}^{5} (x+i)^3\, dx + \int_1^3 (5+iy)^3\, i\, dy \text{ along this path.}$$

$$= \left[\frac{(x+i)^4}{4}\right]_{-2}^{5} + \frac{1}{4}\left[(5+iy)^4\right]_1^3$$

$$= \frac{1}{4}\left[(5+i)^4 - (-2+i)^4\right] + \frac{1}{4}\left[(5+3i)^4 - (5+i)^4\right]$$

Example 1.6: Find $\int_C (z^2 + 3z + 2)\, dz$

where C is the arc of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ between the points $(0, 0)$ and $(\pi a, 2a)$.

Solution: Since $f(z) = z^2 + 3z + 2$ is a polynomial function in z, then $f(z)$ is analytic in z-plane. Therefore, the integral between two points $(0,0)$ and $(\pi a, 2a)$ is independent of the path joining these points.

Then the path of integration of a curve C consists of:

(1) The part of real axis from the $(0, 0)$ to $(\pi a, 0)$ where $z = x$, $dz = dx$ and x varies 0 to πa.

(2) Followed by a line parallel to the imaginary axis $(\pi a, 0)$ to $(\pi a, 2a)$

where $z = \pi a + iy$, $dz = idy$ and y varies from 0 to $2a$.

$$\therefore \quad \int_c (z^2 + 3z + 2)\,dz$$

NOTES

$$= \int_0^{\pi a} (x^2 + 3x + 2)\,dx + \int_0^{2a} \left[(\pi a + iy)^2 + 3(\pi a + iy) + 2\right] idy$$

$$= \left[\frac{x^3}{3} + \frac{3}{2}x^2 + 2x\right]_0^{\pi a} + \left[\frac{(\pi a + iy)^3}{3} + \frac{3(\pi a + iy)^2}{2} + 2iy\right]_0^{2a}$$

$$= \left[\frac{1}{3}(\pi a)^3 + \frac{3}{2}(\pi a)^2 + 2\pi a\right]$$

$$+ \left[\frac{1}{3}(\pi a + 2ia)^3 + \frac{3}{2}(\pi a + 2ai)^2 + 4ai - \frac{(\pi a)^3}{3} - \frac{3}{2}(\pi a)^2\right]$$

$$= 2\pi a + \frac{1}{3}(\pi a + 2ai)^3 + \frac{3}{2}(\pi a + 2ia)^2 + 4ai$$

Example 1.7: Find the value of the integral $\int \frac{1}{z-a}\,dz$ round a circle whose equation is $|z - a| = \gamma$.

Solution: On the circle C, $|z - a| = \gamma$

$z - a = re^{i\theta}$, where θ varies from 0 to 2π

$$\therefore \quad dz = re^{i\theta}\, i\, d\theta$$

$$\therefore \quad \int \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} re^{i\theta}\, i\, d\theta$$

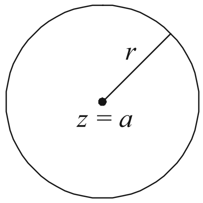
$$= i\int_0^{2\pi} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

Example 1.8: Find the value of the integral $\int_c (x+y)\,dx + x^2 y\,dy$,

(*i*) Along $y = x^2$ having (0, 0), (3, 9) end points.

(*ii*) Along $y = 3x$ between the same points.

Do the values depend upon path?



Solution: We are required to find $\int_c (x+y)\,dx + x^2 y\,dy$

Let $P = x + y$, $Q = x^2 y$; then $\frac{\partial P}{\partial y} = 1$ and $\frac{\partial Q}{\partial x} = 2x$

$$\therefore \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

The integrals are not independent of path:

(*i*) Along the curve $y = x^2$, $dy = 2x\,dx$ and x varies from 0 to 3.

$$\text{Hence } \int_c (x+y)\,dx + x^2 y\,dy = \int_0^3 \left[(x+x^2)\,dx + x^2 \cdot x^2 \cdot 2x\,dx\right]$$

$$= \int_0^3 \left[(x + x^2 + 2x^5)\,dx\right] = \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2}{6}x^6\right]_0^3$$

$$= \frac{9}{2} + 9 + 243 = \frac{513}{2}$$

(*ii*) Now along the curve $y = 3x$, $dy = 3\,dx$ and x varies from 0 to 3.

$$\text{Hence } \int_c (x+y)\,dx + x^2 y\,dy = \int_0^3 (x+3x)\,dx + x^2 . 3x . 3\,dx$$

$$= \int_0^3 (4x + 9x^3)\,dx$$

$$= \left[2x^2 + \frac{9}{4}x^4\right]_0^3 = 18 + \frac{9}{4}\cdot 81 = 18 + \frac{729}{4} = \frac{801}{4}$$

Since values of (*i*) and (*ii*) are not same, hence the value of integration depends upon the path of integration.

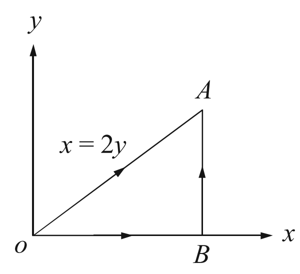
NOTES

Example 1.9: Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$, along (*i*) the line $x = 2y$, (*ii*) along a line parallel to the imaginary axis from $z = 2$ to $z = 2 + i$.

Solution: (*i*) We have, along the line OA,

$$x = 2y,\ z = 2y + iy$$

and $\bar{z} = 2y - iy,\ dz = (2 + i)\,dy$



$$\therefore \int_0^{2+i} (\bar{z})^2\,dz = \int_0^1 (2-i)^2\, y^2\,(2+i)\,dy$$

$$= (4 - i^2)\,(2 - i) \int_0^1 y^2\,dy$$

$$= 5\,(2 - i)\left[\frac{y^3}{3}\right]_0^1 = \frac{5}{3}(2 - i)$$

(*ii*) The line parallel to the imaginary axis from $z = 2$, to $z = 2 + i$ is BA. Along the line BA, $x = 2$, $z = 2 + iy$ and $\bar{z} = 2 - iy$, $dz = i\,dy$, and y varies from 0 to 1.

NOTES

$$\therefore \int_0^{2+i} (\bar{z})^2\, dz = \int_{BA} (\bar{z})^2\, dz$$

$$= \int_0^1 (2-iy)^2\, i\, dy = \int_0^1 \left[4-4yi-y^2\right] i\, dy$$

$$= \int_0^1 \left[(4-y)^2\, i + 4y\right] dy$$

$$= i\left[4y - \frac{y^3}{3}\right]_0^1 + \left[\frac{4y^2}{2}\right]_0^1$$

$$= i\left[4-\frac{1}{3}\right] + 2 = \frac{11}{3} i + 2$$

Cauchy's Theorem

One of the finest results of Complex Analysis is the following theorem:

Theorem 1.1: (Cauchy's Theorem): Let $f(z)$ be an analytic function of z and $f'(z)$ be continuous at each point within and on a closed contour C; then

$$\int_C f(z)\, dz = 0$$

Proof: Let R be the region which consists of all points within and on contour C. We know that if $M(x, y)$, $N(x, y)$, $\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial y}$ are all continuous functions of x and y in the region R, then the Green's theorem states that,

$$\int_C (M\, dx + N\, dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy \qquad ...(1.1)$$

Since $f(z) = u + iv$ is continuous on the simple curve C and $f'(z)$ exists and is continuous in R, then u, v, u_x, v_x, u_y, v_y are all continuous in R. Hence, the conditions of Green's theorem are satisfied.

$$\therefore \int_C f(z)\, dz = \int_C (u+iv)(dx + i\, dy)$$

$$= \int_C (u\, dx - v\, dy) + i\int_C (v\, dx + u\, dy)$$

$$= -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\, dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy \text{ [By (1)]}$$

$= 0$ by Cauchy-Reimann equation $u_x = v_y$, $u_y = -v_z$

Hence $\int_C f(z)\, dz = 0$

1.2.1 Cauchy-Goursat Theorem

Theorem 1.2: Let $f(z)$ be an analytic function of z such that it is single-valued inside and on a simple closed contour C, then

$$\int_C f(z)\, dz = 0.$$

Connected Region: A region R is said to be connected region if any two points of R can be connected by a curve which lies entirely within the region. (Refer Figure 1.1)

NOTES

Simply-Connected Region: A connected region R is said to be a simply-connected region if all the interior points of a closed curve C drawn in R are the points of R.

Multi-Connected Region: If all the points of the area bounded by two or more closed curves drawn in the region R, are the points of R, then the region R is said to be multi-connected region.

Cross-Cut (or Cut): The lines drawn in a multi-connected region, without intersecting any one of the curves which make a multi-connected region a simply-connected one, are called cuts or cross-cuts.

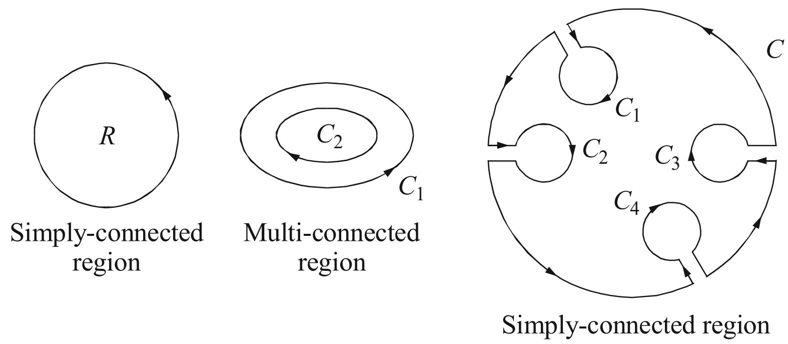


Fig. 1.1 *Connected Regions*

Extension of Cauchy's Theorem to Multi-Connected Region

Theorem 1.3: Let C be a closed curve and C_1, C_2, C_3 ,..., C_n be the other closed curves which lie inside C, and $f(z)$ be an analytic function in the region between these curves and continuous on C, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + ... + \int_{C_n} f(z)\,dz$$

where integral along C_i ($i = 1, 2, ..., n$) is taken in the anti-clockwise direction.

Proof: Beyond the scope of the book.

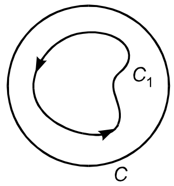
Extension of Cauchy's Integral Theorem

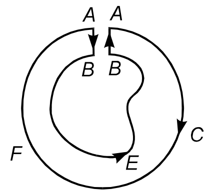
If $f(z)$ is analytic inside and on a multiply connected region whose outer boundary is C and inner boundaries are C_1, C_2, ..., C_n then

$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz,$ where all the integrals are taken in the same sense.

Suppose that $f(z)$ is analytic in the multiply-connected region R enclosed between the two closed curves C and C_1. Now we can convert the multiply-connected region R into a simply-connected region by introducing the strip cut AB.

NOTES





By Cauchy's integral theorem, $\int_{ABEBAFA} f(z)dz = 0$

i.e., $\int_{AB} + \int_{BEB} + \int_{BA} + \int_{ACA} f(z)dz = 0$

i.e., $\int_{AB} f(z)dz + \int_{C_1} f(z)dz - \int_{AB} f(z)dz + \int_{C} f(z)dz = 0$

i.e., $\int_{C_1} f(z)dz = -\int_{C} f(z)dz$

$\Rightarrow \int_{C_1} f(z)dz = \int_{C} f(z)dz$

This proof is for a doubly connected region.

Similarly, we can extend this to a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, ..., C_n$.

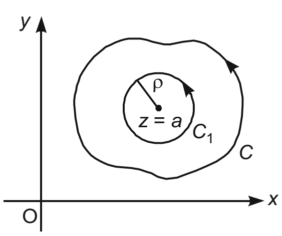
Hence $\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$

1.2.2 Cauchy's Integral Formula

Theorem 1.4: If $f(z)$ is analytic inside and on a simple closed curve C of a simply-connected region R and if 'a' is any point within C, then $f(a) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a}dz,$ where C is described in the anticlockwise sense.

Proof: Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C, except at the point $z = a$.

Hence we draw a small circle C_1 with centre at $z = a$ and radius ρ lying entirely inside C.



Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1.

Hence, by Cauchy's extended theorem,

$$\int_{C} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz \qquad ...(1.2)$$

If z is any point on C_1, then $|z-a|=\rho$ and hence $z-a=\rho e^{i\theta}$ (or) $z=a+\rho e^{i\theta}$

NOTES

$$\therefore \qquad dz = \rho e^{i\theta}\cdot i\,d\theta$$

$$\therefore \quad \int_{C_1}\frac{f(z)}{z-a}dz=\int_{C_1}\frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\rho e^{i\theta}\cdot i\,d\theta = \int_C f(a+\rho e^{i\theta})\cdot i\,d\theta \qquad ...(1.3)$$

In the limit, as $\rho\to 0$ the circle C_1 tends to a point.

Hence, in the limit as $\rho\to 0,$ the Equation (1.3) becomes,

$$\int_{C_1}\frac{f(z)}{z-a}dz = \int_{\theta=0}^{2\pi} f(a)\,i\,d\theta = i\,f(a)\cdot\int_0^{2\pi} d\theta = 2\pi i\,f(a)$$

Hence using in Equation (1.2), we get $\int_C \frac{f(z)}{z-a}dz = 2\pi i\,f(a).$

i.e.,
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

Note: This theorem gives the value of an analytic function at any interior point of a region R in terms of its values on the boundary of the region.

Cauchy's Integral Formulas for the Derivatives of an Analytic Function

By Cauchy's Integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad ...(1.4)$$

Differentiating partially both sides of Equation (1.4) with respect to 'a' and performing the differentiation within the integration symbol in the RHS, we get

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz.$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Example 1.10: Evaluate $\int_C z^2 dz,$ where C is the contour $|z| = 2.$

Solution: $f(z) = z^2$, is analytic everywhere within and on $|z| = 2.$

$\therefore$ By Cauchy's integral theorem, $\int_C f(z)dz = 0$, i.e., $\int_C z^2 dz = 0$.

NOTES

Example 1.11: Evaluate $\int_C (z-2)^n \, dz$ where $n > 0$ and C is the circle whose centre is 3 and radius 1.

Solution: $f(z) = (z-2)^n, n > 0$.

C is the circle $|z-3| = 1$.

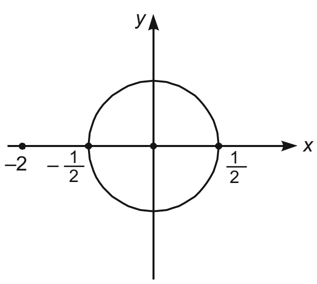
$f(z)$ is analytic everywhere within and on C.

$\therefore$ By Cauchy's integral theorem, $\int_C (z-2)^n \, dz = 0$.

Example 1.12: Evaluate $\int_C \frac{e^{-z}}{z+2} dz$, where C is the circle (*i*) $|z| = \frac{1}{2}$; (*ii*) $|z-1| = 2$.

Solution: Equating denominator to zero, $z + 2 = 0$

$$\Rightarrow \quad z = -2$$



(*i*) $|z| = \frac{1}{2}$ is the circle with centre at z = '0' and radius $\frac{1}{2}$

$z = -2$ lies outside this circle.

$\therefore$ The function $\frac{e^{-z}}{z+2}$ is analytic within and on C.

By Cauchy's integral theorem, $\int_C \frac{e^{-z}}{z+2} dz = 0$.

(*ii*) $|z-1| = 2$ is a circle with centre at $z = 1$ and radius 2.

Put $z = -2$ in $|z-1|$

$$|z-1| = |-2-1| = |-3| = 3 > 2$$

$\therefore$ $z = -2$ lies outside this circle.

$\therefore$ The function $\frac{e^{-z}}{z+2}$ is analytic within and on C.

By Cauchy's integral theorem, $\int_C \frac{e^{-z}}{z+2} dz = 0$.

Example 1.13: Evaluate (*i*) $\int_C \frac{dz}{z-a}$ (*ii*) $\int_C \frac{dz}{(z-a)^n}, n \geq 2$ where C is a simple closed curve and $z = a$ is a point (*i*) outside C (*ii*) inside C.

Solution: (*i*) $z = a$ is a point outside C.

Since $z = a$ is a point outside C both $\frac{1}{(z-a)}$ and $\frac{1}{(z-a)^n}$ are analytic inside and on C.

$\therefore$ By Cauchy's integral theorem,

$$\int_C \frac{dz}{z-a} = 0 \quad \text{and} \quad \int_C \frac{dz}{(z-a)^n} = 0.$$

(*ii*) $z = a$ is inside C.

NOTES

(*a*) Using Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i \cdot f(a),$$

$$\int_C \frac{1}{(z-a)}\,dz = 2\pi i[1]_{z=a}, \quad [\text{Here } f(z) = 1]$$

$$= 2\pi\, i \cdot 1$$

$$= 2\pi i.$$

(*b*) Using Cauchy's integral formula for derivatives,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a),$$

$$\int_C \frac{1}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a) = \frac{2\pi i}{(n-1)!}(0) = 0.$$

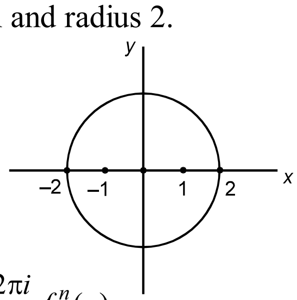
$$[\because f(z) = 1,\ f'(z) = 0,\ f''(z) = 0, \ldots, f^n(z) = 0]$$

Example 1.14: Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz,$ where C is the circle $|z| = 2$.

Solution: $|z| = 2$ is a circle with centre at the origin and radius 2.

Here $f(z) = e^{2z}$.

$$(z+1)^4 = 0 \quad \Rightarrow \quad z + 1 = 0 \quad \Rightarrow \quad z = -1$$



$z = -1$ lies inside the circle $|z| = 2$.

$\therefore$ By Cauchy's integral formula $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a),$

$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \int_C \frac{e^{2z}}{[z-(-1)]^4}\,dz$$

$$= \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{6}[f^{(3)}(z)]_{z=-1}$$

$$= \frac{2\pi i}{6}\left[\frac{d^{(3)}}{dz^3}(e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{6}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{8}{3}\pi i\, e^{-2}$$

NOTES

Example 1.15: Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z-2|=\frac{1}{2}$, using Cauchy's integral formula.

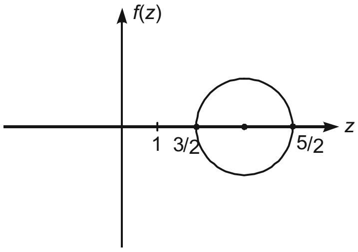
Solution: $|z-2|=\frac{1}{2}$ is a circle with centre at $z = 2$ and radius $\frac{1}{2}$.

Denominator = 0 $\Rightarrow (z-1)(z-2) = 0 \Rightarrow z = 1, 2$

$z = 1$ lies outside the circle C and

$z = 2$ lies inside the circle C.

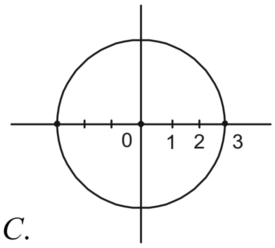
Using Cauchy's integral formula $\int_C \frac{f(z)}{(z-a)} dz = 2\pi i\, f(a)$,



$$\begin{aligned}\int_C \frac{z}{(z-1)(z-2)} dz &= \int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)} dz \\ &= 2\pi i[f(z)]_{z=2} \\ &= 2\pi i\left[\frac{z}{z-1}\right]_{z=2} \\ &= 2\pi i\left[\frac{2}{1}\right] \\ &= 4\pi i\end{aligned}$$

Example 1.16: Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

Solution: $|z| = 3$ is a circle with centre at the origin and radius 3.



$(z-1)(z-2) = 0 \quad \Rightarrow \quad z = 1, 2$

$z = 1, 2$ lies inside C.

$f(z) = \cos \pi z^2$ is analytic everywhere inside and on C.

$$\text{Now} \quad \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 2$, $\quad 1 = 0 + B \Rightarrow B = 1$

Put $z = 1$, $\quad 1 = A(-1) + 0 \Rightarrow A = -1$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = -\frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$\begin{aligned}\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz &= -\int_C \frac{\cos \pi z^2}{(z-1)} dz + \int_C \frac{\cos \pi z^2}{(z-2)} dz \\ &= -2\pi i\,[f(z)]_{z=1} + 2\pi i[f(z)]_{z=2} \\ &= -2\pi i\left[\cos \pi z^2\right]_{z=1} + 2\pi i\left[\cos \pi z^2\right]_{z=2}\end{aligned}$$

$$= -2\pi i[\cos\pi] + 2\pi i[\cos 4\pi]$$

$$= -2\pi i(-1) + 2\pi i(1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

NOTES

Example 1.17: Evaluate $\int_C \frac{z+4}{z^2+2z+5}$ where C is $|z + 1 - i| = 2$, using Cauchy's integral formula.

Solution: $|z + 1 - i| = 2$ can be re-written as,

$$|z - (-1 + i)| = 2$$

This is a circle with centre at $z = -1 + i$ and radius 2.

$z = -1 + i$ means the point $(-1, 1)$.

$$z^2 + 2z + 5 = 0$$

$$\Rightarrow \quad z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$z = -1 + 2i, \ -1 - 2i$$

Put $\quad z = -1 + 2i$

$$|z+1-i| = |i| = 1 < 2$$

Put $\quad z = -1 - 2i$

$$|z + 1 - i| = |-3i| = 3 > 2$$

$\therefore$ Of these two points, $z = -1 + 2i$ lies inside C and $z = -1 - 2i$ lies outside C.

Using Cauchy's integral formula,

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int_C \frac{\left[\dfrac{z+4}{z-(-1-2i)}\right]}{[z-(-1+2i)]}\,dz$$

$$= 2\pi i[f(z)]_{z=-1+2i}$$

$$= 2\pi i\left[\frac{z+4}{z-(-1-2i)}\right]_{z=-1+2i}$$

$$= 2\pi i\left[\frac{-1+2i+4}{-1+2i+1+2i}\right]$$

$$= 2\pi i\left(\frac{3+2i}{4i}\right)$$

$$= \frac{\pi}{2}(3+2i)$$

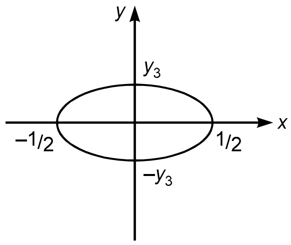
NOTES

Example 1.18: Evaluate $\int_C \frac{z^3+z+1}{z^2-7z+6}\,dz,$ where C is the ellipse $4x^2 + 9y^2 = 1$.

Solution: The ellipse $4x^2+9y^2=1$ can be re-written in the standard form as,

$$\frac{x^2}{\left(\frac{1}{2}\right)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1 \qquad \left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is the standard form}\right]$$

Here $a=\frac{1}{2},\ b=\frac{1}{3}$



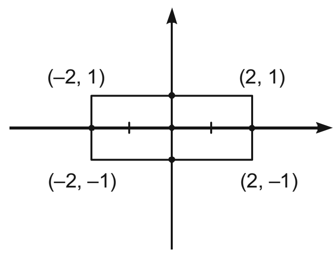
$$z^2-7z+6=(z-1)(z-6)=0$$

$\therefore \quad z = 1, z = 6$

Both these points lie outside the ellipse.

$\therefore$ By Cauchy's integral theorem, $\int_C \frac{z^3+z+1}{z^2-7z+6}\,dz=0$.

Example 1.19: Use Cauchy's integral formula to evaluate $\int_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices $2\pm i,\ -2\pm i$.



Solution: $\quad z^2 - 1 = (z-1)(z+1) = 0$

$\Rightarrow \quad z = 1, -1$

$\therefore$ It is not analytic at $z = 1, -1$.

$z = 1, -1$ lies inside the rectangle.

Also $\frac{1}{(z-1)(z+1)} = \frac{A}{(z-1)}+\frac{B}{(z+1)}=\frac{1}{2(z-1)}-\frac{1}{2(z+1)}$

$$\left[1=A(z+1)+B(z-1) \text{ gives } A=\frac{1}{2}, B=-\frac{1}{2}\right]$$

$$\therefore \int_C \frac{\cos \pi z}{(z^2-1)}\,dz=\frac{1}{2}\int_C \frac{\cos \pi z}{(z-1)}\,dz-\frac{1}{2}\int_C \frac{\cos \pi z}{(z+1)}\,dz$$

$$= \frac{1}{2}2\pi i\left[\cos \pi z\right]_{z=1}-\frac{1}{2}\cdot 2\pi i\left[\cos \pi z\right]_{z=-1}$$

$$= \pi i \cos \pi - \pi i \cos(-\pi)$$

$$= i\pi\cdot(-1)-i\pi(-1) \qquad \because\ \cos(-\theta)=\cos\theta$$

$$= 0$$

Example 1.20: If $f(a) = \int_C \frac{3z^2+7z+1}{(z-a)}\,dz$ where C is $|z|=2$, find (i) $f(3)$ (ii) $f(1-i)$ (iii) $f'(1-i)$ and (iv) $f''(1-i)$.

Solution: (*i*) $f(a) = \int_C \frac{3z^2 + 7z + 1}{(z-a)} dz$

$\therefore f(3) = \int_C \frac{3z^2 + 7z + 1}{(z-3)} dz$ where C is $|z| = 2$.

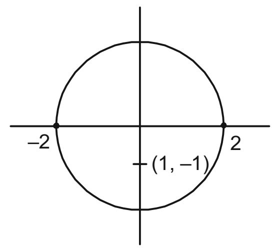
$z = 3$ lies outside C.

$\therefore$ By Cauchy's integral theorem $\int_C \frac{3z^2 + 7z + 1}{(z-3)} dz = 0.$

Hence $f(3) = 0$.

(*ii*) $f(1-i) = \int_C \frac{3z^2 + 7z + 1}{[z-(1-i)]} dz$

$z = 1 - i$ is a point $(1, -1)$,
which lies inside C.



$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{3z^2 + 7z + 1}{[z-(1-i)]} dz = 2\pi i\left[3z^2 + 7z + 1\right]_{z=1-i}$$

$$= 2\pi i\left[3(1-i)^2 + 7(1-i) + 1\right]$$

$$= 2\pi i[8 - 13i]$$

$$= 2\pi(8i + 13)$$

(*iii*) $$f(a) = \int_C \frac{3z^2 + 7z + 1}{(z-a)} dz$$

$$\therefore \quad f'(a) = \int_C (-1)\frac{(3z^2 + 7z + 1)}{(z-a)^2} dz = -\int_C \frac{3z^2 + 7z + 1}{(z-a)^2} dz$$

$$f'(1-i) = -\int_C \frac{3z^2 + 7z + 1}{(z-1)} dz$$

$$= -2\pi i\left[\frac{d}{dz}(3z^2 + 7z + 1)\right]_{z=1-i}$$

$$= -2\pi i[6z + 7]_{z=1-i}$$

$$= -2\pi i[6 - i6 + 7]$$

$$= -2\pi i[13 - 6i]$$

(*iv*) $$f''(a) = (-1)(-2)\int_C \frac{3z^2 + 7z + 1}{(z-a)^3} dz$$

$$= 2\int_C \frac{3z^2 + 7z + 1}{(z-a)^3} dz$$

NOTES

NOTES

$$= 2\cdot\frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(3z^2+7z+1)\right]_{z=a}$$

$$= 2\pi i[6]_{z=a}$$

$$\therefore \quad f''(1-i) = 2\pi i[6]_{z=1-i}$$

$$= 12\pi i$$

Example 1.21: Evaluate $\int_C \frac{e^z}{z(1-z)^3}dz$ if,

(*i*) 0 and 1 lie outside *C*.

(*ii*) 0 and 1 lie inside *C*.

(*iii*) 0 lies inside *C* and 1 lies outside *C*.

(*iv*) 0 lies outside *C* and 1 lies inside *C*.

Solution: $z(1-z)^3$ is not analytic at $z = 0$ and $z = 1$.

(*i*) 0 and 1 lie outside *C*.

$\therefore$ By Cauchy's integral theorem $\int_C \frac{e^z}{z(1-z)^3}dz = 0$.

(*ii*) 0 and 1 lie inside *C*.

$$\frac{1}{z(1-z)^3} = \frac{A}{z}+\frac{B}{(1-z)}+\frac{C}{(1-z)^2}+\frac{D}{(1-z)^3}$$

$$1 = A(1-z)^3 + Bz(1-z)^2 + Cz(1-z) + Dz$$

$$z = 0, \quad 1 = A$$

$$z = 1, \quad 1 = D$$

Coefficient of z^3, $0 = -A + B \Rightarrow \quad B = A = 1$

Coefficient of z^2, $0 = 3A - 2B - C$

$$0 = 3 - 2 - C \quad \Rightarrow \quad C = 1$$

$$\therefore \quad \frac{1}{z(1-z)^3} = \frac{1}{z}+\frac{1}{(1-z)}+\frac{1}{(1-z)^2}+\frac{1}{(1-z)^3}$$

$$= \frac{1}{z}-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}$$

$$\therefore \quad \int_C \frac{e^z}{z(1-z)^3}dz = \int_C \frac{e^z}{z}dz - \int_C \frac{e^z}{(z-1)}dz + \int_C \frac{e^z}{(z-1)^2}dz - \int_C \frac{e^z}{(z-1)^3}dz$$

$$= 2\pi i[e^z]_{z=0} - 2\pi i[e^z]_{z=1} + \frac{2\pi i}{1!}\left[\frac{d}{dz}(e)^z\right]_{z=1} - \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(e^z)\right]_{z=1}$$

$$= 2\pi i(1) - 2\pi i(e) + 2\pi i(e) - \pi i(1)$$

$$= \pi i$$

(*iii*) 0 lies inside *C* and 1 lies outside *C*.

$$\int_C \frac{e^z}{z(1-z)^3}dz = \int_C \frac{\left[\frac{e^z}{(1-z)^3}\right]}{z}dz = 2\pi i\left[\frac{e^z}{(1-z)^3}\right]_{z=0} = 2\pi i$$

NOTES

(*iv*) 0 lies outside *C* and 1 lies inside *C*.

$$\int_C \frac{e^z}{z(1-z)^3}dz = \int_C \frac{\left(\frac{e^z}{z}\right)}{(1-z)^3}dz = -\int_C \frac{\left(\frac{e^z}{z}\right)}{(z-1)^3}dz$$

$$= -\frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)\right]_{z=1}$$

$$= -\pi i\left[\frac{d}{dz}\left(\frac{ze^z - e^z}{z^2}\right)\right]_{z=1}$$

$$= -\pi i\left[\frac{z^2(ze^z + e^z - e^z) - (ze^z - e^z)\cdot 2z}{z^4}\right]_{z=1}$$

$$= -\pi i\left[\frac{z^3e^z - 2z^2e^z + 2ze^z}{z^4}\right]_{z=1}$$

$$= -\pi i(e)$$

$$= -\pi e i$$

Example 1.22: Evaluate $\int_C \frac{e^{az}}{z^{n+1}}dz$ where *C* is any closed contour.

Solution: Case (*i*)- If $z = 0$ lies outside *C* then $\frac{e^{az}}{z^{n+1}}$ is analytic inside and on *C*.

$\therefore$ By Cauchy's integral theorem, $\int_C \frac{e^{az}}{z^{n+1}}dz = 0.$

Case (*ii*)- If $z = 0$ lies inside *C* then by Cauchy's integral formula for derivatives

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a),$$

$$\int_C \frac{e^{az}}{z^{n+1}}dz = \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n}(e^{az})\right]_{z=0} = \frac{2\pi i}{n!}\left[a^n \cdot e^{az}\right]_{z=0} = \frac{2\pi i}{n!}a^n$$

NOTES

Check Your Progress

1. Define multiple point.
2. State Cauchy's theorem.
3. What is a multi-connected region?
4. State Cauchy's integral formula.

1.3 HIGHER ORDER DERIVATIVES

Let f be a function that has a derivative at every point in its domain. We can then define a function that maps every point x to the value of the derivative of f at x. This function is written f' and is called the derivative function or the derivative of f. Sometimes f has a derivative at most, but not all, points of its domain. The function whose value at a equals $f'(a)$ whenever $f'(a)$ is defined and elsewhere is undefined is also called the derivative of f. It is still a function, but its domain is strictly smaller than the domain of f.

Using this idea, differentiation becomes a function of functions: The derivative is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions. If we denote this operator by D, then $D(f)$ is the function f'. Since $D(f)$ is a function, it can be evaluated at a point a. By the definition of the derivative function, $D(f)(a) = f'(a)$.

Higher Derivatives

Let f be a differentiable function, and let f' be its derivative. The derivative of f' (if it has one) is written f'' and is called the second derivative of f. Similarly, the derivative of the second derivative, if it exists, is written f''' and is called the third derivative of f. Continuing this process, one can define, if it exists, the n^{th} derivative as the derivative of the $(n–1)^{\text{th}}$ derivative. These repeated derivatives are called higher-order derivatives. The n^{th} derivative is also called the derivative of order n.

If $x(t)$ represents the position of an object at time t, then the higher-order derivatives of x have specific interpretations in physics. The first derivative of x is the object's velocity. The second derivative of x is the acceleration. The third derivative of x is the jerk. And finally, the fourth through sixth derivatives of x are snap, crackle, and pop; most applicable to astrophysics. A function f need not have a derivative (for example, if it is not continuous). Similarly, even if f does have a derivative, it may not have a second derivative. For example, let

$$f(x) = \begin{cases} +x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x \leq 0. \end{cases}$$

Calculation shows that f is a differentiable function whose derivative at x is given by

$$f'(x) = \begin{cases} +2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x \leq 0. \end{cases}$$

$f'(x)$ is twice the absolute value function at x, and it does not have a derivative at zero. Similar examples show that a function can have a k^{th} derivative for each non-negative integer k but not a $(k + 1)^{th}$ derivative. A function that has k successive derivatives is called k times differentiable. If in addition the k^{th} derivative is continuous, then the function is said to be of differentiability class C^k. A function that has infinitely many derivatives is called infinitely differentiable or smooth.

NOTES

On the real line, every polynomial function is infinitely differentiable. By standard differentiation rules, if a polynomial of degree n is differentiated n times, then it becomes a constant function. All of its subsequent derivatives are identically zero. In particular, they exist, so polynomials are smooth functions.

The derivatives of a function f at a point x provide polynomial approximations to that function near x. For example, if f is twice differentiable, then

$$f(x+h) \approx f(x) + f'(x)h + \tfrac{1}{2}f''(x)h^2$$

In the sense that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h - \frac{1}{2}f''(x)h^2}{h^2} = 0.$$

If f is infinitely differentiable, then this is the beginning of the Taylor series for f evaluated at $x + h$ around x.

1.3.1 Morera's Theorem

Morera's theorem, named after Giacinto Morera, gives an important criterion for proving that a function is holomorphic. Morera's theorem states that a continuous, complex-valued function f defined on an open set D in the complex plane that satisfies

$$\oint_\gamma f(z)\,dz = 0$$

For every closed piecewise C^1 curve γ in D must be holomorphic on D. The assumption of Morera's theorem is equivalent to f having an antiderivative on D.

The converse of the theorem is not true in general. A holomorphic function need not possess an antiderivative on its domain, unless one imposes additional assumptions. The converse does hold e.g. if the domain is simply connected; this is Cauchy's integral theorem, stating that the line integral of a holomorphic function along a closed curve is zero.

NOTES

Proof

There is a relatively elementary proof of the theorem. One constructs an anti-derivative for *f* explicitly. Without loss of generality, it can be assumed that *D* is connected. Fix a point z_0 in *D*, and for any $z \in D$, let $\gamma : [0,1] \to D$ be a piecewise C^1 curve such that $\gamma(0) = z_0$ and $\gamma(1) = z$. Then define the function *F* to be

$$F(z) = \int_\gamma f(\zeta)\, d\zeta.$$

To see that the function is well-defined, suppose $\tau : [0,1] \to D$ is another piecewise C^1 curve such that $\tau(0) = z_0$ and $\tau(1) = z$. The curve $\gamma\tau^{-1}$ (i.e., the curve combining γ with τ in reverse) is a closed piecewise C^1 curve in *D*. Then,

$$\int_\gamma f(\zeta)\, d\zeta + \int_{\tau^{-1}} f(\zeta)\, d\zeta = \oint_{\gamma\tau^{-1}} f(\zeta)\, d\zeta = 0.$$

And it follows that

$$\int_\gamma f(\zeta)\, d\zeta = \int_\tau f(\zeta)\, d\zeta.$$

Then using the continuity of *f* to estimate difference quotients, we get that *F2* (*z*) = *f*(*z*).

1.3.2 Cauchy's Inequality

Let *f* (*z*) be an analytic function within circle *C*, given by $| z - a | = R$ and if $| f(z) \leq M$ on *C*, then

$$| f^n(a) | \leq \frac{M\lfloor n}{R^n}.$$

Proof: We know that $f^n(a) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$

$$\therefore \quad |f^n(a)| = \left| \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz \right|$$

$$\leq \frac{\lfloor n}{|2\pi i|}\int_C \frac{|f(z)|}{|z-a|^{n+1}}|dz|$$

$$\leq \frac{\lfloor n\, M}{2\pi R^{n+1}}\int_0^{2\pi} R\,d\theta$$

(Since $z = Re^{i\theta}$, $dz = Ri\,e^{i\theta}$, $|dz| = |Ri\,e^{i\theta}\,d\theta| = Rd\theta$)

$$= \frac{\lfloor n\, M}{2\pi R^{n+1}} 2\pi R = \frac{M\lfloor n}{R^n}$$

$$\therefore \quad | f^n(a)| \leq \frac{M\lfloor n}{R^n}$$

NOTES

1.3.3 Liouville's Theorem

Liouville's theorem, named after Joseph Liouville, states that every bounded entire function must be constant. That is, every holomorphic function f for which there exists a positive number M such that $|f(z)| \leq M$ for all z in $\mathbb{C}$ is constant. Equivalently, non-constant holomorphic functions on $\mathbb{C}$ have unbounded images. The theorem is considerably improved by Picard's little theorem, which says that every entire function whose image omits two or more complex numbers must be constant.

Proof

The theorem follows from the fact that holomorphic functions are analytic. If f is an entire function, it can be represented by its Taylor's series about 0:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

Where (by Cauchy's integral formula)

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$$

And C_r is the circle about 0 of radius $r > 0$. Suppose f is bounded, i.e., there exists a constant M such that $|f(z)| \leq M$ for all z. We can estimate directly

$$|a_k| \leq \frac{1}{2\pi} \oint_{C_r} \frac{|f(\zeta)|}{|\zeta|^{k+1}} |d\zeta| \leq \frac{1}{2\pi} \oint_{C_r} \frac{M}{r^{k+1}} |d\zeta| = \frac{M}{2\pi r^{k+1}} \oint_{C_r} |d\zeta| = \frac{M}{2\pi r^{k+1}} 2\pi r = \frac{M}{r^k},$$

Where in the second inequality we have used the fact that $|z| = r$ on the circle C_r. But the choice of r in the above is an arbitrary positive number. Therefore, letting r tend to infinity (we let r tend to infinity since f is analytic on the entire plane) gives $a_k = 0$ for all $k \geq 1$. Thus $f(z) = a_0$ and this proves the theorem.

1.3.4 The Fundamental Theorem of Algebra

Theorem 1.5 (Fundamental Theorem of Algebra): Every polynomial of degree n$\geq$ 1 with complex coefficients has a zero in ***C***.

Proof: Let $p(z)=z^n+a_{n-1}z^{n-1}+.........+a_1z+a_0$ be a polynomial of degree n$\geq$1 and assume that $p(z)$`$\neq 0$ for all $z\in\subseteq C$.

By Cauchy's integral theorem we have,

$$\int_{|z|=r} \frac{dz}{zp(z)} = \frac{2\pi i}{p(0)} \neq 0$$

where the circle is traversed anti-clockwise.

Also, since

$$|p(z)| \geq |z|^n |1 - \frac{|a_{n-1}|}{|z|} - ... - \frac{|a_0|}{|z|^n}|$$

$$|\int_{|z|=r} \frac{dz}{zp(z)}| \leq 2\pi r \times \max_{|z|=r} \frac{1}{|zp(z)|} = \frac{2\pi}{\min_{|z|=r} |p(z)|} \to 0$$

as $r\to\infty$, which is a contradiction. Hence the theorem is proved.

NOTES

1.3.5 Taylor's Theorem

Theorem 1.6: If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + ... + \frac{(z-a)^n}{n!}f^n(a) + ...\infty. \quad ...(i)$$

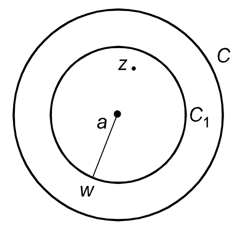
Proof: Let z be any point inside the circle C.

Draw a circle C_1 with centre at 'a' and radius smaller than that of C such that z is an interior point of C_1. Let w be any point on C_1, then

$$|z-a| < |z-w|$$

i.e., $$\left|\frac{z-a}{z-w}\right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left[1-\frac{z-a}{w-a}\right]}$$



$$= \frac{1}{(w-a)}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+...+\left(\frac{z-a}{w-a}\right)^n+....\infty\right] \quad ... (1.5)$$

using binomial theorem as $\left|\frac{z-a}{w-a}\right| < 1.$

This series converges uniformly since $\left|\frac{z-a}{w-a}\right| < 1.$ Multiplying both sides of Equation (1.5) by $\frac{1}{2\pi i}f(w)$ and integrating term by term with respect to w, over C_1, we get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)}dw + \frac{(z-a)}{2\pi i}\int_C\frac{f(w)}{(w-a)^2}dw$$

$$+ \frac{(z-a)^2}{2\pi i}\int_C\frac{f(w)}{(w-a)^3}dw + ...$$

$$+ \frac{(z-a)^n}{2\pi i}\int_C\frac{f(w)}{(w-a)^{n+1}}dw + ...\infty \quad ... (1.6)$$

Since $f(w)$ is analytic inside and on C_1, by Cauchy's formula,

$$\frac{1}{2\pi i}\int_C\frac{f(w)}{w-z}dw = f(z)$$

Also, $$f^n(a) = \frac{n!}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

i.e., $$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw = \frac{f^n(a)}{n!}$$

Using this in Equation (i), we have

$$f(z) = f(a) + (z-a)\cdot f'(a) + \frac{(z-a)^2}{2!} f^n(a) + ... + \frac{(z-a)^n}{n!} f^n(a) + ...\infty$$

This is Taylor's series about $z = a$.

NOTES

Notes:

1. If $a = 0$, then Taylor's series becomes

$$f(z) = f(0) + f'(0) + \frac{z^2}{2!} f''(0) + ... + \frac{z^n}{n!} f^n(0) + ...\infty$$

This series is called **Maclaurin's series.**

2. If $z = a + h$, we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a) + ...\infty$$

Example 1.23: Expand $f(z) = \cos z$ in a Taylor's series at $z = \frac{\pi}{4}$.

Solution: Taylor's series expansion of $f(z)$ about $z = a$ is,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + ...\infty$$

Given, $f(z) = \cos z$, $a = \pi/4$

$f(z) = \cos z$, $f(\pi/4) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z$, $f'(\pi/4) = -\sin \pi/4 = -\frac{1}{\sqrt{2}}$

$f''(z) = -\cos z$, $f''(\pi/4) = -\cos \pi/4 = -\frac{1}{\sqrt{2}}$

$f'''(z) = +\sin z$, $f'''(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$\vdots$

By Taylor's series expansion,

$$\cos z = \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + ...\infty$$

$$= \frac{1}{\sqrt{2}}\left[1 - \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} + \frac{(z-\pi/4)^3}{3!} + ...\infty\right]$$

Example 1.24: Find Taylor's expansion of,

(*i*) $f(z) = \sin z$ about $z = \pi$ (*ii*) $f(z) = e^{2z}$ about $z = 2i$

(*iii*) $f(z) = \frac{\sin z}{(z-\pi)}$ about $z = \pi$

Solution: (*i*) $f(z) = \sin z$, $a = \pi$

$f(z) = \sin z$, $f(\pi) = \sin \pi = 0$

$f'(z) = \cos z$, $f'(\pi) = \cos \pi = -1$

$f''(z) = -\sin z$, $f''(\pi) = -\sin \pi = 0$

NOTES

$$f'''(z) = -\cos z, \qquad f'''(\pi) = -\cos\pi = -(-1) = 1$$
$$f^{iv}(z) = +\sin z, \qquad f^{iv}(\pi) = 0$$
$$f^{v}(z) = \cos z, \qquad f^{v}(\pi) = \cos\pi = -1$$

By Taylor's series expansion,

$$\sin z = 0 + \frac{(z-\pi)}{1!}(-1) + \frac{(z-\pi)^2}{2!}(0) + \frac{(z-\pi)^3}{3!}(1) + \frac{(z-\pi)^4}{4!}(0) + \frac{(z-\pi)^5}{5!}(-1) + ...\infty$$
$$= -\frac{(z-\pi)}{1!} + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + ...\infty$$
$$= -\left[\frac{(z-\pi)}{1!} - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - ....\infty\right]$$

(*ii*) $f(z) = e^{2z}$, $z = 2i$. Here $a = 2i$

$$f(z) = e^{2z}, \qquad f(2i) = e^{2(2i)} = e^{4i}$$
$$f'(z) = 2e^{2z}, \qquad f'(2i) = 2e^{4i}$$
$$f''(z) = 2^2 e^{2z}, \qquad f''(2i) = 2^2 e^{4i}$$
$$f'''(z) = 2^3 e^{2z}, \qquad f'''(2i) = 2^3 e^{4i}$$

By Taylor's expansion,

$$e^{2z} = e^{4i} + \frac{(z-2i)}{1!}2e^{4i} + \frac{(z-2i)^2}{2!}2^2 e^{4i} + \frac{(z-2i)^3}{3!}2^3 e^{4i} + ...\infty$$
$$= e^{4i}\left[1 + \frac{(z-2i)}{1!}\cdot 2 + \frac{(z-2i)^2}{2!}2^2 + \frac{(z-2i)^3}{3!}2^3 + ...\infty\right]$$

(*iii*) $f(z) = \dfrac{\sin z}{z-\pi}$ about $z = \pi$.

Put $t = z - \pi$. Then $z = \pi + t$.

$$\therefore \quad \frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{t} = \frac{-\sin t}{t} = -\frac{1}{t}\cdot\sin t$$
$$= -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - ....\infty\right]$$
$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + ...\infty$$
$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^2}{5!} + ...\infty$$

Example 1.25: Obtain the Taylor's expansion of $f(z) = \dfrac{z}{(z+1)(z+2)}$ in power of $z - 2$.

Solution: In powers of $z - 2$ means about $z = 2$.

NOTES

Put $t = z - 2 \Rightarrow z = 2 + t$

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{2+t}{(2+t+1)(2+t+2)} = \frac{2+t}{(3+t)(4+t)}$$

Resolving into partial fraction,

$$\frac{2+t}{(3+t)(4+t)} = \frac{A}{(3+t)} + \frac{B}{(4+t)} = \frac{-1}{(3+t)} + \frac{2}{(4+t)}$$

$$[2 + t = A(4+t) + B(3+t)$$

$$t = -4, -2 = 0 - B \Rightarrow B = 2$$

$$t = -3, -1 = A + 0 \Rightarrow A = -1]$$

$$\therefore f(z) = -\frac{1}{(3+t)} + \frac{2}{(4+t)}$$

$$= -\frac{1}{3\left(1+\frac{t}{3}\right)} + \frac{2}{4\left(1+\frac{t}{4}\right)}$$

$$= -\frac{1}{3}\left(1+\frac{t}{3}\right)^{-1} + \frac{2}{4}\left(1+\frac{t}{4}\right)^{-1}$$

$$= -\frac{1}{3}\left[1 - \frac{t}{3} + \left(\frac{t}{3}\right)^2 - \left(\frac{t}{3}\right)^3 + \ldots\infty\right] + \frac{2}{4}\left[1 - \frac{t}{4} + \left(\frac{t}{4}\right)^2 - \left(\frac{t}{4}\right)^3 + \ldots\infty\right]$$

$$= \left[-\frac{1}{3} + \frac{1}{3^2}t - \frac{1}{3^3}t^2 + \frac{1}{3^4}t^3 + \ldots\infty\right] + \left[\frac{2}{4} - \frac{2}{4^2}t + \frac{2t^2}{4^3} - \ldots\infty\right]$$

$$= \left(\frac{2}{4} - \frac{1}{3}\right) - \left(\frac{2}{4^2} - \frac{1}{3^2}\right)t + \left(\frac{2}{4^3} - \frac{1}{3^3}\right)t^2 - \left(\frac{2}{4^4} - \frac{1}{3^4}\right)t^3 + \ldots..\infty$$

$$= \left(\frac{2}{4} - \frac{1}{3}\right) - \left(\frac{2}{4^2} - \frac{1}{3^2}\right)(z-2) + \left(\frac{2}{4^3} - \frac{1}{3^3}\right)(z-2)^2 - \ldots.\infty$$

Example 1.26: Find Taylor's expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point $z = i$.

Solution: $f(z) = \frac{2z^3+1}{z^2+z}$

$$= 2z - 2 + \frac{2z+1}{z(z+1)}$$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{(z+1)},$$ [On resolving into partial fraction]

Put $t = z - i \Rightarrow z = i + t$

$$\therefore \quad f(z) = 2(i+t) - 2 + \frac{1}{(i+t)} + \frac{1}{(i+t+1)}$$

$$= (2i-2)+2t+\frac{1}{i\left[1+\frac{t}{i}\right]}+\frac{1}{(1+i)\left[1+\frac{t}{1+i}\right]}$$

NOTES

$$=(2i-2)+2t+\frac{1}{i}\left[1+\frac{t}{i}\right]^{-1}+\frac{1}{(1+i)}\left[1+\frac{t}{1+i}\right]^{-1}$$

$$=(2i-2)+2t+\frac{1}{i}\left[1-\frac{t}{i}+\left(\frac{t}{i}\right)^2-\left(\frac{t}{i}\right)^3+\left(\frac{t}{i}\right)^4-.....\infty\right]$$

$$+\frac{1}{(1+i)}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\frac{t^4}{(1+i)^4}-....\infty\right]$$

$$=(2i-2)+2t+\left[\frac{1}{i}-\frac{t}{i^2}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+....\infty\right]$$

$$+\left[\frac{1}{(1+i)}-\frac{t}{(1+i)^2}+\frac{t^2}{(1+i)^3}-\frac{t^3}{(1+i)^4}+....\infty\right]$$

$$=\left(2i-2+\frac{1}{i}+\frac{1}{1+i}\right)+\left(2-\frac{1}{i^2}-\frac{1}{(1+i)^2}\right)t$$

$$+\left[\frac{1}{i^3}+\frac{1}{(1+i)^3}\right]t^2-\left[\frac{1}{i^4}+\frac{1}{(1+i)^4}\right]t^3+....\infty$$

$$=\left(\frac{i}{2}-\frac{3}{2}\right)+\left(3+\frac{i}{2}\right)(z-i)+\left[\frac{1}{i^3}+\frac{1}{(1+i)^3}\right](z-i)^2$$

$$-\left[\frac{1}{i^4}+\frac{1}{(1+i)^4}\right](z-i)^3+...\infty$$

(or)

$$=\left(\frac{i}{2}-\frac{3}{2}\right)+\left(3+\frac{i}{2}\right)(z-i)+\sum_{n=2}^{\infty}(-1)^n\left[\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right](z-i)^n$$

Example 1.27: Obtain the expansion of log $(1 + z)$ when $|z| < 1$.

Solution: For $z \neq -1$, $f(z)=\log(1+z)$ is analytic.

We can expand each branch of log $(1 + z)$ about $z = 0$ and the expansion is valid for $|z| < 1$.

$$f(z) = \log(1+z), \qquad f(0) = \log 1 = 0$$

$$f'(z)=\frac{1}{1+z} \qquad f'(0)=1$$

$$f''(x)=\frac{(-1)}{(1+z)^2}, \qquad f''(0)=\frac{-1}{1}=-1$$

$$f'''(z) = \frac{(-1)(-2)}{(1+z)^3}, \qquad f'''(0) = 2 = 2!$$

$$f^{iv}(z) = \frac{(-1)(-2)(-3)}{(1+z)^4}, \qquad f^{iv}(0) = -6 = -3!$$

NOTES

In general, $f^n(z) = \dfrac{(-1)^{n-1}\cdot(n-1)!}{(1+z)^n}$

By Taylor's series expansion about $z = a$,

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + ...\infty,$$

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + ...\infty$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \frac{z^4}{4!}(-6) + ...$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...\infty$$

1.3.6 Maximum Modulus Principle

This principle has many uses in complex analysis. It finds applications in the fundamental theorem of algebra and Schwarz's lemma.

Theorem 1.7 (Maximum Modulus Principle): Let $G \subset C$ be a connected open set and $f{:}G \to C$ be analytic. If there is any $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Proof: Let us choose $\delta > 0$, so that $D(a, \delta) \subset G$. Set $0 < r < \delta$ and then by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{z-a}dz$$

In terms of parameterization,

$z = a + re^{i\theta}$ with $0 \leq \theta \leq 2\pi,\ dz = ire^{i\theta}d\theta$

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{ire^{i\theta}}d\theta \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta$$

Hence,

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\,d\theta = |f(a)|$$

Using $|f(a+re^{i\theta})| \leq |f(a)|\ \forall\theta$

We must therefore have equality in the inequalities.

Since, the integrand $|f(a+re^{i\theta})|$ is a continuous function of θ, this implies

$|f(a+re^{i\theta})|=|f(a)|$ for all θ.

Put, $\alpha = \arg(f(a))$

Now,

NOTES

$$|f(a)| = e^{-ia}f(a)$$

$$= \frac{e^{-ia}}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-ia}f(a+re^{i\theta})d\theta$$

$$\Re|f(a)| = |f(a)| = \frac{1}{2\pi}\Re\int_0^{2\pi} e^{-ia}f(a+re^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Re(e^{-ia}f(a+re^{i\theta})d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |e^{-ia}f(a+re^{i\theta})|\,d\theta$$

using $\Re w \leq |w|$ for $w \in \mathbb{C}$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\,d\theta = |f(a)|$$

and so, we must have equality in all inequalities implying,

$\Re(e^{-ia}f(a+re^{i\theta})) = |e^{-ia}f(a+re^{i\theta})| = |f(a)|$ for all θ

Thus,

$\Re(e^{-ia}f(a+re^{i\theta})) = 0$ and

$e^{-ia}f(a+re^{i\theta}) = |f(a)|$ or $f(a+re^{i\theta}) = e^{ia}|f(a)|$

Thus, $f(z)$ is constant for z in the infinite compact set $\{z: |z-a|=r\}$ of G.

Corollary: Suppose $R \subset C$ is a closed bounded region. If f: $R \rightarrow C$ is continuous on R, analytic on the interior of R and not constant, then the maximum value of $|f(z)|$ is attained at a point (or points) on the boundary of R and never at points in the interior of R. Moreover, if we write

$f(x+iy) = u(x, y) + iv(x, y)$,

then the maximum value of $u(x, y)$ is attained at a point (or points) on the boundary of R and never at points in the interior of R.

Proof: The first part follows from the fact that a continuous function on a closed bounded set attains a maximum value, and from the maximum modulus principle this value cannot be attained in the interior of R. The second part follows from the observation that the modulus of the function,

$g(z) = e^{f(z)}$ is $|g(z)| = e^{u(x, y)}$

NOTES

Check Your Progress

5. Define the term higher derivatives in complex integration.
6. What is Morera's theorem?
7. Write the statement of Cauchy's inequality.
8. Define the Liouville's theorem.
9. State the fundamental theorem of algebra.
10. How do you get Maclaurin's series from Taylor's series?
11. State the maximum modulus principle.

1.4 SCHWARZ LEMMA

Theorem 1.8 (Schwarz Lemma): Let D be the open unit disc in the complex plane C, i.e., $D = \{z \in C; |z| < 1\}$. Let f: $D \to D$ be a holomorphic mapping such that, $f(0) = 0$.

Then, the classical Schwarz lemma states that, $|f(z)| \leq |z|$ for $z \in D$ and $|f'(0)| \leq 1$, and the equality $|f'(0)| = 1$ or the equality $|f(z) = |z|$ at a single point $z \neq 0$ implies $f(z) = \varepsilon z$ with $|\varepsilon| = 1$.

Proof: Drop the assumption, $f(0) = 0$. If f: $D \to D$, is an arbitrary holomorphic mapping, we fix an arbitrarily chosen point $z \in D$ and consider the automorphisms g and h of D defined by,

$g(\zeta) = \zeta + z/1 + \bar{z}\zeta$, for $\zeta \in D$

$h(\zeta) = \zeta - f(z)/1 - \overline{f(z)\zeta}$, for $\zeta \in D$

Then, the composed mapping, $F = h.f.g$ is a holomorphic mapping of D into itself which sends 0 into itself.

Since,

$F(0) = 0$ and,

$F'(0) = h'(f(z))f'(z)g'(0)$,

We obtain,

$$F'(0) = \frac{1-|z|^2}{1-|f(z)|^2} f'(z)$$

Hence,

$$\frac{1-|z|^2}{1-|f(z)|^2} |f'(z)| \leq 1$$

Or

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \text{for } z \in D$$

Theorem 1.9: Let f be a holomorphic mapping of the unit disc D into itself. Then,

$$\frac{|df|}{1-|f|^2} \le \frac{|dz|}{1-|z|^2}, \quad \text{for } z \in D$$

and, the equality at a single point z implies that f is an automorphism of D.

NOTES

This result, which we derived from the Schwarz lemma, is actually equivalent to the Schwarz lemma. In fact, if f: $D \to D$ is a holomorphic mapping such that, $f(0) = 0$, then by setting $z = 0$ in the inequality above we obtain,

$$|f'(0)| \le 1,$$

And if,

$$|f'(0)| = 1,$$

Then,

f is an automorphism of D.

Moreover,

$$\int_0^{|f(\zeta)|} \frac{|df|}{1-|f|^2} \le \int_0^{|\zeta|} \frac{|dz|}{1-|z|^2}$$

Hence,

$\log 1+|f(\zeta)|/1-|f(\zeta)| \le \log 1+|\zeta|/1-|\zeta|$,

Which implies,

$2/1-|f(\zeta)-1| = 1+|f(\zeta)|/1-|f(\zeta)| \le 1+|\zeta|/1-|\zeta| = 2/(1-|\zeta|)-1$.

It follows that,

$|f(\zeta)| \le |\zeta|$

The equality $|f(\zeta)|=|\zeta|$ at a single point $\zeta \neq 0$ implies the equality,

$$\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$$

For all, z lying in the line segment joining 0 and ζ. By Theorem 1.9, f is an automorphism of D, and hence, $f(z) = \varepsilon z$ for some ε with $|\varepsilon| = 1$. This proves that Theorem 1.9 implies the classical Schwarz lemma.

1.4.1 Laurent's Series

If C_1 and C_2 are two concentric circles with centre at 'a' and radii r_1 and r_2 ($r_1 > r_2$) and if $f(z)$ is analytic on C_1 and C_2 and in the annular region R between them, then at each point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

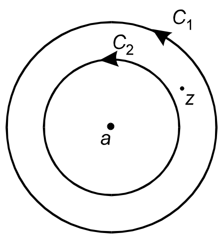
where, $$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, \quad n = 0, 1, 2, 3, ...$$

and $$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw,\ n = 1, 2, 3, ...$$

NOTES

Proof: Let z be any point in the region R, then by Cauchy's integral formula for doubly connected region,

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw \quad ...(1.7)$$



For the first integral in Equation (1.7), w lies on C_1.

$\therefore \quad |z-a| < |w-a|$

i.e., $\left|\frac{z-a}{w-a}\right| < 1$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+......\infty\right]$$

Multiplying both sides by $\frac{1}{2\pi i}f(w)$ and integrating term by term with respect to 'w', along the circle C_1, we get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}dw + ...$$
$$+\frac{(z-a)^n}{2\pi i}\int_{C}\frac{f(w)}{(w-a)^{n+1}}dw + ...\infty$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_n(z-a)^2 + ...\infty$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n \quad ...(1.8)$$

where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}aw$ for $n = 0, 1, 2, ...$.

For the second integral in Equation (1.7), w lies on C_2.

$\therefore \quad |w-a| < |z-a|$

i.e., $\left|\frac{w-a}{z-a}\right| < 1.$

Now $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\frac{1}{(z-a)}\left[1-\frac{w-a}{z-a}\right]^{-1}$$

$$= \frac{-1}{(z-a)}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+...+\left(\frac{w-a}{z-a}\right)^n+...\infty\right]$$

NOTES

Multiplying both sides by $\frac{1}{2\pi i} f(w)$ and integrating term by term with respect to w, along the circles C_2, we get

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw = \frac{1}{(z-a)}\cdot\frac{1}{2\pi i}\int_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\int_{C_2}(w-a)f(w)dw + \ldots$$

$$\ldots + \frac{1}{(z-a)^{n+1}}\int_{C_2}(w-a)^n f(w)dw + \ldots\infty$$

$$= b_1(z-a)^{-1} + b_2(z-a)^{-2} + b_3(z-a)^{-3} + \ldots + b_n(z-a)^{-n} + \ldots\infty$$

$$= \sum_{n=1}^{\infty} b_1(z-a)^{-n} \qquad \ldots(1.9)$$

where, $b_n = \frac{1}{2\pi i}\int_{C_2}(w-a)^{n-1} f(w)\,dw$

(or) $b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$ for $n = 1, 2, 3, \ldots$

From Equations (1.7), (1.8) and (1.9), we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

Notes:

1. As $f(z)$ is not analytic inside C_1,

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}dw \neq \frac{f^n(a)}{n!}$$

2. However, if $f(z)$ is analytic inside C_1, then

$$b_n = 0 \text{ and } a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw = \frac{f^n(a)}{n!}$$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^n(a)$$

$$= f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\infty$$

which is Taylor's series.

$\therefore$ In this case Laurent's series reduces to Taylor's series.

3. To obtain Taylor's or Laurent's series, simply expand $f(z)$ by binomial theorem, instead of finding a_n by complex integration.

4. The part $\sum_{n=0}^{\infty} a_n(z-a)^n$, consisting of positive integral powers of $(z-a)$, is called the *analytic part* of the Laurent's series and the part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called the *principal part* of the Laurent's series.

Example 1.28: Find the Laurent's series about the indicated points for each of the following function:

(*i*) $f(z) = z^2 e^{\frac{1}{z}}$ about $z = 0$ (*ii*) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$

NOTES

Solution: (*i*) $f(z)$ is not analytic at $z = 0$.

$\therefore$ for $|z| > 0$

$$f(z) = z^2 e^{\frac{1}{z}} = z^2\left[1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \ldots\infty\right]$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2} + \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \ldots\infty$$

(*ii*) Put $z - 1 = t \Rightarrow z = 1 + t$.

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+t)}}{t^3} = \frac{e^2 \cdot e^{2t}}{t^3}$$

$$= \frac{e^2}{t^3}\left[1 + \frac{(2t)}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \ldots\infty\right]$$

$$= e^2\left[\frac{1}{t^3} + 2\cdot\frac{1}{t^2} + 2\cdot\frac{1}{t} + \frac{4}{3} + \frac{2}{3}t + \ldots\infty\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2}{3}(z-1) + \ldots\infty\right]$$

This is valid for $|z - 1| > 0$.

Example 1.29: Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region,

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 2$ (*iii*) $|z| > 2$ (*iv*) $0 < |z - 1| < 1$

Solution: $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{(z-2)} - \dfrac{1}{(z-1)}$

(*i*) If $|z| < 1$, then $\left|\dfrac{z}{2}\right| < 1$.

$$\therefore \quad f(z) = \frac{1}{2\left(\frac{z}{2} - 1\right)} - \frac{1}{-(1-z)} = -\frac{1}{2\left(1 - \frac{z}{2}\right)} + \frac{1}{(1-z)}$$

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right) + (1-z)^{-1}$$

$$= -\frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\infty\right] + \left[1 + z + z^2 + \ldots\infty\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n.$$

NOTES

(*ii*) If $1 < |z| < 2$, then $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$ and $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$.

$$\therefore\ f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\infty\right] - \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots\infty\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

This expansion is valid only if $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$, i.e., if $|z| < 2$ and $1 < |z|$, i.e., if $1 < |z| < 2$.

