M.Sc. Previous Year
Mathematics, MM-03

TOPOLOGY

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल
MADHYA PRADESH BHOJ (OPEN) UNIVERSITY - BHOPAL

COURSE WRITERS

Dr. Pratiksha Saxena, Assistant Professor, School of Applied Sciences, Gautam Buddha University, Greater Noida
Units: (1.0-1.2, 1.3-1.3.1, 1.3.3, 1.4, 1.6-1.6.1, 1.7, 1.8, 1.9-1.12, 1.13-1.14, 1.15-1.20, 2.0-2.1, 2.2, 2.3-2.6, 2.7-2.12, 3.0-3.3, 3.4-3.8, 3.9, 3.10, 3.11-3.11.1, 3.12, 3.13-3.17, 4.0-4.2, 4.3, 4.4-4.8, 4.9, 4.10-4.16, 5)

Dr. Harendra Singh, Assistant Professor & Head Department of Mathematics, Post Graduate College, Ghazipur (U.P.)
Units: (1.3.2, 1.5, 1.6.2, 3.11.2)

Published by Registrar, MP Bhoj (Open) University, Bhopal in 2020

VIKAS®

Vikas® is the registered trademark of Vikas® Publishing House Pvt. Ltd.

VIKAS® PUBLISHING HOUSE PVT. LTD.
E-28, Sector-8, Noida - 201301 (UP)
Phone: 0120-4078900 • Fax: 0120-4078999
Regd. Office: A-27, 2nd Floor, Mohan Co-operative Industrial Estate, New Delhi 1100 44
• Website: www.vikaspublishing.com • Email: helpline@vikaspublishing.com

SYLLABI-BOOK MAPPING TABLE

Topology

Syllabi	Mapping in Book
UNIT - I: Countable and Uncountable Sets. Infinite Sets and the Axiom of' Choice. Cardinal Numbers and Its Arithmetic. Schröder-Bernstein Theorem. Cantor's Theorem and the Continuum Hypothesis. Zorn's Lemma. Well-Ordering Theorem. Definition and Examples of Topological Spaces. Closed Sets. Closure. Dense Subsets. Neighbourhoods. Interior, Exterior and Boundary. Accumulation Points and Derived Sets. Bases and Sub-Bases. Subspaces and Relative Topology.	**Unit-1:** Countable and Uncountable Sets **(Pages 3-73)**
UNIT - II: Alternate Methods of Defining a Topology in Terms of Kuratowski Closure. Operator and Neighbourhood Systems. Continuous Functions and Homeomorphism. First and Second Countable Spaces, Lindelöf's Theorem. Separable Spaces. Second Countability and Separability.	**Unit-2:** Continuous Functions and Homeomorphism **(Pages 75-99)**
UNIT - III: Separation Axioms T_0, T_1, T_2, $T_{31/2}$, T_4; Their Characterizations and Basic Properties. Urysohn's Lemma, Tietze Extension Theorem. Compactness. Continuous Functions and Compact Sets. Basic Properties of Compactness. Compactness and Finite-Intersection Property. Sequentially and Countably Compact Sets. Local Compactness and One Point Compactification. Stone-Čech Compactification, Compactness in Metric Spaces. Equivalence of Compactness, Countable Compactness and Sequential Compactness in Metric Spaces.	**Unit-3:** Separation Axioms: Characterizations and Basic Properties **(Pages 101-151)**
UNIT - IV: Connected Spaces. Connectedness on the Real Line. Components. Locally Connected Spaces. Tychonoff Product. Topology in Terms of Standard Sub-Base and Its Characterizations. Projection Maps. Separation Axioms and Product Spaces. Connectedness and Product Spaces. Compactness and Product Spaces {Tychonoff's Theorem}. Countability and Product Spaces. Embedding and Metrization. Embedding Lemma and Tychonoff Embedding. The Urysohn Metrization Theorem.	**Unit-4:** Connected Spaces **(Pages 153-192)**
UNIT - V: Nets and Filters. Topology and Convergence of Nets. Hausdorffness and Nets. Compactness and Nets. Filters and Their Convergence. Canonical Way of Converting Nets to Filters and vice-versa. Ultra-Filters and Compactness. Metrization Theorem and Paracompactness - Local Finiteness. The Nagata-Smirnov Metrization Theorem. Paracompactness. The Smirnov Metrization Theorem. The Fundamental Group and Covering Spaces - Homotopy of Paths. The Fundamental Group. Covering Spaces. The Fundamental Group of the Circle and the Fundamental Theorem of Algebra.	**Unit-5:** Nets and Filters **(Pages 193-244)**

CONTENTS

INTRODUCTION

NOTES

Topology can be defined as the study of qualitative properties of certain objects, known as *topological spaces*, which are invariant under certain kind of transformations, particularly those properties that are invariant under homeomorphism. Topology is a major area of mathematics concerned with properties that are preserved under continuous deformations of objects, such as deformations that involve stretching. It emerged through the development of concepts from geometry and set theory, such as space, dimension and transformation. The term topology is also used to refer to a structure imposed upon a set X, a structure that essentially distinguishes the set X as a topological space by considering properties, such as convergence, connectedness and continuity, upon transformation. Topological spaces appear unsurprisingly in almost every branch of mathematics; hence this has made topology one of the great unifying concepts of mathematics. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together.

A function or map from one topological space to another is called continuous if the inverse image of any open set is open. If the function maps the real numbers to the real numbers (both spaces with the standard topology), then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto, and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. If two spaces are homeomorphic, they have identical topological properties, and are considered topologically the same.

This book is divided into five units which explains countable and uncountable sets, infinite sets and the axiom of' choice, Schröder-Bernstein theorem, Cantor's theorem and the continuum hypothesis, Zorn's lemma, well-ordering theorem, definition and examples of topological spaces, bases and sub-bases, subspaces and relative topology, Kuratowski closure. operator and neighbourhood systems, continuous functions and homeomorphism, first and second countable spaces, Lindelöf's theorem, separable spaces, second countability and separability, Urysohn's lemma, Tietze extension theorem, compactness, continuous functions and compact sets, compactness and finite-intersection property, sequentially and countably compact sets, Stone–Čech compactification, equivalence of compactness, countable compactness and sequential compactness in metric spaces, connected spaces, connectedness on the real line, components, locally connected spaces, Tychonoff product, separation axioms and product spaces, connectedness and product spaces, countability and product spaces, embedding and metrization, embedding lemma and Tychonoff embedding, Urysohn metrization theorem, nets and filters, topology and convergence of nets, Hausdorffness and nets, compactness and nets, filters and their convergence, ultra-filters and compactness, metrization theorem and paracompactness - local finiteness, Nagata-

NOTES

Smirnov metrization theorem, paracompactness, Smirnov metrization theorem, the fundamental group and covering spaces, the fundamental theorem of Algebra.

The book follows the self-instruction mode or the SIM format where in each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The content is presented in a simple and structured form interspersed with Answers to 'Check Your Progress' for better understanding. A list of 'Summary' along with a 'Key Terms' and a set of 'Self-Assessment Questions and Exercises' is provided at the end of each unit for effective recapitulation.

UNIT 1 COUNTABLE AND UNCOUNTABLE SETS

NOTES

Structure

1.0 INTRODUCTION

A set is countable if it is finite, or it has the same cardinality (size) as the set of natural numbers (i.e., denumerable). Equivalently, a set is countable if it has the same cardinality as some subset of the set of natural numbers. Otherwise, it is uncountable.

NOTES

In mathematics, the set of all those natural numbers are defined as infinite or limitless, whose existence can be typically postulated through the axiom of infinity. Characteristically, the axiom of infinity directly uses the set of such natural numbers for mathematical analysis and also the axioms precisely require them to be infinite. The axiom of choice is precisely defined as an axiom of set theory which is equivalent to the mathematical notion or statement which uniquely specifies that a Cartesian product which typically includes a collection or group of non-empty sets is also non-empty.

Fundamentally, in mathematics, the term cardinal numbers or simply the cardinals are defined as the generalisation of the natural numbers specifically used for measuring the cardinality of sets.

The Schröder-Bernstein theorem also called Cantor-Schröder-Bernstein theorem it is the fundamental theorem of set theory. Basically, it states that if two sets are such that each one has at least as many elements as the other then the two sets have equally many elements.

Cantor's intersection theorem uniquely defines that the intersections are used to decrease the nested sequences or structures of non-empty compact sets. In mathematics, the term Continuum Hypothesis (CH) is typically defined as a unique hypothesis which defines the possible sizes of the infinite sets. It states that there is no set whose cardinality is precisely between the integers and the real numbers.

Zorn's lemma, also termed as the Kuratowski–Zorn lemma, uniquely states that a partially ordered set which contains the upper bounds for every chain, i.e., every totally or completely ordered subset, essentially contains at least one maximal (maximum) element. In mathematics, the well-ordering theorem also termed as the Zermelo's theorem, precisely states that every single set can be well-ordered. Characteristically, a topological space is referred as a geometrical space in which the closeness property is uniquely defined but cannot essentially be measured by means of a numeric distance.

In the field of geometry, topology and other associated and related fields of mathematics, a closed set is considered as a set whose complement is an open set and a set which typically contains all its limit points. In mathematical analysis, the closure of a subset S of specific points in a precise topological space typically contains all points in S which is exclusively organized with all limit points of S. The closure of S may be equivalently defined as the union of S and its unique boundary, in addition to the intersection of all closed sets which typically contains S. In topological analysis, a neighbourhood of a point is precisely defined as a set that belongs to the neighbourhood system at that point.

In topology and related areas of mathematics, a subset A of a topological space X is termed as dense (in X) if every point x in X either belongs to A or is a limit point of A, i.e., the closure of A constitutes the whole set X.

Fundamentally in mathematical analysis and more specifically in the topology, the interior of a subset S belonging to a topological space X is considered as the union of all subsets of S that are open in X, which is a point in the interior of S and is therefore termed as an interior point of S. In topological

NOTES

analysis, the exterior of a subset S of a topological space X is considered as the union of all open sets of X which typically are disjoint from S. Alternatively, the boundary of a subset S of a precise topological space X is considered as the set of points which can be uniquely managed both from the inside of S and also from the outside of S.

Bases in topology are considered as ubiquitous. The sets in topology are precisely defined as a normal base and are typically termed as the basic open sets, which can be often described and used rather than the arbitrary open sets. In topology, a subbase for a topological space X along with topology T is defined as a subcollection B of T that uniquely generates T, such that T is defined as the smallest topology containing B. A subspace of a topological space X is defined as a subset S of X which is uniquely equipped with a topology that is typically induced from that of X termed as the subspace topology.

In this unit, you will study about the countable and uncountable sets, infinite sets and the axiom of choice, cardinal numbers, Schröder-Bernstein theorem, Cantor's theorem and the continuum hypothesis, Zorn's lemma, well-ordering theorem, topological spaces, closed sets, closure, and neighbourhoods, dense subsets, interior, exterior, and boundary operations, bases and subbases, subspaces and relative topology.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic properties of countable and uncountable sets
- Comprehend on the infinite sets and the axiom of choice
- Discuss about the cardinal numbers and its arithmetic
- State the Schröder-Bernstein theorem
- Elaborate on the Cantor's theorem and the continuum hypothesis
- Analyse the Zorn's lemma
- State the well-ordering theorem
- Explain about the topological spaces
- Discuss the closed sets, closure and neighbourhoods
- Comprehend on the dense subsets
- Understand the interior, exterior and boundary operations
- Define the bases and subbases
- Explain about the subspaces and relative topology

1.2 COUNTABLE AND UNCOUNTABLE SETS

A set X is defined as countable if it is finite or it can be positioned in 1, –1 correspondence with the positive integers. The non-negative integers are countable by mapping n to $n + 1$. The even numbers are countable; map n to $n/2$. The

NOTES

integers are countable. Map n to $2n$ for $n - 0$, and map n to $1–2n$ for $n < 0$. To check that whether given set is countable or not, we must check that:

- Every element of set is represented in order,
- No element is repeated,
- X is infinite.

All the finite sets of integers are termed as countable, but not for the infinite subsets. Here is a simple diagonalization argument. If the infinite sets are countable then the correspondence builds a list of all possible subsets.

Lemma 1: If a set A is countable and infinite, then there is a bijection between the set A and set of natural numbers N.

Lemma 2: If set A is countable and $A^{'} \subset A$, then A' is also countable.

Let us consider few examples. Set of integers Z form a countable set as there is a bijection from Z to N and it is defined as, $Z \rightarrow N : f(n) \rightarrow f(n) + 2n, n \geq 0$ and $f(n) \rightarrow 1 - 2n, n \prec 0$. From this bijection, f maps 0,1,2,3,…. to 0,2,4,6,…… and -1,-2,-3…. to 1,3,5,7…. And in this way set of integers maps on set of natural numbers, that satisfies definition of countable sets.

If a set is not countable, then it is called uncountable set. For example, set of binary sequences forms uncountable sets.

1.3 INFINITE SETS AND THE AXIOM OF CHOICE

If a set has an infinite number of elements it is an infinite set then the elements of such a set cannot be counted by a finite number. A set of points along a line or in a plane is called a point set. A finite set has a finite subset. An infinite set may have an infinite subset.

Examples of Infinite Sets: A few examples of infinite sets are given below:

(a) A set X comprising of all points in a plane is an infinite set.

(b) Z = {……… -2, -1, 0, 1, 2, ……….},i.e., set of all integers is an infinite set.

(c) Set of all positive integers which are multiple of 5 is an infinite set.

Equivalent Sets: Two sets A and B are said to be equivalent if these exist a one-one onto mapping from A to B. If A and B are equivalent, we denote this relation by the symbol '~'.

Thus $A \sim B \Leftrightarrow A$ & B are equivalent.

For Example: The sets $N = \{1, 2, 3,......\}$ of natural numbers and $E = \{2, 4, 6, ...\}$ of all even natural number are equilavent because these exists the mapping.

$$F : N \longrightarrow E \text{ defined by,}$$

$$f(n) = 2n, \ \forall \, n \in N$$

which is one-one and onto from N to E.

NOTES

1.3.1 Infinite Set Theorem

Theorem 1.1: Let X be a set such that:

(a) There exists an injective function $f: Z_+ \to X$.

(b) There exists a bijection of X with a proper subset of itself.

(c) X is infinite.

Proof : In order to prove the theorem we will first prove that if (a) is true it implies (b), then we will prove that if (b) is true it implies (c), then finally if (c) is true it implies (a).

Let f be an injective function defined as,

$f: Z_+ \to X$. Let the image set $f(Z_+)$ be denoted by Y; and let $f(n)$ be denoted by x_n. Now since f is injective, for all $n \neq m$, $x_n \neq x_m$.

Now let us assume a function f^1

$f^1 : X \to X - \{x_1\}$ such that

$f^1(x_n) = x_{n+1}$ for all $x_n \in Y$

$f^1(a) = a$ for all $a \in X - Y$

It can be seen from the above that the function f^1 is a bijection.

Now for the second step; let us assume that Y is a proper subset of X and

$f: X \to Y$ is a bijection. By assumption, there is a bijection f: $X \to \{1, 2, ...n\}$ for some n. Then the composite $h \, o f^1$ is a bijection of Y with $\{1, 2, ...n\}$. This is a contradiction and cannot be true.

For the third step, we assume that X is infinite and define an injective function $f: Z_+ \to X$ by principle of induction. The set X is a non-empty set, so let us assume an element x_1 of X such that it is equal to $f(1)$.

The set $X - f(\{1, 2, ...n-\})$ is a non-empty set because if it was empty, then the function f: $\{1, 2, ... n-1\} \to X$ would be a surjective function and X would be finite. Now we can select an element of set $X - f(\{1, 2, ... n-1\})$ and accept $f(n)$ to be this element. Therefore, utilising the principle of induction we have defined the function f for all $n \in Z_+$.

It can be noted that the function f is injective. Let $m < n$. Then $f(m)$ belongs to the set $f(\{n-1\})$, whereas $f(n)$, by definition, does not. Therefore, $f(n) \neq f(m)$.

1.3.2 Axiom of Choice

'Axiom of Choice' is a very significant axiom which is extensively utilised in mathematics. The axiom can be stated in many ways. It is also true that many seemingly unrelated statements on closer analysis appear to be equivalent to it.

Definition. Let X be a non-empty collection of non-empty and disjoint sets, then there exists a set Y consisting of exactly one element from each element of X. In other words, a set Y such that Y is contained in the union of the elements of X, and for each $X \in X$, the set $X \cap Y$ has only one element.

NOTES

Another way of stating this axiom is that for any family X of non-empty and disjoint sets, there exists a set that consists of exactly one element from each element of X.

Example 1.1

Let X be the set of countries in the world. Now each country can be considered as a set with the cities of that country as its elements. Then the union of X is the set of all cities in the world. A function which picks up the capital city and matches it with the country is a choice function. Similarly the function that matches the most densely populated city or the largest city of each country with its country is also a choice function.

Example 1.2

Let *S* be a set which has all pairs of shoes in the world as its elements. Then there exists a function that picks the left shoe out of each pair. This is a choice function for *S*.

It may be observed that in the examples quoted above and in any other example that can be stated, the choice function can be defined as a rule that identifies the element to be selected, i.e., select the capital city, select the largest city, select the left shoe of the pair, in the above example. The rule that is defined seems to be fairly simple and straight forward no matter how dense or large the set might be. In fact no axiom is required for such decisions, a simple rule is adequate. Therefore, what the 'Axiom of Choice' does by a consequence is that it ensures existence of a rule or function that will meet the choice.

Axiom of Choice

Choice Function: A choice function is a function *f*. defined on a collection *X* of non-empty sets, such that for every set *A* in *X*, *f* (*A*) is an element of *A*.

Axiom: For any set *X* of non-empty sets, there exist a choice function *f* such that when is defined on *X* it maps each set of *X* to an element of the set. It can be expressed as follows:

$$\forall X\,[\phi \notin X \;\Rightarrow\; \exists F : X \longrightarrow UX\; \forall \in X\;(f\,(A) \in A)]$$

Each choice function on a collection *X* of non-empty sets is an element of the Cartesian product. Given any family of non-empty sets their Cartesian product is a non-empty sets.

Remark: There are many other equivalent statements of the axiom of choice.

- Given any set *X* of pairwise disjoint non-empty sets, these exist at least one set *C* that contains exactly one element in common with each of the sets in *X*.
- Every set has a choice function, which is equivalent, For any set *A* there is a function *F* such that for any non-empty subset *B* of *A*, F (*B*) lies in *B*.

For any set *A*, the power set of *A* (empty set removed) has a choice function.

The basic property of the individual non-empty sets in the collection may make it possible to avoid the axiom of choice even for certain infinite collections.

Example 1.3: Suppose X is the collection of non-empty subsets of natural numbers. Every such subset has a smallest element, therefore to specify the choice function we can simply say that it maps each set to the last element of that set. This gives us a definite choice of an element from each set, and makes it essential to apply the axiom of choice.

Remark: The difficulty appears when there is no natural choice of elements from each set. If we can not make explicit choices then how do we know that our set exists?

Example 1.4: Suppose that X is the collection of non-empty subsets of real numbers. If we try to choose one element from each set then our choice procedure will never come to an end because X is an infinite set. We shall never be able to provide a choice function for all of x.

Zermelo's Postulate: Let $\{A : i \in I\}$ be any non-empty family of disjoint non-empty sets. Then there exists a subset B of the union $\bigcup_{i \in I} A_{0\,i}$ such that the intersection of B and each set A_i consists of exactly one element.

Remark: In Zermelo's postulate the sets are disjoint whereas in the axiom of choice they may not be disjoint.

Theorem 1.2: Show that the axiom of choice is equivalent to Zermelo's postulate.

Proof: Axiom of Choice $\Rightarrow$ Zermelo's Postulate

Let $\{Ai : i \in I\}$ be a non-empty family of disjoint non-empty sets and set f be a choice function on $\{Ai : i \in T\}$. Set $B = \{f(Ai) : i \in I\}$ Then

$$Ai \cap B = \{f(A_i)\}$$

Consists of exactly one element since the A_i are disjoint and f is a choice function.

Zermelo's Postulate $\Rightarrow$ Axiom of Choice

Set $\{A_i : i \in I\}$ be a non-empty family of non-empty sets which may or may not be disjoint.

Set $A_i^k = \{A_i\} \times \{i\}$ or every $i \in I$

Then $\{A_i^*\}$ is a disjoint family of sets since $i \pm i$ implies $A_i \times \{i\} \pm Aj \times \{i\}$ even if $Ai = Aj$.

By Zermelo's postulote, there exists a subset B of $\cup \{A_i^* : i \in I\}$ such that $B \cap A_i^* = \{(ai, i)\}$ consists of exactly one element. then $ai \in Ai$, and so the function f on $\{Ai : i \in I\}$ defined by $f(Ai) = ai$ is a choice function.

1.3.3 Existence of a Choice Function

Lemma. Let X be a non-empty collection of non-empty sets, then there exists a function f such that it selects one member of each of the sets which are elements of X.

$$f : \mathrm{X} \to \mathrm{U\,X} \text{ for all } \mathrm{X} \in \mathrm{X}, \text{ such that } f(\mathrm{X}) \text{ is an element of X}$$

The function f is called choice function for X.

Note the difference between this lemma and axiom of choice: sets are not required to be disjoint.

NOTES

Let X be an element of X. Then let $X^I = \{(X,x) : x \in X)$.

In other words we have constructed X^I as a collection of ordered pairs, where the first element is the set X, and the second element is an element of X. That is the set X^I is a subset of the Cartesian product $X^I \bullet U X$ for all $X \in X^I$.

Since X is a collection of non-empty sets, means X has at least one element x. Therefore, X^I contains at least one element given as (X, x) implying it is non-empty. It can now be seen that if X^I_1 and X^I_2 are two different sets belonging to X then there corresponding elements (X^I_1, x_1) and (X^I_2, x_2) are different sice there coordinates are different.

Now let us define another collection of sets Y such that,

$$Y = \{ X^I : X \in X];$$

Please note that Y by the definition above is a collection of disjoint non-empty subsets of,

$$X^I \bullet U X \text{ for all } X \in X^I$$

Now from the choice axiom we know that there exists a set Y consisting of exactly one element from each element of Y. Now we need to prove that Y is the rule for the desired choice function.

It can be seen that Y is a subset of $X^I \bullet U X$ for all $X \in X^I$.

Further, Y contains exactly one element from each set X^I, therefore, for each $X \in X$, the set Y contains only one ordered pair (X, x). Therefore, Y is the rule for a function from the collection X to the set U X such that $X \in X$.

Also, if $(X, x) \in Y$, then x is an element of X, thus Y(X) is an element of X.

1.4 CARDINAL NUMBERS AND ITS ARITHMETIC

The two sets A and B are said to be **equivalent** or **equipotent** if there exists a bijective map $f: A \to B$ and we write $A{\sim}B$. It is easy to see that the relation of equipotence ~ is an equivalence relation on P[X]. Therefore, this relation must divide P[X] into equivalence classes and we shall use the term cardinal number (or power) to designate the property that equipotent sets have in common. The cardinal number will be, therefore, a measure of the number of points in sets. It can be that all equipotent sets have the same cardinal number. The cardinal number of a set A will be denoted by $|A|$.

It follows that $A{\sim}B \Leftrightarrow |A| = |B|$.

For finite sets, the concept of cardinal number is easy to grasp. We say that any set which is equipotent to the set,

$$\{1, 2,..., n\}$$

has the cardinal number n. We postulate that $|0| = 0$.

Among infinite sets, we denote by **a** or $\mathbf{N_0}$ (read 'Aleph Nought') the cardinal number of all denumerable sets and by c the cardinal number of all those sets which are equipotent to the set R of all real numbers. The cardinal c is often called the cardinal number of the linear continuum.

1.4.1 Ordering

We write $A \leq B$ if A is equipotent to a subset of B. We shall now define an ordering for cardinal numbers, denoted as usual by $\leq$. We will say that,

$|A| = |B|$ if A and B are equipotent.

$|A| \leq |B|$ if A is equipotent to a subset of B.

$|A| < |B|$ if A is equipotent to a subset of B and $|A| \neq |B|$.

The relation $\leq$ is evidently reflexive and transitive on the set of all cardinal numbers.

According to Schröder–Bernstein theorem if A and B are two sets, such that

$$|A| \leq |B| \text{ and } |B| \leq |A|$$

Then $$|A| = |B|.$$

We can now say that the relation $\leq$ is a partial order relation in any set of cardinal numbers.

1.4.2 Addition of Cardinal Numbers

Let λ and μ be cardinal numbers and let A and B be disjoint sets with,

$$|A| = \lambda \text{ and } |B| = \mu.$$

Then we define the addition of λ and μ, written as $\lambda + \mu$, by

$$\lambda + \mu, = |A \cup B| \; i.e. \; |A| + |B| = |A \cup B|.$$

The operation of addition of cardinal numbers as defined earlier is a well-defined operation. For if C and D are disjoint sets with $|C| = \lambda$ and $|D| = \mu$, then $A \sim C$ and $B \sim D$ and consequently $A \cup B \sim C \cup D$. It follows that,

$|A \cup B| = |C \cup B|$ so that $|A| + |B| = |C| + |D|$

It can be easily seen that addition of cardinal numbers is commutative and associative.

1.4.3 Multiplication of Cardinal Numbers

Let λ, μ be two cardinal numbers and let A, B be sets such that,

$|A| = \lambda, |B| = \mu.$

Then the product of λ and μ written as $\lambda\mu$ is defined by,

$\lambda\mu = |A \times B|$ *i.e.,* $|A| \; |B| = |A \times B|.$

It is easy to see that the product of cardinal numbers is well-defined.

If $\{A_\lambda : \lambda \in \Lambda\}$ is any arbitrary collection of sets, then we write, by definition, their product as,

$$II \{|A_\lambda| : \lambda \in \Lambda\} = \{|A_\lambda| : \lambda \in \Lambda\}.$$

Multiplication of cardinal numbers is (i) Commutative (ii) Associative (iii) Distributive over Addition.

Also 1 is the identity element for multiplication.

NOTES

Theorem 1.3: Each of the following subsets of **R** has the cardinal number c of the linear continuum.

(*i*) $]0, 1[= \{\boldsymbol{x} \in \mathbf{R} : 0 < x < 1\}$

(*ii*) $[0, 1[= \{\boldsymbol{x} \in \mathbf{R} : 0 \le x < 1\}$

(*iii*) $[0, 1] = \{\boldsymbol{x} \in \mathbf{R} : 0 \le x \le 1\}$

(*iv*) $[0, 2] = \{\boldsymbol{x} \in \mathbf{R} : 0 \le x < 2\}$

(v) $[0, \infty[= \{\boldsymbol{x} \in \mathbf{R} : x \ge 0\}$

Proof: The mapping,

$$f :]0,1[\rightarrow R : f(x) = \frac{1-2x}{x(1-x)} \; \forall \, x \in]0,1[$$

is evidently one-one and onto so that]0, i[~R.

Therefore $\quad |]0, 1[| = |R| = c$

Again it is obvious that

$$c = |]0, 1[| \le |[0, 1[| \le |[0, 1]|$$
$$\le |[0, 2[| \le |[0, \infty[| \le |R| = c.$$

Now applying the Schröder-Bernstein theorem, we observe that each of the sets in (ii), (iii), (iv) and (v) also has the cardinal number c of the linear continuum.

The unit segment S_1 =]0, 1[on the real line the unit square S_2=]0, 1[×]0, 1[in the plane and the unit cube

$$S_3 =]0, 1[\times]0, 1[\times]0, 1[$$

in space all have cardinal number c.

Theorem 1.4: The cancellation laws of addition and multiplication do not hold for infinite cardinal numbers.

Proof: Recall that we denote the cardinal number of denumerable sets by *a*.

$$a = a + 1$$

But $\quad a + 1 = (a + 1) + 1 = a + (1 + 1)$

[By Associative Law]

Since $1 \ne 2$, we see that the left cancellation law does not hold.

Similarly, right cancellation law cannot hold.

Again the sets $N_0 = \{1, 3, 5,...\}$ and $N_c = \{2, 4, 6...\}$ of odd and even positive integers are disjoint, have each the cardinal number *a* and have as their union the set of positive integers,

$$N = \{1, 2, 3,.......\}$$

which also has the cardinal number *a*.

Thus $\quad N_0 \cup N_c = N$

So that $\quad |N_0 \cup N_c| = |N|$

Or $\quad |N_0| + |N_c| = |N|$ [By Definition of Addition]

Or $\quad a + a = a.$

Hence by distributive law, we have

$$(1 + 1)\, a = 1.\, a = \text{or } 2a = 1a.$$

Since $2 \neq 1$, the cancellation laws for multiplication of cardinal numbers do not hold.

Theorem 1.5: $n < a < c$,

where n is any finite cardinal, and a, c have their usual meanings.

Proof: It is evident that $n < a$. Since N is equipotent with N which is subset of R, we have $\boldsymbol{a} \leq \boldsymbol{c}$. It only remains to show that $\boldsymbol{a} \neq \boldsymbol{c}$. For this purpose it is enough to show that there exists no one-one mapping of $\boldsymbol{N}$ onto the unit interval [0, 1]. Suppose if possible that such a function

$$f : N \rightarrow [0, 1]$$

exists and let the decimal expansion of $f(n), \in \boldsymbol{N}$, be

$$0.a_1\ a_{n2}\ a_{n3}......$$

We define $0.b_1\, b_2\, b_3$ by taking b_i to be 5 if $a_{ij} \neq 5$ and b_i to be 7 if $a_{ii} = 5$. Then $0.b_1\, b_2\, b_3$ is an element of [0, 1] but can never be among the values of f since it differs from $f(n)$ in the nth place, for each $n \in N$. This contradicts our assumption that f is onto. Thus, no such mapping f can exist. Hence, we must have $a \neq c$, and this together with $a \geq c$ gives $a < c$, thus completing the proof of the theorem.

1.4.4 Exponentiation

We now define the final arithmetic operation of taking powers for cardinal numbers, if A and B are sets, we denote by A^B the set of all mappings $f : B \rightarrow A$. Before, we give the formal definition, let us consider the case when A and B are finite sets consisting of m and n elements, respectively, so that $|A| = m$ and $|B| = n$. To count the number of all possible distinct mappings $f: B \rightarrow A$, we observe that the first element b_1 *of* B has just $|A|$ possible images and for each of these there are $|A|$ choices for the images of the second element b_2 of B and so on. Hence the total number of ways of choosing all $|B|$ images is $|A|$ multiplied by itself $|B|$ times, that is, $|A|\ |B|$. It follows that the total number of functions from B into A is $|A|\ |B|$. This combinatorial characterization of $|A|\,|B|$ can be extended to infinite cardinal numbers. Thus, we are led to the following definition.

Let A, B be any sets, finite or infinite. Then we write by definition

$$|A^B| = |A|^{|B|}.$$

It is evident that if $A \sim C$ and $B \sim D$, then

$$|A|^{|B|} = |C|^{|D|}$$

So that the preceding definition is unambiguous.

NOTES

Let us pay particular attention to the cardinal number $2^{|A|}$ where A is any set. The symbol 2 here stands both for the number 2 and any two element sets, say {0, 1}.

Hence $2^{|A|} = 2^A \,| = |[0, 1]^A|$

is the cardinal number of the set of all mappings $f: A \to \{0, 1\}$.

Theorem 1.6: If A is any set, then

(i) $|P(A)| = 2^{|A|}$ where $P(A)$ denotes the power set of A.

(ii) $|A| < |P(A)|$.

(iii) $|A| < 2^{|A|}$

i.e., $\mu < 2^{\mu}$ for any cardinal number μ.

Proof: (i) As remarked earlier $2^{|A|}$ is the cardinal number of the set $\{|0; 1\}^4$ of all mappings of A into the two element set {0, 1). So we have to establish a one-one correspondence between $P(A)$ and $\{0, 1\}^A$. Recall that the characteristic function X_B of a subset B of A is a function on A defined by,

$$X_B(x) = \begin{cases} 1 & \text{when } x \in B \\ 0 & \text{when } x \in A - B \end{cases}$$

Now consider the function,

$$F : P(A) \to \{0, 1\}^A : F(B) = X_B \forall B \in P(A)$$

F is one-one. For we have,

$$F(B) = F(C) \Rightarrow X_B = X_C \Rightarrow B = C.$$

F is onto. Let f be any member of $[0, 1]^A$, *i.e.,* any function from A into {0, 1} and let $f^{-1}\{1\} = E$. Then evidently $XE = f$. Hence f is onto. Thus f establishes a one-one correspondence between P(A) and $\{0, 1\}^A$ and consequently,

j p {A) | = | |0,]}^{A} | = 2 '%' .

(ii) To prove this, we must establish a one-one correspondence between A and a proper subset of P (A). The mapping,

$$g : A \to P(A) : g(x) = \{x\} \;\forall\; x \in A$$

is evidently one-one and into. It follows that $|A| \leq |P(A)$. We now show that $|A| \neq |P(A)|$ *i.e, A* and $P(A)$ are not equipotent. Suppose, if possible, there exists a one-one mapping f of A onto $P(A)$. Thus f assigns to each point $x \in A$, a subset $f(x)$ of A. Let B denote the subset of A consisting of all those points x for which $x \in f(x)$. Since f is a mapping of A onto $P(A)$ and $B \in P(A)$, there must exist some point y in A such that $f(y) = B$. Now either $y \in B$ or $y \in A - B$. If $y \in B$, then by the definition of B. we have $y \notin f(y) = B$. On the other hand if $y \notin B$, then $y \in f(y) = B$. Thus in their case, we arrive at a contradiction.

Hence we must have $|A| \neq |P(A)|$ and consequently,

$$|A| < |P(A)|.$$

(iii) This follows from (i) and (ii).

Theorem 1.7: $2^a = c$

Proof: Since a is the cardinal number of the set,

$$N = \{1, 2, 3,...\}$$

of natural numbers we may write,

$$2^a = 2^{|N|}$$

By the previous Theorem 1.6,

$$2^{|N|} = |P(N)|.$$

Also c is the cardinal number of the interval,

$$I = [0, 1].$$

Our proof then amounts to showing that there exists a one-one correspondence between P(N) (*i.e.* the power set on **N**) and *I*. First we define a mapping $f : P(N) \to I$ as follows:

If $A \in P(N)$, then $f[A]$ is the real number x in I whose decimal expansion. $x = 0.\ \alpha_1\alpha_2\alpha_3...$ is defined by the rale that $\alpha_n = 3$ if $n \in A$ and $a_n = 5$ if $n \notin A$. We can use any other two digits provided none of them is 9. The mapping is evidently one-one so that **P(N)** is equipotent to a subset of *I*.

Again we construct a one-one mapping g of I into **P(N)** as follows. Let x be any real number in I and let its binary expansion be $x = 0.\ \beta_1\beta_2\beta_3...$ so that β_n is either 0 or 1. Then $g(x)$ is defined to be that member A of **P(N)** consisting of positive integers n for which $\beta_n = 1$, that is,

$$A = \{n \in N : \beta_n = 1\}.$$

It is evident that the mapping g is also one-one so that I is equipotent to a subset of **P(N)**. We now use the Schröder-Bernstein theorem to conclude that **P(N)** and I are equipotent sets, that is, **P(N)~I** and consequently,

$$|\mathbf{P(N)}| = |I|.$$

But $|\mathbf{P\ (N)}| = 2^{|N|}$ and $|I| = c$

Hence $\quad 2^{|N|} = c$

So, $\quad 2^a = c.$

Theorem 1.8: The set of all real valued functions defined on the closed unit interval [0, 1] has the cardinal number 2^c.

Proof: Let $I = [1, 0]$. Recall that the set of all real functions defined on I is denoted by R^I. Hence,

$$|\mathbf{R}^I| = |\mathbf{R}|^{|I|} = c^c$$

Also by $2^a = c$. It follows that,

$$|\mathbf{R}^I| = (2^a)^c = 2^c$$

Thus the cardinal number of the set of all real valued functions defined on [0, 1] is 2^c.

Continuum Hypothesis: There exists no cardinal number λ such that,

$$a < \lambda < c.$$

The generalized continuum hypothesis asserts that there is no cardinal number strictly between λ and 2^{λ} for any cardinal number λ.

NOTES

Check Your Progress

1. Define the term countable.
2. What is uncountable?
3. Define an infinite set.
4. State about the axiom of choice.
5. What do you understand by cardinal number?
6. Define the cardinal number of linear continuum.

1.5 SCHRÖDER-BERNSTEIN THEOREM

In set theory, the Schröder–Bernstein theorem specifically states that, if there exists injective functions $f: A \to B$ and $g: B \to A$ between the sets A and B, then there also exists a bijective function $h: A \to B$.

According to the cardinality property of the two sets, this classically implies that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$, i.e., A and B are **equipotent**. This is a significant feature in the ordering of cardinal numbers.

The theorem is named after **Felix Bernstein** and **Ernst Schröder** and is also sometimes termed as **Cantor–Bernstein theorem**, or **Cantor–Schröder–Bernstein**, named after Georg Cantor who first published it without proof.

Proof

The following proof is attributed to **Julius König**.

Assume that A and B are disjoint. For any a in A or b in B we can form a unique two-sided sequence of elements that are alternately in A and B, by repeatedly applying f and g^{-1} to go from A to B and g and f^{-1} to go from B to A, where the inverses f^{-1} and g^{-1} are considered as the partial functions.

$$\ldots \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \ldots$$

For any particular a, this sequence may terminate to the left, at a point where f^{-1} or g^{-1} is not defined.

By the fact that f and g are injective functions, each a in A and b in B is in exactly one such sequence to within identity: if an element occurs in two sequences, then all elements to the left and to the right must be the same in both, by the definition of the sequences. Therefore, the sequences form a partition of the disjoint union of A and B. Hence it is essentially sufficient to produce a bijection between the elements of A and B in each of the sequences separately, as follows:

A sequence is called an *A-stopper* if it stops at an element of A, or a B-stopper if it stops at an element of B. Otherwise, it is called doubly infinite if all the elements are distinct or cyclic if it repeats. Refer Figure (1.1). As per the König's definition of a bijection $h: A \to B$ from given example injections $\boldsymbol{f}: A \to B$ and $\boldsymbol{g}: B \to A$.

An element in A and B is denoted by a number and a letter, respectively. The sequence $3 \to e \to 6 \to \ldots$ is an *A-stopper*, leading to the definitions $h(3) = f(3) = e$, $h(6) = f(6), \ldots$. The sequence $d \to 5 \to f \to \ldots$ is a *B-stopper*, leading to $h(5) = \boldsymbol{g}^{-1}(5) = d, \ldots$. The sequence $\ldots \to a \to 1 \to c \to 4 \to \ldots$ is doubly infinite, leading to $h(1) = \boldsymbol{g}^{-1}(1) = a$, $h(4) = \boldsymbol{g}^{-1}(4) = c, \ldots$. The sequence $b \to 2 \to b$ is cyclic, leading to $h(2) = \boldsymbol{g}^{-1}(2) = b$.

- For an *A-stopper,* the function $\boldsymbol{f}$ is a bijection between its elements in A and its elements in B.
- For a *B-stopper*, the function $\boldsymbol{g}$ is a bijection between its elements in B and its elements in A.
- For a doubly infinite sequence or a cyclic sequence, either $\boldsymbol{f}$ or $\boldsymbol{g}$.

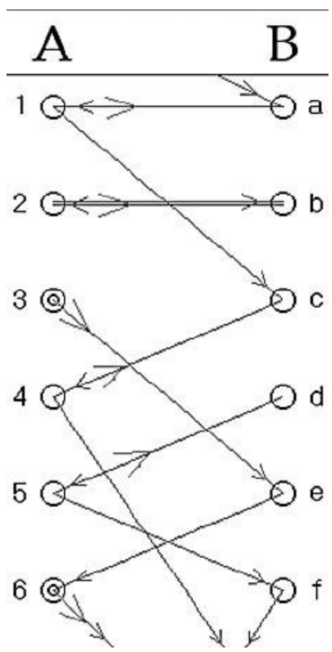


Fig. 1.1 *König's Definition of a Bijection h: A → B*

1.5.1 Schröder-Bernstein Theorem and Cardinal Numbers

Cardinal Numbers: We want to know the size of a given set without necessarily comparing it to another set. For finite set there is no difficulty. For example, the set A= {1, 2, 3} has 3 elements. Any other set with 3 elements is equipotent to A. On the other hand, for infinite sets it is not sufficient to just say that the set has infinitely many element since not all infinite ses are equiptent. To solve this problem we introduce the concept of a coordinal number.

Each set A is assigned a symbol in such a way that two sets A and B are assigned the same symbol if and only if they are equipotent. This symbol is called cardinality or cardinal number of A and it is denoted by

$$|A| \text{ or card } (A)$$

NOTES

Thus

$$|A| = |B| \text{ iff } A \sim B.$$

Finite Cardinal Numbers: For finite cardinal numbers the obvious symbols are used like 0 is assigned to empty set ϕ, and n is assigned to the set $\{1, 2, ...n\}$

Infinite Cardinal Numbers: Infinite sets cardinal numbers are called infinite cardinal numbers.

The cardinal number of the infinite set L of postitue insegers is α_0. Thus

$$|A| = \alpha_0 \text{ iff } A \sim L$$

In particular $|Z| = \alpha_0$ and $|\theta| = \alpha_0$. Cardinal number of $I = [0,1]$ is denoted by C. Thus $|A| = C$ iff $A \sim T$

In particular $|R| = C$.

Schröder-Bernstein's Theorem

If $|A| \leq |B|$ and, then $|A| = |B|$.