(*iii*) $|z| > 2$, i.e., $2 < |z|$

$$\Rightarrow \quad \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \qquad \left[\because \left|\frac{2}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < 1\right]$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\ldots\infty\right]$$

$$-\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots\infty\right]$$

This expansion is valid only if $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$, i.e., if $2 < |z|$ and $1 < |z|$, i.e., if $|z| > 2$.

$$\therefore \quad f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}(2^n - 1)\cdot\frac{1}{z^{n+1}}$$

(*iv*) $0 < |z - 1| < 1$

Put, $t = z - 1 \Rightarrow z = 1 + t$

$0 < |z - 1| < 1 \Rightarrow 0 < |t| < 1$

NOTES

$$\therefore \quad f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{(1+t-2)} - \frac{1}{t}$$

$$= \frac{1}{(t-1)} - \frac{1}{t}$$

$$= -\frac{1}{(1-t)} - \frac{1}{t}$$

$$= -\frac{1}{t} - (1-t)^{-1}$$

$$= -\frac{1}{t} - \left[1 + t + t^2 + t^3 + \ldots\infty\right]$$

$$= -\frac{1}{(z-1)} - \left[1 + (z-1) + (z-1)^2 + \ldots\infty\right]$$

Example 1.30: Expand $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ in a Laurent's series if,

(*i*) $|z| < 2$ (*ii*) $|z| > 3$ and (*iii*) $2 < |z| < 3$

Solution: $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)} = A + \dfrac{B}{(z+2)} + \dfrac{C}{(z+3)}$

$$= 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} \quad \text{[On resolving into partial fraction]}$$

(*i*) $|z| < 2 \Rightarrow \left|\dfrac{z}{2}\right| < 1$ and so $\left|\dfrac{z}{3}\right| < 1$

$$f(z) = 1 + \frac{3}{2\left(1 + \frac{z}{2}\right)} - \frac{8}{3\left(1 + \frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \ldots\infty\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \ldots\infty\right]$$

This expansion is valid only if $\left|\dfrac{z}{2}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$

i.e., $|z| < 2$ and $|z| < 3$

i.e., $|z| < 2$

$$f(z)=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

NOTES

(*ii*) $|z|>3 \quad \Rightarrow \quad 3<|z| \quad \Rightarrow \quad \left|\frac{3}{z}\right|<1$

$$f(z) = 1+\frac{3}{(z+2)}-\frac{8}{(z+3)}$$

$$= 1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)} \qquad \left[\because \left|\frac{3}{z}\right|<1 \Rightarrow \left|\frac{2}{z}\right|<1\right]$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\ldots\infty\right]-\frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\ldots\infty\right]$$

This expansion is valid only if $\left|\frac{2}{z}\right|<1$ and $\left|\frac{3}{z}\right|<1$, i.e., $2<|z|$ and $3<|z|$, i.e., if $|z|>2$ and $|z|>3$, i.e., if $|z|>3$.

(*iii*) $2<|z|<3$.

$2<|z| \quad \Rightarrow \quad \left|\frac{2}{z}\right|<1$ and $|z|<3 \quad \Rightarrow \quad \left|\frac{z}{3}\right|<1$

$$\therefore \qquad f(z) = 1+\frac{3}{(z+2)}-\frac{8}{(z+3)}$$

$$= 1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

Example 1.31: Find Laurent's expansion of $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ in $1<|z+1|<3$.

Solution: $\quad f(z) = \frac{7z-2}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{(z-2)}+\frac{C}{(z+1)}$

$$7z-2 = A\,(z-2)\,(z+1)+Bz\,(z+1)+Cz\,(z-2)$$

$z=2, \quad 12=6B \quad \Rightarrow \quad B=2$

$z=0, \quad -2=-2A \quad \Rightarrow \quad A=1$

$z=-1, \quad -9=3C \quad \Rightarrow \quad C=-3$

$$f(z)=\frac{1}{z}+\frac{2}{(z-2)}-\frac{3}{(z+1)} \qquad \text{[On resolving into partial fraction]}$$

Put $z + 1 = t \quad \Rightarrow \quad z = t - 1$

$1 < |z + 1| < 3 \Rightarrow 1 < |t| < 3$

$1 < |t| \Rightarrow \left|\frac{1}{t}\right| < 1$ and $|t| < 3 \Rightarrow \left|\frac{t}{3}\right| < 1$

NOTES

$$
\begin{aligned}
f(z) &= \frac{1}{(t-1)} + \frac{2}{(t-1-2)} - \frac{3}{t} \\
&= \frac{1}{(t-1)} + \frac{2}{(t-3)} - \frac{3}{t} \\
&= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{3\left(\frac{t}{3}-1\right)} - \frac{3}{t} \\
&= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t} \\
&= \frac{1}{t}\left[1+\frac{1}{t}+\left(\frac{1}{t}\right)^2+\ldots\infty\right] - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\ldots\infty\right] - \frac{3}{t} \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\left(\frac{1}{t}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n - \frac{3}{t} \\
&= \sum_{n=0}^{\infty}\frac{1}{t^{n+1}} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n - \frac{3}{t} \\
&= \sum_{n=0}^{\infty}\frac{1}{(1+z)^{n+1}} - \frac{2}{3}\sum\frac{(1+z)^n}{3^n} - \frac{3}{(1+z)}
\end{aligned}
$$

1.4.2 Isolated Singularities

Zeros of an Analytic Function

A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.

If $f(z_0) = 0$ and $f'(z_0) \neq 0$ then $z = z_0$ is called a zero of the first order or a simple zero of $f(z)$.

If $f(z_0) = f'(z_0) = f''(z_0) = \ldots. = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$ then z_0 is called zero of order n.

For example, $f(z) = z^2 \sin z$ has $z = 0$ and a zero of 3rd order and it has $z = n\pi$, a simple zero.

Singular Points

A point at which a function $f(z)$ ceases to be analytic is called a singular point or singularity of $f(z)$. For example, $z = 3$ is a singular point of $f(z) = \frac{1}{z-3}$.

Types of Singularities

1. Isolated singularity

2. Poles
3. Essential Singularity
4. Removable Singularity

NOTES

Isolated Singularity

A singular point $z = a$ of a function $f(z)$ is called an isolated singular point if there exists a circle with centre at $z = a$ which contains no other singular point of $f(z)$.

For example, $z = -2, 3$ are two isolated singular points of the function $f(z) = \dfrac{z^2 + z}{(z+2)(z-3)}$.

The function $f(z) = \dfrac{1}{\sin z}$ has an infinite number of isolated singular points at $z = \pm n\pi,\ n = 0, 1, 2, ...$.

When $z = a$ is an isolated singular point of $f(z)$, we can expand $f(z)$ in a Laurent's series about $z = a$.

$$\text{i.e.,}\quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + b_1 \cdot \frac{1}{(z-a)} + b_2 \frac{1}{(z-a)^2} + b_3 \frac{1}{(z-a)^3} + ...$$

Pole

If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z - a| < r_1$ has only a finite number of terms then $z = a$ is called a pole.

$$\text{i.e., if in}\qquad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$b_m \neq 0,\ b_{m+1} = b_{m+2} = ... = 0$, then $z = a$ is called a pole of order m.

A pole of order one is called a simple pole.

For example,

$z = 0$ is a simple pole of $f(z) = \dfrac{1}{z(z-1)^2}$,

$$\text{since}\quad f(z) = \frac{1}{z(z-1)^2} = \frac{1}{z}, (1-z)^{-2} = \frac{1}{z}(1 + 2z + 3z^2 + ...\infty)$$

$$= \frac{1}{z} + (2 + 3z + 4z^2 + ...\infty)$$

and the principal part contains only one term $\dfrac{1}{z}$.

Essential Singularity

If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's series of $f(z)$ at $z = a$, valid in $0 < |z - a| < r$, has an infinite number of terms then $z = a$ is called an essential singularity.

For example, $z = 3$ is an essential singularity of $f(z) = e^{\frac{1}{z-3}}$.

Since, $$e^{\frac{1}{z-3}} = 1 + \frac{1}{1!}\frac{1}{(z-3)} + \frac{1}{2!}\frac{1}{(z-3)} + \frac{1}{3!(z-3)^3} + \ldots\infty.$$

Similarly $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!z^2} + \ldots\infty$ has an essential singularily at $z = 0$.

Removable Singularity

If a single-valued function $f(z)$ is not defined at $z = a$, but $\left[\lim_{z\to a} f(z)\right]$ exists, then $z = a$ is called a removable singularity.

For example, $f(z) = \frac{\sin z}{z}$ is not defined at $z = 0$ but $\lim_{z\to 0}\frac{\sin z}{z} = 1$. The Laurent's series of $f(z)$ is given by,

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\infty\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\infty.$$

1.4.3 Meromorphic Functions

Entire Function or Integral function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

For example, e^z, $\cos z$, $\sin z$ are all integral functions.

Meromorphic Function

A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

For example,

$f(z) = \frac{z+3}{(z-1)(z+4)^3}$ is a meromorphic function since it fails to be analytic at $z = 1$ and -4.

1.5 THE ARGUMENT PRINCIPLE

Any function f is meromorphic in domain D, if it is analytic throughout D, except for possible poles. Let f be meromorphic in the domain interior to a positively oriented simple closed contour C and let it be analytic and non-zero on C. The image Γ of C under the transformation $w = f(z)$ is a closed contour, not necessarily simple in the w plane as shown in Figure 1.2. As a point z traverses C in the positive direction, its image w traverses Γ in a particular direction that determines the orientation of Γ. Now, since there are no zeros in f, the contour Γ does not pass through the origin in the w

NOTES

plane. Let w and w_0 be points on Γ, where w_0 is fixed and φ_0 is a value of Arg (w_0). Then, let Arg (w) vary continuously, starting with the value φ_0, as the point w begins at the point w_0 and traverses Γ once in the direction of orientation assigned to it by the mapping $w = f(z)$. When w returns to the point w_0, where it started, Arg (w) assumes a particular value of Arg (w_0), denoted by φ_1. Thus, the change in Arg (w) as w describes Γ once in its direction of orientation is $\varphi_1 - \varphi_0$. This change is independent of the point w_0 chosen to determine it. Since, w–$f(z)$, the number $\varphi_1 - \varphi_0$ is the change in argument of $f(z)$ as z describes C once in the positive direction, starting with a point z_0.

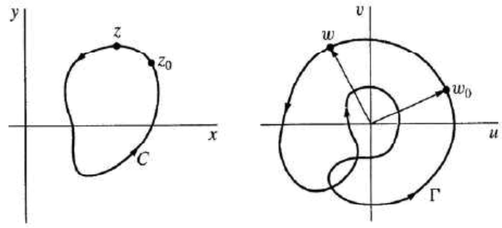


Fig. 1.1 *Closed Contour*

Therefore,

$$\Delta_C \arg f(z) = \varphi_1 - \varphi_0$$

The value of $\Delta_C \arg f(z)$ is an integral multiple of 2π and the integer,

$$\frac{1}{2\pi}\Delta_C \arg f(z)$$

represents the number of times that the point w winds around the origin in the plane. This integer is sometimes called the winding number of Γ with respect to the origin $w = 0$. It is positive, if Γ winds around the origin in the anti-clockwise direction and negative, if it winds clockwise around that point. The winding number is always zero, when Γ does not enclose the origin. We can determine winding number from the number of zeros and poles of f interior to C. The number of poles is necessarily finite since the accumulation points of the poles must not be isolated singular points. Likewise, the zeros of f are finite in number. Now, suppose that, f has Z zeros and P poles in the domain interior to C, where we agree that, f has m_0 zeros at a point z_0 if it has a zero of order m_0 there and if f has a pole of order m_p at z_0, that pole is to be counted m_p times. The argument principle states that the winding number is simply the difference Z–P.

Theorem 1.10: Suppose that a function $f(z)$ is meromorphic in the domain interior to a positively oriented simple contour C, $f(z)$ is analytic and non zero on C, and counting multiplicities, Z is the number of zeros and P is the number of poles of $f(z)$ inside C. Then,

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P \qquad \ldots(1.10)$$

Proof: To prove this, we will evaluate the integral of $f'(z)/f(z)$ around C in two ways.

First, let $z = z(t)$ $(a \le t \le b)$ be a parametric representation for C, so that,

NOTES

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]}dt \quad \text{...(1.11)}$$

Since, under the transformation $w = f(z)$, the image Γ of C never passes through the origin in the w plane, the image of any point $z = z(t)$ on C can be expressed in exponential form as,

$$w = \rho(t)\exp[i\varphi(t)]$$

Thus,

$$f[z(t)] = \rho(t)\exp^{i\varphi(t)} \qquad (a \le t \le b) \quad \text{...(1.12)}$$

and along each of the smooth arcs making up the contour Γ, it follows that,

$$f'[z(t)]z'(t) = \frac{d}{dt}f[z(t)] = \frac{d}{dt}[\rho(t)e^{i\phi(t)}] = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\theta(t)}\phi'(t)$$

$$\text{... (1.13)}$$

Since, $\rho'(t)$ and $\varphi'(t)$ are piecewise continuous on the interval $a \le t \le b$, from Equations (1.12) and (1.13) we can write integral of Equation (1.11) as follows:

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{\rho'(t)}{\rho(t)}dt + i\int_a^b \phi'(t)dt = \ln\rho(t)\Big|_a^b + i\phi(t)\Big|_a^b$$

But, $\quad \rho(b) = \rho(a)$ and $\phi(b) - \phi(a) = \Delta_C \arg f(z)$

Hence, $\quad \displaystyle\int_C \frac{f'(z)}{f(z)}dz = i\Delta_C \arg f(z)$ ...(1.14)

Another way to evaluate the integral in Equation (1.14) is to use Cauchy's residue theorem.

If f has a zero of order m_0 at z_0, then

$$f(z) = (z - z_0)^{m_0} g(z) \quad \text{...(1.15)}$$

where $g(z)$ is analytic and non-zero at z_0.

Hence,

$$f'(z_0) = m_0(z - z_0)^{m_0 - 1} g(z) + (z - z_0)^{m_0} g'(z)$$

And so,

$$\frac{f'(z)}{f(z)} = \frac{m_0}{z - z_0} + \frac{g'(z)}{g(z)} \quad \text{...(1.16)}$$

Since $g'(z)/g(z)$ is analytic at z_0, it has a Taylor series representation about that point, and so Equation (1.16) implies that $f'(z)/f(z)$ has a simple pole at z_0, with residue m_0.

If, on the other hand, f has a pole of order m_p at z_0, then,

$$f(z) = (z - z_0)^{-m_p} \varphi(z) \quad \ldots(1.17)$$

NOTES

where, $\varphi(z)$ is analytic and non-zero at z_0. Equations (1.17) and (1.15) have the same form, with the positive integer m_0 in Equation (1.15) replaced by $-m_p$. Hence, it is clear from Equation (1.16) that $f'(z)/f(z)$ has a simple pole at z_0, with residue $-m_p$.

Thus, we observe that, the integrand $f'(z)/f(z)$ is analytic inside and on C except at the points inside C at which the zeros and poles of f occur.

Let $\alpha_1, \alpha_2, ..., \alpha_s$ be the zeros of f of orders $m_1, m_2, ..., m_s$, respectively inside C, and $\beta_1, \beta_2, ..., \beta_t$ be the poles of f of orders $n_1, n_2, ..., n_t$, respectively, inside C.

From residue theorem,

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i[\sum_{k=1}^{t} \operatorname{Res}_{z=\alpha_k} \frac{f'(z)}{f(z)} + \sum_{k=1}^{t} \operatorname{Res}_{z=\beta_k} \frac{f'(z)}{f(z)}]$$

$$= 2\pi i[\sum_{k=1}^{t} m_k + \sum_{k=1}^{t}(-n_k)] \quad \ldots(1.18)$$

$$= 2\pi i(Z - P)$$

Equation (1.10) follows by equating the right hand sides of Equations (1.14) and (1.18).

Hence, the theorem is proved.

Example 1.32: The only singularity of the function $1/z^2$ is a pole of order 2, at the origin, and there are no zeros in the finite plane. In particular, this function is analytic and non-zero on the unit circle,

$$z = e^{i\theta} (0 \le \theta \le 2\pi)$$

If we let, C denote that positively oriented circle, then from the argument principle,

$$\frac{1}{2\pi} \Delta_C \arg\left(\frac{1}{z^2}\right) = -2$$

So, the image Γ of C under the transformation $w = 1/z^2$ winds around the origin $w = 0$ twice in the clockwise direction. This can be verified directly by noting that Γ has the parametric representation,

$$w = e^{-i2\theta} (0 \le \theta \le 2\pi)$$

1.5.1 Rouche's Theorem

The Rouche's theorem is a consequence of the argument principle and is useful in locating regions of the complex plane in which, a given analytic function has zeros.

Theorem 1.11 (Rouche's Theorem): If f and g are analytic inside and on a simple closed contour C, and $|f(z)| > |g(z)|$ at each point on C, then f and $f + g$ have the same number of zeros, counting multiplicities inside C.

NOTES

Proof: Let the orientation be positive. Assume that neither the function $f(z)$ nor the sum of $f(z) + g(z)$ have a zero on C.

Since,

$|f(z)| > |g(z)| \geq 0$ and $|f(z) + g(z)| \geq ||f(z)| - |g(z)|| \geq 0$

where z is on C.

If Z_f and Z_{f+g} denote the number of zeros, counting multiplicities of $f(z)$ and $f(z) + g(z)$ respectively inside C, then from the argument principle we have,

$$Z_f = \frac{1}{2\pi}\Delta_C \arg f(z)$$

and,

$$Z_{f+g} = \frac{1}{2\pi}\Delta_C \arg\,[f(z)+g(z)]$$

Hence since,

$$\Delta_C \arg[f(z)+g(z)] = \Delta_C \arg\left\{f(z)\left[1+\frac{g(z)}{f(z)}\right]\right\}$$

$$= \Delta_C \arg f(z) + \Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right]$$

Therefore,

$$Z_{f+g} = Z_f + \frac{1}{2\pi}\Delta_C \arg F(z) \qquad \ldots (1.19)$$

Where, $F(z) = 1 + \dfrac{g(z)}{f(z)}$

But,

$$|F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1$$

So, under the map $w = F(z)$, the image of C lies in the open disc $|w-1| < 1$ but does not enclose the origin $w = 0$. Hence, $\Delta_C \arg F(z) = 0$. Since, Equation (1.19) reduces to $Z_{f+g} = Z_f$, the proof is completed.

For example, in order to determine the number of roots of the equation, $z^7 - 4z^3 + z - 1 = 0$ inside the circle $|z|-1$, write $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$.

Now, observe that, when $|z|=1$, we have

$|f(z)| = 4|z|^3 = 4$

and,

$|g(z)| \le |z|^7 + |z| + 1 = 3$

The conditions in Rouche's theorem are now satisfied.

NOTES

Consequently, since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z|=1$, so does $f(z) + g(z)$. Thus, the equation has three roots there.

1.5.2 Inverse Function Theorem

Theorem 1.12: If f is derivable at x and is one-one on some neighbourhood of x then the inverse function of f is derivable at $f(x)$ and $(f^{-1})'(f(x)) = 1/f'(x)$, provided $f'(x) \neq 0$.

Proof: For $f'(x)$ ¹ 0, $$\frac{1}{f'(x)} = \lim_{h\to 0} \frac{h}{f(x+h)-f(x)}$$

$$= \lim_{h\to 0} \frac{f^{-1}(f(x+h)-f^{-1}(f(x))}{f(x+h)-f(x)}$$

In view of the continuity of $f(x)$ at x and its one-oneness f^{-1} exists on some neighbourhood of $f(x)$ and $y = f(x+h) \to f(x)$ as $h \to 0$. Consequently, $(f^{-1})'(f(x))$ exists and

$$\frac{1}{f'(x)} = \lim_{y\to f(x)} \frac{f^{-1}(y)-f^{-1}(f(x))}{y-f(x)} = (f^{-1})'(f(x))$$

From the Figure 1.3, note that the tangent at the point $(x, f(x))$ on $y = f(x)$, if makes an angle 'α' to the line $y = x$ then so does tangent at $(f(x), x)$ on $y = f^{-1}(x)$ as shown.

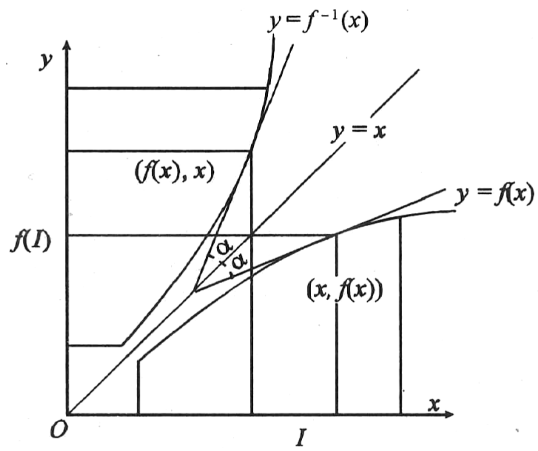


Fig. 1.3

Thus,

$$f'(x) = \tan\left(\frac{\pi}{4} - \alpha\right), \{f^{-1}(y)\}'_{y=f(x)} = \tan\left(\frac{\pi}{4} + \alpha\right)$$

i.e., $f'(x)\left\{f^{-1}(y)\right\}'_{y=f(x)} = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}+\alpha\right)=1$

Hence, $\left\{f^{-1}(y)\right\}'_{y=f(x)} = \frac{1}{f'(x)}$, provided $f'(x) \neq 0$.

The above provides an interpretation for the value of the derivative of the inverse function f^{-1}.

Example 1.33

(i) $\frac{d}{dx}\log x = \frac{1}{x}\,\forall x \in \mathbf{R}^+$,

(ii) $\frac{d}{dx}x^{\lambda} = \lambda\, x^{\lambda-1}\;\forall\;\lambda \in \mathbf{R}$ and $x \neq 0$, $\lambda > 1$ when $x = 0$

Solution: (i) Since $(e^x)' = e^x\;\forall\; x \in \mathbf{R}$ and e^x is one-one on $\mathbf{R}$ with range $(0, \infty)$, therefore, its inverse function $\log x$ is derivable at all points e^x and

$\left(\frac{d}{dy}\log y\right)_{y=e^x} = \frac{1}{e^x}$, which is same as $\left(\frac{d}{dy}\log x\right)_{y=e^x} = \frac{1}{x}\forall x \in \mathbf{R}^+$.

(ii) We have $(x^{\lambda})' = (e^{\lambda \log x})' = e^{\lambda} \log x\, \lambda\, \frac{1}{x}$

$\lambda x^{\lambda-1}\forall\lambda \in \mathrm{R}$ and $x \# 0$.

If $x = 0$ and $\lambda > 1$, then $(x^{\lambda})' = \lim_{h\to 0}\frac{h^{\lambda}-0}{h} = 0$

It can now be seen that there are functions which have discontinuous derivatives.

Example 1.34: Let $f(x) = \tan x$, $0 \leq x < \pi/2$. Then at $y = \tan x$.

$$(f^{-1})'y = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+y^2}$$

$\therefore \quad (\tan^{-1} x)' = \frac{1}{1+x^2}$, $0 \leq x < \infty$

Example 1.35: If f is defined on $\mathbf{R}$ by $f(x) = x^2 \sin 1/x$, if $x \neq 10$ and $f(0) = 0$, then $f'(x)$ exists $\forall\; x \in \mathbf{R}$. But $f'(x)$ is discontinuous at $x = 0$.

Solution : It easily follows that $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, if $x \neq 0$ and

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Thus, $f'(x)$ is continuous on $\mathbf{R}^+ \cup \mathbf{R}^-$ and since $\lim_{x\to 0} f'(x)$ does not exist, $f'(x)$ is discontinuous at $x = 0$.

NOTES

The functions, continuous or discontinuous, which are derivative of some function possess interesting properties.

Check Your Progress

12. Write the statement of Schwarz lemma.
13. Which is the analytic and principal part of Laurent's series?
14. Define the zero of an analytic function.
15. Give examples of integral functions.
16. State Rouche's theorem.
17. Give the statement of inverse function theorem.

1.6 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Let $z = x + iy$ be a point in the Argand's plane where $x = \varphi(t)$, $y = \psi(t)$ are functions of a parameter t. If $\phi(t)$ and $\psi(t)$ are continuous and $t \in [\alpha, \beta] \subset \mathbb{R}$, then z traces out a continuous arc. If the curve crosses itself at a point, i.e., if at two or more values of f, z assumes the same value, the corresponding point is called a multiple point.

2. Let $f(z)$ be an analytic function of z and $f'(z)$ be continuous at each point within and on a closed contour C; then,

$$\int_C f(z)\,dz = 0$$

3. If all the points of the area bounded by two or more closed curves drawn in the region R, are the points of R, then the region R is said to be multi-connected region.

4. If $f(z)$ is analytic inside and on a simple closed curve C of a simply-connected region R and if 'a' is any point within C, then $f(a) = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{z-a}dz,$ where C is described in the anticlockwise sense.

5. Let f be a differentiable function, and let f' be its derivative. The derivative of f' (if it has one) is written f'' and is called the second derivative of f. Similarly, the derivative of the second derivative, if it exists, is written f''' and is called the third derivative of f. Continuing this process, one can define, if it exists, the n^{th} derivative as the derivative of the $(n-1)^{th}$ derivative. These repeated derivatives are called higher-order derivatives. The n^{th} derivative is also called the derivative of order n.

NOTES

6. Morera's theorem, named after Giacinto Morera, gives an important criterion for proving that a function is holomorphic. Morera's theorem states that a continuous, complex-valued function f defined on an open set D in the complex plane that satisfies

$$\oint_{\gamma} f(z)\, dz = 0$$

7. Let $f(z)$ be an analytic function within circle C, given by $| z - a | = R$ and if $| f(z)$ £ M on C, then

$$| f^n(a) | \text{ £ } \frac{M|n}{R^n}.$$

8. Liouville's theorem, named after Joseph Liouville, states that every bounded entire function must be constant. That is, every holomorphic function f for which there exists a positive number M such that, $| f(z) | \leq M$ for all z in $\mathbb{C}$ is constant.

9. Every polynomial of degree $n \geq 1$ with complex coefficients has a zero in C.

10. If $a = 0$, then Taylor's series becomes

$$f(z) = f(0) + f'(0) + \frac{z^2}{2!} f''(0) + ... + \frac{z^n}{n!} f^n(0) + ...\infty$$

This series is called Maclaurin's series.

11. Let $G \subset C$ be a connected open set and $f{:}G \to C$ be analytic. If there is any $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

12. Let D be the open unit disc in the complex plane C, i.e., $D = \{z \in C; |z| < 1\}$. Let f: $D \to D$ be a holomorphic mapping such that $f(0) = 0$.

    Then, the classical Schwarz lemma states that, $|f(z)| \leq |z|$ for $z \in D$ and $|f$¢$(0)|$ £ 1, and the equality $|f$¢$(0)| = 1$ or the equality $|f(z)| = |z|$ at a single point $z \neq 0$ implies $f(z) = \varepsilon z$ with $| \varepsilon | = 1$.

13. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$, consisting of positive integral powers of $(z - a)$, is called the analytic part of the Laurent's series and the part $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ consisting of negative integral powers of $(z - a)$ is called the principal part of the Laurent's series.

14. A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.

NOTES

15. e^z, cos z and sin z are all integral functions.
16. If f and g are analytic inside and on a simple closed contour C, and $|f(z)| > |g(z)|$ at each point on C, then f and $f + g$ have the same number of zeros, counting multiplicities inside C.
17. If f is derivable at x and is one-one on some neighbourhood of x then the inverse function of f is derivable at $f(x)$ and $(f^{-1})'(f(x)) = 1/f'(x)$, provided $f'(x) \neq 0$.

1.7 SUMMARY

- Let $z = x + iy$ be a point in the Argand's plane where $x = \varphi(t)$, $y = \psi(t)$ are functions of a parameter t. If $\phi(t)$ and $\psi(t)$ are continuous and $t \in [\alpha, \beta] \subset \mathbb{R}$, then z traces out a continuous arc.
- If at two or more values of f, z assumes the same value, the corresponding point is called a *multiple point*
- A continuous arc without multiple points is called a Jordan arc.
- A Jordan curve is one which is made of a continuous chain of finite number of Jordan arcs.
- A contour is a closed Jordan curve, i.e., a Jordan curve whose starting point is the same as its end point.
- A region R is said to be connected region if any two points of R can be connected by a curve which lies entirely within the region.
- A connected region R is said to be a simply-connected region if all the interior points of a closed curve C drawn in R are the points of R.
- If all the points of the area bounded by two or more closed curves drawn in the region R, are the points of R, then the region R is said to be multi-connected region.
- The lines drawn in a multi-connected region, without intersecting any one of the curves which make a multi-connected region a simply-connected one, are called cuts or cross-cuts.
- If $f(z)$ is analytic inside and on a simple closed curve C of a simply-connected region R and if 'a' is any point within C, then

  $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz,$ where C is described in the anticlockwise sense.
- An analytic function at any interior point of a region R in terms of its values on the boundary of the region.
- A function that maps every point x to the value of the derivative of f at x. This function is written f' and is called the derivative function or the derivative of f.

NOTES

- Morera's theorem states that a continuous, complex-valued function f defined on an open set D in the complex plane that satisfies

$$\oint_{\gamma} f(z)\, dz = 0$$

- Liouville's theorem, named after Joseph Liouville, states that every bounded entire function must be constant.
- Every polynomial of degree n$\geq$1 with complex coefficients has a zero in C.
- If $a = 0$, then Taylor's series becomes

$$f(z) = f(0) + f'(0) + \frac{z^2}{2!} f''(0) + ... + \frac{z^n}{n!} f^n(0) + ...\infty$$

This series is called Maclaurin's series.
- Maximum Modulus principle has many uses in complex analysis. It finds applications in the fundamental theorem of algebra and Schwarz's lemma.
- To obtain Taylor's or Laurent's series, simply expand $f(z)$ by binomial theorem, instead of finding a_n by complex integration.
- A point at which a function $f(z)$ ceases to be analytic is called a singular point or singularity of $f(z)$. For example, $z = 3$ is a singular point of $f(z) = \frac{1}{z-3}$.
- A singular point $z = a$ of a function $f(z)$ is called an isolated singular point if there exists a circle with centre at $z = a$ which contains no other singular point of $f(z)$.
- If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z - a| < r_1$ has only a finite number of terms then $z = a$ is called a pole.
- If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's series of $f(z)$ at $z = a$, valid in $0 < |z - a| < r$, has an infinite number of terms then $z = a$ is called an essential singularity.
- If a single-valued function $f(z)$ is not defined at $z = a$, but $\left[\lim_{z \to a} f(z)\right]$ exists, then $z = a$ is called a removable singularity.
- A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.
- Rouche's Theorem is a consequence of the argument principle and is useful in locating regions of the complex plane in which, a given analytic function has zeros.

NOTES

1.8 KEY TERMS

- **Continuous arc:** Let $z = x + iy$ be a point in the Argand's plane where $x = \varphi(t)$, $y = \psi(t)$ are functions of a parameter t. If $\phi(t)$ and $\psi(t)$ are continuous and $t \in [\alpha, \beta] \subset \mathbb{R}$, then z traces out a *continuous arc.* If at two or more values of f, z assumes the same value, the corresponding point is called a *multiple point.*

- **Cauchy's theorem:** Let $f(z)$ be an analytic function of z and $f'(z)$ be continuous at each point within and on a closed contour C; then

$$\int_C f(z)\,dz = 0$$

- **Connected region:** A region R is said to be connected region if any two points of R can be connected by a curve which lies entirely within the region.

- **Maclaurin's series:** If $a = 0$, then Taylor's series becomes

$$f(z) = f(0) + f'(0) + \frac{z^2}{2!} f''(0) + \ldots + \frac{z^n}{n!} f^n(0) + \ldots\infty$$

  This series is called Maclaurin's series**.**

- **Singular point:** A point at which a function $f(z)$ ceases to be analytic is called a singular point or singularity of $f(z)$. For example $z = 3$ is a singular point of $f(z) = \dfrac{1}{z-3}$.

- **Pole:** If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z - a| < r_1$ has only a finite number of terms then $z = a$ is called a pole.

- **Essential singularity**: If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's series of $f(z)$ at $z = a$, valid in $0 < |z - a| < r$, has an infinite number of terms then $z = a$ is called an essential singularity.

- **Removable singularity:** If a single-valued function $f(z)$ is not defined at $z = a$, but $\left[\lim_{z \to a} f(z)\right]$ exists, then $z = a$ is called a removable singularity.

- **Meromorphic function:** A function $f(z)$ which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

1.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Define a closed contour.
2. State Cauchy-Goursat theorem.
3. Distinguish Cauchy's integral formula and Cauchy's inequality.
4. Define the higher derivatives in complex integration.
5. What is Morera's theorem?
6. State Liouville's theorem.
7. Where is the fundamental theorem of algebra used?
8. State the Taylor's theorem.
9. What is the application of maximum modulus principle?
10. State a use of Schwarz lemma.
11. What is the relation between Laurent's series and Taylor's series?
12. Find the nature and location of singularities of the function $e^{1/z}$.
13. What is the use of Rouche's theorem?
14. Where is inverse function theorem applied?

Long-Answer Questions

1. Describe analytic function giving examples.
2. State the sufficient conditions that will ensure the analyticity of a function, $w = f(z) = u + iv$.
3. State C-R equations in polar coordinates satisfied by an analytic function giving appropriate examples.
4. Show that, $w = \log z$ is analytic everywhere except at the origin and find its derivative giving appropriate examples.
5. If $u + iv$ is analytic show that $v - iu$ and $-v + iu$ are also analytic.
6. Prove that Cauchy-Riemann equations are satisfied along the curve $x - y = 1$ for the function $f(z) = (x - y)^2 + 2i(x + y)$.
7. Describe the higher derivatives in complex integration. Give appropriate examples.
8. Explain the Morera’s theorem giving examples.
9. Elaborate on the Liouville’s theorem.
10. Discuss about the maximum modulus principle.

NOTES

11. Explain Schwarz lemma with the help of examples.
12. Briefly explain the argument principle.
13. Describe Rouche' theorem with the help of examples.
14. Discuss the applications of inverse function theorem.

1.10 FURTHER READING

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Ahlfors, Lars V. 1978. *Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Lang, Serge. 1998. *Complex Analysis*, 4th Edition. NY: Springer-Verlag New York Inc.

Shastri, Anant R. 2000. *An Introduction to Complex Analysis*, India: Macmillan Publishers India Ltd.

UNIT 2 RESIDUES

NOTES

Structure

2.0 INTRODUCTION

In mathematics, more specifically complex analysis, the residue is a complex number proportional to the contour integral of a meromorphic function along a path enclosing one of its singularities. Where's real analysis is rich with plenty of results that give an inside into the topological character of function, complex analysis highlights a multitude of stunning results that suit the natural phenomena. So far engineering and technology point of view the study of complex analysis is of immense importance. In complex analysis, the residue theorem, sometimes called Cauchy's residue theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals and infinite series as well. It generalizes the Cauchy integral theorem and Cauchy's integral formula. From a geometrical perspective, it can be seen as a special case of the generalized Stokes' theorem.

In this unit, you will learn about the residues, Cauchy's residue theorem, evaluation of integrals and branches of many valued functions.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic concept of residue
- Know Cauchy's residue theorem
- Evaluate integrals
- Discuss about the branches of many valued functions

2.2 RESIDUES

If $z = a$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z = a$.

i.e., $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

NOTES

The coefficient b_1 of $\frac{1}{(z-a)}$ in the Laurent's series of $f(z)$, is called the residue of $f(z)$ at $z = a$.

From Laurent's series,

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

$$\therefore \quad b_1 = \frac{1}{2\pi i} \int_{C_2} f(z)\, dz$$

$\therefore$ $[\text{Res } f(z)]_{z=a} = \frac{1}{2\pi i} \int_C f(z)\, dz$ where C is any closed curve around 'a' such that $f(z)$ is analytic inside and on C execpt at $z = a$.

Formulae for the Evaluation of Residues

1. If $z = a$ is a simple pole of $f(z)$, then

$$[\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z-a)\, f(z)$$

Since $z = a$ is a simple pole of $f(z)$, then the Laurent's series of $f(z)$ is of the form,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)}$$

$$\therefore \quad (z-a)\, f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$\therefore$ $\lim_{z \to a} (z-a)\, f(z) = 0 + b_1$ = Residue of $f(z)$ at $z = a$.

$$\therefore \quad [\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z-a)\, f(z)$$

2. If $z = a$ is simple pole of $f(z) = \frac{P(z)}{Q(z)}$, then

$$[\text{Res } f(z)]_{z=a} = \lim_{z \to a} \left[\frac{P(z)}{Q'(z)} \right] = \frac{P(a)}{Q'(a)}.$$

By the previous formula,

$$[\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z-a)\, f(z)$$

$$= \lim_{z \to a} (z-a) \frac{P(z)}{Q(z)} \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \to a} \left[\frac{(z-a)\, P'(z) + P(z)}{Q'(z)} \right], \text{ by L' Hopital rule}$$

NOTES

$$= \frac{P(a)}{Q'(a)} \text{ or } \lim_{z \to a} \left[\frac{P'(z)}{Q'(z)} \right]$$

3. If $z = a$ is a pole of order 'm', then

$$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right\}$$

Since $z = a$ is a pole of order m, then the Laurent's expansion of $f(z)$ is of the form,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_{m-1}}{(z-a)^{m-1}} + \frac{b_m}{(z-a)^m}$$

$$\therefore \quad (z-a)^m f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_1 (z-a)^{m-1} + b_2 (z-a)^{m-2} + \ldots$$

$$+ b_{m-1}(z-a) + b_n$$

$$\therefore \quad \lim_{z \to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right\} = (m-1)!\, b_1$$

$$\therefore \; b_1 = [\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right\}$$

Note: The residue at an essential singularity of $f(z)$ is found out using the Laurent's expansion of $f(z)$ directly.

2.2.1 Cauchy's Residue Theorem

Theorem 2.1: If $f(z)$ is analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $a_1, a_2, \ldots, a_n$ inside C, then

$$\int_C f(z)\, dz = 2\pi i \text{ [Sum of the residues of } f(z) \text{ at } a_1, a_2, \ldots, a_n]$$

$$= 2\pi i [R_1 + R_2 + \ldots + R_n]$$

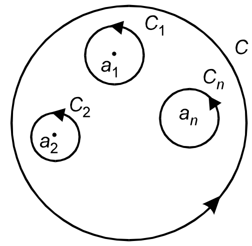
where $R_1, R_2, \ldots, R_n$ are the residues of $f(z)$ at $z = a_1, z = a_2, \ldots, z = a_n$ respectively.

Proof: We enclose the singularities $a_1, a_2, \ldots, a_n$ by small non-intersecting circles $C_1, C_2, \ldots, C_n$ with centres at $a_1, a_2, \ldots, a_n$ and raidii $r_1, r_2, \ldots, r_n$ lying wholly inside C.

Then $f(z)$ is analytic in the multiply connected region enclosed by the curves $C, C_1, C_2, \ldots, C_n$. Hence by Cauchy's extension of integral theorem,

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \ldots + \int_{C_n} f(z)\, dz$$

NOTES



$$= 2\pi i\, R_1 + 2\pi i\, R_2 + \ldots + 2\pi i\, R_n,$$

(By the property of Residue at a point)

$$= 2\pi i\,[R_1 + R_2 + \ldots + R_n].$$

Example 2.1: Find the nature of singularity for the following functions:

$(i)\ \dfrac{e^z}{(z-1)^4}$ $\quad (ii)\ \sin\left(\dfrac{1}{z+1}\right)$ $\quad (iii)\ \dfrac{1}{z(1-e^z)}$ $\quad (iv)\ \dfrac{\cot \pi z}{(z-a)^3}$

Solution: (i) Put $z - 1 = t \Rightarrow z = 1 + t$

$$\frac{e^z}{(z-1)^4} = \frac{e^{1+t}}{t^4} = e \cdot \frac{e^t}{t^4} = \frac{e}{t^4}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\infty\right]$$

$$= e\left[\frac{1}{t^4} + \frac{1}{t^3} + \frac{1}{2!}\cdot\frac{1}{t^2} + \frac{1}{3!}\cdot\frac{1}{t} + \frac{1}{4!} + \frac{t}{5!} + \ldots\infty\right]$$

$$= e\left[\frac{1}{(z-1)^4} + \frac{1}{(z-1)^3} + \frac{1}{2!}\cdot\frac{1}{(z-1)^2} + \frac{1}{3!}\cdot\frac{1}{(z-1)} + \frac{1}{4!} + \frac{(z-1)}{5!} + \ldots\infty\right]$$

Since there are finite number of terms containing negative powers of $(z-1)$,

$z = 1$ is a pole of order '4'.

$$(ii)\ \sin\left(\frac{1}{z+1}\right) = \frac{1}{(z+1)} - \frac{\left(\frac{1}{z+1}\right)^3}{3!} + \frac{\left(\frac{1}{z+1}\right)^5}{5!} - \ldots\infty$$

$$= \frac{1}{(z+1)} - \frac{1}{3!}\frac{1}{(z+1)^3} + \frac{1}{5!}\frac{1}{(z+1)^5} - \ldots\infty.$$

Since there are an infinte number of terms containing negative powers of $z + 1$, $z = -1$ is an essential singularity.

(iii) $\dfrac{1}{z(1-e^z)}$. Poles are obtained by equating the denominator (Dr) to zero.

$$\text{Dr} = 0 \quad\Rightarrow\quad z(1-e^z) = 0$$

$$\Rightarrow z = 0 \quad\text{and}\quad 1 - e^z = 1$$

$$1 - e^z = 0 \quad\Rightarrow\quad e^z = 1$$

$$\Rightarrow \quad z = i2n\pi \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

Already $z = 0$.

$\therefore$ $z = 0$ is a pole of order 2 and

$z = i2n\pi$, $\quad n = \pm 1, \pm 2, \ldots$ are simple poles.

$(iv)\quad f(z)=\dfrac{\cot \pi z}{(z-a)^3}=\dfrac{\cos \pi z}{\sin \pi z\,(z-a)^3}$

$$(z-a)^3 \sin \pi z = 0 \quad \Rightarrow \quad (z-a)^3 = 0,\ \sin \pi z = 0$$

$z = a$ is a pole of order 3.

$\sin \pi z = 0 \ \Rightarrow\ \pi z = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \qquad z = n$

i.e., $\qquad z = 0, \pm 1, \pm 2, \ldots$ which are simple poles.

Example 2.2: Find the nature and location of singularities of the following functions:

$(i)\ \dfrac{z-\sin z}{z^3} \qquad (ii)\ (z+1)\sin\left(\dfrac{1}{z-3}\right) \qquad (iii)\ \dfrac{1}{\cos z-\sin z}$

Solution: (i) Equating to zero the Dr,

$$z^3 = 0 \quad \Rightarrow \quad z = 0$$

$z = 0$ is a singularity.

Also
$$\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\infty\right)\right]$$
$$=\frac{1}{z^3}\left[\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}+\ldots\infty\right]$$
$$=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\ldots$$

Since there are no negative powers of z in this expansion, $z = 0$ is a removable singularity.

$(ii)\ f(z)=(z+1)\sin\left(\dfrac{1}{z-3}\right).$

Put, $t = z-3 \Rightarrow z = t+3$

$$\therefore \qquad f(z)=(z+1)\sin\left(\frac{1}{z-3}\right)=(t+4)\sin\left(\frac{1}{t}\right)$$

$$=(t+4)\left[\frac{1}{t}-\frac{\left(\frac{1}{t}\right)^3}{3!}+\frac{\left(\frac{1}{t}\right)^5}{5!}-\frac{\left(\frac{1}{t}\right)^7}{7!}+\ldots\infty\right]$$

$$=\left[1-\frac{1}{3!}\frac{1}{t^2}+\frac{1}{5!}\frac{1}{t^4}-\frac{1}{7!}\frac{1}{t^6}+\ldots\infty\right]+\left[\frac{4}{t}-\frac{4}{3!}\frac{1}{t^3}+\frac{4}{5!}\frac{1}{t^5}-\frac{4}{7!}\frac{1}{t^7}+\ldots\infty\right]$$

$$=1+\frac{4}{t}-\frac{1}{3!}\frac{1}{t^2}-\frac{4}{3!}\frac{1}{t^3}+\frac{1}{5!}\frac{1}{t^4}+\ldots\infty$$

$$=1+\frac{4}{(z-3)}-\frac{1}{3!}\frac{1}{(z-3)^2}-\frac{4}{3!}\frac{1}{(z-3)^3}+\ldots\infty$$

Since there are infinite number of terms in the negative powers of $(z-3)$, $z = 3$ is an essential singularity.

NOTES

(*iii*) $f(z) = \dfrac{1}{\cos z - \sin z}$. Poles are given by equating the denominator to zero,

$$\cos z - \sin z = 0$$

$$\cos z = \sin z$$

$$1 = \frac{\sin z}{\cos z} = \tan z$$

$$\Rightarrow \qquad z = \pi/4$$

$z = \dfrac{\pi}{4}$ is a simple poles of $f(z)$.

Example 2.3: Expand each of the following functions in Laurent's series about $z = 0$. Identify the type of singularity also.

Solution: (*i*) $z^2 e^{-z}$ (*ii*) $\dfrac{1}{z} e^{-2z}$ (*iii*) $(z-1)\cos\left(\dfrac{1}{z}\right)$ (*iv*) $\dfrac{\sin z}{z}$

(*i*)
$$z^2 e^{-z} = z^2\left[1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + ...\infty\right]$$

$$= z^2 - \frac{z^3}{1!} + \frac{z^4}{2!} - \frac{z^5}{3!} + ..\infty \qquad ...(1)$$

The Laurent's series of Equation (1) does not contain negative powers of z and the circle of convergence $|z| = \infty$ does not include any singularity.

$\therefore$ $z = 0$ is an ordinary point of $z^2 e^{-z}$.

(*ii*)
$$\frac{1}{z} e^{-2z} = \frac{1}{z}\left[1 - \frac{2z}{1!} + \frac{(2z)^2}{2!} - \frac{(2z)^3}{3!} + ...\infty\right]$$

$$= \frac{1}{z} - \frac{2}{1!} + \frac{2^2 z}{2!} - \frac{2^3 z^2}{3!} + ...\infty \qquad ...(2)$$

The principal part of the Laurent's series contains the only term $\dfrac{1}{z}$.

$\therefore$ $z = 0$ is a simple pole of $z^{-1} e^{-2z}$.

(*iii*)
$$(z-1)\cos\left(\frac{1}{z}\right) = (z-1)\left[1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \frac{\left(\frac{1}{z}\right)^6}{6!} + ...\infty\right]$$

$$= \left[z - \frac{1}{2!}\left(\frac{1}{z}\right) + \frac{1}{4!}\left(\frac{1}{z^3}\right) - \frac{1}{6!}\left(\frac{1}{z^5}\right) + ...\infty\right]$$

$$- \left[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \frac{1}{6!}\frac{1}{z^6} + ...\infty\right]$$

$$= -1 + z - \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} - \frac{1}{4!}\cdot\frac{1}{z^4} + ...\infty.$$

This is the required Laurent's series.

The principal part consists of an infinite numbers of terms in the Laurent's series.

$\therefore$ $z = 0$ is an essential singularity.

NOTES

(*iv*) $\dfrac{\sin z}{z} = \dfrac{1}{z}\left[z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + ...\infty\right]$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^5}{7!} + ...\infty .$$

This is the required Laurent's series.

Though $z = 0$ appears to be a singularity of $\dfrac{\sin z}{z}$, the Laurent's series of $\dfrac{\sin z}{z}$ at $z = 0$ does not contain negative powers of z.

$\therefore$ $z = 0$ is a removable singularity of $\dfrac{\sin z}{z}$.

Example 2.4: Find the singularities of $f(z) = \dfrac{z^2 + 4}{z^3 + 2z^2 + 2z}$ and the corresponding residues.

Solution: The singularities of $f(z)$ are given by Dr = 0.

$$z^3 + 2z^2 + 2z = 0$$

$$z(z^2 + 2z + 2) = 0 \qquad\qquad z = \frac{-2 \pm \sqrt{4-8}}{2}$$

i.e., $\quad z = 0, -1 \pm i \qquad\qquad = -1 \pm i$

$z = 0, -1 + i, -1 - i$ are simple poles of $f(z)$.

$$[\text{Res}\, f(z)]_{z=a} = \lim_{z \to a} (z - a)\, f(z)$$

$$\therefore \quad [\text{Res}\, f(z)]_{z=0} = \lim_{z \to 0} (z - 0) \cdot \frac{z^2 + 4}{z(z^2 + 2z + 2)} = 2$$

$$[\text{Res}\, f(z)]_{z=-1+i} = \lim_{z \to -1+i} [z - (-1+i)] \cdot \frac{z^2 + 4}{z[z - (-1+i)][z - (-1-i)]}$$

$$= \lim_{z \to -1+i} \frac{z^2 + 4}{z[z - (-1+i)]} .$$

$$= \frac{1 - 1 - 2i + 4}{(-1+i)[-1+i+1+i]} = \frac{2(2-i)}{2i(-1+i)}$$

$$= \frac{1}{i} \frac{(2-i)(-1-i)}{(1+1)} = \frac{1}{2i}[-2 - 2i + i - 1]$$

$$= \frac{1}{2i}[-3 - i]$$

$$= \frac{-i}{2}[-3-i] = \frac{1}{2}[-1+3i]$$

NOTES

$$[\text{Res}\, f(z)]_{z=-1-i} = \lim_{z\to -1-i} [z-(-1-i)] \frac{z^2+4}{z[z-(-1+i)][z-(-1-i)]}$$

$$= \lim_{z\to -1-i} \frac{z^2+4}{z[z+1-i]} = -\frac{1}{2}(1+3i)$$

Example 2.5: Evaluate the following integrals, using Cauchy's residue theorem:

1. $\int_C \frac{e^z}{(z+1)^3} dz$, where C is the circle

   (*i*) $|z-3| = 3$ (*ii*) $|z-1| = 3$

2. $\int_C \frac{z-3}{z^3+2z+5} dz$, where C is the circle

   (*i*) $|z| = 1$ (*ii*) $|z+1-i| = 2$

Solution: 1. Poles $f(z)$ are given by Dr = 0.

$(z+1)^3 = 0 \Rightarrow z = -1, -1, -1.$ $z = -1$ is a pole of order 3.

(*i*) $|z-3| = 3$

$z = -1$ lies outside the circle $|z-3| = 3$.

$\therefore$ By Cauchy's inregral theorem,

$$\int_C \frac{e^z}{(z+1)^3} dz = 0.$$

(*ii*) C is the circle $|z-1| = 3$.

$z = -1$ lies inside this circle.

$$\therefore \quad [\text{Res}\, f(z)]_{z=-1} = \frac{1}{2!} \lim_{z\to -1} \frac{d^2}{dz^2} \cdot \left[(z+1)^3 \frac{e^z}{(z+1)^3}\right]$$

$$= \frac{1}{2} \cdot \lim_{z\to -1} \frac{d^2}{dz^2} \cdot [e^z]$$

$$= \frac{1}{2} \cdot e^{-1} = \frac{1}{2e}$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_C \frac{e^z}{(z+1)^3} dz = 2\pi i \times \left(\frac{1}{2e}\right) = \frac{\pi i}{e}$$

2. Poles are given by $z^2 + 2z + 5 = 0$.

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

Both are simple poles.

(*i*) C is the circle $|z| = 1$.

Both poles lie ouside the circle $|z| = 1$.

$\therefore$ By Cauchy's integral theorem, $\int_C \frac{z-3}{z^2+2z+s} dz = 0.$

(*ii*) When $z = -1 - 2i$, $|z + 1 - i| = |-3i| = 3 > 2$

When $z = -1 + 2i$, $|z + 1 - i| = |i| = 1 < 2$

$\therefore$ This pole $z = -1 + zi$ lies inside C.

NOTES

$$[\text{Res } f(z)]_{z=-1+2i} = \lim_{z \to -1+2i} [z-(-1+2i)] \frac{z-3}{[z-(-1+2i)][z-(-1-2i)]}$$

$$= \frac{i-2}{2i}$$

$\therefore$ By Residue theorem,

$$\int_C \frac{z-3}{z^2+2z+5} dz = 2\pi i \left(\frac{i-2}{2i}\right) = \pi(i-2)$$

Example 2.6: Evaluate the following integral using Cauchy's residue theorem:

$$\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz \qquad \text{where } C \text{ is } |z| = 3.$$

Solution: Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i \times [\text{Sum of the residues at its poles}]$$

Poles are obtained by equating the denominator to zero.

$(z + 1)(z + 2) = 0 \quad \Rightarrow \quad z = -1, -2$

These are simple poles lying inside the circle $|z| = 3$.

$$\therefore \quad [\text{Res } f(z)]_{z=-1} = \lim_{z \to -1} [z-(-1)] \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}$$

$$= \cos \pi + \sin \pi = -1$$

$$[\text{Res } f(z)]_{z=-2} = \lim_{z \to -2} [z-(-2)] \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{-1} = -1$$

$\therefore$ By Cauchy's residue theorem.

$$\int_C f(z) dz = 2\pi i[-1-1] = -4\pi i$$

Example 2.7: Evaluate $\int_C \frac{dz}{z \sin z}$, where C is $|z| = 1$ by using residue theorem.

Solution: Poles are given by, denominator = 0

$z \sin z = 0$

$\Rightarrow \quad z = 0 \quad$ and $\quad \sin z = 0$

$\sin z = 0 \Rightarrow z = \pm n\pi$, $n = 0, 1, 2, 3, ...$

i.e., $\quad z = 0, \pm \pi, \pm 2\pi, \ldots$

NOTES

Here $z = 0$ is a double pole and others are simple poles.

Of these poles $z = 0$ lies inside $|z| = 1$ and others lie outside C.

$$\therefore \qquad [\text{Res } f(z)]_{z=0} = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left\{(z-0)^2\cdot\frac{1}{z\sin z}\right\}$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{z}{\sin z}\right]$$

$$= \lim_{z\to 0}\left[\frac{\sin z - z\cos z}{\sin^2 z}\right] \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{z\to 0}\frac{\cos z - \cos z + z\sin z}{2\sin z}, \text{ using L' Hopital's Rule}$$

$$= \lim_{z\to 0}\frac{z\sin z}{2\sin z} = 0$$

$\therefore$ By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i[0] = 0$$

Example 2.8: Using Residue theorem evaluate $\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$, where C is $|z| = 4$.

Solution: $(z^2+\pi^2)^2 = 0$

$\Rightarrow \qquad z^2+\pi^2 = 0$, twice

$z = \pm i\pi$, twice

i.e., $z = i\pi,\ -i\pi$ are poles of order 2.

Both lie inside $|z| = 4$. $[\because \pi = 3.14]$

$$[\text{Res } f(z)]_{z=i\pi} = \frac{1}{1!}\lim_{z\to i\pi}\frac{d}{dz}\left[(z-i\pi)^2\frac{e^z}{(z-i\pi)^2(z+i\pi)^2}\right]$$

$$= \lim_{z\to i\pi}\frac{d}{dz}\left[\frac{e^z}{(z+i\pi)^2}\right]$$

$$= \lim_{z\to i\pi}\frac{(z+i\pi)^2 e^z - e^z\cdot 2(z+i\pi)}{(z+i\pi)^4}$$

$$= \lim_{z\to i\pi}\frac{(z+i\pi)e^z - 2e^z}{(z+i\pi)^3}$$

$$= \frac{2i\pi e^{i\pi} - 2e^{i\pi}}{(2i\pi)^3} = \frac{2e^{i\pi}[i\pi-1]}{-8i\pi^3}$$

$$= -\frac{(-1)(i\pi-1)}{4\pi^3 i} = \frac{-i(i\pi-1)}{4\pi^3}$$

$$= \frac{\pi+i}{4\pi^3}$$

Similarly, $[\text{Res}\, f(z)]_{z=-i\pi} = \dfrac{1}{4\pi^3}(\pi - i).$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{4\pi^3}[\pi + i + \pi - i] = \frac{i}{\pi}$$

2.2.2 Evaluation of Integrals

Certain types of real definite integrals can be evaluated using residue theorem and properly chosen contours. The contours chosen consist of straight lines and circular arcs. Before the evaluation of these real integrals, we will see some Lemmas which will be used in evaluation of these integrals.

Cauchy's Lemma I

If $f(z)$ is a uniform continuous function such that $|(z-a)f(z)| \to 0$ as $|z-a| \to 0$, then $\int_C f(z)\,dz \to 0$ where C is the circle $|z-a| = r$

Cauchy's Lemma II

If $f(z)$ is a uniform continuous function such that $|(z-a)f(z)| \to 0$ as $|z-a| \to \infty$, then $\int_C f(z)\,dz \to 0$, as $R \to \infty$, where C is the circle $|z-a| = R$.

Jordan's Lemma

If $f(z)$ is a uniform continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, then $\int_C e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, where C is the semi-circle $|z| = R$ above the real axis and $m > 0$.

Type I: Integration Around the Unit Circle: Integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$, where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

In such type of problems, we take the unit circle $|z| = 1$ as the contour.

On $|z| = 1$, $z = re^{i\theta} = e^{i\theta}$, $\because\ r = 1$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

$$dz = ie^{i\theta}d\theta = iz\,d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

As θ varies from 0 to 2π, z moves once round the unit circle in the anti-clockwise direction.

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

where C is the unit circle $|z| = 1$. The integral on the right side can be evaluated by using the residue theorem.

NOTES

Example 2.9: Show that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta + a^2} = \frac{2\pi a^2}{1-a^2}$, $(a^2 < 1)$.

Solution: This is of the type $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

Take the contour C as the unit circle $|z| = 1$.

On $|z| = 1$, $z = e^{i\theta}$

$$dz = e^{i\theta} \cdot i d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) = \frac{z^4+1}{2z^2}$$

$$\therefore \quad I = \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2}\, d\theta$$

$$= \int_C \frac{(z^4+1)}{2z^2\left[1-2a\left(\frac{z^2+1}{2z}\right)+a^2\right]} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2\left[\frac{z-az^2-a+a^2z}{z}\right]} \frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2[z-az^2-a+a^2z]}\, dz$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2[(z-a)(1-az)]}\, dz$$

$$= \frac{1}{2i}\int_C f(z)\, dz.$$

$$= \frac{1}{2i} \times 2\pi i \text{ [Sum of the residues at its poles which lie inside } C\text{], using Residue theorem} \quad \ldots (1)$$

$$f(z) = \frac{z^4+1}{z^2(z-a)(1-az)}.$$

$$\text{Dr} = 0 \Rightarrow z^2 = 0; \quad z - a = 0; \quad 1 - az = 0$$

$$\Rightarrow \quad z = 0,\, 0;\, z = a;\; z = \frac{1}{a}.$$

NOTES

$z = 0$ is a pole of order '2' and $z = a, \dfrac{1}{a}$ are poles of order '1'.

Since $a^2 < 1, a < 1 \Rightarrow \dfrac{1}{a} > 1, \therefore |z| = \dfrac{1}{a}$ lies outside the unit circle $|z| = 1$.

$$R_1 = [\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z - a) f(z)$$

$$= \lim_{z \to a} (z - a) \frac{z^4 + 1}{z^2 (z - a)(1 - az)}$$

$$= \lim_{z \to a} \frac{z^4 + 1}{z^2 (1 - az)} = \frac{a^4 + 1}{a^2 (1 - a^2)}$$

$$R_2 = [\text{Res } f(z)]_{z=0} = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left\{ (z - 0)^2 f(z) \right\}$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[(z^2) \frac{z^4 + 1}{z^2 (z - a)(1 - az)} \right]$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[\frac{z^4 + 1}{(z - a)(1 - az)} \right]$$

$$= \lim_{z \to 0} \frac{(z - a)(1 - az) \cdot 4z^3 - (z^4 + 1)(1 - 2az + a^2)}{(z - a)^2 (1 - az)^2}$$

$$= \frac{-(1 + a^2)}{a^2}$$

Substituting in Equation (1),

$$I = \frac{1}{2i} 2\pi i [R_1 + R_2] = \pi \left[\frac{a^4 + 1}{a^2 (1 - a^2)} - \frac{(1 + a^2)}{a^2} \right] = \frac{2\pi a^2}{(1 - a^2)}.$$

Example 2.10: Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ using contour integration.

Solution: It is of the type $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta.$

Take the unit circle as the contour C.

On $|z| = 1, \quad z = e^{i\theta}$

$$dz = e^{i\theta} i\, d\theta = z\, i\, d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

NOTES

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\cos 3\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right) = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) = \frac{z^6+1}{2z^3}$$

$$\therefore \quad I = \int_C \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \int_C \frac{z^6+1}{2z^3\left[5 - 4\left(\frac{z^2+1}{2z}\right)\right]} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^6+1}{z^3[5z - 2z^2 - 2]}\, dz$$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3[2z^2 - 5z + 2]}\, dz$$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

$$= -\frac{1}{2i}\int_C f(z)\, dz$$

$= -\dfrac{1}{2i} \times 2\pi i$ [Sum of the residues at its poles which lie inside C] ... (1)

Poles are given by $z^3 = 0$, $2z - 1 = 0$ and $z - 2 = 0$

$\Rightarrow$ $z = 0$ is a pole of order 3.

$z = \dfrac{1}{2}$ is a pole of order 1.

$z = 2$ is a pole of order 1.

Out of these poles $z = 2$ lies outside the unit circle.

$$R_1 = [\text{Res}\, f(z)]_{z=\frac{1}{2}} = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$$

$$= \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^6+1}{z^3(2z-1)(z-2)}$$

$$= \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^6+1}{z^3 2\left(z - \frac{1}{2}\right)(z-2)}$$

$$= \lim_{z \to \frac{1}{2}} \frac{z^6+1}{2z^3(z-2)}$$

$$= -\frac{65}{24}$$

NOTES

$$R_2 = [\text{Res}\, f(z)]_{z=0}\ \frac{1}{2!} = \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 \cdot \frac{z^6+1}{z^3\,(2z-1)\,(z-2)}\right]$$

$$= \frac{1}{2}\lim_{z\to 0} \frac{d}{dz}\left[\frac{(2z-1)\,(z-2)\,6z^3 - (z^6+1) + (4z-5)}{(2z-1)^2\ (z-2)^2}\right]$$

$$= \frac{1}{2}\lim_{z\to 0} \frac{1}{(2z-1)^4\ (z-2)^4}$$

$$\left\{\begin{array}{l}(2z-1)^2\ (z-2)^2[(2z-1)\,(z-2)18z + (2z-1)\cdot 6z^3 + 2\,(z-2)\,6z^3 \\ \quad - (z^6+1)4 - (4z-5)\,6z^5\,] - [(2z-1)\,(z-2)\,6z^3 \\ \quad - (z^6+1)(4z-5)]\,[4\,(2z-1)\,(z-2)^2 + (2z-1)^2\cdot 2\,(z-2)]\end{array}\right\}$$

$$= \frac{1}{2}\times\frac{1}{16}\{(1)(4)[0+0+0-4-0] - 5[4\,(-1)\,4 + (1)\,(2)\,(-2)]\}$$

$$= \frac{84}{2\times 16} = \frac{21}{8}$$

Substituting in Equation (1), we get

$$I = -\frac{1}{2i}2\pi i\left[-\frac{65}{24} + \frac{21}{8}\right] = -\pi\left[-\frac{1}{12}\right] = \frac{\pi}{12}$$

Another method to find R_2.

Residue at $z = a$ is the coefficient of $\frac{1}{z-a}$ in the Laurent's series.

$$f(z) = \frac{z^6+1}{z^3(2z-1)(z-2)} = \frac{\left(z^3 + \frac{1}{z^3}\right)}{(2z-1)(z-2)} = \frac{\left(z^3 + \frac{1}{z^3}\right)}{-(1-2z)(-2)\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)(1-2z)^{-1}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)(1+2z+4z^2+8z^3+\ldots\infty)\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\infty\right)$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\left[1+\left(2+\frac{1}{2}\right)z + 1\cdot\frac{z^2}{4} + (2z)\left(\frac{z}{2}\right) + (4z^2)(1) + \ldots\infty\right]$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\left[1+\frac{5}{2}z + \left(\frac{1}{4}+1+4\right)z^2 + \ldots\infty\right]$$

$$\frac{1}{z}\ \text{term} = \frac{1}{2}\cdot\frac{1}{z^3}\cdot\left(\frac{21}{4}\right)z^2 = \frac{21}{8}\cdot\frac{1}{z}$$

Coefficient of $\frac{1}{z} = \frac{21}{8}$.

$$\therefore [\text{Res}\, f(z)]_{z=0} = \frac{21}{8}$$

Example 2.11: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}d\theta$, using contour integration.

NOTES

Solution: On $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z},\quad \sin\theta = \frac{z^2-1}{2iz} \quad\Rightarrow\quad \sin^2\theta = \frac{(z^2-1)^2}{-4z^2}$$

$$\therefore \qquad I = \int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}d\theta = \int_C \frac{(z^2-1)^2}{-4z^2\left[5-3\left(\frac{z^2+1}{2z}\right)\right]}\frac{dz}{iz}$$

$$= +\frac{1}{i}\int_C \frac{(z^2-1)^2}{-2z^2[10z-3z^2-3]}dz$$

$$= +\frac{1}{2i}\int_C \frac{(z^2-1)^2}{z^2(3z^2-10z+3]}dz$$

$$= \frac{1}{2i}\int_C \frac{(z^2-1)^2}{z^2(3z-1)(z-3)}dz$$

$$= \frac{1}{2i}\int_C f(z)\,dz$$

$$= \frac{1}{2i}\cdot 2\pi i \quad \text{[Sum of the residues at poles which lie inside } C] \quad \dots (1)$$

Dr = 0 $\Rightarrow$ $z^2 = 0$; $3z - 1 = 0$ and $z - 3 = 0$

$z = 0, 0$ is a pole of order 2

$z = \frac{1}{3}$ is a simple pole and $z = 3$ is also a simple pole.

Out of these poles $z = 3$ lies outside the unit circle $|z| = 1$.

$$R_1 = [\text{Res } f(z)]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)\frac{(z^2-1)^2}{z^2.3\left(z-\frac{1}{3}\right)(z-3)} = \frac{-8}{9}$$

$$R_2 = [\text{Res } f(z)]_{z=0} = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{(z^2-1)^2}{z^2(3z-1)(z-3)}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{(z^2-1)^2}{(3z-1)(z-3)}\right]$$

$$= \lim_{z\to 0}\frac{(3z^2-10z+3)\cdot 2(z^2-1)\cdot 2z^2-(z^2-1)^2(6z-10)}{(3z-1)^2(z-3)^2}$$

$$= \frac{0-(1)(-10)}{9} = \frac{10}{9}$$

Substituting in Equation (1)

$$I = \frac{1}{2!} \cdot 2\pi i \left[-\frac{8}{9} + \frac{10}{9} \right] = \frac{2\pi}{9}$$

NOTES

Example 2.12: Evaluate $\int_0^{2\pi} \frac{d\theta}{1 - 2a \sin\theta + a^2}$, $0 < a < 1$.

Solution: This is of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

On the circle $|z| = 1$, $z = e^{i\theta}$, $dz = e^{i\theta} i\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z} \right) = \frac{z^2 - 1}{2iz}$$

$$\therefore \quad I = \int_0^{2\pi} \frac{d\theta}{1 - 2a \sin\theta + a^2} = \int_C \frac{1}{1 - 2a \frac{(z^2 - 1)}{2iz} + a^2} \frac{dz}{iz}$$

$$= \int_C \frac{1}{1 - \frac{a}{iz}(z^2 - 1) + a^2} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{1}{iz - az^2 + a + ia^2 z} \cdot dz$$

$$= \int_C \frac{1}{i(1 + a^2)z - az^2 + a} dz$$

$$= -\int_C \frac{1}{az^2 - i(1 + a^2)z - a} dz$$

$$= -\int_C \frac{1}{(az - i)(z - ia)} dz$$

$= -2\pi i$. [Sum of the Residues at its poles which lie inside C] ... (1)

Dr = 0 $\Rightarrow$ $az - i = 0$; $z - ia = 0$

$$z = \frac{i}{a} \;;\; z = ia$$

Since $a < 1$, $\frac{1}{a} > 1$.

$\therefore$ $z = \frac{i}{a}$ lies outside the unit circle $|z| = 1$.

$z = ia$ is a simple pole.

$$[\text{Res } f(z)]_{z = ia} = \lim_{z \to ia} (z - ia) \cdot \frac{1}{(az - i)(z - ia)}$$

$$= \frac{1}{ia^2 - i} = \frac{1}{i(a^2 - 1)}$$

Substituting in Equation (1),

NOTES

$$I = -2\pi i\left[\frac{1}{i(a^2 - i)}\right] = \frac{2\pi}{1 - a^2}$$

Example 2.13: Show that $\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right]$, $(a > b > 0)$.

Solution: It is of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

Take unit circle $|z| = 1$ as the contour C.

On $|z| = 1, \quad z = e^{i\theta}$.

$$dz = e^{i\theta}.\ i\,d\theta = iz\,d\theta \Rightarrow \theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} = \text{Real part of } \frac{1}{2}\left[1 - e^{i2\theta}\right]$$

$$= \text{Real part of } \frac{1}{2}(1 - z^2)$$

$$\therefore \quad I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta$$

$$= \text{Real part of } \int_C \frac{(1 - z^2)}{2\left[a + b\left(\frac{z^2 + 1}{2z}\right)\right]} \frac{dz}{iz}$$

$$= \text{Real part of } \frac{1}{i}\int_C \frac{(1 - z^2)}{2az + bz^2 + b} \cdot dz$$

$$= \text{Real part of } \frac{1}{i}\int_C \frac{(1 - z^2)}{bz^2 + 2az + b} dz$$

$= \text{Real part of } \frac{1}{i} \cdot 2\pi i$ [Sum of the residues at its poles which lie inside C]

$= \text{Real part of } 2\pi$ [Sum of the residues at its poles which lie inside C] ... (1)

Poles are given by Dr = 0.

$$bz^2 + 2az + b = 0$$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

NOTES

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

$$\alpha\beta = \frac{1}{b^2}\left[\left(-a+\sqrt{a^2-b^2}\right)\left(-a-\sqrt{a^2-b^2}\right)\right]$$

$$= \frac{1}{b^2}\left[(-a)^2 - \left(\sqrt{a^2-b^2}\right)^2\right] = \frac{1}{b^2}\left[a^2-(a^2-b^2)\right] = 1$$

$$\alpha\beta = 1$$

Since $0 < b < a$, $|\beta| > 1$.

$\therefore$ β lies outside the unit circle $|z| = 1$.

Also $z = \alpha$ is a simple pole which lies inside $|z| = 1$.

Now $bz^2 + 2az + b = b\left(z^2 + \frac{2a}{b}z + 1\right) = b(z-\alpha)(z-\beta)$.

$$[\text{Res } f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)\cdot\frac{(1-z^2)}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{1-\alpha^2}{b\left(\alpha-\frac{1}{\alpha}\right)} \qquad \because \ \alpha\beta = 1$$

$$= \frac{(1-\alpha^2)\alpha}{b(\alpha^2-1)} = -\frac{\alpha}{b} = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

Substituting in Equation (1)

$$I = \text{Real part of } 2\pi\left[\frac{a-\sqrt{a^2-b^2}}{b^2}\right] = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

Example 2.14: Evaluate $\int_0^{\pi}\frac{d\theta}{a+b\cos\theta}$ where $a > |b|$.

Solution: Since $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$,

$$\therefore \qquad \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = 2\int_0^{\pi}\frac{d\theta}{a+b\cos\theta} \to \text{(I)}$$

Take unit circle as the contour C.

On $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$.

$$\therefore \qquad I = \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \int_C \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz}$$

NOTES

$$= \frac{1}{i} \int_C \frac{2}{2az + bz^2 + b} dz$$

$$= \frac{2}{i} \int_C \frac{1}{bz^2 + 2az + b} dz$$

$$= \frac{2}{i} \cdot 2\pi i \cdot \text{ [Sum of the residues at its poles which lie inside } C]$$

$$= 4\pi \text{ [Sum of the residues at its poles which lie inside } C] \quad ...(1)$$

Dr = 0 $\Rightarrow$ $bz^2 + 2az + b = 0$

$$z = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ and $\beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b}$.

$$\alpha\beta = 1$$

Since $a > |b|$, $|\beta| > 1$.

$\therefore$ $z = \beta$ lies out side the unit circle $|z| = 1$.

Also $z = \alpha$ is a simple pole which lies inside $|z| = 1$.

Now $bz^2 + 2az + b = b\left(z^2 + \dfrac{2a}{b} z + 1\right) = b(z - \alpha)(z - \beta)$

$$[\text{Res } f(z)]_{z=\alpha} = \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{b(z - \alpha)(z - \beta)}$$

$$= \frac{1}{b(\alpha - \beta)} = \frac{1}{\frac{b}{b}\left[-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}\right]}$$

$$= \frac{1}{2\sqrt{a^2 - b^2}}$$

Substituting in Equation (1),

$$I = 4\pi\left[\frac{1}{2\sqrt{a^2 - b^2}}\right] = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad ... (2)$$

From Equation (1) and Equation (2)

$$2\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

$$\therefore \quad \int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

Type : II: Integration of the form $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ where $P(x)$, $Q(x)$ are polynomials and degree of $Q(x) >$ (degree of $P(x) + 1$).

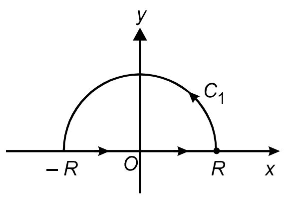
Example 2.15: Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$, using contour integration, where $0 < b < a$.

NOTES

Solution: This is of the type $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials and deg $Q(x) >$ [deg $P(x) + 1$].

Consider $\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz$, where C is the contour consisting of the segment of the real axis from $-R$ to R and the semicircle C, above the real axis having the radius as R.

$$\therefore \qquad \int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_1} f(z)dz$$



$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

Poles are given by $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ia, z = \pm ib$, which are simple poles.

Of these poles $z = ia$ and $z = ib$ lie inside C, and $z = -ia$ and $z = -ib$ lie outside C.

[$\because z = -i\,a, -i\,b$ lie below the real axis].

$$R_1 = [\text{Res}\, f(z)]_{z=ia} = \lim_{z \to ia} (z - ia) \cdot \frac{z^2}{(z-ia)(z+ia)(z^2+b^2)}$$

$$= \frac{-a^2}{2ia(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

$$R_2 = [\text{Res}\, f(z)]_{z=ib} = \lim_{z \to ib} (z - ib) \frac{z^2}{(z^2+a^2)(z-ib)(z+ib)}$$

$$= \frac{-b^2}{(-b^2+a^2)2ib} = \frac{-b}{2i(a^2-b^2)}$$

By Cauchy's residue theorem,

$$\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz = 2\pi i(R_1 + R_2)$$

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)} \right]$$

$$= \frac{\pi(a-b)}{(a^2-b^2)}$$

NOTES

$$= \frac{\pi}{a+b}$$

i.e., $\int_{-R}^{R} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx + \int_{C_1} \frac{z^2}{(z^2+a^2)(z^2+b^2)} dz = \frac{\pi}{a+b}.$ ... (1)

[$\because$ on the real axis, $z = x$ and so $dz = dx$]

Since $z\, f(z) = \frac{z^3}{(z^2+a^2)(z^2+b^2)} = O\left(\frac{1}{z}\right) \to 0$ as $|z| = R \to \infty$.

$\therefore$ By Cauchy's lemma, $\int_{C_1} f(z) dz \to 0$ as $R \to \infty$. ... (2)

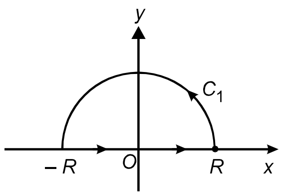
Using Equation (2) in Equation (1) and letting $R \to \infty$, we get

$$\int_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a+b}$$

Example 2.16: Show that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}$.

Solution: Consider $\int_C \frac{z^2-z+2}{z^4+10z^2+9} dz,$ where C is the contour shown in Figure.

$$\int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_1} f(z)dz$$



The singularities of $f(z) = \frac{z^2-z+2}{z^4+10z^2+a}$ are given by,

$z^4 + 10z^2 + 9 = 0$

$(z^2 + 1)(z^2 + 9) = 0$

$z = \pm i, \pm 3i$, which are simple poles.

The poles $z = i$ and $3i$ lie inside C and $z = -i$ and $-3i$ lie outside C.

$$[\text{Res } f(z)]_{z=i} = \lim_{z \to i} (z-i) \frac{z^2-z+2}{(z+i)(z-i)(z^2+9)} = \frac{1-i}{(2i)(8)} = \frac{1-i}{16i}$$

$$[\text{Res } f(z)]_{z=3i} = \lim_{z \to 3i} (z-3i) \frac{z^2-z+2}{(z^2+1)(z-3i)(z+3i)} = \frac{-7-3i}{(-8)(6i)} = \frac{7+3i}{48i}.$$

By residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1-i}{16i}+\frac{7+3i}{48i}\right]=\frac{5\pi}{12}$$

NOTES

But $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_C f(z)\,dz \qquad \ldots (1)$$

$$z\,f(z) = \frac{z^3 - z^2 + 2z}{z^4 + 10z + 9} = O\left(\frac{1}{z}\right) \to 0 \text{ as } |z| = R \to \infty$$

$\therefore$ By Cauchy's lemma, $\int_{C_1} f(z)\,dz \to 0$ as $|z| = R \to \infty$ ... (2)

Letting $R \to \infty$ and using Equation (2) in Equation (1), we get

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx, \qquad [\because \text{ On the real axis } z = x]$$

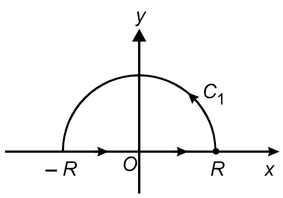
$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{(x^4 + 10x^2 + 9)}\,dx = \frac{5\pi}{12}$$

Example 2.17: Use contour integration to prove that,

$$\int_0^{\infty} \frac{x^2}{x^4 + a^4}\,dx = \frac{\pi}{2\sqrt{2a}}, \text{ where } a > 0.$$

Solution: Consider $\int_C \frac{z^2}{z^4 + a^4}\,dz$, where C is the contour shown in Figure.

$$\int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_1} f(z)dz$$



$$f(z) = \frac{z^2}{z^4 + a^4}$$

Poles are given by $z^4 + a^4 = 0$

$$z^4 = -a^4 = (-1)a^4 e^{i(2n+1)\pi} \cdot a^4$$

$$\therefore \qquad z = e^{i(2n+1)\pi/4} \cdot a, \text{ where } n = 0, 1, 2, 3$$

i.e., $z = ae^{i\pi/4}$, $z = ae^{i3\pi/4}$, $z = ae^{i5\pi/4}$, $z = ae^{i7\pi/4}$, all of which are simple poles.

The poles $z = a\,e^{i\pi/4}$ and $z = a\,e^{i3\pi/4}$ lie inside C and

$z = ae^{i5\pi/4}$ and $z = ae^{i7\pi/4}$ lie outside C.

$[\because 5\pi/4 \text{ and } 7\pi/4 \text{ are} > \pi]$

NOTES

$$[\text{Res}\, f(z)]_{z = ae^{i\pi/4}} = \lim_{z \to ae^{i\pi/4}} \left[\frac{z^2}{4z^3}\right]$$

$$\left[\text{If } f(z) = \frac{P(z)}{Q(z)} \text{ then } [\text{Res}\, f(z)]_{z=a} = \frac{P(a)}{Q'(a)}\right]$$

$$= \lim_{z \to ae^{i\pi/4}} \left(\frac{1}{4z}\right) = \frac{1}{4ae^{i\pi/4}} = \frac{1}{4a} e^{-i\pi/4}$$

$$= \frac{1}{4a}[\cos \pi/4 - i \sin \pi/4] = \frac{1}{4a}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4\sqrt{2}a}(1 - i)$$

Similarly R_2 $[\text{Res}\, f(z)]_{z = ae^{i3\pi/4}} = \lim_{z \to ae^{i3\pi/4}} \left(\frac{z^2}{4z^3}\right)$

$$= \frac{1}{4ae^{i3\pi/4}} = \frac{1}{4a} e^{-i3\pi/4}$$

$$= \frac{1}{4a}[\cos 3\pi/4 - i \sin 3\pi/4]$$

$$= \frac{1}{4a}[\cos(\pi - \pi/4) - i \sin(\pi - \pi/4)]$$

$$= \frac{1}{4a}[-\cos \pi/4 - i \sin \pi/4]$$

$$= \frac{-1}{4a\sqrt{2}}(1 + i)$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_C \frac{z^2}{z^4 + a^4} dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i \cdot \frac{1}{4a\sqrt{2}}[1 - i - 1 - i]$$

$$= \frac{\pi}{a\sqrt{2}}$$

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_C f(z)\, dz \qquad \ldots (1)$$

$[\because$ On the real axis $z = x]$

$$|z f(z)| = \left|\frac{z^3}{z^4 + a^4}\right| = O\left(\frac{1}{z}\right) \to 0 \text{ as } R \to \infty$$

By Cauchy's lemma, $\int_{C_1} f(z)\,dz = 0.$ ... (2)

Letting $R \to \infty$ and using Equation (2) in Equation (1), we get

NOTES

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a\sqrt{2}}$$

i.e., $$\int_{-\infty}^{\infty} \frac{x^2}{x^4+a^4}\,dx = \frac{\pi}{a\sqrt{2}}$$

i.e., $$2\int_0^{\infty} \frac{x^2}{x^4+a^4}\,dx = \frac{\pi}{a\sqrt{2}}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{x^2}{x^4+a^4}\,dx = \frac{\pi}{2a\sqrt{2}}$$

Example 2.18: Evaluate $\int_{-\infty}^{\infty} \frac{x^4}{x^6-a^6}\,dx$, using contour integration, where $a > 0$.

Soluiton: Consider $\int_C \frac{z^4}{z^6-a^6}\,dz.$

Poles are given by $z^6 - a^6 = 0$

$$z^6 = a^6 = 1 \cdot a^6 = e^{i2n\pi} \cdot a^6, \; n = 0, 1, 2, \ldots$$

$$\therefore \quad z = e^{i\frac{2n\pi}{6}} \cdot a, \; n = 0, 1, 2, 3, 4, 5.$$

i.e., $z = a,\; ae^{i\pi/3},\; ae^{i2\pi/3},\; ae^{i\pi},\; ae^{i4\pi/3},\; ae^{i5\pi/3}.$

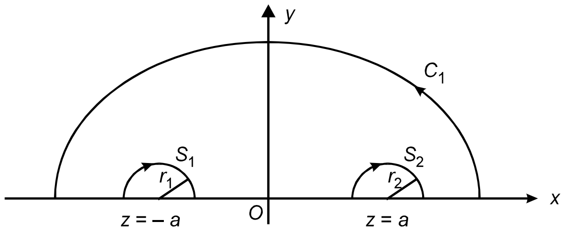
All are simple poles.

$z = ae^{i\pi/3}$ and $ae^{i2\pi/3}$ lie inside c.

$z = a,\; a\,e^{i\pi} = -a$ lie on the x-axis.

$z = ae^{i4\pi/3}$ and $ae^{i5\pi/3}$ lie outside C.

For the evaluation of this type of integrals, no singularity of $f(z)$ should lie on the real axis. To avoid them, we modify C by introducing small semicircles of small radius at $z = \pm a$, as shown in Figure.



Now the modified contour C contains only the simple poles $z = ae^{i\pi/3}$ and $z = ae^{i2\pi/3}$.

NOTES

$$R_1 = [\text{Res}\, f(z)]_{z=ae^{i\pi/3}} = \lim_{z \to ae^{i\pi/3}} \left[\frac{z^4}{6z^5}\right]$$

$$= \frac{1}{6ae^{i\pi/3}} = \frac{1}{6a} e^{-i\pi/3}$$

$$= \frac{1}{6a}[\cos \pi/3 - i \sin \pi/3] = \frac{1}{6a}\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

Similarly $R_2 = [\text{Res}\, f(z)]_{z=ae^{i2\pi/3}} = \dfrac{1}{6ae^{i2\pi/3}} = \dfrac{1}{6a} e^{-i2\pi/3}$

$$= \frac{1}{6a}\left[\cos\frac{2\pi}{3} - i \sin\frac{2\pi}{3}\right]$$

$$= \frac{1}{6a}\left[\cos\left(\pi - \frac{\pi}{3}\right) - i \sin\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{6a}\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

By Cauchy's residue theorem,

$$\int_C \frac{z^4}{z^6 - a^6} dz = \frac{2\pi i}{6a}\left[\frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = \frac{\pi}{a\sqrt{3}}$$

i.e.,
$$\int_{-R}^{-a-r_1} \frac{x^4}{x^6 - a^6} dx + \int_{S_1} \frac{z^4}{z^6 - a^6} dz + \int_{-a+r_1}^{z-r_2} \frac{x^4}{x^6 - a^6} dx + \int_{S_2} \frac{z^4}{z^6 - a^6} dz$$

$$+ \int_{a+r_2}^{R} \frac{x^4}{x^6 - a^6} dx + \int_{C_1} \frac{z^4}{z^4 - a^6} dx = \frac{\pi}{a\sqrt{3}} \qquad \text{... (1)}$$

where r_1 and r_2 are the radii of the semicircles S_1 and S_2 whose equations are $|z + a| = r_1$ and $|z - a| = r_2$. These two circles taken along S_1 and S_2 vanish as $r_1 \to 0$ and $r_2 \to 0$.

Also $\displaystyle\int_{C_1} \frac{z^4}{z^6 - a^6} = 0$ as $R \to \infty$, by Cauchy's lemma.

Now, letting $r_1 \to 0$, $r_2 \to 0$ as $R \to \infty$ in Equation (1), we get

$$\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \frac{x^4}{x^6 - a^6} dx = \frac{\pi}{a\sqrt{3}}.$$

i.e.,
$$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - a^6} dx = \frac{\pi}{a\sqrt{3}}.$$

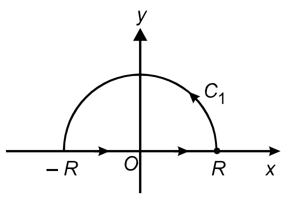
Type III:

Example 2.19: Evaluate $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi}{4b^3}[1+ab]e^{-ab}$, where $a > 0$ and $b > 0$.

NOTES

Solution: Consider $\int_C \frac{e^{iaz}}{(z^2+b^2)^2}dz$, where C is the contour shown in Figure.

The poles of $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$ are given by $(z^2 + b^2)^2 = 0$



$z = \pm\, ib$, twice.

$\therefore$ $z = \pm\, ib$ are poles of order 2.

$z = ib$ lies inside C and $z = -\, ib$ lies outside C.

$$\therefore \quad [\text{Res}\, f(z)]_{z=ib} = \frac{1}{1!}\lim_{z\to ib}\frac{d}{dz}\left[\frac{(z-ib)^2\cdot e^{iaz}}{(z-ib)^2(z+ib)^2}\right]$$

$$= \lim_{z\to ib}\frac{d}{dz}\left[\frac{e^{iaz}}{(z+ib)^2}\right]$$

$$= \lim_{z\to ib}\left[\frac{(z+ib)^2\cdot iae^{iaz} - e^{iaz}\cdot 2(z+ib)}{(z+ib)^4}\right]$$

$$= \lim_{z\to ib}\left[\frac{(z+ib)iae^{iaz} - 2e^{iaz}}{(z+ib)^3}\right]$$

$$= \frac{2ib\cdot iae^{-ab} - 2e^{-ab}}{(2ib)^3}$$

$$= \frac{(ab+1)e^{-ab}}{4ib^3}$$

By Cauchy's residue theorem,

$$\int_C \frac{e^{iaz}}{(z^2+b^2)^2}dz = 2\pi i \times \frac{1}{4ib^3}(ab+1)e^{-ab}$$

i.e., $$\int_{-R}^{R}\frac{e^{iax}}{(x^2+b^2)^2}dx + \int_{C_1}\frac{e^{iaz}}{(z^2+b^2)^2}dz = \frac{\pi}{2b^3}(ab+1)e^{-ab} \quad \ldots (1)$$

Now, $$\left|\frac{1}{(z^2+b^2)^2}\right| \le \frac{1}{(R^2-b^2)^2}$$

Since the RHS $\to 0$ as $R\to\infty$, LHS also $\to 0$ as $R\to\infty$ on $|z| = R$.

NOTES

$\therefore$ By Jordan's lemma,

$$\int_{C_1} \frac{e^{iaz}}{(z^2+b^2)^2} dz \to 0 \text{ as } R \to \infty \text{ on } |z| = R \quad \text{... (2)}$$

Letting $R \to \infty$ in Equation (1) and using Equation (2), we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2} dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}$$

Equating real and imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}$$

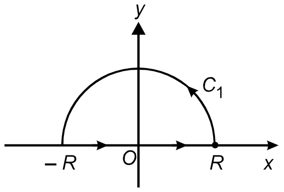
$$2\int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}$$

[$\because$ The integrand is an even function]

$$\Rightarrow \int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi}{4b^3}(ab+1)e^{-ab}$$

Example 2.20: Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$, using contour integration, where $a > b > 0$.

Solution: Consider $\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} dz$, where C is the contour shown in Figure.



The poles of $f(z)$ are given by $z^2 + a^2 = 0$
and $z^2 + b^2 = 0$

i.e., $z = \pm ia, \pm ib$, which are simple poles.

Of these poles $z = ia$, and $z = ib$ lie inside C.

$$R_1 = [\text{Res } f(z)]_{z=ia} = \lim_{z \to ia} = (z-ia) \cdot \frac{e^{iz}}{(z-ia)(z+ia)(z^2+b^2)} = \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$R_2 = [\text{Res } f(z)]_{z=ib} = \lim_{z \to ib} (z-ib) \cdot \frac{e^{iz}}{(z^2+a^2)(z-ib)(z+ib)} = \frac{e^{-b}}{2ib(-b^2+a^2)}$$

By Cauchy's residue theorem,

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz = 2\pi i\left[\frac{e^{-a}}{-2ia(a^2-b^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

NOTES

i.e., $$\int_{-R}^{R} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx + \int_{C_1} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right] \quad ...(1)$$

Now $\left|\frac{1}{(z^2+a^2)(z^2+b^2)}\right| \le \frac{1}{(R^2-a^2)(R^2-b^2)}$

RHS $\to 0$ as $R \to \infty$. Hence

$$\lim_{R\to\infty}\left|\frac{1}{(z^2+a^2)(z^2+b^2)}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Jordan's lemma, $\int_{C_1} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz \to 0$ as $R \to \infty$. ... (2)

Letting $R \to \infty$ in Equation (1), and using Equation (2), we get

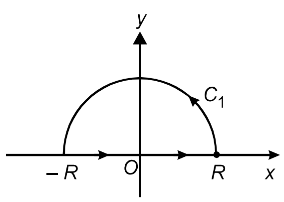
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Example 2.21: Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx$, by contour integration.

Solution: Consider $\int_C \frac{ze^{iz}}{z^2+a^2}dz$, where C is the contour shown in Figure.



The poles are given by $z^2 + a^2 = 0$

$\Rightarrow z = \pm ia$. Both are simple poles.

$z = ia$ lies inside C and $z = -ia$ lies outside C.

$$[\text{Res } f(z)]_{z=ia} = \lim_{z \to ia} (z - ia) \frac{ze^{iz}}{(z+ia)^2 (z-ia)} = \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}$$

NOTES

By Cauchy's residue theorem,

$$\int_C \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \left(\frac{e^{-a}}{2} \right) = \pi i e^{-a}$$

i.e., $$\int_{-R}^{R} \frac{xe^{ix}}{x^2 + a^2} dx + \int_{C_1} \frac{ze^{iz}}{z^2 + a^2} dz = \pi i e^{-a} \qquad \text{... (1)}$$

Now $$\left| \frac{z}{z^2 + a^2} \right| \le \frac{R}{R^2 - a^2}$$

Since the limit of the RHS is zero as $R \to \infty$

$$\lim_{R \to \infty} \left| \frac{z}{z^2 + a^2} \right| = 0 \text{ on } |z| = R$$

$\therefore$ By Jordan's lemma, $\int_{C_1} \frac{ze^{iz}}{z^2 + a^2} dz \to 0$ as $R \to \infty$.

Letting $R \to \infty$ in Equation (1) and using Equation (2), we get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx = i\pi e^{-a}$$

i.e., $$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{x^2 + a^2} dx = i\pi e^{-a}$$

Equating the imaginary parts on both sides,

we get $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}$$

$$2\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a} \qquad [\because \text{ The integrand is an even function of } x]$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$$

NOTES

2.3 BRANCHES OF MANY VALUED FUNCTIONS

Let n be an integer, then, $e^{1+2n\pi i} = e$. … (2.1)

If we write,

$(e^{1+2n\pi i})^{1+2n\pi i} = e$, … (2.2)

And, $(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2} = e\, e^{-4n^2\pi^2}$, … (2.3)

It follows that, $e^{-4n^2\pi^2} = 1$. … (2.4)

There also exist a number of paradoxes involving square roots. For example,

$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\,\sqrt{-1} = i\, i = -1$ … (2.5)

$1/-1 = -1/1$

$1/\sqrt{-1} = \sqrt{-1}/1$ … (2.6)

$1/i = i/1$

$i^2 = 1$

Let us look at some properties of elementary transcendental functions.

For, $z = x + iy$, the complex exponential function is defined by, $e^z = e^x$ $(\cos y + i \sin y)$. It satisfies the property $e^{(z+w)} = e^z e^w$, but does it satisfy the property $(e^z)^w = e^{(zw)}$? For answering this, we use the complex logarithm function. We define the principal argument by, $z = |z|\, e^{i\text{Arg}\,(z)}$ and Arg (z) $\in (-\pi, \pi]$. We do not define the principal argument of 0. Assume that z is different from 0. We have defined the principle argument on the negative axis also, but notice that it is not continuous there.

We then define the principal logarithm Log (z) by Log (z) = Log $|z|$ + i Arg (z), where log $|z|$ denotes the usual real logarithm of $|z|$. We have $e^{\log(z)}$ $= z$ but not Log $(e^z) = z$. For the reason, following terminology is introduced:

Definition 1: Define the imaginary remainder Imr (z) and the imaginary quotient Imq (z) by,

Imr (z) = Imr (z) + 2π Imq (z), where Imr $(z) \in (-\pi, \pi]$ and Imq (z) $\in Z$.

Here,

Imq (z) = $\Gamma(\text{Img}\,(z) + \pi)/2\pi\,\rceil$, where $\Gamma\ \rceil$ is the ceiling function.

Now, prove the following.

Theorem 2.2: We have, Log (e^z) = Re (z) + i Imr (z) or Log $(e^z) = z$, if and only if Im $(z) \in (-\pi, \pi]$.

Proof: We have,

Log (e^z) = Log | e^z | + i Arg(e^z) = x + i Arg ($e^{i\text{Im}(z)}$ = x + i Arg ($e^{i(\text{Imr}(z)}$ + 2 π Imq (z)))

NOTES

= x + i Arg ($e^{i\text{Imr}(z)}$) = x + i Imr (z).

To find whether the complex logarithm satisfies the property,

Log (zw) = Log (z) + Log (w).

We define,

Arg (zw) = Arg (z) + Arg (w) + $d2\pi$, where d is 0 or $\pm$ 1.

Definition 1: Define the principal product excess (ppe), ppe (u, v) of two complex numbers by,

ppe (z, w) = (Arg (zw) – Arg (z) – Arg (w))/ (2 π).

Definition 2: Define the complex sign (csgn), csgn (z) of a complex number by,

$$\text{Csgn}(z) = \begin{cases} 1, \text{ if Re}(z) > 0 \text{ or } (\text{Re}(z) = 0 \text{ and Im}(z) > 0) \\ 0, \text{if } z = 0 \\ -1, \text{if Re}(z) < 0 \text{ or } (\text{Re}(z) = 0 \text{ and Im}(z) < 0). \end{cases}$$

The right (left) half-plane is the set of points where csgn (z) is positive (negative).

Lemma

1. ppe (z, w) is always 0 or $\pm$ 1.
2. If either z or w is positive, then ppe (z, w) = 0.
3. If both z and w lie in the right half-plane, then ppe (z, w) = 0.
4. If both z and w lie in the left half-plane, then ppe (z, w) $\neq$ 0.
5. ppe (z, w) = 0 if and only if z lies in the right half-plane.

Theorem 2.3: We have, Log (zw) = Log (z) + Log (w) + 2 π $i$ppe (z, w).

In particular, Log (z^2) = 2 Log (z), if and only if z lies in the right half-plane.

Proof: Log (zw) = log |zw| + i Arg (zw)

= log |z| + log |w| + i (Arg (z) +Arg (w) + 2 π $i$ppe (z, w))

=Log (z) + Log (w) + 2 π $i$ppe (z, w)

Theorem 2.4: We have,

$$\text{Arg}(1/z) = \begin{cases} -\text{Arg}(z), \text{if } z \text{ is not negative} \\ -\text{Arg}(z) + 2, \text{if } z \text{ is negative} \end{cases}$$

Hence,

NOTES

$$\text{Log}\,(1/z) = \begin{cases} -\text{Log}\,(z), \text{ if } z \text{ is not negative} \\ -\text{Log}\,(z) + 2\,i, \text{ if } z \text{ is negative} \end{cases}$$

Proof

We have, $1/z = \bar{z}/|z|^2$, but Arg $(\bar{z})$ = –Arg z, unless z is negative, in which case both Arg (z) and Arg ($1/z$) are equal to π.

We will now define the complex power and exponential functions.

Definition: We define the complex power and exponential functions by,

$z^a = e^{\text{Log}\,(z)a}$, and $a^z = e^{\text{Log}\,(a)z}$ for $a \neq e$.

Now, we will check whether $(e^z)^w$ equals e^{zw}. Here, $(e^z)^w$ involves the exponential function with base e^z and not just e. So, while e^{zw} is a single-valued function, we need to choose a branch in order to make $(e^z)^w$ single valued.

Theorem 2.5: $(e^z)^w = e^{zw}\,e^{-w2\pi i\text{Imq}\,(z)}$.

Proof

$$(e^z)^w = e^{\text{Log}\,(ez)w}$$

$$= e^{(\text{Re}\,(z) + i\text{Imr}\,(z))w}$$

$$= e^{(z-i2\pi\,\text{Imq}\,(z))w}$$

$$= e^{zw}\,e^{-wi2\pi\,\text{Imq}\,(z)}$$

Using Theorem 2.5, we can easily resolve Clausen's paradox.

In Equation (2.3) we said that,

$$(e^{1+2n\pi i})^{1+2n\pi i} = \text{e}^{1+4n\pi i-4n^2\pi^2}.$$

Replace this by,

$$(e^{1+2n\pi i})^{1+2n\pi i} = \text{e}^{1+4n\pi i-4n^2\pi^2}\text{e}^{-(1+2n\pi i)2\pi\,\text{Imq}\,(1+2n\pi i)}$$

$$= e^{1-4n^2\pi^2}e^{-(1+2n\pi i)2\pi in}$$

$$= e^{1-4n^2\pi^2}\,e^{2n\pi i+4n^2\pi^2}$$

$$= e,$$

This agrees with the Equation (2.2).

We can also prove the following corollary.

Corollary: $(e^z)^{1/2} = (-1)^{\text{Imq}\,(z)}\text{e}^{z/2}$ or $(e^z)^{1/2} = e^{z/2}$, if and only if Im $(z) \in ((4n-1)\pi, (4n+1)\pi]$, $n \in Z$.

Theorem 2.6: $(a^z)^w = a^{zw}e^{-w2\pi i\text{Imq}\,(z\,\text{Log}\,(a))}$.

Theorem 2.7: $(zw)^a = z^a w^a e^{a2\pi\,\text{ippe}\,(z,\,w)}$.

Proof

$$(zw)^a = e^{a\text{Log}\,(zw)}$$
$$= e^{a(\text{Log}\,(z) + \text{Log}\,(w) + 2\pi\text{ ippe}\,(z,\,w))}$$
$$= z^a w^a e^{a2\pi\text{ ippe}\,(z,\,w)}.$$

NOTES

Theorem 2.8: $\sqrt{zw} = (-1)^{\text{ppe}\,(z,\,w)}\sqrt{z}\;\sqrt{w}$ or $\sqrt{z^2} = \text{csgn}\,(z)\,z$.

So,

$\sqrt{z^2} = z$, if and only if z lies in the right half-plane.

Theorem 2.9: We have,

$$\sqrt{1/z} = \begin{cases} 1/\sqrt{z}, & \text{if z is not negative} \\ -1/\sqrt{z}, & \text{if z is negative} \end{cases}$$

In particular, if z is real, then,

$$\sqrt{1/z} = \text{sgn (z)}/\sqrt{z},$$

Proof

If z is negative, we have,

$$\sqrt{1/z} = e^{\text{Log}\,(1/z)/2}$$
$$= e^{(-\text{Log}\,(z) + 2\pi i)/2}$$
$$= -e^{-\text{Log}\,(z)/2}$$
$$= -1/\sqrt{z}.$$

Hence, the two square root paradoxes are resolved.

Check Your Progress

1. What is a residue?
2. State Cauchy's residue theorem.
3. State Jordan's lemma.
4. List some paradoxes involving square roots.

2.4 ANSWERS TO 'CHECK YOUR PROGRESS'

1. If $z = a$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z = a$.

   i.e.,
   $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

   The coefficient b_1 of $\frac{1}{(z-a)}$ in the Laurent's series of $f(z)$, is called the residue of $f(z)$ at $z = a$.

NOTES

2. If $f(z)$ is analytic at all points inside and on a simple closed curve C, except for a finite number of isolated singularities $a_1, a_2, \ldots, a_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \text{ [Sum of the residues of } f(z) \text{ at } a_1, a_2, \ldots, a_n]$$

$$= 2\pi i\,[R_1 + R_2 + \ldots + R_n]$$

3. If $f(z)$ is a uniform continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, then $\int_C e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, where C is the semi-circle $|z| = R$ above the real axis and $m > 0$.

4. Following are some paradoxes which includes the square roots:

$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\,\sqrt{-1} = i\,i = -1$

$1/-1 = -1/1$

$1/\sqrt{-1} = \sqrt{-1}/1$

$1/i = i/1$

$i^2 = 1$

2.5 SUMMARY

- If $z = a$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z = a$.

  i.e., $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

  The coefficient b_1 of $\frac{1}{(z-a)}$ in the Laurent's series of $f(z)$, is called the residue of $f(z)$ at $z = a$.
- If $z = a$ is a pole of order 'm', then

  $$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right\}$$
- The residue at an essential singularity of $f(z)$ is found out using the Laurent's expansion of $f(z)$ directly.
- We enclose the singularities $a_1, a_2, \ldots, a_n$ by small non-intersecting circles $C_1, C_2, \ldots, C_n$ with centres at $a_1, a_2, \ldots, a_n$ and raidii $r_1, r_2, \ldots, r_n$ lying wholly inside C.
- If $f(z)$ is a uniform continuous function such that $|(z-a) f(z)| \to 0$ as $|z-a| \to 0$, then $\int_C f(z)\,dz \to 0$ where C is the circle $|z-a| = r$
- If $f(z)$ is a uniform continuous function such that $|(z-a) f(z)| \to 0$

as $|z-a| \to \infty$, then $\int_C f(z)\,dz \to 0$, as $R \to \infty$, where C is the circle $|z-a| = R$.

NOTES

- If $f(z)$ is a uniform continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, then $\int_C e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, where C is the semi-circle $|z| = R$ above the real axis and $m > 0$.
- We define the complex power and exponential functions by, $z^a = e^{\text{Log}(z)a}$, and $a^z = e^{\text{Log}(a)z}$ for $a \neq e$.

2.6 KEY TERMS

- **Residues:** If $z = a$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z = a$.

  i.e., $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

  The coefficient b_1 of $\frac{1}{(z-a)}$ in the Laurent's series of $f(z)$, is called the residue of $f(z)$ at $z = a$.
- **Jordan's Lemma:** If $f(z)$ is a uniform continuous function such that $|f(z)| \to 0$ as $|z| \to \infty$, then $\int_C e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, where C is the semi-circle $|z| = R$ above the real axis and $m > 0$.
- **Complex Power:** We define the complex power and exponential functions by, $z^a = e^{\text{Log}(z)a}$, and $a^z = e^{\text{Log}(a)z}$ for $a \neq e$.

2.7 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write the various formulae for the evaluation of the residue.
2. Find the nature and location of signification of the function $e^{1/2}$.
3. State the Cauchy's lemma for the evaluation of integral.
4. Define the term complex power.

Long-Answer Questions

1. Classify the nature of singularities of the following functions. Find also the residue at each point.

   (i) $f(z) = \frac{z+2}{(z-2)(z+1)^2}$. (ii) $f(z) = \frac{1}{(z^2+1)^2}$.

(iii) $f(z) = \dfrac{z^2}{(z-1)^2\,(z+2)}$. (iv) $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$.

(v) $f(z) = \dfrac{z^2}{(z-2)^2\,(z^2+9)}$. (vi) $f(z) = \dfrac{z^2}{z^4+a^4}$.

NOTES

2. Find the residue at the essential singularity of each of the following functions using Laurent's expansion:

(i) $e^{1/z}$. (ii) $\dfrac{1-e^z}{z^2}$.

(iii) $\dfrac{\cos z}{z}$. (iv) $\dfrac{1-\cosh z}{z^3}$.

3. Find the residues at the isolated singularities of each of the following functions:

(i) $\cot z$ (at $z = 0$). (ii) $\dfrac{ze^z}{(z-1)^2}$.

(iii) $\dfrac{z^2}{z^2+a^2}$. (iv) $\dfrac{1}{z^4+16}$

4. Evaluate the following integrals using Cauchy's residue theorem:

(i) $\displaystyle\int_C \frac{2z-1}{z(z+1)\,(z-3)}\,dz$, where C is the circle $|z| = 2$

(ii) $\displaystyle\int_C \frac{z-3}{z^2+2z+5}\,dz$, where C is $|z+1+i| = 2$.

(iii) $\displaystyle\int_C \frac{e^z}{\cos \pi z}\,dz$, where C is the unit circle $|z| = 1$.

(iv) $\displaystyle\int_C \frac{z^2+2z-2}{z-4}\,dz$, where C is a closed curve containing the point $z = 4$ in its interior.

5. Evaluate the following integrals by contour integration technique:

(i) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$. (ii) $\displaystyle\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}\,d\theta$ $(a > 0)$.

(iii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a > b > 0$. (iv) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, $a > b > 0$.

(v) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$, $a > 0$. (vi) $\displaystyle\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$, $a > 0$.

(vii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (viii) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$.

(ix) $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta} d\theta.$ (x) $\int_0^{\infty} \frac{dx}{1+x^4}.$

NOTES

(xi) $\int_0^{\infty} \frac{dx}{(1+x^2)^2}.$ (xii) $\int_0^{\infty} \frac{x^2\, dx}{(x^2+a^2)^3},\ a>0.$

(xiii) $\int_0^{\infty} \frac{x^2}{(x^2+4)^2\,(x^2+9)} dx.$ (xiv) $\int_0^{\infty} \frac{x^2}{(x^2+1)^3} dx.$

6. Discuss the branches of many valued functions.

2.8 FURTHER READING

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Ahlfors, Lars V. 1978. *Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Lang, Serge. 1998. *Complex Analysis*, 4th Edition. NY: Springer-Verlag New York Inc.

Shastri, Anant R. 2000. *An Introduction to Complex Analysis*, India: Macmillan Publishers India Ltd.

UNIT 3 BILINEAR TRANSFORMATIONS

NOTES

Structure

3.0 INTRODUCTION

In mathematics, bilinear transform (also known as Tustin's method) is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice versa.

A conformal map is a function that locally preserves angles, but not necessarily lengths. Let U and V be open subset of R^n. A function f: $U \rightarrow V$ is called conformal (or angle-preserving) at a point $u_0 \in U$ if it preserves angles between directed curves through u_0, as well as preserving orientation. Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature. Let $z = x + iy$ and $w = u + iv$, then the point z in the z-plan or xy-plane correspond to the points w in the w-plane or uv plane. The corresponding points of the two plans are called images of each other. The correspondence between z-point and w-point is required to be one-one. Hence under suitable conditions the mapping $w = f(z)$ maps a region R of the z- plane to a region R of w-plane and a curve C of the z- plane is mapped to a curve g of the w-plane.

In this unit, you will learn about the bilinear transformation, conformal mappings, spaces of analytic functions and Riemann mapping theorem.

3.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Explain bilinear transformation
- Define conformal mapping
- Describe the spaces of analytic function
- State Montel's theorem and Hurwitz's theorem
- Elaborate on Riemann mapping theorem and Weirstrass' factorization theorem

3.2 BILINEAR TRANSFORMATIONS

The transformation of the form

$$w = \frac{az+b}{cz+d} \qquad ...(3.1)$$

where z,w are complex variables, a, b, c, d are complex constants and $ad - bc \neq 0$ is called a *bilinear transformation*.

Equation (3.1) can be written as

$cwz + dw - az - b = 0$

which is linear in w and as well as in z; that is why the relation in Equation (3.1) is called a bilinear transformation. It is also sometimes called as linear transformation. Bilinear transformation is known as Mobius transformation after the name of A.F. Mobius (August Ferdinand Mobius) (1790 – 1868).

From Equation (3.1), we get

$$w = \frac{a}{c} \cdot \frac{z + b/a}{z + d/c} \qquad ...(3.2)$$

We see that if $b/a = d/c$, i.e., if $ad - bc = 0$, then we get the same values of w for the different value of z and if $ad - bc \neq 0$, then we get the different values of w for different values of z.

$\therefore$ $ad - bc$ is called the determinant of the transformation.

From Equation (3.1), we get $z = \dfrac{dw-b}{-cw+a} = -\dfrac{d}{c} \cdot \dfrac{w - \frac{b}{d}}{w - \frac{a}{c}}$ ...(3.3)

From Equation (3.2), every point of the z-plane is mapped into a unique point in the w-plane except $z = -d/c$.

From Equation (3.3), every point of the w-plane is mapped into a unique point in the z-plane except $w = a/c$.

For example, the transformation $w = \dfrac{3z+5}{z+4}$ is a bilinear transformation because $(3 . 4 - 5 . 4) = 12 - 5 = 7 \neq 0$.

Theorem 3.1: Every bilinear transformation is the resultant of three basic bilinear transformations.

NOTES

Proof: Let the bilinear transformation be,

$$w = \frac{az+b}{cz+d} \qquad ...(3.4)$$

where $ad - bc \neq 0$ and $c \neq 0$.

or $$w = \frac{a}{c}\,\frac{z+\frac{b}{a}}{z+\frac{d}{c}} = \frac{a}{c} + \frac{a}{c}\left[\frac{z+\frac{b}{a}}{z+\frac{d}{c}} - 1\right]$$

$$= \frac{a}{c} + \left[\frac{z+\frac{b}{a}-z-\frac{d}{c}}{z+\frac{d}{c}}\right]\frac{a}{c}$$

$$= \frac{a}{c} + \frac{\left(\frac{b}{a}-\frac{d}{c}\right)}{z+\frac{d}{c}}\,\frac{a}{c}$$

$$= \frac{a}{c} + \frac{(bc-ad)}{ac\left(z+\frac{d}{c}\right)}\cdot\frac{a}{c}$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

This transformation is the resultant of the three transformations,

$$z_1 = z + \frac{d}{c}, \quad z_2 = \frac{1}{z_1}, \quad z_3 = \frac{bc-ad}{c^2} z_2$$

$\therefore$ $$w = \frac{a}{c} + z_3$$

which can be effected in the same way as $z_1 = \frac{d}{c} + z$ is effected. The above three auxiliary transformations are of the form,

$$w = z + \alpha,\ w = \frac{1}{z},\ w = \beta z$$

which are bilinear transformations.

Hence the given bilinear transformation is the resultant of bilinear transformations of the form,

$$w = z + \alpha,\ \ w = \beta z,\ \ w = \frac{1}{z}$$

Basic Transformations

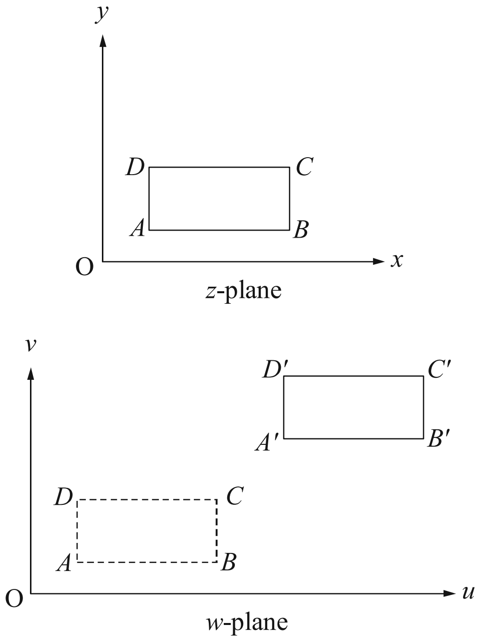
Translation: The transformation $w = z + \alpha$ is called translation, where $\alpha = a + ib$.

$\therefore$ $$w = u + iv = (x + iy) + (a + ib) = (x + a) + i\,(y + b)$$

$\therefore$ $$u = x + a,\ \ v = y + b$$

Therefore, the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + a, y + b)$ in the w-plane. Similarly, other points of z-plane are mapped onto the w-plane. Thus if the w-plane is superposed on the z-plane, the figure of w-plane is shifted through a vector α.

NOTES



Magnification and Rotation

The transformation $w = \beta z$ is called magnification and rotation where w, β, z are complex numbers.

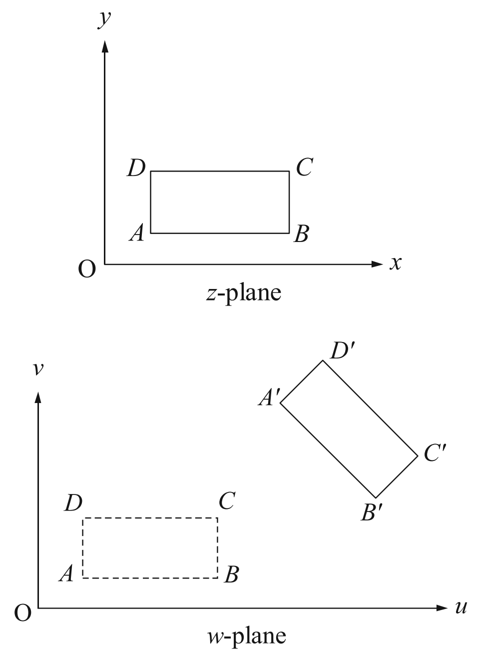
Let $w = Re^{i\phi}$, $\beta = ae^{i\alpha}$, $z = re^{i\theta}$

Then we get from $w = \beta z$,

$$Re^{i\phi} = (ae^{i\alpha})\,(re^{i\theta}) = (ar)\,e^{i(\theta+\alpha)}$$

$$\therefore \qquad R = ar \text{ and } \phi = \theta + \alpha$$

This shows that the transformation $w = \beta z$ corresponds to a rotation together with magnification.



Inversion

The transformation $w = \frac{1}{z}$ is called *inversion*.

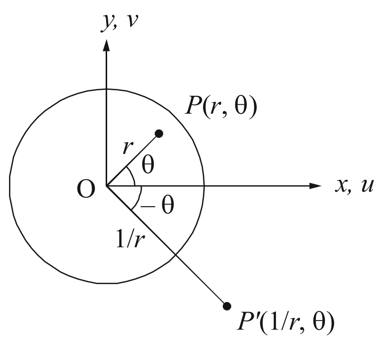
Let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then, we get

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\therefore \quad R = \frac{1}{r} \text{ and } \phi = -\theta$$

Therefore the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'\left(\frac{1}{r}, -\theta\right)$ in the w-plane.

Hence, the transformation is an inversion of z and followed by reflection into the real axis. The points inside the unit circle ($|z| = 1$) map onto the points outside it, and points outside the unit circle into points inside it.



Theorem 3.2: The bilinear transformation,

$$w = \frac{az+b}{cz+d}$$

transforms the circle $\arg \frac{z-z_1}{z-z_2} = \lambda$ into a similar circle $\arg \frac{w-w_1}{w-w_2}$ = Constant where w_1, w_2 correspond to z_1, z_2 respectively.

Proof: Here $w = \frac{az+b}{cz+d}$

Since w_1, w_2 correspond to z_1, z_2 respectively, then

$$w_1 = \frac{az_1+b}{cz_1+d} \text{ and } w_2 = \frac{az_2+b}{cz_2+d}$$

$$\therefore \quad \frac{w-w_1}{w-w_2} = \frac{\dfrac{az+b}{cz+d} - \dfrac{az_1+b}{cz_1+d}}{\dfrac{az+b}{cz+d} - \dfrac{az_2+b}{cz_2+d}} = \frac{cz_2+d}{cz_1+d} \; \frac{z-z_1}{z-z_2}$$

$$= \beta \frac{z-z_1}{z-z_2} \text{ where } \beta = \frac{cz_2+d}{cz_1+d}$$

$$\therefore \quad \arg\left(\frac{w-w_1}{w-w_2}\right) = \arg\left(\beta \cdot \frac{z-z_1}{z-z_2}\right)$$

$$= \arg\beta + \arg\left(\frac{z - z_1}{z - z_2}\right)$$

$$= \arg\beta + \lambda$$

NOTES

$$\therefore \quad \arg\left(\frac{w - w_1}{w - w_2}\right) = k$$

where k is real which is a circle in a w-plane passing through two fixed points w_1, w_2 which are the images of z_1, z_2.

Cross-Ratio

Let z_1, z_2, z_3, z_4 be the four points taken in order, then the ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called the cross-ratio of z_1, z_2, z_3, z_4 which is denoted by (z_1, z_2, z_3, z_4).

Theorem 3.3: Every bilinear transformation preserves the cross-ratio.

Proof: Let $w = \frac{az + b}{cz + d}$ be a bilinear transformation where w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively, then we shall prove that the cross-ratio of w_1, w_2, w_3, w_4 is equal to the cross ratio of z_1, z_2, z_3, z_4, i.e.,

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

Since w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively, then

$$w_1 = \frac{az_1 + b}{cz_1 + d},\ w_2 = \frac{az_2 + b}{cz_2 + d},\ w_3 = \frac{az_3 + b}{cz_3 + d}$$

and $$w_4 = \frac{az_4 + b}{cz_4 + d}$$

$$\therefore \quad w_1 - w_2 = \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \quad ...(3.5)$$

Similarly, $$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \quad ...(3.6)$$

$$w_3 - w_1 = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)} \quad ...(3.7)$$

and $$w_4 - w_1 = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)} \quad ...(3.8)$$

From Equations (3.5), (3.6), (3.7) and (3.8), we get

$$(w_1, w_2, w_3, w_4) = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_4 - z_1)}$$

or $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$.

Example 3.1: Find the bilinear transformation which transforms the points z_1, z_2, z_3 of the z-plane respectively into the points w_1, w_2, w_3 of the w-plane.

Solution: Let $w = \frac{az + b}{cz + d}$ be a bilinear transformation.

Since w_1, w_2, w_3, are the image of z_1, z_2, z_3, respectively, then,

$$w_1 = \frac{az_1 + b}{cz_1 + d},\ w_2 = \frac{az_2 + b}{cz_2 + d} \text{ and } w_3 = \frac{az_3 + b}{cz_3 + d}$$

NOTES

$$\therefore \quad w - w_1 = \frac{az + b}{cz + d} - \frac{az_1 + b}{cz_1 + d} = \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)} \quad ... (1)$$

Similarly, $w_1 - w_2 = \dfrac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$ ...(2)

$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \quad ...(3)$$

and $w_3 - w = \dfrac{(ad - bc)(z_3 - z)}{(cz_3 + d)(cz + d)}$ ...(4)

From Equations (1), (2), (3) and (4) we get

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

This is the required transformation. This transformation can be written in the form,

$w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where α, β, γ and δ are complex constants.

Example 3.2: Find the bilinear transformation which maps the points $z = \infty$, i, 0 into the points $w = 0$, i, ∞ respectively.

Solution: We know that the bilinear transformation, mapping $z = z_1, z_2, z_3$ into
$w = w_1, w_2, w_3$ respectively, is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Here $z_2 = i,\ z_3 = 0,\ w_1 = 0,\ w_2 = i,\ z_1 \to \infty$ and $w_3 \to \infty$.

$$\therefore \quad \frac{(w - o)(i - w_3)}{(o - i)(w_3 - w)} = \frac{(z - z_1)(i - o)}{(z_1 - i)(o - z)}$$

or
$$\frac{w\left(\frac{i}{w_3} - 1\right)}{(-i)\left(1 - \frac{w}{w_3}\right)} = \frac{\left(\frac{z}{z_1} - 1\right)(i)}{\left(1 - \frac{i}{z_1}\right)(0 - z)}$$

or
$$\frac{w}{i} = \frac{i}{z} \qquad (\because z_1 \to \infty, w_3 \to \infty)$$

or
$$w = -\frac{1}{z}$$

which is the required transformation.

Example 3.3: Find the bilinear transformation which transforms the points $z = 2, 1, 0,$ into $w = 1, 0, i$.

Solution: We know that the bilinear transformation which transforms the points $z = z_1, z_2, z_3$ resectively into $w = w_1, w_2, w_3$ is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

NOTES

or $$\frac{i-iw}{i-w} = \frac{2-z}{z}$$

$$(i - iw)z = (i - w)(2 - z)$$

or $$iz - iwz = i(2 - z) - w(2 - z)$$

or $$\{-iz + 2 - z\}w = i(2 - z) - iz = 2i - 2iz$$

or $$w = \frac{2i - 2iz}{2-(1+i)z} = \frac{2i(z-1)}{z(1+i)-2}$$

which is the required transformation.

Example 3.4: Find the bilinear transformation which maps the points $z = 1$, $z = i$, and $z = -1$ into the points $w = i$, $w = 0$ and $w = -i$.

Solution: We know that the bilinear transformation which transforms the points $z = z_1, z_2, z_3$ respectively into $w = w_1, w_2, w_3$ is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Here $z_1 = 1, z_2 = i, z_3 = -1, w_1 = i, w_2 = 0$ and $w_3 = -i$

$$\therefore \quad \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

or $$\frac{i(w-i)}{-i(w+i)} = \frac{(z-1)(1+i)}{(z+1)(i-1)}$$

or $$(w - i)(z + 1)(i - 1) = -(w + i)(z - 1)(1 + i)$$

or $$(wi - i^2 - w + i)(z + 1) = (1 - z)(w + wi + i^2 + i)$$

or $$(wi - w + 1 + i)(z + 1) = (1 - z)(w + wi + i - 1)$$

or $$w\{(i - 1)(z + 1) - (1 - z)(1 + i)\} = \{(i - 1)(1 - z) - (z + 1)(1 + i)\}$$

or $$w\{zi - z + i - 1 - 1 - i + z + zi\} = \{i - 1 - zi + z - z - zi - 1 - i\}$$

or $$w\{2zi - 2\} = \{-2 - 2zi\}$$

or $$w = \frac{-(zi+1)}{zi-1}$$

which is the required transformation.

Bilinear Transformation of a Circle

Theorem 3.4: The bilinear transformation $w = \frac{az+b}{cz+d}$ transforms a circle of the z-plane into a circle of the w-plane and inverse points transform into inverse points.

Proof: Here the transformation is $w = \frac{az+b}{cz+d}$ ...(3.9)

We know that $\left|\frac{z-p}{z-q}\right| = r$ ...(3.10)

represents a circle in the z-plane with inverse points p, q. If $r = 1$, the equation represents a line which is the right bisector of the join of the points p, q.

From Equations (3.9), we see that the points p, q in the z-plane correspond respectively to the points $\frac{ap+b}{cp+d}$ and $\frac{aq+b}{cq+d}$ in the w-plane.

From Equation (3.9) and (3.10), we get

$$\left|\frac{w-\dfrac{ap+b}{cp+d}}{w-\dfrac{aq+b}{cq+d}}\right| = k\left|\frac{cq+d}{cp+d}\right| \quad ...(3.11)$$

where $z = \dfrac{dw-b}{-cw+a}$

This equation shows that it represents a circle in the w-plane, whose inverse points are,

$$\frac{ap+b}{cp+d} \text{ and } \frac{aq+b}{cq+d}$$

Hence a circle in the z-plane transforms into a circle in the w-plane, and the inverse points transform into the inverse points.

Example 3.5: Find the condition that the transformation $w = \dfrac{az+b}{cz+d}$ transforms the unit circle in the w-plane into a straight line in the z-plane.

Solution: Here the transformation is,

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\cdot\frac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}}$$

Therefore the unit circle $|w| = 1$ in the w-plane gives,

$$|w| = 1 = \left|\frac{a}{c}\right|\left|\frac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}}\right|$$

or $$\left|\frac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}}\right| = \left|\frac{c}{a}\right|$$

which represents a line when $\left|\dfrac{c}{a}\right| = 1$ or $|a| = |c|$.

Hence the required condition is $|a| = |c|$.

Notes:

1. The equation $\left|\dfrac{z-p}{z-q}\right| = k$

represents a line or a circle according as $k = 1$ or $k \neq 1$.

2. The bilinear transformation $w = \frac{az+b}{cz+d}$ transforms a circle in the z-plane into a straight line in the w-plane and the inverse points transform into points symmetrical about this line.

Example 3.6: Find the bilinear transformation which transforms the plane $I(z) > 0$ into the unit circle $|w| \leq 1$.

Solution: The bilinear transformation is

$$w = \frac{az+b}{cz+d} = \frac{a}{c} \cdot \frac{z + \frac{b}{a}}{z + \frac{d}{c}} \qquad ...(1)$$

NOTES

which transforms $I(z) = 0$ into $|w| = 1$.

Therefore, the real axis in the z-plane transforms into the unit circle in the w-plane. Hence, the points w, $1/\bar{w}$ (inverse with respect to the unit circle) transform respectively into the points z, $\bar{z}$ (inverse with respect to the real axis in the z-plane). The points $w = 0$, $w = \infty$ correspond to the point α, $\bar{\alpha}$.

Therefore from Equation (1) we get $\frac{-b}{a} = \alpha$, $-\frac{d}{c} = \bar{\alpha}$

$$\therefore \qquad w = \frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}} \qquad ...(2)$$

The point $z = 0$ corresponds to the point $|w| = 1$; then from Equation (2), we get,

$$|w| = 1 = \left|\frac{a}{c}\right| \left|\frac{-\alpha}{-\alpha}\right| \text{ or } 1 = \left|\frac{a}{c}\right| \left|\frac{\alpha}{\bar{\alpha}}\right|$$

$$\text{or} \qquad 1 = \left|\frac{a}{c}\right| \qquad (\because \ |\alpha| = |\bar{\alpha}|)$$

$$\text{or} \qquad \frac{a}{c} = e^{i\lambda} \qquad ...(3)$$

where λ is real.

From Equations (2) and (3) we get

$$w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}$$

which is the required transformation for mapping $I(z) = 0$ into $|w| = 1$.

$$\text{Again} \quad w\bar{w} - 1 = \frac{z-\alpha}{z-\bar{\alpha}} e^{i\lambda} \frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha} e^{-i\lambda} - 1$$

$$\text{or} \quad |w|^2 - 1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\bar{\alpha})(\bar{z}-\alpha)} - 1 = \frac{(z-\bar{z})(\alpha-\bar{\alpha})}{|z-\bar{\alpha}|^2}$$

$$= \frac{2i\,I(z)\,2i\,I(\alpha)}{|z-\bar{\alpha}|^2} = -\frac{4I(z)\,I(\alpha)}{|z-\bar{\alpha}|^2}$$

Since $w = 0$ corresponds to α, then $I(\alpha) > 0$.

Hence $|w|^2 - 1 < 0$ for $I(z) > 0$.

or $|w|^2 < 1$ corresponds to $I(z) > 0$,

Hence $w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}$ which is the required transformation.

Example 3.7: Find the bilinear transformation which transforms the half plane $Re(z) \geq 0$ into the unit circle $|w| \leq 1$.

Solution: The bilinear transformation is,

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\frac{z+\frac{b}{a}}{z+\frac{d}{c}} \quad ...(1)$$

which transforms $Re(z) = 0$ into $|w| = 1$. Therefore, the imaginary axis in z-plane transforms into the unit circle in w-plane.

Hence the points w, $1/\bar{w}$ (inverse with respect to the unit circle) in w-plane transform into the points z, $-\bar{z}$ (inverse with respect to the imaginary axis) in z-plane.

The points $w = 0, \infty$ correspond to the points $\alpha, -\bar{\alpha}$.

Hence from Equation (1) we get $\frac{-b}{a} = \alpha$, $\frac{-d}{c} = -\bar{\alpha}$

$$\therefore \quad w = \frac{a}{c}\frac{z-\alpha}{z+\bar{\alpha}} \quad ...(2)$$

The point $z = 0$ corresponds to the point $|w| = 1$, then from Equation (2), we get

$$|w| = 1 = \left|\frac{a}{c}\right| \left|\frac{-\alpha}{\bar{\alpha}}\right|$$

or $\quad 1 = \left|\frac{a}{c}\right| \quad (\because |\alpha| = |\bar{\alpha}|)$

or $\quad \left|\frac{a}{c}\right| = e^{i\lambda} \quad ...(3)$

where λ is real.

From Equations (2) and (3), we get

$$w = e^{i\lambda}\frac{z-\alpha}{z+\bar{\alpha}}$$

which is the required transformation for mapping $Re(z)$ into $|w| = 1$.

Since $w = 0$ corresponds to α, then $Re(z) \geq 0$.

$$\therefore \quad w\bar{w} - 1 = \frac{z-\alpha}{z+\bar{\alpha}}e^{i\lambda} \cdot \frac{\bar{z}-\bar{\alpha}}{\bar{z}+\alpha} e^{-i\lambda} - 1$$

$$\text{or} \quad |w|^2 - 1 = \frac{(z-\alpha)}{(z+\bar{\alpha})}\frac{(\bar{z}-\bar{\alpha})}{(\bar{z}+\alpha)} - 1$$

$$= -\frac{(z-\bar{z})(\alpha+\bar{\alpha})}{|(z+\bar{\alpha}|^2}$$

$$= -\frac{2\,\mathrm{Re}(z)\,2\,\mathrm{Re}(\alpha)}{|(z+\bar{\alpha}|^2} \text{ where } \mathrm{Re}(\alpha) > 0$$

$\therefore \quad |w|^2 - 1 < 0$ for $\mathrm{Re}(z) > 0$

That is, $|w| < 1$ corresponds to $\mathrm{Re}(z) > 0$.

Hence $w = \frac{z-\alpha}{z+\bar{\alpha}}$ is the required transformation.

Example 3.8: Find all the bilinear transformations which transform the unit circle $|z| \leq 1$ into unit circle $|w| \leq 1$.

Solution: The bilinear transformation is

$$w = \frac{az+b}{cz+d} = \frac{a}{c}\,\frac{z+\dfrac{b}{a}}{z+\dfrac{d}{c}} \qquad ...(1)$$

NOTES

which transforms the circle $|z| = 1$ into the circle $|w| = 1$.

Hence the points w, $1/\bar{w}$ (inverse with respect to the circle $|w| = 1$) in w-plane correspond to the points z, $\frac{1}{\bar{z}}$ (inverse with respect to the circle $|z| = 1$) in z-plane.