Other equivalent from is,

Let X, Y, X be sets such that $X \supseteq Y \supseteq X_1$ and $X \sim X_1$ then $X \sim Y$.

Proof: Since $X \sim X_1$, there exist a mapping $f : x \longrightarrow X_1$.

Which is, one-one and onto.

Since $X \supseteq Y$, then restuiction of f to Y is also one-one. Let $f(Y) = Y_1$.

Then Y and Y_1 are equivalent, i.e.

$$X \supseteq Y \supseteq X_1 \supseteq Y_1$$

And $f . Y \to Y_1$ is one-one and onto.

We now see that $Y \supseteq X_1$ and for similar reason $X_1 \sim X_2$ where

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2$$

And $f : X_1 \to X_2$ is one-one and onto.

In this way there exists sets,

$$X_1, X_2, X_3, \text{.....} \text{ and } Y_1, Y_2, Y_3 \text{........ such that}$$

$$X \supseteq Y \supseteq X_1 \supseteq X_2 \supseteq Y_2 \supseteq Y_3 \supseteq Y_3 \supseteq$$

And $\quad f : X_i \to X_{i+1}$ and

$$f : Y_i \to Y_{i+1} \text{ are one-one and onto.}$$

Let $\quad B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap$..........

Then $\quad X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cup ..... \cup B$

$$Y = (Y_1 - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \ldots\ldots \cup B$$

Also $(X - Y) \sim (X_1 - Y_1) \sim (X_2 - Y_2) \sim \ldots\ldots\ldots$

NOTES

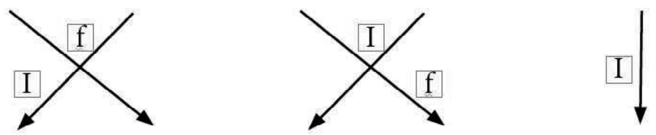
Specially, the function

$f : (X_n - Y_n) \rightarrow (X_{n+1} - Y_{n+1})$ is one-one and onto.

consider the function

g : $X \rightarrow Y$ defined by the following figure.

$$X = (x - y) \bigcup (y - x_1) \bigcup (x_1 - y_1) \bigcup (y_1 - x_2) \bigcup \quad \ldots \bigcup B$$



$$Y = (y - k_1) \bigcup (x_1 - y_1) \bigcup (y_1 - x_2) \bigcup (x_2 - y_2) \bigcup \quad \ldots \bigcup B$$

i.e. $g(x) = \begin{cases} f(x), & \text{if } x \in (X_i - Y_i) \quad \text{or} \quad x \in (X - y) \\ x & \text{if } x \in (Y_i - X_i) \quad \text{or} \quad X \in B \end{cases}$

Therefore, g is one-one and onto.

i.e. $X \sim Y$.

Proof completed.

Cantor's Theorem: The bower set $P(B)$ of any set B has cardinality greater than B. i.e., $|B| < |P(B)|$.

Proof: The function

h : $B \rightarrow P(B)$ such ta

$h(b) = \{b\}, \forall b \in B$

i.e., sends each element $b \in B$ to set containing b alone. then h is one-one and hence

$$|B| \leq |P(B)| \qquad \ldots (1.1)$$

If we show that B is not equivalent to $P(B)$, i.e., $B \psi P(B)$ then the theorem will proved.

Now suppose $B \sim P(B)$, then there exist a mapping

f: $B \rightarrow P(B)$ which is one-one and onto. Let us call $b \in B$ a 'bod' element if $b \notin f(b)$.

Let A be the set of all 'bad' elements. i.e. $A = \{x : \in xB \;\&\; x \notin f(x)\}$.

Hence we see that A is the subset of B and hence $A \in P(B)$.

NOTES

Since $f: B \to P(B)$ is into $\exists$ an element $a \in B$ sit.

$$f(a) = A$$

Now the question is whether a is 'bad' or a is a good element.

If $a \in B$ then $a \notin f(a) = A$, which is a contradiction.

If $a \notin A$ then $a \in f(a) = A$

which is again a contradiction

Then our assumption that $B \sim P(B)$ is wrong and consequentry $B \psi P(B)$.

By Equation (1.1)

$$|B| \leq |P(B)|$$

And $B \psi P(B)$

Hence $B < P(B)$ so that

$$|B| < |P(B)|$$

Proof completed.

1.6 CANTOR'S THEOREM AND THE CONTINUUM HYPOTHESIS

A sequence $<x_n>$ in a metric space (X, d) is said to converge to $x_0 \, \varepsilon \, X$ if it is eventually in every *nhd* of x_0, i.e., if for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that,

$$n \geqslant n(\varepsilon) \Rightarrow d(x_n, x_0) < \varepsilon.$$

Another definition states that 'A sequence $<x_n>$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer n (ε)' such that

$$m, n \geq n(\varepsilon) \Rightarrow d(x_m, x_n) < \varepsilon.$$

It is easy to prove that every convergent sequence in a metric space is Cauchy sequence.

A metric space *X* is said to be complete if every Cauchy sequence in *X* converges to a point in *X*.

Theorem 1.9: (Cantor's Intersection Theorem): Let (X, d) be a complete metric space and let $<F_n>$ be a nested sequence of non-empty closed subsets of *X* such that $\delta(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Proof: For each n, we choose $x_n \in F_n$. Since $\delta(F_n) \to 0$, for every $\varepsilon > 0$, there exists a positive integer m_0 such that $\delta(F_{m0}) < \varepsilon$. Again since $<F_n>$ is nested,we have,

$$n, m > m_0 \Rightarrow F_n, F_m \, F_{m0} \Rightarrow x_n, x_m \in F_{m0} \Rightarrow d(x_n, x_m) < \varepsilon.$$

NOTES

Thus $< x_n >$ is a Cauchy sequence. Since X is complete so $x_n \to x_0$ for some $x_0 \in X$. We assert the $x_0 \in \bigcap_{n=1}^{\infty} F_n$. To prove this, let m be any positive integer. Then $n > m \Rightarrow x_n \in F_m$ ... (1.2)

$[\because \; x_n \in F_n \; and \; n > m \Rightarrow F_n \subset F_m]$.

Since $x_n \to x_0$, the sequence is eventually in every *nhd* of x_0 and by Equation (1.2), every *nhd* of x_0 contains an infinite number of points of F_m. Thus x_0 is a limit point of F_m. Since F_m is closed, $x_0 \in F_m$ and since m is arbitrary, we have $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

Now suppose there is another point $x_0^* \in \bigcap_{n=1}^{\infty} F_n$. Then d $(x_0, x_0^*) < \delta (F_n)$ for every n. Therefore, $\delta (x_0, x_0^*) = 0$ since $\delta (F_n) \to 0$ as $n \infty$. Hence $x_0 = x_0^*$ and so $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.

1.6.1 Continuum Hypothesis

In mathematics, the term Continuum Hypothesis (CH) is uniquely defined as a hypothesis which defines the possible or probable sizes of the infinite sets. It states that there is not any set whose cardinality can be uniquely defined between the integers and the real numbers.

In **Zermelo–Fraenkel Set Theory** and the **Axiom of Choice (ZFC or Zermelo–Fraenkel Continuum)** is considered equivalent to the equation in aleph numbers: $\mathbf{2^{N0} = N_1}$.

The term 'Continuum Hypothesis (CH)' was precisely improved by Georg Cantor in 1878, who established its truth or false property specifically in the Hilbert's first 23 problems described in 1900. The answers to these problems are independent of the ZFC (Zermelo–Fraenkel Continuum), therefore the continuum hypothesis and its negation property can be included as an axiom to ZFC set theory that defines that the theory is consistent and constant if and only if ZFC is consistent and constant. This independence property was proved in the year 1963 by Paul Cohen, who basically complemented the work done by Kurt Gödel in the year 1940.

The name continuum hypothesis is precisely given to this hypothesis which comes from the term the ***continuum*** uniquely defined for the real numbers.

Cardinality of Infinite Sets

Two sets are considered to have the same or equivalent ***cardinality*** or ***cardinal number*** if there typically exists a bijection, i.e., a one-to-one correspondence between them. Spontaneously, for two sets S and T to have the same or equivalent cardinality implies that it is feasible to 'Pair Off' elements of S with elements of T in such a manner that every element of S is paired off with precisely or exactly one element of T and vice versa.

Specifically for the infinite sets, such as the set of ***integers*** or ***rational numbers***, the existence of a bijection condition between the two sets can hardly be demonstrated. Characteristically, the rational numbers apparently form a counter example to the continuum hypothesis, stating that the integers form a proper and appropriate subset of the rational, which themselves form a proper and appropriate subset of the reals, therefore automatically and intuitively there are more rational numbers as compared to integers and more real numbers as compared to rational numbers. Even though, this precise intuitive analysis is inconsistent as it does not consider the fact that all three sets are infinite. Consequently, it can be stated that essentially the rational numbers can be actually placed in one-to-one correspondence with the integers, and therefore the set of rational numbers has the same size the '*Cardinality*' as the set of integers, proving that they both are countable sets.

Cantor has given two proofs stating that the cardinality of the set of integers is precisely and truly less significant than that of the set of real numbers and the Cantor's first uncountability proof which states that the cardinality of the integers is comparatively less than that of the real numbers. Cantor proposed and recommended the term 'Continuum Hypothesis' as a possible and feasible solution to these problems.

The continuum hypothesis states that, 'The set of real numbers has minimal possible cardinality which is greater than the cardinality of the set of integers. i.e., every set, *S*, of real numbers can either be mapped one-to-one into the integers or the real numbers can be mapped one-to-one into *S*'. Because the real numbers are equinumerous with the power set of the integers, therefore, $|\mathbf{R}|=\mathbf{2}^{\aleph_0}$ and the continuum hypothesis states that there is no set $\boldsymbol{S}$ for which $\aleph_0<|\mathbf{S}|<\mathbf{2}^{\aleph_0}$.

Assuming the axiom of choice, we can state that there is a smallest cardinal number $\aleph_1$ greater than $\aleph_0$ and the continuum hypothesis is then equivalent to the equality $\mathbf{2}^{\aleph_0} = \aleph_1$.

Independence from ZFC

The independence property of the Continuum Hypothesis (CH) given by Zermelo–Fraenkel (ZF) set theory is defined on the basis of the combined work of Kurt Gödel and Paul Cohen.

Gödel precisely explained and demonstrated that 'Continuum Hypothesis (CH)' cannot be disproved from 'Zermelo–Fraenkel (ZF)', even if the 'Axiom of Choice' (AC) is typically adopted and implemented producing the 'Zermelo–Fraenkel Continuum (ZFC)'. Gödel's proof demonstrates and reveals that both the CH and AC hold in the constructible universe 'L', an inner model of ZF set theory, by precisely assuming only the axioms of ZF. The existence of an inner model of ZF in which uniquely the additional axioms hold demonstrates and reveals that the additional axioms are uniquely consistent with ZF, provided that the ZF itself is consistent. Due to Gödel's incompleteness theorems, the concluding condition was not possible to be proved in ZF itself, but it is widely considered to be true and can be proved in the stronger set theories.

Cohen demonstrated and revealed that CH cannot be proved on the basis of the ZFC axioms by completing and specifying the overall complete independence proof. To prove and establish his result, Cohen precisely developed and established the method of forcing, which has now considered a standard tool in the set theory. Fundamentally, this method essentially started with a model of ZF in which CH holds and constructs or structures another model which typically contains more sets as compared to the original, in a manner that CH does not hold in the new model.

Consequently, the independence property uniquely states that CH is independent of ZFC. The mathematicians precisely define that CH is independent of all known **large cardinal axioms** in the context of ZFC. Subsequently, it was proved that the cardinality of the continuum can be any cardinal consistent with '**König's Theorem**'. Solovay, proved after Cohen's result on the independence property of the continuum hypothesis that in any model of ZFC, if κ is a cardinal of uncountable cofinality, then there is a forcing extension in which $\mathbf{2^{\aleph_0}} = \kappa$. According to the König's theorem, though it is not possibly consistent to assume that $\mathbf{2^{\aleph_0}}$ is $\mathbf{\aleph_\omega}$ or $\mathbf{\aleph_{\omega 1+\omega.}}$ or any cardinal with cofinality ω.The continuum hypothesis is closely related to many statements in analysis, point set topology and measure theory.

The independence property from ZFC implies that either proving or disproving the CH within ZFC is not possible. Even though, the negative results or consequences of Gödel and Cohen are not universally accepted for determining the continuum hypothesis. The continuum hypothesis and the axiom of choice were considered as the first mathematical statements that demonstrated to be independent of ZF set theory.

Generalized Continuum Hypothesis

The Generalized Continuum Hypothesis (GCH) states that if an infinite set's cardinality lies between that of an infinite set *S* and that of the power set ***P*(*S*)** of *S*, then it has the equivalent cardinality as either *S* or ***P*(*S*)**.That is, for any infinite cardinal λ there is no cardinal κ such that $\lambda < \kappa < 2^\lambda$ GCH is equivalent to:

$\mathbf{\aleph_{\alpha+1} = 2^{\aleph_\alpha}}$ **for every ordinal** α (called '**Cantor's Aleph Hypothesis'**).

1.6.2 Continuum Hypothesis based on Cantor's Theorem

By Cantor's theoremn, $\alpha_0 < 2^{\alpha_0}$ and also we know that, $\alpha_0 < C$.

In next theorem we will prove relationship between 2^{α_0} and C.

Theorem 1.10: $2^{\alpha_0} = C$.

Proof: Let $P(Q)$ be the power set of rational number and R be the set of real numbers.

Let the function

$$f : R \to P(Q) \text{ defined by,}$$

$$f(a) = \{x : x \in Q, x < a\}$$

i.e., f is a function which maps each real number a to the set of rational number less than a. Now, will show f is one-one.

Let $a_1, a_2 \in R, a_1 \pm a_2$ and for simplicity $a_1 < a_2$.

Since rationals are dense into the set of real numbers. These exist a rational number r_1 such that,

$$a_1 < r_1 < a_2$$

NOTES

Then by the definition of f $r_1 \in f\ (a_2)$ and $r_1 \notin f\ (a_1)$, hence,

$$f(a_1) \pm f(a_2)$$

Therefore f is one-one. Thus,

$$|R| \leq |P(Q)| \qquad \text{... (1.3)}$$

Since,

$$|R| = C \qquad \text{and } |Q| = \alpha_0$$

Then by Equation (1.3), we get

$$C \leq 2^{\alpha_0} \qquad \text{... (1.4)}$$

Let C (L), where L = {1, 2, 3, ... } be the family of characteristics function,

$f : L \to \{0, 1\}$ is equivalent to $P(L)$.

Let $I = [0, 1]$ and let the function

$H : C[L] \to$ I be defined by

$H(f) = 0.f(1)f(2)\ f(3)f(4)....$

on infinite decimal consisting of zeros or one. Suppose $f.g \in C(L)$ and $f \pm g$.

Then the decimal would be different and therefore $H(f) \pm H(g)$.

So, H is one-one. Therefore,

$$|P(Q)| = |C(L)| \leq |I| \qquad \text{... (1.5)}$$

Since $|Q|\alpha_0$ and $|I| = C$, then by Equation (1.5), we can write

$$Z^{\alpha_0} \leq C. \qquad \text{... (1.6)}$$

By Equations (1.4) and (1.6), we get

$$\boxed{2^{\alpha_0} \leq C.}$$

It is natural to ask if there exists a cardinal number B which lies between α_0 and C.

Initially, cantor supported the conjecture, whicyh is known as the continuum hypothesis that the answer to the above question is in the negative.

Continuum Hypothesis: There exist no cardinal number B such that

$$\boxed{\alpha_0 < B < C}$$

Zorn's Lemma and Well-Ordering Theorem

Partially Order Sets: A relation $(\leq)$ on a set A is called a partial order relation if for every a, b, c in A.

NOTES

I. $a \leq a$

II. $a \leq b$ and $a \leq a \Rightarrow a = b$

III $a \leq b$ and $b \leq c \Rightarrow a \leq c$

A set 'A' together with the partial order relation, i.e., the pair (A, C) is a partial ordered set. Partial order relation is repeexive, antigymmetric and transitive.

For example, the set inclusion relation in any family of sets is a partial order relation, i.e.,

(i) $A \subset A$ for every set A.

(ii) If $A \subset B$ and $B \subset A \Rightarrow A = B$

(iii) If $A \subset B$ and $B \subset C \Rightarrow A \subset C$

Totally ordered set. A partially ordered set 'A' is said to be totally ordered if for each $a, b \in A$ either $a \leq b$ or $b \leq a$.

Example 1.5: (I) The set of real numbers with its natural ordering (<) is a totally ordered set.

(II) Let $A = \{2, 4, 8, 16, 32\}$ be ordered by $a \leq b$ to mean a divide b then A is totally ordered.

Minimal and Maximal Elements

Let 'Q' be a partially ordered set. An element $a_0 \in A$ is called a minimal element of A if no element of A stoictly proceedes a_0, i.e., if $x \leq a_0 \Rightarrow x = a_0$

Similarly, an element $b_0 \in A$ is called a maximal element of A is no element of A strictly succeeds b_0, i.e., if

$$x \geq b_0 \Rightarrow x = b_0.$$

Geometrically interpretation is that, a_0 is a minimal element of A if no edge enters a_0 (from below), and b_0 is a maximal element of A if no edge leaves b_0 (in an upward direction) So a set A can have more than one minimal and more than one maximal element.

If A is infinite, then A may have no minimal and no maximal element. If A is finite, then A has at least one minimal element and one maximal element.

First and Last element. An element $a_0 \in A$ is called a first element of A if

$$a_0 \leq x \qquad \forall x \in A$$

Similarly an element $b_0 \in A$ is called a last element of A if

$$x \leq b_0 \qquad \forall x \in A$$

NOTES

Remark: A set 'A' can have at most one first element which must be a minimal element of A and 'A' can have at most one last element which must be a maximal element of A. In general, 'A' may have neither a first nor a last element, even when A is finite.

If A is finite totally ordered. Set, then A has both a first element and a last element.

Example 1.6: Let A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} be ordered by the relation "x divides y"

A has two maximal elements, 18 and 24, and neither is a last elekent. A has only one minimal element, 1, which is also a first element.

Well-Ordered Set: A partially ordered set (x_1, <) is said to be well-ordered if every non-empty subset of X has a first element.

Examples 1.7: (I) N with its natural ordering $\leq$ is well-ordered.

(II) Q and R with its natural ordering $\leq$ is not well-ordered.

Theorem 1.11: Every well-ordered set is totally ordered.

Proof: Let (A $\leq$) be a well-ordered set Let a & b be may two elements of A. Then $\{a, b\}$ CA.

Since A is well-ordered and $\{a, b\}$ CA therefore $\{a, b\}$ has a first element. If a is first element of $\{a, b\}$ then $a \leq b$.

If b is first element of $\{a, b\}$ then $b \leq a$.

Then we see that for any two elements a & b of A either $a \leq b$ or $b \leq a$.

Hence (A, $\leq$) is totally ordered.

Well-Ordering Theorem. Every set can be well-ordered.

Proof: Let X be a non-empty set. Let P (X) be the collection of all-subsets of X and f be a choice function given by,

$$f : P\,(X) \to X \text{ with}$$

$$f\,(A) \in B \qquad \forall\; A \subset X$$

A subset A of X will be called normal if it has well-ordering as well as $\forall\; a \in A\;\; f : (X - S_A(a)) = -a$ where $S_A(a) = \{x_6 A : x < a\}$

i.e., $S_A(a)$ is the initial segement of a. Set $x_0 = f(x), x_1 = f\;(x - \{x_0\}),$ $x_2 - f(x - \{x_0, x_1\})$

Then $A = \{x_0, x_1, x_2\}$ is normal.

Claim: If A & B are normal subsets of X then either $A = B$ or one is initial segment of the other.

Thus there exist a similarity mapping $\lambda : A \to B$.

Set

$$A^* = \{x \in A : \lambda(x) \pm x\}$$

NOTES

If $A^* = \phi$ then $A = B$ or A is an initial segment of B.

Suppose $A^* \neq \phi$. Let a_0 be the first element of A^*. Then,

$$S_A(a_0) = S_B(\lambda(a_0))$$

But A and B are normal, so

$$a_0 = f(X - S_A(a_0)) = f(X - S_B(dla_0)) = \lambda(a_0)$$

which contradict the definition of A^* and so $A = B$ or A is an initial segment of B. In particular. If $a \in A$ and $b \in B$ then either $a, b \in A$ or $a, b \in B$. Further more, if $a, b \in A$ and $a, b \in B$ then $a \leq b$ as element of A iff $a \leq b$ as elements of B.

Now let y consists of all those elements in X which belongs to at least one normal set if $a_1 b \in y$, then $a \in A$ and $b \in B$ where A and B are normal and so $a, b \in A$ or $a, b \in B$.

We define an order in Y as follows. $a \leq b$ as elements of Y iff $a \leq b$ as elements of A or as elements of B. This order is well defined and it is a totally order.

Claim: We Y is well-ordered

Let Z be any non-empty subset of Y and let a be an arbitrary element element in Z. Then a belong to a normal set A. Hence $A \cap Z$ is a non-empty subset of a well-ordered set Also contains a first element a_0. Since $a_0 \in A \cap Z$

$\Rightarrow a_0 \in Z$

i.e., a_0 is a first element of Z.

Therefore Y is well-ordered.

Now, we will show that Y is normal.

If $a \in Y$, then $a \in A$, where A is a normal set furthermore, $S_A(a) = S_Y(a)$ and so

$$f(X - S_Y(a) = f(X - S_A(a)) = a$$

i.e. Y is normal

Claim: $Y = X$

Suppose $Y \pm X$ then $X - Y \pm \phi$ and say $a = f(X - Y)$

Set $Y^* = Y \cup \{a\}$ and let Y^* be ordered by the order in Y together with a dominating every element in Y. Then $f(X - S_1(a) = f(X - Y) = a$ and so Y^* is normal. Thus $a \in Y$. But this conttradict the fact that f is a a choice function i.e. $f(X - Y) = a \in X - Y$ which is disjoint from Y.

Hence $Y = X$

NOTES

Since we have already proved that *Y* is well-ordered. Therefore *X* is well-ordered.

Remarks: Let *X* be a partially ordered set. Then *X* contains a maximal totally ordered subset.

Zorn's Lemma: Let *X* be a non-empty partially ordered set in which every totally ordered subset has an upper bound in *X*. Then *X* contains at least one maximal element.

Proof: We will prove Zorn's Lemma using well-ordering theorem.

Let *X* be a partially ordered set in which every totally ordered subset has an upper bound. We need to show that *X* has a maximal element by previous remarks *X* will have a maximal totally ordered subset let it is denoted by *Y*. By our assumption *Y* will have an upper bound in *X*. Let this upper bound is denoted by *a*.

Claim: '*a*' is maximal element of *X*.

Let '*a*' is not maximal element of X. Then there exist $b \in X$ such that *b* dominates '*a*'. It follows that $b \notin Y$ since '*a*' is an upper bound for *Y*. So $Y \bigcup \{b\}$ is a totally ordered set.

Which is, a contradiction because *Y* is maximal totally ordered subset of *X*. So our assumption is wrong.

Therefore, '*a*' is maximal element of *X*.

Hence Zorn's lemma is proved.

Theorem 1.12: Show that Zorn's lemma implies axiom of choice.

Proof: We have already shown that axiom of choice and Zermelo's Posulate are equivalent to each-other. Therefore it is sufficient to show that Zorn's lemma implies Zermelo's postulate.

Let $\{Ai\}$ be a non-empty family of disjoint non-empty sets. Let ρ be the collection of all subsets of $\bigcup Ai$ which intersect each *Ai* in at most one element. We partially order ρ by set inclusion. Let B,} be a totally ordered subset of ρ.

We claim that $B = \bigcup B$, belong to ρ.

If not, then *B* intersects some *A* is in more than one element i.e. $a, b \in B \bigcap Ai_0$ where $a \pm b$.

Since $a, b \in B$. These exist B_{J_1} and B_{J_2} such that $a \in B_{J_1}$ and $b \in B_{J_2}$. But $\{\{B_J\}$ is a totally ordered by set inclusion, hence a & b belongs to either B_{J_1} or B_{J_2}. This implies that BJ_1 or BJ_2 intersects A_{i_0} in more than one element, a contradiction. Accordingly $B \in \rho$ so *B* is an upper bound for the totally ordered set $\{B_J\}$.

We have shown that every totally ordered set in ρ has an upper bound. By Zorn's lemma, ρ has a maximal elements. If *S* does not intersect each A_i in exactly one point, then *S* and some A_{i_0} are disjoint. Let $d \in A_{i_0}$ then $S \bigcup \{d\}$

belongs to ρ which contradict the maximality of B. Thus B intersects each A_i in exactly one point.

Therefore Zorn's lemma implies Zermelo's postulate and hence axiom of choice.

1.7 ZORN'S LEMMA

Zorn's lemma, also known as the **Kuratowski–Zorn lemma**, named after mathematicians Max Zorn and Kazimierz Kuratowski, is a proposition of the set theory. It states that a partially ordered set containing upper bounds for every chain, i.e., every totally ordered subset essentially contains at least one maximal element. As per the Tychonoff's theorem in topology, every product of compact spaces is compact and the theorems in abstract algebra states that a maximal ideal and every field has an algebraic closure.

Zorn's lemma is equivalent to the **well-ordering theorem** and also to the **axiom of choice**, specified that any one of the three, along with the Zermelo–Fraenkel axioms of set theory, is sufficient to prove and establish the other two. The **Zorn's lemma** is **Hausdorff's Maximum Principle (HMP)** which states that every totally ordered subset of a given partially ordered set is uniquely contained in a maximal totally ordered subset of that partially ordered set.

Statement of the Lemma

- The set P has a reflexive, antisymmetric and transitive binary relation $\leq$ is said to be (partially) ordered by $\leq$. Given two elements x and y of P with $x \leq y$, y is said to be greater than or equal to x. The word 'Partial' specifies that 'Not every pair of elements of a partially ordered set is required to be comparable under the order relation', i.e., in a partially ordered set P with order relation $\leq$ there may be elements x and y with neither $x \leq y$ nor $y \leq x$. An ordered set in which every pair of elements is comparable is termed as totally ordered set.
- Every subset S of a partially ordered set P can itself be seen as partially ordered by restricting the order relation inherited from P to S. A subset S of a partially ordered set P is called a chain (in P) if it is totally ordered in the inherited order.
- An element m of a partially ordered set P with order relation $\leq$ is maximal (with respect to $\leq$) if there is no other element of **P** greater than m, i.e., if there is no s in P with $s \neq m$ and $m \leq s$. Depending on the order relation, a partially ordered set may have any number of maximal elements. Even though, a totally ordered set can have at most one maximal element.

Zorn's Lemma Can Then Be Stated As Following Form

Zorn's Lemma: Assume that a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

NOTES

Zorn's Lemma (for Non-Empty Sets): Assume that a non-empty partially ordered set ***P*** has the property that every non-empty chain has an upper bound in ***P***. Then the set ***P*** contains at least one maximal element.

Fundamentally, this formulation is considered weaker since it places on ***P*** the additional condition of being non-empty, but obtains the same conclusion about ***P***, and precisely the two formulations are equivalent. To verify this, first assume that ***P*** satisfies the condition that every chain in ***P*** has an upper bound in ***P***. Then the empty subset of ***P*** is a chain, as it satisfies the definition therefore the hypothesis implies that this subset must have an upper bound in ***P***, and this upper bound shows that ***P*** is uniquely non-empty. Conversely, if ***P*** is assumed to be non-empty and satisfies the hypothesis that every non-empty chain has an upper bound in ***P***, then ***P*** also satisfies the condition that every chain has an upper bound, as an arbitrary or random element of ***P*** considered as an upper bound for the empty chain.

Example Applications for Zorn's Lemma

Zorn's lemma can be used to explain that every non-trivial ring ***R*** with unity contains a maximal ideal.

Let ***P*** be the set consisting of all (two-sided) ideals in ***R*** except ***R*** itself. The ideal ***R*** was excluded because maximal ideals by definition are not equal to ***R***. Since ***R*** is non-trivial, the set ***P*** contains the trivial ideal **{0},** and therefore ***P*** is non-empty. Furthermore, ***P*** is partially ordered by set inclusion. Finding a maximal ideal in ***R*** is the same as finding a maximal element in ***P***.

To apply Zorn's lemma, take a chain ***T*** in ***P*** i.e., ***T*** is a subset of ***P, i.e.***, totally ordered). If ***T*** is the empty set, then the trivial ideal **{0}** is an upper bound for ***T*** in ***P***. Assume then that ***T*** is non-empty. It is necessary to show that ***T*** has an upper bound, that is, there exists an ideal $I \subseteq R$ which is bigger than all members of ***T*** but still smaller than ***R*** (otherwise it would not be in ***P***). Take ***I*** to be the union of all the ideals in ***T***. We wish to show that ***I*** is an upper bound for ***T*** in ***P***. We will first show that ***I*** is an ideal of ***R***, and then that it is a proper ideal of ***R*** and so is an element of ***P***. Since every element of ***T*** is contained in ***I***, this will show that ***I*** is an upper bound for ***T*** in ***P***, as required.

Because ***T*** contains at least one element, and that element contains at least 0, the union ***I*** contains at least 0 and is not empty. To prove that ***I*** is an ideal, note that if ***a*** and ***b*** are elements of ***I***, then there exist two ideals $J, K \in T$ such that a is an element of ***J*** and ***b*** is an element of ***K***. Since ***T*** is totally ordered, we know that $J \subseteq K$ or $K \subseteq J$. In the first case, both ***a*** and ***b*** are members of the ideal ***K***, therefore their sum $a + b$ is a member of ***K***, which shows that $a + b$ is a member of ***I***.

Now, an ideal is equal to ***R*** if and only if it contains 1. (It is clear that if it is equal to ***R***, then it must contain 1; on the other hand, if it contains 1 and ***r*** is an arbitrary element of ***R,*** then $r1 = r$ is an element of the ideal, and so the ideal is equal to R.) So, if ***I*** were equal to ***R***, then it would contain 1, and that means one of the members of ***T*** would contain 1 and would thus be equal to ***R*** – but ***R*** is explicitly excluded from ***P***.

The proof depends on the fact that the ring ***R*** has a multiplicative unit 1. Without this, the proof would not work and indeed the statement would be false. For example, the ring with ***Q*** as additive group and trivial multiplication (i.e., ***ab*** =0 for all ***a,b***) has no maximal ideal (and of course no 1): Its ideals are precisely the additive subgroups. The factor group ***Q/A*** by a proper subgroup ***A*** is a divisible group, hence certainly not finitely generated, hence has a proper non-trivial subgroup, which gives rise to a subgroup and ideal containing ***A***.

1.8 WELL-ORDERING THEOREM

A binary relation R on a set X is a subset of the product X × X. A relation R is written as x R y for all $x, y \in$ X.

1.8.1 Properties

(a) Reflexivity: x R x is true for all $x \in$ X

(b) Symmetry: if x R y then y R x for all $x \in$ X

(c) Antisymmetry: if x R y and y R x then $x = y$ for all $x, y \in$ X

(d) Asymmetry: if x R y then not y R x for all $x, y \in$ X

(e) Transitivity: if x R y and y R z then x R z

(f) Connex or Comparability: either x R y or y R x.

Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

A relation which is reflexive, transitive and antisymmetric is called a **partial order**. A relation which is antisymmetric, transitive and comparable is called to be a **total order**. It may be stated that comparability implies reflexivity. Therefore, total order is a special type of partial order but partial order need not be total order.

A set X along with a binary relation R is called **totally ordered** if the relation R is a total order. Similarly a set X along with a binary relation R is called **partially ordered** if R is a partial order.

Now let us consider a set Y which is a subset of a partial order X and $y \in Y$ is such that, for every element $y_1 \in Y$, $y \leq y_1$, then y is called the least element of Y.

A set X which is totally ordered set is called well-ordered if its each non-empty subset has a minimum element. It may be observed that every finite set with a total order is well-ordered.

Example 1.8: Let us consider the Cartesian product of Z_+ with the set {A, B}

Z_+ X {a, b} = {a1, a2, a3, ...b1, b2, b3..........}

Alphabetical ordering has been applied on the product set above. In accordance with the alphabetical ordering it can be assumed that the element that comes first, or that is to the left is smaller than the one which comes later or the one that is on the right. It can be noticed that each non-empty subset of the above set will have a smallest element.

NOTES

Example 1.9: The set of positive integers is well-ordered. However, set of integers is not well-ordered because the subset containing negative integers does not have a smallest element.

Example 1.10: Consider a set $A = Z_+ \text{ x } Z_+$. The set A can be represented by dictionary order of an infinite sequence of infinite sequences.

Consider set B to be a subset of A, and C be the subset of Z_+ such that its elements are all first coordinates of elements of set B. Now let c_0 be the smallest element of C. Then we have a set Y as a non-empty subset of Z_+ given as:

$\{y : c_0 \text{ x } y\}$ for all y belonging to B

Now if y_0 is the smallest element of Y then by dictionary order $c_0 \text{ x } y_0$ is the smallest element of B. Therefore A is well-ordered.

From the discussion above following can be summarised:-

(a) Every subset of a well-ordered set A is well-ordered for the restriction of order on A.

(b) If set X and Y are well-ordered sets, then $X \times Y$ is well-ordered in the dictionary order.

Theorem 1.13: Every non-empty finite ordered set has the order type of a section of the set of positive integers Z_+, so it is well-ordered.

Proof: In order to prove the theorem we first show that every finite ordered set X has a smallest element.

If X has only one element, then that element is the smallest. Now, let it be true for a subset having m - 1 elements, then let set X have m number of elements and assume element x_0 belongs to X. Then the set $X – \{x_0\}$ has the smallest element as x_1 then the smaller of $\{x_0, x_1\}$ is the smallest element of X.

Next we prove that there exists an order-preserving bijection of X with $\{1, 2, ...m \}$ for all m. Once again if X has one element, this proof is trivial. Now let us assume that it is true for subsets having m-1 elements. Now assume x_m is the largest element of X. By definition there exists an order preserving bijection such that $g : X – \{x\} \rightarrow \{1, 2, ... m\text{-}1\}$

Define an order preserving bijection $f: X \rightarrow \{1, 2, ... m\}$ by letting

$$f(x) = g(x) \text{ for } x \neq x_m.$$
$$F(x_m) = m.$$

Hence, a finite ordered set has only one order type.

1.8.2 Well-Ordering Theorem

Well-Ordering Theorem (WOT) states that for any set X there exists an order relation on X which is well-ordering. This theorem is also termed as Zermelo's theorem and was proved by Zermelo in 1904. It is equivalent to 'Axiom of Choice'. As per the notion of Georg Cantor's the **well-ordering theorem is unobjectionable principle of thought**. The concept of well-ordering defines that any random or arbitrary uncountable set without any positive procedure has been contested by many mathematicians.

Corollary: There exist an uncountable number of well-ordered sets.

Lemma:There exists a well-ordered set X having a largest element L such that the section S_L of X by L is uncountable but every other section of X is countable.

Proof: Let us assume that set A is an uncountable and well-ordered set.

Let B = {1, 2} x A in the alphabetical order; then some section of B. Let L be the smallest element of B for which the section of B by is uncountable. Then let X have this set and L as its elements. It needs to be born in mind that S_L is an uncountable well-ordered set all sections of which are countable. It is called a minimal uncountable well-ordered set. One of the very important properties of this set is discussed in the next theorem.

Theorem 1.14: If X is a countable subset of S_L, then X has an upper bound in S_L.

Proof: Assume that set X is a countable subset of S_L. For each $x \in X$, the Section S_x is countable. Therefore, the union $Y = U\ S_x$ for all $x \in X$, is also countable. Now as S_L is uncountable, the set Y does not include all elements of S_L. Let p be an element of S_L that is not in Y. Then the element p is an upper bound for X, since if $p < x$ for any $x \in X$, then p belongs to S_x and hence to Y in contradiction of our selection.

1.9 DEFINITION AND EXAMPLES OF TOPOLOGICAL SPACES

The word *topology* is typically used to define a family of sets which uniquely has definite properties that are used to define a topological space, a basic object of topology. *Topological spaces* are mathematical structures that provide the formal definition of concepts, such as convergence, connectedness and continuity. Hence, the area of mathematics that studies topological spaces is called topology.

Definition

A topological space is a set X together with τ, a collection of subsets of X, satisfying the following axioms:

1. The empty set and X are in τ.
2. τ is closed under arbitrary union.
3. τ is closed under finite intersection.

The collection τ is called a **topology** on X. The elements of X are usually called *points*, though they can be any mathematical objects. A topological space in which the *points* are functions is called a function space. The sets in τ are called the open sets and their complements in X are called closed sets. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both **closed** and **open** is called a **clopen set**. The following are examples of topological sets:

- $X = \{1, 2, 3, 4\}$ and collection $\tau = \{\{\ \}, \{1, 2, 3, 4\}\}$ of only the two subsets of X required by the axioms form a topology, the trivial topology (indiscrete topology).

- $X = \{1, 2, 3, 4\}$ and collection $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ of six subsets of X forms a topology.
- $X = \{1, 2, 3, 4\}$ and collection $\tau = P(X)$ (the power set of X) form a third topology, the discrete topology.
- $X = \boldsymbol{Z}$, the set of integers and collection τ equal to all finite subsets of the integers plus $\boldsymbol{Z}$ itself is not a topology, because the union of all finite sets not containing zero is infinite but is not all of $\boldsymbol{Z}$, and so is not in τ.

Equivalent Definitions

There are various additional equivalent ways to define a topological space. Consequently, each of the following defines a category equivalent to the category of topological spaces above. For example, using de Morgan's laws, the axioms defining open sets above become axioms defining closed sets:

1. The empty set and X are closed.
2. The intersection of any collection of closed sets is also closed.
3. The union of any pair of closed sets is also closed.

Using these axioms, another way to define a topological space is as a set X together with a collection τ of subsets of X satisfying the following axioms:

1. The empty set and X are in τ.
2. The intersection of any collection of sets in τ is also in τ.
3. The union of any pair of sets in τ is also in τ.

Under this definition, the sets in the topology τ are the closed sets and their complements in X are the open sets. Topological space can also be defined using the Kuratowski closure axioms, which define the closed sets as the fixed points of an operator on the power set of X. A neighbourhood of a point x is any set that has an open subset containing x. The *neighbourhood system* at x consists of all neighbourhoods of x. A topology can be determined by a set of axioms which specifies all neighbourhood systems.

Various types of topologies can be placed on a set to form a topological space. When every set in a topology τ_1 is also in a topology τ_2, we say that τ_2 is better than τ_1 and τ_1 is coarser than τ_2. A proof that relies only on the existence of certain open sets will also hold for any better topology and similarly a proof that relies only on certain sets not being open applies to any coarser topology. The collection of all topologies on a given fixed set X forms a complete lattice: if $F = \{\tau_á : \alpha \text{ in } A\}$ is a collection of topologies on X, then the meet of F is the intersection of F and the join of F is the meet of the collection of all topologies on X that contain every member of F.

Continuous Functions

A function between topological spaces is termed as **continuous** if the inverse image of every open set is open. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are called *homeomorphic* if there exists a homeomorphism between them. The unique property of of topology, defines that homeomorphic spaces are essentially identical.

NOTES

Examples of Topological Spaces

A given set may have many different topologies. If a set is given a different topology, it is viewed as a different topological space. Any set can be given the discrete topology in which every subset is open. The only convergent sequences in this topology are those that are eventually constant. Also, any set can be given the trivial topology also termed as the indiscrete topology in which only the empty set and the whole space are open. Every sequence in this topology converges to every point of the space. This example demonstrates that in general topological spaces, limits of sequences need not be unique. However, often topological spaces must be Hausdorff spaces where limit points are unique.

There are several methods of defining a topology on **R**, the set of real numbers. The standard topology on **R** is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set. More generally, the Euclidean spaces $\mathbf{R}^n$ can be given a topology. In the usual topology on $\mathbf{R}^n$ the basic open sets are the open balls. Similarly, **C** and $\mathbf{C}^n$ have a standard topology in which the basic open sets are open balls. Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the standard topology on any normed vector space. On a finite dimensional vector space this topology is the same for all norms.

Many sets of linear operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function. Any local field has a topology native to it and this can be extended to vector spaces over that field. Every manifold has a natural topology since it is locally Euclidean. Similarly, every simplex and every simplicial complex inherits a natural topology from $\mathbf{R}^n$.

The **Zariski topology** is defined algebraically on the spectrum of a ring or an algebraic variety. On $\mathbf{R}^n$ or $\mathbf{C}^n$, the closed sets of the Zariski topology are the solution sets of systems of polynomial equations. A linear graph has a natural topology that generalizes many of the geometric aspects of graphs with vertices and edges. The **Sierpiñski space** is the simplest non-discrete topological space. It has important relations to the theory of computation and semantics.

There exist numerous topologies on any given finite set. Such spaces are termed as finite topological spaces. Any set can be given the cofinite topology in which the open sets are the empty set and the sets whose complement is finite. This is the smallest $\mathbf{T}_1$ topology on any infinite set.

Any set can be given the cocountable topology, in which a set is defined as open if it is either empty or its complement is countable. When the set is uncountable, this topology serves as a counterexample in many conditions. The **cocountable topology** or **countable complement topology** on any set X consists of the empty set and all cocountable subsets of X, i.e., all sets whose complement in X is countable. It follows that the only closed subsets are X and the countable subsets of X. Every set X with the cocountable topology is Lindelöf, since every non-empty open set omits only countably many points of X. It is also $\mathbf{T}_1$, as all singletons are closed. The only compact subsets of X are the finite subsets, so X has the property that all compact subsets are closed, even though it is not Hausdorff if uncountable.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on **R** is strictly finer than the Euclidean topology defined above; a sequence converges to a point in this topology if and only if it converges from above in the Euclidean topology. This example shows that a set may have many distinct topologies defined on it.

If Γ is an ordinal number, then the set $\Gamma = [0, \Gamma)$ may be endowed with the order topology generated by the intervals (a, b), $[0, b)$ and (a, Γ) where a and b are elements of Γ.

Topological Constructions

Every subset of a topological space can be given the subspace topology in which the open sets are the intersections of the open sets of the larger space with the subset. For any indexed family of topological spaces, the product can be given the product topology, which is generated by the inverse images of open sets of the factors under the projection mappings. For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows:

If X is a topological space and Y is a set, and if $f : X \to Y$ is a surjective function, then the quotient topology on Y is the collection of subsets of Y that have open inverse images under f. In other words, the quotient topology is the finest topology on Y for which f is continuous.

A common example of a quotient topology is when an equivalence relation is defined on the topological space X. The map f is then the natural projection onto the set of equivalence classes.

The Vietoris topology on the set of all non-empty subsets of a topological space X, named for Leopold Vietoris, is generated on the following basis:

For every n-tuple $U_1, ..., U_n$ of open sets in X, we construct a basis set consisting of all subsets of the union of the U_i that have non-empty intersections with each U_i.

Classification of Topological Spaces

Topological spaces can be precisely classified up to homeomorphism by their topological properties. A topological property is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic find a topological property not shared by them. Examples of such properties include connectedness, compactness and various separation axioms.

NOTES

Topological Spaces with Algebraic Structure

For any algebraic objects we can introduce the discrete topology, under which the algebraic operations are continuous functions. For any such structure that is not finite, we often have a natural topology compatible with the algebraic operations in the sense that the algebraic operations are still continuous. This leads to concepts, such as topological groups, topological vector spaces, topological rings and local fields.

Topological Spaces with Order Structure

- **Spectral:** A space is spectral if and only if it is the prime spectrum of a ring (Hochster theorem).
- **Specialization preorder:** In a space the **specialization** (or **canonical**) **preorder** is defined by $x \leq y$ if and only if $\text{cl}\{x\} \subseteq \text{cl}\{y\}$. Here cl denotes canonical preorder.

1.9.1 Euclidean Spaces

For each $n \in N$, let R denote the set of all ordered n-tuples $x = (x_1, x_2,..., x_n)$, where, $x_1,..., x_n$ are real numbers, called the coordinates of x. The elements of $\mathbf{R}^n$ will be called points or vectors and will always be denoted by letters, a, b, c, x, y, z, etc. Let $y = (y_1, y_2,..., y_n)$ and let a be a real number. We define the addition of vectors and multiplication of a vector by a real number (called scalar) as follows:

$$x + y = (x_1 + y_1,..., x_n + y_n);\ \alpha x = (\alpha\, x_1,...,\alpha\, x_n).$$

These definitions show that $x + y \in \mathbf{R}^n$ and $\alpha x \in \mathbf{R}^n$. It is easy to see that for these operations the commutative, associative and distributive laws hold and thus $\mathbf{R}^n$ is a vector space over the real field $\mathbf{R}$. The zero element of $\mathbf{R}^n$ (usually called the origin or the null vector) is the point 0, all of whose coordinates are zero.

The scalar product or inner product of two vectors x and y is defined by,

$$x, y = \sum_{i=1}^{n} x_i y_i,$$

Also the norm of x is defined by $|x| = (x, x)^{1/2} = \left(\sum_{i=1}^{n} x_i^{\,2}\right)$

The vector space $\mathbf{R}^n$ with the preceding inner product and norm is called Euclidean n-space.

In the sequel, we shall need the following inequalities.

(i) If $z_1, z_2,..., z_n$ are complex numbers (or in particular real numbers), then

NOTES

$$|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|.$$

(ii) If z_i, ω_i $(i = 1,...,n)$ are complex numbers, then,

$$\left|\sum_{i=1}^{n} z_i \overline{\omega_i}\right| \le \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\omega_i|^2\right)^{1/2}$$ **(Schwarz Inequality)**

If a_i, $b_i (i = 1,...,n)$ are real numbers, then the above inequality

reduces to $\left|\sum_{i=1}^{n} a_i b_i\right| \le \left(\sum_{i=1}^{n} a_i^{\,2}\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$

(iii) If z_i, $\omega_i (i = 1,...,n)$ are complex numbers, then

$$\left[\sum_{i=1}^{n} |z_i + \omega_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |z_i|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |\omega_i|^2\right]^{1/2}$$ **(Minkowskis Inequality)**

Theorem 1.15: Let $x, y, z \in \mathbf{R}^n$ and let α be real. Then,

(i) $|x| \ge 0$, (ii) $|x| = 0$ if $x = 0$,

(iii) $|\alpha x| = |\alpha|\ |x|$, (iv) $|x, y| \le |x||y|$,

(v) $|x + y| \le |x| + |y|$, (vi) $|x - y| \le |x - y| + |y - z|$.

Proof: The proofs of Cases (i), (ii) and (iii) are trivial but obvious. To prove (iv), by Schwarz inequality, we have

$$\left|\sum_{i=1}^{n} x_i y_i\right| \le \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} \left[\sum_{i=1}^{n} y_i^2\right]^{1/2}$$

Hence, by the definition of scalar product and norm, we at once obtain $|x.y| \le |x|\ |y|$.

Again using Case (iv), we have

$|x + y|^2 = (x + y) \,.\, (x + y) = x.x + 2x.y + y.y$

$\le |x|^2 + 2|x|\,|y| + |y|^2 = (|x| + |y|)^2$

so that $|x + y| \le |x| + |y|$. Thus Case (v) is proved.

Finally, Case (vi) follows from Case (v) by replacing x by $x - y$ and y by $y - z$.

1.9.2 Metric Spaces

Let X be a non-empty set. A function d from $X \times X$ into $\mathbf{R}$ (the set of reals) is called a metric (or distance function) if for all $x, y, z \in X$, the following conditions are satisfied.

[**m 1**]: $d(x, y) \ge 0$.

[**m 2**]: $d(x, y) = 0$ if and only if $x = y$.

[**m 3**]: $d(x, y) = d(y, x)$. **(Symmetry)**

[**m 4**]: $d(x, z) \leq d(x, y) + d(y, z)$. **(Triangle Inequality)**

(i) The pair (X, d) is called a metric space and $d(x, y)$ is called the distance between the points x and y.

Another definition is, the diameter of a subset A of a metric space X, denoted by $\delta(A)$, is defined by,

$\delta(A) = \sup \{d(x, y): x, y \in A\}$.

(ii) The distance between two subsets A, B, of X, denoted by $d(A, B)$, is defined by,

$d(A, B) = \inf \{d(x, y): x \in A, y \in B\}$.

(iii) The distance between a point $a \in X$ and a set $A \subset X$ is defined by,

$d(a, A) = \inf \{d(a, x): x \in A\}$.

(iv) A subset A of X is said to be bounded if $\delta(A)$ is finite. It follows that A is bounded if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.

The most important examples of metric spaces, for our purposes, are the Euclidean spaces $\mathbf{R}^n$, in particular the real line $\mathbf{R}$ and the complex plane $\mathbf{R}^2$. The distance in $\mathbf{R}^n$ is defined by,

$$d(x, y) = |x - y| \; (x, y \in \mathbf{R}^n). \qquad ...(1.7)$$

The conditions [m1], [m2], [m3] and [m4] are satisfied by Theorem 1.11 and thus $\mathbf{R}^n$ is a metric space.

Another example is Let X be any non-empty set. For $x, y \in X$, define

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y. \end{cases}$$

Then it is easy to see that d is a metric on X called the discrete metric.

It can also be defined as,

(i) Let $a_i < b_i$ for $i = 1, ..., n$. Then the set of all points $x = (x_i, ..., x_n)$ in $\mathbf{R}^n$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq n)$ is a called n-cell.

Thus 1-cell is an interval and 2-cell is a rectangle, etc.

(ii) Let a $\in \mathbf{R}^n$ and let $r > 0$. An open (or closed) ball with centre at a and radius r is defined to be set of all $x \in \mathbf{R}^n$ such that $|x - a| < r$ (or $|x - a| \leq r$ and shall be denoted by,

$B(a, r)$ (or $B[a, r]$), Thus

$B(a, r) = \{x \in \mathbf{R}^n: |x - a| < r\}$

and $B[a, r] = \{x \in \mathbf{R}^n: |x - a| \leq r\}$,

(iii) A subset A of $\mathbf{R}^n$ is called to be convex if,

$\lambda x + (1 - \lambda) y \in A$

whenever $x \in A, y \in A$ and $0 < \lambda < 1$.

A metric space is a kind of topological space. In a metric space any union of open sets is open and any finite intersection of open sets is open. Consequently a metric space meets the axiomatic requirements of a topological space and is thus a topological space. It was, in fact, this particular property of a metric space that was used to define a topological space.

Theorem 1.16: The collection of open spheres in a set X with metric d is a base for a topology on X.

Proof : Let d be a metric on a non-empty set X. The topology τ on X generated by the collection of open spheres in X is called the **metric topology**.

The set X together with the topology x induced by the metric d is a metric space. A metric space then can be viewed as a topological space in which the topology is induced by a metric.

Lemma 1: Let d be the usual metric in three dimensional space **R**. Then the set of open spheres in $\mathbf{R}^3$ constitute a base for a topology on $\mathbf{R}^3$. Thus the usual metric on $\mathbf{R}^3$ induces the usual topology on $\mathbf{R}^3$, the collection of all open sets.

Lemma 2: Let d be the usual metric on the real line **R**, i.e., $d(a, b) = |a - b|$. Then the open spheres in **R** correspond to the finite open intervals in **R**. Thus the usual metric on **R** induces the usual topology, the set of all open intervals, on **R**.

Lemma 3: Let d be the trivial metric on some set X. Note that for any $p \varepsilon X$, $S(p, 1/2) = \{p\}$. Thus every singleton set (set consisting of only one element) is open and consequently every set is open. Hence the trivial metric induces the discrete topology on X.

1.9.3 Properties of Metric Topologies

Theorem 1.17: Let p be a point is a metric space X. Then the countable class of open spheres $\{S(p, 1), S(p, 1/2), S(p, 1/3), S(p, 1/4), ...\}$ is a local base at p.

Theorem 1.18: The closure $\overline{A}$ of a subset A of a metric space X is the set of points whose distance from A is zero.

In a metric space all singleton sets $\{p\}$ are closed.

Theorem 1.19: In a metric space, all finite sets are closed.

Following is an important 'Separation' property of metric spaces.

Theorem 1.20: (Separation Axiom). Let A and B be closed disjoint subsets of a metric space. Then there exist disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

One might think that the distance between two disjoint closed sets would be greater than zero. However, this is not necessarily the case as the following example shows.

For example, the two sets:

$$A = \{(x, y): xy \geq 1, x < 0\}$$

$$B = \{(x, y): xy \geq 1, x > 0\}$$

are closed and they are disjoint. However, $d(A, B) = 0$.

Consider another example where let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane $\mathbf{R}^2$. The usual metric d and the two metrics d_1 and d_2 defined by,

$$d_1(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2)$$
$$d_2(P, Q) = |x_1 - x_2| + |y_1 - y_2|$$

All Induce the usual topology on the plane $\mathbf{R}^2$ since the collection of open spheres of each metric is a base for the usual topology on $\mathbf{R}^2$. The open spheres of each of the three metrics. Thus, the three metrics are equivalent.

Isometric Metric Spaces

A metric space (X, d) is isometric to a metric space (Y, e) if and only if there exists a one-to-one onto function $f: X \to Y$ which preserves distances, i.e., for all $p, q \in X$,

$$d(p, q) = e(f(p), f(q))$$

Theorem 1.21: If the metric space (X, d) is isometric to (Y, e), then (X, d) is also homeomorphic to (Y, e).

Check Your Progress

7. State the Schröder-Bernstein theorem.
8. Differentiate between finite and infinite cardinal number.
9. What is a Cauchy sequence?
10. When is a metric space X said to be complete?
11. What do you mean by continuum hypothesis?
12. Define cardinality of infinite sets.
13. What do you understand by Zorn's lemma?
14. State the well-ordering theorem.
15. Define the topological space.

1.10 CLOSED SETS, CLOSURE AND NEIGHBOURHOODS

Let X be a metric space. All points and sets mentioned here are understood to be elements and subsets of X.

(i) If $r > 0$, the set $N(p, r) = \{x \in X: d(p, x) < r\}$

is called a neighbourhood of a point p. The number r is called the radius of $N(p, r)$.

In the metric space $\mathbf{R}, \in N(p, r) = \{x \in \mathbf{R} : |x - p| < r)$

$= \{x \in \mathbf{R} : p - r < x < p + r\} =]p - r, p + r[$

Thus in this case, neighbourhood of p is an open interval with p as a mid-point.

(ii) A point p is said to be a limit point of the set A if every neighbourhood of p contains points of A other than p.

NOTES

Derived Set: The set of all limit points of A is called the **derived set** of A and shall be denoted by **D** (A).

The subset $A = \{1, \frac{1}{2}, \frac{1}{2} ...\}$ of **R** has 0 as a limit point since for each $r > 0$, we can choose a positive integer n_0 such that $1/n_0 < r$ and since $1/n_0 \neq 0$, we see that every neighbourhood $N(0, r)$ of 0 contains a point of A other than 0.

The set 1 of integers has no limit point whereas the set of limit points of **Q** (the set of rationals) is all of **R** as the reader can easily verify.

(iii) A point p is called isolated point of a set A if $p \in A$ but p is not a limit point of A.

(iv) Set A is said to be closed if $\mathbf{D}(A) \subset A$, that is, if A contains all its limit points.

(v) A point p is called an interior point of A if there exists a neighbourhood N of p such that $N \subset A$.

The set of all interior points of A is called the interior of A and shall be denoted by $A°$.

For example, if $A = [0, 1]$, then $A° =]0, 1[$.

(vi) A set A is said to be open if it contains a neighbourhood of each of its points, that is, if to each $p \in A$, there exists a neighbourhood $N(p)$ of p such that $N(p) \subset A$.

Thus A is open if and only if every point of A is an interior point of A.

Thus for every $r > 0$, the open interval $]p - r, p + r[$ is a neighbourhood of a point $p \in \mathbf{R}$ so a subset A of **R** is open if and only if to each $p \in A$, there exists $r > 0$ such that

$$]p - r, p + r[\subset A.$$

In particular, every open interval $]a, b[$ is an open set.

For if $p \in]a, b[$, take

$r = \min \{p - a, b - p\}$. Then $]p - r, p + r [\subset] a, b[$ showing that $]a, b[$ is open.

(vii) A set A is said to be perfect if A is closed and if every point of A is a limit point of A.

(viii) The closure of a set A is the union of A with its derived set $\mathbf{D}(A)$ and shall be denoted by $\overline{A}$.

(ix) A set A is said to be dense in another set B if $\overline{A} \supset B$.

Also, A is said to be dense in X or everywhere dense if $\overline{A} = X$.

(x) A set A is said to be nowhere dense or non-dense if $\overline{A}$ contains no neighbourhoods.

It is easy to see that a set A is nowhere dense if and only if $(\overline{A})^{o} = \emptyset$.

NOTES

(xi) A set A is said to be bounded if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.

Theorem 1.22: In a metric space, every neighbourhood is an open set.

Proof: Let $N(a, r)$ be a neighbourhood of any point $a \in X$ so that $N(a, r) = \{x \in X : d(a, x) < r\}$.

If $p \in N(a, r)$ be arbitrary, then $d(a, p) < r$.

Let $\delta = r - d(a, p) > 0$. We shall show that,

$N(p, \delta) \subset N(a, r)$.

Indeed $y \in N(p, \delta)$ implies $d(p, y) < \delta$ and the triangle inequality shows that,

$$d(a, y) \leq d(a, p) + d(p, y) < d(a, p) + \delta$$
$$= d(a, p) + r - d(a, p) = r.$$

This implies that $y \in N(a, r)$.

Thus we have shown that,

$y \in N(p, \delta) \Rightarrow y \in N(a, r)$ and so $N(p, \delta) \subset N(a, r)$. Hence, $N(a, r)$ is neighbourhood of p and since p was any point in $N(a, r)$, we conclude that $N(a, r)$ is an open set.

Theorem 1.23: Let p be a limit point of a subset A of a metric space. Then every neighbourhood of p contains infinitely many points of A.

Proof: Suppose p has a neighbourhood N which contains only a finite number of points of A. Let $q_1, ..., q_n$ be those points of $N \cap A$, which are distinct from p. Let,

$r = \min \{d(p, q) : 1 \leq i \leq n\}$.

Then $r > 0$ being the minimum of a finite set of positive numbers. The neighbourhood $N(p, r)$ contains no point of A different from p so that p is not a limit point of A which is contradiction. Hence, every neighbourhood of p must contain infinitely many points of A, thus establishing the theorem.

Corollary: A finite point set has no limit points.

Theorem 1.24: A set A is open if and only if its complement is closed.

Proof: Suppose A is open. To show that its complement A' is closed. Let x be any limit point of A'. Then every neighbourhood of x contains a point of A'. This implies that no neighbourhood of x can be contained in A and so x is not an interior point of A. Since A is open, this means that $x \in A'$ and consequently A' is closed.

Conversely, let A' be closed and let x be an arbitrary point of A. Then $x \notin A'$. Since A' is closed x cannot be limit point of A'. Hence, there exists a neighbourhood N of x such that N contains no point of A', that is, $N \subset A$. Thus, A contains a neighbourhood of each of its points and so A is open.

Theorem 1.25: Let A be a closed subset of **R** which is bounded above. If u be the least upper bound or lub of A, then $u \in A$.

NOTES

Proof: Suppose $u \notin A$. For every $h > 0$, there is a point $x \in A$ such that $u - h < x \leq u$, for otherwise $u - h$ would be an upper bound of A. Thus every neighbourhood of u contains a point x of A. Also $x \neq u$ since $u \notin A$. It follows that u is a limit point of A, which is not a point of A. Therefore, A is not closed which contradicts the hypothesis. Hence $u \in A$ as desired.

Note: If A is closed and bounded below and if l is greatest lower bound or glb of A, then $l \in A$.

Proof is similar to that of the preceding theorem.

Theorem 1.26: Let (X, d) be a metric space. Then,

(i) The empty set $\varnothing$ and the whole space X are open as well as closed.

(ii) Union of an arbitrary collection of open sets is open.

(iii) Union of finite number of closed sets is open.

(iv) Intersection of an arbitrary collection of closed sets is closed.

Proof: To prove case (i), let $\{A_\lambda : \lambda \in A\}$ be an arbitrary collection of open sets.

Let, $\quad A = \cup \{A_\lambda : \lambda \in A\}$.

$x \in A \Rightarrow x \in A$ for some $\lambda \in A$

$\Rightarrow$ There exists $\varepsilon > 0$ such that $N(x, \varepsilon) \subset A_\lambda \quad [\because A\lambda$ is open$]$

$\Rightarrow N(x, \varepsilon) \subset A\ \varepsilon \quad [\because A\lambda \subset A]$

$\Rightarrow A$ is open (By Definition).

Proofs of Cases (ii) and (iii) at once follow by using De Morgan laws for complements. Thus to prove Case (ii) let A_i, $i = 1, 2, ..., n$ be a finite collection of closed sets. Then,

A_i is closed $\quad \Rightarrow X - A_i$ is open $\forall\ i = 1, 2, ..., n$.

$\Rightarrow \cap \{X - A_i : i = 1, 2, ..., n\}$ is open by Case (iii).

$\Rightarrow X - \cup \{Ai : i = 1, 2, ..., n\}$ is open.

[By De Morgan Law]

$\Rightarrow \{A_i : i = 1, 2, ..., n\}$ is closed.

1.10.1 Neighbourhoods

A subset N of **R** is called a neighbourhood of a point $p \in$ **R** if N contains an open interval containing p and contained in N, that is, if there exists on open interval $]a, b[$ such that,

$$p \subseteq]a, b[\subseteq N.$$

It immediately follows that an open interval is a neighbourhood of each of its points. For practical purposes, it will therefore suffice to take open intervals containing a point of its neighbourhoods.

Note if]a, b[is an open interval containing a point p so that $a < p < b$, we can always find an $\varepsilon > 0$, such that $]p - \varepsilon, p + \varepsilon[\subseteq]a, b[$. Choose any ε less than the smaller of the two numbers $p–a$ and $b–p$]. Clearly $]p–\varepsilon, p + \varepsilon[$ is an open interval containing p and so it is a neighbourhood of p. We shall use this form of neighbourhood of p, usually called an ε-neighbourhood of p and shall denote it by $N(p, \varepsilon)$. We call ε as the radius of $N(p, \varepsilon)$. The point p itself is a mid-point of centre of $N(p, \varepsilon)$. It is evident that,

$$x\,\text{I}\;, (p, \varepsilon) \text{ if } |x - p| < \varepsilon,$$

We shall use the abbreviated form '*nhd*' for the word 'neighbourhood'.

(i) The closed interval [1, 3] is an *nhd* of 2 since $]\frac{3}{2}, \frac{5}{2}[$ is an open interval such that $2 \subseteq][\frac{3}{2}, \frac{5}{2}[\subseteq [1, 3]$. But [1, 3] is not an *nhd* of 1 since there exists no open interval which contains 1 and contained in [1, 3]. Similarly [1, 3] is not an *nhd* of 3.

(ii) The set N of natural numbers is not *nhd* of any of its points since no open interval can be a subset of N.

1.10.2 Open Sets

A subset G of **R** is called open if for every point $p \in G$, there exists an open interval I such that $p \in I \in G$.

This is equivalent to saying that G is open if for every $p \in G$, there exists ε-*nhd* $N(p,\varepsilon) =]p–\varepsilon, p + \varepsilon[$such that $N(p, \varepsilon) \in G$.

For example,

(i) Every open interval is an open set.

(ii) The empty set ϕ and the whole real line **R** are open sets. Since ϕ contains no points, the preceding definition is satisfied. Hence ϕ is open.

To show that **R** is open, we observe that for every $p \in \mathbf{R}$ and every $\varepsilon > 0$, we have $]p–\varepsilon, p + \varepsilon[\in \mathbf{R}$. Hence, **R** is open.

Now consider the following examples,

(i) The closed open interval [2, 3[is not open, since there exists no ε-*nhd* of 2 contained in [2, 3[.

(ii) The set $A=\{1/n : n \in N\}$ is not open since no point of A has an ε-*nhd* contained in A.

Theorem 1.27: The union of any collection of open sets is an open set.

Proof: Let C be any collection of open sets and let S be their union, that is, let $S = \bigcup \{G : G \in C\}$. Let $p \in S$. Then p must belong to at least one of the sets in C, say $p \in G$. Since G is open there exists and ε-*nhd* $N(p, \varepsilon)$ of p such that $N(p, \varepsilon) \in G$. But $G \in S$, and so $N(p, \varepsilon) \in S$. Hence S is open.

Theorem 1.28: The intersection of a finite collection of open sets is open.

Proof: Let $S = \bigcap_{i=1}^{n} G_i$, where each G_i is open. Assume $p \in S$.

NOTES

If S is empty, there is nothing to prove. Then $p \in G_i$ for every i=1, 2,....,n. Since G_i is open, there exists $\varepsilon_i > 0$ such that $N(p, \varepsilon_i) \in G_i$, for every i = 1, 2,..., n. Let $\varepsilon = \min [\varepsilon_1, \varepsilon_2,...., \varepsilon_n]$.

Then $N(p, \varepsilon) \in \bigcap_{i=1}^{n} G_i = S$. Hence S is open.

Note: The intersection of an infinite collection of open sets is not necessarily open. For example, if $G_n =]-1/n, 1/n]$ $(n \in N)$, then each G_n is open (being an open interval), but $\bigcap_{i=1}^{n} G_n = \{0\}$ which is not open since there exists no $\varepsilon > 0$ such that] $-\varepsilon, \varepsilon[$ I $\{0\}$.

Example 1.11: Show that complement of every singleton set in **R** is open. More generally, the complement of a finite set is open.

Solution: Let $\{x)$ be a singleton set in **R**. To show that its complement $\{x)'$ is open. Let $y \in \{x\}'$. If $\{x\}' = \phi$, there is nothing to prove. Then $y \neq x$. Set $|x - y| = r > 0$. Let $0 < \varepsilon < r$. Then $N(y, \varepsilon) =]\, y - \varepsilon, y + \varepsilon[$ does not contain x and therefore $N(y, \varepsilon) \in \{x)'$. Hence $\{x\}'$ is open. Again, if **A**=$\{x_1, x_2..., x_n\}$ is any finite subset of **R**, then we can write **A**=$\{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}$. Then **A'**=$[\{x_1\} \cup ... \cup \{x_n\}]' = \{x_1\}' \cap ... \cap \{x_n\}'$. Since each $\{x_1\}'$ is open.

1.10.3 Closed Sets

Definition: A set F in **R** is called closed if its complement F is open.

For example, (i) Every closed interval $[a,b]$ is closed, since its complement $[a,b]'$ $=]-\infty, a[\cup]\, b, \infty[$is open, being a union of two open intervals.

(ii) Every singletion set in **R** is closed. More generally every finite set in **R** is closed.

(iii) The closed open interval $[a,b[$ is closed.

Theorem 1.29: (i) The union of a finite collection of closed sets is closed.

(ii) The intersection of an arbitary collection of closed sets is closed.

Proof: (i) Let F_i $(i = 1,2,..., n)$ be a finite collection of closed sets. Then each F_i' is open. By De Morgan law, we have F_i' which is open, being the intersection of finite collection of open sets. Hence F_i is closed.

Let C be an arbitrary collection of closed sets. Then each $F \in C$ is closed and so its complement F' is open. By De Morgan law, we have $(\bigcap F)' = (F \in C)$, which is open. Hence $\cap F\, (F \in C)$ is closed.

For example if A is open and B is closed, then show that AB is open sets and $BA = B \cap A'$, which is the intersection of two closes sets. Hence, AB is open and BA is closed.

NOTES

Accumulation Points: Adherent Points

If $A \in \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an accumulation point (or a limit point) of A if every ε-*nhd* $N(p, \varepsilon)$ of p contains a point of A distinct from p.

The set of all accumulation points of A is called the first derived set (or simply the derived set) of A and is denoted by $\mathbf{D}(A)$. The first derived set of $\mathbf{D}(A)$ is called the second derived set of A and is denoted by $\mathbf{D}^2(A)$.In general, nth derived set of A is denoted by $\mathbf{D}^n(A)$.

For example,

(i) Every point of the closed interval $[a,b]$ is an accumulation point of the set of points in the open interval $]a,[b$. So in this case $\mathbf{D}]a,b[=a,b]$.

(ii) 0 is the only accumulation point of the set $A=\{1/n:n \in N\}$.

Hence $\mathbf{D}(A)=\{0\}$.

(iii) Every real number is an accumulation point of the set Q of rational numbers and so $\mathbf{D}(\mathbf{Q}) = \mathbf{R}$.

(iv) If $A = [2,3[$, then $\mathbf{D}(A) = [2,3]$.

If $A \in \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an adherent point of A if every ε-*nhd* of p contains a point of A. The set of all adherent points of A is called the adherence of A denoted by $Adh(A)$.

Theroem 1.30: If p is an accumulation point of A,then every ε-*nhd* of p contains infinitely many points of A.

Proof: Assume the contrary, that is suppose there exists an ε-*nhd* $(b, \varepsilon) = p+\varepsilon[$of p which cantains only a finite number of point distinct from p, say $p_1, p_2, \ldots p_n$. Let r denote the smallest of the positive numbers.

$$|p - p_1|, |p - p_a|, ..., |p - p_n|.$$

Then $N(p, r/2) =]p-r/2, p+r/2[$ is an ε-*nhd* of p which contains no points of A distinct from p. This is a contradiction. Hence every ε-*nhd* of p must contain infinitely many points of A.

A set is said to be of **first species** if it has only a finite number of derived sets. It is said to be of **second species** if the number of its derived sets is infinite.

Note that if a set is of first species, then its last derived set must be empty.

A set whose $(n+1)$th derived set is empty is called a set of ***n*th order**.

For example, the set $\mathbf{Q}$ of all rational numbers is of second species, since

$$\mathbf{D}(\mathbf{Q}) = \mathbf{R},\ \mathbf{D}^2(\mathbf{Q}) = \mathbf{D}(\mathbf{R}) = \mathbf{R}.$$

$$\mathbf{D}^3(\mathbf{Q}) = \mathbf{D}(\mathbf{R}) = \mathbf{R}, \text{ etc.}$$

Hence, all the derived sets of $\mathbf{Q}$ are equal to $\mathbf{R}$.

Consider that $A = \{1/n : n \in N\}$. Then $\mathbf{D}(A) = \{0\}$. Since $\mathbf{D}(A)$ consists of a single point, it cannot have any limit point and so $\mathbf{D}^2(A) = \phi$. Therefore A is of first species and first order.

NOTES

Again let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}+\frac{1}{3}, \frac{1}{4}, \frac{1}{2}+\frac{1}{4}, \frac{1}{5}, \frac{1}{2}+\frac{1}{5} ....\}$. Then A has the limit point 0 and also the limit point $\frac{1}{2}$, for the subset $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ is dense at the origin and the subset $\{\frac{1}{2}, \frac{1}{2}+\frac{1}{3}, \frac{1}{2}+\frac{1}{4}, ...\}$ is dense at $\frac{1}{2}$. Therefore $\mathbf{D}(A) = \{0, \frac{1}{2}\}$ and $\mathbf{D}^2(A) = \phi$. Hence A is of first species and first order.

For example, let $A = \{1, \frac{1}{2}, (\frac{1}{2})^2, \frac{1}{2} + (\frac{1}{2})^2, (\frac{1}{2})^3, \frac{1}{2} + (\frac{1}{2})^3, (\frac{1}{2})^2 + (\frac{1}{2})^3, (\frac{1}{2})^4, \frac{1}{2} + (\frac{1}{2})^4, (\frac{1}{2})^2 + (\frac{1}{2})^4, (\frac{1}{2})^3 + (\frac{1}{2})^4, (\frac{1}{2})^5, ...\}$.

Then it is clear by inspection that,

$$\mathbf{D}(A) = \{0, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, ...\}, \mathbf{D}^2(A) = \{0\} \text{ and } \mathbf{D}^3(A) = \phi.$$

Hence in this case, A is of first species and second order.

Let the points of a set A be given by,

$$\frac{1}{3^{S_1}} + \frac{1}{5^{S_2}} + \frac{1}{7^{S_3}} + \frac{1}{11^{S_4}},$$

Where S_1, S_2, S_3, S_4 each have all positive integral values.

Hence $\mathbf{D}(A)$ consists of the four sets of points given by,

$$\frac{1}{3^{S_1}} + \frac{1}{5^{S_2}} + \frac{1}{7^{S_3}}, \frac{1}{3^{S_1}} + \frac{1}{5^{S_2}} +, \frac{1}{11^{S_4}}, \frac{1}{3^{S_1}} + \frac{1}{7^{S_3}}, \frac{1}{11^{S_4}}$$

$$\frac{1}{5^{S_2}} + \frac{1}{7^{S_3}} + \frac{1}{11^{S_4}}$$ and the six sets of the points,

$$\frac{1}{3^{S_1}} + \frac{1}{5^{S_2}}, \frac{1}{3^{S_1}} + \frac{1}{7^{S_2}}, \frac{1}{3^{S_1}} + \frac{1}{11^{S_4}}, \frac{1}{5^{S_2}} + \frac{1}{7^{S_3}}$$

$$\frac{1}{5^{S_2}} + \frac{1}{11^{S_4}}, \frac{1}{7^{S_3}} + \frac{1}{11^{S_4}}$$ and four sets of the points,

$$\frac{1}{3^{S_1}}, \frac{1}{5^{S_2}}, \frac{1}{7^{S_3}}, \frac{1}{11^{S_4}}$$ together with the single point 0.

$\mathbf{D}^2(A)$ consists of the last ten of these sets and the point 0, $\mathbf{D}^3(A)$ consists of the last four sets and the point 0 and $\mathbf{D}^4(A)$ consists of the point 0 only. Thus, the set A is of the first species and fourth order.

NOTES

Example 1.12: Show that, zeros of sin $(1/x)$ form a set of first order, zeros of sin $\left(\dfrac{1}{\sin\frac{1}{x}}\right)$ form a set of second order; zeros of sin $\left(\dfrac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right)$ form a set of third order, and so on.

Solution: The zeros of $\sin\frac{1}{x}$ are given by $\sin\frac{1}{x} = 0$, which gives $\frac{1}{x} = n\pi$ or $x = \frac{1}{np}$, where n is an integer. Hence if A is the set of zeros of $\sin\frac{1}{x}$, then A consists of all of the form $\frac{1}{np}$ where n is an integer. Clearly $\mathbf{D}(A) = \{0\}$ and therefore A is of first order.

Again $\sin\dfrac{1}{\sin\frac{1}{x}} = 0$, gives $\dfrac{1}{\sin\frac{1}{x}} = n\pi$, where n is an integer.

Then $\sin\frac{1}{x} = \frac{1}{np}$, from which the general value $\frac{1}{x}$ is given by,

$$\frac{1}{x} = n\pi + \sin^{-1} = \frac{1}{np} \text{ where } m \text{ is an integer}$$

Or
$$x = \frac{1}{m\pi + \sin^{-1}\left(\frac{1}{n\pi}\right)} \qquad ....(1)$$

Thus the zeros of $\sin\left(\dfrac{1}{\sin\frac{1}{x}}\right)$ form a set, say B, of points of the form Equation (1). It is clear that $\mathbf{D}(B)$ consists of points of the form $\frac{1}{m\pi}$ and the point 0 and so $\mathbf{D}^2(B)$ consists of the single point 0. Hence B is of second order.

Similarly we can show that the zeros of $\sin\left(\dfrac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right)$ form a set of third order, and so on.

Theorem 1.31 (Bolzano-Weierstrass Theorem): A bounded infinite set of real numbers has at least one limit point.

NOTES

Proof: Let A be an infinite bounded subset of **R**. Then, there exist finite constants m and M such that $m \geq a \geq$ M for all $a \in A$. At least one of the intervals [m, (m+M), M] must contain an infinite number of points of A. We rename such an interval as [a_1+b_1]. Similarly, one of the intervals [$a_{1,} \frac{1}{2}$ $(a_1 + b_1)$, $[a_1 + b_1]$, $[\frac{1}{2}(a_1 + b_1), b_1]$ contains infinitely many points of A and we desingnate it as $[a_2, b_2]$. Now proceed with $[a_2, b_2]$ as we did with $[a_1, b_1]$. Continuing in this way, we obtain a sequence of closed intervals, $\{I_n\}=\{[a_n, b_n]\}$ such that $I_n \supseteq I_{n+1}$ and $[I_n] = b_n - a_n = \frac{M-m}{2^n} \to 0$ as $n \to \propto$. Hence I_n consists of a single point, say x_0. We choose n such that $b_n - a_n < \varepsilon$.

Then $I_n \subset]x_o - \varepsilon, x_0 + \varepsilon[$ and consequently the interval $]x_o - \varepsilon, x_0 + \varepsilon[$contains an infinite number of points of A. It follows that x_0 is a limit point of A. This completes the proof.

1.10.4 Closed Sets and Accumulation Points

A closed set was defined to be the complement of an open set. The Theorem 1.18 describes closed sets in another way.

Theorem 1.32: A set A is closed if and only if it contains all its accumulation points**.**

Proof: Assume A closed and P is an accumulation point of A. We have to prove that $P \in A$. suppose, if possible, $P \in A$. Then $P \in A$*,* and since A*,* is an open, there exists an ε-*nhd* $N(p,\varepsilon)=]sP-, P + \varepsilon[$such that $N(p, \varepsilon) \subset A$. Consequently $N(p, \varepsilon)$ contains no point of A, contradicting the fact that p is an accumulation point of A. Hence, we must have $P \in A$.

Conversely, we assume that A contains all its accumulation points and show that A is closed. Let $P \in A'$. Then $P \in A$, so by hypothesis p is not an accumulation point of A. Hence there exists an ε-*nhd* $N(p, \varepsilon)$ of p which contains no point of A other then p. Since $P \in A$, $N((p, \varepsilon)$ contains no point of A and consequently N $(p, \varepsilon) \subset \forall A'$. Therefore A' is open, and hence A is closed.

Theorem 1.33: All the derived sets **D** (A), $\mathbf{D}^2$ (A), ... $\mathbf{D}^n$ (A),... of a given set A are closed sets and each of these derivatives, after the first, consists of points belonging to the preceding one and therefore to **D** (A).

Proof: We first prove that **D** (A) is closed. Let p be any accumulation point (limit point) of **D**. Then every ε-*nhd* of p contains infinitely many points of **D** (A) and since point of **D** (A) is an accumulation point of A, every ε-*nhd* of p must contain infinitely many points of A. Thus, p is also a limit point of A and so $p \in$ **D** (A). Therefore, **D** (A) contains all its accumulation points and so **D** (A) is closed. Again since $\mathbf{D}^2$ (A) is the derived set of **D** (A), and so it must be closed as proved earlier. Similarly $\mathbf{D}^2$ (A), ..$\mathbf{D}^n$ (A), ... are closed sets.

To prove the second part of the theorem, let $x \in \mathbf{D}^n (A)$, $n \geq 2$ and suppose, if possible, that $p \in \mathbf{D} (A)$. Then there can be found an ε-*nhd* of x which contains only a finite number of points of A or no such points and this *nhd* would therefore contain no points of $\mathbf{D} (A)$ and consequently it can contain no point of $\mathbf{D}^2 (A)$. $\mathbf{D}^3 (A)$, ..., $\mathbf{D}^n (A)$ which is contrary to the hypothesis that $x \in \mathbf{D}^n (A)$. Hence every point of $\mathbf{D}^n (A)$ $(n \geq 2)$ must belong to $\mathbf{D} (A)$.

1.10.5 Closure

The closure of a set A in **R** is the smallest closed set containing A and is denoted by $\overline{A}$.

By this definition , $A \in \overline{A}$. Also $\overline{A}$ is always a closed set of A.

Theorem 1.34: Let A be a set in **R**. Then $\overline{A} = \text{A} \bigcup \text{D} (A)$, i.e., A is the set of all adherent points of A.

Proof: We first observe that $A \bigcup \mathbf{D} (A)$ is closed. For if p is any limit of $A \bigcup \mathbf{D} (A)$. then either p is a limit point of A or a limit of $\mathbf{D} (A)$. If p is a limit point of A, then $p \in \mathbf{D} (A)$. If p is a limit point of $\mathbf{D}(A)$, then since $\mathbf{D} (A)$ is closed $p \in \mathbf{D} (A)$. So on either case, $p \in \mathbf{D} (A)$ and surely then $p \in A \bigcup \mathbf{D} (A)$. Hence $A \bigcup \mathbf{D} (A)$ is closed.

Now $A \bigcup \mathbf{D} (A)$ is a closed set containing A, and since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \in A \bigcup \mathbf{D} (A). \quad ...(1.8)$$

Also $A \in \overline{A} \Rightarrow \mathbf{D} (A) \in \mathbf{D}(A)$ .. (1.9)

Since $\overline{A}$ is closed, we have $\mathbf{D} (\overline{A}) \in \overline{A}$. ... (1.10)

∴ From Equations (1.9) and (1.10), $\mathbf{D} (A) \in \overline{A}$.

Moreover, $A \in \overline{A}$, and $\mathbf{D} (A)\, A \in \overline{A} \Rightarrow A \bigcup \mathbf{D} (A) \in \overline{A}$ ... (1.11)

∴ From Equations (1.8) and (1.9), $\overline{A} = A \bigcup \mathbf{D} (A)$.

Theorem 1.35: Let A, B be subsets of **R**, Then:

(i) $A \cap B \Rightarrow \overline{A} \cap \overline{B}$, (ii) $\overline{A \bigcup B} \cap \overline{A} \cup \overline{B}$.

(iii) $\overline{A \bigcup B} \cap \overline{A} \cup \overline{B}$, (iv) $\overline{\overline{A}} = \overline{A}$.

Proof: (i) Given $A \cap B$. But $B \cap \overline{B}$ always. Hence $A \cap B$, Thus $\overline{B}$ is a closed set containing A. But $\overline{A}$ is the smallest closed set containing A. Hence $\overline{A} \cap \overline{B}$.

(ii) $A \cap A \bigcup B \Rightarrow \overline{A} \cap \overline{A \bigcup B}$ and $B \cap A \bigcup B \Rightarrow \overline{B}\ \overline{A \bigcup B}$ by Case (i).

Hence, $\overline{A} \cup \overline{B} \cap \overline{A \bigcup B}$. ... (1.12)

Again $A \cup \overline{A}$ and $B \cup \overline{B} \Rightarrow \text{A} \cup B \cup \overline{A} \cup \overline{B}$. But $\overline{A} \cup \overline{B}$ is closed, being the union of two closed sets. Thus $\overline{A} \cup \overline{B}$ is a closed set containing $A \bigcup B$. But $\overline{A \bigcup B}$ is the smallest closed set containing $A \bigcup B$. Hence $\overline{A \bigcup B} \cup \overline{A} \cup \overline{B}$ ... (1.13)

NOTES

$\therefore$ From Equations (1.12) and (1.13), $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iii) $A \cap B \cup A \Rightarrow \overline{A \cap B} \cup \overline{A}$ and $A \cap B \cup B \Rightarrow A \cap B \cup \overline{B}$ by Case (*i*).