The points $w = 0, \infty$ correspond to the points α, $\frac{1}{\bar{a}}$, then from Equation (1) we get

$$\frac{-b}{a} = \alpha, \quad \frac{-d}{c} = \frac{1}{\bar{\alpha}}$$

$$\therefore \qquad w = \frac{a}{c}\,\frac{z-\alpha}{z-\dfrac{1}{\bar{\alpha}}} \qquad ...(2)$$

The point $z = 1$ corresponds to the point $|w| = 1$; then from Equation (2) we get

$$|w| = 1 = \left|\frac{a}{c}\cdot\frac{1-\alpha}{1-\dfrac{1}{\bar{\alpha}}}\right| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{1-\alpha}{1-\bar{\alpha}}\right|$$

or $$1 = \left|\frac{a\bar{\alpha}}{c}\right| \qquad (\because |1-\alpha| = |1-\bar{\alpha}|)$$

or $$\left|\frac{a\bar{\alpha}}{c}\right| = e^{i\lambda} \text{ where } \lambda \text{ is real.} \qquad ...(3)$$

From Equations (2) and (3), we get

$$w = e^{i\lambda}\,\frac{z-\alpha}{z\bar{\alpha}-1}$$

This is the transformation which maps $|z| = 1$ into $|w| = 1$.

Again $$w\bar{w} - 1 = e^{i\lambda}\frac{z-\alpha}{z\bar{\alpha}-1}\, e^{-i\lambda}\frac{\bar{z}-\bar{\alpha}}{\bar{z}\alpha-1} - 1$$

or $$|w|^2 - 1 = \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z\bar{\alpha}-1)(\bar{z}\alpha-1)} - 1$$

$$= -\frac{(1-z\bar{z})(1-\alpha\bar{\alpha})}{|z\bar{\alpha}-1|^2}$$

$$= -\frac{(1-|z|^2)(1-|\alpha|^2)}{|z\bar{\alpha}-1|^2} \text{ where } |\alpha| < 1$$

or $|w| < 1$ corresponds to $|z| < 1$.

Hence $w = e^{i\lambda}\,\frac{z-\alpha}{z\bar{\alpha}-1}$ is the required transformation.

Example 3.9: Show that the transformation $w = \frac{2z+3}{z-4}$ transform the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$ where $w = u + iv$.

NOTES

Solution: Here the transformation is

$$w = \frac{2z+3}{z-4}$$

or $\quad wz - 4w = 2z + 3$

or $\quad (wz - 2z) = 4w + 3$

or $$z = \frac{4w+3}{w-2}$$

$\therefore$ $$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$$

The given circle is $x^2 + y^2 - 4x = 0$

or $(x + iy)(x - iy) - 4x = 0$

or $\quad z\bar{z} - 2(z + \bar{z}) = 0 \qquad [\because z + \bar{z} = 2x]$

or $$\frac{4w+3}{w-2} \cdot \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

or $\quad (4w + 3)(4\bar{w} + 3) - 2[(4w + 3)(\bar{w} - 2) + (4\bar{w} + 3)(w - 2)] = 0$

or $16w\bar{w} + 12w + 12\bar{w} + 9 - 8w\bar{w} + 16w - 6\bar{w} + 12 - 8w\bar{w} + 16\bar{w} - 6w + 12 = 0$

or $\quad 22w + 22\bar{w} + 33 = 0$

or $\quad 2(2 + \bar{w}) + 3 = 0$

or $\quad 4u + 3 = v$, writing $w = u + iv$.

which is a straight line in the w-plane.

Example 3.10: Find the mapping of x-axis under the transformation $w = \frac{i-z}{i+z}$, onto the w-plane.

Solution: The given transformation is $w = \frac{i-z}{i+z}$

or $$u + iv = \frac{i - x - iy}{i + x + iy} = \frac{-x - i(y-1)}{x + i(y+1)}$$

$$= \frac{-x - i(y-1)}{x + i(y+1)} \cdot \frac{x - i(y+1)}{x - i(y+1)}$$

$$= \frac{-x^2 + ix + ix - y^2 + 1}{x^2 + (y+1)^2} = \frac{-x^2 - y^2 + 1 + i(2x)}{x^2 + (y+1)^2}$$

Equating real and imaginary parts, we get

$$u = \frac{-x^2 - y^2 + 1}{x^2 + (y+1)^2}, \quad v = \frac{2x}{x^2 + (y+1)^2}$$

To get the mapping of x-axis, we put $y = 0$ in the above equation; and thus get

$$u = \frac{-x^2 + 1}{x^2 + 1} \qquad ...(1)$$

NOTES

and $$v = \frac{2x}{x^2+1} \qquad ...(2)$$

From Equation (1), we get $ux^2 + u = -x^2 + 1$

or $x^2(u+1) = 1 - u$

or $$x^2 = \frac{1-u}{1+u}$$

Putting the value of x in Equation (2), we get

$$v = \frac{\sqrt{\frac{1-u}{1+u}}}{\frac{1-u}{1+u}+1} = \sqrt{\frac{1-u}{1+u}}\cdot\frac{1+u}{2} = \frac{1}{2}\sqrt{(1-u)(1+u)}$$

or $$v^2 = \frac{1-u^2}{4}$$

or $4v^2 + u^2 = 1$

or $$\frac{u^2}{4} + \frac{v^2}{1} = 1$$

which represents an ellipse.

Critical Point: A point at which $f'(z) = 0$ is called a critical point of the transformation.

Some Special Transformations

1. The transformation $w = z^n$ where n is a positive integer:

Here $w = z^n$, then $\frac{dw}{dz} = nz^{n-1}$

$\therefore$ $\frac{dw}{dz} = 0$ at $z = 0$.

Hence the transformation is conformal at all points except at $z = 0$.

Let $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $w = z^n$ gives,

$$Re^{i\phi} = r^n e^{in\theta}$$

$\therefore$ $R = r^n \qquad ...(3.12)$

and $\phi = n\theta \qquad ...(3.13)$

From the Equations (3.12) and (3.13), we conclude that:

(*i*) The circle $r = a$ = constant about the origin in the z-plane is transformed on the circle $R = a^n$ = constant about the origin in the w-plane.

(*ii*) The lines $\theta = \beta$ = constant about the origin in z-plane is transformed into the lines $\phi = n\beta$ = constant about the origin in the w-plane and the slope of ϕ-line is n times the slope of θ-line.

(*iii*) The circular sector with its vertex at origin in the z-plane is transformed into a circular sector with its vertex at origin and n times the central angle.

(*iv*) The interior of the circular sector with central angle π/n is transformed conformably upon the upper half plane $I(w) > 0$.

2. The transformation $w = z^2$:

Here, $w = z^2$, then $\frac{dw}{dz} = 2z$

NOTES

$\therefore$ $\frac{dw}{dz} = 0$ for $z = 0$

Hence the transformation is conformal at all points except at $z = 0$.

Now $w = z^2$ gives $u + iv = (x + iy)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + i\,(2xy)$

$\therefore$ $u = x^2 - y^2$ ...(3.14)

and $v = 2xy$ ...(3.15)

From the Equations (3.14) and (3.15), we may include the following facts of the transformation.

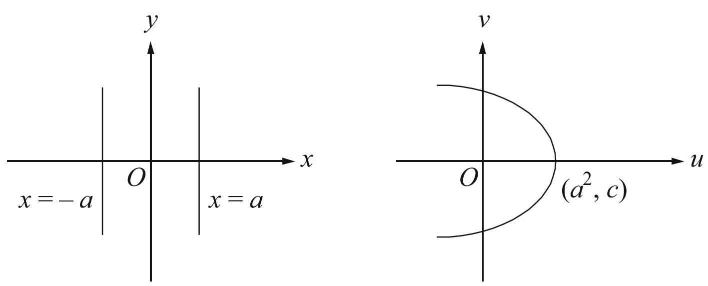
(*i*) When x = constant = a, then from Equations (3.14) and (3.15), we get

$$u = a^2 - y^2 \text{ and } v = 2ay$$

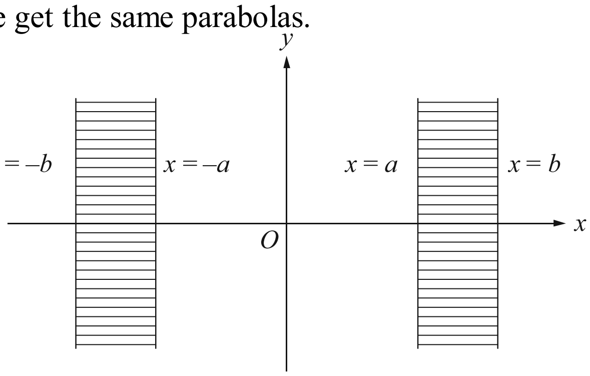
Eliminating y from the above relations, we get

$$v^2 = 4a^2y^2 = 4a^2\,[a^2 - u^2]$$

Therefore the line $x = a$ in the z-plane is transformed into the parabola in the w-plane whose vertex is at (a^2, c) and focus is at the origin. For $a = -a$, we get the same parabola.

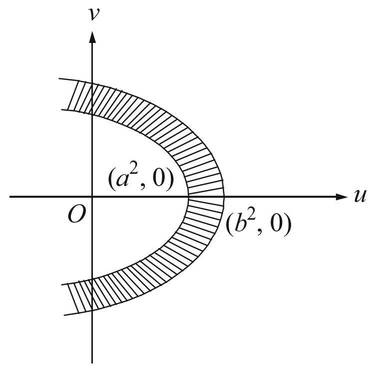


(*ii*) For a strip between the lines $x = a$ and $x = b$ in the z-plane, the line $x = \lambda$ where $a \leq \lambda \leq b$ is transformed into the parabola $v^2 = 4\lambda^2(\lambda^2 - u)$ in the w-plane. Hence, the area enclosed by the lines $x = a$ and $x = b$ in the z-plane is transformed to the area enclosed by the parabola $v^2 = 4a^2\,(a^2 - u)$ and $v^2 = 4b^2\,(b^2 - u)$ in the w-plane. For $x = -a$ and $\lambda = -b$, we get the same parabolas.



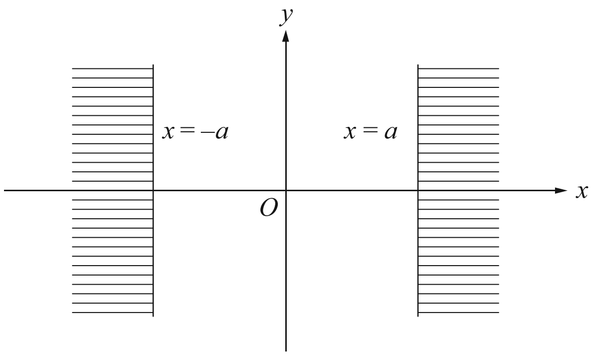
z-plane

NOTES

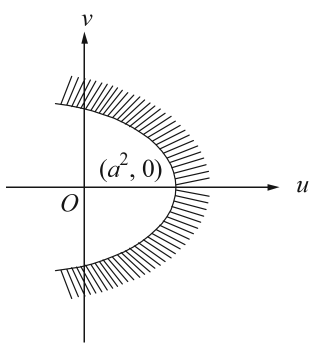


w-plane

(*iii*) The region $x \geq a$ and $x \leq -a$ in the z-plane is transformed into the exterior of the parabola $v^2 = 4a^2(a^2 - u)$ in the w-plane.

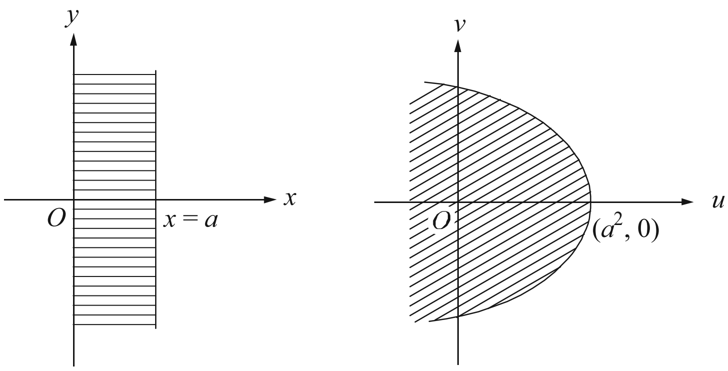


z-plane



w-plane

(*iv*) The region $0 \leq x \leq a$ in the z-plane is transformed into the interior of the parabola $v^2 = 4a^2(a^2 - u)$ in the w-plane.



z-plane *w*-plane

3. The transformation $z = \sqrt{w}$ (which is inverse mapping of $w = z^2$):

Here, $z = \sqrt{w}$, then $w = z^2$ or $u + iv = (x + iy)^2 = x^2 - y^2 + i\,(2xy)$

$$\therefore \quad u = x^2 - y^2 \quad ...(3.16)$$

$$\text{and} \quad v = 2xy \quad ...(3.17)$$

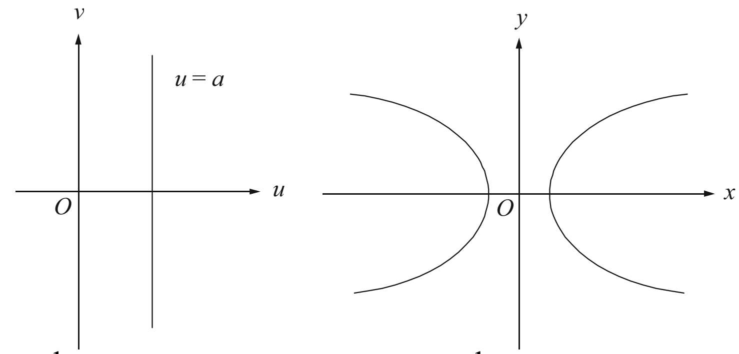
From the Equations (3.16) and (3.17), we may conclude the following facts of the transformation.

(*i*) When u = constant = a (> 0), then from Equation (3.16) we get

$$x^2 - y^2 = a$$

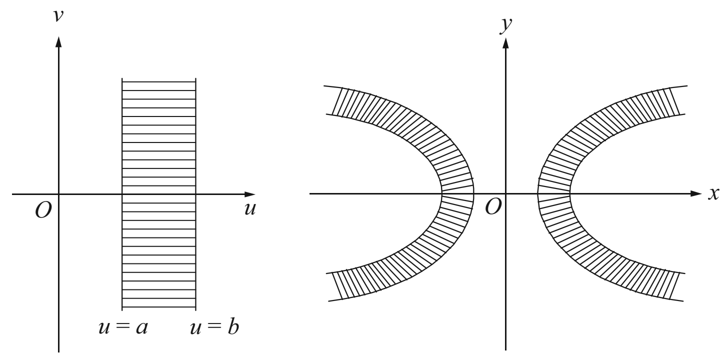
which is a rectangular hyperbola.

Therefore the line $u = a$ in the w-plane is transformed into the rectangular hyperbola $x^2 - y^2 = u$ in the z-plane.



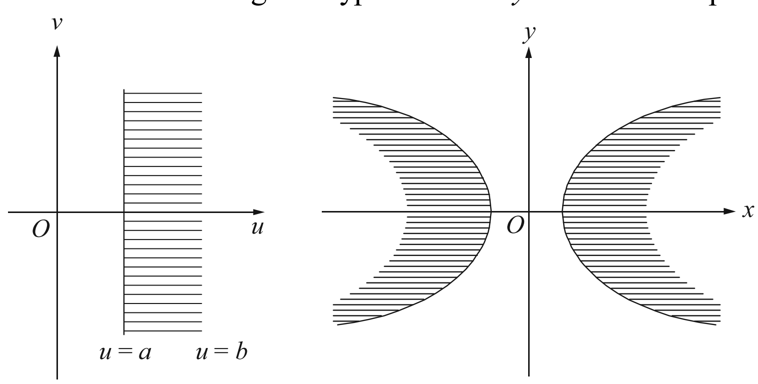
w-plane *z*-plane

(*ii*) The strip between the lines $u = a$ and $u = b$ in the w-plane is transformed to the region enclosed between the rectangular hyperbolas $x^2 - y^2 = a$ and $x^2 - y^2 = b$ in the z-plane.



w-plane *z*-plane

(*iii*) The region $u \geq a$ in the w-plane is transformed to the region of interior of the rectangular hyperbola $x^2 - y^2 = a^2$ in the z-plane.

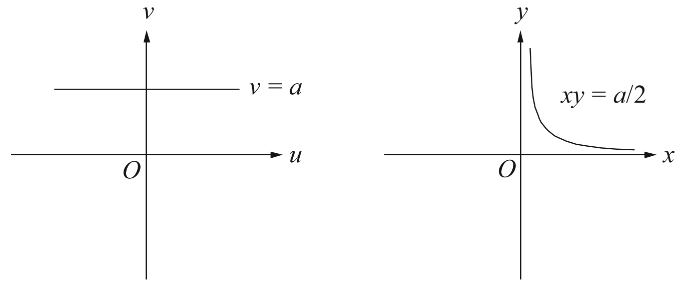


w-plane *z*-plane

NOTES

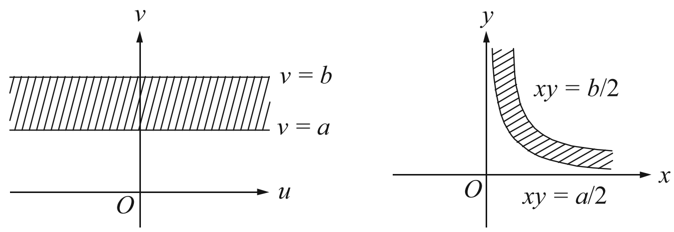
(*iv*) When v = constant = a (> 0), then we get from Equation (3.17), $xy = a/2$ which is a rectangular hyperbola.

Hence the line $v = a$ in the w-plane is transformed to the rectangular hyperbola $xy = a/2$ in the z-plane.



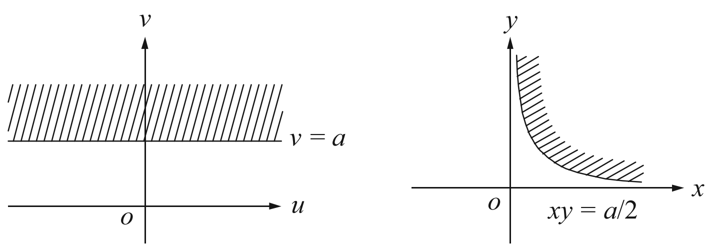
w-plane z-plane

(*v*) The region between the lines $v = a$ and $v = b$ in the w-plane is transformed to the region enclosed by the rectangular hyperbolas $xy = a/2$ and $xy = b/2$ in the z-plane.



w-plane z-plane

(*vi*) The region $v \geq a$ in the w-plane is transformed to the region of interior of the rectangular hyperbola $xy = a/2$ in the z-plane.



w-plane z-plane

4. The transformation $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$:

Here $\quad w = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then $\frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$

$\therefore \quad \frac{dw}{dz} = 0$ for $z = \pm 1$

Hence, the given transformation is conformal except at $z = \pm 1$.

Let $z = re^{i\theta}$, then $w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left[re^{i\theta} + \frac{1}{r}e^{-i\theta}\right]$

or $\quad u + iv = \frac{1}{2}\left[r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)\right]$

NOTES

$$= \left[\frac{1}{2}\cos\theta\left(r+\frac{1}{r}\right)+\frac{i}{2}\left(r-\frac{1}{r}\right)\sin\theta\right]$$

$\therefore \qquad u = \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta \qquad ...(3.18)$

and $\qquad v = \frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta \qquad ...(3.19)$

Then we consider the following cases of the transformation:

Case (i) When r = constant, then from Equations (3.18) and (3.19), we get

$$\cos^2\theta + \sin^2\theta = \frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2}$$

or $$\frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2} = 1$$

When $r > 1$, the circle $|z| = r$ in the z-plane is transformed to the ellipse

$$\frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2} = 1$$

in the w-plane.

When $r = 1$, the circle $|z| = 1$ in the plane is transformed to the part of the real axis between –1 to 1 in the w-plane described twice.

Case (ii) When θ = constant, then from Equations (3.18) and (3.19), we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \frac{1}{4}\left(r+\frac{1}{r}\right)^2 - \frac{1}{2}\left(r-\frac{1}{r}\right)^2 = 1$$

Hence the radial line θ = constant in the z-plane is transformed to the hyperbola in the w-plane.

5. The transformation $w = e^z$:

Here, $\qquad w = e^z$, then $\frac{dw}{dz} = e^z$

$\therefore \qquad \frac{dw}{dz} \neq 0$ for all z

Hence the given transformation is conformal for all values of z in the z-plane.

Let $w = Re^{i\phi}$ and $z = x + iy$, then $Re^{i\phi} = e^{x+iy} = e^x . e^{iy}$

$\therefore \qquad R = e^x \qquad ...(3.20)$

and $\qquad \phi = y \qquad ...(3.21)$

Now we consider the following cases of the transformation:

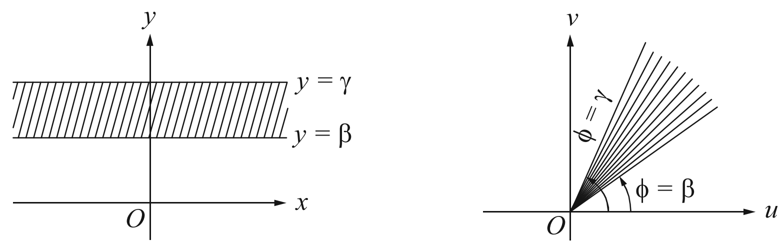
(i) When y = constant = β, then from Equation (3.31), we get

$$\phi = \beta$$

NOTES

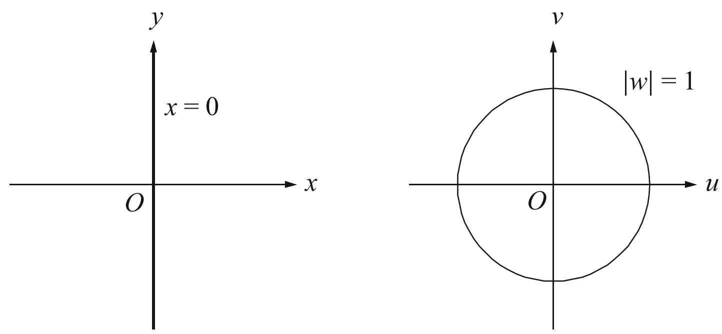
Thus, the line $y = \beta$ in the z-plane is transformed to the line $\phi = \beta$ in the w-plane.

(*ii*) The region enclosed by the lines $y = \beta$ and $y = \gamma$ in the z-plane is transformed to the region bounded by the radial lines $\phi = \beta$ and $\phi = \gamma$ in the w-plane.



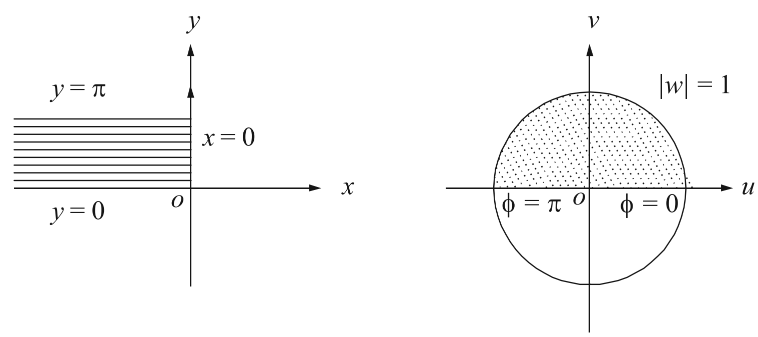
z-plane *w*-plane

(*iii*) When $x = 0$, then from Equation (3.21) we get $R = e^0 = 1$, i.e., $|w| = 1$. Hence, the imaginary axis (i.e., $x = 0$) in the z-plane is transformed to the unit circle $|w| = 1$ in the w-plane.



z-plane *w*-plane

(*iv*) The region bounded by the lines $y = 0$, $y = \pi$ and $x = 0$ in the z-plane is transformed to the region bounded by $\phi = 0$, $\phi = \pi$ and $|w| = 1$ in the w-plane.



z-plane *w*-plane

Example 3.11: Find the bilinear transformation which transforms the circle $|z| = 1$ onto $|w| = 1$ and makes the point $z = 1, -1$ correspond to $w = 1, -1$ respectively.

Solution: We know that the transformation which transform the circle $|z| = 1$ into $|w| = 1$ is,

$$w = e^{i\lambda} \frac{z - \alpha}{z\bar{\alpha} - 1} \text{ where } \lambda \text{ is real and } |\alpha| < 1$$

Since the points $z = 1, -1$ maps into the points $w = 1, -1$, we get

NOTES

$$1 = e^{i\lambda} \frac{1-\alpha}{\bar{\alpha}-1} \qquad ...(1)$$

and $$-1 = e^{i\lambda} \frac{-1-\alpha}{-\bar{\alpha}-1}$$

or $$1 = e^{i\lambda} \frac{(-1-\alpha)}{\bar{\alpha}+1} \qquad ...(2)$$

From Equations (1) and (2), we get

$$e^{i\lambda} \frac{(1-\alpha)}{\bar{\alpha}-1} = e^{i\lambda} \frac{(-1-\alpha)}{\bar{\alpha}+1}$$

or $$\frac{1-\alpha}{\bar{\alpha}-1} = \frac{-(1+\alpha)}{1+\bar{\alpha}}$$

or $1 + \bar{\alpha} - \alpha - \alpha\bar{\alpha} = -\bar{\alpha} + 1 - \alpha\bar{\alpha} + \alpha$

or $$2\bar{\alpha} = 2\alpha \Rightarrow \alpha = \bar{\alpha}$$

Hence, the required transformation is $w = e^{i\lambda} \dfrac{z-\alpha}{z\alpha - 1}$.

Example 3.12: Find the image of $|z - 2i| = 2$ under the mapping $w = \dfrac{1}{z}$.

Solution: Here, $w = \dfrac{1}{z}$; then $z = \dfrac{1}{w}$

or $$x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} + \frac{-v}{u^2+v^2} i$$

∴ $$x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2} \qquad ...(1)$$

Now $|z - 2i| = 2$ or $|x + iy - 2i| = 2$

or $|x + i(y-2)| = 2$

or $x^2 + (y-2)^2 = 4$

or $$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2} - 2\right)^2 = 4 \qquad \text{[By Equation (1)]}$$

or $u^2 + [v + 2(u^2+v^2)]^2 = 4(u^2+v^2)^2$

or $u^2 + v^2 + 4u^2v + 4v^3 + 4(u^2+v^2)^2 = 4(u^2+v^2)^2$

or $u^2 + v^2 + 4u^2v + 4v^3 = 0$

or $u^2 + v^2 + 4v(u^2+v^2) = 0$

or $(u^2+v^2)(1+4v) = 0 \qquad (\because u^2 + v^2 \neq 0)$

Hence, $1 + 4v = 0$ is the required equation of the image.

Example 3.13: Find the image of the straight line $x + y = 1$ under the transformation $w = z^2$.

Solution: Here $w = z^2$, then $u + iv = (x+iy)^2 = x^2 - y^2 + 2xyi$

∴ $$u = x^2 - y^2 \text{ and } v = 2xy \qquad ...(1)$$

Now $x + y = 1$ or $(x+y)^2 = 1$

or $x^2 + y^2 + 2xy = 1$ or $x^2 + y^2 = 1 - 2xy = 1 - v$ [By Equation (1)]

or $(x^2+y^2)^2 = (`1 - v)^2$ [By Squaring] or $(x^2 - y^2)^2 + 4x^2y^2 = (1 - v)^2$

or $\quad u^2 + v^2 = 1 - 2u + v^2$ [By Equation (1)]

or $\quad u^2 + 2v = 1$

NOTES

Hence, the required transformation is $u^2 + 2v = 1$.

Example 3.14; Find the image of the infinite strip $1/6 < y < 1/3$ under the transformation $w = 1/z$ and draw the graphical region.

Solution: Here, $w = \dfrac{1}{z}$

$$\therefore \quad z = \frac{1}{w} = \frac{1}{u + iv}$$

$$\text{or} \quad x + iy = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2} i$$

$$\therefore \quad x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

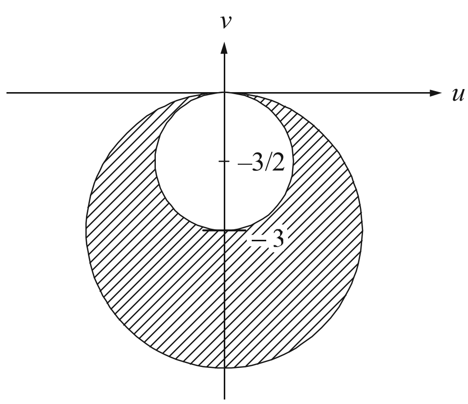
When $y < \dfrac{1}{3} : \dfrac{-v}{u^2 + v^2} < \dfrac{1}{3}$ or $u^2 + v^2 + 3v > 0$ or $u^2 + \left(v + \dfrac{3}{2}\right)^2 > \dfrac{9}{4}$

which represent the outer region of the circle with radius 3/2 and centre (0, –3/2).

When $y > \dfrac{1}{6} : \dfrac{-v}{u^2 + v^2} > \dfrac{1}{6}$ or $u^2 + v^2 + 6v < 0$ or $u^2 + (v + 3)^2 < 9$

which represent the inner region of the circle with radius 3 and centre (0, –3).

The graphical region is,



Example 3.15: Prove that the circle $|w| = 1$ corresponds to the circle $x^2 + y^2 \pm 2y - 1 = 0$ under the transformation $w = \dfrac{1}{2}(z + z^{-1})$.

Solution: The given transformation is $w = \dfrac{1}{2}(z + z^{-1})$

Here, $\quad |w| = 1$

$$\text{or} \quad \left|\frac{1}{2}\left(z + z^{-1}\right)\right| = 1$$

$$\text{or} \quad \left|\left(z + \frac{1}{z}\right)\right| = 2$$

or $$\left|\left(z+\frac{1}{z}\right)\right|^2 = 4$$

or $$\left(z+\frac{1}{z}\right)\overline{\left(z+\frac{1}{z}\right)} = 4 \qquad [\because z\bar{z} = |z|^2]$$

or $$\left(z+\frac{1}{z}\right)\left(\bar{z}+\frac{1}{\bar{z}}\right) = 4$$

or $$(z^2+1)(\bar{z}^2+1) = 4z\bar{z}$$

or $$z^2\bar{z}^2 + z^2 + \bar{z} - 4z\bar{z} + 1 = 0$$

or $$(z^2\bar{z}^2 - 2z\cdot\bar{z} + 1) + (z^2 + \bar{z}^2 - 2z\bar{z}) = 0$$

or $$(z\bar{z} - 1)^2 + (z - \bar{z}^2) = 0$$

or $$(x^2+y^2-1)^2 + (2iy)^2 = 0 \qquad [\because z - \bar{z} = 2yi]$$

or $$(x^2+y^2-1) - 4y^2 = 0$$

or $$(x^2+y^2-1)^2 = 4y^2 = (2y)^2$$

or $$x^2+y^2-1 = \pm 2y$$

or $$x^2+y^2 \pm 2y - 1 = 0$$

Hence the result.

Example 3.16: Prove that the transformation $w = \sin z$ maps the families of lines x = constant and y = constant into two families of confocal conics.

Solution: Here, $w = \sin z$

or $$u + iv = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\therefore \quad u = \sin x \cosh y \qquad ...(1)$$

and $$v = \cos x \sinh y \qquad ...(2)$$

Eliminating y from Equations (1) and (2), we get

$$1 = \cosh^2 y - \sinh^2 y = \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x}$$

or $$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

This shows that the straight lines x = constant in the z-plane are mapped into confocal hyperbolas in w-plane.

Again, eliminating x from Equations (1) and (2), we get

$$\sin^2 x + \cos^2 x = 1$$

or $$\frac{u^2}{\cosh^2 xy} + \frac{v^2}{\sinh^2 y} = 1$$

This shows that the straight line y = constant in the z-plane are mapped into confocal ellipse in w-plane.

Example 3.17: Find the transformation of $w = \cosh z$.

Solution: Here $w = \cosh z$

or $$u + iv = \cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$$

$$\therefore \quad u = \cosh x \cos y \qquad ...(1)$$

and $$v = \sin x \sin y \qquad ...(2)$$

NOTES

Eliminating y from Equations (1) and (2), we get

$$\sin^2 y + \cos^2 y = 1$$

$$\text{or} \frac{v^2}{\sinh^2 x} + \frac{u^2}{\cosh^2 x} = 1$$

This shows that the lines parallel to y-axis (i.e., x = constant) in the z-plane transform into ellipse in the w-plane.

Again, eliminating x from Equations (1) and (2), we get

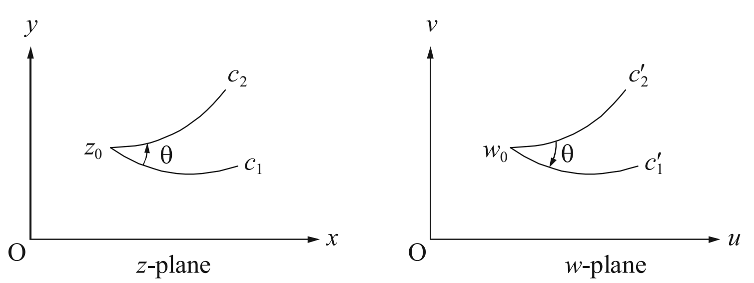
$$\cosh^2 x - \sinh^2 x = 1$$

$$\text{or} \quad \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$$

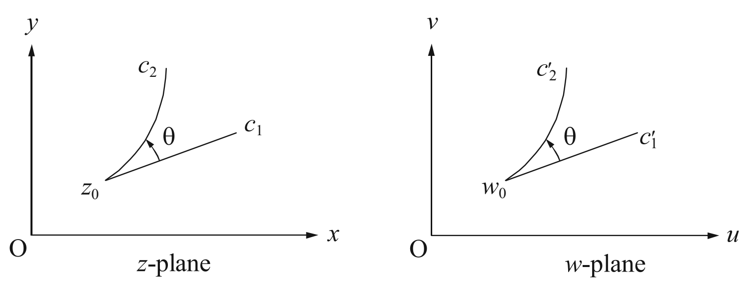
This shows that the lines parallel to x-axis (i.e., y = constant) in the z-plane transform into hyperbola in the w-plane.

3.2.1 Conformal Mappings

A transformation is said to be *isogonal* if two curves in the z-plane intersecting at the point z_0 at an angle θ are transformed into two corresponding curves in the w-plane intersecting at the point w_0 which corresponds to the point z_0 at the same angle θ. Hence if only the magnitude of the angle is preserved the transformation is called isogonal.



If the sense of the rotation as well as the magnitude of the angle is preserved, then the transformation is called *conformal*.

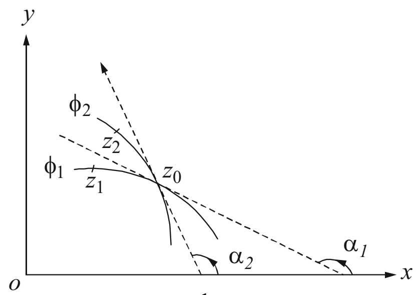


Theorem 3.5: If $f(z)$ is analytic, then the mapping is conformal.

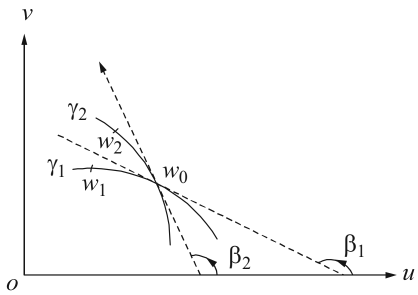
Proof: Let ϕ_1 and ϕ_2 be the two continuous curves in the z-plane, intersecting at the point z_0 and let the tangents at this point make angles α_1 and α_2 with the real axis. Let z_1 and z_2 be the points on the curves ϕ_1 and ϕ_2 near to z_0 and at the same distance r from z_0; so we have,

$$z_1 - z_0 = re^{i\theta_1}, \; z_2 - z_0 = re^{i\theta_2}$$

When $r \to 0$, then $\theta_1 \to \alpha_1$ and $\theta_2 \to \alpha_2$



z - plane



w-plane

NOTES

Let w_0 be the point in the *w*-plane corresponding to z_0, and let z_1 and z_2 correspond to points w_1 and w_2 in the *w*-plane which describes the curves γ_1 and γ_2 in the *w*-plane.

Let, $\qquad w_1 - w_0 = \rho e^{i\varphi 1},\ w_2 - w_0 = \rho e^{i\varphi 2}$

$$\therefore \qquad f'(z_0) = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0}$$

or $\qquad Re^{i\lambda} = \lim \dfrac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}}$ (Since $f'(z_0)$ may be written as $Re^{i\lambda}$)

i.e., $\qquad Re^{i\lambda} = \lim \dfrac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$

Hence, $\lim \left[\dfrac{\rho_1}{r}\right] = R = |f'(z_0)|$ and $\lim (\phi_1 - \theta_1) = \lambda$ or $\lim \phi_1 - \lim \theta_1 = \lambda$

or $\qquad \beta_1 - \alpha_1 = \lambda$ or $\beta_1 = \alpha_1 + \lambda$

Similarly it can be proved that $\beta_2 = \alpha_2 + \lambda$.

Therefore the curves γ_1 and γ_2 have tangents at w_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ with real axis and the angle between γ_1 and γ_2 at w_0 is $\beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 + \lambda) = \alpha_1 - \alpha_2$ which is same as the angle between ϕ_1 and ϕ_2 at z_0. Hence the curves γ_1 and γ_2 intersect at the same angle as the curve ϕ_1 and ϕ_2; also the angle between γ_1 and γ_2 has the same sense as angle between ϕ_1 and ϕ_2.

Hence the transformation is conformal.

Theorem 3.6 (Converse): If a mapping $w = f(z)$ is conformal, then it is analytic.

Proof: Let $u = u(x, y)$ and $v = v(v, y)$ be two conformal transformations from *xy*-plane to *uv*-plane.

NOTES

Let dr and ds be the elementary length in uv-plane and xy-plane respectively and $w = u + iv = f(z)$ where $z = x + iy$ and u, v are differentiable functions of x and y, then

$$ds^2 = dx^2 + dy^2 \qquad ...(3.22)$$

$$dr^2 = du^2 + dv^2$$

Since u and v both are functions of x, y, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \text{ and } dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\therefore \qquad du^2 + dv^2 = \left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right]^2 + \left[\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right]^2$$

$$\text{or, } dr^2 = \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}dx^2 + 2\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\right\}dx\,dy + \left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \qquad ...(3.23)$$

Since the mapping is conformal, then the ratio $dr : ds$ is independent of direction; then from Equations (3.22) and (3.23) we get

$$\therefore \quad \frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}}{0} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} \qquad ...(3.24)$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \qquad ...(3.25)$$

$$\text{and} \quad \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} = 0 \qquad ...(3.26)$$

Equation (3.25) is satisfied when,

$$u_x = v_y,\ u_y = -v_x \qquad ...(3.27)$$

(Cauchy-Riemann equations)

and Equation (3.26) is satisfied if $u_x = -v_y$, $v_x = u_y$ ...(3.28)

Equations (3.28) reduce to Equations (3.27) by writing $-v$ for v, that is, by taking as image figure by reflection in the real axis of the w-plane. Hence, Equations (3.28) correspond to an isogonal but not conformal transformation.

We see that if the mapping of z-plane to w-plane is conformal, the only form of transformation is $w = f(z)$ where $f(z)$ is an analytic function of z.

Transformations which are Isogonal but not Conformal

In this case, the magnitude of the angles of a transformation is conserved but their sign is changed. For example, consider the transformation,

$$w = x - iy \text{ and } z = x + iy.$$

Therefore, $w = x - iy$ is the reflection of z in the real axis where the angles are conserved but their signs are changed.

In general, this is true for every transformation of the form

$$w = f(\bar{z}) \qquad ...(3.29)$$

where $f(z)$ is analytic because such a transformation is combination of the two transformations,

$$\xi = \bar{z} \quad \text{...(3.30)}$$

and $$w = f(\xi) \quad \text{...(3.31)}$$

In Equation (3.30) the angles are conserved but their signs are changed, in Equation (3.31) angles and their signs are conserved. Hence, in the resultant transformation Equation (3.29), the angles are conserved and their signs are changed.

Hence, the transformation is isogonal but not conformal.

Check Your Progress

1. What is bilinear transformation?
2. Define the term isogonal transformation.
3. What is translation?

3.3 SPACES OF ANALYTIC FUNCTIONS

Let Ω be an open subset of C. Then $A(\Omega)$ will denote the space of analytic functions on Ω, while $C(\Omega)$ will denote the space of all continuous functions on Ω. For $n = 1, 2, 3, \ldots$, let

$$K_n = \overline{D}(0,n) \cap \{z : |z - w| \geq 1/n \text{ for all } w \in \mathbb{C} \setminus \Omega\}$$

By basic topology of the plane, the sequence $\{K_n\}$ has the following three properties:

1. K_n is compact.
2. $K_n \subseteq K^o_{n+1}$, where K^o_{n+1} is the interior of K_{n+1}.
3. If $K \subseteq \Omega$ is compact, then $K \subseteq K_n$ for n sufficiently large.

3.3.1 Montel's Theorem

Definition: A conformal one-to-one map of a domain D_1 onto D_2 is said to be a conformal isomorphism, while the domains D_1 and D_2 that admit such a map are called isomorphic or conformally equivalent. Isomorphism of a domain onto itself is called conformal automorphism.

It is easy to see that the set of all automorphisms $\phi: D \to D$ of a domain D forms a group that is denoted by A at D. The group operation is the composition $\phi_1 \circ \phi_2$, the unity is the identity map and the inverse is the inverse map $z = \phi^{-1}(w)$.

Theorem 3.7: Let f_0: $D_1 \to D_2$ be a fixed isomorphism. Then any other isomorphism of D_1 onto D_{2}- has a form

$$f = \phi \circ f_0 \quad \text{...(3.32)}$$

where φ is an automorphism of D_2.

NOTES

Proof: First, it is clear that all maps of the form of the right side of Theorem 3.7 are isomorphisms from D_1 onto D_2. Furthermore, if f: $D_1 \rightarrow D_2$ is an arbitrary isomorphism then

$\phi = f \circ f_0^{-1}$ is a conformal map of D_2 onto itself, that is, an automorphism of D_2. This completes the proof.

Theorem 3.8: Any conformal automorphism of a canonical domain is a fractional linear transformation.

Proof: Let ϕ be automorphism of $\overline{C}$. There exists a unique point z_0 that is mapped to infinity. Therefore ϕ is holomorphic everywhere in C except at z_0 where it has a pole. This pole has multiplicity one, since in a neighborhood of a pole of higher order the function ϕ could not be one-to-one. Therefore, since the only singularities of ϕ are poles, ϕ is a rational function. Since it has only one simple pole, ϕ should be of the form $\phi(z) = A/z–z_0 + B$ if $z_0 \neq \infty$ and $\phi(z) = \infty$. The case of the open complex plane C is similar.

Let ϕ be an arbitrary automorphism of the unit disc U. Let us denote $w_0 = \phi(0)$ and consider a fractional linear transformation,

$$\lambda: w \rightarrow \omega - \omega_0/1 - \overline{w_0}\, w$$

of the disc U, that maps w_0 into 0. The composition $f = \lambda \circ \phi$ is also an automorphism of U so that $f(0) = 0$. Moreover, $|f(z)| < 1$ for all $z \in U$. Therefore the Schwarz lemma implies that $|f(z)| \leq |z|$ for all $z \in U$. However, the inverse map $z = f^{-1}(w)$ also satisfies the assumptions of the Schwarz lemma and hence $|f^{-1}(w)| \leq |w|$ for all $w \in U$ that in turn implies that $|z| \leq |f(z)|$ for all $z \in U$. Thus, $|f(z)| = |z|$ for all $z \in U$ so that, the Schwarz lemma implies that $f(z) = e^{i\alpha}z$.

Then, $\phi = \lambda^{-1} \circ f = \lambda^{-1}(e^{i\alpha}z)$ is also a fractional-linear transformation.

We obtain the complete description of all conformal automorphisms of the canonical domains as:

(1) The closed complex plane:

Aut $= \{z \rightarrow az + b/cz + d,\ ad–bc \neq 0\}$. …(3.33)

(2) The open plane:

Aut $C = \{z \rightarrow az + b,\ a \neq 0\}$. ...(3.34)

(3) The unit disc:

Aut $U = \{z \rightarrow e^{i\alpha} z–a/1–a,\ |a| < 1,\ \alpha \in R\}$. …(3.35)

It is easy to see that different canonical domains are not isomorphic to each other. Indeed, the closed complex plane $\overline{C}$ is not even holomorphic to C and U and hence it may not be mapped conformally onto these domains. The domains C and U are holomorphic but there is no conformal map of C onto U since, such a map would have to be realized by an entire function such that $|f(z)| < 1$ which has then to be equal to a constant by the Liouville theorem.

A domain that has no boundary coincides $\overline{C}$ with . Domains with boundary that consists of one point are the plane $\overline{C}$ without a point which are clearly conformally equivalent to C.

Theorem 3.9: If a domain D is conformally equivalent to the unit disc U then, the set of all conformal maps of D onto U depends on three real parameters. In particular, there exists a unique conformal map f of D onto U normalized by,

$$f(z_0) = 0, \text{Arg} f'(z_0) = \theta \qquad \ldots(3.36)$$

where z_0 is an arbitrary point of D and θ is an arbitrary real number.

Proof: The first statement follows from Theorem 3.7 since, the group Aut U depends on three real parameters: two coordinates of the point a and the number α.

In order to prove the second statement, let us assume that there exist two maps f_1 and f_2 of the domain D onto U normalized as in Equation (3.36). Then $\phi = f_1 \text{ o} f^{-1}{}_2$ is an automorphism of U such that $\phi(0) = 0$ and Arg $f'(0) = 0$. Equation (3.35) implies that then $a = 0$ and $\alpha = 0$,

that is $\phi(z) = z$ and $f_1 = f_2$.

In order to prove the Riemann theorem we need to develop some methods that are useful in other areas of the complex analysis.

The Compactness Principle

Definition: A family $\{f\}$ of functions defined in a domain D is locally uniformly bounded if for any domain K properly contained in D there exists a constant $M = M(K)$ such that,

$$|f(z)| \leq M \text{ for all } z \in K \text{ and all } f \in \{f\}. \qquad \ldots(3.37)$$

A family $\{f\}$ is locally equicontinuous if for any $\varepsilon > 0$ and any domain K properly contained in D there exists $\delta = \delta(\varepsilon, K)$ so that,

$$|f(z') - f(z'')| < \varepsilon \qquad \ldots(3.38)$$

for all $z', z'' \in K$ so that $|z' - z''| < \delta$ and all $f \in \{f\}$.

Theorem 3.10: If a family $\{f\}$ of holomorphic functions in a domain D is locally uniformly bounded then it is locally equicontinuous.

Proof: Let K be a domain properly contained in D. Let us denote the distance between the closed sets $\overline{K}$ and ∂D^5 by 2ρ and let $K^{(\rho)} = U_{z0\in K}\{z: |z-z_0| < \rho\}$ be a ρ-enlargement of K. The set $K^{(\rho)}$ is properly contained in D and thus there exists a constant M so that $|f(z)| \leq M$ for all $z \in K^{(\rho)}$ and $f \in \{f\}$. Let z' and z'' be arbitrary points in K so that $|z' - z''| < \rho$. The disc $U_p = \{z: |z - z'| < \rho\}$ is contained in $K^{(\rho)}$ and hence $|f(z) - f(z')| < 2M$ for all $z \in U_p$. The mapping $\zeta = 1/\rho(z - z')$ maps U_p onto the disc $|\zeta| < 1$ and the function $g(\zeta) = 1/2M \{f(z' + \zeta\rho) - f(z')\}$ satisfies the assumptions of the Schwarz lemma.

This lemma implies that $g(\zeta) \leq |\zeta|$ for all ζ, $|\zeta| < 1$, which means

$$|f(z) - f(z') \leq 2M/\rho \, |z - z'| \text{ for all } z \in U_p. \qquad \ldots(3.39)$$

Given $\in > 0$ we choose $\delta = \min(\rho, \varepsilon\rho/2M)$ and obtain from Equation (3.39) that $|f(z') - f(z'')| < \varepsilon$ for all $f \in \{f\}$ provided that $|z' - z''| < \delta$.

Definition: A family of functions $\{f\}$ defined in a domain D is compact in D if any sequence f_n of functions of this family has a subsequence f_{nk} that converges uniformly on any domain K properly contained in D.

NOTES

Theorem 3.11 (Montel Theorem): If a family of functions $\{f\}$ holomorphic in a domain D is locally uniformly bounded then it is compact in D.

Proof: (*a*) We first show that if a sequence $f_n \subset \{f\}$ converges at every point of an everywhere dense set $E \subset D$ then it converges uniformly on every compact subset K of D. We fix $\varepsilon > 0$ and the set K. Using equicontinuity of the family $\{f\}$ we may choose a partition of D into squares with sides parallel to the coordinate axes and so small that for any two points z', $z'' \in K$ that belong to the same square and for any $f \in \{f\}$ we have,

$$|f(z') - f(z'')| < \varepsilon/3. \qquad \ldots(3.40)$$

The set K is covered by a finite number of such squares q_p, $p = 1,\ldots, p$. Each q_p contains a point $z_p \in E$ since the set E is dense in D. Moreover, since the sequence $\{f_n\}$ converges on E there exists N so that,

$$|f_m(z_p) - f_n(z_p)| < \varepsilon/3. \qquad \ldots(3.41)$$

for all m, $n > N$ and all z_p, $p = 1,\ldots,p$.

Let now z be an arbitrary point in K. Then there exists a point z_p that belongs to the same square as z. We have for all m, $n > N$:

$$|f_m(z) - f_n(z)| \leq |f_m(z) - f_m(z_p)| + |f_m(z_p) - f_n(z_p)| + |f_n(z_{-p}) - f_n(z)| < \varepsilon$$

due to Equations (3.40) and (3.41). The Cauchy criterion implies that the sequence $\{z_n\}$ converges for all $z \in K$ and convergence is uniform on K.

(*b*) Let us show now that any sequence $\{f_n\}$ has a subsequence that converges at every point of a dense subset E of D. We choose E as the set $z = x + iy \in D$ with both coordinates x and y rational numbers. This set is clearly countable and dense in D; let $E = \{Zv\}_v^{\infty} = 1$.

The sequence $f_n(z_1)$ is bounded and hence it has a converging subsequence $f_{k1} = f_{nk}(z_1)$,

$k = 1, 2,\ldots$The sequence $f_{n1}(z_2)$ is also bounded so we may extract its subsequence

$f_{k2} = f_{n_k 1}$, $k = 1, 2,\ldots$The sequence f_{n2} converges at least at the points z_1 and z_2.

Then we extract a subsequence $f_{k3} = f_{n_k 2}$ of the sequence $f_{n2}(z_3)$ so that, f_{n3} converges at least at z_1, z_2 and z_3. We may continue this procedure indefinitely. It remains to choose the diagonal sequence,

$f_{11}, f_{22},\ldots, f_{mn},\ldots$

This sequence converges at any point $z_p \in E$ since by construction all its entries after index p belong to the subsequence f_{np} that converges at z_p.

Parts (*a*) and (*b*) together imply the statement of the theorem.

Note: The Montel's theorem is often called the compactness theorem.

Definition: A functional J of a family $\{f\}$ of functions defined in a domain D is a mapping J: $\{f\} \rightarrow C$, that is, $J(f)$ is a complex number. A functional J is continuous if given any sequence of functions $f_n \in \{f\}$ that converges uniformly to a function $f_0 \in \{f\}$ on any compact set $K \subset D$ we have,

$$\lim_{n\to\infty} J(f_n) = J(f_0)$$

Example 3.18: Let $O(D)$ be the family of all functions f holomorphic in D and let a be an arbitrary point in D. Consider the p-th coefficient of the Taylor series in a:

$$c_p(f) = f^{(p)}(a)/p!$$

This is a functional on the family $O(D)$. Let us show that it is continuous. If $f_n \to f_0$ uniformly on every compact set $K \subset D$ we may let K be the circle $\gamma = \{|z - a| = r\} \subset D$.

Then given any $\varepsilon > 0$ we may find N so that $|f_n(z) - f_0(z)| < \varepsilon$ for all $n>N$ and all $z \in \gamma$. The Cauchy formula for c_p,

$$c_p = 1/2\pi i \int_\gamma f(z)/(z-a)^{n+1}\, dz$$

This implies that,

$$|c_p(f_n) - c_p(f_0)| \le \varepsilon/r^n$$

for all $n > N$ which in turn implies the continuity of the functional $c_p(f)$.

Definition: A compact family of functions $\{f\}$ is sequentially compact if the limit of any sequence f_n that converges uniformly on every compact subset $K \subset D$ belongs to the family $\{f\}$.

Theorem 3.12: Any functional J that is continuous on a sequentially compact family $\{f\}$ is bounded and attains its lowest upper bound. That is, there exists a function $f_0 \in \{f\}$ so that we have $|J(f_0)| \ge |J(f)|$, for all $f \in \{f\}$.

Proof: We let $A = \sup_{f\in\{f\}} |J(f)|$. This is a number that might be equal to infinity.

By definition of the supremum there exists a sequence $f_n \in \{f\}$ so that $|J(f_n)| \to A$. Since $\{f\}$ is a sequentially compact family there exists a subsequence f_{nk} that converges to a function $f_0 \in \{f\}$. Continuity of the functional J implies that,

$$|J(f_0)| = \lim_{k\to\infty} |J(f_{nk})| = A.$$

This means that first $A < \infty$ and second, $|J(f_0)| \ge |J(f)|$ for all $f \in \{f\}$.

3.3.2 Hurwitz's Theorem

We will consider below families of univalent functions in a domain D. The following theorem is useful to establish sequential compactness of such families.

Theorem 3.13 (Hurwitz Theorem): Let a sequence of functions f_n holomorphic in a domain D converge uniformly on any compact subset K of D to a function $f \ne$ constant. Then if $f(z_0) = 0$ then given any disc $U_r = \{|z-z_0| < r\}$ there exists N so that all functions f_n vanish at some point in U_r when $n > N$.

Proof: The Weierstrass theorem implies that f is holomorphic in D. The uniqueness theorem implies that there exists a punctured disc $\{0 < |z-z_0| \le \rho\} \subset D$ where $f \ne 0$ (we may assume that $\rho < r$). We denote $\gamma = \{|z - z_0| = \rho\}$ and $\mu = \min_{z\in\gamma} |f(z)|$, and observe that $\mu > 0$. However, f_n converges uniformly to f on γ and hence there exists N so that, $|f_n(z) - f(z)| < \mu$ for all $z \in \gamma$ and all $n > N$. The Rouche's theorem implies that for such n the function $f_n = f + (f_n - f)$

has as many zeroes (with multiplicities) as f inside γ, that is, f_n has at least one zero inside U_p.

Corollary 1: If a sequence of holomorphic and univalent functions f_n in a domain D converges uniformly on every compact subset K of D then, the limit function f is either a constant or univalent.

Proof: Assume that $f(z_1) = f(z_2)$ but $z_1 \neq z_2$, $z_1, z_2 \in D$ and $f \neq$ constant. Consider a sequence of functions $g_n(z) = f_n(z) - f_n(z_2)$ and a disc $\{|z-z_1| < |r|\}$ with $r < |z_1 - z_2|$. The limit function $g(z)$ vanishes at the point z_1. Hence, according to the Hurwitz theorem all functions f_n starting with some N vanish in this disc. This however contradicts the assumption that $f_n(z)$ is univalent.

3.3.3 The Riemann Theorem

Theorem 3.14: Any simply connected domain D with a boundary that contains more than one point is conformally equivalent to the unit disc U.

Proof: Consider the family S of holomorphic and univalent functions f in D bounded by one in absolute value, that is, those that map D into the unit disc U. We fix a point $a \in D$ and look for a function f that maximizes the dilation coefficient $|f'(a)|$ at the point a. Restricting ourselves to a sequentially compact subset S_1 of S and using continuity of the functional $J(f) = |f'(a)|$ we may find a function f_0 with the maximal dilation at the point a. Finally, we check that f_0 maps D onto U and not just into U as other function in S.

Such a variational method when one looks for a function that realizes the extremum of a functional is often used in analysis.

(*i*) Let us show that there exists a holomorphic univalent function in D that is bounded by one in absolute value. By assumption the boundary ∂D contains at least two points α and β. The square root $\sqrt{z-\alpha/z-\beta}$ admits two branches ϕ_1 and ϕ_2 that differ by a sign. Each one of them is univalent in D^8 since the equality $\phi_\nu(z_1) = \phi_\nu(z_2)$ ($v = 1$ or 2) implies,

$$z_1 - \alpha/z_1 - \beta = z_2 - \alpha/z_2 - \beta \qquad \text{...(3.42)}$$

which implies $z_1 = z_2$ since fractional linear transformation are univalent. The two branches ϕ_1 and ϕ_2 map D onto domains $D*_{1} = \phi_1(D)$ and $D*_{2} = \phi_2(D)$ that have no overlap. Otherwise there would exist two points $z_1, z_2 \in D$ so that $\phi_1(z_1) = \phi_2(z_2)$ which would in turn imply Equation (3.42) so that $z_1 = z_2$ and then $\phi_1(z_1) = -\phi_2(z_2)$. This is a contradiction since $\phi_v(z) \neq 0$ in D.

The domain $D*_2$ contains a disc $\{|w - w_0| < \rho\}$. Hence ϕ_1 does not take values in this disc. Therefore the function,

$$f_1(z) = \rho/\phi_1(z) - w_0 \qquad \text{...(3.43)}$$

is clearly holomorphic and univalent in D and takes values inside the unit disc: we have $|f_1(z)| \leq 1$ for all $z \in D$.

(*ii*) Let us denote the family of functions that are holomorphic and univalent in D, and are bounded by one in absolute value by S. This family is not empty since it contains the function f_1. It is compact by the Montel's theorem. The subset S_1 of the family S that consists of all functions $f \in S$ such that,

$$|f'(a)| \geq |f'_1(a)| > 0 \qquad \text{...(3.44)}$$

at some fixed point, $a \in D$ is sequentially compact. Indeed Corollary 1 implies that the limit of any sequence of functions f_n ε S_1 that converges on any compact subset K of D may be only a univalent function or be a constant but the latter case is ruled out by Equation (3.44).

Consider the functional $J(f) = |f'(a)|$ defined on S_1. Therefore, there exists a function $f_0 \in S$ that attains its maximum, that is, such that

$$|f'(a)| \leq |f'(a)| \quad \text{...(3.45)}$$

for all $f \in S$.

(*iii*) The function $f_0 \in S_1$ maps D conformally into the unit disc U. Let us show that,

$f_0(a) = 0$. Otherwise, the function $g(z) = f_0(z) - f_0(a)/1-\overline{f_0(a)}\ f_0(z)$ would belong to S_1 and have $|g'(a)| = \dfrac{1}{1-|f_0(a)|^2}\ |f'_0(a)| > |f'(a)|$, contrary to the extremum property of the function f.

Finally, let us show that f_0 maps D onto U. Indeed, let f_0 omit some value $b \in U$. Then $b \neq 0$ since $f_0(a) = 0$. However, the value $b* = 1/b$ is also not taken by f_0 in D since $|b*| > 1$. Therefore, one may define in D a single valued branch of the square root

$$\Psi(z) = \sqrt{\frac{f_0(z)-b}{1-\bar{b}f_0(z)}} \quad \text{...(3.46)}$$

That also belongs to S: it is univalent for the same reason as in the square root in part (*i*), and $|\psi(z)| \leq 1$. However, then the function

$$H(z) = \frac{\Psi(z)-\Psi(a)}{1-\Psi(a)\Psi(z)}$$

also belongs to S. We have,

$$|b'(a)| = \frac{1+|b|}{2\sqrt{|b|}}|f'_0(a)|$$

However, $1+|b| > 2\sqrt{|b|}$ since $|b|<1$ and thus $h \in S_1$ and $|h'(a) > |f'(a)|$ contrary to the extremal property of f_0.

The Riemann theorem implies that any two simply connected domains D_1 and D_2 with boundaries that contain more than one point are conformally equivalent. Indeed, as we have shown there exists conformal isomorphism $f_j : D_j \to U$ of these domains onto the unit disc. Then $f = f^{-1}{}_2 \circ f_1$ is a conformal isomorphism between D_1 and D_2.

3.3.4 Riemann Mapping Theorem

In complex analysis, the Riemann mapping theorem states that if U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic (bijective and holomorphic) mapping f from U onto the open unit disc,

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is known as a Riemann mapping.

NOTES

Theorem 3.15 (Riemann Mapping Theorem): Let $U \subsetneq \mathbb{C}$ be a simply connected domain. Then there is a biholomorphic map $f : U \to B_1(0)$.

Note: Holomorphic functions are very rigid. They are determined by a small amount of information. But they can be used to transform any simply connected domain to a simple open disc.

Corollary: Every simply connected domain is homeomorphic to the open unit disc.

Proof: Let $U \subset \mathbb{C}$ be a domain. When $U \neq \mathbb{C}$, then this follows from the Riemann mapping theorem, since a biholomorphic map is a homeomorphism. We know that an injective holomorphic map automatically is biholomorphic onto its image. An injective holomorphic function has non-vanishing derivative, hence the inverse function is again holomorphic. So it suffices to show that there is an injective holomorphic map from U onto $B_1(0)$. To show this, we prove the following lemma:

Lemma 1: Let $U \subsetneq \mathbb{C}$ be a simply connected domain. Then there is an injective holomorphic function $f : U \to B_1(0)$.

Proof: First, we want to map U biholomorphically to a domain whose complement contains an open ball. This need not be the case for U itself; an example for this is the slit plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Let $a \in \mathbb{C} \setminus U$. The function $z - a$ does not vanish on U. Since U is simply connected, there is a "square root" s of this function of U, i.e., a holomorphic function $s : U \to \mathbb{C}$ such that $s(z)^2 = z - a$. This function s is injective: $s(z) = s(w)$ implies $z - a = s(z)^2 = s(w)^2 = w - a$ and hence $z = w$. Let $U_1 = s(U)$ be the image of U under s. Since non-constant holomorphic maps are open, there is $b \in U_1$ and $r > 0$ such that $B_r(b) \subset U_1$. Claim that $B_r(-b) \subset \mathbb{C} \setminus U_1$. Assume that there is $w \in U_1 \cap B_r(-b)$. Then $w = s(z)$ for some $z \in U$; also $-w = s(z')$ for some $z' \in U$ (since $-w \in B_r(b) \subset U$). But this implies $z = s(z)^2 + a = w^2 + a = s(z')^2 + a = z'$, so $w = s(z) = s(z') = -w$, which is a contradiction, since $0 \notin U_1$.

Now we let,

$$f(z) = \frac{r}{s(z) - b}$$

then f is injective and holomorphic on U and $f(U) \subset B_1(0)$.

Lemma 2: Let $U \subseteq \mathbb{C}$ be a simply connected domain and let $a \in U$. Then there is an injective holomorphic function,

$$f : U \to B_1(0)$$

such that $f(a)=0$.

Proof: By Lemma 1, $f_0 : U \to B_1(0)$ is holomorphic and injective. Then, we want

NOTES

$$f(z) = \frac{1}{2}\left(f_0(z) - f_0(a)\right)$$

Now let,

$F = \{f : U \to B_1(0) : f \text{ holomorphic and injective}, f(a) = 0\}$.

F is non-empty. Choose some point $b \in U \setminus \{a\}$. Let $\rho = \sup\{|f(b)| : f \in F\}$. Note that $\rho > 0$ since F is non-empty and $f(b) \neq 0$ for every $f \in F$.

Claim 1: There is some $f \in F$ such that $|f(b)| = \rho$.

Proof: By definition of ρ there is a sequence (f_n) in F such that $|f_n(b)| \to \rho$ as $n \to \infty$. Also, F is bounded, so by Montel's theorem, there is a compactly convergent subsequence, and without loss of generality, we can assume that (f_n) itself converges compactly. Let $f = \lim_{n\to\infty} f_n$ be the limit function. Then clearly, $|f(b)| = \rho$. Now we will prove that $f \in F$.

f is not constant. f is injective as the limit of a sequence of injective holomorphic functions. Also, $f(U)$ is open and contained in $\overline{B_1(0)}$, so $f(U) \subset B_1(0)$.

Lemma 3: Let $w \in B_1(0)$ and define,

$$\phi_w(z) = \frac{w - z}{1 - \overline{w}z}$$

Then ϕ_w is an involutory automorphism of $B_1(0)$ that interchanges 0 and w.

Proof: For $z \in B_1(0)$, we have

$(1 - |w|^2)(1 - |z|^2) > 0$, so $|w|^2 + |z|^2 < 1 + |w|^2|z|^2$

Therefore,

$|w - z|^2 = |w|^2 - \overline{w}z - w\overline{z} + |z|^2 < 1 - \overline{w}z - w\overline{z} + |w|^2|z|^2 = |1 - \overline{w}z|^2$

Hence,

$|\phi_w(z)| < 1$

Also,

$|\phi_w(\phi_w(z)) = z$

This implies that ϕ_w is an automorphism. We can say that, $\phi_w(0) = w$ and $\phi_w(w) = 0$.

Claim 2: The function *f* above satisfies $f(U) = B_1(0)$.

Proof: Assume that the claim is false. Then there is $w \in B_1(0) \setminus f(U)$. We construct another function $g \in F$ such that $|g(b)| > |f(b)|$, which contradicts the choice of *f*.