Hence $\overline{A \cap B} \cup \overline{A} \cap \overline{B}$.

(iv) Since $\overline{A}$ is closed, hence $\overline{\overline{A}} = \overline{A}$.

Consider the following examples:

(i) If **Q** is the set of rational numbers, then

$\mathbf{D}(\mathbf{Q}) = \mathbf{R}$ and so $\overline{Q} = \mathbf{Q} \cup \mathbf{D}(\mathbf{Q}) = \mathbf{Q} \cup \mathbf{R} = \mathbf{R}$.

(ii) Let $A =]2, 3[$. Then $\mathbf{D}(A) = [2, 3]$ and so,

$\overline{A} = A \cup \mathbf{D}(A) = [2, 3]$.

(iii) Let A be any finite set in **R**. Since every finite set of real numbers is closed, we have $\overline{A} = A$.

1.11 DENSE SUBSETS

In topology, a subset A of a topological space X is called **dense** (in X) if any point x in X belongs to A or is a limit point of A. Generally, for every point in X the point is either in A or arbitrarily 'Close' to a member of A, for example every real number is either a rational number or has one arbitrarily close to it. Formally, a subset A of a topological space X is dense in X if for any point x in X, any neighborhood of x contains at least one point from A. Equivalently, A is dense in X if and only if the only closed subset of X containing A is X itself. This can also be expressed by saying that the closure of A is X or that the interior of the complement of A is empty. The **density** of a topological space X is the least cardinality of a dense subset of X.

Definition: Let A, B be subset of **R**. Then.

(i) A is said to be dense in B if $B \subset \overline{A}$.

(ii) A is said to be **dense in R** or everywhere dense if $\overline{A} = \mathbf{R}$.

(iii) A is said to be **non-dense** or **nowhere dense** if $(\overline{A})^\circ = \phi$, that is, if the interior of the closure is empty.

Thus, A is non-dense if $\overline{A}$ contains no (non-empty) open interval. It follows that a closed set is non-dense if it contains no open intervals.

(iv) A is said to be dense-in-itself if $A \subset D(A)$, that is, if every point of A is limit point of A.

(v) A is said to be perfect if A is dense-in-itself and closed.

Let A be a subset of **R** and let $p \in A$. Then p is called an isolated point of A if there exists an ε-*nhd* of a which contains no point of A other than p itself. A set A is called isolated or discrete if all its points are isolated points.

NOTES

An isolated point of a set is characterized by the fact that it is not a limit point of that set.

It follows from this definition that a set is perfect if and only if it is closed and has no isolated points.

Theorem 1.36: A set A is perfect if and only if $A = \mathbf{D}(A)$.

Proof: A is perfect $\Leftrightarrow A$ is dense-in-itself and closed.

$\Leftrightarrow$ Every point of A is a limit point of A and every limit point of A belongs to A.

$\Leftrightarrow A \subset \mathbf{D}(\text{A})$ and $\mathbf{D}(A) \subset A$

$\Leftrightarrow A = \mathbf{D}(A)$.

Theorem 1.37: Every countable set is of first category.

Proof: Let A be a countable set, so that we may write it as,

$$A = \{x_1, x_0, x_3, ..., x_n, ...\}.$$

A can also be written as a countable union of one point sets as,

$$A = \bigcup_{i=1}^{\infty} \{x_i\}.$$

Since every one point set $\{x_n\}$ is nowhere dense, we see that A is of first category.

Consider the following examples:

(i) Since $\overline{Q} = \mathbf{R}$, the set **Q** of all rational points is everywhere dense. It is also dense-in-itself since $\mathbf{Q} \subset \mathbf{D}(\text{Q}) = \mathbf{R}$. It is not, however, perfect since it is not closed. It is of first catagory since it is countable.

(ii) Let $A = [0, 1]$. Then A is perfect since $A = \mathbf{D}(A)$.

(iii) Let $A = \{1/n : n \in N\}$. Then every point of A is an isolated point since it is not a limit point of A. Hence A is discrete. A is also non-dense, since $\overline{A} = A \cup \{0\}$ and so no open interval can be a subset of $\overline{A}$.

Theorem 1.38: If A and B are sets of first category, then $A \cup B$ is also of first category.

Proof: Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, where each A_n and each B_n is nowhere dense. Then $A \cup B$ is the union of all A_n's and B_n's. But all the sets A_n and B_n form a countable collection. Hence $A \cup B$ is of first category.

Thus the assumption was inadmissible and as per **reduction ad absurdum**, the theorem follows.

For the converse, the same proof will apply with the necessary difference in the first paragraph.

NOTES

Theorem 1.39: If a set A is dense-in-itself, then its first derived set $\mathbf{D}(A)$ is perfect.

Proof: Since A is dense-in-itself, we have $A \cup \mathbf{D}\ (A)$. This implies that:

$A \cup \mathbf{D}\ (A) = \mathbf{D}(A)$.

$\therefore \quad \mathbf{D}\ [A \cup \mathbf{D}(A)]$

$= \mathbf{D}(\mathbf{D}(A))$

Or $\quad \mathbf{D}(A) \cup \mathbf{D}(\mathbf{D}(A))$

$= \mathbf{D}(\mathbf{D}(A)$

Or $\quad \mathbf{D}(A) \cup \mathbf{D}^2(A)$

$= \mathbf{D}^2(A)$.

Since $\mathbf{D}(A)$ is closed, we have $\mathbf{D}^2(A) \cup \mathbf{D}(A) \cup \mathbf{D}(A)$

which implies that $\mathbf{D}(A) \cup \mathbf{D}^2(A) = \mathbf{D}(A)$.

Thus, $\mathbf{D}(A) = \mathbf{D}^2(A)$.

Hence $\mathbf{D}(A)$ is perfect.

Theorem 1.40: Every isolated set of point is countable.

Proof: Let A be an isolated set. Then each point of A is an isolated point and hence can be enclosed in an interval containing no other point of the set. If any two of these intervals intersect, one or both may be shortened so that they become disjoint. We now show that the collection **C** of these intervals is countable. To show this, let $\{x_1, x_2, x_3 ...\}$ denote the countable set of rational numbers. In each interval of the collection **C**, there will be infinitely many x_n, but among these there will be exactly one with smallest index n. We now define the mapping $f{:}\rightarrow N$ by the equation $f(I)=n$, if x_n is the rational number in I with the smallest index n. This mapping is one-one since I, $J \in \mathbf{C}$ and $f(I)=F(J)=n$ implies that I and J have x_n in common and this implies that $I=J$. Note that since the intervals in **C** are disjoint, they cannot have a point in common unless they are identical.

Thus f established a one-one correspondence between the intervals of **C** and a subset of N (the set of natural numbers). Hence **C** is countable.

Since each intervals in **C** contains one and only one point of A, it follows that A is countable.

NOTES

1.12 INTERIOR, EXTERIOR AND BOUNDARY OPERATIONS

The sets of all interior points, all exterior points and all boundary points are respectively called the interior, the exterior and the boundary.

(i) Let A be a subset of **R** and let $p \in A$. Then p is called an interior point of A if there exists and an ε-*nhd* of p contained in A, i.e., if there exists an $\varepsilon > 0$ such that $]p-\varepsilon, p+\varepsilon[\, \subset A$. The set of all interior points of A is called the interior of A and is denoted by $A°$ or by **int** A.

(ii) A point p is called an exterior point of A if there exists an ε-*nhd* of p contained in the complement A' of A. The set of all exterior points of A is called exterior of A and is denoted by **ext** A.

(iii) A point p is called a boundary point (or frontier point) of A if it is neither an interior nor an exterior point of A. The set of all boundary points of A called the boundary (or frontier) of A and is denoted by $b(A)$ [or Fr (A)].

Theorem 1.41: Let A be a subset of **R**, then

(i) $A°$ is an open set.

(ii) $A°$ is the largest open set contained in A.

(iii) A is open if and only if $A° = A$.

The easy proof is left for the reader.

Theorem 1.42: Let A, B be any subsets of **R**. Then prove that,

$(A \cap B)° = A° \cap B°$,

Proof: The following is the proof of the theorem:

$A \cap B \subset A \Rightarrow (A \cap B)° \subset A°$ and $A \cap B \subset B \Rightarrow (A \cap B)° \subset B°$.

$$\therefore\ (A \cap B)° \subset A° \cap B°. \qquad ......(1.14)$$

Again $A° \subset A$ and $B° \subset B \Rightarrow A° \cap B° \subset A \cap B$.

Also $A° \cap B°$ is open, being the intersection of two open sets.

Thus $A° \cap B°$ is an open set contained in $A \cap B$. But $(A \cap B)°$ is the largest open set contained in $A \cap B$. Hence,

$$A° \cap B° \subset (A \cap B). \qquad ....(1.15)$$

From Equation (1.15) $(A \cap B)° = A° \cap B°$.

For example, (i) Let $A =]0, 1[$. Since A is open $A° = A =]0, 1[$.

Since $A' =]-\infty, 0] \cup [1, \infty,[$, **ext** $A = (A')° =]-\infty, 0[\cup]1, \infty[$.

Note that 0 and 1 are not the interior points of A'.

NOTES

$\therefore$ b (A) = {0, 1}, since boundary of A consists of those points of A which are neither interiror points of A nor exterior points of A.

(ii) Let **Q** be the set of all rational points. Then $\mathbf{Q}^\circ = \phi$, since no point of **Q** can have an ε-*nhd* contained in **Q**.

ext Q = $(\mathbf{Q}')^\circ = \phi$, since $\mathbf{Q}'$ consists of all irrational points and no ε-*nhd* of an irrational point can be contained in $\mathbf{Q}'$. Note that every ε-*nhd* of a point always consists of an infinite number of rational as well as irrational points. Hence $b(\mathbf{Q}) = \mathbf{R}$.

1.13 BASES AND SUBBASES

Consider a set X and a proper subset, $\phi \neq A \subset X$. Then {ϕ, A, X} is a topology containing A. Actually, it is the smallest topology. Further, suppose $\phi \neq B \neq A \subset X$. The smallest possible topology that contains both A, B is clearly {ϕ, A, B, $A \cap B$, $A \cup B$, X}. Similarly, if there are several different sets $A_1, \ldots, A_n$ in the topology, many other sets must be also there. This is the idea behind generating a topology.

Definition 1: Let (X, **T**) be a topological space. A set $\mathcal{B} \subset \mathbf{T}$ is called a base for **T** if,

$T = \{\cup A : A \subset B\}$

Any $G \in \mathcal{B}$ is called a basic open set. We will obtain the topology T by taking all arbitrary unions of sets in $\mathcal{B}$.

Let us consider the following collection of subsets in **R**,

$S = \{(a, \infty) : a \in \mathbf{R}\} \cup \{(-\infty, b) : b \in \mathbf{R}\}$.

It is not a base for the standard topology of **R**, because the intersection of two such subsets may not be in itself. However, these finite intersections are all the open intervals and hence they form a base. This becomes the typical example for a subbase.

Definition 2: Let (X, T) be a topological space. A set $S \subset \mathbf{T}$ is called a subbase if $\{S_1 \cap \ldots\ldots \cap S_n : S_j \in S, n \in \mathbf{Z}\}$ is a base for **T**. Equivalently, $\mathbf{T} = \bigcup \{B \subset X : \mathcal{B} \bigcap \mathcal{A}$ for some finite subset $\mathcal{A} \subset S\}$.

Here, we can also say that **T** is generated by S. However, the same **T** may be generated by different subbases. A topological space (X, **T**) may have different bases or subbases.

Theorem 1.43: Any non-empty collection $\mathbf{C} \subset \mathcal{P}(X)$ defines a topology **T** such that **C** is a subbase for **T**.

Proof: Let $\mathcal{B} = \{\bigcap \mathcal{F} : \mathcal{F} \subset \mathbf{C}, \#(\mathcal{F}) < \infty\}$. Then, the topology is generated by **C**, $\mathbf{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$. In order to verify that **T** is a topology, use the De Morgan's law.

However, not every C can be a base; additional conditions are needed. The rest of the proof is left as an exercise.

1.14 SUBSPACES AND RELATIVE TOPOLOGY

Topological Spaces: If X is a set, a family $\mathcal{U}$ of subsets of X defines a topology on X if,

(i) $\phi \in \mathcal{U}$, $X \in \mathcal{U}$.

(ii) The union of any family of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

(iii) The intersection of a finite number of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

If $\mathcal{U}$ defines a topology on X, then we say that X is a topological space. The sets in $\mathcal{U}$ are called open sets. The sets of the form $X \backslash U$, $U \in \mathcal{U}$, are called closed sets. If Y is a subset of X, then the closure of Y is the smallest closet set in X that contains Y.

Let Y be a subset of a topological space X. Then we may define a topology $\mathcal{U}_Y$ on Y, called the subspace or relative topology or the topology on Y induced by the topology on X, by taking

$$\mathcal{U}_Y = \{Y \cap U \mid U \in \mathcal{U}\}.$$

A system $\mathcal{B}$ of subsets of X is called a basis (or base) for the topology $\mathcal{U}$ if every open set is the union of certain sets in $\mathcal{B}$. Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{U}$ such that $x \in B \subset U$.

For example, the set of all bounded open intervals in the real line **R** forms a basis for the usual topology on **R**.

Let $x \in X$. A neighbourhood of x is an open set containing x. Let $\mathcal{U}_x$ be the set of all neighborhoods of x. A subfamily $\mathcal{B}_x$ of $\mathcal{U}_x$ is a basis or base at x, a neighborhood basis at x, or a fundamental system of neighbourhoods of x, if for each $U \in \mathcal{U}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset U$. A topology on X may be specified by giving a neighbourhood basis at every $x \in X$.

If X and Y are topological spaces, then there is a natural topology on the Cartesian product $X \times Y$ that is defined in terms of the topologies on X and Y, called the product topology. Let $x \in X$ and $y \in Y$. The sets $U_x \times V_y$ as U_x ranges over all neighbourhoods of x and V_y ranges over all neighbourhoods of y, forms a neighbourhood basis at the point $(x, y) \in X \times Y$, for the product topology.

If X and Y are topological spaces, a function $f: X \to Y$ is continuous if whenever U is an open set in Y, the set $f^{-1}(U) = \{x \in X | f(x) \in U\}$ is an open set in X. A function $f: X \to Y$ is a homeomorphism of X onto Y if f is bijective and both f and f^{-1} are continuous functions.

An **open covering** of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family. A **subcover** of an open covering is an open covering of X which consists of sets belonging to the open covering. A topological space X is compact if every open covering of X contains a finite subcover.

NOTES

A subset Y of a topological space X is compact if Y is compact in the subspace topology. A topological space X is locally compact if for each $x \in X$ there exists a neighbourhood of x whose closure is compact.

A topological space X is Hausdorff (or $\mathbf{T}_2$) if given distinct points x and $y \in X$, there exist neighbourhoods U of x and V of y such that $U \cap V = \phi$. A closed subset of a locally compact Hausdorff space is locally compact.

Topological Groups

A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \to g_1g_2$ from $G \times G \to G$ and $g \to g^{-1}$ from G to G are continuous maps. In this definition, $G \times G$ has the product topology.

Lemma: Let G be a topological group. Then

(i) The map $g \to g^{-1}$ is a homeomorphism of G onto itself.

(ii) Fix $g_0 \in G$. The maps $g \to g_0g$, $g \to gg_0$ and $g \to g_0gg\ g_0^{-1}$ are homeomorphisms of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G and let $p: G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $U_{G/H}$ on G/H, called the quotient topology, by $\mathcal{U}_{G/H} = \{p(U) \mid U \in \mathcal{U}_G\}$. Here, $\mathcal{U}_G$ is the topology on G. By definition the canonical map p is open and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a closed subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map f: $G \to G'$ is a continuous homomorphism of G into G' if f is a homomorphism of groups and f is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H.

A topological group G is a locally compact group if G is locally compact as a topological space.

Theorem 1.44: Let G be a locally compact group and let H be a closed subgroup of G. Then,

(i) H is a locally compact group in the subspace topology.

(ii) If H is normal in G, then G/H is a locally compact group.

(iii) If G' is a locally compact group, then $G \times G$ is a locally compact group in the product topology.

NOTES

Check Your Progress

16. Define the term derived set.
17. What happens when p is called an interior point of A?
18. What are closed and open sets?
19. Define the term dense subset in topology.
20. When a point p is called an exterior point?
21. What is base?
22. What is topological group?

1.15 SOLVED EXAMPLES

Examples 1: If X and Y are countable sets, then XY is also a countable Sets.

Solution: Suppose that X and Y are countable.

By theorem : If X is a non-empty set then,

1. X is countable.

2. There exists a surjection $f : \mathbb{N} \to X$

3. There exists a injection $g : X \to \mathbb{N}$.

By the above theorem,

There exists surjection $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$.

Define a function $h = \mathbb{N} \to (X \cup Y)$ by,

$$h(k) = \begin{cases} f\left(\frac{k}{2}\right) k \text{ is even} \\ g\left(\frac{k+1}{2}\right) k \text{ is odd.} \end{cases}$$

Since f and g are subjective, we have that,

$h(\mathbb{N}) = f(\mathbb{N}) \cup g(\mathbb{N}) = X \cup Y$

And thus h is a subjective function.

Hence, by the above Theorem we have that $X \cup Y$ is countable.

Example 2: Show that $\mathbb{Q}$ is Countably Infinite

Solution: As we know that $\mathbb{Z}$ is countable since there is an injection f: $\mathbb{Z} \setminus \{0\} \to \mathbb{Z}$. By definition $f(x) = x$ for all x

By theorem: if X is a non-empty set, and Y is a countably set then:

1. X is countable

2. There exists a surjection function f: $Y \to X$

3. There exists an injection function g: $X \to Y$.

By above Theorem, we also have That $\mathbb{Z} \setminus \{0\}$ is countable.

NOTES

Theorem: Let $n \in \mathbb{N}$, and let $X_1, X_2, ...X_n$ be non-empty countable sets.

Then $\prod_{I=1}^{n} X_i = X_1 \times X_2 \times ...... \times X_n$ is countable

By the above theorem we have that $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$ is countable

Consider the function $g = \mathbb{Z}\times(\mathbb{Z}\setminus\{0\}) \rightarrow Q$

defined by $g(a,b)=a\backslash b$

Note that this is well defined, as $b \neq 0$. Moreover since carry element of $\mathbb{Q}$ can be expressed in an least one way as a ratio of integers with a non-zero denominator, we have that g is rejective.

But we know that Q is constable by theorem 1 finally, Q is not finite as $\mathbb{N} \leq$ Q and any subset of a finite set must also be finite.

Therefore, Q is countably infinite.

Example 3: Use the choice axiom to show that if *f*: *A* → *B* is surjective, then *f* has a right inverse *h*: *B* → *A*.

Solution: Suppose that $h: B \rightarrow A$ is a right inverse to f so foh = I_B.

The identity map I_B is bijective, hence f is surjective by theorem.

Conversely if f is surjective, then for each $b \in B$, there is some $a \in A$ so that $f(a) = b$.

Thus for each $b \in B$. The set $S_b = \{a \in A: f(a) = b\}$ is non-empty.

So, $X = \{S_b: b \in A\}$ is a set of non-empty sets according to Axiom of choice, there is a choice function is,

$F: X \rightarrow$ UX so that $f(s_b)S$. for each $b \in B$ Notice that since $F(S_b)$ is an element of S_b. We must have $f(F(S_b)) = b$ for any possible choice function. If follows that we may take the choice function $h = F$ itself as a right inverse, and the choice function F exists.

(i.e, the hypothesis of the Axiom of Choice that X is a set of non-empty sets is satisfied)

Because f is a surjection.

Hence, if $f: A \rightarrow B$ is surjective, thus f has right inverse $h: B \rightarrow A$.

Example 4: Let *X* be a set, let *B* be a basis for a topology ζ on *X*. Then ζ equals the collection of all unions of elements of *B*.

Solution: Given a collection elements of B. They are also elements of ζ. Because ζ is a topology. There union is in ζ.

Conversely, given $U \in \zeta$, choose for each $X \in U$ an element Bx of B such that, $x \in B_x \subset$ U. Then = U = $\bigcup_{x \in U} B_x$ so U equals a union of elements of B.

Example 5: Let *X* be a non-empty set and $\mathbb{B} = \{\{X\}: x \in X\}$. Then *B* is a basis for topology on *X*.

Solution: (i) for every $x \in X$ there exists $B = \{X\} \in \mathbb{B}$ such that $x \in B$

(ii) $B_1, B_2 \in \mathbb{B}$ and $X \in B_1 \cap B_2$ implies There exists

$B_3 = \{X\} \in \mathbb{B}$ such that $X \in B_3 \subseteq B_1 \cap B_2$

NOTES

Hence, both (B_1) and (B_2) are satisfied,

This implies that the collection $\mathbb{B}=\{\{X\}: x\in X)$ is a basis for a topology on X.

Now let us find out $\zeta_{\mathbb{B}}$. The topology generated by $\mathbb{B}$

We define $\zeta_{\mathbb{B}}$ as $\zeta_B = \{U \subseteq X\text{: } x\in U$ implies there exists $B\in\mathbb{B}$ such that $X\in B\subseteq U$ is a topology on X.

In this case for any non-empty subset U of X, $x\in U$ implies. There exists $B=\{x\}$ such that $x\in B\leq U$.

Hence, by the definition of $\zeta_{\mathbb{B}}$. $A\in B$ whenever A is a non-empty subset of X. Also the full set $\Phi\in\zeta_{\mathbb{B}}$.

Hence $A\subseteq X$ implies $A\in\zeta_{\mathbb{B}}$ implies $P(X)\subseteq\zeta_{\mathbb{B}}$. Also by the definition, $\zeta_{\mathbb{B}}\subseteq P(X)$, the collection of all subset of X this implies that $\zeta_{\mathbb{B}}$ is same as the discrete $\zeta_{\mathbb{B}}$ defined on X.

Example 6: Let X={1, 2, 3} and ζ={Φ, X, {1}, {2}, {1,2}, {1,3}, {2,3}}, Is ζ a topology on X?

Solution: (i) Let $A=\{1, 3\}$, $B = \{2, 3\}$ here $A\in\zeta$, $B\in\zeta$, but $A\cap B = \{3\}\notin\zeta$ Hence ζ is not a topology on X.

(ii) Let $X = \{1, 2, 3\}$ and $\zeta=\{\Phi, X, \{1\}, \{2\}, \{1,2\}$ then ζ is a topology on X. Now $A=\{2, 3\}$ is a subset of X. $2\in A$ and also there is an open set $U=\{2\}$ such that $2\in U$ and $U\subseteq A$. Hence 2 is an interior point of A. But 3 is not an interior point of A. How to check 3 is an interior point of A or not?

Step 1. First check whether $3\in A$ (if x is an interior point of A. Then it is essential that $x\in A$) so yes here $3\in\{2, 3\}=A$.

Step 2. Now find out all the open sets containing 3. X be the only open set containing 3 but this open set is not contained in A. Hence 3 is not an interior point of A.

What will happen if the given set A is an open subset of a topological space X. Our aim is to check whether an element $x\in X$ is an interior point of A.

Step 1. It is essential that $x\in A$.

Step 2. It is necessary to find out all the open set containing X? Of-course not necessary. It is enough if we find at least an open U such that $x\in U$ and $U\subseteq A$.

In this case the given set A is an open set and hence there exists an open set $U=A$ such that $x\in U$ and $U=A\subseteq A$.

Therefore every element x of A is an interior point of A. That is $A\subseteq A°$.

By definition $A°\subseteq A$, Hence $A° = A$ that is if A is an open set then $A° = A$.

What about the converse?

Suppose for a subset A of X, $A° = A$

Is A an open set? Yes, A is an open subset of X.

Take $x\in A$ Then $x\in A°$. Hence by the definition of $A°$ there exists at least one open set say Ux such that $x\in Ux$ and $Ux\subseteq A$.

This implies that $A = \bigcup_{x \in A} U_x$.

Now by the definition, ζ is closed under arbitrary union. Hence, for each $x \in A$, $Ux \in \zeta$ implies $\bigcup_{x \in A} x \in \zeta$ implies $A \in \zeta$.

NOTES

That is, A is an open set. Thus we have proved.

Example 7: Let X be a topological space. Then prove that following points:

(i) Φ and X are closed.

(ii) Arbitrary intersections of closed sets are closed.

(iii) Finite unions of closed sets are closed.

Solution: (i) Φ and X are closed because they are the complements of the sets X and Φ, respectively

(ii) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply De-Morgan's law

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

Since the set $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets and is thus open. Therefore $\cap A_\alpha$ is closed.

(iii) Similarly if A_i is closed for $i = 1,....n$, consider the equation,

$$X = \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open hence $\cup A_i$ is closed.

Example 8: Let A be a subset of the topological space X, let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'$$

Solution: If x is in A' every neighbourhood of x then intersects A (in a point different from x)

Therefore, by theorem that if A be a subset of the topological space X.

Then $x \in \bar{A}$ iff every open set U containing x intersects A.

So, x belongs to $\bar{A}$ hence $A' \subset \bar{A}$ since by definition $A \subset \bar{A}$, it follows that $A \cup A' \subset \bar{A}$.

To demonstrate the reverse inclusion, we let x be a point of $\bar{A}$ and show that $x \in A \cup A'$.

If x happens to lie in A, it is trivial that $x \in A \cup A'$; suppose that x does not lie in A. Since $x \in \bar{A}$, we know that every neighbourhood U of x intersects A, because $X \notin A$, the set U must intersects A a point different from x. Then $x \in A'$, so that $x \in A \cup A'$, as described.

Example 9: Let A be a subset of the topological space x if

(a) Then $X \in \overline{A}$, iff every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$, iff carry basis element B containing x intersects A.

Solution: Consider the statement in (a). It is a statement of the form $P \Leftrightarrow Q$ let in transform each implication to its contrapositive. Thereby obtaining the logically equivalent statement (not P) $\Leftrightarrow$ (not Q).

$x \notin \overline{A} \Leftrightarrow$ There exists an open set U containing x that does not intersects A.

If x does not in , the set $U = X- \overline{A}$ is an open set containing x that does not intersects A, as desired conversely if there exists an open set U containing x which does not intersects A, then $X–U$ is a closed set containing A. By definition of the closure $\overline{A}$, the set $X–U$ must contain $\overline{A}$, therefore x can not be in $\overline{A}$.

Statement (b) follows readilay if every open set containing x intersects A, so does every basis element B containing x, because B is an open set.

Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

Example 10: (a) Let X be a topological space and suppose that we can write $X = F_1, U...UF_n$, where each F_i is closed. Let Y be another topological space, Let $f: X \rightarrow Y$ and let $f_i: \rightarrow Y$ be the restriction of f to F_i. That is for $X \in F_i$ we set $f_i(X) = f(X)$ show that f is continuous iff F_i is continuous for all i.

Solution: Let $G \subset Y$ be closed observe that,

$$fi^{-1}(G) = f^{1}(G) \cup F_i$$

If f is continous then $f^{-1}(G)$ is closed in X, so $f^{-1}(G) \cap F_i$ is closed in F_i by definition of the subspace topology thus f_i is continous.

Conversely, suppose that f_i is continous for all i because $X = F_i \cup ..., \cup F_n$, we have,

$$\begin{aligned} f^{-1}(G) &= f^{-1} \cap (G)(F_l \cup .... \cup F_n) \\ &= (f^{-1}(G) \cap F_1) \cup .... \cup (f^{-1}(G) \cap F_n) \\ &= f_i^{-1}(G) \cap ..... \cap f_n^{-1}(G). \end{aligned}$$

Because f_i is continous, we know that $f_i^{-1}(G)$ is closed in f_i, here is closed in X.

So, $f^{-1}(G)$ is a finite union of closed sets hence is closed thus f is continous.

(b) Show that if A is closed in Y and Y is closed in X then A is closed in X.

Solution: Since A is closed in Y we can write,

$$A = F \cap Y$$

Where F is closed in x.

Since an interaction of closed set is closed and Y is closed in X, it follows that.

$$F \cap Y = A$$

NOTES

So, A is closed in x.

Example 11: The principle of well-ordering may not be true over real number or negative integers. In general not every set of integers or real numbers must have a smallest element. Explain by giving examples.

Solution: The set $\mathbb{Z}$

The set $\mathbb{Z}$ has no smallest element because given any integers x, it is clear that $x - 1 < x$, and this argument can be repeated indefinitely. Hence, $\mathbb{Z}$ does not have a smallest element

2. The open interval (0,1)

A similar problem occurs in the interval (0,1) If x lies between 0 and 1, then so $\frac{x}{2}$ and $\frac{x}{2}$ lies between 0 and x.

such that

$$0 < x < 1 \Rightarrow 0 < \frac{x}{2} < x < 1$$

This process can be repeated indefinitely. Yielding

$$0 < ... < \frac{x}{2^n} < ... \frac{x}{2^3} < \frac{x}{2^2} < \frac{x}{2} < x < 1$$

We keep getting and smaller numbers. All of them are positive and less than 1. There is no end in sight, hence the interval (0,1) does not have a smallest dement.

1.16 ANSWERS TO 'CHECK YOUR PROGRESS'

1. A set X is defined as countable if it is finite or it can be positioned in 1, –1 correspondence with the positive integers. The non-negative integers are countable by mapping n to $n + 1$. The even numbers are countable; map n to $n/2$. The integers are countable. Map n to $2n$ for $n - 0$, and map n to $1-2n$ for $n < 0$. To check that whether given set is countable or not, we must check that:
   - Every element of set is represented in order,
   - No element is repeated,
   - X is infinite.
2. If a set is not countable, then it is called uncountable set. For example, set of binary sequences forms uncountable sets.
3. If a set has an infinite number of elements it is an infinite set then the elements of such a set cannot be counted by a finite number. A set of points along a line or in a plane is called a point set. A finite set has a finite subset. An infinite set may have an infinite subset.

NOTES

4. 'Axiom of Choice' is a very significant axiom which is extensively utilised in mathematics. The axiom can be stated in many ways. It is also true that many seemingly unrelated statements on closer analysis appear to be equivalent to it.

   Definition. Let X be a non-empty collection of non-empty and disjoint sets, then there exists a set Y consisting of exactly one element from each element of X. In other words, a set Y such that Y is contained in the union of the elements of X, and for each X $\in$ X, the set X $\cap$ Y has only one element.

   Another way of stating this axiom is that for any family X of non-empty and disjoint sets, there exists a set that consists of exactly one element from each element of X.

5. The two sets A and B are said to be equivalent or equipotent if there exists a bijective map $f: A \to B$ and we write $A \sim B$. It is easy to see that the relation of equipotence ~ is an equivalence relation on P[X]. Therefore, this relation must divide P[X] into equivalence classes and we shall use the term cardinal number (or power) to designate the property that equipotent sets have in common. The cardinal number will be, therefore, a measure of the number of points in sets. It can be that all equipotent sets have the same cardinal number. The cardinal number of a set A will be denoted by $|A|$.

   It follows that $A \sim B \Leftrightarrow |A| = |B|$.

6. Among infinite sets, we denote by **a** or $\mathbf{N_0}$ (read 'Aleph Nought') the cardinal number of all denumerable sets and by c the cardinal number of all those sets which are equipotent to the set R of all real numbers. The cardinal c is often called the cardinal number of the linear continuum.

7. In set theory, the Schröder–Bernstein theorem specifically states that, if there exists injective functions $f: A \to B$ and $g: B \to A$ between the sets A and B, then there also exists a bijective function $h: A \to B$.

8. Finite Cardinal Numbers: For finite cardinal numbers the obvious symbols are used like 0 is assigned to empty set ϕ, and n is assigned to the set $\{1, 2, ...n\}$

   Infinite Cardinal Numbers: Infinite sets cardinal numbers are called infinite cardinal numbers.

9. A sequence $< x_n >$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$.

   $$m, n \geq n(\varepsilon) \Rightarrow d(x_m, x_n) < \varepsilon.$$

10. A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

11. In mathematics, the term Continuum Hypothesis (CH) is uniquely defined as a hypothesis which defines the possible or probable sizes of the infinite sets. It states that there is not any set whose cardinality can be uniquely defined between the integers and the real numbers.

12. Two sets are considered to have the same or equivalent *cardinality* or cardinal number if there typically exists a bijection, i.e., a one-to-one

NOTES

correspondence between them. Spontaneously, for two sets S and T to have the same or equivalent cardinality implies that it is feasible to 'Pair Off' elements of S with elements of T in such a manner that every element of S is paired off with precisely or exactly one element of T and vice versa.

Specifically for the infinite sets, such as the set of integers or rational numbers, the existence of a bijection condition between the two sets can hardly be demonstrated.

13. Zorn's lemma, also known as the Kuratowski–Zorn lemma, named after mathematicians Max Zorn and Kazimierz Kuratowski, is a proposition of the set theory. It states that a partially ordered set containing upper bounds for every chain, i.e., every totally ordered subset essentially contains at least one maximal element. As per the Tychonoff's theorem in topology, every product of compact spaces is compact and the theorems in abstract algebra states that a maximal ideal and every field has an algebraic closure.
14. Well-Ordering Theorem (WOT) states that for any set X there exists an order relation on X which is well-ordering. This theorem is also termed as Zermelo's theorem and was proved by Zermelo in 1904. It is equivalent to 'Axiom of Choice'. As per the notion of Georg Cantor's the well-ordering theorem is unobjectionable principle of thought. The concept of well-ordering defines that any random or arbitrary uncountable set without any positive procedure has been contested by many mathematicians.
15. A topological space is a set X together with τ, a collection of subsets of X, satisfying the following axioms:
    (i) The empty set and X are in τ.
    (ii) τ is closed under arbitrary union.
    (iii) τ is closed under finite intersection.

    The collection τ is called a topology on X. The elements of X are usually called *points*, though they can be any mathematical objects.
16. The set of all limit points of A is called the derived set of A and shall be denoted by **D** (A).
17. A point p is called an interior point of A if there exists a neighbourhood N of p such that $N \subset A$.
18. A subset G of **R** is called open if for every point $p \in G$, there exists an open interval I such that $p \in I \in G$.

    A set F in **R** is called closed if its complement F is open.

    For example, (i) Every closed interval $[a,b]$ is closed, since its complement $[a,b]' =]-\infty, a[\cup]\, b, \infty[$ is open, being a union of two open intervals.
19. In topology, a subset A of a topological space X is called dense (in X) if any point x in X belongs to A or is a limit point of A. Generally, for every point in X the point is either in A or arbitrarily 'Close' to a member of A, for example every real number is either a rational number or has one arbitrarily close to it. Formally, a subset A of a topological space X is dense in X if for any point x in X, any neighborhood of x contains at least one point from A.

Equivalently, A is dense in X if and only if the only closed subset of X containing A is X itself. This can also be expressed by saying that the closure of A is X or that the interior of the complement of A is empty. The **density** of a topological space X is the least cardinality of a dense subset of X.

Definition: Let A, B be subset of **R**. Then.

(i) A is said to be dense in B if $B \subset \overline{A}$.

(ii) A is said to be dense in **R** or everywhere dense if $\overline{A} = \mathbf{R}$.

(iii) A is said to be non-dense or nowhere dense if $(\overline{A})^\circ = \phi$, that is, if the interior of the closure is empty.

20. A point p is called an exterior point of A if there exists an ε-*nhd* of p contained in the complement A' of A. The set of all exterior points of A is called exterior of A and is denoted by **ext** A.

21. Let (X, **T**) be a topological space. A set $\mathcal{B} \subset \mathbf{T}$ is called a base for **T** if,

$$T = \{\cup A : A \subset B\}$$

Any $G \in \mathcal{B}$ is called a basic open set. We will obtain the topology T by taking all arbitrary unions of sets in $\mathcal{B}$.

22. A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \to g_1 g_2$ from $G \times G \to G$ and $g \to g^{-1}$ from G to G are continuous maps. In this definition, $G \times G$ has the product topology.

1.17 SUMMARY

- All the finite sets of integers are termed as countable, but not for the infinite subsets. Here is a simple diagonalization argument. If the infinite sets are countable then the correspondence builds a list of all possible subsets.
- If a set A is countable and infinite, then there is a bijection between the set A and set of natural numbers N.
- If set A is countable and $A' \subset A$, then A' is also countable.
- Two sets A and B are said to be equivalent if these exist a one-one onto mapping from A to B. If A and B are equivalent, we denote this relation by the symbol '~'.
- The set $X - f(\{1, 2, ...n-\})$ is a non-empty set because if it was empty, then the function $f: \{1, 2, ... n-1\} \to X$ would be a surjective function and X would be finite. Now we can select an element of set $X - f(\{1, 2, ... n-1\})$ and accept $f(n)$ to be this element. Therefore, utilising the principle of induction we have defined the function f for all $n \in Z_+$.
- A choice function is a function f. defined on a collection X of non-empty sets, such that for every set A in X, $f(A)$ is an element of A.
- Each choice function on a collection X of non-empty sets is an element of the Cartesian product. Given any family of non-empty sets their Cartesian product is a non-empty sets.

- Every set has a choice function, which is equivalent, For any set *A* there is a function *F* such that for any non-empty subset *B* of *A*, F (*B*) lies in *B*.
- The basic property of the individual non-empty sets in the collection may make it possible to avoid the axiom of choice even for certain infinite collections.
- Zermelo's Postulate: Let $\{A : i \in I\}$ be any non-empty family of disjoint non-empty sets. Then there exists a subset *B* of the union $\bigcup_{i \in I} A_{0\,i}$ such that the intersection of *B* and each set A_i consists of exactly one element.
- Show that the axiom of choice is equivalent to Zermelo's postulate.
- Let X be a non-empty collection of non-empty sets, then there exists a function *f* such that it selects one member of each of the sets which are elements of X.
- In other words we have constructed X^I as a collection of ordered pairs, where the first element is the set X, and the second element is an element of X. That is the set X^I is a subset of the Cartesian product $X^I \bullet$ U X for all $X \in X^I$.
- For finite sets, the concept of cardinal number is easy to grasp. We say that any set which is equipotent to the set,

  $\{1, 2,..., n\}$

  has the cardinal number n. We postulate that |0| = 0.
- Among infinite sets, we denote by **a** or $\mathbf{N_0}$ (read 'Aleph Nought') the cardinal number of all denumerable sets and by c the cardinal number of all those sets which are equipotent to the set R of all real numbers. The cardinal c is often called the cardinal number of the linear continuum.
- The operation of addition of cardinal numbers as defined earlier is a well-defined operation. For if C and D are disjoint sets with $|C| = \lambda$ and $|D| = \mu$, then $A \sim C$ and $B \sim D$ and consequently $A \cup B \sim C \cup D$.
- Each of the following subsets of **R** has the cardinal number c of the linear continuum.
- The cancellation laws of addition and multiplication do not hold for infinite cardinal numbers.
- We now define the final arithmetic operation of taking powers for cardinal numbers, if *A* and *B* are sets, we denote by A^B the set of all mappings $f: B \to A$.
- The set of all real valued functions defined on the closed unit interval [0, 1] has the cardinal number 2^c.
- According to the cardinality property of the two sets, this classically implies that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$, i.e., *A* and *B* are equipotent. This is a significant feature in the ordering of cardinal numbers.
- By the fact that ***f*** and ***g*** are injective functions, each *a* in *A* and *b* in *B* is in exactly one such sequence to within identity: if an element occurs in two sequences, then all elements to the left and to the right must be the same in

NOTES

both, by the definition of the sequences. Therefore, the sequences form a partition of the disjoint union of *A* and *B*.

- A sequence is called an *A*-stopper if it stops at an element of *A*, or a *B*-stopper if it stops at an element of *B*. Otherwise, it is called doubly infinite if all the elements are distinct or cyclic if it repeats.
- We want to know the size of a given set without necessarily comparing it to another set. For finite set there is no difficulty. For example, the set A = {1, 2, 3} has 3 elements. Any other set with 3 elements is equipotent to A. On the other hand, for infinite sets it is not sufficient to just say that the set has infinitely many element since not all infinite ses are equiptent. To solve this problem we introduce the concept of a coordinal number.
- The bower set *P* (*B*) of any set *B* has cardinality greater than *B*. i.e., $|B| < |P(B)|$.
- It is easy to prove that every convergent sequence in a metric space is Cauchy sequence.
- Let (*X, d*) be a complete metric space and let $<F_n>$ be a nested sequence of non-empty closed subsets of *X* such that $\delta(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.
- In Zermelo–Fraenkel Set Theory and the Axiom of Choice (ZFC or Zermelo–Fraenkel Continuum) is considered equivalent to the equation in aleph numbers: $2^{N0} = N_1$.
- The name continuum hypothesis is precisely given to this hypothesis which comes from the term the *continuum* uniquely defined for the real numbers.
- The independence property from ZFC implies that either proving or disproving the CH within ZFC is not possible. Even though, the negative results or consequences of Gödel and Cohen are not universally accepted for determining the continuum hypothesis. The continuum hypothesis and the axiom of choice were considered as the first mathematical statements that demonstrated to be independent of ZF set theory.
- Zorn's lemma is equivalent to the well-ordering theorem and also to the axiom of choice, specified that any one of the three, along with the Zermelo–Fraenkel axioms of set theory, is sufficient to prove and establish the other two. The Zorn's lemma is Hausdorff's Maximum Principle (HMP) which states that every totally ordered subset of a given partially ordered set is uniquely contained in a maximal totally ordered subset of that partially ordered set.
- Every subset ***S*** of a partially ordered set ***P*** can itself be seen as partially ordered by restricting the order relation inherited from ***P*** to ***S***. A subset ***S*** of a partially ordered set ***P*** is called a chain (in ***P***) if it is totally ordered in the inherited order.
- Assume that a partially ordered set ***P*** has the property that every chain in ***P*** has an upper bound in ***P***. Then the set ***P*** contains at least one maximal element.