NOTES

Here, notice that $\phi_w \circ f$ is an injective holomorphic function $U \to B_1(0)$ whose image does not contain zero.

Since $U' = (\phi_w \circ f)(U)$ is simply connected, there is then a holomorphic square root function *s* on U', which is injective. So $s \circ \phi_w \circ f$ is an injective holomorphic function that maps *U* into $B_1(0)$ again. We have $(s \circ \phi_w \circ f)(a) = s(w)$, hence if we define $g = \phi_{s(w)} \circ s \circ \phi_w \circ f$, then *g* is an injective holomorphic function that map *U* into $B_1(0)$ and such that $g(0) = 0$, so $g \in F$.

To show that, $|g(b)| > \rho = |f(b)|$, define

$$h(z) = \phi_w(\phi_{s(w)}(z)^2)$$

then *h* is a holomorphic map $B^1(0) \to B_1(0)$, which is not an automorphism

$$|h(z)| < |z| \text{ for all } z \in B_1(0) \setminus \{0\}.$$ In particular,

$$|g(b)| > |h(g(b))| = |f(b)|$$

(since $h \circ g = f$). This is the desired contradiction.

Check Your Progress

4. Define the spaces of analytic function.
5. State the Montel's theorem.
6. Define the Riemann theorem.
7. What is Riemann mapping?

3.4 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The transformation of the form $w = \dfrac{az + b}{cz + d}$

   where *z*,*w* are complex variables, *a*, *b*, *c*, *d* are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.
2. The transformation $w = z + \alpha$ is called translation, where $\alpha = a + ib$.
3. A transformation is said to be isogonal if two curves in the *z*-plane intersecting at the point z_0 at an angle θ are transformed into two corresponding curves in the *w*-plane intersecting at the point w_0 which corresponds to the point z_0 at the same angle θ.
4. Let Ω be an open subset of *C*. Then $A(\Omega)$ will denote the space of analytic functions on Ω, while $C(\Omega)$ will denote the space of all continuous functions on Ω.

5. A family of functions $\{f\}$ defined in a domain D is compact in D if any sequence f_n of functions of this family has a subsequence f_{nk} that converges uniformly on any domain K properly contained in D.
6. Any simply connected domain D with a boundary that contains more than one point is conformally equivalent to the unit disc U.
7. In complex analysis, the Riemann mapping theorem states that if U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic (bijective and holomorphic) mapping f from U onto the open unit disc, $D = \{z \in \mathbb{C} : |z| < 1\}$.

3.5 SUMMARY

- The transformation of the form

  $$w = \frac{az + b}{cz + d}$$

  where z,w are complex variables, a, b, c, d are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.
- The transformation $w = z + \alpha$ is called translation, where $\alpha = a + ib$.
- The transformation $w = \beta z$ is called magnification and rotation where w, β, z are complex numbers.
- The transformation $w = \frac{1}{z}$ is called inversion.
- The transformation is an inversion of z and followed by reflection into the real axis.
- The ratio $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$

  is called the cross-ratio of z_1, z_2, z_3, z_4 which is denoted by (z_1, z_2, z_3, z_4).
- The bilinear transformation $w = \frac{az+b}{cz+d}$ transforms a circle of the z-plane into a circle of the w-plane and inverse points transform into inverse points.
- A point at which $f'(z) = 0$ is called a critical point of the transformation.
- The circle $r = a$ = constant about the origin in the z-plane is transformed on the circle $R = a^n$ = constant about the origin in the w-plane.
- The lines $\theta = \beta$ = constant about the origin in z-plane is transformed into the lines $\phi = n\beta$ = constant about the origin in the w-plane and the slope of ϕ-line is n times the slope of θ-line.

NOTES

- The circular sector with its vertex at origin in the z-plane is transformed into a circular sector with its vertex at origin and n times the central angle.
- The interior of the circular sector with central angle π/n is transformed conformably upon the upper half plane $I(w) > 0$.
- A transformation is said to be isogonal if two curves in the z-plane intersecting at the point z_0 at an angle θ are transformed into two corresponding curves in the w-plane intersecting at the point w_0 which corresponds to the point z_0 at the same angle θ.
- If only the magnitude of the angle is preserved the transformation is called isogonal.
- If a mapping $w = f(z)$ is conformal, then it is analytic.
- The magnitude of the angles of a transformation is conserved but their sign is changed.
- Let Ω be an open subset of C. Then $A(\Omega)$ will denote the space of analytic functions on Ω, while $C(\Omega)$ will denote the space of all continuous functions on Ω.
- A conformal one-to-one map of a domain D_1 onto D_2 is said to be a conformal isomorphism, while the domains D_1 and D_2 that admit such a map are called isomorphic or conformally equivalent.
- The Montel's theorem is often called the compactness theorem.
- Any simply connected domain D with a boundary that contains more than one point is conformally equivalent to the unit disc U.
- Riemann mapping theorem states that if U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic (bijective and holomorphic) mapping f from U onto the open unit disc, $D = \{z \in \mathbb{C} : |z| < 1\}$.

3.6 KEY TERMS

- **Bilinear transformation:** The transformation of the form $w = \frac{az + b}{cz + d}$ where z,w are complex variables, a, b, c, d are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.

  The transformation $w = z + \alpha$ is called translation, where $\alpha = a + ib$.
- **Inversion:** The transformation $w = \frac{1}{z}$ is called inversion.
- **Critical point:** A point at which $f'(z) = 0$ is called a critical point of the transformation.
- **Isogonal:** If only the magnitude of the angle is preserved the transformation is called isogonal.

3.7 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Name the various types of transformation.
2. What is cross-ratio?
3. Give a transformation which is isogonal but not conformal.
4. Define conformal isomorphism.
5. State Hurwitz's theorem.
6. In which problem the Weierstrass factorization theorem is applied?
7. What are the significations of Reimann mapping theorem?

Long-Answer Questions

1. Under the transformation $w = z^2$ show that the circle $|z - \alpha| = \beta$ in the z-plane correspond to the limacon $R = 2\beta\,(\alpha + \beta \cos \phi)$ in the w-plane where α and β are real.
2. Find the transformation of $w = \cos z$.
3. Show that the transformation $w = z + [(a^2 - b^2)/4z]$ transform the circle of radius $\frac{a+b}{2}$ and centre at origin in the z-plane into ellipse of semi-axis a, b, is the w-plane.
4. Prove that in general circle $|z|$ = constant and lines arg z = constant correspond to conic with focii at $w = \pm 1$ in the w-plane by the transformation $w = 1/2\,(z + 1/z)$.
5. Prove that the following transformations are bilinear.

   (*i*) $w = \frac{3z+5}{z+8}$ (*ii*) $w = \frac{z+7}{2z+5}$ (*iii*) $w = \frac{2z+3}{2z+5}$
6. Prove that the following transformation $w = \frac{iz+2}{4z+i}$ is a bilinear transformation. Find the transformation of the line $y = \frac{1}{3}x$ by this transformation.
7. Find the bilinear transformation which maps:

   (*i*) The points $z = 1, i, -1$ into the points $w = 2, i, -2$

   (*ii*) The points $z = 2, i, -2$ into the points $w = 1, i, -1$
8. Find the fixed points (invariant points) and critical point of the following transformation.

   (*i*) $w = \frac{i(1-z)}{1+z}$ (*ii*) $w = \frac{(1+i)-(1-i)z}{2}$ (*iii*) $w = \frac{(2i-6z)}{(iz-3)}$

   (*iv*) $w = z^3$ (*v*) $w = \frac{2z+3}{z+1}$

NOTES

9. Prove that the transformation $w = \frac{i(i-z)}{1+z}$ transforms the circle $|z| = 1$ into the real axis of w-plane and interior of the circle $|z| < 1$ into the upper half of the w-plane.
10. (*i*) Find the image of the circle $|z - 1| = 1$ under the transformation $w = 1/z$.
    (*ii*) Find the image of $|z - 3i| = 3$ under the transformation $w = 1/2$.
    (*iii*) Find the image of the circle $|z| = a$ under the transformation $w = \sqrt{2}(1 + i)z$.
11. Describe the spaces of analytic functions giving examples.
12. Explain the Montel's theorem.
13. Explain Hurwitz's theorem giving examples.
14. Elaborate the applications of Riemann mapping theorem.

3.8 FURTHER READING

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Ahlfors, Lars V. 1978. *Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Lang, Serge. 1998. *Complex Analysis*, 4th Edition. NY: Springer-Verlag New York Inc.

Shastri, Anant R. 2000. *An Introduction to Complex Analysis*, India:

Macmillan Publishers India Ltd.

UNIT 4 WEIERSTRASS FACTORISATION THEOREM, ANALYTIC CONTINUATION, INEQUALITY THEOREM AND FUNCTIONS

Structure

4.0 INTRODUCTION

In mathematics, Weierstrass factorization theorem in complex analysis, a product involving their zeroes can represent the entire functions. In addition, every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.

A second form extended to meromorphic functions allows one to consider a given meromorphic function as a product of three factors: the function's poles, zeroes and an associated non-zero holomorphic function.

In gamma functions a lot of important functions in applied sciences are defined using improper integrals and Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s = \sigma + it$ (here, s, σ and t are traditional notations associated to the study of the ζ-function).

Riemann's Functional Equation implies that $\zeta(s)$ has a simple zero at each even negative integer $s = -2n$. These zeros are the trivial zeros of $\zeta(s)$. Riemann established the functional equation which is used to construct the analytic continuation in the first place. An equivalent relationship was conjectured by Euler for the Dirichlet eta function or the alternating zeta function and Runge's theorem (also known as Runge's approximation theorem) is named after the German mathematician Carl Runge who first proved it in the year 1885 and in Weierstrass factorisation theorem you will learn about analytic continuation. If there exist two functions $f_1(z)$ and $f_2(z)$, such that they are analytic (regular) in domains D_1 and D_2, respectively, and that D_1 and D_2 have a common part, throughout which $f_1(z) = f_2(z)$, then the aggregate of values of $f_1(z)$ and $f_2(z)$ at the interior points of D_1 or D_2, can be regarded as a single regular function (say) $F(z)$. It is obvious that $F(z)$ is regular in the common part say D of the two domains and $F(z) = f_1(z)$ in domain D_1 and $F(z) = f_2(z)$ in domain D_2. We thus regard the function $f_2(z)$ as one, extending the domain in which $f_1(z)$ is defined and so it is called an analytic continuation of $f_1(z)$.

In this unit, you will learn about the Weierstrass factorisation theorem, gamma function and its properties, Riemann zeta function, Riemann's functional equation, Runge's theorem, Mittag-Leffler's theorem, analytic continuation, power series method of analytic continuation, Schwarz reflection principle, monodromy theorem and its consequences, harmonic functions on a disk, Harnack's inequality and theorem, Dirichlet problem and Green's function.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain Weirstrass' factorization theorem
- Discuss gamma function and its properties
- Define Riemann zeta function and Riemann's functional equation
- Elaborate on Runge's theorem and Mittag-Leffler's theorem
- Define analytic continuation
- Describe the power series method of analytic continuation
- Understand Schwarz reflection principle
- Explain monodromy theorem and its consequences
- Describe Harnack's inequality theorem
- Discuss Dirichlet problem and Green's function

4.2 WEIERSTRASS FACTORISATION THEOREM

According to Weierstrass factorization theorem, in complex analysis a product involving their zeroes can represent the entire functions. In addition, every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.

A second form extended to meromorphic functions allows one to consider a given meromorphic function as a product of three factors: the function's poles, zeroes and an associated non-zero holomorphic function. Consider a polynomial $p(z)$ with (all) zeros $z_1, z_2, \ldots, z_n$. Then,

$$\begin{aligned} P(z) &= C(z_1 - z) \ldots (z_n - z) \qquad (C \text{ Constant}) \\ &= Cz_1 \ldots z_n \left(1 - \frac{z}{z_1}\right) \ldots \left(1 - \frac{z}{z_n}\right) \\ &= P(0)\left(1 - \frac{z}{z_1}\right) \ldots \left(1 - \frac{z}{z_n}\right) \end{aligned}$$

Let now $f(z)$ be an entire function with zeros $z_1, z_2, \ldots, z_n, \ldots$ arranged by increasing moduli, i.e.,

$$0 \leq |z_1| \leq |z_2| \leq \ldots \leq |z_n| \leq \ldots$$

By the uniqueness theorem of analytic functions, $\lim_{n\to\infty} |z_m| = \infty$. Assume $z_1 \neq 0$. Then a factorization similar to the polynomial case above is not immediate, since

$$\prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right)$$

may diverge. Therefore, we must somehow modify the situation to ensure the convergence. This may be done by the following:

Theorem 4.1 (Weierstrass Theorem): Let $(z_m)_{n\in N}$ be an arbitrary sequence of complex numbers different from zero arranged by increasing moduli and $\lim_{n\to\infty} |z_n| = \infty$ and let $m \in \mathbb{N} \cup \{0\}$. Then there exist $v \in \mathbb{N} \cup \{0\}$, $v = v(j)$, such that $\sum_{j=1}^{\infty} |z_j|^{-(v+1)}$ converges uniformly in C and that for the polynomial,

$$Q_v(z) := z + \frac{1}{2}z^2 + \ldots + \frac{1}{v}z^v, \qquad v \geq 1; \quad Q_0(z) = 0$$

Also, for an arbitrary function $g(z)$,

$$G(z) := e^{g(z)zm} \prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right) e^{Q_v\left(\frac{z}{z_j}\right)} \qquad \ldots(4.1)$$

is an entire function with a zero of multiplicity m at $z = 0$ and with the other zeros exactly at (z_n).

Note: The sequence (z_n) is not necessarily formed by distinct points.

Before proving the Theorem 4.1, we consider the entire function,

$$E_v(z) := (1 - z)e^{Q_v(z)}, \qquad v \geq 1; \quad E_0(z) := 1 - z$$

also called the Weierstrass factor.

We first prove three basic properties for $E_v(z)$:

(1) $E_v^{'}(z) = -z^v e^{Q_v(z)}$ for $v \geq 1$:

$$E_v^{'}(z) = -e^{Q_v(z)} + (1-z)(1+z+...+z^{v-1})e^{Q_v(z)}$$

$$= e^{Q_v(z)}(-1+1+...+z^{v-1}-z-z^2-...-z^v) = -z^v e^{Q_v(z)}.$$

(2) $E_v(z) = 1 + \sum_{j>v} a_j z^j$ with $\sum_{j>v} |a_j| = 1$ for $v \geq 0$.

For $v = 0$, this is trivial. Since $E_v(z)$ is entire, we may consider its Taylor expansion around $z = 0$:

$$E_v(z) = \sum_{j=0}^{\infty} a_j z^j$$

Differentiating, we get

$$\sum_{j=1}^{\infty} j a_j z^{j-1} = E_v^{'}(z) = -z^v e^{Q_v(z)}$$

Expanding the right hand around $z = 0$, we get $-z^v \sum_{j=0}^{\infty} \beta_j z^j$ with $\beta_j \geq 0$ for all j. Therefore $a_1 = a_2 = ... = a_v = 0$ and $a_j \leq 0$ for $j > v$, hence $|a_v| = -a_v$ for $j > v$. Moreover, $a_0 = E_v(0) = 1$ and

$$0 = E_v(1) = 1 + \sum_{j>v} a_j;$$

Thus

$$\sum_{j>v} a_j = -\sum_{j>v} |a_j| = -1,$$

resulting in the assertion.

(3) If $|z| \leq 1$, then $|E_v(z) - 1| \leq |z|^{v+1}, v \geq 0$. By (2),

$$|E_v(z) - 1| = \left| \sum_{j=v+1}^{\infty} a_j z^j \right| \leq \sum_{j=v+1}^{\infty} |a_j| |z|^j$$

$$= |z|^{v+1} \sum_{j=v+1}^{\infty} |a_j| |z|^{j-(v+1)} \leq |z|^{v+1} \sum_{j>v} |a_j| = |z|^{v+1}.$$

Proof of Theorem 4.1: We consider $E_v\left(\frac{z}{z_j}\right)$ for $j \in \mathbb{N}$. The idea is to determine v so that $\prod_{j=1}^{\infty} E_v\left(\frac{z}{z_j}\right)$ converges absolutely and uniformly for $|z| < R, R$ large enough. To this end, fix $R > 1$ and $0 < \alpha < 1$. Since $\lim_{n \to \infty} |z_m| = \infty$, we find q such that, $|z_q| \leq \frac{R}{\alpha}$, while $|z_{q+1}| > \frac{R}{\alpha}$. Then $\prod_{j=1}^{q} E_v\left(\frac{z}{z_j}\right)$ is an entire function as a finite product of entire functions. Consider now the remainder term

$$\prod_{j=q+1}^{\infty} E_v\left(\frac{z}{z_j}\right)$$

in the disc $|z| \le R$. Since $j > q, |z_j| > \frac{R}{\alpha}$ and so

$$|z/z_j| < \alpha < 1$$

Writing

$$E_v\left(\frac{z}{z_j}\right) = \left(1 - \frac{z}{z_j}\right) e^{Q_v\left(\frac{z}{z_j}\right)} = 1 + U_j(z),$$

we proceed to estimate $U_j(z)$. Since $j > q$, and $|z/z_j| < 1$, (3) above implies

$$|U_j(z)| = \left|E_v\left(\frac{z}{z_j}\right) - 1\right| \le \left|\frac{z}{z_j}\right|^{v+1} \qquad \text{...(4.2)}$$

We now divide our consideration in two cases:

Case I: There exists $p \in \mathbb{N}$ such that $\sum_{j=1}^{\infty} |z_j|^{-p} < \infty$. In this case, we define $v := p - 1$. From Equation (4.2), we obtain

$$|U_j(z)| \le R^p |z_j|^{-p},$$

since $|z| \le R$. Therefore,

$$\sum_{j=1}^{\infty} |U_j(z)| \le R^p \sum_{j=1}^{\infty} |z_j|^{-p} < \infty$$

for $|z| \le R$.

Now,

$$\prod_{j=q+1}^{\infty} (1 + U_j(z)) = \prod_{j=q+1}^{\infty} E_v\left(\frac{z}{z_j}\right) \text{ converges uniformly and absolutely.}$$

Case II: For all $p \in \mathbb{N}$, $\sum_{j=1}^{\infty} |z_j|^{-p} = \infty$. In this case, we take $v = j - 1$, so v depends on j. Then, by Equation (4.2) again

$$|U_j(z)| \le \left|\frac{z}{z_j}\right|^j$$

provided $j > q$ (which means $\left|\frac{z}{z_j}\right| < \alpha < 1$) and $|z| \le R$. Since $|z/z_j| < \alpha < 1$, we have

$$\limsup_{j \to \infty} \sqrt[j]{\left|\frac{z}{z_j}\right|^j} \le \alpha < 1,$$

and therefore, by the root test, which carries over from the (real) analysis word by word, $\sum_{j=q+1}^{\theta} |U_j(z)|$ converges. As above, we get that

$\prod_{j=q+1}^{\infty} E_v\left(\frac{z}{z_j}\right)$ converges absolutely and uniformly for $|z| \leq R$. If we now have

NOTES

proved that $\prod_{j=1}^{\infty} E_v\left(\frac{z}{z_j}\right)$ is analytic in $\mathbb{C}$, then $G(z)$ is entire and has exactly the desired zeros. Therefore, it remains to prove.

Theorem 4.2: If $(f_n(z))$ is a sequence of analytic functions in a domain G and if there exists

$$\lim_{n\to\infty} f_n(z) = f(z) \qquad \ldots(4.3)$$

uniformly in closed subdomains of G, then $f(z)$ is analytic and $f'(z) = \lim_{n\to\infty} f_n'(z)$.

Proof: This is a consequence of the Cauchy integral formula. In fact, fix $z \in G$ arbitrarily and let $B(z, r)$ be a disc, such that,

$\overline{B(z,r)} \subset G$

By the Cauchy formula,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z} d\zeta, \quad n \in \mathbb{N}$$

Since the convergence formula is uniform on ∂B,

$|f_n(\zeta) - f(\zeta)| < \varepsilon$

for $n \geq n_\varepsilon$ and for all $\zeta \in \partial B$. Therefore,

$$\left|\frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta\right|$$

$$\leq \frac{1}{2\pi}\int_{\partial B} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|}|d\zeta| \leq \frac{\varepsilon.2\pi r}{2\pi.r} = \varepsilon,$$

and so

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{\partial B}\int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

By Equation (4.3),

$$f(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Now, $f'(z)$ exists since,

$$\frac{1}{h}[f(z+h) - f(z)] = \frac{1}{2\pi h i}\int_{\partial B}\left(\frac{f(\zeta)}{\zeta - (z+h)} - \frac{f(\zeta)}{\zeta - z}\right)d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)(\zeta - (z+h))} d\zeta \to \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z))} d\zeta$$

Therefore, $f(z)$ is analytic. Since, the limit of Equation (4.3) is uniform in ∂B, we get

$$f'(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i}\int_{\partial B}\left(\lim_{n\to\infty} f_n(\zeta)\right)\frac{d\zeta}{(\zeta - z)^2}$$

$$= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial B} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta .$$

4.2.1 Gamma Function and its Properties

A lot of important functions in applied sciences are defined using improper integrals. One of the most famous among them is the **gamma function**. In the search for a function generalizing the factorial expression for the natural numbers, one will come upon the well-known formula,

$$\int_0^{+\infty} e^{-\gamma} t^n dt = n!$$

By replacing n by x in the improper integral, we generate the function

$$f(x) = \int_0^{+\infty} e^{-\gamma} t^x dt$$

The only possible bad points in this definition are 0 and $+\infty$. Now, since, $e^{-\gamma} \sim 1$ when $t \approx 0$, then we have

$$e^{-\gamma} t^x \sim t^x = \frac{1}{t^{-x}}$$

when $t \approx 0$. We have convergence around 0 if and only if $-x < 1$ (or equivalently $x > -1$). Alternatively, the improper integral is convergent at $+\infty$ regardless of the value of x. Therefore, the domain of $f(x)$ is $(-1, +\infty)$. If we want $(0, +\infty)$ as a domain, we will need to translate the x-axis to get the new function,

$$\Gamma(x) = f(x-1) = \int_0^{+\infty} e^{-t} t^{x-1} dt$$

Now the domain of this new function called the gamma function is $(0, +\infty)$.

The above formula is also known as **Euler's second integral**.

Basic Properties of Gamma Function

- $\Gamma(n) = (n-1)!, n = 1, 2, ...$
- One of the most important formulas satisfied by the gamma function is,

  $$\Gamma(x+1) = x\Gamma(x)$$

  for any $x > 0$. To prove this formula from the definition of $\Gamma(x)$, we will use the following identity,

  $$\int_a^b e^{-\gamma} t^x dt = \left[-e^{-\gamma} t^x\right]_a^b + x\int_a^b e^{-\gamma} t^{x-1} dt \quad \text{[Integration by parts]}$$

  If we let a tend to 0 and b to $+\infty$, we get the desired identity. In particular, we get

  $$\Gamma(x+n) = (x+n-1)(x+n-2)...(x+1)x\Gamma(x)$$

  for any $x > 0$ and any integer $n \geq 1$. This formula makes it possible for the function $\Gamma(x)$ to be extended to $(-\infty, 0)$ except for the negative integers. In particular, it is enough to know $\Gamma(x)$ on the interval (0, 1] to know the function for any $x > 0$. Note that since,

$$\int_0^{+\infty} e^{-\gamma} dt = 1$$

we get $\Gamma(1) = 1$. Combined with the above identity, we get what we expected before:

$$\Gamma(n) = (n-1)! \text{ for } n = 1, 2, \ldots$$

- If we notice that in $(\Gamma(x))$ is a convex function then we get the inequality,

$$\frac{n^x n!}{x(x+1)\ldots(x+n)} \le \Gamma(x) \le \frac{n^x n!}{x(x+1)\ldots(x+n)} \frac{x+n}{n}$$

or

$$\Gamma(x)\frac{n}{x+n} \le \frac{n^x n!}{x(x+1)\ldots(x+n)} \le \Gamma(x)$$

for every $n \ge 1$ and $x > 0$. If we let n tend to $+\infty$, then we get the identity

$$\Gamma(x) = \lim_{n\to+\infty} \frac{n^x n!}{x(x+1)\ldots(x+n)}$$

- **Weierstrass Identity:** A simple algebraic manipulation gives,

$$\frac{n^x n!}{x(x+1)\ldots(x+n)} = e^{x(\ln(n)-1-1/2-\ldots-1/n)} \frac{1}{x} \frac{e^{x/1}}{1+x/1} \frac{e^{x/2}}{1+x/2} \cdots \frac{e^{x/n}}{1+x/n}$$

Knowing that the sequence $(\ln(n)-1-1/2-\ldots-1/n)$ converges to the constant, C, where

$$C = \lim_{n\to+\infty} 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n)$$

is the Euler's constant, we get,

$$\Gamma(x) = e^{-Cx} \frac{1}{x} \lim_{n\to+\infty} \prod_{k=1}^{k=n} \frac{e^{x/k}}{1+x/k}$$

or

$$\Gamma(x) = e^{-Cx} \frac{1}{x} \prod_{n=1}^{+\infty} \frac{e^{x/n}}{1+x/n}.$$

- For the logarithmic derivative of the gamma function we consider that, $\Gamma(x) > 0$ for any $x > 0$, we can take the logarithm of the above expression to get,

$$\ln(\Gamma(x)) = -Cx - \ln(x) + \sum_{n=1}^{+\infty} \left(\frac{x}{n} - \ln\left(1 + \frac{x}{n}\right) \right)$$

By taking the derivative, we get

$$\frac{d}{dx}\left(\ln(\Gamma(x))\right) = -C - \frac{1}{x} + \sum_{n=1}^{+\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right)$$

Or

$$\frac{\Gamma'(x)}{\Gamma(x)} = -C - \frac{1}{x} + \sum_{n=1}^{+\infty}\frac{x}{n(x+n)}$$

Lemma 1: $0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}$ for every $n \in \mathbb{N}$ and $t \in [0,n]$.

Proof: The function $\chi_n(t) = e^t\left(1-\frac{t}{n}\right)^n$ has the property that $\chi_n(0) = 1, \chi_n(n) = 0$ and

$$\chi'_n(t) = e^t\left[\left(1-\frac{t}{n}\right)^n + n\left(1-\frac{t}{n}\right)^{n-1}\left(-\frac{1}{n}\right)\right] = e^t\left(1-\frac{t}{n}\right)^{n-1}\left(-\frac{t}{n}\right) \le 0 .$$

For the other inequality, we consider $\theta_n(t) = 1 - e^t\left(1-\frac{t}{n}\right)^n - \frac{t^2}{n}$. Clearly $\theta_n(0) = 0$ and

$$\theta'_n(t) = \frac{t}{n}e^t\left(1-\frac{t}{n}\right)^{n-1} - \frac{2t}{n} = \frac{t}{n}\left[e^t\left(1-\frac{t}{n}\right)^{n-1} - 2\right].$$

Let $g_n(t) = e^t\left(1-\frac{t}{n}\right)^{n-1} - 2$.

Then,

$$g'_n(t) = e^t\left[\left(1-\frac{t}{n}\right)^{n-1} - \frac{n-1}{n}\left(1-\frac{t}{n}\right)^{n-2}\right] = e^t\left(1-\frac{t}{n}\right)^{n-2}\frac{1-t}{n}$$

$g'_n(t) = 0$ only when $t = 1$, and $g_n(1)$ is a maximum for $g_n(t)$. One checks that for $n \ge 2$, $\left(1-\frac{1}{n}\right)^{n-1} \le 1/2$, hence $g_n(1) \le e/2 - 2 < 0$. This implies that $\theta_n(t)$ is always decreasing when $n \ge 2$. For $n = 1$, we know that $g_1(t) \le 0$ when $t \in [0, \log 2]$, $g_1(t) \ge 0$ when $t \in [\log 2, 1]$. Hence $\theta_1(t)$ is decreasing on $[0, \log 2]$, increasing on $[\log 2, 1]$. Nevertheless $\theta_1(1) = 0$.

Hence the lemma is proved.

Let us define the auxiliary functions φ_n for $n \in \mathbb{N}$ by

$$\varphi_n(z) := \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt$$

Lemma 2: The functions φ_n converge to Γ locally uniformly in the half-plane $\{z \mid \operatorname{Re} z > 0\}$.

Proof: Let $x = \operatorname{Re} z$. If $0 < x \le 1$,

$$\left|\Gamma(z) - \int_0^n e^{-t}t^{z-1}dt\right| \le \int_n^\infty e^{-t}t^{x-1}dt \le n^{x-1}\int_n^\infty e^{-t}dt \le \frac{e^{-n}}{n^{1-x}}$$

If $x > 1$ we can integrate by parts and obtain

$$\int_n^\infty e^{-t} t^{x-1} dt = e^{-n} n^{x-1} + (x-1)\int_n^\infty e^{-t} t^{x-2} dt$$

It is clear that for any $0 < A < B < \infty$ there is a constant C such that whenever $A \le \operatorname{Re} z \le B$ we have

$$\left|\Gamma(z) - \int_0^n e^{-t} t^{z-1} dt\right| \le C n^{B-1} e^{-n} \to 0 \quad \text{as} \quad n \to \infty$$

On the other hand,

$$\left|\varphi_n(z) - \int_0^n e^{-t} t^{z-1} dt\right| \le \int_0^n t^{x-1}\left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) dt \le \int_0^n t^{x-1} \frac{t^2 e^{-t}}{n} dt$$

$$\le \frac{1}{n}\int_0^n e^{-t} t^{x+1} dt \le \frac{\Gamma(x+2)}{n} \to 0 \quad \text{as} \quad n \to \infty .$$

Hence the lemma is proved.

Let us make the change of variable $t = n\tau$ in φ_n:

$$\varphi_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau = n^z \frac{n}{z}\int_0^1 (1-\tau)^{n-1} \tau^z d\tau = \ldots$$

$$= n^z . \frac{n!}{z(z+1)\ldots(z+n-1)} \int_0^1 \tau^{z+n-1} d\tau = \frac{n^z n!}{z(z+1)\ldots(z+n)}$$

$$= \frac{n^z}{z\left(\frac{z}{1}+1\right)\left(\frac{z}{2}+1\right)\ldots\left(\frac{z}{n}+1\right)}$$

Define the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty}(\sum_{j=1}^{\infty}\frac{1}{j} - \log n)$, and let $\gamma_n = \sum_{j=1}^{n}\frac{1}{j} - \log n - \gamma$. Rewriting n^z as

$$n^z = \exp\left(\left[\sum_{j=1}^{n}\frac{1}{j} - \gamma - \gamma_n\right] z\right).$$

Putting it into the last expression $\varphi_n(z)$, let $n \to \infty$

$$\Gamma(z) = \lim_{n\to\infty}\left(z e^{\gamma z}\prod_{j=1}^{n}\left(1 + \frac{z}{j}\right)e^{-\frac{z}{j}}\right)^{-1}$$

when Re $z > 0$. Also, note that the product is a Weierstrass product. Hence it is an entire function and coincides with $1/\Gamma$ everywhere. This proves that Γ never vanishes and has simple poles only at all non-positive integers.

From the Weirestrass product of the gamma function, we get

$$\Gamma(z)\Gamma(-z) = -\frac{1}{z^2}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)^{-1}$$

Put,

$$f(z) = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$$

Hence, f is also a Weierstrass product. Taking the logarithmic differentiation of f term by term, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \pi \cot \pi z = \frac{d(\sin \pi z)}{\sin \pi z}$$

We know that there is a constant C such that $f(z) = C \sin \pi z$. Dividing both sides by πz and let z tend to 0, we get $C = 1$. Therefore,

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \ldots(4.4)$$

Hence,

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin \pi z} \qquad \ldots(4.5)$$

Putting $z = 1/2$ and noting that $\Gamma(x)$ is positive when x is positive, we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4.2.2 Riemann Zeta Function

The Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s = \sigma + it$ (here, s, σ and t are traditional notations associated to the study of the ζ-function).

The following infinite series converges for all complex numbers s with real part greater than 1 and defines $\zeta(s)$ in this case:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \quad \sigma = \Re(s) > 1$$

The Riemann zeta function is defined as the analytic continuation of the function defined for $\sigma > 1$ by the sum of the preceding series.

Riemann showed that the function defined by the series on the half-plane of convergence can be continued analytically to all complex values $s \neq 1$.

For $s = 1$ the series is the harmonic series which diverges to $+\infty$, and

$$\lim_{s \to 1} (s-1)\zeta(s) = 1$$

Thus the Riemann zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.

NOTES

For any positive even number 2n,

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where B_{2n} is a Bernoulli number; for negative integers, one has

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

for $n \geq 1$, so in particular ζ vanishes at the negative even integers because $B_m = 0$ for all odd m other than 1. No such simple expression is known for odd positive integers.

If $\Re e\, z \geq 1+\varepsilon$ where $\varepsilon > 0$ then

$$\sum_{k=m}^{n} |k^{-z}| = \sum_{k=m}^{n} |k^{-\Re e\, z}| \leq \sum_{k=m}^{n} k^{-1-\varepsilon} \qquad \text{.... (4.6)}$$

implies $\sum_{n=1}^{\infty} |n^{-z}|$ converges uniformly on $\{z \in \mathbb{C} \,|\, \Re e\, z \geq 1+\varepsilon\}$.

Thus the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad \text{...(4.7)}$$

converges normally in the half plane $H = \{z \in \mathbb{C} \,|\, \Re e\, z > 1\}$ and so defines an analytic function ζ in H. The function ζ is called the *Riemann zeta function.*

Substituting nt for t yields,

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = n^z \int_0^{\infty} e^{-nt} t^{z-1} dt \qquad \text{.... (4.8)}$$

Therefore,

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} t^{z-1} dt \qquad \text{... (4.9)}$$

for $\Re e\, z > 1$.

Now,

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = (e^t - 1)^{-1} \qquad \text{... (4.10)}$$

if $t > 0$. If $z = x + iy$ then,

$$\int_0^{\infty} \sum_{n=1}^{\infty} |e^{-nt} t^{z-1}|\, dt = \int_0^{\infty} (e^t - 1)^{-1} t^{x-1} dt \qquad \text{... (4.11)}$$

For large t we have $(e^t - 1)^{-1} \approx e^{-t}$ and for small t we have $(e^t - 1)^{-1} \approx t^{-1}$. It follows that the integral in Equation (4.11) converges if $x > 1$. Now, we can interchange the order of integration in Equation (4.9).

Thus,

$$\zeta(z)\Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt \qquad \text{... (4.12)}$$

The integral in Equation (4.12) is badly behaved for Re z near 1 since then the integrand behaves roughly for small t. Riemann therefore considers a related contour integral where we avoid the origin.

$$I(z) = \int_\gamma (e^w - 1)^{-1}(-w)^z \frac{dw}{w}. \qquad \dots (4.13)$$

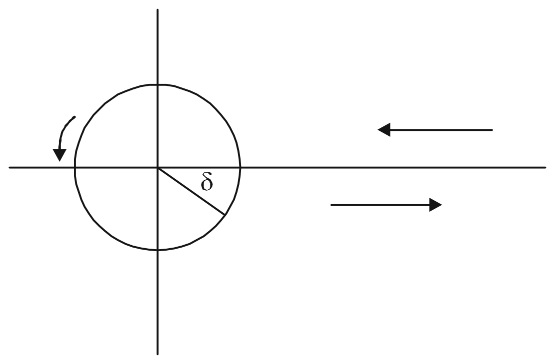


Fig. 4.1 *Countour along the Real Axis from ∞ to δ > 0*

In Figure 4.1, the γ is the contour along the real axis from ∞ to $\delta > 0$, counterclockwise around the circle of radius δ with center at the origin, and then along the real axis from δ to ∞. We take $-w$ to have argument $-\pi$ when we are going towards the origin and argument π when we are going towards ∞. (Strictly speaking we should open this contour up a little and then pass to a limit, or else view it as lying in the appropriate Riemann surface.

The integral in Equation (4.13) converges for all z and defines an entire function. Moreover, by Cauchy's theorem it is independent of the choice of $\delta > 0$. Additionally, $w(e^w - 1)^{-1}$ has a removable singularity at the origin and so by Cauchy's theorem,

$$I(k) = 0 \qquad \text{for} \qquad k = 2, 3, 4, \ldots \qquad \dots (4.14)$$

Since, when z is an integer, the integrals along the real axis in Equation (4.13) cancel and so we may regard γ as just the circle of radius δ.

Now,

$$\begin{aligned} I(z) &= \int_\infty^\delta (e^t - 1)^{-1} e^{z(\log(t) - i\pi)} \frac{dt}{t} \\ &+ \int_{|w|=\delta} (e^w - 1)^{-1}(-w)^z \frac{dw}{w} \\ &+ \int_\delta^\infty (e^t - 1)^{-1} e^{z(\log(t) + i\pi)} \frac{dt}{t} \end{aligned} \qquad \dots (4.15)$$

We cannot use the Cauchy's formula to evaluate the middle integral in Equation (4.15), but with $w = \delta e^{i\theta}$, we have $\frac{dw}{w} = id\theta$ and so since $w(e^w - 1)^{-1}$ has a removable singularity at the origin we see the integral is bounded by

$C\delta^{\Re e\, z-1}$. In particular, the integral goes to 0 as $\delta \to 0$ provided that Re $z > 1$. Thus, letting $\delta \to 0$ we obtain,

$$\begin{aligned} I(z) &= (e^{\pi iz} - e^{-\pi iz})\int_0^\infty (e^t - 1)^{-1} t^{z-1} dt \\ &= 2i\sin(\pi z)\zeta(z)\Gamma(z) \quad \text{if} \quad \Re e\, z > 1. \end{aligned} \qquad \ldots (4.16)$$

Recalling the functional equation for the gamma function,

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad ... (4.17)$$

we obtain,

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} I(z). \qquad \ldots (4.18)$$

Now, Equation (4.18) has been proved for Re $z > 1$, but the right side is analytic in the whole plane, except that $\Gamma(1-z)$ has simple poles at z =1, 2, 3, ….On the other hand, $I(z)$ has zeros at z = 2, 3, … Thus $\zeta(z)$ is actually analytic in $\mathbb{C} \sim \{1\}$. At $z = 1$, there is at worst a simple pole. We can see the pole is actually there by computing the residue.

$$\begin{aligned} \lim_{z\to 1} \frac{(z-1)\Gamma(1-z)}{2\pi i} I(z) &= -\frac{1}{2\pi i} I(1) \\ &= -\frac{1}{2\pi i}\int_{|w|=\delta} (e^w - 1)^{-1} \frac{dw}{w} \\ &= -\lim_{w\to 0}(e^w - 1)^{-1}(-w) \\ &= 1 \end{aligned} \qquad \ldots (4.19)$$

Theorem 4.3: The zeta function ζ continues analytically to a meromorphic function in C with a simple pole at z = 1. The residue at the pole is 1.

Riemann now goes on to deduce the functional equation for the zeta function, but first he remarks that ζ vanishes at the negative even integers. This fact may be seen as follows:

If $n \geq 0$ is an integer then,

$$\zeta(-n) = \frac{n!}{2\pi i} I(-n) \qquad \ldots (4.20)$$

$$\begin{aligned} I(-n) &= \int_\gamma (e^w - 1)^{-1}(-w)^{-n} \frac{dw}{w} \\ &= (-1)^n \int_{|w|=\delta} \frac{w}{e^w - 1} \frac{dw}{w^{n+2}} \end{aligned} \qquad \ldots (4.21)$$

Now $w(e^w - 1)^{-1}$ is analytic in a neighborhood of the origin.

Thus,

$$\frac{w}{e^w - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n w^n \quad \dots (4.22)$$

for $|w| < 2\pi$. The numbers B_n defined by Equation (4.22) are called the Bernoulli numbers. Since,

$$\frac{w}{e^w - 1} + \frac{w}{2} \quad \dots (4.23)$$

is an even function we see that $B_1 = -1/2$ and the other odd Bernoulli numbers all vanish. We can easily compute the even ones: for example

$$B_2 = \frac{1}{6}$$

$$B_4 = \frac{-1}{30}$$

$$B_6 = \frac{1}{42} \quad \dots. (4.24)$$

$$B_8 = \frac{-1}{30}$$

Now by Cauchy's integral formula we have,

$$\frac{(n+1)!}{2\pi i} \int_{|w|=\delta} \frac{w}{e^w - 1} \frac{dw}{w^{n+2}} = B_{n+1} \quad \dots (4.25)$$

And therefore,

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \quad \dots (4.26)$$

for each integer $n \geq 0$. It follows,

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \dots = 0.$$

These roots are called the trivial zeros of the zeta function. The remaining roots are called the nontrivial zeros or critical roots of the zeta function.

4.2.3 Riemann's Functional Equation

The functional equation implies that $\zeta(s)$ has a simple zero at each even negative integer $s = -2n$. These zeros are the trivial zeros of $\zeta(s)$. Riemann established the functional equation which is used to construct the analytic continuation in the first place. An equivalent relationship was conjectured by Euler for the Dirichlet eta function or the alternating zeta function.

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

This relation exhibits $\zeta(s)$ as a Dirichlet series of the η-function which is convergent although non-absolutely in the larger half-plane $\sigma > 0$ and not just

$\sigma > 1$, up to an elementary factor. Riemann also found a symmetric version of the functional equation, given by first defining

$$\xi(s) = \frac{1}{2}\pi^{-s/2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The functional equation is then given by,

$$\xi(s) = \xi(1-s)$$

Riemann defined a similar but different function which he called $\xi(t)$.

Let us define the contour γ_n consisting of two circles centered at the origin and a radius segment along the positive real as shown below in Figure 4.2.

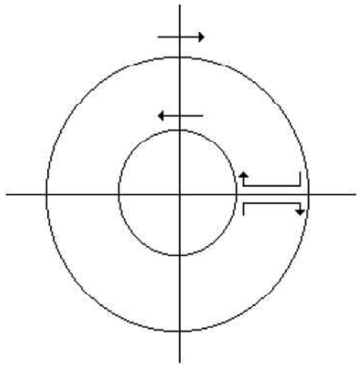


Fig. 4.2 *Contour of Two Circles* γ_n

The outer circle has radius $(2n+1)\pi$ and the inner circle has radius $\delta < \pi$. The outer circle is traversed clockwise and the inner one counterclockwise. The radial segment is traversed in both directions. If we open the contour a little bit along the real axis, we can employ the residue theorem and then pass to a limit, to obtain

$$-\frac{1}{2\pi i}\int_{\gamma_n}(e^w-1)^{-1}\frac{(-w)^z}{w}dw$$

$$= \sum_{k=-n\cdots n, k\neq 0} \Re es\left((e^w-1)^{-1}\frac{(-w)^z}{w}, w = 2\pi ik\right) \qquad \ldots(4.27)$$

$$= -\sum_{k=1}^{n}\left((2\pi ik)^{z-1} + (-2\pi ik)^{z-1}\right)$$

Since,

$$i^{z-1} + (-i)^{z-1} = \frac{1}{i}(e^{z\log(i)} - e^{z\log(-i)})$$

$$= \frac{1}{i}\left(e^{\frac{z\pi i}{2}} - e^{-\frac{z\pi i}{2}}\right) \qquad \ldots(4.28)$$

$$= 2\sin\left(\frac{\pi z}{2}\right)$$

We get,

$$\frac{1}{2\pi i}\int_{\gamma_n}(e^w-1)^{-1}\frac{(-w)^z}{w}dw = 2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\sum_{k=1}^{n}k^{z-1} \quad \ldots(4.29)$$

On the circle, $|w|=(2n+1)\pi$, we have $|e^z-1|$ is bounded independently of n and we have $|(-w)^z/w|\leq|w|^{\Re e\, z-1}$. Thus, if Re $z<0$, the integral over the large circle tends to 0 as n→∞. Therefore,

$$\frac{1}{2\pi i}\int_{\gamma}(e^w-1)^{-1}\frac{(-w)^z}{w}dw = 2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}n^{z-1} \quad \ldots(4.30)$$

But,

$$\zeta(z)=\frac{\Gamma(1-z)}{2\pi i}I(z)$$

Hence,

$$\zeta(z)=2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z) \quad \ldots(4.31)$$

for Re $z<0$.

Now, by uniqueness of analytic continuation, Equation (4.31) is valid for all $z\neq 1$. Note that $\zeta(1-z)$ has a simple pole at $z=0$ and roots at the positive odd integers greater than 1, $\Gamma(1-z)$ has simple poles at the positive integers and $\sin(\pi z/2)$ has roots at the even integers. Hence, we can conclude that all the singularities on the right hand side of Equation (4.31), except at $z=1$, are removable.

4.2.4 Runge's Theorem

If K is a compact subset of **C** (the set of complex numbers), A is a set containing at least one complex number from every bounded connected component of **C**\\K, and f is a holomorphic function on an open set containing K, then there exists a sequence (r_n) of rational functions all of whose poles are in A such that the sequence (r_n) approaches the function f uniformly on K.

Theorem 4.4 (Runge's Theorem): For any compact set $K\subset C$ we have $R(K)=A(K)$ and $P(K)=A(K)$ provided $C–K$ is connected.

Proof: For the first result, suppose $f(z)$ is analytic on a smoothly bounded neighborhood U of K. Then we can write

$$f(z)=\frac{1}{2\pi i}\int_{\partial U}\frac{f(t)dt}{t-z}=\int_{\partial U}F_z(t)dt$$

Since $d(z,\partial U)\geq d(K,\partial)>0$, the functions $\{F_z\}$ range in a compact subset of $C(\partial U)$. Thus we can replace this integral with a finite sum at the cost of an error that is small and independent of z. But the terms $f(t_i)/(t_i-z)$ appearing in the sum are rational functions of z, so $R(K)=A(K)$.

The second result is proved by pole-shifting. By what we have just done, it suffices to show that $f_p(z)=1/(z-p)\in P(K)$ for every $p\notin K$. Let $E\subset C-K$ denote the set of p for which this is true.

NOTES

Clearly E contains all p which are sufficiently large, because then the power series for $f_p(z)$ converges uniformly on K. Also E is closed by definition. To complete the proof, it suffices to show E is open.

The proof that E is open is by 'Pole Shifting'. Suppose $p \in E$, $q \in B(p, r)$ and $B(p, r) \cap K = \theta$. Note that $f_q(z)$ is analytic on $\mathbb{C} - B(p, r)$ and tends to zero as $|z| \to \infty$. Thus $f_q(z)$ can be expressed as a power series in $1/(z-p)$:

$$(z-q)^{-1} = \sum_{0}^{\infty} a_n (z-p)^{-n} = \sum a_n f_p(z)^n,$$

convergent for $|z-p| > |z-q|$, and converging uniformly on K. Compare the expression,

$$\frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{z^n},$$

valid for $|z| > 1$.) Since $(z-q)^{-1} \to 0$ as $|z| \to \infty$, only terms with $n \geq 0$ occur on the right. But $f_p \in A(K)$ by assumption, and $A(K)$ is an algebra, so it also contains f_p^n. Thus, $f_q \in A(K)$ as well.

This completes the proof.

We can use Runge's theorem to show that there is a sequence of polynomials $f_n(z)$ that converge pointwise but whose limit is not even continuous.

4.2.5 Mittag-Leffler's Theorem

This concerns the existence of meromorphic functions with prescribed poles and asserts the existence of holomorphic functions with prescribed zeros.

Suppose f is meromorphic in a region Ω with a pole at $b \in \Omega$. Then, recalling the Laurent expansion,

$$f(z) = \frac{c_n}{(z-b)^n} + \frac{c_{n-1}}{(z-b)^{n-1}} + \ldots + \frac{c_1}{(z-b)} + a_0 + a_1(z-b) + a_2(z-b)^2 + \ldots,$$

for z near b. The sum of the first n terms,

$$S_b(z) = \frac{c_n}{(z-b)^n} + \frac{c_{n-1}}{(z-b)^{n-1}} + \ldots + \frac{c_1}{(z-b)}$$

is called singular part of f at b. If f is rational, then by a partial fraction expansion

$$f(z) = \sum_{k=1}^{n} S_{b_k}(z) + p(z),$$

where p is a polynomial and $\{b_k\}$ are the poles of f. If f is meromorphic in a region Ω with only finitely many poles $\{b_k\}$ and singular parts S_{b_k}, $k = 1, \ldots, n$, then

$$f(z) = \sum_{k=1}^{n} S_{b_k}(z) + g(z),$$

where g is analytic in Ω. This follows because $f(z) - \sum S_{b_k}(z)$ is analytic at each b_k and therefore in all of Ω. In this section, we will find a similar expansion for meromorphic functions in Ω with infinitely many poles. We say that an infinite sequence $b_k \in \Omega \to \partial\Omega$ as $k \to \infty$ if each compact $K \subset \Omega$ contains only finitely many b_k.

Theorem 4.5 (Mittag-Leffler Theorem): Suppose $b_k \in \Omega \to \partial\Omega$

Set $S_k(z) = \sum_{j=1}^{n_k} \frac{c_{j,k}}{(z-b_k)^j}$

where each n_k is a positive integer and $c_{j,k} \in \mathbb{C}$. Then there is a function meromorphic in Ω with singular parts S_k at b_k, $k = 1, 2, \ldots$, and no other singular parts in Ω.

Proof: Let,

$$K_n = \{z \in \Omega : \text{dist}\,(z, \partial\Omega) \geq \frac{1}{n} \text{ and } |z| \leq n\}.$$

Then K_n is a compact subset of Ω such that each component of $\mathbb{C} \setminus K_n$ contains a point of $\mathbb{C} \setminus \Omega$ and $K_n \subset K_{n+1} \subset \bigcup K_n = \Omega$.

By Runge's theorem we can find a rational function f_n with poles in C/Ω so that,

$$\left| \sum_{b_k \in K_{n+2} \setminus K_{n+1}} S_k(z) - f_n(z) \right| < 2^{-n}$$

for all $z \in K_n$.

Then,

$$\sum_{n \geq m} \sum_{b_k \in K_{n+2} \setminus K_{n+1}} S_k(z) - f_n(z)$$

converges uniformly on K_m to an analytic function on K_m by the Weierstrass M-test and Weierstrass's Theorem.

Thus,

$$f(z) = \sum_{b_k \in K_2} S_k(z) + \sum_{n=1}^{\infty} \sum_{b_k \in K_{n+2} \setminus K_{n+1}} S_k(z) - f_n(z) \qquad \ldots (4.32)$$

is the desired function.

Suppose we have to find a function meromorphic in $\mathbb{C}$ with singular part $a_k/(z-b_k)$ at $\{b_k\}$, where $|b_k| \to \infty$. Then,

$$\sum_{k=1}^{\infty} \frac{a_k}{z-b_k} \qquad \ldots (4.33)$$

will work, provided the sum converges. When does it converge? If $|z| \leq R < \infty$, write

NOTES

$$\sum_{k=1}^{\infty} \frac{a_k}{z-b_k} = \sum_{\{k:|b_k|\le 2R\}} \frac{a_k}{z-b_k} + \sum_{\{k:|b_k|>2R\}} \frac{a_k}{z-b_k} \qquad \dots (4.34)$$

The first sum has only finitely many terms. For the second sum, $|z| \le R < |b_k|/2$, so that,

$$\left|\frac{1}{z-b_k}\right| \le \frac{2}{|b_k|}$$

Thus if,

$$\sum_{k=1}^{\infty} \frac{|a_k|}{|b_k|} < \infty . \qquad \dots (4.35)$$

Then the second sum in Equation (4.34) converges uniformly on $\{|z| \le R\}$ to an analytic function, by Weierstrass again. The right side of Equation (4.34) then is meromorphic in $|z| < R$ with singular part $a_k/(z-b_k)$ at $\{b_k : |b_k| < R\}$. Since R is arbitrary, the sum in Equation (4.34) is meromorphic.

Mittag-Leffler's idea was to subtract a polynomial from each term so that the result converges.

$$\frac{1}{z-b_k} = \frac{1}{-b_k\left(1-\frac{z}{b_k}\right)} = \frac{1}{-b_k}\sum_{k=0}^{\infty}\left(\frac{z}{b_k}\right)^j .$$

So it is natural to subtract a few terms of the expansion to make it smaller:

$$\left|\frac{1}{z-b_k} - \left(\frac{1}{-b_k}\right)\sum_{j=0}^{n_k}\left(\frac{z}{b_k}\right)^j\right| = \left|\frac{1}{-b_k}\sum_{j=n_k+1}^{\infty}\left(\frac{z}{b_k}\right)^j\right|$$

$$\le \left|\frac{1}{b_k}\right|\left|\frac{z}{b_k}\right|^{n_k+1} \frac{1}{1-\left|\frac{z}{b_k}\right|^j}$$

provided $|z| < |b_k|$.

For example, the following proposition holds.

Proposition 4.1: If $b_k \to \infty$ and if for some $n < \infty$

$$\sum \frac{|a_k|}{|b_k|^{n+2}} < \infty$$

Then,

$$f(z) = \sum_{k=1}^{\infty}\left(\frac{a_k}{z-b_k} - \left(\frac{a_k}{-b_k}\right)\sum_{j=0}^{n}\left(\frac{z}{b_k}\right)^j\right)$$

is meromorphic in $\mathbb{C}$ with singular part $S_{b_k}(z) = \dfrac{a_k}{z-b_k}$ at b_k, $k = 1, 2, \dots$, and no other poles.

To prove Proposition 4.1,

if $|z|< R$, split the sum into two pieces: a finite sum of the terms with $|b_k|\leq 2R$ and a convergent sum of the terms with $|b_k|>2R$.

For example,

$$\frac{\pi^2}{\sin^2 \pi z}=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2} \qquad \ldots (4.36)$$

The right side of Equation (4.36) converges uniformly on compact subsets of $\mathbb{C}$ and the limit is meromorphic with singular parts $S_n(z)=1/(z-n)^2$ at $z = n$ and has no other poles.

Fix $R<\infty$ and shift the sum into two pieces: a finite sum of terms with $|n|<2R$ and the remaining infinite sum. In the second sum when $|z|\leq R$, the *nth* term is uniformly bounded by $1/(|n|-R)^2\leq 4/n^2$. Thus the second (infinite) sum converges uniformly and absolutely on $|z|\leq R$ to an analytic function by Weierstrass's Theorem.

The function $\pi z/\sin \pi z$, *h* as a removable singularity at $z = 0$ and is an even function

So that,

$$\frac{\pi z}{\sin \pi z}=1+\mathrm{O}(z^2),$$

near 0. By squaring and dividing by z^2 we conclude that the the singular part of $\pi^2/\sin^2 \pi z$ at $z = 0$ is $1/z^2$. Since $\sin^2 \pi(z-n)=\sin^2 \pi z$, we conclude that the singular part of the left side of Equation (4.36) is the same as the singular part of the right side of Equation (4.36) at $z = n$ for each n. Set $F(z)=\frac{\pi^2}{\sin^2 \pi z}-\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$.

Then F is entire and $F(z + 1) = \mathrm{F}(z)$.

Write $z = x + iy$ and suppose $0\leq x\leq 1$. We claim that

$$|F(z)|\to 0, \qquad \ldots (4.37)$$

as $|y|\to\infty$. If so, F is bounded in the strip $0\leq x\leq 1$ and since $F(z+1)=F(z)$, the function F is bounded in C. By Liouvulle's Theorem, F is constant and by

Equation (4.37) $F = 0$, proving the Example. To see Equation (4.37) first observe that,

$$|\sin \pi z|=\left|e^{-\pi y+i\pi x}-e^{\pi y-i\pi x}\right|/2\to\infty,$$

as $|y|\to\infty$. Thus the left side of (4.36) tends to 0 as $|y|\to\infty$, with $0\leq x\leq 1$. Likewise in the strip $0\leq x\leq 1$, the right side of Equation (4.36) is dominated by

$$\sum\left|\frac{1}{(|n|-1)^2+y^2}\right|$$

which also tends to 0 as $|y| \to \infty$ by comparison with the integral $\int_1^\infty dx/(x^2+y^2)$. This proves Equation (4.37) and the example.

Again consider an example.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right). \qquad \ldots(4.38)$$

First observe that $\frac{d}{dz}\pi \cot \pi z = -\frac{\pi^2}{\sin^2 \pi z} = -\sum \frac{1}{(z-n)^2}$.

But, $\qquad \pi \cot \pi z \neq \sum \frac{1}{z-n}$,

because the latter sum does not converge. The difficulty is that for $|z| \leq R$, the nth term behaves like $1/n$. However,

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n} \sim \frac{1}{n^2},$$

for large n and $\sum n^{-2} < \infty$. To prove this example suppose $|z| \leq R$ and split the sum on the right side of Equation (4.37) into the sum of the terms with $|n| \leq 2R$ and the sum of the terms with $|n| > 2R$. The first sum is finite and the second sum has terms satisfying

$$\left|\frac{1}{z-n} + \frac{1}{n}\right| = \left|\frac{z}{(z-n)n}\right| \leq \frac{R}{\frac{n}{2}.n} = \frac{2R}{n^2}.$$

Since $\sum 2R/n^2 < \infty$, the right side of Equation (4.37) is meromorphic in $|z| \leq R$ with poles only at the integers and prescribed singular parts $1/(z-n)$, for each $R < \infty$. Furthermore by Weierstrass's Theorem and previous example the right side of Equation (4.38) has derivative

$$-\frac{1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} = \frac{d}{dz}\pi \cot \pi z.$$

Thus the two sides of Equation (4.38) differ by a constant.

The convergence of the right side of Equation (4.38) is absolute, so that we can rearrange the terms and the sum will converge to the same limit. This allows us to conclude that the right side of Equation (4.38) is an odd function. Since the left side of Equation (4.38) is also odd, the difference between the left and right sides is odd and constant and hence is identically 0.

Check Your Progress

1. State Weierstrass' factorization theorem.
2. What is the logarithmic derivative of the gamma function?
3. Define Riemann zeta function.
4. What does the functional equation implies?
5. What does Runge's theorem state?
6. What does Mittag-Leffler theorem state?

4.3 ANALYTIC CONTINUATION

Suppose that V is a connected, open subset of C and that $f_1: V \to C$ and $f_2: V \to C$ are holomorphic functions. If there is an open, non-empty subset U of V such that $f_1 \equiv f_2$ on U, then $f_1 \equiv f_2$ on all of V. We can also say that, if we are given f holomorphic on U, then there is at most one way to extend f to V so that the extended function is holomorphic.

Example 4.1: Define $f(z) = \sum_{j=0}^{\infty} z^j$. This series converges normally on the disc $D = \{z \,\varepsilon\, C: |z| < 1\}$.

It diverges for $|z| > 1$. On summing the series, we observe that

$$f(z) = \frac{1}{1-z}$$

In this formula, the natural domain of definition for f is a large set $C \setminus \{1\}$.

Again consider the gamma function, $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt$

The size of the term t^{z-1} in this function is $|t^{z-1}| = t^{\text{Re } z-1}$. Thus, the singularity at the origin will be integrable when Re $z > 0$. Because of the presence of the exponential factor, the integrand will certainly be integrable at infinity. The function Γ is holomorphic on the domain $U_0 \equiv \{z: \text{Re } z > 0\}$. The functions,

$$\int_a^{1/a} t^{z-1} e^{-t}\, dt,$$

where $a > 0$ are holomorphic by differentiation under the integral sign and $\Gamma(z)$ is the normal limit of these integrals as $a \to 0^+$. The given definition of gamma function makes no sense when Re $z \le 0$ because the improper integral diverges at 0.

We can write,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt = \frac{1}{z} t^z e^{-t} \Big|_0^{\infty} + \frac{1}{z} \int_0^{\infty} t^z e^{-t}\, dt$$

This function is holomorphic on $U_1 \equiv \{z: \text{Re } z > -1\} / \{0\}$.

Again integrating by parts,

$$\Gamma(z) = \frac{1}{z(z+1)} \int_0^{\infty} t^{z+1} e^{-t}\, dt$$

which converges on $U_2 \equiv \{z: \text{Re } z > -2\} / \{0, -1\}$.

Hence, the gamma function can be analytically continued to $U = \{z \,\varepsilon\, C: z \neq 0, -1, -2, \ldots\}$.

There can be atmost one way to affect the analytic continuation process. Any attempt of analytic continuation along two different paths results in ambiguities in the process.

Function Elements: A function element is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic function defined on U. If W is an open set, then a function element in W is a pair (f, U) such that $U \subseteq W$.

Direct Analytic Continuation: Let (f, U) and (g, V) be function elements. We say that (g, V) is a direct analytic continuation of (f, U) if $U \cap V \neq \varphi$, and f and g are equal on $U \setminus V$. Obviously (g, V) is a direct analytic continuation of (f, U) if and only if (f, U) is a direct analytic continuation of (g, V).

Analytic Continuation of a Function: If $(f_1, U_1), \ldots, (f_k, U_k)$ are function elements and if each (f_j, U_j) is a direct analytic continuation of $(f_{j-1}, U_{j-1}), j = 2, \ldots, k$, then we say that (f_k, U_k) is an analytic continuation of (f_1, U_1). Clearly, (f_k, U_k) is an analytic continuation of (f_1, U_1) if and only if (f_1, U_1) is an analytic continuation of (f_k, U_k). Also if (f_k, U_k) is an analytic continuation of (f_1, U_1) via a chain $(f_1, U_1), \ldots, (f_k, U_k)$ and if (f_{k+1}, U_{k+1}) is an analytic continuation of (f_k, U_k) via a chain $(f_k, U_k), (f_{k+1}, U_{k+1}), \ldots, (f_{k+1}, U_{k+1})$, then stringing the two chains together into $(f_1, U_1), \ldots, (f_{k+1}, U_{k+1})$ exhibits (f_{k+1}, U_{k+1}) as an analytic continuation of (f_1, U_1). Obviously, any function element (f, U) is an analytic continuation of itself.

Analytic Continuation Along a Curve

Let $\gamma : [0,1] \rightarrow C$ be a curve and let (f, U) be a function element with $\Upsilon(0)$ the centre of the disc U. An analytic continuation of (f, U) along the curve is a collection of function elements (f_t, U_t), $t \in [0, 1]$, such that $(f_0, U_0) = (f, U)$, for each $t \in [0, 1]$, the centre of the disc U_t is $\gamma(t)$, $0 \leq t \leq 1$ and for each $t \in [0, 1]$, there is an $\varepsilon>0$, such that for each $t' \varepsilon [0,1]$ with $|t' - t|<\varepsilon$, it holds that: $\gamma(t') \in U_t$ and hence $U_{t'} \cap U_t \neq \emptyset$ and $f_t \equiv f_{t'}$ on $U_{t'} \cap U_t$.

Let U be a disc with function element (f, U) and centre P. Let Υ be a curve such that $\gamma(0) = P$. Now, if (f_m, U_m) is the terminal element of one analytic continuation (f_t, U_t) and if $\left(\tilde{f}_{\tilde{m}}, \tilde{U}_{\tilde{m}}\right)$ is the terminal element of another analytic continuation $\left(\tilde{f}_{\tilde{t}}, \tilde{U}_{\tilde{t}}\right)$ then f_m and $\tilde{f}_{\tilde{m}}$ are equal on $U_m \cap \tilde{U}_{\tilde{m}}$. This result says that analytic continuation of a given element along a curve is unique.

4.3.1 Power Series Method of Analytic Continuation

A series of geometrically increasing numbers,

$$S_n = 1 + x + x^2 + x^3 + \ldots + x^n$$

can be expressed in terms of just the second and the last number by noting that,

$$1 + xS_n = 1 + x + x^2 + x^3 + \ldots + x^n + x^{n+1} = S_n + x^{n+1}$$

Solving for S_n gives,

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + x^3 + \ldots + x^n \qquad \ldots (4.39)$$

Now, if the magnitude of x is less than 1, the quantity x^{n+1} goes to zero as n increases, so we immediately have the sum of the infinite geometric series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \ldots (4.40)$$

Archimedes evaluated the area enclosed by a parabola and a straight line essentially by determining the sum of such a series. This is perhaps the first example of a function being associated with the sum of an infinite number of terms. To illustrate, if we set x equal to 1/2, this equation gives,

$$2 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

There is, of course, a very significant difference between Equations (4.39) and (4.40), because the former is valid for any value of x, whereas the latter is not—at least not in the usual sense of finite arithmetical quantities. For example, if we set x equal to 2 in Equation (4.41) we get,

$$-1 = 1 + 2 + 4 + 8 + \ldots$$

which is surely not a valid arithmetic equality in the usual sense, because the right hand side does not converge on any finite value. This shows that the correspondence between a function and an infinite series, such as Equation (4.40) may hold good only over a limited range of the variable. Generally speaking, an analytic function $f(z)$ can be expanded into a power series about any complex value of the variable z by means of Taylor's expansion, which can be written as

$$f(z_0+z) = f(z_0) + \frac{f'(z_0)}{1!}z + \frac{f''(z_0)}{2!}z^2 + \frac{f'''(z_0)}{3!}z^3 + \ldots$$

but the series will converge on the function only over a circular region of the complex plane centered on the point z_0 and extending to the nearest pole of the function (i.e., a point where the function goes to infinity). For example, the function $f(z) = 1/(1-z)$ has a pole at $z = 1$; so the disc of convergence of the power series for this function about the origin ($z = 0$) has a radius of 1. Hence, the series given by Equation (4.41) converges unconditionally only for values of x with magnitude less than 1.