NOTES

- Zorn's Lemma (for Non-Empty Sets): Assume that a non-empty partially ordered set ***P*** has the property that every non-empty chain has an upper bound in ***P***. Then the set ***P*** contains at least one maximal element.
- A relation which is reflexive, transitive and antisymmetric is called a partial order. A relation which is antisymmetric, transitive and comparable is called to be a total order. It may be stated that comparability implies reflexivity. Therefore, total order is a special type of partial order but partial order need not be total order.
- A set X which is totally ordered set is called well-ordered if its each non-empty subset has a minimum element. It may be observed that every finite set with a total order is well-ordered.
- The word topology is typically used to define a family of sets which uniquely has definite properties that are used to define a topological space, a basic object of topology. Topological spaces are mathematical structures that provide the formal definition of concepts, such as convergence, connectedness and continuity. Hence, the area of mathematics that studies topological spaces is called topology.
- There are various additional equivalent ways to define a topological space. Consequently, each of the following defines a category equivalent to the category of topological spaces above.
- The Zariski topology is defined algebraically on the spectrum of a ring or an algebraic variety. On $\mathbf{R}^n$ or $\mathbf{C}^n$, the closed sets of the Zariski topology are the solution sets of systems of polynomial equations. A linear graph has a natural topology that generalizes many of the geometric aspects of graphs with vertices and edges. The Sierpiñski space is the simplest non-discrete topological space. It has important relations to the theory of computation and semantics.
- The set of all limit points of A is called the derived set of A and shall be denoted by **D** (A).
- A subset N of **R** is called a neighbourhood of a point $p \in \mathbf{R}$ if N contains an open interval containing p and contained in N, that is, if there exists on open interval $]a, b[$ such that,

$$p \subseteq]a,\ b[\subseteq N.$$

- A closed set was defined to be the complement of an open set.
- The closure of a set A in **R** is the smallest closed set containing A and is denoted by $\overline{A}$.
- The sets of all interior points, all exterior points and all boundary points are respectively called the interior, the exterior and the boundary.
- Consider a set X and a proper subset, $\phi \neq A \subset X$. Then $\{\phi, A, X\}$ is a topology containing A. Actually, it is the smallest topology. Further, suppose $\phi \neq B \neq A \subset X$. The smallest possible topology that contains both A, B is clearly $\{\phi, A, B, A \cap B, A \cup B, X\}$. Similarly, if there are several different sets $A_1, \ldots, A_n$ in the topology, many other sets must be also there. This is the idea behind generating a topology.

- If $\mathcal{U}$ defines a topology on X, then we say that X is a topological space. The sets in $\mathcal{U}$ are called open sets. The sets of the form $X\backslash U$, $U \in \mathcal{U}$, are called closed sets. If Y is a subset of X, then the closure of Y is the smallest closet set in X that contains Y.
- A system $\mathcal{B}$ of subsets of X is called a basis (or base) for the topology $\mathcal{U}$ if every open set is the union of certain sets in $\mathcal{B}$. Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{U}$ such that $x \in B \subset U$.

  For example, the set of all bounded open intervals in the real line **R** forms a basis for the usual topology on **R**.
- An open covering of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family. A subcover of an open covering is an open covering of X which consists of sets belonging to the open covering. A topological space X is compact if every open covering of X contains a finite subcover.
- A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G and let $p: G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $U_{G/H}$ on G/H, called the quotient topology, by $\mathcal{U}_{G/H} = \{p(U) \mid U \in \mathcal{U}_G\}$.

1.18 KEY TERMS

- **Uncountable set:** If a set is not countable, then it is called uncountable set, i.e., set of binary sequences forms uncountable sets.
- **Infinite set:** If a set has an infinite number of elements then it is an infinite set.
- **Axiom of choice:** It states that for any family of no-empty and disjoint sets, there exists a set that consists of exactly one element from each element of that family.
- **Well-ordering theorem:** It states that for any set there exists an order relation on the set which is in well-ordering.
- **Zorn's lemma:** Consider a set X that is strictly partially ordered. If every simply ordered subset of X has an upper bound in X, then X has a maximal element.
- **Topological spaces:** Topological spaces are mathematical structures that authorize the formal definition of concepts, such as convergence, connectedness and continuity.

1.19 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Distinguish between countable and uncountable sets.
2. What do you understand by infinite sets?

NOTES

3. Give some examples of infinite sets.
4. Define the term axiom of choice.
5. What do you mean by total order relation?
6. Define the term cardinal number.
7. State about the addition and multiplication of cardinal numbers.
8. What is exponentiation?
9. State the Schröder-Bernstein theorem.
10. What do you understand by Cantor's intersection theorem?
11. What is continuum hypothesis?
12. State the generalized continuum hypothesis.
13. Define the term Zorn's lemma.
14. What is well-ordering set?
15. State the well-ordering theorem.
16. Give the equivalent definition of topological spaces.
17. What are Euclidean spaces?
18. What are the metric spaces?
19. Give the properties of metric topologies.
20. Define the terms neighbourhood, limit points, open sets and closed sets.
21. What are dense subsets?
22. Define the terms interior, exterior and boundary.
23. What are bases and subbases?
24. What is a subspace?
25. Define the term topological groups.

Long-Answer Questions

1. Briefly explain about the countable and uncountable sets with the help of examples.
2. Describe the infinite set theorem and axiom of choice giving suitable examples.
3. Explain in detail about the ordering, addition and multiplication of cardinal numbers giving appropriate examples.
4. State and prove Schröder-Bernstein theorem with the help of examples.
5. Elaborate on the Cantor's intersection theorem and continuum hypothesis.
6. Show that Zorn's Lemma implies Kuratowski lemma which states that if X is a collection of sets and if for every sub-collection Y of X that is simply ordered by proper inclusion, the union of the elements of Y is contained in X. Then X has an element that is properly contained in no other element of Y.
7. Show that the well-ordering theorem implies the choice axiom and also prove that a well-ordered set satisfies least upper bound property.
8. Define topology and topological sets with the help of definitions and

examples.

9. How topological spaces are classified and constructed? Explain giving examples.
10. Briefly explain about the various types of topological spaces with the help of definitions and examples.
11. Discuss about the Euclidean and metric spaces with the help of theorems.
12. Explain about the neighbourhood, closure and closed sets on the basis of topology with the help of relevant examples.
13. Describe the dense subset in topology with the help of theorems and examples.
14. Discuss about the interior points, exterior points and all boundary points in topology with the help of theorems and examples.
15. Explain in detail about the bases and subbases with reference to topological spaces. Give examples.
16. Discuss the significance of relative topology and topological groups with the help of examples.

NOTES

1.20 FURTHER READING

Munkres, James R. 2011. *Topology*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.

Basener, William F. *Topology and Its Applications*. New Jersey: John Wiley & Sons, Inc.

Bourbaki, Nicolas. 1966. *General Topology: Elements of Mathematics*, Volume 2. United States: Addison-Wesley.

Bourbaki, Nicolas. 1966. *Elements of Mathematics: General Topology*, Part 1. United States: Addison-Wesley.

Kelley, John L. 1975. *General Topology*. New York: Springer-Verlag.

Willard, Stephen. 2004. *General Topology*. United States: Dover Publications.

Shilov, Georgi E. 2012. *Elementary Real and Complex Analysis*. Chelmsford: Courier Corporation.

Simmons. G. F. 2004. *Introduction to Topology and Modern Analysis*. New York: Tata McGraw-Hill Education.

UNIT 2 CONTINUOUS FUNCTIONS AND HOMEOMORPHISM

NOTES

Structure

2.0 INTRODUCTION

In mathematical analysis, a continuous function is typically defined as form or type of function which does not has any unexpected or sudden changes in the value, i.e., the discontinuities. Even though, a function is considered as continuous if there are certain arbitrarily insignificant changes in the output which can be ensured or reassured by constraining or restricting to reasonably minor changes in the input. The mathematical field of topological domain, a homeomorphism, topological isomorphism or continuous function is precisely defined as a unique continuous function, which is characteristically specified for the topological spaces having continuous inverse function. In the field of mathematics and topological spaces, the term homeomorphism is uniquely defined as the isomorphism, i.e., these terms are typically defined as the unique mappings that preserve the entire or complete topological properties of a given space.

In topological analysis and precisely the other related or associated fields, the Kuratowski closure axioms are described as unique set of axioms that are used for defining the topological structure of a set. The Kuratowski closure axioms are named after Kazimierz Kuratowski who first formalised this concept. Characteristically, the Kuratowski closure axioms are equivalent or similar to more general and universally applied open set definition.

The first-countable space is described as a topological space which uniquely satisfies the 'First Axiom of Countability'. Exclusively, a space X is termed as the first-countable space if there is a countable neighbourhood base (basis) or local base to each point. The term second countability holds more stronger notion in comparison to the first countability. A topological space is considered as the first-countable if every single point holds a countable local base. When the base for a topological space and a point X is given, then it can be stated that the set of entire or complete basis sets which contains X uniquely forms a local base at X.

The Lindelöf's theorem, in mathematics, is named after the Finnish mathematician Ernst Leonard Lindelöf. In topology, the Lindelöf's theorem is defined based on the conclusion or result obtained in the complex analysis. It uniquely explains about the holomorphic function on a half-strip in the complex plane which is precisely in bounded or confined direction on the boundary of the given strip and the growth is not 'Too Fast', while in the unbounded direction of the given strip it should continue as bounded on the entire given strip.

A topological space, in mathematical analysis, is termed as the separable space if it precisely comprises of a countable dense subset, i.e., there exists a unique sequence (series) of precise elements related to the topological space such that every single non-empty open subset of the given topological space uniquely holds at least one element of the said sequence.

The terms 'Axioms of Countability' and the 'Axioms of Separability' is defined on the basis of the notation 'Limitation on Size' specifically in the topological perception, and not essentially on the basis of cardinality concept. Particularly, every continuous function whose image is a subset of a Hausdorff space on a separable space is precisely determined by its values on the countable dense subset.

In this unit, you will study about the alternate methods of defining a topology in terms of Kuratowski closure operator and neighbourhood systems, continuous functions and homeomorphism, first and second countable spaces, Lindelöf's theorem, separable spaces, second countability and separability.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Define the alternate methods of defining a topology.
- Understand the Kuratowski closure operator and neighbourhood systems
- Elaborate on the continuous functions and homeomorphism
- Explain about the first and second countable spaces
- Analyse the Lindelöf's theorem
- Discuss about the separable spaces, second countability and separability

NOTES

2.2 ALTERNATE METHODS OF DEFINING A TOPOLOGY IN TERMS OF KURATOWSKI CLOSURE OPERATOR AND NEIGHBOURHOOD SYSTEMS

In mathematical analysis and the field of topology, the Kuratowski closure axioms is named after the Polish mathematician Kazimierz Kuratowski, who first formalized this axiom. The Kuratowski closure axioms are a set of axioms that are typically used for defining a topological structure on a set.

The Kuratowski closure axioms characteristically developed a radically different approach to specifying a topology for a set. Kuratowski considered particular functions from the set of subsets of **K** to the set of subsets of **K** as explained below.

A topology for a set **K** is a collection of subsets of **K** such that,

- The union of any arbitrary subcollection is also a member of the collection.
- The intersection of finite numbers of members of the collection is also a member of the collection.
- The null set belongs to the collection.
- The whole set **K** belongs to the collection.

In topology, the elements of the collection are termed as the open sets. The openness of a set is not a property of the set itself but it refers only to the membership of the set in the collection of subsets which is called the topology.

A set is defined as being closed with respect to a topology if its complement is open with respect to the topology, i.e., if its complement belongs to the topology. At least two sets, the null set and the whole set **K** are both open and closed in any topology of **K**.

If the closed sets of a topology are given the open sets can easily be constructed since they are simply the complements of the closed sets.

2.2.1 Kuratowski Closure Axioms

Every topological space consists of the following:

- A set of points.
- A class of subsets defined axiomatically as open sets.
- The set operations of union and intersection.

The class of open sets must be defined in such a way that the intersection of any finite number of open sets is itself open and the union of any infinite collection of open sets is likewise open. A point p is called a limit point of the set S if every open set containing p also contains some point (s) of S (points other than p, should p happen to lie in S). The concept of limit point is fundamental to topology and it can be used axiomatically to define a topological space by specifying limit points for each set according to rules known as the **Kuratowski closure axioms.** Any set of objects can be made into a topological space in various ways, but the usefulness of the concept depends on the manner in which the limit points are separated from each other.

NOTES

The **Kuratowski closure axioms** are referred as a set of axioms that are used to define a topology on a set. In mathematical analysis of topology, the **Kuratowski closure axioms** are precisely referred as a set of axioms that are typically used for defining a topology on a set. They were first introduced by Kuratowski, in a slightly different form that applied only to Hausdorff spaces. In general topology, if X is a topological space and A is a subset of X, then the **closure** of A in X is defined to be the smallest closed set containing A or equivalently the intersection of all closed sets containing A. The **closure operator** C that assigns to each subset of A its closure $C(A)$ is thus a function from the power set of X to itself. The closure operator satisfies the following axioms:

1. **Isotonicity:** Every set is contained in its closure.
2. **Idempotence:** The closure of the closure of a set is equal to the closure of that set.
3. **Preservation of Binary Unions:** The closure of the union of two sets is the union of their closures.
4. **Preservation of Nullary Unions:** The closure of the empty set is empty.

Definition: An operator C of $\rho(X)$ into itself which satisfies the following four properties mentioned in Theorem 2.1 is called a closure operator on the set X.

Theorem 2.1: In the topological space (X, **T**), the closure operator has the following properties.

($\mathbf{K}_1$) $C(\phi) = \phi$

($\mathbf{K}_2$) $E \subseteq C(E)$

($\mathbf{K}_3$) $C(C(E)) = C(E)$

($\mathbf{K}_4$) $C(A \cup B) = C(A) \cup C(B)$

Proof: ($\mathbf{K}_1$): Because the void set is closed and also we know that a set A is closed if and only if,

$A = C(A)$, therefore it follows that $C(\phi) = \phi$.

($\mathbf{K}_2$): It follows from the definition as $C(E)$ is the smallest closed set containing E.

($\mathbf{K}_3$): Since $C(E)$ is the smallest closed set containing E, we have $C(C(E)) = C(E)$ by the result that a set is closed if and only if it is equal to its closure.

($\mathbf{K}_4$): Since $A \subset A \cup B$ and $B \subset A \cup B$, therefore

$C(A) \subset C(A \cup B)$ and $C(B) \subset C(A \cup B)$ and so

$$C(A) \cup C(B) \subset C(A \cup B) \quad ...(2.1)$$

By ($\mathbf{K}_2$), we have

$A \subset C(A)$ and $B \subset C(B)$

Therefore, $A \cup B \subset C(A) \cup C(B)$

Since $C(A)$ and $C(B)$ are closed sets and so $C(A) \cup C(B)$ is closed. By the definition of closure, we have

$$C(A \cup B) \subset C[C(A) \cup C(B)] \qquad ...(2.2)$$

From Equations (2.1) and (2.2), we have

$C(A \cup B) = C(A) \cup C(B)$.

Note: $C(A \cap B)$ may not be equal to $C(A) \cap C(B)$. For example, if $A = (0, 1)$, $B = (1, 2)$, then $C(A) = [0, 1]$, $C(B) = [1, 2]$.

Therefore,

$C(A) \cap C(B) = \{1\}$ where $A \cap B = \phi$

But $C(\phi) = \phi$. Therefore $C(A \cap B) = \phi$ and thus $C(A \cap B) \neq C(A) \cap C(B)$.

Note: The closed sets are simply the sets which are fixed under the closure operator.

Theorem 2.2: Let C^* be a closure operator defined on a set X. Let F be the family of all subsets F of X for which $C^*(F) = F$ and let **T** be a family of all complements of members of F. Then **T** is a topology for X and if C is the closure operator defined by the topology **T**. Then $C^*(E) = C(E)$ for all subsets $E \subseteq X$.

Proof: Suppose $G_\lambda \in \mathbf{T}$ for all λ. We must show that $\bigcup_\lambda G_\lambda \in \mathbf{T}$, i.e., $\left(\bigcup_\lambda G_\lambda\right)^C \in F$.

Thus, we must show that,

$C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right] = \left(\bigcup_\lambda G_\lambda\right)^C$

By (K_2) $\left(\bigcup_\lambda G_\lambda\right)^C \subseteq C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right]$

So we need only to prove that,

$C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right] \subseteq \left(\bigcup_\lambda G_\lambda\right)^C$

By De Morgan's Law, this reduces to the form,

$C^* \bigcap_\lambda \left[(G_\lambda)^C\right] \subseteq \bigcap_\lambda (G_\lambda)^C$

Since $(\bigcap_\lambda (G_\lambda)^C) \subseteq ((G_\lambda)^C)$ for each particular λ.

$C^*[\bigcap_\lambda (G_\lambda)^C] \subseteq C^*[(G_\lambda)^C]$ for each λ.

So, $C^*[\bigcap_\lambda (G_\lambda)^C] \subseteq \bigcap_\lambda C^*[(G_\lambda)^C]$

But, $G_\lambda \in \mathrm{T} \Rightarrow (G_\lambda)^C \in F$.

Hence, $C^*[(G_\lambda)^C] = (G_\lambda)^C$

Thus we have $C^*[\bigcap_\lambda (G_\lambda)^C] \subseteq \bigcap_\lambda (G_\lambda)^C$

Consequently, if $G_\lambda \in \mathbf{T}$, then $\bigcup_\lambda G_\lambda \in \mathbf{T}$.

To check that $\phi, X \in \mathbf{T}$, we observe that by Kuratowski closure axiom ($\mathbf{K}_2$),

$X \subseteq C^*(X) \subseteq X \Rightarrow C^*(X) = X \Rightarrow X \in F$

Hence $X^C = \phi \in \mathbf{T}$.

Also by (Kurtowaski closure axiom $\mathbf{K}_1$) we have $C^*(\phi) = \phi \Rightarrow \phi \in F$

$\Rightarrow \phi^C = X \in \mathbf{T}$.

Finally consider that $G_1 G_2 \in \mathbf{T}$. Then by hypothesis,

$C^*(G_1)^C = G_1{}^C$ and $C^*(G_2)^C = G_2{}^C$

We may now calculate,

$$\begin{aligned} C^*[(G_1 \cap G_2)^C] &= C^*[\, G_1{}^C \cup G_2{}^C \,] \\ &= C(\, G_1{}^C) \cup C^*(\, G_2{}^C) \\ &= G_1{}^C \cup G_2{}^C = (\, G_1 \cap G_2)^C \\ &\Rightarrow (\, G_1 \cap G_2)^C \in F \\ &\Rightarrow G_1 \cap G_2 \in \mathbf{T}. \end{aligned}$$

Because all the axioms for a topology are satisfied hence **T** is a topology. We will now prove that $C^* = C$.

As per the analysis and discussion remember that **T** is a topology for X. Consequently, the members of **T** are open sets and therefore the closed sets are simply the members of the family F.

By (K_3), $C^*[\, C^*(E)] = C^*(E)$

This implies that $C^*(E) \in F$. Now by axiom ($\mathbf{K}_2$) $E \subseteq C^*(E)$. Thus $C^*(E)$ is a closed set containing E and hence $C^*(E) \supseteq C(E)$ ...(2.3)

The consder that C^* (E) is the smallest closed set containing E. On the other hand by axiom ($\mathbf{K}_2$),

$E \subseteq C(E) \in F$.

So, $C^*(E) \subseteq C^*(C(E)) = C(E)$ ...(2.4)

Thus by Equations (2.3) and (2.4),

$C^*(E) = C(E)$ for any subset $E \subseteq X$

2.2.2 Dense Subsets

Definition: Let A be a subset of the topological space (X, T). Then A is said to be dense in X if $\overline{A} = X$.

Trivially the entire set X is always dense in itself. **Q** is dense in **R** since,

$\overline{\mathbf{Q}} = \mathbf{R}$ ·

Let **T** be finite complement topology on **R**. Then every infinite subset is dense in **R**.

Theorem 2.3: A subset A of topological space $(X, \mathbf{T})$ is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \phi$.

Proof: Suppose A is dense in X and B is a non-empty open set in X. If $A \cap B = \phi$, then $A \subseteq X–B$ implies that $\overline{A} \subset X–B$ since $X–B$ is closed. But then $X–B \underset{\neq}{\subset} X$ contradicting such that $\overline{A} = X$.

Because $\overline{A} \subset X–B \underset{\neq}{\subset} X$.

Conversely assume that A meets every nonempty open subset of X. Thus the only closed set containing A in X and consequently $\overline{A} = X$. Hence A is dense in X.

Theorem 2.4: In a topological space (X, **T**)

(i) Any set C, containing a dense set **D**, is a dense set.

(ii) If A is a dense set and B is dense on A, then B is also a dense set.

Proof: (i) Since $\mathbf{D} \subset C \Rightarrow \overline{\mathbf{D}} \subset \overline{C}$

But $\overline{\mathbf{D}} = X$, hence $X \subset \overline{C}$ also $\overline{C} \subset X$ so that $\overline{C} = X$.

Thus C is dense in (X, **T**).

(ii) Since A is dense, $\overline{A} = X$

Also B is dense on A.

$\Rightarrow A \subset \overline{B} \Rightarrow \overline{A} \subset \overline{\overline{B}} = \overline{B}$ (By Closure Property)

$\Rightarrow \overline{A} \subset \overline{B}$

$\Rightarrow X = \overline{A} \subset \overline{B}$

Thus B is dense in (X, **T**).

2.3 CONTINUOUS FUNCTIONS AND HOMEOMORPHISM

Topological equivalences are said to be redirected here. Fundamentally, for two spaces to be homeomorphic, there is no necessity of a continuous deformation. The mathematical analysis precisely for the field of topology, states that the term homeomorphism or topological isomorphism refers to the deformation of morphe which is a bicontinuous function occurring between the two topological spaces. The term homeomorphism is taken from the Greek word '*homoios*' which means 'similar' and the Latin word '*morphe*' which means 'shape'. Characteristically, the 'Homeomorphisms' are considered as the 'Isomorphisms' as per the notation of topological space category theory, i.e., the homeomorphism and isomorphism are the mappings which uniquely preserve the entire topological properties of a given space. When there is homeomorphism between the two spaces then it is termed as 'Homeomorphic', and according to the topological perspective they are considered equivalent.

Principally, the topological space is defined as a structure of geometric object and the homeomorphism is considered as a continuous stretching and bending of the geometric object into a new shape. Consequently, a square and a circle are considered as homeomorphic to each other, but a sphere and a donut are not as the shape is not equivalent.

Definition: According to the Encyclopaedia Britannica, in mathematics and topology, the term homeomorphism is defined as. "A correspondence between two figures or surfaces or other geometrical objects is defined by a one-to-one mapping that is continuous in both directions is termed as homeomorphism".

A homeomorphism characteristically maps the points in the 'First Object' which are sufficiently 'Close Together' to the points in the 'Second Object' which are also sufficiently 'Close Together', and the points in the first object which are not precisely close together to the points in the second object which are also precisely not close together. Subsequently, the 'Topology' is the appropriate study of those properties of objects which precisely not change when the homeomorphisms are applied.

A function 'f' between two topological spaces 'X' and 'Y' is called a homeomorphism if it has the following properties:

- 'f' is a bijection (1-1 and onto).
- 'f' is continuous.
- The inverse function 'f' $^{-1}$ is continuous (f is an open mapping).

In topology, a function having all these three properties is also sometimes termed as 'Continuous'. When such type of function exists, then we state that 'X' and 'Y' are homeomorphic. Additionally, a self-homeomorphism is defined as a homeomorphism of a topological space and also itself. Characteristically, the homeomorphism structure and develop an equivalence relation precisely on the class of all topological spaces. The resulting or subsequent 'Equivalence Classes' are termed as the 'Homeomorphism Classes'. For example,

- The unit 2-disc $\mathbf{D}^2$ and the unit square in $\mathbf{R}^2$ are homeomorphic.
- The open interval (–1, 1) is homeomorphic to the real numbers $\mathbf{R}$.
- The product space S^1 and the two-dimensional torus are homeomorphic.
- Every uniform isomorphism and isometric isomorphism is a homeomorphism.
- Any 2-sphere with a single point removed is homeomorphic to the set of all points in $\mathbf{R}^2$ (a 2-dimensional plane).
- Let A be a commutative ring with unity and let S be a mutiplicative subset of A.
- An exceptional example of a continuous bijection form which is not a homeomorphism can be defined as the map which holds the half-open interval [0,1) and covers it precisely around the circle. In this instance, the inverse — even though it exists — may not be continuous. The preimage of certain specific sets which are of real open form in the relative topology stating the half-open interval [0,1) are then not open in the other natural or usual topology stating the circle as they are half-open intervals.

Homeomorphisms, as already discussed, are the isomorphisms according to the topological space category. Essentially, the composition or structure of two homeomorphisms is yet again a homeomorphism and subsequently the set of all self-homeomorphisms 'X' forms or structures a group termed as the homeomorphism group of 'X'.

For certain reasons, the homeomorphism groups specifically occur to be too large, but because of the isotope relation the homeomorphism groups can be reduced to the mapping class groups.

NOTES

1. Two homeomorphic spaces typically contribute to or share the identical or equivalent topological properties. For example, if one of the homeomorphism groups is compact, then the other is also; if one of the homeomorphism groups is connected, then the other is also; if one of the homeomorphism groups is Hausdorff, then the other is also; and their homology groups will uniquely overlap or coincide. Remember that, however this cannot extend and continue having the properties defined by means of a metric; there are metric spaces which are considered as homeomorphic despite the fact that one of the homeomorphisms is complete while the other is not.
2. A homeomorphism is simultaneously an open mapping and a closed mapping, that is it maps open sets to open sets and closed sets to closed sets.
3. Every self-homeomorphism in S^1 can be extended to a self-homeomorphism.

2.4 FIRST AND SECOND COUNTABLE SPACES

The countable complement topology on any set X consist of the empty set and all countable subsets of X, i.e., all sets whose complement in X is countable.

2.4.1 First Countable Space

Characteristically, a topological space X is termed as the **'First Countable Space'** if it precisely satisfies or fulfils the following given axiom termed as the **'First Axiom of Countability'**:

For each point $p \in X$ there exists a countable class $\mathbf{B}_p$ of open sets containing p such that every open set G containing p also contains a member of $\mathbf{B}_p$. In other words, a topological space X is a **first countable space** if and only if there exists a countable local base at every point $p \in X$.

Fundamentally, the first countable and precisely the separable Hausdorff space, specifically a separable metric space takes as a maximum the continuum cardinality c. In such specific space, the closure is uniquely determined and accurately defined by means of 'Limits of Sequences' and any sequence precisely has at the most just one limit, consequently there is a surjective map from the set of convergent sequences having unique values in the countable dense subset to the points of X. Essentially, a separable Hausdorff space holds cardinality at the most 2^c where c is referred as the cardinality of the continuum. Consequently, this closure is distinguished and characterized according to the limits of filter bases: 'If

Y is a subset of X and z is a point of X, then z is typically in the closure of Y if and only if there exists a filter base B which consists of subsets of Y that uniquely converges to z'. The cardinality of the set $S(Y)$ of such filter bases is at most $2^{2^{[Y]}}$. Moreover, in a Hausdorff space, there is at most one limit to every filter base. Therefore, there is a surjection $S(Y) \to X$ when $\overline{Y} = X$. The same arguments establish a more general result: suppose that a Hausdorff topological space X contains a dense subset of cardinality k. Then X has cardinality at most 2^{2k} and cardinality at most 2^k if it is first countable. The consequence or product of at most continuum many separable spaces is a separable space. In particular the space R^R of all functions from the real line to itself, endowed with the consequence or product topology, is a separable Hausdorff space of cardinality 2^c. More generally, if k is any infinite cardinal, then a consequence or product of at most 2^k spaces with dense subsets of size at most k has itself a dense subset of size atmost k.

Example 2.1: Let X be a metric space. Let $p \in X$. The countable class of open spheres $\{S(p, 1), S(p, 1/2), S(p, 1/3),...\}$ with center at p is a local base at p. Thus every metric space satisfies the first axiom of countability.

Example 2.2: Let X be any discrete space and let $p \in X$. A local base at point p is the singleton set $\{p\}$ — which is countable. Thus every discrete space satisfies the first axiom of countability.

The first uncountable ordinal ω_1 in its order topology is not separable.

The Banach space l^∞ of all bounded real sequences, with the supremum norm, is not separable.

2.4.2 Second Countable Space

In the field of topological analysis, a topological space X precisely with the topology τ is termed as a '**Second Countable Space**' if it exceptionally satisfies the following given axiom termed as the '**Second Axiom of Countability**'.

There exists a countable base **B** for the topology τ.

Consider that given the class of open intervals (a, b) together with rational end points, i.e., $a, b \in Q$ where Q is referred as the set of precise rational numbers — which is countable and is defined as a base for the normal topology on the real line R. Accordingly, R precisely satisfies the second axiom of countability and is therefore a second countable space.

Consider the example for real line R together with the discrete topology **D**. Now the only unique base specially for a discrete topology on a specific set X is that distinctive collection **B** of all or entire singleton sets of X. Remember that R is typically non-countable and consequently the class of singleton sets $\{p\}$ of R is also typically non-countable. Therefore, the topological space R together with the discrete topology **D** does not satisfy or comply with the second axiom of countability. By the similar identical logic, the topological space Q of unique rational numbers together with the discrete topology **D** satisfies the second axiom of countability.

If **B** is a countable base for a space X, and if $\mathbf{B}_p$ consists of the members of **B** which contain the point $p \in X$, then $\mathbf{B}_p$ is a countable local base at p Any second-countable space is separable: if $\{U_n\}$ is a countable basis, choosing any $x_n \in U_n$ gives a countable dense subset. Conversely, a metrizable space is separable if and only if it is second countable if and only if it is Lindelöf.

An arbitrary or random subspace of a unique second countable space is referred as a second countable subspaces of typical separable spaces essentially may not be separable.

In topology, a consequence of at the most continuum various separable spaces are defined as the 'Separable'. Additionally, the countable consequence of the unique second countable spaces is typically defined as the second countable, but an uncountable consequence of the unique second countable spaces essentially may not even be first countable.

Characteristically, any continuous image of a unique and distinct separable space is defined as 'Separable' even though the quotient of a unique second countable space essentially may not be second countable.

The unique property of separability essentially does not provide any limitations on the cardinality typically of a distinct topological space, any set capable with the trivial topology is considered as separable and also second countable, quasi-compact and connected. The 'Problem' with the trivial topology is referred as its inadequate separation properties, its Kolmogorov quotient is considered as the one-point space.

Theorem 2.5: A function defined on a first countable space X is continuous at $p \in X$ if and only if it is sequentially continuous at p.

In other words, if a topological space X satisfies the first axiom of countability, then $f: X \rightarrow Y$ is continuous at $p \in X$ if and only if for every sequence $\{a_n\}$ in X converging to p, the sequence $\{f(a_n)\}$ in Y converges to $f(p)$, i.e.,

$$a_n p \Rightarrow f(a_n) \rightarrow f(p)$$

Theorem 2.6: A second countable space is also first countable.

Let S be a set. Let A be a subset of S. Then a collection C of subsets of S is a **cover of** A if A is a subset of the union of the members of C, i.e.,

$$A \subset \bigcup\{c : c \in C\}$$

If each member of C is an open subset of S, then C is called an **open cover** of A. If C contains a countable subclass which also is a cover of A, then C is said to be **reducible to a countable cover** of A.

Theorem 2.7: Let A be any subset of a second countable space X. Then every open cover of A is reducible to a countable cover.

Theorem 2.8: Let X be a second countable space. Then every base **B** for X is reducible to a countable base for X.

Lindelöf Space: A topological space X is called a Lindelöf space if every open cover of X is reducible to a countable cover.

Thus, every second countable space is Lindelöf space.

Check Your Progress

1. Define the term set.
2. Give the Kuratowski closure axioms.
3. Which axioms are satisfied by closure operator?
4. Define homeomorphism.
5. When is homeomorphisms considered isomorphisms?
6. What is homeomorphic?
7. What do you understand by first countable space?
8. What is second axioms of countability?
9. Define Lindelöf's space.

2.5 LINDELÖF'S THEOREMS

Let $C=\{A_\lambda:\lambda\in A\}$ be a collection of sets of real numbers. We say that C is a cover (or covering) of a set A of real numbers if each point of A belongs to A_λ for same $\lambda\in A$, that is, if $A \subset \bigcup\{A_\lambda:\lambda\in A\}$. If C is a collection of open sets, then C is called an open covering of A.

If C' is a cover of A such that $C' \subset C$, is called a subcover of A.

A set A is said to be compact if each open cover of A has a finite subcover.

Example 2.3: Let $C=\{]-n, n[:n\in N\}$

And $C'=\{]-3n, 3n[:n\in N\}$.

Then C and C' are both open covers of R. Also C' is a subcover of C.

Theorem 2.9 (Lindelöf): Let C be a collection of open sets of real numbers. Then there exists a countable subcollection $\{G_1\}$ of C such that,

$$\bigcup\{G:G\in C\}=\bigcup_{n=1}^{\infty} G_i.$$

Proof: Let $S=\bigcup\{G:G\in C\}$ and let $x\in S$. Then there is at least one $G\in C$ such that $x\in G$. Since G is open, there exists an open interval I_x with centre at x such that $I_x \subset G$. Since every interval I_x contains infinitely many rational points we can find an open interval J_x with rational end points such that $x\in J_x\subset I_x$. Since the collection $\{J_x\}$, $x\in S$, is countable, and

$$S=\bigcup\{J_x:x\in S\}.$$

For each interval in $\{J_x\}$, choose a set G in C which contains it. This gives a countable subcollection

$$\{G_i\}_{i=1}^{\infty} = \text{of C and } S=\bigcup_{i=1}^{\infty} G_i$$

Hence $\cup\{G : G \in C\} = S = \bigcup_{i=1}^{\infty} G_i$.

Theorem 2.10 (Lindelöf Covering Theorem): Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.

Proof: Since C is an open cover of A, we have

$$A \subset \cup\{G : G \in C\}.$$

Now show that there exists a countable subcollection $\{G_i\}_{i=1}^{\infty}$ of C such that

$$\cup\{G : G \in C\} = S = \bigcup_{i=1}^{\infty} G_i$$

And so $A = \bigcup_{i=1}^{\infty} G_i$ Hence the theorem.

The Heine-Borel covering theorem states that from any open covering of an arbitrary set A of real numbers, we can extract a countable covering. The Heine-Borel theorem tells us that if, in addition, we know that A is closed and bounded, then we can reduce the covering to a finite covering. The proof makes use of the nested interval theorem.

Theorem 2.11 (Heine-Borel): Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering. That is, if C is a collection of open sets such that $F \subset \cup\{G : G \in C\}$, then there exists a finite subcollection $(G_1 G_2,...,G_n)$ of C such that $F \subset \bigcup_{i=1}^{n} G_i$.

In other words, every closed and bounded set of real numbers is compact.

Proof: Since F is bounded, it is contained in some closed and bounded interval $[a, b]$. Let C^* be the collection obtained by adding F' (the complement of F) to C; that is, $C^* = C \cup \{F'\}$. Since F is closed, F' is open and so C' is a collection of open sets. By hypothesis, $F \subset \cup\{G : G \in C\}$, and so

$$R = F \cup F' \subset [\cup\{G : G \in C\}] \cup F' = \cup\{G : G \in C^*\}$$

We now show that $[a, b]$ is covered by a finite sub-collection of C. Suppose the contrary to be true. Now if $I_0 = [a, b]$ cannot be covered by a finite subcollection of C^*, then one of the intervals $[a, \frac{1}{2}(a + b)]$, $[\frac{1}{2}(a + b), b]$ cannot be so covered. We rename such an interval as $[a_1, b_2]$. Similarly, one of the intervals $\left[a_1, \frac{1}{2}(a_1 + b_1)\right], \left[\frac{1}{2}(a_1 + b_1), b_1\right]$ cannot be covered by a finite subcollection of C^*. We designate such an interval as $[a_2, b_2]$. Continuing in this way, we obtain a sequence of closed intervals

$$\{I_n\} = \{[a_n, b_n]\} \text{ such that } I_n \subset I_{n+1}$$

NOTES

And $\left|I_n\right| = b_n - a_n = \dfrac{b-a}{2^n} \to C$ as $n \to \infty$.

Hence by the nested interval theorem, $\bigcup_{n=1}^{\infty} I_n$ consists of a single point, say x_0. This point x_0 must belong to one of the members, say G_0 of C*. Since G_0 is an open set and $x_0 \in G_0$, there exists ε > 0 such that $]x_0 - \varepsilon, x_0 + \varepsilon[\subset G_0$. Also we can choose k so large that $\dfrac{b-a}{2^k} < \varepsilon$. Then $I_k \subset]x_0 - \varepsilon, x_0 + \varepsilon[\subset G_0$ and so I_k is covered by a single member of C*. But this contradicts the constructed property of the members of $\{I_n : n \in N\}$. Hence $[a, b]$ must be covered by a finite subcollection of C* and hence F must also be covered by a finite subcollection of C*. If this finite subcollection does not contain F', it is a subcollection of C and the conclusion of our theorem holds. If the subcollection contains F', denote it by $\{G_1\ G_2...,G_n,\ F'\}$. Then $F \subset F' \cup G_1 \cup G_2 \cup ... \cup G_n$. Since no point of F is contained in F', we have $F \subset G_1 \cup G_2 \cup ..... \cup G_n$, and the collection $\{G_1, G_2,..., G_n\}$ is a finite subcollection of C which covers F.

This proves the theorem.

The converse of the preceding theorem is given in the next theorem.

Theorem 2.12: Compact subsets of R are closed and bounded.

Proof: Let A be any compact subset of R. If $A_n =]-n, n[$, then the collection $C = \{A_n : n \in N\}$ is evidently an open cover of R and so an open cover of A. Since A is compact, there exist finitely many positive integers $n_1, n_2 ... n_k$ such that the subcollection $\left\{A_{n_1}, A_{n_2}, ..., A_{n_k}\right\}$ of C covers A. Let $n_0 = \max\ \{n_1, n_2 ..., n_k\}$, then evidently $A \subset A_{n_0} =]-n_0, n_0[$. This implies that A is bounded.

If we can show that no point of $R - A$ can be a limit point of A, then A will be closed. So let $a \in R$. Then $a \notin A$. Consider the family of closed sets $F_n = [a - 1/n, a + 1]$ for each $n \in N$. Then $C' = \{R - F_n : n \in N\}$ is a collection of open sets. Also evidently $\bigcap_{n=1}^{\infty} F_n = \{a\}$. Since $a \notin A$, we have,

$$A \subset [R - \cap\{F_n : n \in N\}] = \cup\{R - F_n : n \in N\} \text{ [De Morgan law]}.$$

Thus A is covered by the collection C. Hence by compactness of A, there exists finitely many positive integers $m_1; m_2,...,m_s$ such that every point of A is contained in one of the open sets $R - F_{m_1}, R - F_{m_2}, ..., R - F_{m_s}$. Hence if $x \in A$, then for some $i \in \{1, 2, 3,...,s\}, x \in R - F_{m_i}$, which implies that no point of A is contained in $F_{m_i} - [a - 1/m_i, a + 1/m_i]$. This implies that a is not a limit point of A. Hence A is closed.

NOTES

A Characterization of Compact Sets on *R*

Theorem 2.13: A subset of R is compact if and only if it is closed and bounded.

Theorem 2.14: R is not compact.

Proof: Let $A_n =]-n, n[$. Then clearly $C = \{A_n : n \in N\}$ is an open cover of R. If $\{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$ be any finite subfamily of C, let $n_0 = \max \{n_1, n_2, ..., n_k\}$. Then $n_0 \notin A_{n_i}$, for any $i = 1, 2, ..., k$.

It follows that no finite subfamily of C can cover R. Hence R is not compact.

Theorem 2.15: Show that open intervals on 1 are not compact.

Proof: Let $I =]a, b[$ be any open interval on R.

If $A_n =]a + 1/n, b[$, then evidently the collection $C = \{A_n : n \in N\}$ is an open cover of $]a, b[$ since $\bigcup_{n=1}^{\infty} A_n =]a, b[$. But it is not possible to find a finite subcollection of C which covers A. For, if $C' = \{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$ be any finite subcollection of C, let $n_0 = \max \{n_1\ n_2, ..., n_k\}$. Then it is evident that the subset $]a, a + 1/n_0]$ of A is not covered by C'. Thus we have shown that there exists an open cover of A which does not admit of a finite subcover. Hence $]a, b[$ is not compact.

2.6 SEPARABLE SPACES, SECOND COUNTABILITY AND SEPARABILITY

A topological space is called **separable** if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every non-empty open subset of the space contains at least one element of the sequence. In particular, every continuous function on a separable space whose image is a subset of a Hausdorff space is determined by its values on the countable dense subset.

In general, **separability** is a technical hypothesis on a space which is quite useful and among the classes of spaces studied in geometry and classical analysis generally considered to be quite mild. It is important to compare separability with the related notion of **second countability**, which is in general stronger but equivalent on the class of metrizable spaces. Following are some examples of separable space.