The analytic function $f(z) = 1/(1-z)$ can also be expanded into a power series about any other point (where the function and its derivatives are well behaved). The derivatives of $f(z)$ are,

$$f'(z) = \frac{1}{(1-z)^2},\ f''(z) = \frac{2}{(1-z)^3},\ f'''(z) = \frac{6}{(1-z)^4}, \text{ and so on.}$$

Inserting these into the expression for Taylor's series, we get

$$f(z_0+z) = \frac{1}{(1-z_0)} + \frac{z}{(1-z_0)^2} + \frac{z^2}{(1-z_0)^3} + \frac{z^3}{(1-z_0)^4} + \ldots$$

Hence, the power series for this function about the point $z_0 = 2$ is,

$f(2 + z) = -1 + z - z^2 + z^3 - z^4 + ...$

Each of the power series obtained in this way is convergent only on the circular region of the complex plane centered on z_0 and extending to the nearest pole of the function. For example, since the function $f(z) = 1/(1-z)$ has a pole at $z = 1$, the power series with $z_0 = 2$ is convergent only in the shaded region as shown in the Figure 4.3.

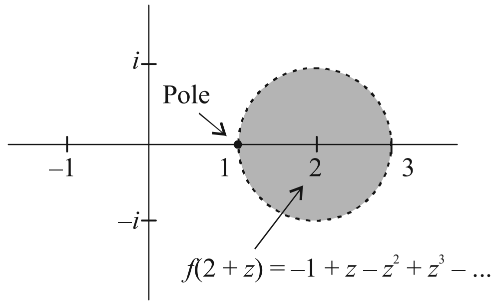


Fig. 4.3 *Power Series Convergent in Shaded Region*

So far we have discussed only the particular function $f(z) = 1/(1-z)$ and have shown how this known analytic function is equal to certain power series in certain regions of the complex plane. However, in some circumstances we may be given a power series having no explicit closed-form expression for the analytic function it represents in its region of convergence. In such cases, we can often still determine the values of the underlying analytic function for arguments outside the region of convergence of the given power series by a technique called analytic continuation. To illustrate with a simple example, suppose we are given the power series, $f(z) = 1 + z + z^2 + z^3 + \ldots$, and suppose we do not know the closed-form expression for the analytic function represented by this series. As noted above, the series converges for values of z with magnitudes less than 1, but it diverges for values of z with magnitudes greater than 1. Nevertheless, by the process of analytic continuation we can determine the value of this function at any complex value of z (provided the function itself is well behaved at that point). To do this, consider again the region of convergence for the given power series as shown below in Figure 4.4.

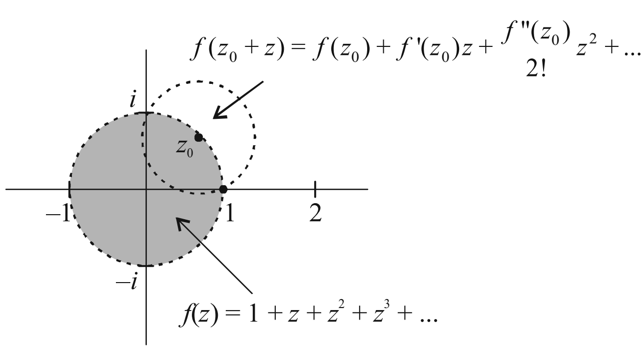


Fig. 4.4 *Region of Convergence*

Since the known power series equals the function within its radius of convergence, we can evaluate $f(z)$ and its derivatives at any point in that region. Therefore, we can choose a point, such as z_0 shown in the Figure and determine the power series expression for $f(z_0 + z)$, which will be convergent within a circular region centered on z_0 and extending to the nearest pole. Thus, we can now evaluate the function at values that lie outside the region of convergence of the original power series. Once we have determined the power series for $f(z_0 + z)$, we can repeat the process by selecting a point z_1 inside the region of convergence and determining the power series for $f(z_1 + z)$, which will be convergent in a circular region centered on z_1 and extending to the nearest pole (which is at $z = 1$ in this example). This is illustrated in the following Figure 4.5.

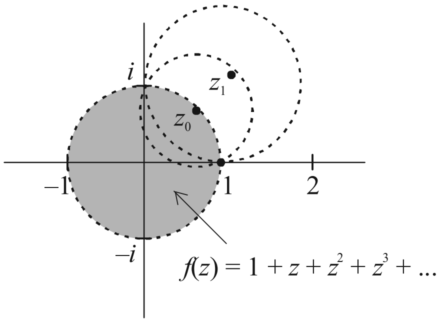


Fig. 4.5 *Convergence in Circular Region*

Continuing in this way, we can analytically extend the function throughout the entire complex plane, except where the function is singular, i.e., at the poles of the function. In general, given a power series of the form,

$$f(z) = a_0 + a_1(z - \alpha) + + a_2(z - \alpha)^2 + a_3(z - \alpha)^3 + ...$$

where the a_j are complex coefficients and α is a complex constant, we can express the same function as a power series centered on a nearby complex number β as,

$$f(z) = b_0 + b_1(z - \beta) + b_2(z - \beta)^2 + b_3(z - \beta)^3 + ...$$

where the b_j are complex coefficients. In order for these two power series to be equal for arbitrary values of z in this region, we must equate the coefficients of powers of z, so we must have

$$\begin{aligned}
a_0 - \alpha a_1 + \alpha^2 a_2 - \alpha^3 a_3 + ... &= b_0 - \beta b_1 + \beta^2 b_2 - \beta^3 b_3 + ... \\
a_1 - 2\alpha a_2 + 3\alpha^2 a_3 - 4\alpha^3 a_4 + ... &= b_1 - 2\beta b_2 + 3\beta^2 b_3 - 4\beta^3 b_4 + ... \\
a_2 - 3\alpha a_3 + 6\alpha^2 a_4 - 10\alpha^3 a_5 + ... &= b_2 - 3\beta b_3 + 6\beta^2 b_4 - 10\beta^3 b_5 + ...
\end{aligned}$$

In matrix form these conditions can be written as,

$$\begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 & \alpha^4 & \cdots \\ 0 & 1 & -2\alpha & 3\alpha^2 & -4\alpha^3 & \cdots \\ 0 & 0 & 1 & -3\alpha & 6\alpha^2 & \cdots \\ 0 & 0 & 0 & 1 & -4\alpha & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \beta^2 & -\beta^3 & \beta^4 & \cdots \\ 0 & 1 & -2\beta & 3\beta^2 & -4\beta^3 & \cdots \\ 0 & 0 & 1 & -3\beta & 6\beta^2 & \cdots \\ 0 & 0 & 0 & 1 & -4\beta & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{bmatrix}$$

Multiplying through by the inverse of the right hand coefficient matrix, this gives

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & \cdots \\ 0 & 1 & 2\varepsilon & 3\varepsilon^2 & 4\varepsilon^3 & \cdots \\ 0 & 0 & 1 & 3\varepsilon & 6\varepsilon^2 & \cdots \\ 0 & 0 & 0 & 1 & 4\varepsilon & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} \qquad \ldots (4.41)$$

where ε = β – α. Naturally, this is equivalent to applying Taylor's expansion. Now, it might seem as if this precludes any extension of the domain of the original power series. For example, suppose the original function was the power series for $1/(1-z)$ about the point $\alpha = 0$, so the power series coefficients $a_0, a_1, \ldots$ would all equal 1. According to the above matrix equation, the coefficient b_0 for the power series about the point β would be simply $b_0 = 1 + \beta + \beta^2 + \beta^3 + \ldots$

This of course converges only over the same region as the original power series. In addition, it is of no help to split up the series transformation into smaller steps, because the compositions of the coefficient matrix are given by,

$$\begin{bmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & \cdots \\ 0 & 1 & 2\varepsilon & 3\varepsilon^2 & 4\varepsilon^3 & \cdots \\ 0 & 0 & 1 & 3\varepsilon & 6\varepsilon^2 & \cdots \\ 0 & 0 & 0 & 1 & 4\varepsilon & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & (n\varepsilon) & (n\varepsilon)^2 & (n\varepsilon)^3 & (n\varepsilon)^4 & \cdots \\ 0 & 1 & 2(n\varepsilon) & 3(n\varepsilon)^2 & 4(n\varepsilon)^3 & \cdots \\ 0 & 0 & 1 & 3(n\varepsilon) & 6(n\varepsilon)^2 & \cdots \\ 0 & 0 & 0 & 1 & 4\varepsilon & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus, the net effect of splitting ε into n segments of size ε/n and applying the individual transformation n times is evidently identical to the effect of performing the transformation in a single step. From this we might conclude that it is impossible to analytically continue the power series $1 + z + z^2 + \ldots$ to any point such as $3i/2$ with magnitude greater than 1. However, it actually is possible to analytically continue the geometric series, but only because of conditional convergence. This is most easily explained with an example. Beginning with the power series,

$$f(z) = 1 + z + z^2 + z^3 + .. \qquad \ldots (4.42)$$

centered on the origin, we can certainly express this as a power series centered on the complex number $\varepsilon = 3i/4$, because the power series $f(z)$ and it derivatives are all convergent at this point as it is inside the unit circle of convergence. By Equation (4.41) with $a_0 = a_1 = a_2 = \ldots = 1$, the coefficients of

$$f(z) = b_0 + b_1(z - \varepsilon) + b_2(z - \varepsilon)^2 + b_3(z - \varepsilon)^3 + \ldots$$

are,

$$b_0 = 1 + \left(\frac{3}{4}i\right) + \left(\frac{3}{4}i\right)^2 + \left(\frac{3}{4}i\right)^3 + .. \quad = \frac{16+12i}{25}$$

$$b_1 = 1 + 2\left(\frac{3}{4}i\right) + 3\left(\frac{3}{4}i\right)^2 + 4\left(\frac{3}{4}i\right)^3 + .. \quad = \frac{112+384i}{625}$$

$$b_2 = 1 + 3\left(\frac{3}{4}i\right) + 6\left(\frac{3}{4}i\right)^2 + 10\left(\frac{3}{4}i\right)^3 + .. \quad = \frac{-2816+7488i}{15625}$$

The absolute values of these coefficients are $b_n = (4/5)^{n+1}$. Now if we take these as the a_n values and apply the same transformation again, shifting the center of the power series by another $\varepsilon = 3i/4$, so that the resulting series is centered on $3i/2$, we find that the zeroth coefficient given by Equation (4.41) is,

$$b_0 = \left[\frac{16+12i}{25}\right] + \left(\frac{3}{4}i\right)\left[\frac{112+384i}{625}\right] + \left(\frac{3}{4}i\right)^2\left[\frac{-2816+7488i}{15625}\right] + \ldots = \frac{4+6i}{13}$$

in agreement with the analytic expression for the function. This series clearly converges, because each term has geometrically decreasing magnitude. Similarly, we can compute the higher order coefficients for the power series centered on the point $3i/2$, which is well outside the radius of convergence of the original geometric series centered on the origin. We have essentially just multiplied the unit column vector by the coefficient vector for ε twice, which we know gives the divergent result

$$b_0 = 1 + \left(\frac{3}{2}i\right) + \left(\frac{3}{2}i\right)^2 + \left(\frac{3}{2}i\right)^3 + \left(\frac{3}{2}i\right)^4 \ldots$$

To examine this more closely, let us expand the quantities in the square brackets in the preceding expression for b_0. This gives,

$$\begin{aligned} b_0 = {} & 1 + \left(\frac{3}{4}i\right) + \left(\frac{3}{4}i\right)^2 + \left(\frac{3}{4}i\right)^3 + \ldots \\ & + \left(\frac{3}{4}i\right) + 2\left(\frac{3}{4}i\right)^2 + 3\left(\frac{3}{4}i\right)^3 + 4\left(\frac{3}{4}i\right)^4 + \ldots \\ & + \left(\frac{3}{4}i\right)^2 + 3\left(\frac{3}{4}i\right)^3 + 6\left(\frac{3}{4}i\right)^4 + 10\left(\frac{3}{4}i\right)^5 + \ldots \\ & + \left(\frac{3}{4}i\right)^3 + 4\left(\frac{3}{4}i\right)^4 + 10\left(\frac{3}{4}i\right)^5 + 20\left(\frac{3}{4}i\right)^6 + \ldots, \text{etc.} \end{aligned}$$

Each individual row is convergent and the rows converge on geometrically decreasing values; so the sum of the sums of the rows is also convergent. However, if we sum the individual values by diagonals we get,

$$b_0 = 1+\left[\left(\frac{3}{4}i\right)+\left(\frac{3}{4}i\right)\right]+\left[\left(\frac{3}{4}i\right)^2+2\left(\frac{3}{4}i\right)^2+\left(\frac{3}{4}i\right)^2\right]$$

$$+\left[\left(\frac{3}{4}i\right)^3+3\left(\frac{3}{4}i\right)^3+3\left(\frac{3}{4}i\right)^3+\left(\frac{3}{4}i\right)^3\right]+...$$

$$= 1+\left(\frac{3}{2}i\right)+\left(\frac{3}{2}i\right)^2+\left(\frac{3}{2}i\right)^3+\left(\frac{3}{2}i\right)^4+...$$

Thus, the terms for b_0 are divergent if we sum them diagonally, but they are convergent if we sum them by rows. In other words, the series is conditionally convergent, which is to say, the sum of the series— even whether it sums to a finite value at all—depends on the order in which we sum the terms. The same applies to the series for the other coefficients.

Since the terms of a conditionally convergent series can be rearranged to give any value we choose, whether analytic continuation—which is based so fundamentally on conditional convergence—really gives a unique result? To show that we can also continue the geometric series to points on the other side of the singularity using this procedure. Consider again the initial power series in Equation (4.42), and this time suppose we determine the sequence of series centered on points located along the unit circle centered on the point $z = 1$ as indicated in the Figure 4.6.

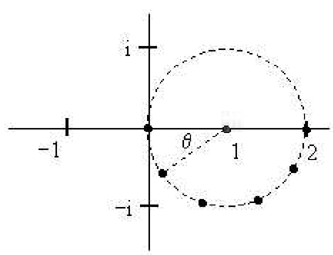


Fig. 4.6 *Sequence of Series*

Thus, letting $\alpha = e^{i\theta}$, we wish to carry out successive shifts of the power series center by the increments,

$$\varepsilon_1 = 1-\alpha \quad \varepsilon_2 = \alpha(1-\alpha) \quad \varepsilon_3 = \alpha^2(1-\alpha) \quad \varepsilon_4 = \alpha^3(1-\alpha)$$

and so on. Applying Equation (4.41) with $\varepsilon = \varepsilon_1$ to perform the first of these transformations we get the sequence of coefficients,

$$b_0 = 1+(1-\alpha)+(1-\alpha)^2+(1-\alpha)^3+... = \frac{1}{\alpha}$$

$$b_1 = 1+2(1-\alpha)+3(1-\alpha)^2+4(1-\alpha)^3+... = \frac{1}{\alpha^2}$$

$$b_2 = 1+3(1-\alpha)+6(1-\alpha)^2+10(1-\alpha)^3+... = \frac{1}{\alpha^3}$$

Now if we call these the a_j coefficients and perform the next transformation using Equation (4.42) with $\varepsilon = \varepsilon_2$, we get

NOTES

$$b_0 = \frac{1}{\alpha} + \left[\alpha(1-\alpha)\right]\frac{1}{\alpha^2} + \left[\alpha(1-\alpha)\right]^2\frac{1}{\alpha^3} + \left[\alpha(1-\alpha)\right]^3\frac{1}{\alpha^4} + ... = \frac{1}{\alpha^2}$$

$$b_1 = \frac{1}{\alpha^2} + 2\left[\alpha(1-\alpha)\right]\frac{1}{\alpha^3} + 3\left[\alpha(1-\alpha)\right]^2\frac{1}{\alpha^4} + 4\left[\alpha(1-\alpha)\right]^3\frac{1}{\alpha^5} + ... = \frac{1}{\alpha^4}$$

$$b_2 = \frac{1}{\alpha^3} + 3\left[\alpha(1-\alpha)\right]\frac{1}{\alpha^4} + 6\left[\alpha(1-\alpha)\right]^2\frac{1}{\alpha^5}10\left[\alpha(1-\alpha)\right]^3\frac{1}{\alpha^6} + ... = \frac{1}{\alpha^6}$$

Each of these sums is clearly convergent, because $|\alpha| = 1$ and $|1-\alpha| < 1$. Continuing in this way, the *n*th power series in this sequence is,

$$f_n(z) = e^{-ni\theta} + e^{-2ni\theta}(z-\mu_n) + e^{-3ni\theta}(z-\mu_n)^2 + e^{-4ni\theta}(z-\mu_n)^3 + ...$$

where $\mu_n = (1 - e^{ni\theta})$ is the *n*th point around the circle. This can also be written in the form,

$$f_n(z) = e^{-ni\theta}\left(1 + \left(\frac{z-\mu_n}{e^{ni\theta}}\right) + \left(\frac{z-\mu_n}{e^{ni\theta}}\right)^2 + \left(\frac{z-\mu_n}{e^{ni\theta}}\right)^3 + ...\right)$$

$$= \frac{e^{-ni\theta}}{1 - \frac{z-\mu_n}{e^{ni\theta}}} = \frac{1}{1-z}$$

These examples demonstrate that Equation (4.41) can be used consistently to give the analytic continuations of power series; although in cases where the sums cannot be explicitly identified by closed-form expressions there is a problem of sensitivity to the precision of the initial conditions and the subsequent computations. At each stage, we need to evaluate infinite series and the higher order coefficients tend to require more and more terms before they converge, and there are infinitely many coefficients to evaluate. Smaller incremental steps require fewer terms for convergence of each sum, but they also require more transformations to reach any given point, and this necessitates carrying a larger number of coefficients. So, in practice, the pure numerical transformation of series using Equation (4.41) often leads to difficulties. Also note that many power series possess a 'Natural Boundary', i.e., the region of convergence is enclosed by a continuous locus of points at which the function is singular or not well-behaved in some other sense (e.g., not differentiable), and this prevents analytic continuation of the series. Nevertheless, it is interesting that an analytic function can, at least formally, be represented by a field of infinite-dimensional complex vectors, and that the process of analytic continuation can be represented by non-associative matrix multiplication. The failure of associativity is because the convergence of the conditionally convergent series depends on the order in which we add the terms and this depends on the order in which the matrix multiplications are performed.

Incidentally, in each when analytically continuing the geometric series $f(z) = 1 + z + z^2 + z^3 + \ldots$ by the procedure described above, we could have noted that the transformed functions centered on the point z_0 are expressible in the form,

NOTES

$$f(z) = f(z_0)\, f\left(\frac{z - z_0}{1 - z_0}\right)$$

This is a simple functional equation and can be applied recursively to give the analytic continuation of the function to all points on the complex plane except for the pole at $z = 1$. For any z we can choose a value of z_0 that is close enough to z so that the absolute value of $(z - z_0)/(1 - z_0)$ is less than 1 and hence the function f of that value converges. Of course, to apply the above equation we must also be able to evaluate $f(z_0)$, even if the magnitude of z_0 exceeds 1, but we can do this by applying the formula again. For example, if we wish to evaluate $f(3i)$ we could use the power series centered on $z_1 = 7i/4$, which requires us to evaluate $f(7i/4)$, and this can be done using the power series centered on $z_0 = 3i/4$. Thus, we can write

$$f(z) = f\left(\frac{z_0 - 0}{1 - 0}\right) f\left(\frac{z_1 - z_0}{1 - z_0}\right) f\left(\frac{z - z_1}{1 - z_1}\right)$$

The argument of each of the right hand side functions has magnitude less than 1, so they can each be evaluated using the original geometric series to give $f(3i) = 0.1 + 0.3i$, which naturally agrees with the value $1/(1-3i)$. In general, to evaluate $f(z_n)$ for any arbitrary value of z_n, we could split up a path from the origin to z_n into n small increments Δz and then multiply together the values of $f(\Delta z/(1-z))$ to give the overall result. If we take the natural log of both sides, the expression could be written in the form,

$$\ln\left[f(z_n)\right] = \ln\left[f\left(\frac{\Delta z_0}{1 - z_0}\right)\right] + \ln\left[f\left(\frac{\Delta z_1}{1 - z_1}\right)\right] + \ldots + \ln\left[f\left(\frac{\Delta z_{n-1}}{1 - z_{n-1}}\right)\right]$$

In the limit as the increments become arbitrarily small we can replace Δz with dz and integrate the right hand side. In this limiting case only the first-order term of the geometric series is significant, so we have

$$f\left(\frac{dz}{1 - z}\right) \to 1 + \frac{dz}{1 - z} \qquad \ln\left[1 + \frac{dz}{1 - z}\right] \to \frac{dz}{1 - z}$$

Therefore, the integral of the right hand side reduces to,

$$\ln\left[f(z_n)\right] = \int_0^{z_n} \frac{dz}{1 - z} = \ln\left(\frac{1}{1 - z_n}\right)$$

from which it follows that,

$$f(z_n) = \frac{1}{1 - z_n}$$

Another important aspect of analytic continuation is the fact that the continuation of a given power series to some point outside the original region of convergence can lead to different values depending on the path taken. This phenomenon did not arise in our previous examples, because the analytic function $1/(1-z)$ is single-valued over the entire complex plain, but some functions are found to be multi-valued when analytically continued. To illustrate, consider the power series

$$f(z)=(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\frac{1}{4}(z-1)^4+\ldots$$

which of course equals ln(z) within the region of convergence. This series is centered about the point $z = 1$ and at $z = 0$ it yields the negative of the harmonic series, which diverges. So the function is singular at $z = 0$. Now suppose we analytically continue this power series to a sequence of power series centered on points on the unit circle around the origin, i.e., the sequence of points $e^{i\theta}$, $e^{2i\theta}$, $e^{3i\theta}$, … for some constant angle θ. Noting that the nth derivative of ln(z) is,

$$\frac{d^n}{dz^n}\ln(z)=(-1)^{n-1}\frac{(n-1)!}{z^n}$$

We see that the power series centered on the point $e^{i\theta}$ is given by the Taylor series expansion,

$$\begin{aligned}f_1(z) &= f\left(e^{i\theta}\right)+\frac{1}{1!}f'\left(e^{i\theta}\right)\left(z-e^{i\theta}\right)+\frac{1}{2!}f''\left(e^{i\theta}\right)\left(z-e^{i\theta}\right)^2+\frac{1}{3!}f'''\left(e^{i\theta}\right)\left(z-e^{i\theta}\right)^3+\ldots\\ &= i\theta+e^{-i\theta}\left(z-e^{i\theta}\right)-\frac{e^{-2i\theta}}{2}\left(z-e^{i\theta}\right)^2+\frac{e^{-3i\theta}}{3}\left(z-e^{i\theta}\right)^3-\ldots\\ &= i\theta+\left(\frac{z}{e^{i\theta}}-1\right)-\frac{1}{2}\left(\frac{z}{e^{i\theta}}-1\right)^2+\frac{1}{3}\left(\frac{z}{e^{i\theta}}-1\right)^3-\ldots\\ &= i\theta+f\left(\frac{z}{e^{i\theta}}\right)\end{aligned}$$

Repeating this calculation for each successive point, we find

$$f_n(z)=ni\theta+f\left(\frac{z}{e^{ni\theta}}\right)$$

This converges provided $|z-e^{ni\theta}|<1$. For $n\theta = 2\pi k$ we have $f_n(z) = (2\pi i)k + f(z)$. So each time we circle the singularity at the origin, the value of the function increases by $2\pi i$. This is consistent with the fact that the natural log function (i.e., the inverse of the exponential function) of any given complex number has infinitely many values, separated by $2\pi i$. Notice that, in this case, our functional equation is

$$f(z)=f(z_0)+f\left(\frac{z}{z_0}\right)$$

which can be used in a way analogous to how the functional equation for the geometric series was used to analytically continue the power series to all non-singular points. In this regard, it is interesting to recall that the matrix formulation given by Equation (4.41) is entirely generic, and applies to all power series, represented as infinite dimensional vectors. So whether or not a certain power series continues to a single-valued function like the geometric series, a multi-valued function like the series for the natural log, or cannot be continued at all, depends entirely on the initial vector.

4.3.2 Schwarz Reflection Principle

Theorem 4.6 (Schwarz Principle): Given a piecewise continuous function $U(\theta)$ on $0 \le \theta \le 2\pi$, the Poisson integral,

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{e^{i\theta}+z}{e^{i\theta}-z}U(\theta)\,d\theta$$

is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$ provided U is continuous at θ_0.

Proof: The function, $f(x) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\zeta+z}{\zeta-z}U(\zeta)\frac{d\zeta}{\zeta}$ is analytic on $|z| < 1$ and hence $P_U(z)$ is harmonic for $|z| < 1$. P is a linear operator which maps piecewise continuous functions U on the unit circle to harmonic functions P_U on the open unit disc. It satisfies $P_{U_1+U_2} = P_{U_1} + P_{U_2}$ and $P_{cU} = cP_U$ for constant c. Now, $P_c = c$ and $m < U < M$ implies $m \le P_U \le M$. Let $U(\theta_0) = 0$; otherwise we can take $U - U(\theta_0)$ instead. Now, take a short arc $C_2 \subset \partial D$ containing θ_0 in its interior, such that $|U(\theta)| < \varepsilon$ on C_2. Let C_1 be its complement in ∂D. Let U_i, $i = 1, 2$, equal U on C_i and zero elsewhere. Then $U = U_1 + U_2$. P_{U_1} is harmonic away from C_1 or we can say that it is harmonic in a neighbourhood of $e^{i\theta_0}$.

On the other hand, P_{U_2} satisfies $|P_{U_2}| < \varepsilon$ since $|U_2| < \varepsilon$. Adding up P_{U_1} and P_{U_2} we see that $|P_U|$ can be made arbitrarily small in a neighbourhood of $e^{i\theta_0}$.

Therefore, there is a 1-1 correspondence between continuous functions on the unit circle and continuous functions on the closed unit disk that are harmonic on the open unit disk. The correspondence is given by $U \mapsto P_U$, and its inverse map is just the restriction $P_U|_{\partial D}$.

Goal of Schwarz Reflection Principle: The goal of the Schwarz reflection principle is to extend or continue an analytic function $f : \Omega \to C$ to a larger domain. The ultimate goal is to find the maximal domain on which f can be defined.

Observation: If $f(z)$ is analytic on Ω, then $\overline{f(\bar{z})}$ is analytic on $\Omega' = \{\bar{z} \mid z \in \Omega\}$. If $f(z)$ is an analytic function, defined on a region Ω which is symmetric about the x-axis, and $f(z) = \overline{f(\bar{z})}$, then $f(z)$ is real on the x-axis. We have the following converse:

Theorem 4.7: Let Ω be a symmetric region about the x-axis and let $\Omega^+ = \Omega \cap \{Imz > 0\}$, $\sigma = \Omega \cap \{Im\, z = 0\}$. If $f(z)$ is continuous on $\Omega^+ \cup \sigma$, analytic on Ω^+ and real for all $z \in \sigma$, then $f(z)$ has an analytic extension to all of Ω such that $f(z) = \overline{f(\bar{z})}$.

Proof: Given $f(z) = u(z) + iv(z)$ on Ω^+, extend $f(z)$ to Ω^- by defining $f(z) = \overline{f(\bar{z})} = u(\bar{z}) - iv(\bar{z})$ for $z \in \Omega^-$. From above, $v(z)$ is extended to a harmonic function $V(z)$ on all of Ω as above. Since $-u(z)$ is the harmonic conjugate of $v(z)$ on Ω^+, we define $U(z)$ to be a harmonic conjugate of $-V(z)$ (at least in a neighborhood $D_\delta(z_0)$ of $z_0 \in \sigma$). Adjust $U(z)$ (by adding a constant) so that $U(z) = u(z)$ on the upper half disc.

We prove that, $g(z) = U(z) - U(\bar{z}) = 0$ on $D_\delta(z_0)$. Indeed, $U(z) = U(\bar{z})$ on σ, so $\frac{\partial g}{\partial x} = 0$ on σ. Also, $\frac{\partial g}{\partial y} = 2\frac{\partial u}{\partial y} = -2\frac{\partial v}{\partial x} = 0$ on σ. Therefore, the analytic function $\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}$ vanishes on the real axis, and hence is constant. Since $g(z) = 0$ on σ, $g(z)$ is identically zero. This implies that $U(z) - U(\bar{z})$ on all of $D_\delta(z_0)$, hence proving the theorem.

Theorem 4.8: Suppose $v(z)$ is continuous on $\Omega^+ \cup \sigma$, harmonic on Ω^+ and zero on σ. Then v has a harmonic extension to Ω satisfying $v(z) = -v(\bar{z})$.

Proof: Define $V(z)$ to be $v(z)$ for $z \in \Omega^+$, 0 for $z \in \sigma$, and $-v(\bar{z})$ for $z \in \Omega^- = \Omega \cap \{\operatorname{Im} z < 0\}$. We want to prove that $V(z)$ is harmonic. For each $z_0 \in \sigma$, take an open disc $D_\delta(z_0) \subset \Omega$.

Now, we define P_V to be the Poisson integral of V with respect to the boundary $\delta D_\delta(z_0)$. P_V is harmonic on $D_\delta(z_0)$ and continuous on $\bar{D}_\delta(z_0)$. Now, we will show that $V = P_V$.

On the upper half disc, V and P_V are both harmonic, so $V - P_V$ is harmonic. $V - P_V = 0$ on the upper semicircle, since $V(z) = v(z)$ by definition and $P_V(z) = v(z)$ by the continuity of P_V (here z is on the semicircle). Also, $V - P_V = 0$ on $\sigma \cap D_\delta(z_0)$, since $v(z) = 0$ by definition and $P_V(z) = \frac{1}{2\pi}\int_{|\zeta|=\delta} \frac{\delta^2 - |z|^2}{|\zeta - z|^2} V(\zeta)\, d\theta$, and we note that the contributions from the upper semicircle cancel those from the lower semicircle.

Summarizing, $V - P_V$ is harmonic on the upper half disc $D_\delta(z_0) \cap \Omega^+$, continuous on its closure and zero on its boundary. Therefore, $V = P_V$ on the upper half disc.

4.3.3 Monodromy Theorem and its Consequences

In complex analysis, the monodromy theorem is an important result about analytic continuation of a complex-analytic function to a larger set. The idea is that one can extend a complex-analytic function (from here on called simply analytic function) along curves starting in the original domain of the function and ending in the larger set. A potential problem of this analytic continuation along a curve strategy is there are usually many curves, which end up at the same point in the larger set. The monodromy theorem gives sufficient conditions for analytic continuation to give the same value at a given point regardless of the curve used to get there, so that the resulting extended analytic function is well defined and single-valued.

Before stating this theorem, it is necessary to define analytic continuation along a curve and study its properties.

Analytic Continuation Along a Curve

The definition of analytic continuation along a curve is a bit technical, but the basic idea is that one starts with an analytic function defined around a point, and one extends that function along a curve via analytic functions defined on small overlapping discs covering that curve.

Formally, consider a curve (a continuous function) $\gamma:[0,1]\to\mathbb{C}$. Let f be an analytic function defined on an open disc U centered at γ (0). An analytic continuation of the pair (f, U) along γ is a collection of pairs (f_t, U_t) for $0\le t\le 1$ such that,

- $f_0 = f$ and $U_0 = U$.
- For each $t \in [0, 1]$, U_t is an open disc centered at $\gamma(t)$ and $f_t : U_t \to \mathbb{C}$ is an analytic function.
- For each $t \in [0, 1]$ there exists $\varepsilon > 0$ such that for all $t' \in [0, 1]$ with $|t-t'|<\varepsilon, \gamma(t')\in U_t$ (which implies that U_t and $U_{t'}$ have a non-empty intersection) and the functions f_t and $f_{t'}$ coincide on the intersection $U_t \cap U_{t'}$.

Properties of Analytic Continuation along a Curve

Analytic continuation along a curve is essentially unique in the sense that given two analytic continuations (f_t, U_t) and (g_t, V_t) $(0 \le t \le 1)$ of (f, U) along γ, the functions f_1 and g_1 coincide on $U_1 \cap V_1$. Informally, this says that any two analytic continuations of (f, U) along γ will end up with the same values in a neighborhood of γ (1).

If the curve γ is closed, i.e., $\gamma(0) = \gamma(1)$ one need not have f_0 equal f_1 in a neighborhood of $\gamma(0)$. For example, we start at a point $(a, 0)$ with $a > 0$ and the complex logarithm defined in a neighborhood of this point, and let γ be the circle of radius a centered at the origin travelled counterclockwise from $(a, 0)$, then by doing an analytic continuation along this curve we will end up with a value of the logarithm at $(a, 0)$ which is $2\pi i$ plus the original value.

Theorem Statement

Two analytic continuations along the same curve yield the same result at the curve's endpoint. However, given two different curves branching out from the same point around which an analytic function is defined, with the curves reconnecting at the end, it is not true in general that the analytic continuations of that function along the two curves will yield the same value at their common endpoint.

Consider the complex logarithm defined in a neighborhood of a point $(a, 0)$, and the circle centered at the origin and radius a. Then, it is possible to travel from $(a, 0)$ to $(-a, 0)$ in two ways, counterclockwise, on the upper half-plane arc of this circle and clockwise, on the lower half-plane arc. The values of the logarithm at $(-a, 0)$ obtained by analytic continuation along these two arcs will differ by $2\pi i$.

If, however, one can continuously deform one of the curves into another while keeping the starting points and ending points fixed, and analytic continuation is possible on each of the intermediate curves, then the analytic continuations along the two curves will yield the same results at their common endpoint. This is called the monodromy theorem.

Let U be an open disc in the complex plane centered at a point P and $f : U \to \mathbb{C}$ be a complex-analytic function. Let Q be another point in the complex plane. If there exists a family of curves $\gamma_s : [0,1] \to \mathbb{C}$ with $s \in [0,1]$ such that $\gamma_s(0) = P$ and $\gamma_s(1) = Q$ for all $s \in [0,1]$ the function $(s,t) \in [0,1] \times [0,1] \to \gamma_s(t) \in \mathbb{C}$ is continuous, and for each $s \in [0,1]$ it is possible to do an analytic continuation of f along γ_s, then the analytic continuations of f along γ_0 and γ_1 will yield the same values at Q.

The monodromy theorem makes it possible to extend an analytic function to a larger set via curves connecting a point in the original domain of the function to points in the larger set. The theorem below is called the monodromy theorem.

Let U be an open disc in the complex plane centered at a point P and f: $U \to C$ be a complex-analytic function. If W is an open simply-connected set containing U, and it is possible to perform an analytic continuation of f on any curve contained in W which starts at P, then f admits a direct analytic continuation to W, meaning that there exists a complex-analytic function g: $W \to C$ whose restriction to U is f.

4.3.4 Harmonic Functions on a Unit Disk

In mathematics, a positive harmonic function on the unit disc in the complex numbers is characterized as the Poisson integral of a finite positive measure on the circle. The result of the Herglotz-Riesz representation theorem was proved independently by Gustav Herglotz and Frigyes Riesz in 1911. It can be used to give a related formula and characterization for any holomorphic function on the unit disc with positive real part.

Such functions had already been characterized in 1907 by Constantin Carathéodory in terms of the positive definiteness of their Taylor coefficients.

Herglotz-Riesz Representation Theorem for Harmonic Functions

A positive function f on the unit disk with $f(0) = 1$ is harmonic if and only if there is a probability measure μ on the unit circle, such that,

$$f(re^{i\theta}) = \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2}\, d\mu(\varphi)$$

The formula clearly defines a positive harmonic function with $f(0) = 1$.

Conversely if f is positive and harmonic and r_n increases to 1, we can define,

$$f_n(z) = f(r_n z).$$

Then,

$$f_n(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} f_n(\varphi)\, d\phi$$

$$= \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} d\mu_n(\varphi)$$

Where,

$d\mu_n(\varphi) = \frac{1}{2\pi} f(r_n e^{i\varphi})\, d\varphi$ is a probability measure.

By a compactness argument, a subsequence of these probability measures has a weak limit which is also a probability measure μ.

Since r_n increases to 1, so that $f_n(z)$ tends to $f(z)$, the Herglotz formula follows.

4.4 HARNACK'S INEQUALITY THEOREM

Theorem 4.9: Let u be a harmonic function on an open neighborhood of the compact disc $U_R(a)$. Assume that $u(z) \geq 0$ for $|z-a| \leq R$. For any number r such that, $0 < r < R$ and for all z such that $|z-a| = r$, the Harnack inequality

$$\frac{R-r}{R+r}u(a) \leq u(z) \leq \frac{R-r}{R+r}u(a) \text{ holds.}$$

The proof follows from the Poisson formula applied to $U(z) = u(a+Rz)$, together with the trivial inequality,

$$\frac{|w|-|z|}{|w|+|z|} \leq K(w,z) \leq \frac{|w|+|z|}{|w|-|z|} \qquad (|z|<|w|)$$

Corollary 1: Let u be a harmonic function on an open neighborhood of the compact disc $U_R(a)$ such that

(a) $u(a) = 0$;

(b) $m \leq u(z) \leq M$ for $z \in U_R(a)$ (where m, M are real constants).

Then,

$$m\frac{2r}{R+r} \leq u(z) \leq M\frac{2r}{R+r} \text{ for } |z-a| - r < R$$

Proof: We apply Harnack's inequality to the functions $u(z) - m$ and $M - u(z)$. Because of the maximum principle, it is enough to know that the inequality (b) holds on the boundary of the disc. The most important application of Harnack's inequality is the following statement.

Harnack's Principle

Theorem 4.10: Let U_n be a monotonically increasing sequence of harmonic functions,

$$u_n : D \to R, D \subset \mathbb{C} \text{ open},$$
$$u_1(z) \le u_2(z) \le \ldots \text{for } z \in D .$$

The set of all points $z \in D$ for which the sequence $(U_n(z))$ remains bounded is open and closed in D.

Corollary 2: Let D be a (connected) domain. When the sequence $(U_n\,(z_0))$ converges for some $z_0 \in D$, then it converges for all $z \in D$ and the convergence is locally uniform. In particular, the limit function is harmonic.

Proof: Since $u_n(z)$ can be replaced by $u_n(z) - u_1(z)$, we may assume that

$$u_n(z) \ge 0 \text{ for all } z \in D$$

Now let $a \in D$ be a point such that $(u_n(a))$ is bounded, i.e.,

$$u_n(a) \le C$$

It follows from Harnack's inequality that

$$u_n(z) \le C\frac{R+r}{R-r} \qquad (r = |z-a|)$$

for all z in a full neighborhood of a. Hence, the set of all $z \in D$ on which (u_n) remains bounded is open in D. By means of an estimation of $u\,(z)$ from below, it can be shown analogously that the set of all points $z \in D$ on which (u_n) is unbounded is open as well.

It remains to prove the locally uniform convergence as stated in the corollary. Again this follows from Harnack's inequality, applied to the functions.

$$u_m(z) - u_n(z), \quad m \ge n$$

Namely, let $a \in$ D be a given point; it then follows from Harnack's inequality that there exists a neighborhood U (say $U = U_{\frac{1}{2}R}(a)$) such that each $\varepsilon > 0$ admits an $N \in \mathbb{N}$ with

$$0 \le u_m(z) - u_n(z) \le \varepsilon \text{ for } m \ge n \ge N \text{ and } z \in U.$$

Hence the sequence (u_n) is a locally convergent Cauchy sequence.

Theorem 4.11 (Harnack's Inequality): If $u\,(z)$ is a positive harmonic function on the open unit disc D, then

$$\frac{1-r}{1+r}u(0) \le u(re^{i\theta}) \le \frac{1+r}{1-r}u(0), \qquad re^{i\theta} \in D$$

NOTES

Proof:

$$\frac{1-r}{1+r} = \frac{1-r^2}{1+r^2+2r} \le P_r(\theta)\frac{1-r^2}{1+r^2-2r} = \frac{1+r}{1-r} \qquad \textbf{...} (4.43)$$

We can always approximate $u(z)$ by a dilate $u(\rho z)$, $\rho < 1$, and assume that $u(z)$ extends harmonically across the unit circle. Then $u(z)$ is represented by the Poisson integral formula.

Substituting the estimates for $p_r(\theta)$ into Poisson integral formula and using the positivity of $u(z)$, we obtain

$$\frac{1-r}{1+r}\int_0^{2\pi} u\left(e^{i(\theta-\varphi)}\right)\frac{d\varphi}{2\pi} \le \int_0^{2\pi} u\left(e^{i(\theta-\varphi)}\right)P_r(\varphi)\frac{d\varphi}{2\pi} \le \frac{1+r}{1-r}\int_0^{2\pi} u\left(e^{i(\theta-\varphi)}\right)\frac{d\varphi}{2\pi}$$

Since, the average value of $u\left(e^{i(\theta-\varphi)}\right)$ is $u(0)$, this becomes Equation (4.43).

It becomes an equality for the function,

$$u(z) = Re((1+z)/(1-z)) \text{ at } z = \pm r.$$

If we scale and translate Harnack's inequality to an arbitrary disc, of radius $R > 0$ and center z_0, we obtain

$$\frac{R-r}{R+r}u(z_0) \le u(z) \le \frac{R+r}{R-r}u(z_0), \; |z-z_0| \le r, r < R \qquad \ldots(4.43)$$

for all positive harmonic functions $u(z)$ on the disk $|z-z_0| < R$.

The Equation (4.44) is also referred to as Harnack's inequality. From the left hand inequality, it follows that if $\{u_n(z)\}$ is a sequence of positive harmonic functions on the disc $\{|z-z_0| < R\}$ such that $u_n(z_0) \to +\infty$, then $u_n(z) \to +\infty$ on any subdisc $\{|z-z_0| \le \rho\}$, $\rho < R$. From the right hand inequality it follows that if $\{u_n(z_0)\}$ is bounded then $u_n(z)$ is uniformly bounded on any subdisc $\{|z-z_0| \le \rho\}$,
$\rho < R$.

Lemma 1: Suppose $\{u_n(z)\}$ is a sequence of positive harmonic functions on a domain D. If there is $z_0 \in D$ such that $u_n(z_0) \to +\infty$, then $u_n(z) \to +\infty$ uniformly on each compact subset of D. If there is $z_0 \in D$ such that $\{u_n(z_0)\}$ is bounded, then $\{u_n(z)\}$ is uniformly bounded on each compact subset of D.

To see this, let U be the set of $z \in D$ such that $u_n(z) \to +\infty$. The remarks preceding the lemma show that both U and $D\backslash U$ are open. Since D is a domain, either $U = D$ or U is empty. In the case $U = D$, we can cover any compact set by a finite number of discs on which $u_n(z) \to +\infty$ uniformly, and we see that $u_n(z) \to +\infty$ uniformly on compact subsets of D. If U is empty, we cover any compact set by a finite number of discs, and we see that $\{u_n(z)\}$ is uniformly bounded on each compact subset of D.

If we combine the preceding lemma and the compactness theorem for families of harmonic functions, we obtain immediately the following compactness theorem for families of positive harmonic functions.

Theorem 4.12: Let F be a family of positive harmonic functions on a domain D. Every sequence in F has a subsequence that either converges uniformly on compact subsets of D to a harmonic function or converges uniformly on compact subsets of D to $+\infty$.

The compactness theorem for positive harmonic functions can be applied to monotone sequences of harmonic functions.

Theorem 4.13: Let $\{u_n(z)\}$ be an increasing sequence of harmonic functions on a domain D. Then $\{u_n(z)\}$ converges uniformly on compact subsets of D, either to a harmonic function or to $+\infty$.

Consider the functions $v_n(z) = u_n(z) - u_1(z)$. For each fixed z the sequence $\{v_n(z)\}$ is increasing, hence convergent either to a finite value or to $+\infty$. The preceding theorem shows that $\{v_n(z)\}$ has a subsequence that converges uniformly on compact subsets of D, either to a harmonic function or to $+\infty$. It follows that $\{u_n(z)\}$ also converges uniformly on compact subsets of D.

4.4.1 Dirichlet Problem

Assume that we have a 2π periodic function f and a summation of trigonometric functions of varying frequencies that converges to it. Thus we have,

$$f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) = b_k \cos(k\theta)$$

But, by Euler's identity we have,

$$e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$$

So, with the correct complex, C_kS, we have,

$$f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta) = \sum_{k=-\infty}^{\infty} C_k e^{ik\theta}$$

Assuming that this sum converges uniformly, we now must determine the actual values for the C_kS. Consider the functions which are periodic in [0, 1] which prevents us from dividing out constants later. So, solving for the $C_k S$ in $\sum_{k=-\infty}^{\infty} C_k e^{2\pi ikx}$, we have

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ikx}$$

Multiplying each side by $e^{-2\pi imx}$ and integrating, we get

$$\int_0^1 f(x)e^{-2\pi imx}dx = \sum_{k=-\infty}^{\infty} c_k \int_0^1 e^{2\pi ikx}e^{-2\pi imx}dx$$

Then, we note that the $e^{2\pi ikx}$ are orthogonal with respect to the inner product, $\langle g,h\rangle = \int_0^1 g(x)\overline{h(x)}dx$. It can easily be verified that,

$$\int_0^1 e^{2\pi ikx}e^{-2\pi imx}dx = \begin{cases} 0 & (k \neq m) \\ 1 & (k = m) \end{cases}$$

Thus,

$$C_m = \int_0^1 f(x)e^{-2\pi imx}dx$$

Now, let

$$\hat{f}(k) = \int_0^1 f(x)e^{2\pi ikx}dx$$

be the kth Fourier coefficient of f. Then the Fourier series is defined as,

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$$

Dirichlet's Problem on the Disc

Given a connected open set Ω and a function f defined on the boundary of $\Omega, \partial\Omega$ the solution to the Dirichlet's problem is a function, u, such that

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = f & x \in \partial\Omega \end{cases}$$

Here, for the open disc, we will find a function that is harmonic on the interior of the disc and periodic on the circle. First we will show that the solution will be unique, and then we will explicitly solve the Dirichlet problem on the disc.

Theorem 4.14: Let Ω be any bounded domain and let $u(x, y)$ in $C^0(\Omega) \cap C^2(\Omega)$ be harmonic in Ω. Then u attains its maximum value on Ω somewhere on $\partial\Omega$.

Corollary: Let Ω be any bounded domain and let $u(x, y)$ in $C^0(\Omega) \cap C^2(\Omega)$ be harmonic in Ω and let $u(x,y) = 0\ \forall (x,y) \in \partial\Omega$. Then, $u(x,y) = 0\ \forall (x,y) \in \Omega$.

Proof: By the theorem, $u(x,y) \leq 0\ \forall (x,y) \in \Omega$. Then, since $-u(x,y) = 0\ \forall (x,y) \in \partial\Omega$, we also have that $-u(x,y) \leq 0\ \forall (x,y) \in \Omega$. Thus, $u(x,y) = 0\ \forall (x,y) \in \Omega$.

Now we will prove the uniqueness of the Dirichlet problem.

Theorem 4.15: Suppose we have two functions u and v such that,

$$\begin{cases} \Delta u = \Delta v = 0 & x \in \Omega \\ u = v = f & x \in \partial\Omega \end{cases}$$

Then,

$$u(x) = v(x) \forall x \in \Omega$$

Proof: Consider $w(x)=u(x)-v(x)$. The function w is harmonic in Ω, since $\Delta w = \Delta u - \Delta v = 0 - 0 = 0$, and $w(x)=u(x)-v(x)=f(x)-f(x)=0 \ \forall x \in \partial\Omega$. So, by the Corollary, $w(x)=0 \ \forall x \in \Omega$. Thus, $u(x) = v(x) \forall x \in \Omega$.

Solution of Unit Disc

When solving the Dirichlet problem on the unit disc, we first observe that we are looking for a harmonic function that approximates the function, f, from the interior from the disc. Noting that f is periodic, it would be sufficient to find a harmonic function that is equivalent to the Fourier series of f on the boundary of the disc. First, we note that the Fourier series of f is equivalent to the limit as $r \to 1^-$ of,

$$u(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=-\infty}^{-1} \hat{f}(k) z^{|k|} \text{ where } z = re^{2\pi i\theta}$$

In order for this expression to be useful in this situation, we need to know that this function is harmonic. For this, we will use the fact that every holomorphic function is harmonic. Thus, since u is the sum of a holomorphic function, $\sum_{k=0}^{\infty} \hat{f}(k) z^k$ and an antiholomorphic function, $\sum_{k=-\infty}^{-1} \hat{f}(k) z^{|k|}$ in the unit disc, we know that u is harmonic.

Combine the two sums in the definition of u to get,

$$u\left(re^{2\pi i\theta}\right) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi ik\theta}$$

If this converges nicely enough, then

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi ik\theta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ik(\theta - t)} dx$$

Now, we will find a closed form for this convolution operator,

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ik(t)} = \sum_{k=0}^{\infty} r^{|k|} e^{2\pi ik(t)} + \sum_{k=1}^{\infty} r^{|k|} e^{-2\pi ik(t)}$$

Since each of the above terms is a geometric series,

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ik(t)} = \frac{1}{1 - re^{2\pi it}} + \frac{re^{-2\pi it}}{1 - re^{-2\pi it}} = \frac{1 - re^{-2\pi it} + re^{-2\pi it} - \left(re^{-2\pi it}\right)\left(re^{2\pi it}\right)}{1 - \left(re^{2\pi it} + re^{-2\pi it}\right) + \left(re^{2\pi it}\right)\left(re^{-2\pi it}\right)}$$

So, we finally have

$$\sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ik(t)} = \frac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2}$$

This convolution operator is known as the Poisson kernel for the unit disc and is denoted by,

NOTES

$$P_r(t) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$$

Thus, the solution is

$$u\left(re^{2\pi i\theta}\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) P_r(\theta - t)\, dt$$

4.4.2 Green's Function

Consider a linear differential equation written in the general form,

$$L(x)u(x) = f(x) \qquad \ldots (4.45)$$

where $L(x)$ is a linear, self-adjoint differential operator, $u(x)$ is the unknown function and $f(x)$ is a known non-homogeneous term. Operationally, we can write a solution to Equation (4.5) as,

$$u(x) = L^{-1}(x) f(x) \qquad \ldots (4.46)$$

where L^{-1} is the inverse of the differential operator L. Since L is a differential operator, it is reasonable to expect its inverse to be an integral operator. We expect the usual properties of inverses to hold,

$$LL^{-1} = L^{-1}L = I \qquad \ldots (4.47)$$

where I is the identity operator. More specifically, we define the inverse operator as,

$$L^{-1} f = \int G(x; x') f(x')\, dx' \qquad \ldots (4.48)$$

where the kernel $G(x; x')$ is the Green's function associated with the differential operator L. Note that $G(x; x')$ is a two-point function which depends on x and x'. To complete the idea of the inverse operator L, we introduce the Dirac delta function as the identity operator I. The properties of the Dirac delta function are,

$$\int_{-\infty}^{\infty} \delta(x - x') f(x')\, dx' = f(x)$$

$$\int_{-\infty}^{\infty} \delta(x')\, dx' = 1 \qquad \ldots (4.49)$$

The Green's function $G(x; x')$ then satisfies,

$$L(x) G(x; x') = \delta(x - x') \qquad \ldots (4.50)$$

The solution to Equation (4.45) can then be written directly in terms of the Green's function as,

$$u(x) = \int_{-\infty}^{\infty} G(x; x') f(x')\, dx' \qquad \ldots (4.51)$$

To prove that Equation (4.51) is indeed a solution to Equation (4.45), simply substitute as follows:

$$\begin{aligned} Lu(x) &= L\int_{-\infty}^{\infty} G(x;x')f(x')dx' \\ &= \int_{-\infty}^{\infty} LG(x;x')f(x')dx' \\ &= \int_{-\infty}^{\infty} \delta(x-x')f(x')dx' \\ &= f(x) \end{aligned} \quad \dots (4.53)$$

Note that we have used the linearity of the differential and inverse operators in addition to Equations (4.48), (4.49) and (4.50) to arrive at the final answer. The Green's function can be interpreted physically for a variety of differential operators encountered in mathematical physics. For example, consider the two-dimensional Laplace's equation,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

The Green's functions for this particular differential operator is known to be,

$$G(x;x') = -\frac{1}{2\pi}\ln r$$

where,

$$r = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}$$

In basic physics, the Green's function gives the potential at the point x due to a point charge at the point x' (the source point) and only depends on the distance between the source and field points. In electrostatics, the Green's function represents the displacement in the solid due to the application of a unit force. In heat transfer, the Green's function represents the temperature at the field point due to a unit heat source applied at the source point.

Free-Space and Region Dependent Green's Functions

In the discussion above concerning the solution of a differential equation with a Green's function, no mention was made of boundary conditions for the problem. This is true when we are seeking a particular solution to Equation (4.50),

$$L(x)G(x;x') = \delta(x-x')$$

NOTES

The particular solution is independent of any boundary conditions for the problem. However, we can always add homogeneous solutions to the Green's function,

$$G(x;x') = G_0(x;x') + G_R(x;x')$$

where,

$$L(x)G_R(x;x') = 0$$
$$L(x)G_0(x;x') = \delta(x-x')$$

Here, G_0, the particular solution, is termed the free-space Green's function and is also referred to as the fundamental solution for the differential operator $L(x)$. As we have seen with the example of Laplace's equation given in the previous section, the free-space Green's function is singular. The homogenous solution G_R is non-singular. Since G_R is a homogenous solution, it will contain constants, which can be evaluated to satisfy any boundary conditions for the problem. We term the full Green's functions $G(x; x')$ a region-dependent Green's function since, in general, it contains not only the particular solution, but also the necessary terms to satisfy any boundary conditions for the problem.

Green's Function for a Partial Differential Equation

Consider the Helmholtz equation in three dimensions,

$$(\Delta + k^2)u = 0$$

where Δ is the Laplace operator, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. In this case,

$$L(x)G(x;x') = -\delta(x-x')$$

and we seek the Green's function,

$$L(x)G(x;x') = -\delta(x-x').$$

Note that the three-dimensional Dirac delta function is simply a compact representation for the product of delta functions in each coordinate,

$$\delta(x-x') = \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3')$$

To obtain the free-space Green's function for this example problem, we will use a Fourier transform method. Since we will only be calculating the free-space component of the Green's function, we can use a single variable $r = x - x'$, as the free-space Green's function will only depend on the relative distance between the source and field points, and not on their absolute positions. The Fourier transform pair is,

$$\hat{u}(q) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} u(r)e^{-iqr}\,dr$$

$$u(r) = \int_{-\infty}^{\infty} \hat{u}(q) e^{iqr} dq$$

Applying the forward transform to the differential equation for the Green's function, we have

$$\left(q_1^2 + q_2^2 + q_3^2 - k^2\right)\hat{G}(q) = \frac{1}{(2\pi)^3}$$

Now, let $q^2 = q_1^2 + q_2^2 + q_3^2$. Then

$$\left(q^2 - k^2\right)\hat{G}(q) = \frac{1}{(2\pi)^3}$$

In transform space the Green's function is then,

$$\hat{G}(q) = \frac{1}{(2\pi)^3\left(q^2 - k^2\right)}$$

In physical space, the Green's function is then given through the inversion integral,

$$G(r) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{e^{iqr}}{\left(q^2 - k^2\right)} dq$$

The integral is an isotropic Fourier integral since it depends only on the magnitude of **q**, which is *q*. The general result for isotropic Fourier integrals in three dimensions is,

$$\int_{-\infty}^{\infty} f(q) e^{iqr} dq = \frac{4\pi}{R}\int_{0}^{\infty} qf(q)\sin(qR)\, dq$$

where R is the magnitude of ***r***. Utilizing this result, the inversion integral seek is then

$$G(r) = \frac{4\pi}{(2\pi)^3}\frac{1}{R}\int_{0}^{\infty} \frac{q}{\left(q^2 - k^2\right)}\sin(qR)\, dq$$

Since the integrand is even,

$$G(r) = \frac{4\pi}{(2\pi)^3}\frac{1}{2R}\int_{-\infty}^{\infty} \frac{q}{\left(q^2 - k^2\right)}\sin(qR)\, dq$$

This integral can be evaluated by contour integration. First, the sine term is written in terms of complex exponentials as,

$$\sin(qR) = \frac{e^{iqR} - e^{-iqR}}{2i}$$

and the integral is written as,

$$G(r) = \frac{4\pi}{(2\pi)^3}\frac{1}{4iR}\left\{\int_{-\infty}^{\infty}\frac{qe^{iqR}}{(q-k)(q+k)}dq - \int_{-\infty}^{\infty}\frac{qe^{-iqR}}{(q-k)(q+k)}dq\right\}$$

$$= \frac{4\pi}{(2\pi)^3}\frac{1}{4iR}\{I_1 - I_2\}$$

The first integral will be evaluated by considering a contour in the complex q plane. Since the denominator of the integrand has poles on the real axis, we introduce a small imaginary part to offset the poles from the real q axis,

$$I_1 = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{qe^{iqR}}{(q-[k+i\varepsilon])(q+[k+i\varepsilon])}dq$$

We next take a contour in the upper half-plane due to the behavior of the numerator of the integrand as q becomes large. Using the theory of integration by residues, we then have

$$I_1 = 2\pi i \sum_{\substack{\text{Res} \\ \text{Im}\, q>0}} \frac{qe^{iqR}}{(q+[k+i\varepsilon])(q-[k+i\varepsilon])}$$

$$= \pi i e^{i(k+i\varepsilon)R}$$

Taking the limit as ε tends to zero we then have, $I_1 = \pi i e^{ikR}$

Similarly, for I_2, we take a contour in the lower half plane and obtain,

$$I_2 = \pi i e^{ikR}$$

The Green's function is then,

$$G(r) = \frac{4\pi}{(2\pi)^3}\frac{1}{4iR}\left\{\pi i e^{ikR} + \pi i e^{ikR}\right\}$$

$$= \frac{1}{4\pi R}e^{ikR}$$

Check Your Progress

7. Define function element.
8. Define direct analytic continuation.
9. Define the analytic continuation of a curve.
10. Define the goal of Schwarz reflection principle.
11. When the Harnack's inequality hold?
12. When u attains its maximum value on Ω?
13. What is region-dependent Green's function?

4.5 ANSWERS TO 'CHECK YOUR PROGRESS'

1. According to Weierstrass factorization theorem in complex analysis, a product involving their zeroes can represent the entire functions. In addition, every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.

2. For the logarithmic derivative of the gamma function we consider that, $\Gamma(x) > 0$ for any $x >$ 0, we can take the logarithm of the above expression to get,

$$\ln(\Gamma(x)) = -Cx - \ln(x) + \sum_{n=1}^{+\infty}\left(\frac{x}{n} - \ln\left(1+\frac{x}{n}\right)\right)$$

3. The Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s = \sigma + it$ (here, s, σ and t are traditional notations associated to the study of the ζ-function).
   The Riemann zeta function is defined as the analytic continuation of the function defined for $\sigma > 1$ by the sum of the preceding series.
4. The functional equation implies that $\zeta(s)$ has a simple zero at each even negative integer $s = -2n$. These zeros are the trivial zeros of z(s). Riemann established the functional equation which is used to construct the analytic continuation in the first place.
5. Runge's theorem states that for any compact set $K \subset C$ we have $R(K) = A(K)$ and $P(K) = A(K)$ provided C–K is connected.
6. The Mittag-Leffler's theorem state the following:
   Suppose $b_k \in \Omega \to \partial\Omega$
   Set $S_k(z) = \sum_{j=1}^{n_k} \frac{c_{j,k}}{(z-b_k)^j}$
   where each n_k is a positive integer and $c_{j,k} \in \mathbb{C}$. Then there is a function meromorphic in Ω with singular parts S_k at b_k, $k = 1, 2, ...,$ and no other singular parts in Ω.
7. A function element is an ordered pair (f, U), where U is a disc D (P, r) and f is a holomorphic function defined on U. If W is an open set, then a function element in W is a pair (f, U) such that $U \subseteq W$.
8. Let (f, U) and (g, V) be function elements. We say that (g, V) is a direct analytic continuation of (f, U) if $U \cap V \neq \varphi$, and f and g are equal on $U \setminus V$. Obviously (g, V) is a direct analytic continuation of (f, U) if and only if (f, U) is a direct analytic continuation of (g, V).
9. Let $\gamma : [0,1] \to$ C be a curve and let (f, U) be a function element with $\Upsilon(0)$ the centre of the disc U. An analytic continuation of (f, U) along the curve is a collection of function elements (f_t, U_t), $t \in$ [0, 1], such that (f_0, U_0) = (f, U), for each $t \in$ [0, 1], the centre of the disc U_t is $\gamma(t)$, $0 \leq t \leq 1$ and for each $t \in$ [0, 1], there is an ε>0, such that for each $t'\varepsilon$ [0,1] with $|t' - t| < \varepsilon$, it holds that: $\gamma(t') \in U_t$ and hence $U_{t'} \cap U_t \neq \emptyset$ and $f_t \equiv f_{t'}$ on $U_{t'} \cap U_t$.
10. The goal of the Schwarz reflection principle is to extend or continue an analytic function $f : \Omega \to C$ to a larger domain. The ultimate goal is to find the maximal domain on which f can be defined.

11. Let u be a harmonic function on an open neighborhood of the compact disc $U_R(a)$. Assume that $u(z) \geq 0$ for $|z-a| \leq R$. For any number r such that, $0 < r < R$ and for all z such that $|z-a| = r$, the Harnack inequality $\frac{R-r}{R+r}u(a) \leq u(z) \leq \frac{R-r}{R+r}u(a)$ holds.
12. Let Ω be any bounded domain and let $u(x, y)$ in $C^0(\Omega) \cap C^2(\Omega)$ be harmonic in Ω. Then u attains its maximum value on Ω somewhere on $\partial\Omega$.
13. We term the full Green's functions $G(x; x')$ a region-dependent Green's function, since in general, it contains not only the particular solution, but also the necessary terms to satisfy any boundary conditions for the problem.

4.6 SUMMARY

- According to Weierstrass factorization theorem in complex analysis, a product involving their zeroes can represent the entire functions. In addition, every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.
- If $(f_n(z))$ is a sequence of analytic functions in a domain G and if there exists $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly in closed subdomains of G, then $f(z)$ is analytic and $f'(z) = \lim_{n\to\infty} f_n'(z)$.
- A lot of important functions in applied sciences are defined using improper integrals. One of the most famous among them is the gamma function.
- The Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s = \sigma + it$ (here, s, σ and t are traditional notations associated to the study of the ζ-function).
- The Riemann zeta function is defined as the analytic continuation of the function defined for $\sigma > 1$ by the sum of the preceding series.
- Riemann showed that the function defined by the series on the half-plane of convergence can be continued analytically to all complex values $s \neq 1$.
- The functional equation implies that $\zeta(s)$ has a simple zero at each even negative integer $s = -2n$.
- These zeros are the trivial zeros of $\zeta(s)$. Riemann established the functional equation which is used to construct the analytic continuation in the first place.
- An equivalent relationship was conjectured by Euler for the Dirichlet eta function or the alternating zeta function.
- A function element is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic function defined on U. If W is an open set, then a function

element in W is a pair (f, U) such that $U \subseteq W$.

- Let (f, U) and (g, V) be function elements. We say that (g, V) is a direct analytic continuation of (f, U) if $U \cap V \neq \varphi$, and f and g are equal on $U \setminus V$. Obviously (g, V) is a direct analytic continuation of (f, U) if and only if (f, U) is a direct analytic continuation of (g, V).
- A series of geometrically increasing numbers,

  $S_n = 1 + x + x^2 + x^3 + \ldots + x^n$
- Given a piecewise continuous function $U(\theta)$ on $0 \leq \theta \leq 2\pi$, the Poisson integral,

  $$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{e^{i\theta}+z}{e^{i\theta}-z}U(\theta)\,d\theta$$

  is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$ provided U is continuous at θ_0.
- The goal of the Schwarz reflection principle is to extend or continue an analytic function $f : \Omega \to C$ to a larger domain. The ultimate goal is to find the maximal domain on which f can be defined.
- In complex analysis, the monodromy theorem is an important result about analytic continuation of a complex-analytic function to a larger set.
- Given a connected open set Ω and a function f defined on the boundary of $\Omega, \partial\Omega$ the solution to the Dirichlet's problem is a function, u, such that

  $$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = f & x \in \partial\Omega \end{cases}$$
- Let Ω be any bounded domain and let $u(x, y)$ in $C^0(\Omega) \cap C^2(\Omega)$ be harmonic in Ω. Then u attains its maximum value on Ω somewhere on $\partial\Omega$.
- Let Ω be any bounded domain and let $u(x, y)$ in $C^0(\Omega) \cap C^2(\Omega)$ be harmonic in Ω and let $u(x,y) = 0 \,\forall (x,y) \in \partial\Omega$. Then, $u(x,y) = 0 \,\forall (x,y) \in \Omega$.
- We term the full Green's functions $G(x; x')$ a region-dependent Green's function since, in general, it contains not only the particular solution, but also the necessary terms to satisfy any boundary conditions for the problem.

4.7 KEY TERMS

- **Weierstrass factorization theorem:** According to Weierstrass factorization theorem, in complex analysis a product involving their zeroes can represent the entire functions. In addition, every sequence tending to infinity has an associated entire function with zeroes at precisely the points of that sequence.