- Every compact metric space (or metrizable space) is separable.
- Any topological space which is the union of a countable number of separable subspaces is separable. Together, these first two examples give a different proof that n-dimensional Euclidean space is separable.
- The space of all continuous functions from a compact subset of R^n into R is separable.
- It follows easily from the Weierstrass approximation theorem that the set $Q[t]$ of polynomials with rational coefficients is a countable dense subset of

the space $C([0,1])$ of continuous functions on the unit interval [0, 1] with the metric of uniform convergence. The Banach-Mazur theorem asserts that any separable Banch space is isometrically isomorphic to a closed linear subspace of $C([0, 1])$.

- The Lebesgue spaces L^p are separable for any $1 \le p \le \infty$.
- A Hilbert space is separable if and only if it has countable orthonormal basis; it follows that any separable, infinite-dimensional Hilbert space is isometric to ℓ^2.
- An example of a separable space that is not second-countable is $R_{llt,}$ the set of real numbers equipped with the lower limit topology.
- A topological space X is separable if and only if there exists a finite or denumerable subset A of X such that the closure of A is the entire space i.e., $\bar{A} = X$.
- The real line R with the usual topology is separable since the set Q of rational numbers is denumerable and is dense in R, i.e., $\bar{Q} = R$.
- Every **second countable space** is separable but not every separable space is second countable. For example, the real line R with the topology generated by the closed-open intervals $[a, b)$ is a classic example of a separable space which does not satisfy the second axiom of countability.

Check Your Progress

10. State the Lindelöf covering theorem.
11. Define Heine-Borel covering theorem.
12. What is a separable space?
13. Define is saparability.
14. Give some examples of separable space.

2.7 SOLVED EXAMPLES

Example 1: Prove that characteristics function of $X \subset Y$ is continuous on Y iff X is both open and closed in Y.

Solution: Let (Y, T) be a space and let $X \subset Y$. The characteristic function f or X is defined as,

$$f(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \text{ or } x \in X^1 \end{cases}$$

Let X is both open and closed then to prove that f is continuous,

$f: y \to R$ let G be an open subset of R.

Then by definition, $f^{-1}(G) = \{x \in Y: f(x) \in G$

$$\text{And } f^{-1}(G) = \begin{cases} X & \text{if } 1 \in G, 0 \notin G \\ X^1 & \text{if } 0 \in G, 0 \notin G \\ Y & \text{if } 0,1 \in G \\ \phi & \text{if } 0,1 \notin G \end{cases}$$

In all of the cases, $f^{-1}(G)$ is an open set,

Therefore f is continuous.

Now conversely let f is continuous, then to prove that X is both open and closed.

Let G can be an open subset of R such $0 \in G$, $1 \notin G$.

Then $f^{-1}(G)=X^1$, f is continuous.

$\Rightarrow f^{-1}(G) = X^1$ is open X is closed.

Now let H is an open subset of R such that $1 \in H$, $0 \notin H$,

Then $f^{-1}(H) = X$, H is an open in R.

$\Rightarrow F^{-1}(H) = X$ is open in Y.

Thus $\Rightarrow X$ is both open and closed.

Example 2: Let $f: \to R$ be a constant map, prove that f is continuous.

Solution: Let $f: \to \mathbf{R}$ be a map defined as $f(x)=C_1 \forall\, x \in \mathbf{R}$ …(1)

To prove that f is continuous.

Let $G \subset \mathbf{R}$ be an arbitrary open set, then by definition,

$f^{-1}(G) = \{X \in \mathbf{R}: f(x) \in G\}$ …(2)

From Equations (1) and (2) we have,

$$f^{-1}(G) = \begin{cases} \mathbf{R} & \text{if } c \in G \\ \phi & \text{if } c \notin G \end{cases}$$

ϕ and $\mathbf{R}$ both are open sets in $\mathbf{R}$ and hence $f^{-1}(G)$ is open in $\mathbf{R}$.

$\Rightarrow f$ is a continuous map.

Example 3: Prove that the interval $X=[a,b]$ is homeomorphic to the unit interval [0,1].

Solution: Let the map $f: X[0,1]$ is given by,

$$fx = 0 + \left(\frac{1-0}{a-b}\right)(x-a),$$

Or,

$$f(x) = \left(\frac{x-a}{b-a}\right)$$

Therefore, f is one-one, onto and bicontinuous.

Example 4: If S={ϕ, {1}, {2}, {1,2}, {2, 3, 4}, A} be a topology on A = {1, 2, 3, 4} and $f: A \to B$ is defined as $f(1)=2$, $f(2)=4$, $f(3)=2$, $f(4)=3$. Prove that f is not continuous at 3 and is continuous at 4.

Solution: Given that $B = \{f(1), f(2), f(3), f(4)\}$

Or $\quad B = \{2, 4, 3\}$

$= \{2, 3, 4\}$

NOTES

Let $H = \{G \cap B: G \in S\}$ be a topology on B.

Then $H = \{\phi, B, \{2\}\}$

A map $f: A \to B$ is continuous at $x \in A$ if,

$\forall$ H-open J set containing $f(x)$, $f^{-1}(J)$ is S-open set containing X.

Now to test continuity at $x = 3$,

H open sets containing $f(3) = 2$ and $B\{2\}$

Also, $f^{-1}(B) = A$, $f^{-1}\{2\} = \{1, 3\} \notin S$ but $\{2\} \in H$

$\therefore f$ is not continuous at $x = 3$

To test continuity at $x = 4$

H open set, which is contain only **B** so that, $f(4) = 3$

And $f^{-1}(B) = A$ which contains 4.

Therefore, f is continuous at $x = 4$.

Example 5: Prove that a constant map from (R,U) into (R,U) is a closed map, where U- denotes usual topology on R.

Solution: By definition $f: (R, U) \to (R, U)$ such that,

$f(x) = c, \forall x \in R$

Let X, be any U closed set, then

$F(x) = \{f(x): x \in X\}$

$= \{c: x \in A\}$

$= \{c\}$

$= a$ finite set

$= a$ U-closed set

Therefore, f is closed.

Example 6: Let (X, U) and (Y, U) are topological spaces and $H \subset X$, and f: (X, T) $\to$ (Y, U) be continous. Prove that f_H: (H, T_H) $\to$ (Y, U) is continuous where f_H is the restriction of f to H.

Solution: Let $f(X, T) \to (Y, T)$ is a continuous map and $H \subset X$. To prove that $f_H: (H, T_H) \to (Y, U)$ is continuous,

Let $G \subset Y$ is open that $G \in U$

If we prove that $f_H^{-1}(G) \in T_H$, the result will be proven.

$G \in U, f$ is continuous $\Rightarrow f^{-1}(G) \in T$

$\Rightarrow H \cap f^{-1}(G) \in T_H$ for $(H, T_H) \subset (X, T)$

$\Rightarrow f_H^{-1}(G) = H \cap f^{-1}(G) \in T_H$

$\Rightarrow f_H^{-1}(G) \in T_H$.

2.8 ANSWERS TO 'CHECK YOUR PROGRESS'

NOTES

1. A set is defined as being closed with respect to a topology if its complement is open with respect to the topology; i.e., if its complement belongs to the topology. At least two sets, the null set and the whole set K are both open and closed in any topology of K.
2. The concept of limit point is fundamental to topology and it can be used axiomatically to define a topological space by specifying limit points for each set according to rules known as the Kuratowski closure axioms.
3. The closure operator satisfies the following axioms:
   (i) Isotonicity: Every set is contained in its closure.
   (ii) Idempotence: The closure of the closure of a set is equal to the closure of that set.
   (iii) Preservation of Binary Unions: The closure of the union of two sets is the union of their closures.
   (iv) Preservation of Nullary Unions: The closure of the empty set is empty.
4. The term homeomorphism is taken from the Greek word '*homoios*' which means 'similar' and the Latin word '*morphe*' which means 'shape'. Characteristically, the 'Homeomorphisms' are considered as the 'Isomorphisms' as per the notation of topological space category theory, i.e., the homeomorphism and isomorphism are the mappings which uniquely preserve the entire topological properties of a given space.

   According to the Encyclopaedia Britannica, in mathematics and topology, the term homeomorphism is defined as. "A correspondence between two figures or surfaces or other geometrical objects is defined by a one-to-one mapping that is continuous in both directions is termed as homeomorphism".
5. Homeomorphisms are considered the isomorphisms in the field of topological spaces, when, they are defined as the mappings which preserve all the topological properties of a given space.
6. Two spaces with a homeomorphism between them are termed as homeomorphic and from a topological perspective they are the same.
7. A first countable, separable Hausdorff space (in particular, a separable metric space) has at most the continuum cardinality c. In such a space, closure is determined by limits of sequences and any sequence has at most one limit, so there is a surjective map from the set of convergent sequences with values in the countable dense subset to the points of X.
8. A topological space X with topology τ is termed as a second countable space if it satisfies the following axiom, termed as the second axiom of countability.
9. A topological space X is called a Lindelof space if every open cover of X is reducible to a countable cover. Thus, every second countable space is a Lindelöf space.

10. Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.
11. The Heine-Borel covering theorem states that from any open covering of an arbitrary set A of real numbers, we can extract a countable covering. The Heine-Borel theorem tells us that if, in addition, we know that A is closed and bounded, than we can reduce the covering to a finite covering.
12. A topological space is called separable if it contains a countable dense subset.
13. In general, separability is a technical hypothesis on a space which is quite useful and among the classes of spaces studied in geometry and classical analysis.
14. Let us consider some examples of separable space.
    - Every compact metric space (or metrizable space) is separable.
    - Any topological space which is the union of a countable number of separable subspaces is separable. Together, these first two examples give a different proof that n-dimensional Euclidean space is separable.
    - The space of all continuous functions from a compact subset of R^n into R is separable.
    - The Lebesgue spaces L^p are separable for any $1 \leq p \leq \infty$
    - A Hilbert space is separable if and only if it has countable orthonormal basis; it follows that any separable, infinite-dimensional Hilbert space is isometric to ℓ^2.
    - A topological space X is separable if and only if there exists a finite or denumerable subset A of X such that the closure of A is the entire space i.e., $\overline{A} = X$.
    - Every second countable space is separable but not every separable space is second countable. For example, the real line R with the topology generated by the closed-open intervals $[a, b)$ is a classic example of a separable space which does not satisfy the second axiom of countability.

2.9 SUMMARY

- In mathematical analysis and the field of topology, the Kuratowski closure axioms is named after the Polish mathematician Kazimierz Kuratowski, who first formalized this axiom.
- The Kuratowski closure axioms are a set of axioms that are typically used for defining a topological structure on a set.
- The Kuratowski closure axioms characteristically developed a radically different approach to specifying a topology for a set. Kuratowski considered particular functions from the set of subsets of **K** to the set of subsets of **K**.

NOTES

- In topology, the elements of the collection are termed as the open sets. The openness of a set is not a property of the set itself, but it refers only to the membership of the set in the collection of subsets which is called the topology.
- A set is defined as being closed with respect to a topology if its complement is open with respect to the topology, i.e., if its complement belongs to the topology. At least two sets, the null set and the whole set **K** are both open and closed in any topology of **K**.
- If the closed sets of a topology are given the open sets can easily be constructed since they are simply the complements of the closed sets.
- The Kuratowski closure axioms are referred as a set of axioms that are used to define a topology on a set.
- In mathematical analysis of topology, the Kuratowski closure axioms are precisely referred as a set of axioms that are typically used for defining a topology on a set. They were first introduced by Kuratowski, in a slightly different form that applied only to Hausdorff spaces.
- In general topology, if X is a topological space and A is a subset of X, then the closure of $\boldsymbol{A}$ in $\boldsymbol{X}$ is defined to be the smallest closed set containing A or equivalently the intersection of all closed sets containing A.
- The closure operator $\boldsymbol{C}$ that assigns to each subset of A its closure $C(A)$ is thus a function from the power set of X to itself.
- A subset A of topological space $(X, \mathbf{T})$ is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \phi$.
- Topological equivalences are said to be redirected here. Fundamentally, for two spaces to be homeomorphic, there is no necessity of a continuous deformation.
- The mathematical analysis precisely for the field of topology, states that the term homeomorphism or topological isomorphism refers to the deformation of morphe which is a bicontinuous function occurring between the two topological spaces.
- The term homeomorphism is taken from the Greek word '*homoios*' which means 'similar' and the Latin word '*morphe*' which means 'shape'.
- Characteristically, the 'Homeomorphisms' are considered as the 'Isomorphisms' as per the notation of topological space category theory, i.e., the homeomorphism and isomorphism are the mappings which uniquely preserve the entire topological properties of a given space.
- When there is homeomorphism between the two spaces then it is termed as 'Homeomorphic', and according to the topological perspective they are considered equivalent.
- Principally, the topological space is defined as a structure of geometric object and the homeomorphism is considered as a continuous stretching and bending of the geometric object into a new shape. Consequently, a square and a circle are considered as homeomorphic to each other, but a sphere and a donut are not as the shape is not equivalent.

NOTES

- According to the Encyclopaedia Britannica, in mathematics and topology, the term homeomorphism is defined as. "A correspondence between two figures or surfaces or other geometrical objects is defined by a one-to-one mapping that is continuous in both directions is termed as homeomorphism".
- In topology, a function having all these three properties is also sometimes termed as 'Continuous'. When such type of function exists, then we state that 'X' and 'Y' are homeomorphic. Additionally, a self-homeomorphism is defined as a homeomorphism of a topological space and also itself.
- Characteristically, the homeomorphism structure and develop an equivalence relation precisely on the class of all topological spaces. The resulting or subsequent 'Equivalence Classes' are termed as the 'Homeomorphism Classes'.
- Homeomorphisms, as already discussed, are the isomorphisms according to the topological space category. Essentially, the composition or structure of two homeomorphisms is yet again a homeomorphism and subsequently the set of all self-homeomorphisms 'X' forms or structures a group termed as the homeomorphism group of 'X'.
- Characteristically, a topological space X is termed as the 'First Countable Space' if it precisely satisfies or fulfils the following given axiom termed as the 'First Axiom of Countability':
- In the field of topological analysis, a topological space X precisely with the topology τ is termed as a 'Second Countable Space' if it exceptionally satisfies the following given axiom termed as the 'Second Axiom of Countability'. There exists a countable base **B** for the topology τ.
- A function defined on a first countable space X is continuous at $p \in X$ if and only if it is sequentially continuous at p.
- A second countable space is also first countable.
- Let A be any subset of a second countable space X. Then every open cover of A is reducible to a countable cover.
- Let X be a second countable space. Then every base **B** for X is reducible to a countable base for X.
- An arbitrary or random subspace of a unique second countable space is referred as a second countable subspaces of typical separable spaces essentially may not be separable.
- In topology, a consequence of at the most continuum various separable spaces are defined as the 'Separable'. Additionally, the countable consequence of the unique second countable spaces is typically defined as the second countable, but an uncountable consequence of the unique second countable spaces essentially may not even be first countable.
- Characteristically, any continuous image of a unique and distinct separable space is defined as 'Separable' even though the quotient of a unique second countable space essentially may not be second countable.

- Lindelöf space: A topological space X is called a Lindelöf space if every open cover of X is reducible to a countable cover. Thus, every second countable space is a Lindelöf space.
- Lindelöf Covering Theorem: Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.
- The Heine-Borel covering theorem states that from any open covering of an arbitrary set A of real numbers, we can extract a countable covering. The Heine-Borel theorem tells us that if, in addition, we know that A is closed and bounded, we can reduce the covering to a finite covering.
- Compact subsets of R are closed and bounded.
- A subset of R is compact if and only if it is closed and bounded.
- A topological space is called separable if it contains a countable dense subset.
- In particular, every continuous function on a separable space whose image is a subset of a Hausdorff space is determined by its values on the countable dense subset.
- Every compact metric space (or metrizable space) is separable.
- Any topological space which is the union of a countable number of separable subspaces is separable. Together, these first two examples give a different proof that n-dimensional Euclidean space is separable.
- The space of all continuous functions from a compact subset of R^n into R is separable.
- The Lebesgue spaces L^p are separable for any $1 \le p \le \infty$.
- A topological space X is separable if and only if there exists a finite or denumerable subset A of X such that the closure of A is the entire space.
- Every second countable space is separable but not every separable space is second countable. For example, the real line R with the topology generated by the closed-open intervals $[a, b)$ is a classic example of a separable space which does not satisfy the second axiom of countability.

2.10 KEY TERMS

- **Kuratowski closure axioms:** These are referred as a set of axioms that are used to define a topology on a set.
- **Homeomorphic:** Two spaces with a homeomorphism between them are formed as homeomorphic.

  Lindelöf space: A topological space X is called a Lindelöf space if every open cover of X is reducible to a countable cover. Thus, every second countable space is a Lindelöf space.
- **Lindelöf covering theorem:** Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.

- **Open sets:** In topology, the elements of the collection are termed as the open sets.
- **Homeomorphism:** The term homeomorphism is taken from the Greek word '*homoios*' which means 'similar' and the Latin word '*morphe*' which means 'shape'.
- **Heine-Borel covering theorem:** The Heine-Borel covering theorem states that from any open covering of an arbitrary set A of real numbers, we can extract a countable covering. The Heine-Borel theorem tells us that if, in addition, we know that A is closed and bounded, we can reduce the covering to a finite covering.
- **Separability:** Separability is a technical hypothesis on a space which is quite useful and among the classes of spaces studied in geometry and classical analysis.

2.11 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define Kuratowski closure operator.
2. What is Kuratowski closure axioms?
3. State about dense subsets.
4. What is continuous function?
5. Define the term homeomorphism.
6. What is do you mean by topological spaces?
7. State the first countable space.
8. Differentiate between the first and second countable spaces.
9. State Lindelöf's theorem.
10. Give the Heine-Borel covering theorem.
11. What are separable spaces?
12. Define the term separability.

Long-Answer Questions

1. Briefly explain about the alternate methods of defining a topology in terms of Kuratowski closure operator and neighbourhood systems with the help of relevant examples.
2. Describe the significance of continuous functions and homeomorphism giving significant examples.
3. Explain in detail about the first and second countable spaces with the help of appropriate examples.
4. Discuss in detail about the Lindelöf's and Heine-Borel covering theorems giving the characteristic featurs and proof.
5. Elaborate on the separable spaces, second countability and separability giving appropriate examples.

2.12 FURTHER READING

NOTES

Munkres, James R. 2011. *Topology*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
Basener, William F. *Topology and Its Applications*. New Jersey: John Wiley & Sons, Inc.
Bourbaki, Nicolas. 1966. *General Topology: Elements of Mathematics*, Volume 2. United States: Addison-Wesley.
Bourbaki, Nicolas. 1966. *Elements of Mathematics: General Topology*, Part 1. United States: Addison-Wesley.
Kelley, John L. 1975. *General Topology*. New York: Springer-Verlag.
Willard, Stephen. 2004. *General Topology*. United States: Dover Publications.
Shilov, Georgi E. 2012. *Elementary Real and Complex Analysis*. Chelmsford: Courier Corporation.
Simmons. G. F. 2004. *Introduction to Topology and Modern Analysis*. New York: Tata McGraw-Hill Education.

UNIT 3 SEPARATION AXIOMS: CHARACTERIZATIONS AND BASIC PROPERTIES

Structure

3.0 INTRODUCTION

In the field of topology, the 'Separation Axioms' are typically used in the topological analysis and specifications for specifying and distinguishing the disjoint sets and the distinct points. Characteristically, in topological analysis the Urysohn's lemma is defined as a specific lemma which states that, 'A topological space is considered as normal iff and only iff any two disjoint closed subsets can be uniquely separated by means of a distinct continuous function'.

In topology, the Tietze extension theorem is also known as the Tietze–Urysohn–Brouwer extension theorem which states that continuous functions on a closed subset of a normal topological space can be extended to the entire space, preserving boundedness if necessary.

In mathematical analysis and specifically in particular in the field of general topology, the term 'Compactness' is defined as a unique and distinctive property which typically generalises the concept and conventional notion of a subset of the Euclidean space specifically considered as closed and bounded.

In topological specifications, a function is considered as 'Continuous' if the random or arbitrary slight or insignificant changes or differences in its output can be typically guaranteed by simply restricting and limiting to sufficiently and appropriately slight or insignificant changes in its input. A topological space is compact if every open cover of X has a finite sub cover. In other words, if X is the union of a family of open sets, there is a finite subfamily whose union is X.

In general, in the topological analysis a non-empty family A of subsets of a set X is characterised to have the Finite Intersection Property (FIP) if and only if the intersection over any predetermined or finite subcollection of A is non-empty.

In topological analysis and other associated and related fields in the mathematical evaluation, a sequential space is uniquely defined as a topological space which satisfies an exceptionally weak axiom of countability. In mathematics, a topological space is termed as countably compact if and only if every single countable open cover holds a finite subcover.

In mathematical evaluation and topological analysis, the locally compact spaces that typically holds the Hausdorff properties are specifically studied and analysed which are generally abbreviated as Locally Compact Hausdorff (LCH) spaces. The LCH spaces are considered as a exceptional and distinctive topological space in which every single point holds a compact neighbourhood. One point compactification is also sometimes known as Alexandroff compactification. The Alexandroff extension is defined as a specific methodology for precisely extending a non-compact topological space by accurately adjoining and connecting a single point in such a manner that the subsequent and resulting space can be defined as compact.

In the mathematical discipline of general topology, Stone–Čech compactification is a technique for constructing a universal map from a topological space X to a compact Hausdorff space βX.

In this unit, you will study about the separation axioms, Urysohn's lemma, Tietze extension theorem, compactness and basic properties, continuous functions and compact sets, compactness and finite-intersection property, sequentially and countably compact sets, local compactness and one point compactification, Stone-Čech compactification, compactness in metric spaces, countable compactness and sequential compactness in metric spaces.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the separation axioms
- Discuss about the Urysohn's lemma
- Comprehend on the Tietze extension theorem
- Interpret about the compactness and its basic properties

NOTES

- Understand the continuous functions and its compact sets
- Describe the compactness and finite intersection property
- Elaborate on the sequentially and countably compact sets
- Define the local compactness and one point compactification
- State the Stone-Čech compactification
- Know about the compactness in metric spaces
- Explain about the countable compactness and sequential compactness in metric spaces

3.2 SEPARATION AXIOMS: CHARACTERIZATIONS AND BASIC PROPERTIES

The separation axioms T_i specify the degree to which the separate and distinct points or closed sets may be separated by means of open sets. These completeness are statements about the richness of topology.

Definition (T_i axioms)**:** Let $(X, \mathcal{T})$ be a topological space.

***T_0* Axiom:** If a, b are two distinct and separate elements in X, then there exists an open set $U \in \mathcal{T}$ such that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$ (i.e., U contains exactly one of these points).

***T_1* Axiom:** If $a, b \in X$ and $a \neq b$, then there exists open sets U_a, $U_b \in \mathcal{T}$ containing a, b respectively, such that $b \notin U_a$, and $a \notin U_b$.

***T_2* Axiom:** If $a, b \in X$, $a \neq b$, then there exists disjoint open sets U_a, $U_b \in \mathcal{T}$ containing a, b, respectively.

***T_3* Axiom:** If A is a closed set and b is a point in X such that $b \notin A$, then there exist to separate or disjoint open sets U_A, $U_b \in \mathcal{T}$ containing A and b, respectively.

***T_4* Axiom:** If A and B are disjoint closed sets in X, then there exists disjoint open sets U_A, $U_B \in \mathcal{T}$ containing A and B, respectively.

***T_5* Axiom:** If A and B are separated sets in X, then there exist to disjoint open sets U_A , $U_B \in \mathcal{T}$ containing A and B, respectively.

If $(X, \mathcal{T})$ satisfies a T_i axiom, then X is called a T_i space. A T_0 space is sometimes called a Kolmogorov space and a T_1 space is called a **Frechet space**. A T_2 space is called a **Hausdorff space.**

Each of axioms in the Definition given above is independent of the axioms for a topological space; in fact there exists examples of topological spaces which fail to satisfy any T_i. But they are not independent of each other, for instance, axiom T_2 implies axiom T_1, and axiom T_1 implies axiom T_0.

More significantly the separation axioms are typically defined successively stronger properties. Remember that if a space is both T_3 and T_0 then it is T_2, while a space that is both T_4 and T_1 then it must be T_3. The former spaces are called regular and the latter spaces are termed as normal.

Specifically a space X is said to be **regular** if and only if it is both a T_0 and a T_3 space, **normal** if and only if it is both a T_1 and T_4 space, **completely normal** if and only if it is both a T_1 and a T_5 space. Consequently it holds the following implications:

Completely Normal $\Rightarrow$ Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff $\Rightarrow$ T_1 $\Rightarrow$ T_0

The terms 'Regular' and 'Normal' are typically used as per the standard notations and the properties that it holds. While some authors use these terms interchangeably with 'T_3 Space' and 'T_4 Space', respectively, some others use T_3 Space as a 'Regular' space and vice versa, and similarly transpose 'T_4 Space' and 'Normal'. This helps to define the successively stronger specified properties that are parallel to increasing T_i axioms.

Hausdorff Spaces

Theorem 3.1: Let $(X, \mathcal{T})$ be a topological space. Then the following statements are considered equivalent:

1. T_2 axiom: Any two distinct points of X have disjoint neighbourhoods.
2. The intersection of the closed neighborhoods of any point of X consists of that point alone.
3. The diagonal of the product space $X \times X$ is defined as a closed set.
4. For every set I, the diagonal of the product space $Y + X^I$ is close in Y.
5. No filter on X has more than one limit point.
6. If a filter $\mathcal{F}$ on X converges to x, then x is the only cluster point of $\mathcal{F}$.

Proof: We will proof the following implications:

$(1) \Rightarrow (2) \Rightarrow (6) \Rightarrow (5) \Rightarrow (1)$

$(1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1)$

(1) $\Rightarrow$ (2) : If $x \neq y$ there is an open neighbourhood U of x and an open neighbourhood V of y such that $U \cap V = \varnothing$; hence $y \notin \overline{U}$.

(2) $\Rightarrow$ (6): Let $x \neq y$; then there is a closed neighbourhood V of x such that $y \notin V$, and by hypothesis there exists $M \in F$ such that $M \subset V$; therefore $M \cap CV = \varnothing$. But CV is a neighbourhood of y; hence y is not a cluster point of $\mathcal{F}$.

(6) $\Rightarrow$ (5): Obvious, since every limit point of a filter is also a **cluster point**.

(5) $\Rightarrow$ (1): Suppose $x \neq y$ and that every neighbourhood V of x meets every neighbourhood W of y. Then the sets $V \cap W$ form a basis of a filter, which has both x and y as limits points, which is contrary to the hypothesis.

(1) $\Rightarrow$ (4): Let $(x) = (x_i)$ be a point of X^I which does not belong to the diagonal Δ. Then there are at least two indices λ, μ such that $x_\lambda \neq x_\mu$. Let V_λ (respectively V_μ) be a neighbourhood of x_λ (respectively x_μ) in X, such that $V_\lambda \cap V_\mu = \varnothing$; then the set $W = V_\lambda \times V_\mu \times \prod_{i \neq \lambda, \mu} X_i$ where $X_i = X$ if $i \neq \lambda, \mu$ is a neighbourhood of x in X^I which does not meet Δ. Hence Δ is closed in X^I.

(4) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): If $x \neq y$ then $(x, y) \in X \times X$ is not in the diagonal Δ, hence there is a neighbourhood V of x and a neighbourhood W of y in X such that $(V \times W) \cap \Delta = \varnothing$, which means that $V \cap W = \varnothing$.

Let $f : X \to Y$ be a mapping of a set X into a Hausdorff space Y; then it follows immediately from Theorem 3.1 that f has at the most one limit with respect to a filter $\mathcal{F}$ on X, and that if f has y as a limit with respect to $\mathcal{F}$, then y is defined as the only cluster point of f with respect to $\mathcal{F}$.

Theorem 3.2: Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y; then the set of all $x \in X$ such that $f(x) = g(x)$ is closed in X.

Corollary 1 (Principle of Extension of Identities): Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.

In other words, a continuous map of X into Y (Hausdorff) is uniquely determined by its values at all points of a dense subset of X.

Corollary 2: If f is a continuous mapping of a topological space X into a Hausdorff space Y, then the growth of f is closed in $X \times Y$.

This graph is the set of all $(x, y) \in X \times Y$ such that $f(x) = y$ and the two mappings $(x, y) \to f(x)$ are continuous.

The invariance properties of Hausdorff topologies are defined in Thorem 3.3.

Theorem 3.3:

1. Hausdorff topologies are invariant under closed bijections.
2. Each subspace of a Hausdorff space is also a Hausdorff space.
3. The Cartesian product $\prod \{X_\alpha \mid \alpha \in A\}$ is Hausdorff if and only if each X_α is Hausdorff.

Proof:

1. Since a closed bijection is also an open map, the images of disjoint neighborhoods are disjoint neighbourhoods and the result follows.

2. Let $A \subset X$ and $p, q \in A$; since there are disjoint neighbourhoods $U(p)$, $U(q)$ in X, the neighbourhoods $U(p) \cap A$ and $U(q) \cap A$ in A are also disjoint.

3. Assume that each X_α is Hausdorff and that $\{p_\alpha\} \neq \{q_\alpha\}$; then $p_\alpha \neq q_\alpha$ for some α. Therefore, considering and taking the disjoint neighbourhoods $U(p_\alpha)$, $U(q_\alpha)$ will give the required disjoint neighbourhoods $(U(p_\alpha))$, $(U(q_\alpha))$ in $\prod_\alpha X_\alpha$. Conversely, if $\prod_\alpha X_\alpha$ is Hausdorff, then each X_α is homeomorphic to some segment in $\prod_\alpha X_\alpha$, therefore by Condition (2), X_α is Hausdorff (since the Hausdorff property is a topological invariant.

Theorem 3.4: If every point of a topological space X has a closed neighbourhood, then X is Hausdorff which is a Hausdorff subspace of X.

3.3 URYSOHN'S LEMMA

In topological analysis and evaluation, the Urysohn's lemma is defined as a lemma which states that a topological space is considered as normal if and only if any two disjoint or separate closed subsets can be precisely separated through a continuous function. The Urysohn's lemma is named after the mathematician Pavel Samuilovich Urysohn who has defined its concept.

Characteristically, the Urysohn's lemma is generally used for constructing the continuous functions in conjunction with various properties on the normal spaces. The Urysohn's lemma is extensively applicable and appropriate because all the metric spaces and all the compact Hausdorff spaces are defined as normal. In addition, the Urysohn's lemma is generalized through the Tietze extension theorem and is commonly used in the proof of Tietze extension theorem.

Theorem 3.5 (Urysohn's Lemma): Let X be a topological space and let any two disjoint closed sets A, B in X can be separated by open neighbourhoods then there is a continuous function $f : X \to [0, 1]$ such that $f \mid A \equiv 1$ and $f \mid B \equiv 0$.

Proof: Here, define f as the pointwise limit of a sequence of functions. Any collection of sets $\mathcal{U}_r = (A_0, A_1, \ldots, A_r)$ is considered as an admissible chain (Refer Figure 3.1) if $A = A_0 \subset A_1 \subset, \ldots, \subset A_r \subset X \backslash B$ and $\overline{A}_{k-1} \subset \overset{0}{A}_k, 0 \leq k \leq r$. The set $\overset{0}{A}_{k+1} \backslash \overline{A}_{k-1}$ is the kth step domain of U_r, where $A_{r+1} = X$ and $A_{-1} = \phi$.

NOTES

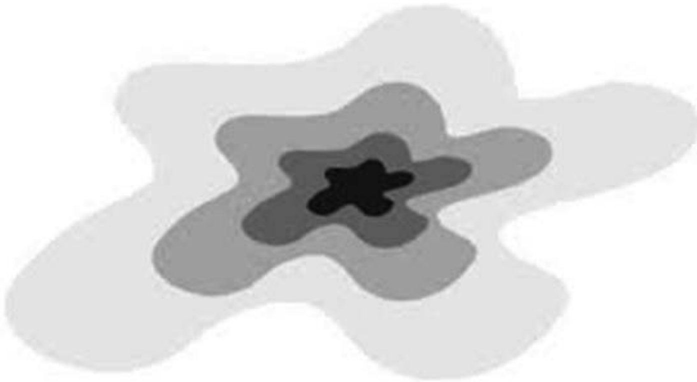


Fig. 3.1 *An Admissible Chain*

Figure 3.1 illustrates an advisible chain in which each pair of the adjacent shaded regions uniquely represent a step domain. This Theorem with 3.5 can be provedthe help of the following lemmas.

Lemma 1: Each $x \in X$ lies in some step domain for any U_r.

Proof: Take $x \in X$ and any admissible chain U_r. Let k, $0 \le k \le r+1$, be the smallest number such that $x \in \overset{0}{A}_k$. Then $x \in \overset{0}{A}_k \setminus \overline{A}_{k-2}$.

Lemma 2: Each step domain is open.

Proof: Since $\overset{0}{A}_{k+1} \setminus \overline{A}_{k-1} = \overset{0}{A}_{k+1} \cap \left(X \setminus \overline{A}_{k-1}\right)$ is the finite intersection of open sets, hence it is open.

For any U_r, define the uniform step function $f_r : X \to [0, 1]$ as,

$f_r|A \equiv 1, f_r|(X \setminus A_r) \equiv 0$ and $f_r|(A_k \setminus A_{k-1}) \equiv 1-k/r,\ 1 \le k \le r$

Lemma 3: If x and y are in the same step domain, then $|f_r(x) - f_r(y)| \le 1/r$.

Proof: Let $x, y \in \overset{0}{A}_{k+1} \setminus \overline{A}_{k-1}$. If both x and y are in $\overset{0}{A}_{k+1}$ or A_k, then by definition of f_r, $f_r(x) = f_r(y)$. Hence $|f_r(x) - f_r(y)| = 0$. If $x \in \overset{0}{A}_{k+1}$ and $y \in A_k$, then $f_r(x) = 1-(k+1)/r$ and $f_r(y) = 1 - k/r$. So $|f_r(x) - f_r(y)| = 1/r$.

The above three lemmas will be used in the last step of the proof.

Now, let the admissible chain $\mathcal{U}_{2r-1} = (A_0, A'_1, \ldots, A'_r, A_r)$ be a refinement of the admissible chain $\mathcal{U}_r = (A_0, A_1, \ldots, A_r)$.

We can say that, the refinement $\mathcal{U}_{2r-1}$ of the admissible chain $\mathcal{U}_r$ contains every set in $\mathcal{U}_r$, and for every $i \ge 1$ contains a set A'_i such that $A_{i-1} \subset A'_i \subset A_i$. Naturally, refinements place new sets between each pair of sets in the original admissible chain.

NOTES

Lemma 4: Every admissible chain has a refinement.

Proof: It is appropriate to show that for any subsets M, N of X, with $\overline{M} \subset \overset{0}{N}$, there exists $L \subset X$ with $\overline{M} \subset \overset{0}{L} \subset \overline{L} \subset \overset{0}{N}$. Now since $\overline{M} \subset \overset{0}{N}, \overline{M} \cap \left(X \setminus \overset{0}{N} = \varphi \right)$ and since (X \ $\overset{0}{N}$) is the complement of an open set and therefore closed, there exist disjoint open sets U, V, with $\overline{M} \subset U$ and (X \ $\overset{0}{N}$) $\subset V$. Again since U and V are disjoint, $U \subset (X \setminus V)$. Also because $(X \setminus V)$ is closed and $\overline{U}$ is contained in every closed set containing U, so $\overline{U} \subset (X \setminus V)$. Moreover, $(X \setminus \overset{0}{N}) \subset V$ means $(X \setminus V) \subset \overset{0}{N}$. Putting all this collectively gives, $\overline{M} \subset \overset{0}{U} \subset \overline{U} \subset (X \setminus V) \subset \overset{0}{N}$. Let $L = U$ and the proof is complete.

Lemma 5: Let $\mathcal{U}_r$ be an admissible chain with $r + 1$ elements and $\mathcal{U}_s$ be a refinement with $2r+1$ elements. Then $|f_r(x) - f_s(x)| \leq 1/(2r)$.

Proof: Suppose $x \in A_k \setminus A_{k-1}$, where $A_k, A_{k-1} \in U_r$. Then, $f_r(x) = 1 - k/r$. Also, $\mathcal{U}_s = (A_0, A'_1, A_1, \ldots, A'_j, A_j, \ldots, A'_k, A_k) = (A_0, A_{1'}, A_{2'}, \ldots, A_{(2k-1)'}, A_{2k}, A_{(2r-1)'}, A_{2r'})$. Now, x is either in $A_k \setminus A'_k = A_{2k} \setminus A_{(2k-1)}$ or in $A'_k \setminus A_{k-1} = A_{(2k-1)}$. Therefore either $f_s(x) = 1 - (2k)/(2r) = f_r(x)$ or $f_s(x) = 1 - (2k-1)/(2r) = f_r(x) + 1/(2r)$. By both the methods we get the desired result.

Now we will define the sequence. Let $\mathcal{U}_0 = (A, X \setminus B)$ and let $\mathcal{U}_{n+1}$ be a refinement of $\mathcal{U}_n$.

By Lemma 4, every admissible chain has a refinement. Therefore, we get a sequence of admissible chains. Let f_n be the uniform step function on the nth admissible chain.

Lemma 6: The sequence $\{f_n(x)\}$ converges for each $x \in X$.

Proof: From the definition of the uniform step functions, the sequence is bounded above by 1. Now it remains to prove that the sequence is non-decreasing. Note first that $\mathcal{U}_0$ contains one term excluding A itself and therefore by definition of refinement, $\mathcal{U}_1$ will contain 2 terms excluding A.

Applying induction, $\mathcal{U}_n$ contains 2^n terms excluding A. Notice that for $x \notin A_j \setminus A_{j-1} \;\forall\; A_j, A_{j-1} \in \mathcal{U}_r, f_k(x)$ is either 0 or 1, i.e., constant and constant sequences converge. Suppose $x \in A_j \setminus A_{j-1}$, where $A_j, A_{j-1} \in U_r$. Then $f_k(x) = 1 - j/k$. Furthermore, $\mathcal{U}_{k+1} = (A_0, A'_1, A_1, \ldots, A'_j, A_j, \ldots, A'_k, A_k) = (A_0, A_{1'}, A_{2'}, \ldots, A_{(2j-1)}, A_{2j}, \ldots, A_{(2k-1)'}, A_{2k'})$ and x is either in $A_j \setminus A'_j = A_{2j} \setminus A_{(2j-1)'}$ or in $A'_j \setminus A_{j-1} = A_{(2j-1)'} \setminus A_{(2j-2)'}$. This implies,

$$f_{k+1}(x) = 1-(2j)\backslash(2k) = f_k(x)$$

Or

$$f_{k+1} = 1-(2j-1)\backslash(2k) \geq f_k(x)$$

This proves that the sequence is nondecreasing and hence convergent, because bounded monotonic sequences converge.

For each x, let $f(x) = \lim_{n\to\infty} f_n(x)$. Because each f_n is constantly 1 on A and 0 on B, f will also have this property. For proving that f is continuous, it is sufficient to show that if we take any $f(x) \in [0, 1]$ and any open set $(a, b) \subset [0, 1]$ containing $f(x)$, then there is an open set $U \subset X$ such that $x \in U$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. To prove this, we have to show one more lemma.

Lemma 7: For fixed x and any n, $|f(x) - f_n(x)| \le 1/2^n$.

Proof: $|f(x) - f_n(x)| = |\lim_{\substack{k\to\infty \\ k\ge n}} f_k(x) - f_n(x)|$

$$= |\lim_{\substack{k\to\infty \\ k\ge n}} (f_k(x) - f_n(x))| = \lim_{\substack{k\to\infty \\ k\ge n}} |(f_k(x) - f_{k-1}(x)) + (f_{k-1}(x) - f_{k-2}(x)) + (f_{n+1}(x) - f_n(x))|$$

$$\le \lim_{\substack{k\to\infty \\ k\ge n}} (|f_k(x) - f_{k-1}(x)| + |f_{k-1}(x) - f_{k-2}(x)| + \ldots\ldots + |f_{n+1}(x) - f_n(x)|)$$

$$\le \lim_{\substack{k\to\infty \\ k\ge n}} \left(1/2^k + 1/2^{k-1} + \ldots. + 1/2^{n+1}\right)$$

$$= \sum_{k=n+1}^{\infty} 1/2^k = 1/2^n \left(\sum_{k-1}^{\infty} 1/2^k\right) = 1/2^n .$$

Take n large enough so that $3/2^n < \varepsilon$, and suppose x lies in the kth step domain,

$S_k = \overset{0}{A}_{k+1} \setminus \overline{A}_{k-1}$ (Since by Lemma 1, every x lies in some step function).

Furthermore, by Lemma 2, this step domain is an open neighbourhood of x. Take any $y \in S_k$. Then by Lemmas 3 and 6,

$$|f(x) - f(y)|$$

$$= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$
$$\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$\le 1/2^n + 1/2^n + 1/2^n = 3/2^n < \varepsilon$. So every $y \in S_k$ maps into (a, b), which proves that f is continuous.

3.4 TIETZE EXTENSION THEOREM

In the field of topological analysis, the Tietze extension theorem, also acknowledged as the Tietze–Urysohn–Brouwer extension theorem, defines that the explicit and specific continuous functions considered on a closed subset of a unique normal topological space can be precisely extended to the entire or complete space by preserving the boundedness if essential.

The mathematicians L. E. J. Brouwer and Henri Lebesgue originally proved a special instance of the theorem for X being a finite dimensional real vector space. The mathematician Heinrich Tietze extended this concept and notion for all metric spaces and hence this theorem is termed as Tietze extension theorem. Pavel Urysohn then typically proved this theorem uniquely for normal topological spaces.

To explain Tietze extension main theorem, Lemma 8 is considered.