- **Gamma function**: A lot of important functions in applied sciences are defined using improper integrals. One of the most famous among them is the gamma function.
- **Riemann zeta function**: The Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s = \sigma + it$ (here, s, σ and t are traditional notations associated to the study of the ζ-function).
- **Zeta function**: If $\Re e\, z \geq 1+\varepsilon$ where $\varepsilon > 0$ then

$$\sum_{k=m}^{n} | k^{-z} | = \sum_{k=m}^{n} | k^{-\Re e\, z} | \leq \sum_{k=m}^{n} k^{-1-\varepsilon}$$

- **Mittag-Leffler's theorem:** This concerns the existence of meromorphic functions with prescribed poles and asserts the existence of holomorphic functions with prescribed zeros.
- **Function elements:** A function element is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic function defined on U. If W is an open set, then a function element in W is a pair (f, U) such that $U \subseteq W$.
- **Harnack's inequality**: Let u be a harmonic function on an open neighborhood of the compact disc $U_R(a)$. Assume that $u(z) \geq 0$ for $|z-a| \leq R$. For any number r such that $0 < r < R$ and for all z such that $|z-a| = r$, the Harnack inequality

$$\frac{R-r}{R+r}u(a) \leq u(z) \leq \frac{R-r}{R+r}u(a) \text{ holds.}$$

4.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. When is Weierstrass' factorization theorem applied?
2. State a property of gamma function.
3. What is the significance of Riemann zeta function?
4. What is the alternating zeta function?
5. State Runge's theorem.
6. Write the statement of Mittag-Leffler's theorem.
7. Define the analytic continuation of a function.
8. What is power series?
9. What is the goal of Schwarz reflection principle?
10. State monodromy theorem.

11. What is Harnack's inequality?
12. State the Dirichlet's problem.
13. What is Green's function?

Long-Answer Questions

1. Briefly discuss the Weierstrass factorization theorem with the help of examples.
2. Prove that for $z \in \mathbb{C} \setminus S,\ S := \{0, -1, -2, -3, \ldots\}$

   $$\lim_{n\to\infty} \frac{\Gamma(z+n)}{n^z \Gamma(n)} = 1$$

3. Explain Riemann zeta function with the help of examples.
4. Briefly discuss Riemann's functional equation giving examples.
5. Let $K \subset \mathbb{C}$ is compact. If (R_n) and (S_n) are sequences of elements of $C(K)$ which converge uniformly on K to f and g, then prove that $R_n S_n$ converges uniformly on K to fg.
6. Show that there exists a sequence of polynomials (R_n) such that $R_n(0)=1$ for all n while $R_n(z) \to 0$ as $n \to \infty$ if $z \in \mathbb{C}$ and $z \neq 0$.
7. Determine a meromorphic function on the complex plane whose poles are simple poles at the positive integers with residues all equal to 1.
8. Show that the series, $\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \ldots$ represents the function which can be continued analytically outside the circle of convergence.
9. If $f(z) = \sum_{n=0}^{\infty} d(n) z^n$, $|z|<1$, $d(n)$ being the number of divisors of n then prove that the unit circle is a natural boundary of this function.
10. Discuss Schwarz reflection principle giving examples.
11. Describe the consequences of monodromy theorem with the help of examples.
12. Prove from Harnack's inequality that any harmonic function on the whole of C which is bounded from above or below is constant.
13. Let D be a bounded domain with a smooth boundary, and let $u(z)$ and $v(z)$ be smooth functions on $D \cup \partial D$ such that $u(z)$ is harmonic on D and $v(z)= u(z)$ on ∂D. Prove that,

    $$\iint_D |\nabla_v|^2\, dx\, dy = \iint_D |\nabla_u|^2\, dx\, dy + \iint_D |\nabla(v-u)|^2\, dx\, dy$$

14. Let D be the complement of the segment $[a, b]$, including the point at ∞. Find the Green's function for D with singularity at ∞ and evaluate,

    $$K = \lim_{|z|\to\infty} \{G(z;\infty) - \log|z|\}$$

4.9 FURTHER READING

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Ahlfors, Lars V. 1978. *Complex Analysis*, 3rd Edition. London: McGraw-Hill Education – Europe.

Lang, Serge. 1998. *Complex Analysis*, 4th Edition. NY: Springer-Verlag New York Inc.

Shastri, Anant R. 2000. *An Introduction to Complex Analysis*, India: Macmillan Publishers India Ltd.

UNIT 5 CANONICAL PRODUCTS, FUNCTIONS AND THEOREMS

NOTES

Structure

5.0 INTRODUCTION

In mathematics a canonical, normal, or standard form of a mathematical object is a standard way of presenting that object as a mathematical expression. Often, it is one which provides the simplest representation of an object and which allows it to be identified in a unique way. The distinction between 'Canonical' and 'Normal' forms varies from subfield to subfield. In most fields, a canonical form specifies a unique representation for every object, while a normal form simply specifies its form, without the requirement of uniqueness.

The canonical form of a positive integer in decimal representation is a finite sequence of digits that does not begin with zero. More generally, for a class of objects on which an equivalence relation is defined, a canonical form consists in the choice of a specific object in each class.

In Poisson-Jensen formula relates the average magnitude of an analytic function on a circle with the magnitudes of its zeros inside the circle. It forms an important statement in the study of entire functions.

An entire function, also called an integral function, is a complex-valued function that is holomorphic on the whole complex plane and the Bloch theorem

NOTES

states that in large quantum systems, the expectation value of the U(1) current operator averaged over the entire space vanishes. An analytic function is a function that is locally given by a convergent power series. There exist both real analytic functions and complex analytic functions. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not generally hold for real analytic functions. A function is analytic if and only if its Taylor series about x_0 converges to the function in some neighbourhood for every x_0 in its domain.

In this unit, you will study about the canonical products, Jensen's formula, Poisson-Jensen formula, Hadamard's three circles theorem, Hadamard's factorization theorem, order of an entire function, exponent of convergence, Borel's theorem, the range of an analytic function, Bloch's theorem, Picard's theorem, Schottky's theorem, Montel Caratheodory theorem, univalent functions and Bieberbach's conjecture and the ¼ theorem (Koebe's one-quarter theorem).

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe canonical products
- State Poisson-Jensen formula
- Define order of an analytic function
- Describe exponent of convergence
- State and prove Borel's theorem
- Define the range of an analytic function
- Discuss Picard's theorems
- Explain Schottky's theorem
- State and prove Montel Caratheodory theorem
- Explain univalent functions
- Prove Koebe's one-quarter theorem

5.2 CANONICAL PRODUCTS

Infinite Products

We know that $\Pi(1+a_n)$ converges to an element of $\mathbb{C}^*$ if and only if $\sum|a_n|$ converges. In the same way, $\Pi(1+f_n(z))$ defines an entire function on $\mathbb{C}$ if and only if $\sum|f_n(z)|$ converges uniformly on compact sets.

Polynomials work for finitely many zeros. We can try to address the case of infinitely many zeros by defining $f(z)=\Pi(z-a_i)$. But this product has no chance of converging. Assuming $a_i \neq 0$, a better choice is $\Pi(1-z/a_i)$.

NOTES

Theorem 5.1: If $\sum 1/|a_n|$ is finite then $f(z)=\Pi(1-z/a_n)$ defines an entire function with zeros exactly at these points.

Multiplicities of zeros correspond to repetitions of the same number in the sequence a_i. But, this result is too weak to address the case $a_n = n$ needed for constructing $\sin(\pi z)$.

Weierstrass Factors: It is an elegant expression for an entire function with a zero only at $z = 1$ which is also close to 1 for $|z| < 1$. It is known as the Weierstrass factor of order p:

$$E_p(z)=(1-z)\exp\left(z+\frac{z^2}{2}+...+\frac{z^p}{p}\right)$$

By convention, $E_0(z) = (1 - z)$. The basic idea behind this expression is, $\log(1/(1-z)) = z + z^2/2 + z^3/3 + ...$ and hence, the two terms almost cancel to give $(1 - z)/(1 - z) = 1$. Since the term z^p is truncated, it is easy to see that for $|z| < 1/2$, we have

$$|E_p(z)-1| = O\left(|z|^{p+1}\right)$$

Although this bound is sufficient, however it is sometimes useful to have a bound which works for any $z \in \Delta$ and where the implicit constant is explicit.

Theorem 5.2: For $|z| < 1$, we have $|E_p(z)-1| \le |z|^{p+1}$.

Proof: Now,

$$-E_p'(z)/E_p(z) = 1/(1-z)-1-z-...z^{p-1} = z^p/(1-z)$$

Hence, for all z we have

$$-E_p'(z) = z^p \exp\left(z+z^2/2+...z^p/p\right)z^p\sum_0^\infty a_k z^k$$

with $a_k \ge 0$ for all k. Integrating term by term and using the fact that $E_p(0)=1$, we get

$$1-E_p(z) = z^{p+1}\sum_0^\infty b_k z^k$$

with $b_k \ge 0$. We also have $\sum b_k = 1-E_p(1)=1$, and hence for $|z| < 1$ we have $|1-E_p(z)| \le |z|^{p+1}\sum b_k = |z|^{p+1}$

Theorem 5.3: For any sequence of non zero complex numbers $a_n \to \infty$, the formula $f(z)=\Pi_1^\infty E_n(z/a_n)$ converges for all z and defines an entire analytic function with zero set exactly (a_n).

Proof: The previous estimate yields convergence of the tail of the series. For all $z \in B(0,R)$,

$$\sum_{|a_n|>2R}|1-E_n(z/a_n)| \le \sum_1^\infty \left(|z|/|a_n|\right)^{n+1} < \sum_1^\infty (R/2R)^{n+1} < \infty$$

NOTES

Theorem 5.4: If $\sum a/|a_n|^{p+1} < \infty$, then $P(z) = \Pi E_p(z/a_n)$ defines an entire analytic function.

Proof: For $|z| < R$, we have

$$\sum_{|a_n|>2R} \left|1 - E_p(z/a_n)\right| \sum_{1}^{\infty} \left(|z|/|a_n|\right)^{p+1} < \infty \leq R^{p+1} \sum 1/|a_n|^{p+1} < \infty$$

If $p \geq 0$ is the least integer such that $\sum 1/|a_n|^{p+1} < \infty$, then we say $P(z) = \Pi E_p(z/a_n)$ is the canonical product associated to (a_n).

The Counting Function: It is defined as $N(r) = \left|\{n : |a_n| < r\}\right|$. $r^{-\beta} N(r)$ is a rough approximation to $\sum_{|a_n|<r} 1/|a_n|^{\beta}$. Consequently, we have

$$\alpha = \limsup_{r\to\infty} \frac{\log N(r)}{\log r}$$

Or, suppose $N(r) \leq r^{\beta}$. Collecting the points a_n into groups where $2^n < |a_n| \leq 2^{n+1}$,

$$\sum |a_n|^{-\alpha} \leq \sum \left(2^n\right)^{-\alpha} N\left(2^{n+1}\right) \leq 2^{\beta} \sum 2^{n(\beta-\alpha)} < \infty \text{ if } \alpha > \beta.$$

So, β is an upper bound for the critical exponent. Similarly, if $N(r) \geq r^{\beta}$ then β is a lower bound because $\sum |a_n|^{-\alpha} \geq r^{-\alpha} N(r) \geq r^{\beta-\alpha} \to \infty$ if $\alpha < \beta$.

Observe that knowledge of $N(r)$ is the same as knowledge of $r_n = |a_n|$ for all n. Thus we can also express functions of r_n in terms of $N(r)$. A typical example is:

$$\sum |a_n|^{-\alpha} = \int_0^{\infty} N(r) \alpha r^{-\alpha} \frac{dr}{r}$$

A similar expression will arise in connection with Jensen's formula.

Entire Function of Finite Order: An entire function $f : \mathbb{C} \to \mathbb{C}$ is of finite order if there is a $\rho > 0$ such that $|f(z)| = O\left(\exp|z|^{\rho}\right)$. The infimum of all such ρ is the order $\rho(f)$. Denote the maximum and minimum of $|f|$ on $|z| = r$ by $M(r)$ and $m(r)$. Thus, the order f is given by,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

For example, polynomials have order 0; sin(z), cos(z) and exp(z) have order 1; $E_p(z)$ has order p; exp(exp(z)) has infinite order.

5.2.1 Hadamard's Three Circles Theorem

This result applies not only to entire functions, but also to functions analytic in an annulus of the form $r_1 < |z| < r_2$.

NOTES

Theorem 5.5 (Hadamard's): For any analytic function $f(z)$, the quantity $\log M(r)$ is a convex function of $\log r$.

Proof: A function $\phi(s)$ of one real variable is convex if and only if $\phi(s) + ar$ satisfies the maximum principle for any constant a. This holds for $\log M(\exp(s))$ by considering $f(z)z^a$ locally.

Corollary 1: We have $M\left(\sqrt{rs}\right) \le \sqrt{M(r)M(s)}$.

The convex function satisfying $F(\log r) = \log M(r)$ look roughly linear for polynomials, e.g., like $F(x) = \deg(f)x + c$ and look roughly exponential for functions of finite order, e.g., $F(x) = \exp(\rho(f)x)$.

Hadamard's Factorization Theorem

This formula describes every entire function of finite order in terms of its zeros and an additional polynomial.

Theorem 5.6: An entire function $f(z) \neq 0$ of finite order ρ can be uniquely expressed in the form,

$$f(z) = z^m \Pi E_p(z/a_n) e^{Q(z)}$$

where (a_n) are the zeros of f, $p \ge 0$ is the least integer such that $\sum 1/|a_n|^{p+1} < \infty$ and $Q(z)$ is a polynomial of degree q. We have, $p, q \le \rho$.

The number p is called the genus of f. Ordinary polynomials arise when $p = q = \rho = 0$.

Note: This theorem shows that the zeros of f determine f upto finitely many additional constants, namely the coefficients of $Q(z)$. If $f(z)$ has no zeros, it is determined by its values at any $\lfloor \rho + 1 \rfloor$ points. This is not quite true, however, since $f(z)$ only determines $Q(z) \bmod 2\pi i\mathbb{Z}$. For example, $\exp(2\pi iz)$ and the constant function 1 agree on the integers.

But it is true that $f'/f = Q'$ in this case. Hence, knowing the logarithmic derivative at enough points almost determines Q.

Corollary 2: Suppose $f(z)$ and $g(z)$ are entire functions of order ρ with the same zeros and $f'/f = g'/g$ at $\lfloor \rho \rfloor$ distinct points where neither function vanishes. Then f is a constant multiple of g.

Proof of the Hadamard Factorization Theorem: Let $f(z)$ be an entire function of order ρ with zeros (a_i), and let $P(z)$ be the corresponding canonical product. Then P also has order ρ. Since f and P have the same zeros, the quotient f/P is an entire function with no zeros. The lower bound on $m(r)$ just established implies that f/P also has order ρ and hence $f/P = \exp Q(z)$ where $\deg Q \le \rho$.

NOTES

Entire Functions without Zeros

The most simple case of Hadamard's theorem in which there is no canonical product is handled by the following theorem:

Theorem 5.7: Let $f(z)$ be an entire function of finite order with no zeros. Then $f(z) = e^{Q(z)}$ where $Q(z)$ is a polynomial of degree d and $\rho(f) = d$.

For proving this theorem, we strengthen our characterization of polynomials by $M(r) = O(r^d)$.

Lemma 1: Let $Q(z)$ be an entire function satisfying $\operatorname{Re} Q(z) \leq A|z|^d + B$ for some $A, B > 0$. Then Q is a polynomial of degree atmost d.

Proof: There is a constant $C > 0$ such that for $R > 1$, Q maps $\Delta(2R)$ into half-plane $U(R) = \{z: \operatorname{Re} z < CR^d\}$. By Schwarz lemma, Q is distance-decreasing from the hyperbolic metric on $\Delta(2R)$ to the hyperbolic metric on $U(R)$. Since, $\Delta(R) \subset \Delta(2R)$ has bounded hyperbolic diameter, the same is true for $Q(\Delta(R)) \subset U(R)$. So, in the Euclidean metric,

$$diam\, Q(\Delta(R)) = O(d(Q(0), \partial U(R))) = O(R^d)$$

This shows that $|Q(z)| = O(|z|^d)$ for $|z| > 1$ and hence Q is a polynomial of degree atmost d.

Proof of Theorem 5.8: Since, f has no zeros, $f(z) = e^{Q(z)}$ for some entire function $Q(z)$. Since f has finite order, $|f(z)| = O\left(e^{|z|^d}\right)$ for some d and thus $\operatorname{Re} Q(z) \leq |z|^d + O(1)$. Now, applying the above lemma completes the proof.

Functions with Zeros: Now we will analyse entire functions with zeros.

Theorem 5.9: Let $f(z)$ be an entire function of order ρ with zeros (a_n). Then $\sum 1/|a_n|^{\rho+\varepsilon} < \infty$ for all $\varepsilon > 0$.

In general, a sequence $a_n \to \infty$ has a critical exponent α, the least number such that $\sum 1/|a_n|^{\alpha+\varepsilon} < \infty$. The result above states that the critical exponent of the zeros of f satisfies $\alpha \leq \rho(f)$. Thus if $p = \lfloor \rho(f) \rfloor$ then $p + 1 > \rho(f) \geq \alpha$ and hence $\sum |a_n|^{p+1} < \infty$.

Corollary 3: The genus of f satisfies $p \leq \rho(f)$.

Informally, the result above says that if f has many zeros, then $M(r)$ must grow rapidly.

5.2.2 Jensen's Formula

Theorem 5.10 (Jensen's Formula): Let $f(z)$ be a holomorphic function on $B(0, R)$ with zeros $a_1, \ldots, a_n$. Then

$$avg_{S^1(R)}(R)\log\left|f(z)\right| = \log\left|f(0)\right| + \sum \log\frac{R}{|a_i|}$$

Proof: First note that if f has no zeros, then log $|f(z)|$ is harmonic and the formula holds. Moreover, if the formula holds for f and g, then it holds for fg and the case of general R follows from the case $R = 1$. Now, we verify that the formula holds when $f(z) = (z-a)/(1-\bar{a}z)$ on the unit disc, with $|a| < 1$. Indeed, in this case log $|f(z)| = 0$ on the unit circle and log $|f(0)|$ + log $(1/|a|)$ = log $|a/a| = 0$ as well. The general case now follows, since a general function $f(z)$ on the unit disc can be written in the form $f(z) = g(z)\Pi(z-a_i)/(1-\bar{a}_i z)$ where $g(z)$ has no zeros.

Note: The physical interpretation of Jensen's formula is that log $|f|$ is the potential for a set of unit point charges at the zeros of f.

Counting Zeros: Here is another way to write Jensen's formula. Let $N(r)$ be the number of zeros of f inside the circle of radius r. Then,

$$\int_0^R N(r)\frac{dr}{r} = avg_{S^1(R)}(R)\log\left|f(z)\right| - \log\left|f(0)\right|$$

Proof of Theorem 5.11: Since $N(r)$ is an increasing function, by integrating from $r/2$ to r we find,

$$N(r/2)\log(r/2) \le \log M(r) + O(1)$$

and hence,

$$\alpha = \limsup \log N(r)/\log r \le \limsup \log M(2r)/\log(2r) = \rho$$

Functions can grow rapidly without having any zeros. But, then Jensen's formula shows that the average of log $|f|$ is constant over every circle $|z| = R$; so if f is large over much of the circle, it must also be close to zero somewhere on the same circle.

5.2.3 Canonical Products

Canonical Products: Now, we will determine the order of a canonical product. It will be used to complete the proof of Hadamard's theorem to obtain lower bounds for such a product.

Theorem 5.12: Let α be the critical exponent of the sequence $a_n \to \infty$. Then the canonical product $P(z) = \Pi E_p(z/a_n)$ has order $\rho(P) = \alpha$.

Proof: Let $r_n = |a_n|$ and $r = |z|$. Now, p is the least integer such that $\sum(1/r_n)^{p+1} < \infty$. So we also have $\sum 1/r_n^p = +\infty$. This implies $p \le \alpha \le p+1$. For convenience assume $\sum(1/r_n)^\alpha < \infty$. For small z, the Weierstrass factor,

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right)$$

satisfies the inequality $|1 - E_p(z)| \le |z|^{p+1} \le 1/2$ and hence also the inequality, $|\log E_p(z)| = O(|z|^{p+1})$

NOTES

While for large z, we get

$$|\log E_p(z)| = O(|\log(1-z)| + |z|^p)$$

The term log (1–z) can be ignored unless z is very close to 1, in which case $E_p(z)$ is close to zero. So, it can also be ignored when bounding $E_p(z)$ from above.

Combining these estimates for $P(z) = \prod E_p(z/a_n)$, we get the upper bound,

$$\log|P(z)| \le O\left(r^{p+1}\sum_{r_n>2r}\frac{1}{r_n^{p+1}} + r^p\sum_{r_n\le 2r}\frac{1}{r_n^p}\right)$$

In the second term, since $p \le \alpha$, we have

$$\sum_{r_n\le 2r}\frac{1}{r_n^p} = \sum\frac{r_n^{\alpha-p}}{r_n^\alpha} \le (2r)^{\alpha-p}\sum\frac{1}{r_n^\alpha} = O(r^{\alpha-p})$$

Similarly, in the first sum, since $\alpha \le p+1$, we have

$$\sum_{r_n>2r}\frac{1}{r_n^{p+1}} = \sum_{r_n>2r}\frac{1}{r_n^{p+1-\alpha}}\frac{1}{r_n^\alpha} \le (2r)^{\alpha-p-1}\sum\frac{1}{r_n^\alpha} = O(r^{\alpha-p-1})$$

Altogether this gives,

$$\log|P(z)| \le O(r^\alpha)$$

Thus, $\log M(r) = O(r^\alpha)$ and hence $\rho(P) \le \alpha$. By Jensen's theorem, the order of $P(z)$ is equal to α.

The Minimum Modulus: For controlling the result of division by a canonical product, we now estimate its minimum modulus.

Theorem 5.13: The minimum modulus of the canonical product $P(z)$ above satisfies $m(r) \ge \exp(-r^{\alpha+\varepsilon})$ for large r.

Proof: We will show $|\log m(r)| = O(r^{\alpha+\varepsilon})$. The proof follows the same lines as the bound $\log M(r) = O(r^\alpha)$ just obtained, since Theorem 5.1 and Theorem 5.3 give bounds for $|\log E_p(z)|$. We cannot ignore the logarithmic term in Theorem 5.3. We must also decide which values of r to choose, since $m(r) = 0$, whenever $r = |a_n|$.

We fix $\varepsilon > 0$ and exclude from consideration the balls B_n defined by $|z - a_n| < r_n^{\alpha+\varepsilon}$. Since the sum of the radii of the excluded balls in finite, there are plenty of large circles $|z| = r$ which avoid $\bigcup B_n$.

To complete the proof, it suffices to show that for z on such circles, we have

$$\sum_{|z-a_n|<r_n}|\log(1-z/a_n)| = O(r^{\alpha+\varepsilon})$$

Note that the number of terms in the sum above is at most $N(2r) = O(r^\alpha)$. Because we have kept z away from a_n, we have

$$|\log(1-z/a_n)| = O(\log r)$$

Consequently,

$$\sum_{|z-a_n|<r_n} |\log(1-z/a_n)| = O(N(2r)\log r) = O(r^{\alpha+\varepsilon})$$

as desired.

NOTES

Trignometric Functions: We determine the canonical factorization of the sine function:

Theorem 5.14: We have,

$$\sin(\pi z) = \pi z \prod_{n\neq 0}\left(1-\frac{z^2}{n^2}\right)$$

Proof: Indeed, the right hand side is a canonical product, and sin (πz) has order one, so the formula is correct up to a factor exp $Q(z)$ where $Q(z)$ has degree one. But, since sin (πz) is odd, we conclude **Q** has degree zero, and by checking the derivative at $z = 0$ of both sides we get $\mathbf{Q} = 0$.

Use of the Logarithmic Derivative: Some useful properties of the logarithmic derivative f'/f of an entire function $f(z)$ are as follows:

1. We have $(fg)'/fg = f'/f + g'/g$.
2. If $f'/f = g'/g$, then $f = Cg$ for some constant $C \neq 0$.
3. We have $f'\ (az + b)/f\,(az + b) = a\,(f'/f)\,(az + b)$.

Sine, Cotangent and Zeta: The product formula above gives, under logarithmic differentiation,

$$\frac{(\sin(\pi z))'}{\sin(\pi z)}\pi\cot(\pi z) = \frac{1}{z} + \sum_{1}^{\infty}\frac{1}{z-n} + \frac{1}{z+n}$$

This formula shows that π cot (πz) has simple poles at all points of z with residue one. This property can be used, for example, to evaluate $\zeta(2k) = \sum_1^{\infty} 1/n^{2k}$ and other similar sums by the residue calculus.

We note that, the product formula for sine(z) can also be used to prove $\zeta(2) = \pi^2/6$, by looking at the coefficient of z^2 on both sides of the equation. The sine formula also shows $\sum_{a<b} 1/(ab)^2 = \pi^4/5!$, $\sum_{a<b<c} 1/(abc)^2 = \pi^6/7!$, etc. So with some more work it can be used to evaluate $\zeta(2k)$. For example we have,

$$\zeta(4) = \left(\sum\frac{1}{a^2}\right)\left(\sum\frac{1}{b^2}\right) - 2\left(\sum_{a<b}\frac{1}{(ab)^2}\right) = \frac{\pi^4}{36} - \frac{\pi^4}{60} = \frac{\pi^4}{90}$$

Translation and Duplication Formulas: Many of the basic properties of the sine and cosine functions can be derived from the point of view of the uniqueness of an odd entire function with zeros at $z\pi$. For example, equations sin ($z+\pi$) = sin (z) and sin ($2z$) = 2 sin ($z + \pi/2$) hold up to a factor of exp ($az + b$) as a consequence of the fact that both sides have the same zero sets.

NOTES

5.2.4 Poisson-Jensen Formula

Poisson-Jensen formula relates the average magnitude of an analytic function on a circle with the magnitudes of its zeros inside the circle. It forms an important statement in the study of entire functions.

Theorem 5.15 (Poisson-Jensen Formula): If $f(z)$ is meromorphic in $|z| \le R$ and has zeros a_p and poles b_v, and if $\zeta = re^i$, $f(\zeta) \neq 0$, then for $0 \le r \le R$ we have

$$\log f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\log|(\mathrm{Re}^{i\phi})|(R^2 - r^2)dr}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

$$+\sum_{\mu} \log\left|\frac{R(\zeta - a_\mu)}{R^2 - \bar{a}_\mu \zeta}\right| - \sum_{v} \log\left|\frac{R(\zeta - b_v)}{R^2 - \bar{b}_v \zeta}\right| \qquad \ldots(5.1)$$

Proof:

Let $f(z) \neq 0$, in $|z| \le 1$. Then, since we can define an analytic branch of $\log f(z)$ in $|z| \le 1$, we have by the residue theorem

$$\frac{1}{2\pi i}\int_{|z|=1} \log f(z)\frac{dz}{z} = \log f(0)$$

By change of variable,

$$\frac{1}{2\pi}\int_0^{2\pi} \log f(e^{i\phi})d\phi = \log f(0)$$

And now taking the real part on both sides,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\phi})|\,d\phi = \log|f(0)|$$

For any ζ with $|\zeta| < 1$, we effect the conformal transformation $w = \dfrac{z - \zeta}{1 - \bar{\zeta} z}$ for the integral $\displaystyle\int_{|z|=1} \log f(z)\frac{dz}{z}$. This in turn becomes,

$$\frac{1}{2\pi i}\int_{|w|=1} \log\phi(w)\frac{dw}{w} = \log f(\zeta) \text{ where } \phi(w) = f\{z(w)\}$$

so that $\phi(0) = f(\zeta)$. Substituting in the integral $z = e^{i\phi}$ and taking real part we get

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\log|f(e^{i\phi})|}{1 - 2r\cos(\phi - \theta) + r^2}(1 - r^2)d\phi = \log|f(\zeta)|, \zeta = re^{i\theta}$$

Note for the function $f(z)$ with poles b_v and zeros a_μ none of them being on $|z| = 1$, let us define

$$\psi(z) = f(z)\frac{\pi\dfrac{(z - b_v)}{(1 - \bar{b}_v z)}}{\pi\dfrac{(z - a_\mu)}{(1 - \bar{a}_\mu z)}}$$

NOTES

On $|z|=1, |\psi(z)|=|f(z)|$ and the function has no zeros or poles in $|z| \le 1$. By the above result,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(e^{i\theta})|\frac{(1-r^2)}{1-2r\cos(\phi-\theta)+r^2}d\phi = \log|\psi(\zeta)|$$

$$\zeta = re^{i\theta}, r<1$$

Substitution for ψ gives the theorem for $R = 1$. In the case when there are poles and zeros on the circumference of the unit circle we proceed as follows. We have only to show that if $f(z)$ has no zeros or poles in $|z| < 1$, but has poles and zeros on $|z| = 1$, then

$$\frac{1}{2\pi i}\int_{|z|=1} \log f(z)\frac{dz}{z} = \log f(0)$$

for if $f(z)$ has zeros and poles in $|z| < 1$ we can consider $\psi(z)$, in place of $f(z)$. Further we can assume that there is only one zero (the case of pole being treated in the same manner) on $|z| = 1$. For the case when $f(z)$ has a finite number (it can have atmost only a finite number) of zeros (poles) can be treated similarly.

Let therefore $z = a$, $|a| = 1$ be a zero of $f(z)$ on $|z| = 1$. Let p be the point $z = a$ and consider a circle of radius $\rho < 1$ about p, ρ being small. Consider the counter SQR (Refer Figure 5.1). Inside it, $f(z)$ has no zeros or poles. Hence, by the residue theorem $\int \log f(z)dz = \log f(0)$. Thus, it is enough to prove that $\int_{QR} \log f(z)dz$ tends to zero as ρ tends to zero.

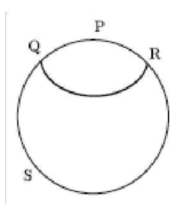


Fig. 5.1

Let $z = a$ be zero of order k. Then $f(z) = (z-a)^k \lambda(z), \lambda(a) \ne 0,$ in a certain neighbourhood of a and we can assume the choice of ρ such that this expansion is valid within and on the circle of radius ρ about p.

$$\int_{QR} \log f(z)\frac{dz}{z} = k\int_{QR} \log(z-a)\frac{dz}{z} + \int \log\lambda(z)\frac{dz}{z}$$

Since $\lambda(z)$ remains bounded the second integral tends to zero. So we have only to prove that $\int_{QR} \log(z-a)\frac{dz}{z}$ tends to zero as $\rho \to 0$. Now,

$$\left| \int_{QR} \log(z-a) \frac{dz}{z} \right| \le \max \left\{ \left| \frac{\log|(z-a)|}{|z|} \right| \right\}$$

NOTES

$$\pi\rho \le [\log(1/\rho) + 0(1)]\pi\rho \to 0 \text{ if } \rho \le \frac{1}{2}$$

This proves the result in case when the function $f(z)$ has zeros or poles on the unit circle. In case $R \neq 1$, we consider the function $f(Rz)$ instead of $f(z)$ and arrive at the result. Hence the theorem is proved completely.

Corollary: In the special case, when $\xi = 0$ we get the Jensen's formula,

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(R\,e^{i\phi})|\,d\phi + \sum_{\mu} \log\left|\frac{a_\mu}{R}\right| - \sum_{v} \log\left|\frac{b_v}{R}\right| \qquad \ldots (5.2)$$

The summation ranging over poles and zeros of $f(z)$ in $|z| \le R$. the above formula does not hold if zero is a pole or a zero of $f(z)$. If $f(0) = 0$ or ∞ and $f(z)$ is not identically constant then $f(z) = C_\lambda z^\lambda + ... + ...$. Consider $f(z)/z^\gamma$. This has neither zero nor pole at zero. Hence we get,

$$\log|C_\lambda| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{f(R\,e^{i\phi})}{R^\lambda}\right| d\phi + \sum \log\frac{|a_\mu|}{R} - \sum \log\frac{|b_v|}{R}$$

$$= \frac{1}{2\pi} \int \log|f(R\,e^{i\phi})|\,d\phi + \sum \log\frac{|a_\mu|}{R} - \sum \log\frac{|b_v|}{R} - \lambda \log R$$

where sums are taken over zeros and poles of $f(z)$ are in $0 \le |z| \le R$.

The Characteristic Function

Set x, real and positive.

$$\log^+ x = \log x \text{ if } x > 1,$$

$$\log^+ x = 0 \quad \text{ if } x \le 1,$$

Then clearly, $\log x = \log^+ x - \log^+(1/x)$.

$$\int_0^{2\pi} \log|f(Re^{i\phi})|\,d\phi = \int_0^{2\pi} \log^+|f(Re^{i\phi})|\,d\phi - \int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\phi})|}\,d\phi.$$

We note that the first term represents the contribution when f is large and the second term when f is small. Let $0 < r_1 \le r_2 \le ... \le r_n \le R$ be the moduli of the poles in the order of increasing magnitude. Let $n(r)$ denote the number of poles in $|z| < r$ of $f(z)$. Then the Riemann-Stieljes integral formula is,

$$\int_0^R \log\frac{R}{t}\,dn(t) = \sum_v \log\frac{1}{|b_v|}$$

Given on integrating by parts,

$$n(t)\log\frac{R}{t}\Bigg|_0^R + \int_0^R n(t)\frac{dt}{t} = \sum_v \log(R/|b_v|).$$

The first term is zero, in the consequence of the fact $n(t) = 0$, near zero.

We write $n(r, f)$ for the number of poles of $f(z)$ in $|z| \le r$, so that $n(r, 1/f)$ is equal to the number of zeros of $f(z)$ in $|z| \le r$. We define $N(r, f)$ to be

NOTES

$$\int_0^r n(t, f)\frac{dt}{t}$$

If $f(0) = \infty$ we define $N(r, f) = \int_0^r [n(t, f) - n(0, f)]\frac{dt}{t} + n(0, f)\log r.$

Then the Equation (5.1) becomes for, $f(0) \ne 0, \infty$

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\theta})|}d\theta + N(r.f)$$
$$-N(r, 1/f).$$

We define,

$$T(r, f) = N(r, f) + m(r, f)$$

where,

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$$

Again Equation (5.1) takes the form, for $f(0) \ne 0, \infty$

$$T(r, f) = T(r, 1/f) + \log|f(0)| \qquad \ldots (5.3)$$

If $f(z) \sim C_\lambda Z^\lambda$ near $z = 0$, where $\lambda \ne 0$, then we obtain $T(r, f) = T(r, 1/f)$ $+ \log|C_\lambda|$. In future such modifications will be taken for granted.

The function $T(r,f)$ is called the characterstic function of $f(z)$. This is the Nevanlinna characteristic function.

Theorem 5.16 (First Fundamental Theorem): For any complex a,

$$T(r, f) = T[r, 1/(f - a)] + \log|f(0) - a| + \varepsilon(a)$$
$$\text{where } |\varepsilon(a)| \le \log^+|a| + \log 2.$$

Proof: Note that,

$$\log^+|z_1 + z_2| \le \log^+|z_1| + \log^+|z_2| + \log 2$$

and $\log^+|z_1 - z_2| \ge \log^+|z_1| - \log^+|z_2| - \log 2$. Whence,

$$\log^+|f(z) - a| - \log^+|f(z)| \le \log 2 + \log^+|a|$$

Integrating we get,

$$-\log 2 - \log^+|a| + m(r, f - a) \le m(r, f) \le \log 2 + \log^+|a| + m(r, f - a)$$

Since $f, f-a$ has the same poles,

$$N(r, f) = N(r, f - a).$$

Therefore,

$$T(r, f - a) - \log^+|a| - \log 2 \le T(r, f) \le \log 2 + \log^+|a| + T(r, f - a)$$

That is, $|T(r,f)-T(r,f-a)|\leq \log 2+\log^{+}|a|$

$T(r,f)+T(r,f-a)+\varepsilon(a)$, where $|\varepsilon(a)|\leq \log 2+\log^{+}|a|$

From Equation (5.3) we have,

$$T(r,f)=T\left(r,\frac{1}{f-a}\right)+\log|f(0)-a|+\varepsilon(a)$$

where $|\varepsilon(a)|\leq \log 2+\log^{+}|a|$. Hence the theorem is proved.

If we write $m(r,a)$, $N(r,a)$ for

$$m\left(r,\frac{1}{f-a}\right),N\left(r,\frac{1}{f-a}\right)$$

then $m(r,a)$ represents the average degree of approximation of $f(z)$ to the value a on the circle $|z|=r$ and $N(r,a)$ the term involving the number of zeros of $f(z)-a$. Their sum can be regarded as the total affinity of $f(z)$ for the value a and we see then apart from a bounded term the total affinity for every value of a. However, the relative size of the two terms m, N remains in doubt.

5.3 ORDER OF AN ENTIRE FUNCTION

Entire function is a function that is analytic in the whole complex plane except, possibly, at the point at ∞. It can be expanded in a power series,

$$f(z)=\sum_{k=0}^{\infty}a_k z^k, \qquad a_k=\frac{f^{(k)}(0)}{k!}, \qquad k\geq 0,$$

which converges in the whole complex plane, $\lim_{k\to\infty}|a_k|^{1/k}=0$.

Now, if $f(z)\neq 0$ everywhere, then $f(z)=e^{P(z)}$, where $P(z)$ is an entire function. If there are finitely many points at which $f(z)$ vanishes and these points are $z_1,...,z_k$ (the zeros of the function), then

$$f(z)=(z-z_1)...(z-z_k)e^{P(z)}$$

Now, when $f(z)$ has infinitely many zeros $z_1,z_2,...,$ there is a product representation,

$$f(z)=z^{\lambda}e^{P(z)}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{z}{z_k}+...+\frac{z^k}{kz_k^k}\right), \qquad ...(5.4)$$

where $P(z)$ is an entire function, $\lambda=0$ if $f(0)\neq 0$, and λ is the multiplicity of the zero $z=0$ if $f(0)=0$.

Let

$$M(r)=\max_{|z|\leq r}|f(z)|$$

If for large r the quantity $M(r)$ grows no faster than r^{μ}, then $f(z)$ is a polynomial of degree not exceeding μ. Therefore, if $f(z)$ is not a polynomial, then

$M(r)$ grows faster than any power of r. To estimate the growth of $M(r)$ in this case, take the exponential function as the comparison function.

By definition, $f(z)$ is an entire function of finite order if there is a finite number such that,

$$M(r) < e^{r^{\mu}}, \qquad r > r_0.$$

The greatest lower bound ρ of the set of numbers μ satisfying this condition is called the order of the entire function $f(z)$. The order can be computed by the formula,

$$\rho = \overline{\lim_{k\to\infty}} \frac{k \ln k}{\ln|1/a_k|}.$$

If $f(z)$ of order ρ satisfies the condition,

$$M(r) < e^{\alpha r^{\rho}}, \qquad \alpha < \infty, \qquad r > r_0, \qquad \ldots(5.5)$$

then we can say that $f(z)$ is a function of order ρ and of finite type. The greatest lower bound σ of the set of numbers α satisfying this condition is called the type of the entire function $f(z)$. It is determined by the formula,

$$\overline{\lim_{k\to\infty}}\ k^{1/\rho}\,|a_k|^{1/k} = (\sigma\ e\ \rho)^{1/\rho}.$$

Among the entire functions of finite type, one distinguishes entire functions of normal type ($\sigma > 0$) and of minimal type ($\sigma = 0$). If the Equation (5.5) does not hold for any $\alpha < \infty$, then the function is said to be an entire function of maximal type or of infinite type. An entire function of order 1 and of finite type, and also an entire function of order less than 1, is said to be of exponential type if it is characterized by the condition,

$$\overline{\lim_{k\to\infty}}\ k\,|a_k|^{1/k} = \beta < \infty,$$

The zeros of an entire function $f(z)$ of order ρ have the property,

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{\rho+\in}} < \infty, \qquad \text{for all } \in > 0$$

Suppose p is the least integer $(p \le \rho)$ such that $\sum_{k=1}^{\infty} |z_k|^{-p-1} < \infty$. Then,

$$f(z) = z^{\lambda} e^{P(z)} \prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right)\exp\left(\frac{z}{z_k} + \ldots + \frac{z^p}{p z_k^p}\right)$$

where $P(z)$ is a polynomial of degree not exceeding ρ.

Theorem 5.17: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that there exist real constants $C_\lambda > 0, \lambda > 0$, such that $\mathrm{Re}(f(z)) \le C(1 + |z|^{\lambda})$ for all $z \in C$. Then f is a polynomial, atmost of degree $[\lambda]$.

Proof:

$$\int_0^{2\pi} \cos(k\theta)\sin(l\theta) = 0$$

and

$$\int_0^{2\pi} \cos(k\theta)\cos(l\theta) = e\int_0^{2\pi} \sin(k\theta)\sin(l\theta) = \begin{cases} 0, & k \neq l \\ \pi, & k = l. \end{cases}$$

NOTES

If $f(0) \neq 0$, then we can just substitute the function $f - f(0)$ for f. Thus we can assume without loss of generality that $f(0) = 0$. Developing f in a power series around 0, we write

$$f(z) = \sum_{n=1}^{\infty} (a_n + ib_n) z^n$$

where a_n and b_n are real numbers. Thererore

$$\mathrm{Re}(f(z)) = \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) - b_n \sin(n\theta))$$

where $z = re^{i\theta}$. So for each $k \in \mathbb{N}$ we have

$$\int_0^{2\pi} \cos(k\theta)\,\mathrm{Re}(f(z))d\theta = a_k r^k \pi.$$

Similarly

$$\int_0^{2\pi} \sin(k\theta)\,\mathrm{Re}(f(z))d\theta = b_k r^k \pi$$

Also,

$$\int_0^{2\pi} \mathrm{Re}(f(z))d\theta = f(0) = 0$$

$$n = 1$$

Therefore,

$$\begin{aligned} |a_k| &\leq \frac{1}{\pi r^k}\int_0^{2\pi} |\mathrm{Re}(f(z))|\, d\theta \\ &= \frac{1}{\pi r^k}\int_0^{2\pi} (|\mathrm{Re}(f(z))| + \mathrm{Re}(f(z)))d\theta \\ &= \frac{2}{\pi r^k}\int_0^{2\pi} \max(\mathrm{Re}(f(z)), 0)d\theta \\ &= \frac{4C(1 + r^{\lambda})}{r^k} \end{aligned}$$

Therefore, taking $r \to \infty$, we see that if $k > \lambda$ then $a_k = 0$. An analogous argument shows also that $b_k = 0$.

Definition: An entire function f is said to have finite order if there exists some real number $\rho > 0$, and a constant $C > 0$, such that

$$|f(z)| \leq Ce^{|z|^{\rho}}$$

for all $z \in \mathbb{C}$. The infimum over all such ρ is the order of f. We can say that α is the order of f if $|f(z)| < Ce^{|z|^{\alpha+\epsilon}}$, for all $\in > 0$ and $z \in \mathbb{C}$. If $|f(z)| \leq Ce^{|z|^{\alpha}}$, for all $z \in \mathbb{C}$ then α is the strict order of f.

5.3.1 Exponent of Convergence

Theorem 5.18 (Weierstrass): Given a nonnegative integer λ and an increasing sequence of non zero complex numbers $\{\zeta_n\}$ converging to infinity, there exists an entire function $f(z)$ whose zeros coincide with the points,

$$\underbrace{0,...,0,}_{\lambda\ \text{times}}\ \zeta_1,...,\zeta_n,... \qquad ...(5.6)$$

Proof: Consider the sequence of entire functions,

$$f_m(z) = z^{\lambda}\prod_{n=1}^{m}\left(1-\frac{z}{\zeta_n}\right)e^{P_n(z)} \qquad (m=1,\,2,...),$$

NOTES

where the $p_n(z)$ are polynomials. Obviously, the zeros of $f_m(z)$ coincide with the first $m + \lambda$ points of the sequence Equation (5.6). In general, $f_m(z)$ has multiple zeros, since Equation (5.6) can contain the same point several times (this possibility is explicitly indicated for the point $z = 0$). The idea of the proof is to choose the polynomials $p_n(z)$ in such a way that the sequence $\{f_m(z)\}$ is uniformly convergent on every compact subset.

Deducing the limit function, we get

$$f(z) = \lim_{m\to\infty} f_m(z) \qquad ...(5.7)$$

Let K_R denote the disc $|z| < R$, and let $N(R)$ be the smallest integer such that $|\zeta_\nu| > 2R$ for all $n > N(R)$. Then, if $z \in K_R$ and $m > N(R)$, we can write $f_m(z)$ in the form,

$$f_m(z) = f_{N(R)}(z)\prod_{n=N(R)+1}^{m}\left(1-\frac{z}{\zeta_n}\right)e^{P_n(z)} \qquad ...(5.8)$$

$$= f_{N(R)}(z)\exp\left\{\sum_{n=N(R)+1}^{m}\left[\ln\left(1-\frac{z}{\zeta_n}\right)+P_n(z)\right]\right\},$$

where every logarithmic term can be expanded as a power series,

$$\ln\left(1-\frac{z}{\zeta_n}\right) = -\frac{z}{\zeta_n} - ... - \frac{z^n}{n\zeta_n^n} - \frac{z^{n+1}}{(n+1)\zeta_n^{n+1}} - ...,$$

since $|z/\zeta_n| < 1/2$ for all $z \in K_R$ and $n > N(R)$.

Choosing $p_n(z)$ so as to cancel the first n terms of this series, i.e.,

$$P_n(z) = \frac{z}{\zeta_n} + ... + \frac{z^n}{n\zeta_n^n} \qquad ...(5.9)$$

we have,

$$\ln\left(1-\frac{z}{\zeta_n}\right) + P_n(z) = -\frac{z^{n+1}}{(n+1)\zeta_n^{n+1}} - ..., \qquad ...(5.10)$$

implying,

$$\left|\ln\left(1-\frac{z}{\zeta_n}\right)+P_n(z)\right| \le \frac{1}{n+1}\left|\frac{z}{\zeta_n}\right|^{n+1} + \frac{1}{n+2}\left|\frac{z}{\zeta_n}\right|^{n+2} + ...$$

$$< \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + ... = \frac{1}{2^n} \qquad ...(5.11)$$

Then, the series

$$\sum_{n=N(R)+1}^{\infty}\left[\ln\left(1-\frac{z}{\zeta_n}\right)+P_n(z)\right] \qquad ...(5.12)$$

is uniformly convergent on K_R, since

$$\sum_{n=N(R)+1}^{\infty}\left|\ln\left(1-\frac{z}{\zeta_n}\right)+P_n(z)\right|<\sum_{n=1}^{\infty}\frac{1}{2^n}=1<\infty$$

NOTES

Comparing Equation (5.7) and (5.8), and using the continuity of the exponential function, we find that

$$f(z)=f_{N(R)}(z)e^{x_R(z)} \qquad (z\in K_R) \qquad \ldots(5.13)$$

This shows that $f(z)$ is analytic on K_R, a fact which also follows from the uniform convergence of the sequence $\{f_m(z)\}K_R$. Since the disc K_R can have arbitrarily large radius, $\{f_m(z)\}$ is uniformly convergent on every compact set and hence the function $f(z)$ is analytic in the whole plane, i.e., $f(z)$ is entire. The fact that the zeros of $f(z)$ coincide with the points of Equation (5.6) is almost obvious, and follows from the representation Equation (5.13) and the arbitrariness of R, since $\exp[X_R(z)]$ is nonvanishing, while, by construction, the zeros of $f_{N(R)}(z)$ in K_R are precisely those points of the sequence Equation (5.6) which lie in K_R. Finally, recalling the definition of $f_m(z)$, we note that

$$f(z)=\lim_{m\to\infty} z^{\lambda}\prod_{n=1}^{m}\left(1-\frac{z}{\zeta_n}\right)e^{P_n(z)} \qquad \ldots(5.14)$$

Corollary 1: The finite product,

$$f(z)=z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{\zeta_n}\right)\exp\left(\frac{z}{\zeta_n}+\ldots+\frac{z^n}{n\zeta_n^n}\right) \qquad \ldots(5.15)$$

is an entire function satisfying the requirements of Theorem 5.18.

Proof: Equation (5.15) is another way of writing Equation (5.14) and hence the proof follows.

Corollary 2: Let $f(z)$ be an entire function with zeros given by the increasing sequence,

$$\underbrace{0,\ldots,0}_{\lambda\text{ times}},a_1,\ldots,a_n,\ldots$$

Then, $f(z)$ can be represented in the form,

$$f(z)=e^{g(z)}z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\ldots+\frac{z^n}{na_n^n}\right), \qquad \ldots(5.16)$$

where $g(z)$ is an entire function.

Proof: The function,

$$\phi(z)=z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\ldots+\frac{z^n}{na_n^n}\right)$$

is entire, with the same zeros as $f(z)$, and hence the quotient $f(z)/\phi(z)$ is entire and non-vanishing. Therefore,

$$\frac{f(z)}{\phi(z)}=e^{g(z)}$$

where $g(z)$ is an entire function.

NOTES

The Exponent of Convergence: Let $\{\zeta_n\}$ be an arbitrary increasing sequence of non zero complex numbers, which converges to infinity and consider the series α_0,

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\alpha}} \qquad \ldots (5.17)$$

where α is non negative. If the series Equation (5.17) converges for some $\alpha_0 > 0$ then it converges for all $\alpha > \alpha_0$ (since the numbers $|\zeta_n|^{-1}$ are all less than 1, starting from some value of n). The greatest lower bound of the values of α for which Equation (5.17) converges is a non negative number τ called the exponent of convergence of the sequence $\{\zeta_n\}$. If Equation (5.17) diverges for all $\alpha > 0$, we set $\tau = \infty$ and say that the exponent of convergence of the sequence $\{\zeta_n\}$ is infinite.

For, example, the exponent of convergence of the sequences,

$\{e^n\}, \{n^{1/\tau}\}$ and $\{\ln(n+1)\}$

are 0, τ and ∞, respectively.

Theorem 5.19: The exponent of convergence τ of the sequence $\{\zeta_n\}$ is given by the formula,

$$\tau = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln|\zeta_n|} \qquad \ldots (5.18)$$

Proof: Suppose τ is infinite. Then the series,

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\alpha}}$$

converges for any $\alpha > \tau$. Since the terms of this series are non increasing, it follows that

$$\overline{\lim_{n\to\infty}} \frac{n}{|\zeta_n|^{\alpha}} = 0$$

Therefore,

$$\alpha > \frac{\ln n}{\ln|\zeta_n|}$$

for all sufficiently large n, implying

$$\alpha \geq \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln|\zeta_n|}$$

or

$$\tau \geq \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln|\zeta_n|} \qquad \ldots (5.19)$$

since α is an arbitrary number exceeding τ.

Next let α' be any number exceeding the right hand side of Equation (5.18). Then there is an integer $N = N(\alpha')$ such that,

$$\frac{\ln n}{\ln|\zeta_n|} < \alpha'$$

for all $n > N$. Therefore

$$\frac{1}{|\zeta_n|} < n^{-1/\alpha'}$$

NOTES

for all $n > N$, meaning that the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\beta}}$$

converges for any $\beta > \alpha'$. It follows from the definition of the convergence exponent that $\tau \leq \alpha'$, and hence

$$\tau \leq \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln |\zeta_n|} \qquad \ldots (5.20)$$

because of the definition of α'. Comparing Equation (5.19) and Equation (5.20), we obtain Equation (5.18). Moreover, if the right hand side of Equation (5.20) is finite, so is τ. In other words, if τ is infinite, so is the right hand side of Equation (5.18) and the proof is complete.

Corollary 1: If the right hand side of Equation (5.18) is finite and equal to τ, then the exponent of convergence of the sequence $\{\zeta_n\}$ equals τ. If the right hand side of Equation (5.18) is infinite, so is τ.

Theorem 5.20: Given a nonnegative integer λ and an increasing sequence of non zero complex numbers $\{\zeta_n\}$ converging to infinity. Let x be the largest nonnegative integer k for which the series,

$$\sum_{n=1}^{\infty} \frac{1}{\zeta_n|^k} \text{ diverges.}$$

Then the expression,

$$f(z) = z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \ldots + \frac{z^x}{x\zeta_n^x}\right)$$

known as a canonical product, where the exponential factors disappear if $x = 0$, represents an entire function whose zeros coincide with the points,

$$\underbrace{0,\ldots,0,}_{\lambda \text{ times}} \quad \zeta_1, \ldots, \zeta_n, \ldots$$

Proof: We have,

$$P_n(z) = \begin{cases} \dfrac{z}{\zeta_n} + \ldots + \dfrac{z^x}{x\zeta_n^x} & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x = 0 \end{cases} \qquad \ldots (5.21)$$

and

$$\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z) = -\frac{z^{x+1}}{(x+1)\zeta_n^{x+1}} - \ldots,$$

instead of Equation (5.9) and Equation (5.10), and

$$\left|\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right| = \left|-\frac{z^{x+1}}{\zeta_n^{x+1}} \sum_{p=1}^{\infty} \frac{z^{p-1}}{(x+p)\zeta_n^{p-1}}\right| < \frac{R^{x+1}}{|\zeta_n|^{x+1}} \sum_{p=1}^{\infty} \frac{1}{2^{p-1}} = \frac{2R^{x+1}}{|\zeta_n|^{x+1}}$$

NOTES

instead of Equation (5.11). Therefore Equation (5.12) is again uniformly convergent on $K_R : |z| < R$, but this time because of the convergence of the series,

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{x+1}}$$

instead of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

The remaining proof is identical with that of Theorem 5.18.

Corollary 1: Let $f(z)$ be an entire function with zeros given by the increasing sequence,

$$\underbrace{0, ..., 0}_{\lambda \text{ times}}, a_1, ..., a_n, ..., \qquad \ldots (5.22)$$

and let x be the largest nonnegative integer k for which the series,

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k} \qquad \ldots (5.23)$$

diverges. Then $f(z)$ can be represented in the form,

$$f(z) = e^{g(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + ... + \frac{z^x}{x a_n^x}\right) \qquad \ldots (5.24)$$

where $g(z)$ is an entire function and the exponential factors disappear if $x = 0$.

Note: Theorem 5.20 and its corollary remain valid if we replace x by any larger integer.

Theorem 5.21: If $f(z)$ is an entire function of finite order ρ with zeros given by Equation (5.19) and if the sequence $\{a_n\}$ has convergence exponent τ, then $\tau \le \rho$.

Proof: Given any $\varepsilon > 0$,

$$\frac{1}{|a_n|^{\rho+\varepsilon}} < \frac{e(\rho + 2\varepsilon)}{n}$$

or

$$\frac{1}{|a_n|^{\rho+2\varepsilon}} < [e(\rho + 2\varepsilon)]^{(\rho+2\varepsilon)/(\rho+\varepsilon)} \frac{1}{n^{(\rho+2\varepsilon)/(\rho+\varepsilon)}}$$

for all sufficiently large n. Therefore the series,

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\alpha}}$$

converges for all $\alpha \ge \rho + 2\varepsilon$, and hence for all $\alpha > \rho$, since $\varepsilon > 0$ is arbitrary, $\tau \le \rho$, by the definition of the convergence exponent.

Note: In particular, if x is the largest non negative integer k for which the series Equation (5.23) diverges, then $x \le [\rho]$.

NOTES

Note: The importance of Equation (5.24) over Equation (5.16) is that polynomials of the same fixed degree x can be used in all the exponential factors, instead of polynomials whose degree becomes arbitrarily large (with n). In the special case where $x = 0$, the exponential factors in Equation (5.24) disappear and we have the particularly simple infinite product expansion,

$$f(z) = e^{g(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

For example, $x = 0$ if $\tau = 1$ and the series,

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|}$$

converges. If $f(z)$ is of finite order ρ, then $x = 0$ if $\rho < 1$, or if $\rho \geq 1$ but $\tau < 1$.

5.3.2 Borel's Theorem

Theorem 5.22 (Borel): If f is a continuous functions on $[a, b]$, then for any $\varepsilon > 0$ the interval can always be divided into a finite number of sub-intervals in each of which the variation of $f(x)$ is less than ε.

Proof: Let the theorem be untrue. Divide $[a, b]$ into two-halves $[a, c]$, $[c, b]$ where $c = \frac{1}{2}(a + b)$. Then at least in one of these the theorem is untrue. If untrue in both, for definiteness, take right hand side half. Denote this half interval by $[a_1, b_1]$. As earlier divide $[a_1, b_1]$ into two halves, at least in one the theorem must be untrue, denote this half by $[a_2, b_2]$. Continuing this process of division indefinitely; we get a sequence of closed interval $\langle [a_n, b_n] \rangle$ such that

$$a \leq a_1 \leq a_2 ... \leq a_n \leq a_{n+1} .... \leq b_{n+1} \leq b_n .... \leq b_2 \leq b_1 \leq b, \text{ with } b_n - a_n = (b - a)/2^n.$$

Hence by the nested interval theorem, $< a_n >$, $< b_n >$, converge to a unique point, say α. Then $\alpha \varepsilon [a_n, b_n] \subset [a, b] \ \forall n \in \mathbf{N}$.

Let us assume that $\alpha \neq a$ or b. Then, on account of continuity of f at α, for $\varepsilon > 0 \ \exists \ \delta > 0$ such that

$x_1, x_2 \in (\alpha - \delta, \alpha + \delta) \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.

Since $b_n - a_n = (b - a)/2^n \rightarrow 0$ as $n \rightarrow \infty$, for $\delta > 0 \ \exists \ m \in \mathbf{N}$ such that

$b_n - a_n < \delta \ \forall \ n \geq m$. In particular, $b_m - a_m < \delta$ with $\alpha \in [a_m, b_m]$

$$\Rightarrow [a_m, b_m] \subset (\alpha - \delta, \delta + \delta).$$

Hence, (*) gives that $x_1, x_2 \in [a_m, b_m] \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

This shows that the variation of f in $[a_m, b_m]$ is less than $\varepsilon > 0$. This contradiction leads to the fact that the theorem must be true.

A slight modification would establish this fact even when $\alpha = a$ or b. The possibility of $\alpha = a$ or b necessitates that the interval from a to b must be closed.

NOTES

Check Your Progress

1. When is an entire function said to be of finite order?
2. State Poisson-Jensen formula.
3. Define the order of an entire function.
4. State Weierstrass theorem.
5. Give the statement of Borel's theorem.

5.4 THE RANGE OF AN ANALYTIC FUNCTION

Here, the range of an analytic function is investigated.

Lemma 1: Let f be analytic in $D = \{z : |z| < 1\}$ such that $f(0) = 0, f'(0) = 1$ and $|f(z)| \le M$ for all z in D. Then $M \ge 1$ and

$$f(D) \supset B\left(0; \frac{1}{6M}\right)$$

Proof: Let $f(z) = z + a_2 z^2 + a_3 z^3 + ...$

Since $f(z)$ is analytic in $D = B(0; 1)$, so by Cauchy's estimate

$|a_n| \le M$ for $n \ge 1$

$|a_1| \le M$

$M \ge 1$ $\qquad [\because a_1 = 1]$

Let $z \in D$ such that $|z| = 1/4M$. Then

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n z^n|$$

$$\ge \frac{1}{4M} - \sum_{n=2}^{\infty} M.\left(\frac{1}{4M}\right)^n$$

$$= \frac{1}{4M} - \frac{1}{16M}\left(1 + \frac{1}{4M} + \frac{1}{16M^2} + ...\right)$$

$$= \frac{1}{4M} - \frac{1}{16M}\frac{1}{\left(1 - \frac{1}{4M}\right)} = \frac{1}{4M} - \frac{1}{16M - 4} = \frac{1}{4M}\left(\frac{12M - 4}{16M - 4}\right) \ge \frac{1}{6M}$$

$$\left[\because \text{Minimum value of } \frac{12M-4}{16M-4} \text{ is } \frac{2}{3} \text{ when } M = 1\right]$$

Suppose $w \in B(0; 1/6M)$ then $|w| < 1/6M$

Consider the function $g(z) = f(z) - w$.

For $|z| = 1/4M$, $|f(z) - g(z)| = |w| < 1/6M \le |f(z)|$

So by Rouche's theorem, f and g have the same number of zeros in $B(0; 1/4M)$. Since $f(0) = 0$, so $g(z_0) = 0$ for some $z_0 \in B(0; 1/4M)$

$\therefore f(z_0) - w = 0$, for some $z_0 \in D$ $\quad [B(0; 1/4M) \subset D]$

i.e., $w = f(z_0)$, for some $z_0 \in D$

NOTES

i.e., $w \in f(D)$

Hence, $B(0; 1/6M) \subset f(D)$.

Lemma 2: Suppose $g(z)$ is analytic on $B(0; R)$, $g(0) = 0$, $|g'(0)| = \mu > 0$ and $|g(z)| \le M$ for all z, then

$$g(B(0; R)) \supset B\left(0; \frac{R^2\mu^2}{6M}\right)$$

Proof: Let $f(z) = \dfrac{g(Rz)}{Rg'(0)}$ for $z \in D$ where $D = \{z: |z| < 1\}$.

Then, f is analytic on D, $f(0) = 0$, $f'(0) = 1$ and

$$|f(z)| = \left|\frac{g(Rz)}{Rg'(0)}\right| = \frac{|g(Rz)|}{R\mu} = \frac{M}{R\mu} \text{ for all } z \text{ in } D.$$

So by Lemma 1,

$$B\left(0; \frac{\mu R}{6M}\right) \subset f(D)$$

To show $B\left(0; \dfrac{R^2\mu^2}{6M}\right) \subset g(B(0; R))$, let $w \in B\left(0; \dfrac{R^2\mu^2}{6M}\right)$

Then $|w| < B\dfrac{R^2\mu^2}{6M}$

$$\Rightarrow \quad \left|\frac{w}{R\mu}\right| < \frac{R\mu}{6M}$$

$$\Rightarrow \quad \frac{w}{R\mu} \in B\left(0; \frac{\mu R}{6M}\right) \subset f(D)$$

$$\Rightarrow \quad \frac{w}{R\mu} = f(z) \text{ for some } z \in D$$

$$\Rightarrow \quad \frac{w}{R\mu} = \frac{g(Rz)}{Rg'(0)} \text{ where } |z| < 1$$

$$\Rightarrow \quad w = g(Rz) \text{ where } |Rz| < R$$

$$\Rightarrow \quad w \in g(B(0; R) \qquad [\because Rz \in B(0; R)]$$

Hence, the result is proved.

Lemma 3: Let f be an analytic function on the disc $B(0; r)$ such that $|f'(z) - f'(a)| < |f'(a)|$ for all z in $B(a; r)$, $z \ne a$, then f is one one.

Proof: Suppose z_1 and z_2 are points in $B(a; r)$ such that $z_1 \ne z_2$. Let γ be the line segment $[z_1, z_2]$ then

$$|f(z_1) - f(z_2)| = \left|\int_\gamma f'(z)dz\right|$$

$$\geq \left|\int_\gamma f'(a)dz\right| - \left|\int_\gamma [f'(z) - f'(a)]dz\right|$$

$$\left[\because \left|\int_\gamma f'(a)dz\right| \leq \left|\int_\gamma [f'(a) - f'(z)]dz\right| + \left|\int_\gamma f'(z)dz\right|\right]$$

$$\geq |f'(a)||z_1 - z_2| - \int_\gamma |f'(z) - f'(a)||dz|$$

$$> |f'(a)||z_1 - z_2| - |f'(a)||z_1 - z_2| = 0$$

$$\Rightarrow \quad f(z_1) \neq f(z_2)$$

Hence *f* is one one.

5.4.1 The Little Picard Theorem

Theorem 5.23 (Little Picard Theorem): An entire function $f: C \to C$, which omits two values, must be constant.

Corollary 1: A meromorphic function on *C* can omit atmost two values on $\widehat{\mathbb{C}}$.

The Little Picard theorem is equivalent to the assertion that there is no solution to the equation $e^f + e^g = 1$ where *f* and *g* are non constant entire functions. Similarly, it implies that there is no solution to Fermat's equation $f^n + g^n = 1,\ n \geq 3$, unless the entire functions *f* and *g* are constant.

Proof of the Little Picard Theorem: Suppose if $f: C \to C$ is nonconstant and omits 0 and 1. Then $f_n(z) = f_n^{1/n}(z)$ omits more and more points on the unit circle. We can rescale in the domain so that the spherical derivative satisfies $\| f_n'(0) \|_\infty \to \infty$. Passing to a subsequence and reparameterizing, we obtain in the limit a non constant entire function that omits the unit circle. This contradicts Liouville's theorem.

Classical Proof: This proof of the little Picard theorem is based on the fact that the universal cover of $C - \{0, 1\}$ can be identified with the upper half plane.

Consider the subgroup $\Gamma_0 \subset \text{Isom}(\Delta)$ generated by reflections in the sides of the ideal triangle *T* with vertices $\{1, i, -1\}$. Now let $\pi : T \to H$ be the Riemann mapping sending *T* to *H* and its vertices to $\{0, 1, \infty\}$. Developing in both the domain and range by Schwarz reflection, we obtain a covering map $\pi : \Delta \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

Given this fact, we lift an entire function $f : C \to C - \{0, 1\}$ to a map $\tilde{f} : \mathbb{C} \to \mathbb{H}$, which is constant by Liouville's theorem.

Uniformization of Planar Regions: Once we know that $\widehat{\mathbb{C}} - \{0, 1, \infty\}$ is uniformized by the disc, it is straight forward to prove.

Theorem 5.24: The universal cover of any region $U \subset \widehat{\mathbb{C}}$ with $|\widehat{\mathbb{C}} - U| \geq 3$ is isomorphic to the unit disc.