Lemma 8: If X is a normal topological space and A is closed in X, then for any continuous function $f: A \to R$ such that $|f(x) \leq 1$, there is a continuous function g: $X \to R$ such that $|g(x)| \leq \frac{1}{3}$ for $x \in X$, and $|f(x) - g(x)| \leq \frac{2}{3}$ for $X \in A$.

Proof: The sets $f^{-1}\left(\left(-\infty - \frac{1}{3}\right]\right)$ and $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ are disjoint and closed in A. Since A is closed, they are also closed in X. Since X is normal, then by Urysohn's lemma and the fact that [0, 1] is homomorphic to $\left[-\frac{1}{3}, \frac{1}{3}\right]$, there is a continuous function g: $X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $g\left(f^{-1}\left(-\infty, -\frac{1}{3}c\right]\right) = -\frac{1}{3}$ and $g\left(f^{-1}\left[\frac{1}{3}, \infty\right)\right) = \frac{1}{3}$. Thus $|g(X)| \leq \frac{1}{3}$ for $x \in X$. Now, if $-\infty \leq f(X) \leq -\frac{1}{3}$, then $|g(x)| \leq -\frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Similarly if $\frac{1}{3} \leq f(x) \leq 1$, then $g(x) = \frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Finally, for $|f(x)| \leq \frac{1}{3}$ we have that $|g(x)| \leq \frac{1}{3}$, and so $|f(x) - g(x)| \leq \frac{2}{3}$. Hence $|f(x) - g(x)| \leq \frac{2}{3}$ holds for all $x \in A$.

Theorem 3.6: First suppose that for any continuous function on a closed subset there is a continuous extension. Let C and d be disjoint and closed in X. Define f: $C \cup D \to R$ by $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$. Now f is continuous and we can extend it to a continuous function $F: X \to R$. By Urysohn's lemma, x is normal because F is continuous function such that $F(x) = 0$ for $x \in C$ and $F(x) = 1$ for $x \in D$.

Conversely, let X be normal and A be closed in X. By the Lemma 8, there is a continuous function g_0: $X \to R$ such that $\left|g_0(x) \leq \frac{1}{3}\right|$ for $x \in X$ and $|f(x) - g_0(x)| \leq \frac{1}{3}$ for $x \in A$. Since $(f - g_0)$: $A \to R$ is continuous, the lemma tells us that there is a continuous function g_1: $X \to R$ such that $|g_1(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)$ for $x \in X$ and $|f(x) - g_0(x) - g_1(x)| \leq \frac{2}{3}\left(\frac{2}{3}\right)$ for $x \in A$. By repeated application of the lemma we can constuct a sequence of continuous

NOTES

function $g_0, g_1, g_2, ...$, such that $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n$ for all $x \in X$, and $\left|f(x) - g_0(x) - g_1(x) - g_2(x) - ......\right| \le \left(\frac{2}{3}\right)^n$ for $x \in A$.

Define $F(x) = \sum_{n=0}^{\infty} g_n(x)$. Since $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n$ and $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$ converges as a geometric series, then $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely and uniformly, so F is a continuous function defined everywhere. Moreover, $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n =$ 1, implies that $|F(x)| \le 1$.

Now, for $x \in A$, we have that $\left|f(x) \sum_{n=0}^{k} g_n(x)\right| \le \left(\frac{2}{3}\right)^{k+1}$ and as k extends to infinity, then the right hand side is zero and hence the sum extends to $F(x)$. Thus, $|f(x) - F(x)| = 0$. Therefore f extends to F.

Notes: If f is considered as a function which uniquely satisfies that $|f(x)| < 1$, then the theorem can be strengthended. Find an extension F of f. The set $B\,F^{-1}(\{-1\} \cup \{1\})$ is considered as closed and disjoint from A because $|F(x)| = |f(x)| < 1$ for $x \in A$. By Urysohn's lemma there is a unique continuous function f such that $\phi(A) = \{1\}$ and $\phi(B) = \{0\}$. Hence $\phi(x)\phi(x)$ is a continuous extension of $\phi(x)$ and has the property that $|F(x)f(x)| < 1$.

If f is unbounded, then, the Tietze extension theorem holds. To verify consider $t(x) = \tan^{-1}(x)/(n/2)$. The function $t\,.\,f$ has the property that $(t\,.\,f)\,(x) <$ for $x \in a$, and therefore it can be extended to a continuous function $h: X \to R$ which has the property $|h(x)| < 1$. Hence $t^{-1}\,.\,h$ is a continuous extension of f.

3.5 COMPACTNESS: BASIC PROPERTIES

An open cover of a subset A of a metric space X, we mean a collection $C = \{G_\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.

In particular, C is said to be an open cover of the metric space X if $X = \cup \{G_\lambda : \lambda \in A\}$.

By a subcover of an open cover C of A, we mean a subcollection C' *of* C' such that C' covers A.

An open cover of A is said to be finite if it consists of finite number of open sets.

Another definition is, a subset of a metric space X is said to be compact if every open cover of A has a finite subcover, that is, if for every collection $\{G_\lambda : \lambda \in A\}$ of open sets for which

$A \subset \cup \{G_\lambda : \lambda \in A\}$,

NOTES

there exist finitely many sets $G_{\lambda 1},...,G_{\lambda n}$ among the G_λ's such that $A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}$.

In particular, the metric space X is said to be compact if for every collection $\{G_\lambda : \lambda \in A;$ of open sets for which

$$X = \cup\{G_\lambda : \lambda \in A\},$$

there exist finitely many sets $G_{\lambda_1},...,G_{\lambda_n}$ among the G_λ's such that $X = G_{\lambda_1} \cup ... \cup G_{\lambda_n}$

Theorem 3.7: Let Y be a subspace of a metric space X and let $A \subset Y$. Then A is compact relative to X if and only if A is compact relative Y.

Proof: Let A be compact relative to X and let $\{V_\lambda, \lambda \in A\}$ be a collection of sets, open relative to Y, which covers A so that $A \subset \cup\{V_\lambda : \lambda \in A\}$. Then there exists G_λ, open relative to X, such that $V_\lambda = Y \cap G_\lambda$ for every $\lambda \in A$. It then follows that,

$$A \subset \cup\{G_\lambda : \lambda \in A\},$$

that is, $\{G_\lambda : \lambda \in A\}$ is an open cover of A relative to X. Since A is compact relative to X, there exist finitely many indices $\lambda_1,...,\lambda_n$ such that

$$A \subset G_{\lambda_1} \cup,...,\cup G_{\lambda_n}.$$

Since $A \subset Y$, we have $A = Y \cap A$.

Hence $A \subset Y \cap \{G_{\lambda_1} \cup,...,\cup G_{\lambda_n}\} = (Y \cap G_{\lambda_1}) \cup ... \cup (Y \cap G_{\lambda_n})$

[By Distributive Law]

Since $Y \cap G_{\lambda_i} = A_{\lambda_i} \{i = 1, 2, ..., n\}$, we obtain

$$A \subset A_{\lambda_1} \cup,...,\cup A_{\lambda_n}. \quad .....(3.1)$$

This shows that A is compact relative to Y.

Conversely, let A be compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of X which cover A so that,

$$A \subset \cup \{G_\lambda : \lambda \in \Lambda\} \quad .....(3.2)$$

Since $A \subset Y$, Equation (3.2) implies that,

$$A \subset Y \cap [\cup \{G_\lambda : \lambda \in \Lambda\} = \cup\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

[By Distributive Law]

Since $Y \cap G_\lambda$ is open relative to Y, the collection

$$\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

is an open cover of A relative to Y. Since A is compact relative to Y, we must have

$$A \subset (Y \cap G_{\lambda 1}) \cup .....\cup (Y \cap G_{\lambda n}) \quad ....(3.3)$$

for some choice of finitely many indices $\lambda_1,...,\lambda_n$. But (Equation 3.3) implies that,

$$A \subset G_{\lambda 1} \cup .....\cup G_{\lambda n}.$$

It follows that A is compact relative to X.

Theorem 3.8: Every compact subset of a metric space is closed.

Proof: Let A be a compact subset of a metric space X. We shall prove that $X-A$ is an open subset of X. Let $p \in X-A$.

For each $q \in A$, let $N_q\,(p)$ and $M(q)$ be neighbourhoods of p and q, respectively, of radius less than $\frac{1}{2}d\;(p, q)$ so that

$$N_q\,(p) \cap M\,(q) = \varnothing$$

Then the collection,

$$\{M\,(q) : q \in A\}$$

is an open cover of A, according to the notion that neighbourhoods are open sets. Since A is compact, there are finitely many points $q_1,...,q_n$ in A such that

$$A \subset M\,(\mathrm{q}_1) \cup, ...,\cup N(q_n) \quad ......(3.4)$$

If $N = N_{q_1}(p) \cap, ...,\cap N_{q_n}(p)$ then N is a neighbourhood of p. Now

$$N \subset N_{q_1}(p) \text{ and } M(q_i) \cap N_{qi}(p) = \varnothing \text{ for } i = 1,...,n$$

$$\Rightarrow M(q_i) \cap N = \varnothing \text{ for } i = 1,...,n$$

$$\Rightarrow \cup\; \{M(q_i) \cap N : i = 1,...,n = \varnothing$$

$$\Rightarrow [\cup\; \{M(q_i) : i = 1,...,n\}] \cap N = \varnothing$$

[By Distributive Law]

$$\Rightarrow\; Y \cap N = \varnothing$$

$$\Rightarrow N \subset X - Y.$$

Thus we have shown that to each $p \in X - Y$, there exists a neighbourhood N of p such that $N \in X - Y$ and consequently $X - Y$ is open. It follows that Y is closed.

Theorem 3.9: Closed subsets of compact sets are compact.

Proof: Let Y be a compact subset of a metric space X and let F be a subset of Y, closed relative to X. To show that F is compact, let

$$C = \{G_\lambda : \lambda \in \Lambda\}$$

NOTES

be an open cover of F. Then the collection,

$$D = \{G_\lambda : \lambda \in \Lambda\} \cup \{X - F\}$$

forms an open cover of Y. Since Y is compact, there is a finite subcollection D' of D which covers Y, and hence F. If $X - F$ is a member of D', we may remove it from D' and still retain an open finite cover of F. We have thus shown that a finite subcollection of C covers F. Hence, F is compact.

Corollary 1: If F is closed and Y is compact then $F \cap Y$ is compact.

Check Your Progress

1. What are the separation axioms?
2. State the principle of extension identities.
3. Define the Urysohn's lemma.
4. State the Tietze extension theorem.
5. Define an open cover of subset.
6. When an open cover A is said to be finite?

3.6 CONTINUOUS FUNCTIONS AND COMPACT SETS

$D \cup R$ is compact if and only if for any given open covering of D we can subtract a finite sucovering. That is, given (G_α), $\alpha \in A$, a collection of open subsets of R (A an arbitrary set of indices) such that $D \subset \cup_{\alpha \in A} G_\alpha$, then there exist finitely many indices $\alpha_1,...,\alpha_n \in A$ such that $D \subset \cup_{i=1}^{N} G_{\alpha_1}$.

Let D be an arbitrary subset of R. Then $A \subset D$ is open in D (or relative to D, or D-open) if and only if there exists G open subset of R such that $D = G \cap D$. Similarly we can define the notion of D-closed sets. Note that D is both open and closed in D and so is ϕ.

$D \subset R$ is connected if and only if ϕ and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \phi$ or $B = \phi$.

Let $D \subseteq R$, $a \in D$ a fixed element and $f: D \to R$ an arbitrary function. By definition, f is continuous at a if and only if the following property holds:

$$\forall \varepsilon > 0, \exists \delta_a(\varepsilon) > 0$$

Such that $|x - a| < \delta_a(\varepsilon)$ and $x \in D \Rightarrow |f(x) - f(a)| < \varepsilon$

The last implication can be rewritten in terms of sets as follows:

$$f\left(B_a\left(\delta_a(\varepsilon)\right) \cap D\right) \subseteq B_{f(a)}(\varepsilon)$$

Here, we use the notation $B_x(r) := (x - r, x + r)$.

Characterization of Continuous Functions Using Preimages

Theorem 3.10: Let $D \subseteq R$ and $f : D \to R$ a function, then the following propositions are equivalent:

- f is continuous (on D).
- $\forall G \subseteq R$ open, $f^{-1}(G)$ is open in D.
- $\forall F \subseteq R$ closed, $f^{-1}(F)$ is closed in D.

Proof: $a \Rightarrow b$. Let $G \subset R$ open. Pick $a \in f^{-1}(G)$. Then $f(a) \in G$ and since G is open, there must exist $\varepsilon > 0$ such that $B_{f(a)}(\varepsilon) \subseteq G$. By continuity, corresponding to this $\varepsilon > 0$ there exists $\delta > 0$ such that $f\left(B_a(\delta) \cap D\right) \subset B_{f(a)}(\varepsilon)$. But this places the entire set $B_0(\delta) \cap D$ inside $f^{-1}(G)$:

$$B_a(\delta) \cap D \subseteq f^{-1}(G)$$

Writing now $\delta = \delta_a$ to mark the dependence of δ on a and varying $a \in f^{-1}(G)$, we obtain

$$f^{-1}(G) = \left(\cup_{a \in f^{-1}(G)} B_a(\delta_a)\right) \cap D$$

Which shows that $f^{-1}(G)$ is open in D.

$b \Leftrightarrow c$. Let $F \subseteq R$ a closed set, which is equivalent to saying that $G = GF$ (the complement in R) is open.

Then

$$f^{-1}(F) = \{x \in D \mid f(x) \in F\} = \{x \in D \mid f(x) \notin G\} = D - f^{-1}(G)$$

Since the complement of a D-open subset of D is D-closed, it means that $f^{-1}(F)$ is closed in D if and only if $f^{-1}(G)$ is open in D.

$c \Rightarrow a$: Left as an exercise.

Using this characterization, we can prove that the composition of continuous functions is a continuous function.

Proposition: Assume $f : D \to R$ is continuous, $g : E \to R$ is continuous and $f(D) \subseteq E$. Then the function $h := gof : D \to M$ defined by $h(x) = g(f(x))$ is continuous.

Proof: Let $G \subseteq R$ an open set. Then $h^{-1}(G) = f^{-1}(g^{-1}(G))$. But $g^{-1}(G) = V \cap E$, for some open set $V \subseteq R$. But then $h^{-1}(G) = f^{-1}(V \cap E) - f^{-1}(V)$ is open in D. So h is continuous.

Theorem 3.11: Assume $f : D \to R$ is a continuous function, such that $f(x) \neq 0$, $\forall x \in D$. Then $h : D \to R$ given by $h(x) = 1/f(x)$, is continuous as well.

Proof: $g : R - \{0\} \to R$ $g(x) = 1/x$ is continuous, $f(D) \subseteq R - \{0\}$, hence $h = g \circ f$ is continuous.

General Properties of Continuous Functions

NOTES

Theorem 3.12: A continuous function maps compact sets into compact sets.

Proof: In other words, assume $f : D \to R$ is continuous and D is compact. Then we need to prove that the image $f(D)$ is a compact subset of R. For that, we consider an arbitrary open covering $f(D) \subseteq \cup_\alpha G_\alpha$ of $f(D)$ and we will try to find a finite subcovering. Taking the preimage we have $D \subseteq \cup_\alpha f^{-1}(G_\alpha)$. But $f^{-1}(G_\alpha)$ is open in D, so there must exist $V_a \subseteq R$ open such that $f^{-1}(G_\alpha) = V_\alpha \cap D$. Then $D \subseteq \cup_\alpha (V_\alpha \cap D)$ which simply means that $D \subseteq \cup_\alpha V_\alpha$. We thus arrived at an open covering of D. So there must exist finitely many indices $\alpha_1, \ldots, \alpha_N$ such that $D \subseteq \cup_{i=1}^{N} G_{\alpha_i}$, which implies the equality $D = \cup_{i=1}^{N} (V_{\alpha_i} \cap D) = \cup_{i=1}^{N} f^{-1}\left(G_{\alpha_i}\right)$. But this implies in turn that $f(D) \subseteq \cup_{i=1}^{N} G_{\alpha_i}$ is compact.

Theorem 3.13: A continuous function maps connected sets into connected sets.

In other words, assume $f : D \to R$ is continuous and D is connected. Then $f(D)$ is connected as well.

Proof: Assume $f(D)$ is not connected. Then there must exist A, B disjoint, non-empty subsets of $f(D)$, both open relative to $f(D)$, such that $f(D) = A \cup B$. Being open relative to $f(D)$ simply means there exists U, $V \subseteq R$ open such that $A = f(D) \cap U$, $B = f(D) \cap V$. So $f(D) \subseteq U \cup V$. But this implies that $D \subset f^{-1}(U) \cap f^{-1}(V)$. Since U, V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open relative to D. But they are also disjoint. Since D is connected, it follows that at least one of them, say $f^{-1}(U)$, is empty. But $A \subseteq U$, so this forces $f^{-1}(A) = \phi$ as well, which is impossible unless $A = \phi$ (note that A is a subset of the image of f), contradiction.

Theorem 3.14: A continuous function on a compact set is uniformly continuous.

Proof: Assume D compact and $f : D \to R$ continuous. Given $\varepsilon > 0$ we need to find $\delta(\varepsilon) > 0$ such that if $x, y \in D$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

From the definition of continuity, given $\varepsilon > 0$ and $x \in D$, there exists $\delta_x(\varepsilon)$ such that if $|y - x| < \delta_x(\varepsilon)$, then $|f(y) - f(x)| < \varepsilon$. Clearly $D \subseteq \cup_{x \in D} B_x\left(\frac{1}{2}\delta\left(\varepsilon/2\right)\right)$. From this open covering we can extract a finite subcovering (D is compact), meaning there must exists finitely many $x_1, x_2, \ldots, x_N$ εD such that $D \subseteq \cup_{i=1}^{N} B_{x_i}(\frac{1}{2}\delta_{x_i}(\varepsilon/2))$.

Let now $\delta(\varepsilon) = \min\{\frac{1}{2}\delta_{x_i}(\varepsilon/2)\}$.

Take $y, z \in D$ arbitrary such that $|y - z| < \delta(\varepsilon)$. The idea is that y will be near some x_j, which in turn places z near that same x_j. But that forces both $f(y)$, $f(z)$ to be close to $f(x_j)$ (by continuity at x_j), and hence close to each other.

Since $y \in D$, there must exist some j, $1 \leq j \leq N$ such that $y \in B_{x_j}(\frac{1}{2}\delta_{x_j}(\varepsilon/2))$. Thus,

- $|y - x_j| < \frac{1}{2}\delta_{x_j}(\varepsilon/2)$
- but $|y - z| < \delta(\varepsilon) \leq \frac{1}{2}\delta_{x_j}(\varepsilon/2)$

By the triangle inequality it follows that $|z - x| < \delta_{x_j}(\varepsilon/2)$. So y, z are within $\delta_{x_j}(\varepsilon/2)$ of x.

This implies that,

- $|f(y) - f(x_j)| < \varepsilon/2$
- $|f(z) - f(x_j)| < \varepsilon/2$

By the triangle inequality once again we have $|f(y) - f(z)| < \varepsilon$.

Alternative Proof using Sequences: Assume f is not uniformly continuous, meaning that there exists $\varepsilon > 0$ such that no $\delta > 0$ completes the proof.

Checking what this means for $\delta = \frac{1}{n}$, we see that for any such $3n \geq 1$ there exist $x_n, y_n \in D$ such that $|x_n - y_n| < \frac{1}{n}$ and yet $|f(x_n) - f(y_n)| > \varepsilon$. However D is compact; in particular any sequence in D has a convergent subsequence whose limit belongs to D. Applying this principle twice we find that there must exist $n_1 < n_2 < \ldots$ such that the subsequences $\left(x_{n_k}\right)_{k\geq 1}$ and $\left(y_{n_k}\right)_{k\geq 1}$ are convergent, and $x = \lim_{x\to\infty} x_{n_k} \in D, y = \lim_{x\to\infty} y_{n_k} \in D$. We have the following:

- By construction, $\left|x_{n_k} - y_{n_k}\right| < \frac{1}{n_k} \leq \frac{1}{k}$. Taking the limit, we find $x = y$.
- By continuity, $\lim_{k\to\infty} f\left(x_{n_k}\right) = f(x)$, since $x \in D$. Also

  $\lim_{k\to\infty} f\left(y_{n_k}\right) = f(y)$.
- Also by construction, $\left|f\left(x_{n_k}\right) - \left(y_{n_k}\right)\right| > \varepsilon$. Hence in the limit, $|f(x) - f(y)| > \varepsilon$.

We thus reach a contradiction.

Proposition: Let $D \subseteq R$. Then the following propositions are equivalent:

(a) D is compact

(b) D is bounded and closed

(c) Every sequence in D has a convergent subsequence whose limit belongs to D.

Proof: $a \Rightarrow b$. $D \subseteq R = \cup_{n=1} \infty(-n, n)$ is an open covering of that D. Hence $\exists N \geq 1$ such that $D \subseteq \cup_{n=1} \infty(-n, n) = (-N, N)$. This shows that D is bounded. To prove D is closed, we prove that $R - D$ is open. Let $y \in R - D$. Then $D \subseteq \cup_{n=1}^{\infty}\left(R - [y - \frac{1}{n}, \frac{1}{n}]\right)$. This open covering must have a finite subcovering, so $\exists N \geq 1$ such that $D \subseteq R - [y - \frac{1}{N}, y + \frac{1}{N}]$. But this implies that $\left(y - \frac{1}{N}, y + \frac{1}{N}\right) \subseteq R - D$. But y was chosen arbitrary in $R - D$, so this set is open, and hence D itself is closed.

$b \Rightarrow c$. This has to do with the fact that every bounded sequence has a convergent subsequence.

$c \Rightarrow b$. Here one shows $D = \bar{D}$ and this has to do with the fact that D is the set of limits of convergent sequences of D, etc.

c $\Rightarrow a$. Let $D \subseteq \cup_{k=1}^{\infty} G_k$ be an arbitrary open covering of D.

Note: A covering by a countable collection of open sets is not the most general infinite open covering one can imagine, of course; we need an intermediate step to prove that from any open covering of D we can extract a countable subcovering, and this has to do with the fact that R admits a countable dense set.

We will now prove that there exists $n \geq 1$ such that $D \subseteq \cup_{k=1}^{\infty} G_k$. Assume this was not the case. Then $\forall N \geq 1$, there exists $x_n \in D - \cup_{k=1}^{\infty} G_k$. But x_n is a sequence in D, so it must have a convergent subsequence; call it $\left(x_{n_j}\right)_{j \geq 1}$, with limit in D. So $\lim_{j \to \infty} x_{n_j} = a \in D$. But a belongs to one of the G_i's, say $a \in G_N$. Since G_N is open, it follows that $x_{n_j} \in G_N$, for $j \geq j_0$ (j large enough). In particular this shows that for j large enough (larger than j_0 and larger than N) we have $x_{n_j} \in G_N \subseteq \cup_{k=1}^{n_j}$, since $n_j \geq j > N$. This contradicts the defining property of x_n's.

Theorem 3.15: R is connected.

Proof: This can be restated as, ϕ and R itself are the only subsets of R which are both open and closed. To prove this, let E be a non-empty subset of R with this property. We will prove that $E = R$. For that, take an arbitrary $c \in R$. To prove

NOTES

that $c \in E$, we assume that $c \notin E$ and look for a contradiction. Since E is non-empty, it follows that E either has points to the left of c or to the right of c. Assume that the former holds.

- Consider the set $S = \{x \in E | x < c\}$. By construction, S is bounded from above (c is an upper bound for S). Therefore we can consider $y = lub(S) \in R$.
- Input: E is closed. Then $S = E \cap (-\infty, c]$ is also closed. Then $y \in \overline{S} = S$, so $y < c$.
- Input: E is open, $y \in S \subseteq E$ and E is open. This means that there exists $\varepsilon >$ 0 such that $(y - \varepsilon, y + \varepsilon) \subseteq E$. Choose ε small enough so that $\varepsilon < c - y$. In that case $z = y + \varepsilon/2 \in (y - \varepsilon, y + \varepsilon) \subseteq E$ is an element of E with the properties,
  - o $z < c$, hence $z \in S$
  - o $z > y$

Which is in contradiction with the defining property of y.

Theorem 3.16: The only connected subsets of R are the intervals, bounded or unbounded, open or closed or neither.

Proof: First we prove that a connected subset of R must be an interval.

Let $E \subseteq R$ be a connected subset. We prove that if $a < b \in E$, then $[a,b] \subseteq E$. In other words, together with any two elements, E contains the entire interval between them. To see this, let c be a real number between a and b. Assume $c \notin E$. Then $E = A \cup B$, where $A = (-\infty, c) \cap E$ and $B = (c, +\infty) \cap E$. Note that A and B are disjoint subsets of D, both open relative to D. Since S is connected, at least one of them should be empty, contradiction, since $a \in A$ and $b \in B$. Thus $c \in E$.

To show that E is actually an interval, consider inf E and sup E. E is bounded. Then $m = \inf E$, $M = \sup E \in R$, and clearly $E \subseteq [m, M]$. On the other hand, for any given $x \in (m, M)$, there exists $a,b \in E$ such that $a < x < b$. That is because $m, M \in E$ and one can find elements of E as close to m (respectively M) as desired (draw a picture with the interval (m, M) and place a point x inside it). But then $[a,b] \subseteq E$, and in particular $x \in E$. Since x was chosen arbitrarily in (m, M), we must have $(m, M) \subseteq E \subseteq [m, M]$, so E is definitely an interval. Case two: E is unbounded. With a similar argument, show that E is an unbounded interval.

Conversely, we need to show that intervals are indeed connected sets. The proof is almost identical to that in the case where the interval is R itself.

Theorem 3.17: Let $D \subseteq R$ be compact and $f: D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.

Proof: $f(D)$ is a compact subset of R, so it is bounded and closed. This implies that glb($f(D)$) $\in f(D)$ and lub($f(D)$) $\in f(D)$ as well. But then there must exist y_1, y_2

$\in D$ such that $f(y_1) = \text{glb}\, f(D)$ and $f(y_2) = \text{lub}\, f(D)$. But this implies $f(D) \subseteq [f(y_1), f(y_2)]$ and we are done.

Note: We use the notation $\sup_{x \in D} f(x)$ to denote the lub of the image of D. In other words, $\sup_{x \in D} f(x) = \text{lub}\{f(y) \mid y \in D\}$. The theorem says that if D is compact and f is continuous, then $\sup_{x \in D} f(x)$ is finite, and moreover that there exists $y_1 \in D$ such that $f(y_1) = \sup_{x \in D} f(y)$. If the domain is not compact, one can find examples of continuous functions such that either i) sup $f = +\infty$ or such that ii) sup f is a real number but not in the image of f.

For case i), take $f(x) = 1/x$ defined on (0,1]. For case ii), take $f(x) = x$ defined on [0,1).

Theorem 3.18: A continuous (real-valued) function defined on an interval in R has the intermediate value property.

Proof: Assume E is an interval in R and $f: E \to R$ a continuous function. Let $a,b \in E$ (say $a < b$) and y a number between $f(a)$ and $f(b)$. The intermediate value property is the statement that there exists c between a and b such that $f(c) = y$. But this follows immediately from the fact that $f(E)$ is an interval. E is an interval in R $\Rightarrow E$ is connected $\Rightarrow f(E)$ is a connected subset of $R \Rightarrow f(E)$ is an interval in R.

3.7 COMPACTNESS AND FINITE INTERSECTION PROPERTY

The Finite Intersection Property or FIP is defined as a unique property of a collection of subsets of a set X. A collection has this property if typically the intersection over any finite subcollection of the collection is non-empty.

A centered system of sets is defined as a collection of sets with the specific finite intersection property.

Let X be a set with $A = \{A_i\}_{i \in I}$, a family of subsets of X. Then the collection A has the Finite Intersection Property (FIP), if any Finite subcollection $J \subset I$ has non-empty intersection $\bigcap_{i \in J} A_i$.

Theorem 3.19: Let X be a precise compact Hausdorff space which typically satisfies the property that every one point set is open. If X has more than one point, then X is uncountable.

To prove the Theorem 3.19, we will first consider the following significant examples:

1. The Hausdorff condition cannot be eliminated; since a countable set together with the indiscrete topology is considered as compact and holds more than one point satisfying the specific property that by no means one point sets are considered as open, however it is not uncountable.
2. The compactness condition cannot be eliminated because the set of all rational numbers shows the properties of discrete topology.
3. The compactness condition cannot be eliminated which specifies that one point sets cannot be considered open because a given finite space shows the properties of discrete topology.

Proof: Consider that X be a precise compact Hausdorff space. If U is a non-empty and precise open subset of X and also if x is precisely a point of X, then there is a distinctive neighbourhood V which is typically contained in U whose closure possibly may not contain x, i.e., x may or may not be contained in U. Initially, consider for y in U that is different from x, if x is contained in U, then there must exist such a y to satisfy the condition for otherwise U would be considered as an open one point set; alternatively, if x is not contained in U, then this is possible because U is non-empty. Consequently, according to the Hausdorff condition, consider the disjoint or separate neighbourhoods W and K of x and y, respectively. Subsequently, $(K \cap U)$ will be considered as a unique neighbourhood of y which is contained in U whose closure does not precisely contains x as required.

Now suppose f is a bijective function from Z (the positive integers) to X. Denote the points of the image of Z under f as $\{x_1, x_2, ...\}$. Consider that X is uniquely the first open set and than select uniquely a neighbourhood U_1 that is contained in X whose closure does not contains x_1. Secondly, uniquely select a neighbourhood U_2 contained in U_1 whose closure does not contains x_2. Continue analysis and evaluation by typically selecting a unique neighbourhood U_{n+1} contained in U_n whose closure does not contains x_{n+1}. Remember, that the collection $\{U_i\}$ for i in the positive integers satisfies the finite intersection property and hence the intersection of their closures is non-empty (by the compactness of X). Therefore there is a point x in this intersection. No x_i can belong to this intersection because x_i does not belongs to the closure of U_i. This means that x is not equal to x_i for all i and f is not surjective; a contradiction. Therefore, X is uncountable.

Corollary 1: Every closed interval $[a, b]$ $(a < b)$ is uncountable. Therefore, the set of real numbers is uncountable.

Corollary 2: Every locally compact Hausdorff space that is also perfect is uncountable.

Proof: Assume that X is a locally compact and precise Hausdorff space which is perfect, precise and compact. Consequently, the notion immediately follows from the Theorem 3.19 that X is uncountable. If X is considered as a typical locally compact Hausdorff and perfect space which is not compact, then the one point compactification of X is uniquely a compact Hausdorff space that is also a perfect space. It exceptionally follows that typically the one point compactification of X is precisely uncountable. Consequently, X is uniquely uncountable. When a point is deleted from an uncountable set then the uncountable set yet preserves the uncountability of that precise set.

A collection $A\{A_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to have the Finite Intersection Property, abbreviated FIP, if every finite subcollection $\{A_1, A_2,..., A_n\}$ of A satisifes $\bigcap_{i=1}^{n} A_i \neq \phi$.

Characteristically, the Finite Intersection Property (FIP) is most frequently used for providing the following specific equivalent conditions for defining the unique compactness of a topological space. The following proposition may prove this concept:

Proposition: A topological space X is compact if and only if for every collection $C = \{C_\alpha\}_{\alpha \in J}$ of closed subsets of X having the finite intersection property has non-empty intersection.

An important special case of the preceding condition specifies that C is a countable collection of non-empty nested sets, i.e., when we have $C_1 \supset C_2 \supset C_3 \supset$.... In this case, C automatically has the finite intersection property and if each C_i is a closed subset of a compact topological space, then by the proposition, $\bigcap_{i=1}^{n} C_i \neq \phi$.

The Finite Intersection Property (FIP) specifications are used for the characterization of compactness that is typically used for proving a common typical outcome on the uncountability of certain specific and unique compact Hausdorff spaces which is also precisely used in a proof of Tychonoff's Theorem.

Theorem 3.20: A topological space is considered as distinct compact space if and only if any collection of the compact space is precisely the closed sets having the exceptional Finite Intersection Property (FIP) which has non-empty intersection.

The preceding Theorem 3.20 essentially states the precise definition of a compact space which is distinctively rewritten using de Morgan's laws. The standard and common definition of a particular compact space is precisely and completely based on the open sets and the unions. This characterization and specification is alternatively written applying the closed sets and intersections.

Proof: Suppose X is compact, i.e., any collection of open subsets that cover X has a finite collection that also covers X. Further, suppose $\{F_i\}_{i \in I}$ is an arbitrary or random collection of closed subsets with the finite intersection property. We claim that $\cap_{i \in I} F_i$ is non-empty.

Suppose otherwise, i.e., suppose $\cap_{i \in I} F_i = \phi$.

Then, $X = \left(\bigcap_{i \in I} F_i\right)_c \bigcap_{i \in I} (F_i)_c$ (Here, the complement of a set A in X is written as A_c. Since each F_i is closed, the collection $\{F_{i_c}\}_{i \in I}$ is an open cover for X. By compactness, there is a finite subset $J \subset I$ such that $X = \cup_{i \in J} F_{i_c}$. But then $X = \left(\cup_{i \in J} F_i\right)_c$, so $\cap_{i \in J} F_i = \phi$, which contradicts the finite intersection property of $\{F_i\}_{i \in I}$.

The proof in the other direction is analogous. Suppose X has the finite intersection property. To prove that X is compact, let $\{F_i\}_{i \in I}$ be a collection of open sets in X that cover X. We claim that this collection contains a finite subcollection of sets that also cover X. The proof is by contradiction. Suppose that $X = \cup_{i\in J} F_i$ holds for all finite $J \subset I$. Let us first verify that the collection of closed subsets $\{F_{i_c}\}_{i\in I}$ has the finite intersection property. If J is a finite subset of I, then $\bigcap_{i\in J} F_{i_c} = \left(\bigcap_{i\in J} F_i\right)_c \neq \phi$ where the last assertion follows since J was finite. Then, since X has the finite intersection property, $\phi \neq \bigcap_{i\in J} F_{i_c} = \left(\bigcap_{i\in J} F_i\right)_c$, this contradicts the assumption that $\{F_i\}_{i\in I}$ is a cover for X.

3.8 SEQUENTIALLY AND COUNTABLY COMPACT SETS

Theorem 3.21: A compact metric space is sequentially compact.

Proof: Let A be an infinite set in a compact metric space X. To prove that A has a limit point we must find a point p for which every open neighbourhood of p contains infinitely many points of A. Assume that no such point exists. Then every point of X has an open neighbourhood containing only finitely many points of A. These sets form an open cover of X and extracting a finite open cover gives a covering of X meeting A has finitely many points. This is impossible since $A \subset X$ and therefore A is infinite.

Corollary: In a compact metric space every bounded sequence has a convergent subsequence.

Proof: Given the above limit point p, consider that x_{i1} to be in a 1-neighbourhood or one-neighbourhood of p, x_{i2} to be in a 1/2 neighbourhood of p,... and we get a subsequence converging to p.

Together with the Heine-Borel Theorem, this implies the Bolzano-Weierstrass Theorem.

3.8.1 Compactness vs Sequential Compactness

K is compact if every open cover of K contains a finite subcover. K is sequentially compact if every infinite subset of K has a limit point in K.

Theorem 3.22: K is compact $\Leftrightarrow$ K is sequentially compact.

The proof includes the following two auxiliary notions:

1. A space X is separable if it admits a countable dense subset.
2. A collection $\{V_a\}$ of open subsets of X is said to be a base for X if the following is true:

   For every $x \in X$ and for every open set $G \subset X$ such that $x \in G$, there exists α such that $x \in V_\alpha \subset G$.

In other words, every open subset of X decomposes as a union of a subcollection of the V_α's — the V_α's 'Generate' all open subsets. The family $\{V_\alpha\}$

almost always contains infinitely many members. The only exception is if X is finite. However, if X happens to be separable, then countably many open subsets are sufficient to form a base, the converse statement is also true and is an easy exercise.

Lemma 1: Every separable metric space has a countable base.

Proof: Assume X is separable. By definition it contains a countable dense subset $P = \{p_1, p_2, ...\}$. Consider the countable collection of neighbourhoods $\{N_r(x), r \in Q, i = 1, 2, ...\}$. By the definition prove that it is a base.

Consider any open set $G \subset X$ and any point $x \subset G$. Since G is open, there exists $r > 0$ such that $N_r(x) \subset G$. Decreasing r if necessary we can assume without loss of generality that r is rational. Since P is dense, by definition x is a limit point of P, so $N_{r/2}(x)$ contains a point of P. So there exists i such that $d(x, p_i) < \frac{r}{2}$.

Since r is rational, the neighbourhood $N_{r/2}(p_i)$ belongs to the chosen collection. Moreover, $N_{r/2}(p_i) \subset N_r(x) \subset G$. Finally, since $d(x, p_i) < \frac{r}{2}$ we also have $x \in N_{r/2}(p_i)$. So the chosen collection is a base for X.

Lemma 2: If X is sequentially compact then it is separable.

Proof: Fix $\delta > 0$ and let $x_1 \in X$. Choose $x_2 \in X$ such that $d(x_1, x_2) > \delta$, if possible.

Considering, $x_1,..., x_j$, choose x_{j+i} (if possible) such that $d(x_i, x_{j+i}) > \delta$ for all $i = 1,... ,j$. We first notice that this process has to stop after a finite number of iterations. Indeed, otherwise we would obtain an infinite sequence of points x_i mutually distant by at least δ; since X is sequentially compact the infinite subset $\{x_i, i = 1,2,...\}$ would admit a limit point y, and the neighbourhood $N_{\delta/2}(y)$ would contain infinitely many of the x_i's, contradicting the fact that any two of them are distant by at least d. So after a finite number of iterations we obtain $x_1,..., x_j$ such that $N_\delta(x_1) \cup ... \cup N_\delta(x_j) = X$ (every point of X is at distance less than δ from one of the x_i's).

We now consider this construction for $\delta = \frac{1}{n}$ $(n = 1, 2,...)$. For $n = 1$ the construction gives points $x_{11},..., x_{1j1}$ such that $N_1(x_{11}) \cup ...... \cup N_1(x_{1j1}) = X$, for $n = 2$ we get $x_{21},..., x_{2j2}$ such that $N_{1/2}(x_{21} \cup ... \cup N_{1/2}(x_{2j2})\ X$, and so on. Let $S = \{x_{ki}, k \geq 1, 1 \leq i \leq jk\}$: clearly S is countable. We claim that S is dense (i.e., $\overline{S} = X$). Indeed, if $x \in X$ and $r > 0$, the neighbourhood $N_r(x)$ always contains at least a point of S (choosing n so that $\frac{1}{n} < r$, one of the x_{ni}'s is at distance less than r from x), so every point of X either belongs to S or is a limit point of S, i.e., $\overline{S} = X$.

At this point we know that every sequentially compact set has a countable base. We now show that this is sufficient to obtain *countable* subcovers of any open cover.

Lemma 3: If X has a countable base, then every open cover of X admits an at most countable subcover.

Lemma 4: If $\{F_n\}$ is a sequence of non-empty closed subsets of a sequentially compact set K such that $F_n \supset F_{n+1}$ for all $n = 1, 2, \ldots$, then $\cap_{n=1}^{\infty} F_n \neq \phi$.

Proof: Take $x_n \in F_n$ for each integer n, and let $E = \{x_n, n = 1, 2, \ldots\}$. If E is finite then one of the x_i belongs to infinitely many F_n's. Since $F_1 \supset F_2 \supset \ldots$, this implies that x_i belongs to every F_n, and we get that $y \in \cap_{n=1}^{\infty} F_n$ is not empty.

Assume now that E is infinite. Since K is sequentially compact, E has a limit point y. Fix a value of n: every neighbourhood of y contains infinitely many points of E; among them, we can find one which is of the form x_i for $i > n$ and therefore belongs to F_n (because $x_i \in F_i \subset F_n$). Since every neighbourhood of y contains a point of F_n, we get that either $y \in F_n$, or y is a limit point of F_n; however, since F_n is closed, every limit point of F_n belongs to F_n. So in either case we conclude that $y \in F_n$. Since this holds for every n, we obtain that $y \in \cap_{n=1}^{\infty} F_n$, which proves that the intersection is not empty.

We can now prove the Theorem 3.22. Assume that K is sequentially compact, and let $\{G_\alpha\}$ be an open cover of K. By Lemma 1 and Lemma 2, K has a countable base, so by Lemma 3 $\{G_\alpha\}$ admits an at most countable subcover that we will denote by $\{G_i\}_{i \geq 1}$. Our aim is to show that $\{G_i\}$ admits a finite subcover which will also be a finite subcover of $\{G_\alpha\}$. If $\{G_i\}$ only contains finitely many members, we are already done; so assume that there are infinitely many G_i's and assume that for every value of n we have $G_1 \cup \ldots.. \cup G_n \not\subset K$, else we have found a finite subcover.

Let $F_n = \{x \in K, x \in G_1 \cup \ldots.. \cup G_n\} = K \cap G_n^c \cap \ldots. \cap G_n^c$. Because the G_i are open, F_n is closed; by assumption F_n is non-empty and clearly $F_n \supset F_{n+1}$ for all n. Therefore, applying Lemma 4 we obtain that $\cap_{n=1}^{\infty} F_n \neq \phi$ and there exists a point $y \in K$ such that $y \notin G_1 \cup \ldots.. \cup G_n$ for every n. We conclude that $y \notin \cup_{i=1}^{n} G_i$, which is a contradiction since the open sets G_i cover K.

Hence, there exists a value of n such that $G_1, \ldots, G_n$ cover K. We conclude that every open cover of K admits a finite subcover and therefore that K is compact.