NOTES

Proof: Consider a base point p in the abstract universal cover $\pi : \widetilde{U} \to U$ and let F be the family of all holomorphic maps,

$$f : (\widetilde{U}, p) \to (\Delta, 0)$$

that are covering maps to their image. Using the uniformization of the triply-punctured sphere, we have that F is non empty. It is also a closed, normal family of functions in O ($\widetilde{U}$) and by the classical square-root trick, it contains a subjective function (which maximizes $|f'(p)|$). By the theory of covering spaces, this external map must be bijective.

5.4.2 Great Picard Theorem

Theorem 5.25 (Great Picard): An analytic function $f: U \to C$ takes on every value in C, with atmost one exception, in every neighborhood of an essential singularity p.

Proof of the Great Picard Theorem: Let $f : \Delta^* \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$ be an analytic function. We will show f does not have an essential singularity at $z = 0$.

Consider a loop γ around the puncture of the disc. If f sends to a contractible loop on the triply-punctured sphere, then f lifts to a map into the universal cover H, which implies by Riemann's removability theorem that f extends holomorphically over the origin.

Otherwise, by the Schwarz lemma, $f(\gamma)$ is a homotopy class that can be represented by an arbitrarily short loop. Thus, it corresponds to a puncture, which we can normalize to be $z = 0$; so again the singularity is not essential.

Lemma 1: If $f_n \to f$ and f is non constant, then any value omitted by all f_n is also omitted by f.

5.4.3 Bloch's Theorem

Theorem 5.26 (Bloch's Theorem): There exists a universal $R > 0$ such that for any $f : \Delta \to C$ with $|f'(0)| = 1$, not necessarily univalent, there is an open set $U \subset \Delta$ such that f maps U univalently to a ball of radius R.

Corollary 2: For any analytic map on Δ, the image $f(\Delta)$ contains a ball of radius $R\,|f'(0)|$.

Note that the ball usually cannot be centered at $f(0)$; for example, $f(z) = \exp(nz)/n$ satisfies $f'(0) = 1$ but the largest ball about $f(0) = 1/n$ in $f(\Delta)$ $C*$ has radius $1/n$.

The optimal value of R is known as Bloch's constant. It satisfies $0.433 < \sqrt{3}/4 \leq R < 0.473$. The best-known upper bound comes from the Riemann surface branched with order 2 over the vertices of the hexagonal lattice.

NOTES

Lemma 1: Let $f \in O(\overline{V})$ be non constant and satisfy $|f'|_v \le 2|f'(a)|$. Then $B_R(f(a)) \subset f(V)$, with $R := (3-2\sqrt{2}r)|f'(a)|$. $(3-2\sqrt{2} > \frac{1}{6})$

Proof: We may assume that, $a = f(a) = 0$. Set $A(z) = f(z) - f'(0)z$. Then, $A(z) = \int_{[0,z]}[f'(\zeta) - f'(0)]d\zeta$, whence $|A(z)| \le \int_0^1 |f'(zt) - f'(0)||z|\,dt$.

For $v \in V$, Cauchy's integral formula and standard estimates give,

$$f'(v) - f'(0) = \frac{v}{2\pi i}\int_{\partial V}\frac{f'(\zeta)d\zeta}{\zeta(\zeta - v)}, \qquad |f'(v) - f'(0)| \le \frac{|v|}{r - |v|}|f'|_v$$

It follows that,

$$|A(z)| \le \int_0^1 \frac{|zt||f'|_v}{r - |zt|}|z|\,dt \le \frac{1}{2}\frac{|z|^2}{r - |z|}|f'|_v \qquad \dots (5.25)$$

Now let $\rho \in (0, r)$. The inequality $|f(z) - f'(0)z| \ge |f'(0)|\rho - |f(z)|$ holds for z such that $|z| = \rho$. Since $|f'|_v \le 2|f'(0)|$, it follows from Equation (5.25) that,

$$|f(z)| \ge \left(\rho - \frac{\rho^2}{r - \rho}\right)|f'(0)|$$

Now $\rho - \rho^2/(r-\rho)$ assumes its maximum value, $(3-2\sqrt{2})r$, at $\rho^* := \left(1 - \frac{1}{2}\sqrt{2}\right)r \in (0, r)$.

It follows that, $|f(z)| \ge (3-2\sqrt{2})r|f'(0)| = R$ for all $|z| = \rho^*$.

Proof of Bloch's Theorem: Given f: $\Delta \to C$, let $\|f'(z)\| = \|f'(z)\|^{\Delta,\mathbb{C}} = (1/2)|f'(z)|(1-|z|^2)$ denote the norm of the derivative from the hyperbolic metric to the Euclidean metric. By assumption, $\|f'(0)\| = 1/2$. We can assume that f is smooth on S^1. Then $\|f'(z)\| \to 0$ as $|z| \to 1$, and thus sup $\||f'(z)\|$ is achieved at some $p \in \Delta$.

Now replace f with $f \notin r$ where $r \in$ Aut (Δ) moves p to zero. Replacing f with $af + b$ with $|a| < 1$, we can also arrange that $f(0) = 0$ and $\|f'(0)\| = 1$. This will only decrease the size of its unramified disc. Then $\|f'(z)\| \le \|f'(0)\| = 1$, thus $f|\Delta(1/2)|$ ranges in a compact family of non constant analytic functions. Thus, the new f has an unramified disc of definite radius, but then the old f does as well.

Theorem 5.27: Let $f_n : C \to C$ be a sequence of non constant entire functions. Thereafter passing to a subsequence, there is a sequence of Mobius transformations A_n and a non constant entire function g: $C \to C$ such that $g = \lim f_n \circ A_n$ uniformly on compact subsets of C.

Note here that the A_n need not fix infinity, so $f_n \circ A_n$ is unidentified at some point $p_n \in \hat{\mathbb{C}}$, but we will have $p_n \to \infty$.

For example, $f_n(z) = z^n$. We can take $f_n \circ A_n(z) = (1 + z/n)^n \to e^z$.

Metrics: As a preliminary to the proof, for $g\colon C \to \hat{C}$ we define

$$\| g'(z) \|_\infty = \frac{| g'(z) |}{(1 + | g(z) |^2)}$$

and $\|g'\|_\infty = \sup \|g'(z)\|$ over all $z \in C$. This is the norm of the derivative from the scaled Euclidean metric $\rho_\infty = 2\, |dz|$ to the spherical metric. Note that $g(z) = \exp(z)$ has $\|g'\|_\infty = 1/2$.

Similarly, for $g\colon \Delta(R) \to \hat{C}$, we define

$$\| g'(z) \|_R = | g'(z) | \frac{1 - | z / R |^2}{1 + | g(z) |^2}$$

This is the derivative from a suitably rescaled hyperbolic metric ρ_R on $\Delta(R)$ to $\hat{\mathbb{C}}$. Clearly, $\rho_R \to \rho_\infty$ uniformly on compact sets. Its key property is that, $\| (g \circ A)' \|_R = \| g' \|_R$ for all $A \in \mathrm{Aut}(\hat{\mathbb{C}})$ stabilizing $\Delta(R)$.

We also note that the set of maps with uniformly bounded derivatives in one of these norms is compact.

Theorem 5.28: Let us first consider an arbitrary non constant analytic function $f(z)$ and a radius $R > 0$. We claim there exists an $S \geq R$ and an $A \in \mathrm{Aut}(\hat{\mathbb{C}})$, such that $g = f \circ A$ is analytic on $\Delta(S)$, and

$$\| (g'(0) \|_S = \| g' \|_S = 1$$

Indeed, by replacing f with $f(az + b)$, we can assume $\| f'(0) \|_R = 1$. Then $\| f' \|_R \geq 1$. On the other hand, the R-norm of the derivative of f tends to zero at the boundary of $\Delta(R)$. Thus we can choose $B \in \mathrm{Aut}(\Delta(R))$, such that

$$M = \| (f \circ B)'(0) \|_R = \| (f \circ B)' \|_R \geq 1$$

Now just let $g(z) = (f \circ B)(z/M)$ and $S = RM$.

Applying this claim to f_n and a sequence $R_n \to \infty$, we obtain $S_n \to \infty$ and maps $g_n = f_n \circ A_n$ with $\|g'_n(0)\|_\infty = 1$ and $\|g'_n\|S_n \leq 1$. Now pass to a convergent subsequence.

5.4.4 Schottky's Theorem

Lemma 1: Let Ω be a connected open set in C. Suppose that there is a function $\lambda \in C^2(\Omega)$, such that

$$\lambda(z) > 0, \qquad z \in \Omega \qquad \Delta^c(\log \lambda) \geq \lambda \text{ on } \Omega$$

Let $f : D_R \to \Omega$ be a holomorphic map. Then,

$$|f'(z)|^2 \lambda(f(z)) \leq \theta_R(z) = 2R^2/(R^2 - |z|^2)^2, \qquad z \in D_R$$

NOTES

Proof: Let $0<r<R$. Consider the function u defined on D_r by,

$$u(z) = |f'(z)|^2 \lambda(f(z))/\theta_r(z), \quad z \in D_r$$

Clearly, $\theta_r(z) \to \infty$ as $|z| \to r$; hence $u(z) \to 0$ as $|z| \to r$. Hence, unless $u \equiv 0$—where the lemma is trivial—there is $a \in D_r$, such that

$$u(a) = \sup_{z \in D_r} u(z) > 0$$

Since, $u(a) > 0$, $\log u$ is C^2 in a neighbourhood of a, so that we have,

$$\Delta^c(\log u)(a) \leq 0$$

This implies,

$$\begin{aligned} 0 &\geq \Delta^c(\log u)(a) = \Delta^c \log \lambda(f(z))|_{z=a} - \Delta^c(\log \theta_r)(a) \\ &= |f'(a)|^2 \Delta^c(\log \lambda)(f(a)) - \theta_r(a) \\ &\geq |f'(a)|^2 \lambda(f(a)) - \theta_r(a) \end{aligned}$$

Since, $\Delta^c(\log \lambda) \geq \lambda$. Hence, $u(a) \leq 1$, so that $u(z) \leq 1$ for $z \in D_r$. Thus,

$$|f'(z)|^2 \lambda(f(z)) \leq 2r^2/(r^2 - |z|^2)^2 \qquad \text{for } |z| < r$$

Putting $r \to R$ proves the lemma.

Corollary 1: Let $f \in H(D), D = D_1$ and suppose that $f(D) \subset D$. Then, for $z \in D$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

Proof: Putting $\lambda(z) = 2/(1 - |z|^2)^2$, $\Omega = D$ in Lemma 1 gives,

$$|f'(z)|^2 \frac{2}{(1 - |f(z)|^2)^2} \leq \frac{2}{(1 - |z|^2)^2}$$

Lemma 2: Let $C_{0,1} = C - \{0\} - \{1\}$ be the complex plane with the points 0 and 1 removed. There exists a function $\lambda \in C^\infty(\mathbb{C}_{0,1})$ with the following properties:

(i) $\lambda(z) > 0$ for $z \in C_{0,1}$

(ii) $\Delta^c(\log \lambda) \geq \lambda$ on $C_{0,1}$

(iii) There exists a constant $c > 0$, such that for $z \in \mathbb{C}_{0,1}$

$$\lambda(z) \geq \frac{c}{(1 + |z|^2)(\log(2 + |z|^2))^2}$$

NOTES

Proof: Let us construct a C^∞ function ϕ on C such that $\phi(z) = |z|^2$ for $|z| \le 1/2$, $\phi(z)=1/2$ for $|z| \ge 3/4$ and $0 < \phi(z) < 1$ for $z \ne 0$; $\phi(0) = 0$.

Now, there exists a C^∞ function α on R such that $\alpha(t) = 1$ for $t \le 1/2$, $\alpha(t) = 0$ for $t \ge 3/4$ and $0 \le \alpha(t) \le 1$ for all $t \in R$.

Let $\beta(t) = \alpha(t).t^2 + (1-\alpha(t)).\dfrac{1}{2}$

We have,

$\beta(t) = t^2$ for $t \le 1/2$, $\beta(t) = 1/2$ for $t \ge 3/4$ and $0 \le \beta(t) \le 1$ for all $t > 0$.

We only have to set,

$\phi(z) = \beta(|z|)$

Similarly construct a function,

$\psi \in C^\infty(\mathbb{C})$ such that $\psi(z) = 2$ for $|z| \le 2$, $\psi(z) = |z|^2$ for $|z| \ge 3$ and $\psi(z) \ge 2$ for all $z \in \mathbb{C}$.

Now, let $0 < \delta < 1$ and put

$$u(z) = \mu_\delta(z) = |z|^{-2} (\log(\delta\phi(z)))^{-2}, \quad z \ne 0$$

Then $u_\delta(z) \to 0$ as $\delta \to 0$ uniformly on $|z| \ge \rho$ for any $\rho > 0$. Since $\Delta^c(\log|z|^2) = 0$ for $z \ne 0$, we have, for $0 < |z| < 1/2$,

$$\Delta^c(\log u)(z) = \frac{\partial^2}{\partial z \partial \bar{z}}\left(-2\log\left(\log\frac{1}{\delta\phi(z)}\right)\right) = \frac{\partial}{\partial z}\left(-\frac{2}{\bar{z}}\frac{1}{\log(\delta z\bar{z})}\right)$$

$$= \frac{2}{|z|^2 (\log(\delta|z|^2))^2} = 2u(z)$$

For $|z| \ge 1/2$, we have

$$\Delta^c(\log u)(z) = \frac{\partial^2}{\partial z \partial \bar{z}}\left(-2\log\left(\log\frac{1}{\delta\phi(z)}\right)\right)$$

$$= \frac{\partial}{\partial z}\left(-\frac{2}{\log(\delta\phi(z))}.\frac{1}{\phi(z)}\frac{\partial\phi}{\partial\bar{z}}\right)$$

$$= \frac{2}{(\log(\delta\phi(z)))^2}\frac{1}{(\phi(z))^2}\left|\frac{\partial\phi}{\partial z}\right|^2 - \frac{2}{\log(\delta\phi(z))}\frac{\partial}{\partial z}\left(\frac{1}{\phi(z)}\frac{\partial\phi}{\partial\bar{z}}\right)$$

Now, since $\phi(z) = 1/2$ for $|z| \ge 3/4$ therefore for $|z| \ge 3/4$, the above expression is equal to 0. Also, the above expression tends to 0 uniformly for $|z| \ge 1/2$ as $\delta \to 0$.

Similarly, set

$$v(z) = v_\delta(z) = \left(\log\frac{\psi(z)}{\delta}\right)^{-2}$$

Then $v(z) \to 0$ as $\delta \to 0$ uniformly in z and we have,

$$\Delta^c(\log v)(z) = \frac{2}{|z|^2}v(z) \quad \text{for } |z| \ge 3$$

NOTES

Whereas

$$\Delta^c(\log v)(z) = \frac{2}{\left(\log\frac{\psi(z)}{\delta}\right)^2}\frac{1}{(\psi(z))^2}\left|\frac{\partial\psi}{\partial z}\right|^2 - \frac{2}{\log\frac{\psi(z)}{\delta}}\frac{\partial}{\partial z}\left(\frac{1}{\psi(z)}\frac{\partial\psi}{\partial \bar{z}}\right)$$

for $|z| \leq 3$. Hence, $\Delta^c(\log v)(z) \to 0$ as $\delta \to 0$, uniformly for $|z| \leq 3$.

Let us now define,

$$\lambda(z) = \lambda_\delta(z) = (1+|z|^2)u(z)u(z-1)v(z), \quad z \neq 0, 1$$

We will now show that if δ is sufficiently small, then λ has the properties stated in the lemma.

Let $K_0 = \{z \in \mathbb{C} : |z| \leq 1/2\}$, $K_1 = \{z \in \mathbb{C} : |z-1| \leq 1/2\}$. Now $\Delta^c \log(1+|z|^2)$ $= (1+|z|^2)^{-2}$. Hence, if $z \in K_0$, we have

$$\Delta^c(\log\lambda)(z) = (1+|z|^2)^{-2} + \Delta^c(\log u)(z-1) + \Delta^c(\log v)(z) + 2u(z)$$

$(\Delta^c \log u = 2u$ on $K_0)$. Since $\Delta^c(\log u)(z-1)$ and $\Delta^c(\log v)(z) \to 0$ as $\delta \to 0$, uniformly on K_0 (and since $(1+|z|^2)^{-2} \geq 1/4$), we have

$$\Delta^c(\log\lambda)(z) > 2u(z), \qquad z \in K_0$$

if δ is sufficiently small. Since, moreover $(1+|z|^2)u(z-1)v(z) < 2$ for $z \in K_0$ and small enough δ, this will give

$$\Delta^c(\log\lambda)(z) > \lambda(z)$$

and

$$\Delta^c(\log\lambda) > \lambda \text{ on } K_1 \text{ if } \delta \text{ is sufficiently small.}$$

Now, let $|z| > 3$. Then,

$$\Delta^c(\log\lambda)(z) = \frac{1}{(1+|z|^2)^2} + \Delta^c(\log v)(z)$$
$$> \frac{2}{|z|^2}v(z)$$

We know that,

$$(1+|z|^2)u(z)u(z-1) = \frac{1+|z|^2}{|z|^2|z-1|^2}\cdot\frac{1}{(\log(\frac{1}{2}\delta))^2}, \quad |z| > 3$$

If δ is sufficiently small, this expression is less than $2/|z|^2$. Thus, $\Delta^c(\log\lambda) > \lambda$ on $|z| > 3$.

Now consider, $K = \{z \in \mathbb{C} : |z| \leq 3\} - K_0 - K_1$. We have, $(1+|z|^2)^{-2} \geq 1/100$ if $z \in K$. Since $u(z)$, $u(z-1)$, $v(z)$, $\Delta^c \log u(z)$, $\Delta^c \log u(z-1)$, $\Delta^c \log v(z)$, all $\to 0$ as $\delta \to 0$ uniformly for $z \in K$, we have

$$\Delta^c(\log\lambda)(z) = \frac{1}{(1+|z|^2)^2} + \Delta^c \log u(z) + \Delta^c \log u(z-1) + \Delta^c \log v(z)$$
$$> \frac{1}{200} > \lambda(z) \quad \text{if } \delta \text{ is sufficiently small}$$

This proves that there exists $\delta_0 > 0$, such that

$$\Delta^c(\log\lambda)(z) > \lambda(z), \qquad z \in \mathbb{C}_{0,1}, \qquad \delta \le \delta_0$$

So that λ satisfies (ii).

NOTES

Now, set the value of δ. Then there exists constants $c_1, c_2 > 0$ depending upon δ such that for $|z| \ge 3$, we have

$$\lambda(z) = \frac{1+|z|^2}{|z|^2|z-1|^2}\frac{1}{\left(\log\left(\frac{1}{2}\delta\right)\right)^2}\frac{1}{\left(\log\frac{|z|^2}{\delta}\right)^2}$$

$$> \frac{c_1}{|z|^2(\log|z|^2)^2} > \frac{c_2}{(1+|z|^2)(\log(2+|z|^2))^2}$$

Since, clearly $u(z) \to \infty$ as $z \to 0$ for fixed δ, and $u(z-1) \to \infty$ as $z \to 1$, while $u(1) \ne 0$, $u(-1) \ne 0$, $v(0) = v(1) \ne 0$, it follows that $\lambda(z) \to \infty$ as $z \to 0$ and as $z \to 1$. This along with the inequality proved above for $|z| > 3$ proves that λ satisfies (iii).

Theorem 5.29 (Landau's): There is a function $\alpha \to R(\alpha)$ from C into the positive real numbers with the following property:

If $f \in H(D_R)$ and $f(0) = \alpha$, $f'(0) = 1$ and if $R > R(\alpha)$ then f assumes one of the values 0, 1 on D_R.

Proof: Let $f \in H(D_R)$, $f(z) = \alpha + z + \ldots$ and suppose $f \ne 0, 1$ on D_R. Then $f(D_R) \subset \mathbb{C}_{0,1}$. Hence, if λ is the function constructed in Lemma 2 then Lemma 1 gives,

$$|f'(z)|^2\lambda(f(z)) \le 2R^2(R^2 - |z|^2)^{-2}, \qquad |z| < R$$

For $z = 0$, we get

$$\lambda(\alpha) \le 2R^2.R^{-4}, \text{ i.e., } R \le (2/\lambda(\alpha))^{1/2}$$

Theorem 5.30 (Schottky's): Let $R > 0$ and $C > 0$. Let F be the family of functions $f \in H(D_R)$ which do not assume the values 0 or 1 and $F_C = \{f \in F : |f(0)| \le C\}$. Then for $0 < r < R$, there exists a constant M depending only on r, R and C, such that for all $f \in F_C$, $|f(z)| \le M$ for $|z| \le r$.

Proof: Let λ be the function constructed in Lemma 2. By Lemma 1, we have

$$|f'(z)|^2\lambda(f(z)) \le 2R^2(R^2 - |z|^2)^{-2} \text{ for } z \in D_R,\ f \in F$$

Hence, if $r_1 < R$, we have

$$|f'(z)|^2\lambda(f(z)) \le M_0 \text{ for } |z| \le r_1, f \in F$$

where M_0 depends only on R and r_1. Now, from

$$\lambda(w) \ge \frac{c}{(1+|w|^2)(\log 2 + |w|^2))^2}$$

we get,

$$|f'(z)| \leq M_1(2+|f(z)|^2)^{1/2}\log(2+|f(z)|^2),\ |z| \leq r_1,\ f \in F$$

where $M_1 = M_0/\sqrt{c}$. Let θ be fixed in the interval $0 \leq \theta \leq 2\pi$ and define,

$u(r) = \log(2+|f(re^{i\theta})|^2)$ for $0 \leq r \leq r_1$

We have,

$$\frac{du}{dr} = \frac{1}{2+|f(re^{i\theta})|^2}.\frac{d}{dr}(f(re^{i\theta})\overline{f(re^{i\theta})})$$

Now,

$$\left|\frac{d(f(re^{i\theta})\overline{f(re^{i\theta})})}{dr}\right| = |e^{i\theta}f'(re^{i\theta})\overline{f(re^{i\theta})} + e^{-i\theta}f(re^{i\theta})\overline{f'(re^{i\theta})}|$$

$$\leq 2|f(re^{i\theta})||f'(re^{i\theta})|$$

$$\leq 2M_1|f(re^{i\theta})|(2+|f(re^{i\theta})|^2)^{1/2}\log(2+|f(re^{i\theta})|^2)$$

$$\leq 2M_1(2+|f(re^{i\theta})|^2)\log(2+|f(re^{i\theta})|^2)$$

Therefore,

$$\left|\frac{du}{dr}\right| \leq 2M_1\log(2+|f(re^{i\theta})|^2) = 2M_1u(r)$$

This implies that,

$$\frac{d\log u(r)}{dr} \leq 2M_1,\quad \log u(r) \leq \log u(0) + 2M_1 r$$

so that $u(r) \leq u(0)e^{2M_1 r}$. Hence, $\log(2+|f(re^{i\theta})|^2) \leq \log(2+C^2)e^{2M_1 R}$ for $f \in F_C$ and $r \leq r_1$. Thus, the theorem is proved.

5.4.5 Montel Caratheodory Theorem

Theorem (Montel Caratheodary) 5.31: If $\Im \subset O(\Omega, \hat{\mathbb{C}})$ omits three values, then $\Im$ is normal.

Proof: Without loss of generality, we may assume that $\Omega = D$, and that $\Im$ omits 0, 1, and ∞. In particular $\Im \subset O(\mathbb{D})$ and each $f \in \Im$ admits a holomorphic n-th root for any $n \in N$. Let us collect all 2^n-th roots of elements of $\Im$ and form the family,

$$\Im_n = \{g \in O(\mathbb{D}) : g^{2n} = f \text{ pointwise for some } f \in \Im\}$$

It is obvious that $\Im_n$ omits 0, ∞ and 2^n-th roots of unity. Expecting a contradiction, suppose that $\Im$ is not normal. Then $\Im_n$ is not normal, because convergence of a sequence implies convergence of the sequence composed of 2^n-th power of the elements from the original sequence. Let $g_n \in O(\mathbb{C}, \hat{\mathbb{C}})$ be the limit function from Zalcman's lemma applied to $\Im_n$. We have $\| g_n^{\#} \| \leq g_n^{\#}(0) = 1$, and g_n omits o, ∞, and the 2^n-th roots of unity. In particular, $\{g_n\}$ is normal by Marty's theorem, and pasing to a subsequence, the limit function $g = \lim g_n \in O(\mathbb{C}, \hat{\mathbb{C}})$ omits $0, \infty$, and the 2^n-th roots of unity for all n. Moreover, g is not constant since $g^{\#}(0) = 1$. It follows from the open mapping theorem that g omits $\partial\mathbb{D}$, hence either $g(\mathbb{C}) \subset \mathbb{D}$ or $g(\mathbb{C}) \subset \mathbb{C} \setminus \mathbb{D}$. Finally, Liouville's theorem applied to either g or $1/g$ implies that g is constant, reaching a contradiction.

NOTES

Theorem 5.32: If $f \in O(\mathbb{D}^{\times})$ omits 2 values in C, then f extends to $f \in O(\mathbb{D}, \hat{\mathbb{C}})$.

Proof: With a positive sequence $\varepsilon_n \to 0$, let $g_n(z) = f(\varepsilon_n z)$ for $z \in \mathbb{D}^{\times}$. Then $\{g_n\}$ omits 3 values in $\hat{\mathbb{C}}$, so it is normal. Passing to a subsequence, let $g = \lim g_n \in O(\mathbb{D}^{\times}, \hat{\mathbb{C}})$. So either g is a meromorphic function on $\mathbb{D}^{\times}$, or $g \equiv \infty$.

If $g \not\equiv \infty$, then there is a circle ∂D_r that does not pass through any pole of g, i.e., such that $\| g \|^{\partial D_r} < M$ for some $M > 0$. This means that $\| g_n \|^{\partial D_r} < M$ for all large n, or in other words, that $|f(z)| < M$ for $|z| = \varepsilon_n r$ for all large n. By the maximum principle, $|f| < M$ on the annulus of inner and outer radii $\varepsilon_{n+1} r$ and $\varepsilon_n r$ respectively, and this is true for all large n; hence 0 is a removable singularity of f. For the case $g \equiv \infty$, we have $|f(z)| \to \infty$ as $z \to 0$, so 0 is a pole of f.

5.5 UNIVALENT FUNCTIONS

Let $D \subset C$ be a domain, i.e., an open and connected non empty subset of the complex plane. We know that a function $f: D \to C$ is analytic at z_0 if it is complex differentiable at every point in some neighborhood of $z_0 \in D$. We say that f is analytic on D if f is analytic at z_0 for every $z_0 \in D$.

Definition 1: A function $f: D \to C$ is called univalent on D if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$.

Note: It follows from Rouche's theorem that if f is analytic on D, then $f'(z_0) \neq 0$ if and only if f is locally univalent at z_0, i.e., if f is univalent in some neighbourhood of z_0.

Definition 2: A function $f: D \to C$ which is both analytic on D and univalent on D is called conformal on D. Such an f is often referred as a conformal mapping of D.

It is important to remember that the underlying domain is an integral part of the definition of a univalent function (or a conformal mapping). Suppose that

$$D = \{z \in \mathbb{C} : 0 < |z| < 1, \operatorname{Im}\{z\} > 0, \operatorname{Re}\{z\} >\} = \{z \in \mathbb{C} : 0 < |z| < 1, 0 < \arg z < \pi/2\}$$

which is that part of the unit disc in the first quadrant. The function $f(z) = z^2$ then maps D conformally onto $D \cap H = \{z \in \mathbb{C} : 0 < |z| < 1, \operatorname{Im}\{z\} > 0\}$. That is, $f: D \to D \cap H$ is analytic and univalent on D, and onto. However, the function $g(z) = z^2$ does not map D conformally onto the unit disc D, although $g(D) = D$. While $g: D \to D$ is analytic, it is not univalent. For instance, $g(1/2) = g(-1/2) = 1/4$. In fact, $g'(0) = 0$ meaning that there is no neighbourhood of 0 in which g is univalent.

Again note that an analytic function may be locally univalent throughout a domain although it need not be univalent in that domain. Consider the domain,

$$D = \{z \in \mathbb{C} : 1 < |z| < 2,\ 0 < \arg z < 3\pi/2\}$$

and the function $f : D \to C$ given by $f(z) = z^2$. It is clear that f is analytic on D and locally univalent at every $z_0 \in D$ since $f'(z_0) = 2z_0 \neq 0$ for all $z_0 \in D$.
However, f is not univalent on D since,

$$f\left(\frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}i\right) = f\left(-\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}}i\right) = \frac{9}{4}i.$$

Suppose that**,**

$$D = \{z \in \mathbb{C} : (\text{Re}\{z\} - 1)^2 + \text{Im}\{z\}^2 < 1/4\} = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1/4\}$$

Let $f : D \to C$ be given by $f(z) = z^2$. We will show that f is univalent on D.

Recall from the Riemann mapping theorem that any simply connected proper subset of the complex plane is conformally equivalent to the unit disc. That is, if $D \subsetneq \mathbb{C}$ is simply connected and $z_0 \in D$, then there exists a unique conformal transformation f: $D \to D$ with $f(0) = z_0$ and $f'(0) > 0$. Therefore, statements about univalent functions on arbitrary simply connected domains can be translated to statements about univalent functions on the unit disc.

Notation: Let S denote the set of analytic, univalent functions on the unit disc D normalized by the conditions that $f(0) = 0$ and $f'(0) = 1$. That is,

$$S = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic and univalent on } \mathbb{D}, f(0) = 0 \text{ and } f'(0) = 1\}.$$

It follows that every $f \in S$ has a Taylor expansion of the form,

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots,\ |z| < 1,$$

where $a_n \in C$, $n = 2, 3, \ldots$. We will often be setting $a_1 = 1$ for $f \in S$.

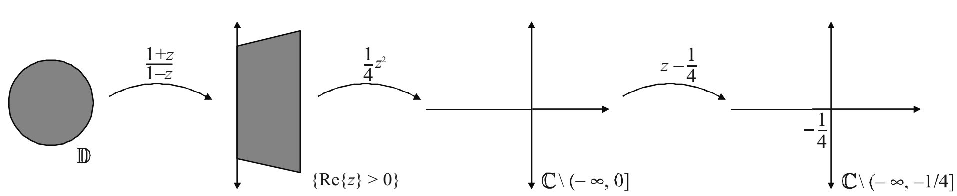
The most important member of S is the Koebe function, which is given by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$$

and maps the unit disc to the complement of the ray $(-\infty, -1/4]$. This can be verified by writing,

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

and noting that $\frac{1+z}{1-z}$ maps the unit disc conformally onto the right half-plane {Re{z}>0}.



Many of the results that we will discuss give bounds for function in S which will be attained only by the Koebe function.

Other examples of functions belonging to S include

NOTES

(i) The identity map, $f(z) = z$.

(ii) $f(z) = \dfrac{z}{1-z} = z + z^2 + z^3 + \ldots$ which maps D onto the half-plane $\{\text{Re}\{z\} > -1/2\}$.

(iii) $f(z) = \dfrac{z}{1-z^2} = z + z^3 + z^5 + \ldots$ which maps D onto the plane minus the two half-lines $[1/2, \infty)$ and $(-\infty, -1/2]$.

(iv) $f(z) = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$ which maps D onto the horizontal strip $\{-\pi/4 < \text{Im}\{z\} < \pi/4\}$.

The following example shows that $f \in S$ and $g \in S$ need not imply that $f + g \in S$ so that S is not closed under addition.

Here, let $f(z) = \dfrac{z}{1-z}$ and $g(z) = \dfrac{z}{1+iz}$

So that $f, g \in S$. However,

$$f'(z) = \frac{1}{(1-z)^2}$$

and

$$g'(z) = \frac{1}{(1+iz)^2}$$

So that,

$$f'(z) + g'(z) = \frac{2-2(1-i)z}{(1-z)^2(1+iz)^2}$$

From which we conclude that $f'(z) + g'(z) = 0$ if,

$$z = \frac{1}{1-i} = \frac{1+i}{2}.$$

This shows that, $f + g \notin S$.

Theorem 5.33: The class S is preserved under the following transformations:

(i) (Rotation) If $f \in S$, $\theta \in R$, and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in S$.

(ii) (Dilation) If $f \in S$, $0 < r < 1$, and $g(z) = \dfrac{1}{r} f(rz)$, then $g \in S$. .

(iii) (Conjugation) If $f \in S$ and $g(z) = \overline{f(\overline{z})}$, then $g \in S$.

Proof: In order to prove that S is preserved under rotation, dilation and conjugation, we consider that the composition of one-to-one mapping is again a one-to-one mapping.

(i) Suppose that $f \in S$. Let $R(z) = e^{i\theta}z$ and $T(z) = e^{-i\theta}z$ so that $R: C \to C$ and $T: C \to C$ are clearly one-to-one. Since $g(z) = e^{-i\theta} f(e^{i\theta}z) = (T \circ f \circ R)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on D. Since,

$$g'(z) = e^{-i\theta} . e^{i\theta} . e^{i\theta} . f'(e^{i\theta} z) = f'(e^{i\theta} z)$$

we see that g is analytic on D. Furthermore, $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$ so that $g \in S$ as required. Note that the Taylor expansion of g is given by,

$$g(z) = e^{-i\theta} (e^{i\theta} z + a_2 e^{2i\theta} z^2 + a_3 e^{3i\theta} z^3 + \cdots) = z + a_2 e^{i\theta} z^2 + a_3 e^{2i\theta} z^3 + \cdots.$$

(ii) Suppose that f Î S and let $0 < r < 1$. Let $R(z) = rz$ and $T(z) = z/r$ so that $R: C \to C$ and $T: C \to C$ are clearly one-to-one. Since $g(z) = \frac{1}{r} f(rz) = (T \circ f \circ R)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on D. Since,

$$g'(z) = \frac{1}{r} \cdot r \cdot f'(rz) = f'(rz)$$

we notice that g is analytic on D. Additionally, $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$ so that $g \in S$ as required.

We also note that the Taylor expansion of g is given by,

$$g(z) = \frac{1}{r}(rz + a_2 r^2 z^2 + a_3 r^3 z^3 + \cdots) = z + a_2 rz^2 + a_3 r^2 z^3 + \cdots.$$

(iii) Suppose that $f \in S$. Let $w(z) = \bar{z}$ so that $w : C \to C$ is clearly one-to-one. Since $g(z) = \overline{f(\bar{z})} = (w \circ f \circ w)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on D. Note that $w(z)$ is not analytic on D and so we cannot simply use the fact that a composition of analytic functions is analytic. Instead, we note that the Taylor series for f, namely

$$z + \sum_{n=2}^{\infty} a_n z^n \qquad \dots (5.26)$$

has radius of convergence. That is, the Taylor series Equation (5.26) converges to $f(z)$ for all $|z| < 1$ with the convergence uniform on every closed disc $|z| \le r < 1$. It then follows that the Taylor series,

$$z + \sum_{n=2}^{\infty} \bar{a}_n z^n \qquad \dots (5.27)$$

has radius of convergence 1 and so Equation (5.27) defines an analytic function on D. Hence, we conclude that

NOTES

$$g(z) = \overline{f(\overline{z})} = \overline{\overline{z} + a_2\overline{z}^2 + a_3\overline{z}^3 + \cdots} = z + \overline{a}_2 z^2 + \overline{a}_3 z^3 + \cdots$$

is analytic on D with $g(0) = 0$ and $g'(0) = 1$. Thus, $g \in S$.

Hence proved.

Theorem 5.34: The class S is preserved under the following transformations,

(i) (Disc automorphism) If $f \in S$ and,

$$g(z) = \frac{f\left(\dfrac{z+z_0}{1+z_0 z}\right) - f(z_0)}{\left(1-|z_0|^2\right) f'(z_0)}$$

for any $|z_0| < 1$, then $g \in S$.

(ii) (Range transformation) If $f \in S$, $\phi : f(D) \to C$ is analytic and univalent on $f(D)$, and

$$g(z) = \frac{(\phi \circ f)(z) - \phi(0)}{\phi'(0)}$$

then $g \in S$.

(iii) (Omitted-value transformation) If $f \in S$ with $f(z) \neq w$ and,

$$g(z) = \frac{wf(z)}{w - f(z)},$$

then $g \in S$.

Proof: In order to prove that S is preserved under disc automorphism, range transformation and omitted value transformation, we again note as in the proof of Theorem 5.34 that the composition of one-to-one mappings is a one-to-one mapping.

(i) Suppose that $f \in S$ and let $w(z) = \dfrac{z+z_0}{1+z_0 z}$ be the mobius transformation which maps the unit disc D conformally onto itself with $w(0) = z_0$. Since $z_0 \in D$, we conclude that

$$g(z) = \frac{f(w(z)) - f(z_0)}{(1-|z_0|^2) f'(z_0)}$$

is univalent on D with $g(0) = 0$. Furthermore,

$$g'(z) = \frac{w'(z) f'(w(z))}{(1-|z_0|^2) f'(z_0)} = \frac{f'(w(z))}{(1-\overline{z}_0 z)^2 f'(z_0)}$$

so that g is analytic on D with $g'(0) = 1$. Thus, $g \in S$ as required.

(ii) Suppose that $f \in S$ and let $\phi : f(D) \to C$ be analytic and univalent on $f(D)$. If

$$g(z) = \frac{(\phi \circ f(z) - \phi(0)}{\phi'(0)}$$

then g is clearly univalent on D with $g(0) = 0$. Furthermore,

$$g'(z) = \frac{f'(z)\phi'(f(z))}{\phi'(0)}$$

so that g is analytic on D with $g'(0) = 1$. Thus, $g \in S$ as required.

(iii) Suppose that $f \in S$ with $f(z) \neq w$ and let

$$g(z) = \frac{wf(z)}{w - f(z)}$$

If $T(\zeta) = \dfrac{w\zeta}{w - \zeta}$ which is clearly one-to-one if $\zeta \neq w$, then it follows that $g(z) = (T \circ f)(z)$ is univalent on D. Furthermore,

$$g'(z) = \frac{w^2 f'(z)}{(w - f(z))^2}$$

and since $w \neq f(z)$ it follows that g is analytic on D with $g'(0) = 1$. Thus, $g \in S$ as required.

Lemma 1: If f is analytic on D with $0 \neq f(D)$, then there exists an analytic function h on D with $h^2 = f$.

Proof: Let $g(0)$ be any complex number with $\exp\{g(0)\} = f(0)$. For any other $w \in D$, let

$$g(w) = g(0) + \int_\gamma \frac{f'(z)}{f(z)} dz$$

where γ: $[0, 1] \to D$ is any C^1 curve from 0 to w. From the fundamental theorem of calculus, it follows that

$$g'(w) = \frac{f'(w)}{f(w)} \qquad \dots (5.28)$$

Note that $f(z) \neq 0$ for $z \in D$ so that $g'(z)$ is well-defined for all $z \in D$ implying that g is analytic on D. It now follows from Equation (5.28) that,

$$[fe^{-g}]'(w) = f'(w)e^{-g(w)} - g'(w)e^{-g(w)}f(w) = e^{-g(w)}[f'(w) - g'(w)f(w)] = 0$$

The equation $[fe^{-g}]'(w) = 0$ implies that $f(w) = e^{g(w)}$. Hence, the proof is completed by taking $h(z) = \exp\left\{\dfrac{g(z)}{2}\right\}$ so that h is analytic on D with $h^2(z) = f(z)$ for every $z \in D$.

NOTES

Note: The analytic function g in the previous lemma is unique upto translation by integral multiples of $2\pi i$. In fact,

$g(z) = \log|f(z)| + i \arg f(z)$

Lemma 2: If $f \in S$, then there exists an odd function $h \in S$ such that $h^2(z) = f(z^2)$ for every $z \in D$.

Proof: If $f \in S$, then $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ so that $\frac{f(z)}{z} = 1 + a_2 z + a_3 z^2 + ...$ is a non zero, analytic function on D. By Lemma 1 there exists an analytic function F on D with,

$$F^2(z) = \frac{f(z)}{z}$$

If we define $h(z) = zF(z^2)$, then it is clear that h is odd with $h^2(z) = f(z^2)$, $h(0) = 0$ and $h'(0) = F(0) = 1$. Let $z_1, z_2 \in D$ and suppose that $h(z_1) = h(z_2)$. The univalent of f implies that $z_1^2 = z_2^2$. Therefore, it must be the case that either $z_1 = z_2$ or $z_1 = -z_2$, then this implies that $h(z_1) = -h(-z_1) = -h(z_2)$ since h is odd. However, this contradicts the assumption that $h(z_1) = h(z_2)$, and so we conclude that $z_1 = z_2$. This shows that $h \in S$, thus completing the proof.

Theorem 5.35: The class S is preserved under the square root transformation. That is, if $f \in S$ and $g(z) = \sqrt{f(z^2)}$, then $g \in S$.

Proof: Suppose that $f \in S$ and $g(z) = \sqrt{f(z^2)}$. Since $f(z) = 0$ only when $z = 0$, it is possible to choose a single-valued branch of the square root by writing,

$g(z) = \sqrt{f(z^2)} = z(1 + a_2 z^2 + a_3 z^4 + a_4 z^6 + ...)^{1/2}) = z + b_3 z^3 + b_5 z^5 + ...$

for $|z| < 1$ for some coefficients $b_n \in C$. It now follows from Lemma 2 that g is univalent on D and that g is analytic on D with $g(0) = 0$ and $g'(0) = 1$. That is, $g \in S$ as required.

5.5.1 Bieberbach's Conjecture

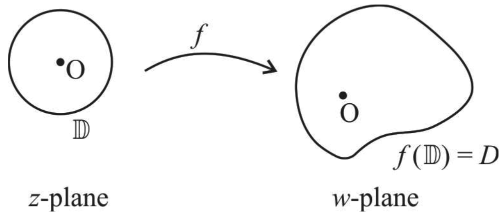
Theorem 5.36 (Area Theorem): If $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal mapping of $\mathbb{D}$ with $f(0)=0$ and $f'(0) > 0$ so that f has a Taylor expansion,

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + ..., \quad |z| < 1 \qquad ...(5.29)$$

with $a_1 \in \mathbb{R}, \quad a_1 > C$ then,

$$\text{Area } (f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n |a_n|^2$$

Proof: Write $D = f(\mathbb{D})$ as shown in the Figure.



NOTES

From Green's theorem,

$$\text{Area}\,(D) = \frac{1}{2i}\int_{\partial D} \overline{w}\, dw$$

that upon changing variables gives,

$$\frac{1}{2i}\int_{\partial D} \overline{w}\, dw = \frac{1}{2i}\int_{\partial D} \overline{f(z)}\, f'(z)\, dz$$

Substituting this in Equation (5.29) and changing to polar coordinates, we get

$$\begin{aligned}\text{Area}\,(D) &= \frac{1}{2i}\int_{\partial D}\left(\sum_{n=1}^{\infty}\overline{a}_n \overline{z}^n\right)\left(\sum_{m=1}^{\infty} m a_m z^{m-1}\right)dz \\ &= \frac{1}{2i}\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\overline{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{i(m-1)\theta}\right) ie^{i\theta}\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\overline{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{im\theta}\right) d\theta\end{aligned}$$

as $\partial\mathbb{D} = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$.

Now, expanding the product of sums, we get

$$\begin{aligned}\text{Area}\,(D) = \frac{1}{2}\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\overline{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{im\theta}\right) d\theta &= \frac{1}{2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty} k\overline{a}_k a_k\right) d\theta \\ &= \left(\sum_{k=1}^{\infty} k\,|a_k|^2\right)\frac{1}{2}\int_0^{2\pi} d\theta \\ &= \pi\sum_{k=1}^{\infty} k\,|a_k|^2\end{aligned}$$

The off-diagonal terms integrate to 0 over the range $0 \le \theta \le 2\pi$. Suppose that $f \in S$ so that $f : \mathbb{D} \to f(\mathbb{D})$ is analytic and univalent on $\mathbb{D}$ with $f(0)=0$ and $f'(0) = 1$. We say that f is bounded if $\text{Area}\,(f(\mathbb{D})) < \infty$. Theorem 5.36 gave the following characterization of $\text{Area}\,(f(\mathbb{D}))$:

$$\text{Area}\,(f(\mathbb{D})) = \pi\sum_{n=1}^{\infty} n\,|a_n|^2 \qquad \ldots(5.30)$$

From the change-of-variables theorem,

$$\text{Area}\,(f(\mathbb{D})) = \iint_{f(\mathbb{D})} dx\, dy = \iint_{\mathbb{D}} |f'(z)|^2\, dx\, dy$$

The expression,

$$\iint_D |f'(z)|^2\, dx\, dy$$

NOTES

is sometimes called the Dirichlet integral of f. In other words, the function $f \in S$ is bounded if and only if it has a finite Dirichlet integral. Equation (5.30) implies that the coefficients a_n of a bounded function in S are $o(n^{-1/2})$. So, if $f \in S$ is bounded then,

$$\lim_{n\to\infty} \frac{|a_n|}{n^{-1/2}} = \lim_{n\to\infty} |a_n| \sqrt{n} = 0$$

We have followed the convention that $a_1 = 1$ for $f \in S$.

Corollary 1: If $f \in S$ then,

$$\iint_{\mathbb{D}} |f'(z)|^2 \, dx\, dy = \pi \sum_{n=1}^{\infty} n |a_n|^2$$

If $f \in S$ is bounded so that

$$\iint_{\mathbb{D}} |f'(z)|^2 \, dx\, dy = \pi \sum_{n=1}^{\infty} n |a_n|^2 < \infty$$

then, $a_n = o(n^{-1/2})$

Theorem 5.37: There exists an absolute constant $\delta > 0$ such that, $a_n = O(n^{-1/2-\delta})$ for every bounded function $f \in S$.

Now we will introduce another area theorem for functions which are analytic and univalent on the domain $\mathbb{U} = \{z : |z| > 1\}$ except for a simple pole at infinity with residue 1. Such functions have a Laurent expansion of the form,

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + ... = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \qquad |z| > 1$$

where $b_n \in \mathbb{C}$. We write $\mathcal{L}$ for denoting the set of such functions. Here, also note that if $g \in \mathcal{L}$, then g maps $\mathbb{U}$ onto the complement of a compact and connected set E. Additionally, if $f \in S$ and g is defined by inversion as,

$$g(z) = \frac{1}{f(1/z)}$$

then,

$$g(z) = z - a_2 + (a_2^2 - a_3)z^{-1} + ..., |z| > 1 \qquad ...(5.31)$$

In fact, inversion establishes a one-to-one correspondence between S and the subclass $\mathcal{L}'$ of $\mathcal{L}$ for which $0 \in E$, i.e., for which $g(z) \neq 0$ in U. Univalence places a strong restriction on the size of the coefficients of $g \in \mathcal{L}$.

Theorem 5.38 (Area Theorem): If $g \in \mathcal{L}$ so that g has a Laurent expansion,

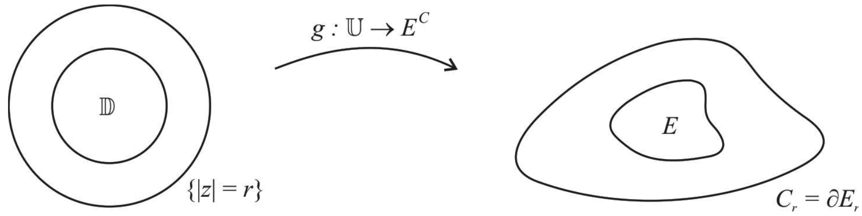
$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \ |z| > 1 \qquad ...(5.32)$$

then,

$$\sum_{n=1}^{\infty} n |b_n|^2 \le 1$$

NOTES

Proof: Let $g \in \mathcal{L}$ and let E denote the compact, connected set of points which are omitted by g. Suppose that r>1 and let C_r be the image of the circle of radius r under g. Since g is a univalent function, it follows that C_r is a Jordan curve. Therefore, C_r encloses a domain E_r which contains E as shown in the Figure.



It now follows from Green's theorem and change-of-variables that,

$$\text{Area}\,(E_r) = \frac{1}{2i}\int_{C_r} \overline{w}\,dw = \frac{1}{2i}\int_{\{|z|=r\}} \overline{g(z)}\,g'(z)\,dz$$

Substituting in Equation (5.32) and changing to polar coordinates, we get

$$\text{Area}\,(E_r) = \frac{1}{2}\int_0^{2\pi}\left(re^{-i\theta} + \sum_{n=0}^{\infty}\overline{b}_n r^{-n} e^{in\theta}\right)\left(1 - \sum_{j=0}^{\infty} jb_j r^{-j-1} e^{-i(j+1)\theta}\right) re^{i\theta}\,d\theta$$

$$= \pi\left(r^2 - \sum_{n=1}^{\infty} n\,|\,b_n\,|^2 r^{-2n}\right)$$

Limit $r \to 1+$ implies,

$$\text{Area}\,(E) = \lim_{r\to 1+}\text{Area}\,(E_r) = \pi\left(1 - \sum_{n=1}^{\infty} n\,|\,b_n\,|^2\right)$$

Now, as Area(E) ≥ 0, so

$$\sum_{n=1}^{\infty} n\,|\,b_n\,|^2 \leq 1$$

as required.

Corollary 2: If $g \in \mathcal{L}$ then

$$|\,b_n\,| \leq n^{-1/2}, \qquad n = 1, 2, 3, ...$$

However, if $n \geq 2$, the upper bound given in this corollary is not sharp because

$$g(z) = z + n^{-1/2} z^{-n}$$

is not univalent on U. In fact,

$$g'(z) = 1 - n^{-1/2} z^{-n-1}$$

which equals zero at certain points in U if $n \geq 2$. But, it can be easily checked that the function,

$$g(z) = z + b_0 + \frac{b_1}{z} \qquad ...(5.33)$$

with $|b_1| = 1$ is univalent on U and so the upper bound is sharp for $n = 1$. In fact, the function g of Equation (5.33) maps $\mathbb{U}$ conformally onto the complement of a line segment of length 4.

Theorem 5.39 (Bieberbach) : If $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in S$, then $|a_2| \leq 2$.

Proof: Suppose that $f \in S$. Apply a square root transformation (Theorem 5.35 of univalent functions) and invert f to give,

NOTES

$$g(z) = [f(z^{-2})]^{-1/2} = z - \frac{a_2}{2} z^{-1} + ...$$

so that $g \in \mathcal{L}$. It therefore follows from the Corollary 2 that, $|b_1| = \left|\frac{a_2}{2}\right| \leq 1$ and so $|a_2| \leq 2$ as required.

Theorem 5.40 (Bieberbach Conjecture-de Brange's Theorem): If $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in S$, then $|a_n| \leq n$ for all $n \geq 2$.

5.5.2 Koebe's One Quarter(1/4)

In complex analysis, the Koebe one-quarter (1/4) theorem is named after Paul Koebe, who conjectured the result in 1907. The theorem was proven by Ludwig Bieberbach in 1916. The example of the Koebe function shows that the constant 1/4 in the theorem cannot be improved (increased).

Theorem 5.41 (Koebe One-Quarter Theorem): The range of every $f \in S$ contains the disc $\{w \in \mathbb{C} : \ |w| < 1/4\}$, i.e., $\text{dist}(0, \partial f(\mathbb{D})) \geq 1/4$.

Proof: Let $f \in S$, so that $f(z) = z + a_2 z^2 + a_3 z^3 + ..., \quad |z| < 1$ and note that $|a_2| \leq 2$ by Bieberbach's theorem. Suppose that $z_0 \notin f(D)$ and consider the omitted-value transformation,

$$g_{z_0}(z) = \frac{z_0 f(z)}{z_0 - f(z)} = z + \left(a_2 + \frac{1}{z_0}\right) z^2 + ...$$

From Theorem 5.34 of univalent functions that, $g_{z_0} \in S$ and so from Bieberbach's theorem we can conclude that,

$$\left| a_2 + \frac{1}{z_0} \right| \leq 2$$

Combined with the fact that $|a_2| \leq 2$, we can conclude that

$$\left| \frac{1}{z_0} \right| \leq 4 \ \text{ or } \ |z_0| \geq \frac{1}{4}$$

In other words, every omitted value of $f \in S$ lies outside the disc of radius 1/4 centered at the origin.

Note: Univalence is crucial to the Koebe one-quarter theorem. If

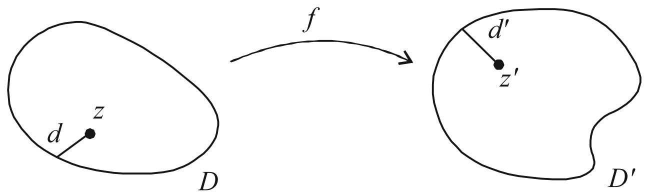
$$f_n(z) = \frac{1}{n}(e^{nz} - 1), \quad n = 1, 2, ...,$$

then $f_n(0) = 0$ and $f'_n(0) = 1$ (so that f_n is locally univalent at 0), but f_n omits the value $-1/n$.

Theorem 5.42: If $F: D \to D'$ is a conformal transformation with $F(z) = z'$, then

$$\frac{d'}{4d} \leq |F'(z)| \leq \frac{4d'}{d}$$

where $d = \text{dist}(z, \partial D)$ and $d' = \text{dist}(z', \partial D')$



Proof: Without loss of generality, suppose that $z = z' = 0$. In order to prove this theorem we will establish the following generalization of the Koebe one-quarter theorem. Let $F: D \to D'$ be a conformal transformation with $F(0) = 0$. It then follows that F has a Taylor expansion converging for all z in the disc of radius $d = \text{dist}(0, \partial D)$ centered 0 which is given by,

$$F(z) = A_1 z + A_2 z^2 + A_3 z^3 + ..., \ |z| < d.$$

The function f defined for $z \in D$ by,

$$f(z) = \frac{F(dz)}{dF'(0)} = \frac{F(dz)}{dA_1} = z + \frac{A_2 d}{A_1} z^2 + \frac{A_3 d^2}{A_1} z^3 + ... = z + a_2 z^2 + a_3 z^3 + ..., \ |z| < 1,$$

is therefore analytic and univalent on D with $f(0) = 0$ and $f'(0) = 1$, i.e., $f \in S$. Suppose that w is an omitted-value of F restricted to the disc of radius d centered at 0. That is, suppose that $w \notin F(dD)$ or, in other words, suppose that $F(z) \neq w$ for any $|z| < d$. This is equivalent to supporting that,

$$\frac{F(dz)}{dF'(0)} \neq \frac{w}{dF'(0)}$$

for any $|z| < 1$,or, that the function f omits the value $w_0 = w/(dF'(0))$. Consider the omitted-value transformation,

$$g(z) = \frac{w_0 f(z)}{w_0 - f(z)} = \frac{wf(z)}{w - dF'(0) f(z)} = z + \left(a_2 + \frac{dF'(0)}{w}\right) z^2 + ..., \ |z| < 1,$$

which belongs to S by Koebe one-quarter theorem. As in the proof of the Koebe one-quarter theorem, we know that Bieberbach's theorem implies that both,

$$|a_2| \leq 2 \quad \text{and} \quad \left| a_2 + \frac{dF'(0)}{w} \right| \leq 2.$$

Hence, we conclude that

$$\left| \frac{dF'(0)}{w} \right| \leq 4 \quad \text{and so} \quad |w| \geq \frac{d\,|F'(0)|}{4}.$$

Thus, if $F: D \to D'$ is a conformal transformation with $F(0) = 0$, then D' necessarily contains the disc of radius $d|F'(0)|/4$ centered at 0. If we write $d' = \text{dist}(0, \partial D')$, then

$$\frac{d\,|F'(0)|}{4} \leq d' \qquad ...(5.34)$$

and the upper bound follows. To derive the lower bound, we consider the inverse function $G(z) = F^{-1}(z)$ and note that G: D $\to D'$ is a conformal transformation of D' onto D. If we then define the function g for $z \in D$ by,

$$g(z) = \frac{G(d'z)}{d'G'(0)}, \qquad |z| < 1,$$

so that $g \in S$, we see that argument as above implies,

NOTES

$$\frac{d'|G'(0)|}{4} \leq d. \qquad \ldots(5.35)$$

Noting that $G'(0) = 1/F'(0)$ so that Equation (5.34) implies $d' \leq 4d|F'(0)|$, gives the lower bound and together with Equation (5.35) this proves the theorem.

Check Your Progress

6. State a necessary condition for an analytic function on the disc to be one-one.
7. State Bloch's theorem.
8. Write the statement of Schottky's theorem.
9. State Montel Caratheodory theorem.
10. Define conformal function.
11. State Koebe's one quarter theorem.

5.6 ANSWERS TO 'CHECK YOUR PROGRESS'

1. An entire function $f : \mathbb{C} \to \mathbb{C}$ is of finite order if there is a $\rho > 0$ such that,

   $|f(z)| = O\left(\exp|z|^{\rho}\right)$.

2. As per Poisson-Jensen formula, if $f(z)$ is meromorphic in $|z| \leq R$ and has zeros a_p and poles b_v, and if $\zeta = re^{i}$, $f(\zeta) \neq 0$, then for $0 \leq r \leq R$ we have

$$\log f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\log|(\mathrm{Re}^{i\phi})|(R^2 - r^2)dr}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

$$+ \sum_{\mu} \log\left|\frac{R(\zeta - a_\mu)}{R^2 - \bar{a}_\mu \zeta}\right| - \sum_{v} \log\left|\frac{R(\zeta - b_v)}{R^2 - \bar{b}_v \zeta}\right|$$

3. $f(z)$ is an entire function of finite order if there is a finite number such that,

   $M(r) < e^{r^{\mu}}, r > r_0$.

   The greatest lower bound ρ of the set of numbers μ satisfying this condition is called the order of the entire function $f(z)$.

4. As per Weierstrass theorem, Given a nonnegative integer λ and an increasing sequence of non zero complex numbers $\{\zeta_n\}$ converging to infinity, there exists an entire function $f(z)$ whose zeros coincide with the points, $\underbrace{0,\ldots,0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_n, \ldots$

NOTES

5. If f is a continuous functions on $[a, b]$, then for any $\varepsilon > 0$ the interval can always be divided into a finite number of sub-intervals in each of which the variation of $f(x)$ is less than ε.
6. Let f be an analytic function on the disc B $(0;\ r)$ such that $|f'(z) - f'(a)| < |f'(a)|$ for all z in B $(a; r)$, $z \neq a$, then f is one one.
7. There exists a universal $R > 0$ such that for any $f: \Delta \to C$ with $|f'(0)| = 1$, not necessarily univalent, there is an open set $U \subset \Delta$ such that f maps U univalently to a ball of radius R.
8. Let $R > 0$ and $C > 0$. Let F be the family of functions $f \in H(D_R)$ which do not assume the values 0 or 1 and $F_C = \{f \in F : |f(0)| \leq C\}$. Then for $0 < r < R$, there exists a constant M depending only on r, R and C, such that for all $f \in F_C$, $|f(z)| \leq M$ for $|z| \leq r$.
9. If $\Im \subset O(\Omega, \hat{\mathbb{C}})$ omits three values, then $\Im$ is normal.
10. A function f: $D \to C$ which is both analytic on D and univalent on D is called conformal on D. Such an f is often referred as a conformal mapping of D.
11. The range of every $f \in S$ contains the disc $\{w \in \mathbb{C} : \ |w| < 1/4\}$, i.e., $\text{dist}(0, \partial f(\mathbb{D})) \geq 1/4$.

5.7 SUMMARY

- It is an elegant expression for an entire function with a zero only at $z = 1$ which is also close to 1 for $|z| < 1$. It is known as the Weierstrass factor of order p:

$$E_p(z) = (1 - z)\exp\left(z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right)$$

- For any sequence of non zero complex numbers $a_n \to \infty$, the formula $f(z) = \Pi_1^{\infty} E_n(z / a_n)$ converges for all z and defines an entire analytic function with zero set exactly (a_n).
- It is defined as $N(r) = \left|\{n : |a_n| < r\}\right|$. $r^{-\beta} N(r)$ is a rough approximation to $\sum_{|a_n|<r} 1/|a_n|^{\beta}$.
- An entire function $f : \mathbb{C} \to \mathbb{C}$ is of finite order if there is a $\rho > 0$ such that $|f(z)| = O\left(\exp|z|^{\rho}\right)$.
- A function $\phi(s)$ of one real variable is convex if and only if $\phi(s) + ar$ satisfies the maximum principle for any constant a. This holds for log $M(\exp(s))$ by considering $f(z)z^a$ locally.
- Hadamard's factorization formula describes every entire function of finite order in terms of its zeros and an additional polynomial.

NOTES

- For controlling the result of division by a canonical product, we now estimate its minimum modulus.
- Many of the basic properties of the sine and cosine functions can be derived from the point of view of the uniqueness of an odd entire function with zeros at $z\pi$.
- Entire function is a function that is analytic in the whole complex plane except, possibly, at the point at ∞. It can be expanded in a power series, $f(z)=\sum_{k=0}^{\infty} a_k z^k, \quad a_k=\frac{f^{(k)}(0)}{k!}, \quad k \geq 0,$
- An entire function f is said to have finite order if there exists some real number $\rho > 0$, and a constant $C > 0$, such that $|f(z)| \leq Ce^{|z|^\rho}$ for all $z \in \mathbb{C}$.
- Given a nonnegative integer λ and an increasing sequence of non zero complex numbers $\{\zeta_n\}$ converging to infinity, there exists an entire function $f(z)$ whose zeros coincide with the points, $\underbrace{0,\ldots,0}_{\lambda \text{ times}}, \zeta_1,\ldots,\zeta_n,\ldots$
- Let $\{\zeta_n\}$ be an arbitrary increasing sequence of non zero complex numbers, which converges to infinity and consider the series α_0, $\sum_{n=1}^{\infty}\frac{1}{|\zeta_n|^\alpha}$
- If f is a continuous functions on $[a, b]$, then for any $\varepsilon > 0$ the interval can always be divided into a finite number of sub-intervals in each of which the variation of $f(x)$ is less than ε.
- An entire function f: $C \to C$, which omits two values, must be constant.
- There exists a universal $R > 0$ such that for any $f : \Delta \to C$ with $|f'(0)| = 1$, not necessarily univalent, there is an open set $U \subset \Delta$ such that f maps U univalently to a ball of radius R.
- The little Picard theorem is based on the fact that the universal cover of $C - \{0, 1\}$ can be identified with the upper half plane.
- If $\mathfrak{I} \subset O(\Omega, \hat{\mathbb{C}})$ omits three values, then $\mathfrak{I}$ is normal.
- A function f: $D \to C$ is called univalent on D if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$.
- A function f: $D \to C$ which is both analytic on D and univalent on D is called conformal on D. Such an f is often referred as a conformal mapping of D.
- If $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal mapping of $\mathbb{D}$ with $f(0)=0$ and $f'(0) > 0$ so that f has a Taylor expansion,
  $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots, \quad |z| < 1$
- If $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S$, then $|a_2| \leq 2$.
- The range of every $f \in S$ contains the disc $\{w \in \mathbb{C} : |w| < 1/4\}$, i.e., $\text{dist}(0, \partial f(\mathbb{D})) \geq 1/4$.

5.8 KEY TERMS

NOTES

- **Weierstrass factors:** It is an elegant expression for an entire function with a zero only at $z = 1$ which is also close to 1 for $|z| < 1$.
- **Counting function:** It is defined as $N(r) = \left|\{n : |a_n| < r\}\right|$. $r^{-\beta}N(r)$ is a rough approximation to $\sum_{|a_n|<r} 1/|a_n|^{\beta}$.
- **Counting zeros:** Here is another way to write Jensen's formula. Let $N(r)$ be the number of zeros of f inside the circle of radius r. Then, $\int_0^R N(r)\frac{dr}{r} = avg_{S^1(R)}(R)\log|f(z)| - \log|f(0)|$
- **Entire function:** It is a function that is analytic in the whole complex plane except, possibly, at the point at ∞. It can be expanded in a power series,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k = \frac{f^{(k)}(0)}{k!}, \qquad k \geq 0,$$

- **Little Picard:** An entire function f: $C \to C$, which omits two values, must be constant.

5.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define canonical products.
2. Give the utility of Poisson-Jensen formula.
3. When is entire function said to be of exponential type?
4. Define exponent of convergence.
5. State an application of Borel's theorem.
6. Define the range of an entire function.
7. State little Picard theorem and great Picard theorem.
8. Where is Schottky's theorem applied?
9. What do you understand by disc automorphism?
10. Define Koebe's one-quarter theorem.

Long-Answer Questions

1. Prove that every entire function, $f(z) = \sum_0^{\infty} a_k z^k$ of order 1 and infinite type must have infinitely many zeros.
2. Explain the Poisson-Jensen formula with the help of examples and illustrations.
3. Find the order of the functions $\sin z$ and $\cos z$.

NOTES

4. Describe exponent of convergence with the help of examples.
5. State and prove Borel's theorem.
6. Discuss the range of analytic function with the help of examples.
7. Prove the little Picard theorem by using Montel's theorem.
8. Prove great Picard theorem using little Picard theorem and Schottky's theorem.
9. State and prove Montel Caratheodory theorem.
10. Describe univalent functions with the help of examples and illustrations.
11. Prove Koebe's quarter theorem and illustrate its uses.

5.10 FURTHER READING

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd edition. London: McGraw-Hill Education – Europe.

Ahlfors, Lars V. 1978. *Complex Analysis*, 3rd edition. London: McGraw-Hill Education – Europe.

Lang, Serge. 1998. *Complex Analysis*, 4th edition. NY: Springer-Verlag New York Inc.

Shastri, Anant R. 2000. *An Introduction to Complex Analysis*, India: Macmillan Publishers India Ltd.