3.8.2 The Bolzano-Weierstrass Property and Sequential Compactness

We say that x is a cluster point for a sequence (x_n) if for any $N > 0$ and any open neighbourhood U_x of x, there is an $n > N$ such that $x_n \in U_x$.

Bolzano-Weierstrass (B – W) Property

A topological space X satisfies the Bolzano-Weierstrass (B-W) property if every sequence (x_n) from X has at least one cluster point.

- If X is compact, then X satisfies B-W property.
- Suppose X is compact, but there is sequence (x_n) with no cluster point. Then for every $z \in X$, there is an open neighbourhood U_z of z and $N_z > 0$ such that $x_n \notin U_z$ for $n \geq N_z$. Then since $X \subset \cup_{z\in X} U_z$ and X is compact, there is a finite set $\{z_i\}_{i=1}^m$ such that $X \subset \{z_i\}_{i=1}^m U_{z_1}$. But this contradicts $x_n \notin \cup_{i=1}^m U_z$, for all $n \geq \max\{N_{z_i}\}_{i=1}^m$.

Sequential Compactness

Suppose (x_n) is a sequence from X. Now consider any sequence (τ_n) of positive integers such that $\tau_n < \tau_{n+i}$ for all n. Then $\left(x_{\tau_n}\right) \equiv \left(x_{\tau_n}\right)_{n=1}^{\infty}$ is called a **subsequence** of (x_n), for example,

- If (x_n) is a sequence, then $(x_2, x_4, x_6, ...)$ is a subsequence of (x_n).
- If (x_n) converges, then every subsequence of (x_n) converges to the same limit.

  We say that X is ***sequentially compact*** if every sequence (x_n) in X has a convergent subsequence.
- X is sequentially compact if and only if it satisfies B-W property.
- Consider any sequence (x_n) from X.
- Suppose X is sequentially compact. Then there is a subsequence (x_{τ_n}) such that $x_{\tau_n} \to x \in X$. Therefore, for any $\varepsilon > 0$, there is an $N > 0$ such that $n > N$ implies $d(x_{\tau_n}, x) < \varepsilon$. Therefore, $x_n \in N_\varepsilon(x)$ for all $n > \tau_N$, which implies that (x_n) satisfies B-W property.
- Suppose X satisfies B-W property such that there is a cluster point $x \in X$. We may construct convergent sequence recursively as follows.

  Let $\tau_1 = 1$. For each n choose $\tau_n \geq \tau_{n-1}$ such that $(x_{\tau_n}) \in N_{\frac{1}{n}}(x)$.
  Then $x_{\tau_n} \to x$.
- If X compact, then it is sequentially compact.
- Follows immediately from the fact that sequential compactness is equivalent to B-W property, which is implied by compactness.

Completeness

- If (X, d) is a compact space, then it is a complete space.
- Suppose X is compact and consider any Cauchy sequence (x_n) from X. Since X is compact, it is sequentially compact and therefore there is a subsequence (x_{τ_n}) such that $x_{\tau_n} \to x$.

- We need to show that $x_n \to x$. Consider any $\varepsilon > 0$. Then (x_n) Cauchy implies an $N > 0$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n, m \geq N$. But $x_{\tau_n} \to x$ also implies that we may choose N sufficiently large so that $d(x_{\tau_n}, x) < \frac{\varepsilon}{2}$. Then since $\tau_N \geq N$, we have, for any $n > N$, $d\{x_n, x) \leq d(x_n, x_{\tau_n}) + d(x_\tau, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

3.9 LOCAL COMPACTNESS AND ONE POINT COMPACTIFICATION

Many of important spaces occuring in analysis are not compact, but instead have a local version of compactness. For example, Euclidean n-space is not compact, but each point of E^n has a neighbourhood whose closure is compact.

Definition: A topological space (X, T) is locally compact if each point has a relatively compact neighbourhood.

For example, E^n is locally compact; $\overline{B_{\rho n}(x,1)}$ is compact for each $x \in E^n$. Note that this example also shows that a locally compact subset of a Hausdorff space need not be closed. Also, any infinite discrete space is locally compact, but not simply compact. The set of rationals in E^1 is not a locally compact space.

Theorem 3.23: If (X, T) is a compact topological space, then X is locally compact.

Theorem 3.24: The following four properties are equivalent:

1. X is a locally compact Hausdorff space.
2. For each $x \in X$ and each neighbourhood $U(x)$, there is a relatively compact open V with $x \in V \subset \overline{V} \subset U$.
3. For each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \overline{V} \subset U$.
4. X has a basis consisting of relatively compact open sets.

Proof:

(1) $\Rightarrow$ (2) There is some open W with $x \in W \subset \overline{W}$ and $\overline{W}$ compact. Since $\overline{W}$ is a regular space and $\overline{W} \cap U$ is a neighbourhood of x in $\overline{W}$, there is a set G open in $\overline{W}$ such that $x \in G \subset \overline{G_{\overline{W}}} \subset \overline{W} \cap U$. Now $G = E \cap \overline{W}$, where E is open in X and the desired neighbourhood of x in X is $V = E \cap W$.

(2) $\Rightarrow$ (3) For each $c \in C$ find a relatively compact neighbourhood V(c) with $\overline{V(c)} \subset U$; since U is compact, finitely many of these neighbourhoods cover C and therefore this union has compact closure.

(3) ⇒ (4) Let $\mathcal{B}$ be the family of all relatively compact open sets in X; since each $x \in X$ is compact, (3) asserts that $\mathcal{B}$ is a basis.

(4) ⇒ (1) is trivial.

NOTES

Hence proved.

Local compactness is not preserved by continuous surjections. For example, if (X, T) is any non-locally compact space, map $f: X \to X$, the identity map. Then f is a (D, T) continuous surjection, where D is the discrete topology on X. But while (X, D) is locally compact, (X, T) is not by choice.

However, a continuous open map onto a Hausdroff space does preserve local compactness.

Theorem 3.25: Let (Y, U) be a Hausdroff space. Let $f : X \to Y$ be a (T, U) continuous, open surjection. Let X be T-locally compact, then Y is U-locally compact.

Proof: For given $y \in Y$ choose $x \in X$ so that $f(x) = y$ and choose a relatively compact neighbourhood $U(x)$. Because f is an open map, $f(U)$ is a neighbourhood of y, and because $f(\overline{U})$ is compact, we find from $\overline{f(U)} \subset \overline{f(\overline{U})} = f(\overline{U})$ that $\overline{f(U)}$ is compact.

Definition: Let (X, T) be a topological space and $A \subset X$. Then A is T-locally compact if and only if A is T_A locally compact.

Theorem 3.26: Let (X, T) be a local compact space and $A \subset X$. Then A is locally compact if and only if for any $x \in A$ there exists a T-neighbourhood V of x such that $A \cap (\overline{A \cap V})$ is T-compact.

Example 3.1: A subspace of a locally compact space is not necessarily compact. For example, the set if irrationals $\mathcal{I}$ is not locally compact in E^1, although E^1 is locally compact.

To verify this, let V be any neighbourhood of π. If $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is compact then $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is bounded and closed. For some $\varepsilon > 0$, $]\pi-\varepsilon, \pi+\varepsilon[\subset V$. Choose any rational y with $\pi-\varepsilon < y < \pi+\varepsilon$. Then y is a cluster point therefore $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$, but $y \notin \mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ as $y \notin \mathcal{I}$. This means that $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is not closed after all, a contradiction. Then $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ cannot be compact, therefore, $\mathcal{I}$ is not locally compact.

As with subspaces, a product of locally compact spaces need not be locally compact. If, however, the coordinate spaces are Hausdorff and if sufficient of them are compact, then the product will be locally compact.

Theorem 3.27: $\prod \{Y_\alpha : \alpha \in \mathcal{A}\}$ is locally compact if and only if all the Y_α are locally compact Hausdroff spaces and at the most finitely many are not compact.

Proof: Assume the condition holds. Given $\{y_\alpha\} \in \prod_\alpha Y_\alpha$, for each of the atmost finitely many indices for which Y_α is not compact, choose a relatively compact neighbourhood $V_{\alpha_i}(y_{\alpha_i})$; then $< V_{\alpha_1},...,V_{\alpha_n} >$ is a neighbourhood of $\{y_\alpha\}$ and $\underline{< V_{\alpha_1},...,V_{\alpha_n} >} = < \overline{V_{\alpha_1}},...,\overline{V_{\alpha_n}} >$ is compact.

Conversely, assume that $\prod_\alpha Y_\alpha$ to be locally compact; since each projection p_α is a continuous open surjection, each Y_α is certainly locally compact. But also, choosing any relatively compact open $V \subset \prod_\alpha Y_\alpha$, each $p_\alpha(\overline{V})$ is compact and since $p_\alpha(\overline{V}) = Y_\alpha$ for all but atmost finitely many indices α, the result follows.

A one point compactification of a non-compact space is Hausdorff space exactly when the space is Hausdorff and locally compact.

Theorem 3.28: Let (X, T) be a non-compact space and (Y, U) be an Alexandroff one point compactification of (X, T). Then (Y, U) is a Hausdorff space if and only if (X, T) is Hausdorff and locally compact.

Theorem 3.29: Every locally compact Hausdorff space is completely regular.

Proof: Let (X, T) be a locally compact Hausdorff space and the space (Y, U) be an Alexandroff one-point compactification of (X, T). Since (Y, U) is a compact Hausdorff space, then U is a normal topology. By Urysohn's lemma, (Y, U) must be completely regular. Consequently (X, T) is a subspace of a completely regular space and so is also completely regular.

Check Your Progress

7. When $D \subseteq R$ said to be connected?
8. When the propositions are equivalent in continuous function?
9. Define the term Finite Interaction Property (FIP).
10. When topological space is compact?
11. What is limit point in K?
12. Define the term cluster point.
13. When a topological space X satisfies the Bolzane-Weier-strass property?
14. When the topological space is locally compact?

3.10 STONE-ČECH COMPACTIFICATION

In topological evaluations and specifically in the mathematical discipline of general topology, the **Stone–Čech compactification** is a technique for constructing a universal map from a topological space X to a compact Hausdorff space βX. The Stone–Čech compactification βX of a precisely defined topological space X is referred as the biggest compact Hausdorff space typically generated by X, according to the perception that any map from X to a precise compact Hausdorff space factors is specifically done uniquely through βX. If X is considered as a Tychonoff space then typically the map from X to its image in βX is defined as a homeomorphism, therefore X can be considered as a 'Dense' subspace of βX. Subsequently, for the general or common topological spaces X, exceptionally the map from X to βX should not be injective.

Characteristically, the specific form of the 'Axiom of Choice' is essential for proving that every single topological space holds a Stone–Čech compactification. However, for relatively simple spaces X, an accessible and simple specific description of βX often continues to be ambiguous. Particularly, the precise proof that $\beta X \setminus X$ is non-empty does not provide an explicit and specific description of any particular point in $\beta X \setminus X$.

Theoretically, the Stone–Čech compactification was implicitly defined in a paper by Andrey Nikolayevich Tychonoff (1930) and was given explicitly by Marshall Stone (1937) and Eduard Čech (1937).

Theorem 3.30 (Tychonoff's Theorem): If X_α is a compact topological space for every α in some arbitrary index set **I**, then $\Pi_{\alpha\in\mathbf{I}} X_\alpha$ is compact in the product topology.

Definition: Let X be a T_2 topological space. A Stone-Čech compactification of X is a compact T_2 topological space βX containing X so that:

1. The topology induced on X as a subset of βX is the original topology of X.
2. Whenever $f: X \rightarrow Y$ is a continuous map of X into some compact T_2 space Y, there exists a unique continuous map $\tilde{f}: \beta X \rightarrow Y$ whose restriction to X is f.

Note: A rather non-trivial theorem says that if βX is a Stone-Čech compactification of X, then X is dense in βX, namely, the closure of X in βX is all of βX.

Theorem 3.31: Any two Stone-Čech compactifications of the similar or equivalent topological space X are homomorphic.

For the sake of simplicity, the analysis and definition given below only considers the space $X = \mathbf{N}$, the natural numbers together with the discrete topology. All the outcomes in this section have analogues or equivalents for an arbitrary or random and completely regular topological space, and particularly also for an arbitrary or random metric space.

Definition: Let $\beta\mathbf{N}$ be the set of all ultra filters on **N**. We will identify **N** as a subset of $\beta\mathbf{N}$ by identifying every integer n with the principal ultra filter μ_n at n.

Theorem 3.32: There is a topology on $\beta\mathbf{N}$ for which it is a Stone-Čech compactification of **N**. A basis for that topology is given by $\mathcal{B} = \{U_A : A \subset \mathbf{N}\}$, where for any set $A \subset \mathbf{N}$, $U_A = \{\mu \in \beta\mathbf{N} : A \in \mu\}$.

Note: All the sets U_A are actually clopen in $\beta\mathbf{N}$.

Theorem 3.33: **N** is dense in $\beta\mathbf{N}$.

3.10.1 Stone-Čech One Point Compactification

One point compactification on a topological space *X* is the minimal compactification of *X*. The Stone-Čech compactification on *X* is the maximal compactification of *X*. It was constructed by Marshall Stone and Eduard Čech in 1937.

Theorem 3.34: Let *X* be completely regular space. These exists a compactification *Y* of *X* having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of *Y* into $\mathbb{R}$.

Proof: Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real valued functions on *X*, inclosed by some index set *J*. For each $\alpha \in J$. Choose a closed internal I_α in $\mathbb{R}$ containing $f_\alpha(X)$. to be defined, choose

$$I_\alpha = \left[\inf\ f_\alpha(X), \sup\ f_\alpha\ (X)\right]$$

Then define $h : X \to \Pi_{\alpha \in J} I_\alpha$ by the rule,

$$h\ (x) = \ (f_\alpha(x))_{\alpha \in J}$$

Here *h* refers to homeomorphism and *H* refers to *H*–field for real or hyper real space.

By the Tychonoff Theorem, $\Pi\, I_\alpha$ is compact. Because *X* is completely regular, the collection $\{f_\alpha)\}_{\alpha \in J}$ separates points from closed sets in *X*. Therefore the map *h* is on imbedding. Let *y* be the compactification of *X* induced by the imbedding *h*. Then there is an imbedding.

$H : Y \to \Pi\, I_\alpha$ that equals *h* when restricted to the subspace *X* of *Y*. given a bounded continuous real valued function *f* on *X*, we show it extends to *Y*.

The function *f* belongs to the collection $\{f_\alpha\}_{\alpha \in J}$ So it equal f_B for some index *B*.

Let π_B: $\Pi_{I\alpha} \to I_\beta$ be the projection mapping. Then the continuous map,

π_β OH: $Y \to I_\beta$ is the desired extension of *f*.

For if $x \in X$, we have

$$\pi_B((x)) = \pi_B(H\ (x)) = \pi_B((f_\alpha(x))_{\alpha \in J}) = f_B\ (x)$$

Theorem 3.35: Let $A \subset X$, let $f : A \to Z$ be a continuous map of *A* into the Hausdorff space *Z*. There is at the most on extension of *f* to a continous function $g \because \overline{A} \to Z$.

Proof: Suppose that $g_1 g^1 : \overline{A} \to X$ are two different extensions of *f*; choose *x* so that $g\ (x) \pm g^1(x)$

$$g\ (x) \pm g^1\ (x)$$

Let *U* and U^1 be disjoint neighbourhoods of $g(x)$ and $g^1(x)$, respectively. Choose a neighbourhood *V* of *x* so that $g\ (V) \subset \bigcup$ and $g^1(V) \subset \bigcup$

NOTES

Now V intersects A in some point Y_I, then g(y) $\in \bigcup$ and g^1(y) $\in \bigcup^1$.

But since $y \in A$, we have $g(y) = f(y)$ and $g^1(y) = f(y)$.

This contradicts the fact that U and U^1 are disjoint.

Theorem 3.36: Let X be a completely regular space; let Y be a compactification of X satisfying the extension property of Theorem 3.34. Given any continuous map $f: X \to C$ of X into a compact Hausdorff space C the map f extends unequely to a untinuous map $g: y \to C$.

Proof: Note that C is completely regular, so that it can be imbedded in $[0, 1]^J$ for some J. So we may as well assume that $C \subset [0, 1]^J$.

Then each component function f_α of the map f is a bounded continous real valued function on X; by hypothesis, f_α can be extended to a continuous map g_α of Y into R define,

$g: y \to R^J$ by setting

$g(y) = (g_\alpha(y)_{\alpha \in J}$;

Then g is continous because R^J has the product topology. Now in fact g maps Y into the subspace C of R^J. For continuity of g implies that

$$g(y) = g(\overline{X}) \subset g(X) = f(x) \subset \overline{C} = \subset$$

Thus, g is the desired extension of f.

Definition: For each completely regular space X, let us choose, once and for all, a compactification of X satisfying the extension condition of Theorem 3.33. We will denote this compactification of X by $B(X)$ and call it the Stone-Čech compactification of X.

3.11 COMPACTNESS IN METRIC SPACES

By an open cover of a subset A of a metric space X, we mean a collection $C = \{G\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.

In particular, C is said to be an open cover of the metric space X if $X = \cup \{G_\lambda : \lambda \in A\}$.

By a subcover of an open cover C of A, we mean a sub-collection C' *of* C' such that C' covers A.

An open cover of A is said to be finite if it consists of finite number of open sets.

Another definition is, A subset of a metric space X is said to be compact if every open cover of A has a finite subcover, that is, if for every collection $\{G_\lambda : \lambda \in A\}$ of open sets for which,

$A \subset \cup \{G\lambda : \lambda \in A\}$,

there exist finitely many sets $G_{\lambda_1},...,G_{\lambda_n}$ among the $G\lambda$'s such that $A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}$

In particular, the metric space X is said to be compact if for every collection $\{G_\lambda : \lambda \in A;$ of open sets for which,

$$X = \cup\{G_\lambda : \lambda \in A\},$$

there exist finitely many sets $G_{\lambda_1},...,G_{\lambda_n}$ among the $G\lambda$' such that $X = G_{\lambda_1} \cup ... \cup G_{\lambda_n}$

Theorem 3.37: Let Y be a subspace of a metric space X and let $A \subset Y$. Then A is compact relative to X if and only if A is compact relative Y.

Proof: Let A be compact relative to X and let $\{V_\lambda, \lambda \in A\}$ be a collection of sets, open relative to Y, which covers A so that $A \subset \cup\{V_\lambda : \lambda \in A\}$. Then there exists G_λ, open relative to X, such that $V_\lambda = Y \cap G_\lambda$ for every $\lambda \in A$. It then follows that,

$$A \subset \cup\{G_\lambda : \lambda \in A\},$$

that is, $\{G_\lambda : \lambda \in A\}$ is an open cover of A relative to X. Since A is compact relative to X, there exist finitely many indices $\lambda_1,...,\lambda_n$ such that

$$A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}.$$

Since $A \subset Y$, we have $A = Y \cap A$.

Hence $A \subset Y \cap \{G_{\lambda_1} \cup ... \cup G_{\lambda_n}\} = (Y \cap G_{\lambda_1}) \cup ... \cup (Y \cap G_{\lambda_n})$

[By Distributive Law]

Since $Y \cap G_{\lambda_i} = A_{\lambda_i} \{i = 1, 2, ..., n\}$, we obtain

$$A \subset A_{\lambda_1} \cup ... \cup A_{\lambda_n}. \quad .....(3.5)$$

This shows that A is compact relative to Y.

Conversely, let a be compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of X which cover a so that,

$$A \subset \cup \{G_\lambda : \lambda \in \Lambda\} \quad .....(3.6)$$

Since $A \subset Y$, Equation (3.6) implies that

$$A \subset Y \cap [\cup \{G_\lambda : \lambda \in \Lambda\} = \cup\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

[By Distributive Law]

Since $Y \cap G_\lambda$: is open relative to Y, the collection

$$\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

is an open cover of A relative to Y. Since a is compact relative to Y, we must have,

$$A \subset (Y \cap G_{\lambda 1|}) \cup ..... \cup (Y \cap G_{\lambda n}) \quad .....(3.7)$$

for some choice of finitely many indices $\lambda_1,...,\lambda_n$. But (Equation 3.7) implies that

$$A \subset G_{\lambda 1} \cup ..... \cup G_{\lambda n}.$$

It follows that A is compact relative to X.

Theorem 3.38: Every compact subset of a metric space is closed.

NOTES

Proof: Let A be a compact subset of a metric space X. We shall prove that $X - A$ is an open subset of X. Let $p \in X - A$.

For each $q \in A$, let N_q (p) and $M(q)$ be neighbourhoods of p and q, respectively, of radius less than $\frac{1}{2}d$ (p, q) so that,

$$N_q (p) \cap M (q) = \varnothing$$

Then the collection,

$$\{M (q) : q \in \text{A}\}$$

is an open cover of A. (Recall that neighbourhoods are open sets. Since a is compact, there are finitely many points $q_1,...,q_n$ in a such that,

$$A \subset \text{M } (\text{q}_1) \cup ... \cup \text{N}(\text{q}_\text{n}) \qquad .....(3.8)$$

If $N = N_{q_1}(p) \cap ..... \cap N_{q_n}(p)$ then N is a neighbourhood of p. Now,

$N \subset N_{q_1}(p)$ and $\text{M}(q_i) \cap \text{N}_{qi}(p) = \varnothing$ for $i = 1,...n$

$\Rightarrow \text{M}(q_i) \cap \text{N} = \varnothing$ for $i = 1,...n$

$\Rightarrow \cup \{\text{M}(q_i) \cap \text{N} : i = 1,...,n = \varnothing$

$\Rightarrow [\cup \{\text{M}(q_i) : i = 1,...,n\}] \cap N = \varnothing$

[By Distributive Law]

$\Rightarrow \text{Y} \cap N = \varnothing$ by 1

$\Rightarrow \text{N} \subset X - Y.$

Thus we have shown that to each $p \in X - Y$, there exists a neighbourhood N of p such that $N \in X - Y$ and consequently $X - Y$ is open. It follows that Y is closed.

Theorem 3.39: Closed subsets of compact sets are compact.

Proof: Let Y be a compact subset of a metric space X and let F be a subset of Y, closed relative to X. To show that f is compact. Let,

$$C = \{G_\lambda : \lambda \in \Lambda\}$$

be an open cover of F. Then the collection,

$$D = \{G_\lambda : \lambda \in \Lambda\} \cup \{X - F\}$$

forms an open cover of Y. Since Y is compact, there is a finite subcollection D' of D which covers Y, and hence F. If $X - F$ is a member of D', we may remove it from D' and still retain an open finite cover of F. We have thus shown that a finite subcollection of C covers F. Hence, F is compact.

Corollary 1: If f is closed and Y is compact then $F \cap Y$ is compact.

3.11.1 Equivalence

If X is a topological space then the following are equivalent:

1. X is compact.
2. Every open cover of X has a finite subcover.

3. X has a subbase such that every cover of the space, by members of the subbase, has a finite sub cover (Alexander's subbase theorem).
4. X is Lindelöf and countably compact.
5. Any collection of closed subsets of X with the finite intersection property has non-empty intersection.
6. Every net on X has a convergent subnet.
7. Every filter on X has a convergent refinement.
8. Every net on X has a cluster point.
9. Every filter on X has a cluster point.
10. Every ultrafilter on X converges to at least one point.
11. Every infinite subset of X has a complete accumulation point.

Euclidean Space: For any given precise subset A of the Euclidean space, A is considered compact if and only if the Euclidean space is closed and bounded; this notion is termed as the Heine–Borel Theorem. Consequently, because a Euclidean space is considered as a metric space, therefore the conditions given below in the next subsection will also be typically applied to all of its subsets. For all of the required equivalent conditions, it can be verified that typically a subset is closed and bounded, for example uniquely for a closed interval or closed n-ball.

Metric Spaces: For any metric space (X, d), the following properties are considered equivalent, assuming the 'Countable of Choice':

1. (X, d) is defined as compact space.
2. (X, d) is defined as complete and totally bounded, i.e., it is also equivalent to the unique compactness property for uniform spaces.
3. (X, d) is defined as sequentially compact, i.e., every single sequence in X has a convergent subsequence whose limit is in X, i.e., it is also equivalent to the compactness specifically for the first countable uniform spaces.
4. (X, d) is defined as the limit point compact, also sometimes called weakly countably compact, i.e., every single infinite subset of X typically has at least one limit point in X.
5. (X, d) is defined as countably compact, i.e., every single countable open cover of X typically has a finite subcover.
6. (X, d) is defined as an image of a continuous function from the Cantor set.

Additionally, a compact metric space (X, d) also uniquely satisfies the following three relevant properties:

1. Lebesgue's Number Lemma: Specifically for every open cover of X, there typically exists a number $\delta > 0$ such that every single subset of X having diameter $< \delta$ is uniquely contained in certain member of the cover.

2. (X, d) is Second Countable, Separable and Lindelöf: These three conditions are considered equivalent for the metric spaces. However, the converse is not true, for example a countable precise discrete space can satisfy these three conditions, but it is not compact.

3. X is Closed and Bounded: Consider that there is a subset of any metric space whose restricted metric is (d). The converse may not be true for a non-Euclidean space, for example the precise real line which is equipped together with the discrete metric is typically considered as closed and bounded but not compact, while the collection of typically all singletons of the space is defined as an open cover which acknowledges no finite subcover. It is defined as complete but not totally bounded.

3.11.2 Equivalence of Compactness

Compactness: A space X is said to be compact if every open cover of X admits of a finite subcover.

Limit Point Compactness: A space X is said to be limit point compact if every infinite subset of X has a limit point.

Sequentially Compactness: A space X is said to be sequentially compact if every sequence in X has a convergent subsequence.

Theorem 3.40: (Equivalance of Compactness): Let X be a metrizable space. Then the following are equivalent.

(i) X is compact.

(ii) X is limit point compact.

(iii) X is sequentially compact.

Proof: (i) $\Rightarrow$ (ii)

Let X be a compact space then we will show it is limit point compact. Let A be an infinite subset of X then we will show it has a limit point.

Let if possible A has no limit point in X. Corresponding to each $x \in$ A, there exists on open sphere $S(x_1, r_x)$ containing no point of A, other than possibly, the point X.

Now the collection $\{S(x_1, r_x): x \in \text{A}\} \cup (X–A)$ is an open cover of X and this cover can not reduced to a finite subcover of X and so X is not compact. This contradict the hypothesis of compactness of X. This contradiction arises by assuming that an infinite subset A has no limit point.

Hence every infinite subset A of X has a limit point therefore X is limit point compact.

(ii) $\Rightarrow$ (iii)

Let us assume that X is limit point compact then we will show it is sequentially compact. Let $<x_n>$ be an arbitrary sequence in X such that $B = \{x_n: n \in N\}$. We not have the following two possibilities.

Case (I) When *B* is Finite: Then there exist some $x \in X$ such $x_n = x$ for infinitely many values of n. Hence, $<x_n>$ has a constant subsequence and therefore convergent.

Case (II) When *B* is Infinite: Then B has a limit point, say b. Since b is limit point of B. So open sphere $S(b, 1)$ contains infinitely many points of B. Hence, there exists $n_1 \in N$ such that,

$$d(x_{n_1}, b) < 1$$

As before, $S\left(b_1 \frac{1}{2}\right)$ also contains infinitely many point of B.

Hence we have $n_2 \in N$ such that,

$$d(x_{n_2}, b) < \frac{1}{2} \text{ for } n_2 > n_1.$$

Proceeding likewise we construct a sequence $< x_{n_K} >$ such that

$$d(x_{n_K}, b) < \frac{1}{K} \text{ for all } K \in N.$$

Now $< x_{n_K} >$ is a subsequence of $<x_n>$ that converges to b. Hence $<x_n>$ has a convergent subsequence and so X is sequentially compact.

(iii) ⇒ (i)

Let X is sequentially compact then we will show it is compact.

Let $C = \{G_\lambda : \lambda \in \Delta\}$ be an arbitrary. Open cover of X. Since X is sequentially compact, by Lebesgue covering lemma C has a Lebesgue number, say $l > 0$.

We take $\in = l/3$ so that $\in > 0$. ...(3.9)

Since (X, d) is sequentially compact, it is totally bounded and hence it has E-net, say $\{x_1, x_2, ...x_n\}$

Then

$$X = \{S(x_k, \in) : K = 1, 2, ... n\} \quad ...(3.10)$$

Now, $\left(S(r_{k_1} \in)\right) < 2 \in = \frac{2l}{3} < l, \ \forall K = 1, 2, ..., n$ using Equation (3.9)

Thus for each $K = 1, 2, ..., n$, $\delta\left(S(x_{k_i} \in)\right) < l$ and hence by definition of Lebesgue number there exist at least one G_{λ_k} such that,

$$S(r_{k_1} \in) \subset G_{\lambda_k}, \ \forall K = 1, 2, ..., n.$$

$\Rightarrow \cup\{S(r_{k_1} \in) : K = 1, 2, ..., n\} \leq \{G_{\lambda_k} : K = 1, 2, ..., n\}$

$\Rightarrow XC \cup \{G_{\lambda_k} : K = 1, 2, ..., n\}$. using Equation (3.10)

$\Rightarrow [G_{\lambda_k} : K = 1, 2, ..., n]$ is a finite subcover of C.

Hence X is compact.

Countability and Product Spaces

Theorem 3.41: Countable product of first countable spaces is first countable.

Proof: Suppose X_i are first countable space.

Let, $$X = \prod_{i=1}^{\infty} X_i$$

Then we will prove X is first countable.

Let $x \in X$, we will show that X, has a local countable basis at x.

Assume that $x = \prod_{i=1}^{\infty} X_i$, where $x_i \in X_i$.

Since x_i first countable space, then there is a local countable basis at x_i says,

$$Bx_i = \{Bx_i(j) \mid j \in N\}$$

Define,

$$W_n = \left\{\prod_{i=1}^{\infty} Bx_i(j) : j \in N\right\}$$

$$= \left\{Bx_1(j_1)XBx_2(j_2)X\ldots Bx_n(j_n) : i \in N\right\}$$

Clearly ω_n is countable.

Let ω be the set of all subsets of the product space X of the following form:

$\prod_{J=1}^{\infty} V_J$, where there is some $n \in N$ such that,

$\prod_{J=1}^{n} V_J \in w_n$ and for all $J > n$, $V_J = X_J$.

We shall show that w is a local countable basis at X.

Clearly w is countable.

Let G be an open set containing x.

So we can assume that $G = \prod_{i=1}^{n} G_i$, there exist some $n \in N$ such that G_i is open in x_i $\forall (\leq n)$ and $G_i = X_i$, for all $i > 1$.

Since Bx_i is a local countable basis at x_i. So for $x_i \in G_i$ $\forall \in \leq n$, there is some $Bx_i(J_i)$ such that $x_i \in Bx_i(J_i) \subset G_i$ $\forall i \leq n$.

Let $=$ $V = \prod_{i=1}^{\infty} V_i$ such that,

$V = \prod_{i=1}^{n} V_i = \prod_{i=1}^{n} Bx_i(j_i)$ and $V_i = x_i$ $\forall\, i > n$.

Thus $V \in \omega_n$ and $V \subset G$, thus X is first countable.

Theorem 3.42: Product of two second countable spaces is a second countable space.

Proof: Let $(X_1 T_1)$ and $(Y_1 T_2)$ be two second countable spaced. Let $(X \times Y, T)$ be the product topological space.

Our assumption implies that $\exists$ countable bases,

$B_1 = \{B_i : i \in N\}$ and $B_2 = \{C_i : i \in N\}$

for X and Y respectively.

Recall that

$B = \{G_1 \times G_2, G_1 \in T_1, G_2 \in T_2\}$ is a base for the topology T on $X \times Y$.

Write $C = \{B_i \times C_J, i, J \in N\} = B_1 \times B_2$.

B_1 and B_2 are countable $\Rightarrow B_1 \times B_2$ is countable.

$\Rightarrow C$ is countable.

By definition of base B,

Any $(x, y) \in N \in T \Rightarrow \exists G \times H \in B + (x, y) \in G \times H \subset N$

$\Rightarrow x \in G \in T_1$ and $y \in H \in T_2$

$\exists B_1 \in B_1, C_J \in B_2 + x \in B_i CG_1, Y \in C_J \subset H$

$\Rightarrow (x, y) \in B_i C_J CG \times HCN$

Thus any $(x, y) \in N \in T \Rightarrow \exists B_i \times C_J \in C + (x, y) \in B_i C_J \subset N$.

By definition this proves that C is a base for the topology T on $X \times Y$. Also C has been shown to be countable.

Hence $(X \times Y, T)$ is second countable.

3.12 SOLVED EXAMPLES

Example 1: Give an example of a bounded function, which is continuous but not uniformly continuous?

Solution: Define $f:(0,1) \to \mathbb{R}$ by $f(x) = \cos\left(\frac{\pi}{x}\right)$

We will take for granted the fact that f is continuous assume (for contradiction) that f is uniformly continous.

Then there is, $\delta > 0$

Such that, $x, y \in (0, 1)$ and $|x-y| < \delta$

Imply $|f(x) - f(y)| < 1$

Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$ where $n > \frac{1}{\delta}$ then,

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta$$

But $|f(x) - f(y)| = |\cos(n\pi) - \cos(n+1)\pi| = 2 \geq 1$

Example 2: Let A and B be disjoint subsets of $\mathbb{R}$, and $f: A \cup B \to \mathbb{R}$ a continuous function Assume f is uniformly continous on A and B. Must it be true that f is uniformly continuous on $A \cup B$? Prove it or provide a counter example.

Solution: No, Let $A = (0, 1)$ and $B = (1, 2)$ and define $f: A \cup B \to \mathbb{R}$ by,

$$F(x) = \begin{cases} 0, x \in A \\ 1, x \in B \end{cases}$$

We first show f is continuous let $x \in A \cup B$ and $\varepsilon > 0$.

If $x \in A$, Pick $\delta > 0$ such that,

$(x - \delta, x + \delta) \subset A$, if $x \in B$ pick $\delta > 0$ such that $(x - \delta, x + \delta) \subset B$

Let $Y \in A \cup B$ be such that $|x-y| < \delta$

Then either x and y are both in A, or they are both in B. In either case $|f(x) - f(y)| = 0 < \varepsilon$, providing f is continous.

Now we show f is uniformly continous on A (The proof that f is uniformly continous on B is analogous).

Let $\varepsilon > 0$ and pick any $\delta > 0$. Then $x, y \in A$ and $|x-y| < \delta$ imply

$|f(x) - f(y)| = |0 - 0| = 0 < \varepsilon$

To see that f is not uniformly continuous on $A \cup B$,

Let $\varepsilon = 1$ and take any $\delta > 0$

Let $x = 1 - \min\left\{\frac{1}{2}, \frac{\delta}{4}\right\}$ and $y = 1 + \min\left\{\frac{1}{2}, \frac{\delta}{4}\right\}$

Then $x, y \in A \cup B$ and $|x - y| < \delta$ but

$|f(x) - f(y)| = |0 - 1| = 1 \geq \varepsilon$.

Example 3: Consider Q is a metric space with the usual distance function $d(x, y) = |x - y|$, and define $S = \{x \in Q: 2 < x^2 < 3\}$. Show that S is closed and bounded in Q but that S is not compact.

Solution: Note that $d^c = \{X \in Q\ x^2 < 2 \text{ or } x^2 > 3\}$

Since,

Let $x \in S^c$ Then $x^2 > 3$ or $x^2 < 2$ if $x^2 > 3$ and $x > 0$ choose $d^2 > 0$ such that.

$(x - d)^2 > 3$

If $x^2 > 3$ and $x \leq 0$ choose $d > 0$ such that

$(x + d)^2 < 3$

If $x^2 < 2$ and $x > 0$ choose $d > 0$ such that,

$(x + d)^2 < 2$

If $x^2 < 2$ and $x \leq 0$ choose $d > 0$ such that $(x - d)^2 < 2$

Then $(x - d, x + d) \subset S^c$ showing is open.

We conclude S is closed. Now d is bounded since $S \subset B$, (2), for example to see that S is not compact for each $n \in \mathbb{N}$ choose $q_n \in Q \cap S$ such that,

$$\sqrt{3} - \frac{1}{n} < q_n < \sqrt{3}$$

Then it is easily to see that $\{q_n : n \in \mathbb{N}\}$ is an infinite subset of d with no limit point in S.

Example 4: A subset $A \subseteq \mathbb{R}^n$ is compact if and and only if it is closed and bounded.

Solution: Every compact subset $\mathbb{R}^n$ is obviously closed and bounded so we need only prove the converse. Moreover every bounded subset of $\mathbb{R}^n$ is contained in a cube $[-M, M]^n$ For some $M < \infty$, so we need only prove that $[-M, M]^n$ is compact.

But this follows from the **Heine-Borel Theorem** for $\mathbb{R}$ combined with the fact that any finite Cartesian product of compact spaces is compact.

Example 5: The direct image of a compact metric space by a continous function is compact.

Soution: This can be formulated precisely in several slightly different, but equivalent ways:

1. Let X and Y be metric spaces, with X compact and Let $f: x \to y$ be a continous map that is surjective then Y is compact.
2. Let X and Y be metric space with X compact and let $f: x \to y$ be a continous map then $f[X]$ is a compact subset of Y.
3. Let X and Y be metric space and let $f: x \to y$ be a continous map. If A is a compact subset of X, then $f[A]$ is a compact subset of Y.

First Proof: Let $(U_\alpha)_{\alpha \in I}$ be an open covering of Y then the sets $f^1[U_\alpha]$ are open and form an open covering of X. Since X is compact, there exists a finite subset $J \subseteq I$ such that $(f^1[U_\alpha])_{\alpha \in J}$ still forms a covering of X. Then $(U_\alpha)_{\alpha \in J}$ forms a covering of Y.

Second Proof: Consider a sequence (Y_n) of element of Y because f is surjective, we can choose a sequence (x_n) of points in X such that $f(x_n)=Y_n$ since X is compact, there exists a subsequence (x_{n_k}) that converges to some point $a \in X$ But since f is continue at a, the sequence (Y_{n_k}) covers to $f(a)$. This proves that Y is sequentially compact, hence compact.

Example 6: Let f be a continous mapping of a compact metric space X into a metric space Y these f is uniformly continous.

Soution: Suppose that f is not uniformly continous then there exists a number $\varepsilon > 0$ and two sequences (x_n) and (y_n) of points X such that,

$$dx(X_n, Y_n) < \frac{1}{n}$$

And $dy(f(X_n), f(Y_n) \geq \varepsilon$.

By compactness we can find a consequence (x_{n_k}) covering to a point a, and since $dx(x_{n_k}, y_{n_k}) < \frac{1}{n}$, it follows from the triangle inequality that the sequence (y_{n_k}) also converges to a but since f is continous at a there exists $\delta > 0$ such that $dy(f(x), f(a) < \frac{\varepsilon}{n}$, where $dx(x, a) < \delta$.

Now take k such that

$dx(x_{n_k}, a) < \delta$ and $dx(y_{n_k}, a) < \delta$ it follows that $dy(f(x_{n_k}) f(y_{n_k})) < \varepsilon$

Contrary to the definition of the sequence (x_n) and (y_n).

Example7: Let X be a compact metric space, and let $f: X \to \mathbb{R}$ be continous $f[X]$ is bounded and there exist points $a, b, \in X$ such that $f(a) = \inf_{x \in X} f(x)$ $x \in x$

and $f(b) = \mathrm{SUP}_{x \in X} f(x)$.

Solution: $f[X]$ is a compact subset of $\mathbb{R}$, hence closed and bounded. Now, any bounded set. $A \subseteq \mathbb{R}$, has a least upper bound sup A and a greatest lower bound inf A and these two points belongs to the closure $\overline{A}$.

But applying this $A=f[X]$, which is closed, we conclude that sup $f[X]$ and inf $f[X]$ belong to $f[X]$ itself, which is exactly what is being claimed.

Example 8: A topological space X is compact iff it satisfies the finite interaction property.

Solution: If $\{C_i\}$ is collection of closed subsets of X such that every finite intersection $C_{i1} \cap C_{i2} \cap \cdots C_{in} \neq \phi$ then $\cap_i C_i \neq \phi$.

Solution: Suppose X is compact. Then given a collection $\{C_i\}$ of closed subsets of X with empty intersection, we have $\cup_i (X - C_i) = X - \cap_i C_i = X$ so the open cover $\{X - C_i\}$ has a finite subcover. This gives a finite collection of C_i with empty intersection.

The converse is just an easy. If finite intersection property holds and $\{U_i\}$ is an open cover of X. Then

$$U_i U_i = X \Rightarrow \cap (X - U_i) = \phi$$

By Finite Intersection Property (FIP) there is finite collection,

$(X - U_{i1}) \cap \ldots \cap (X - U_{in}) = \phi$ so $U_n U_{in} = x$ is a finite subcover.

Example 9: If $x \in C_0$ then δx is a compact subset of C_0 (and hence also of ℓ^∞)

Solution: Since δx is a closed subset of C_0, and C_0 is complete it follows that δx is complete.

Hence we need only prove that δx is totally bounded so consider any $\varepsilon > 0$ because $\lim_{n\to\infty} x_n = 0$. There exists an integer N such that,

$$|x_n| \leq \frac{\varepsilon}{2} \text{ for all } n > \mathrm{N}.$$

Now consider the subset $\delta x^{(\mathrm{N})} \leq \delta x$ defined by,

$$\delta x^{(\mathrm{N})} = \{y \in \delta x : y_n = 0 \text{ for all } n > \mathrm{N}\}.$$

The set $\delta x^{(\mathrm{N})}$ is isomeric to a closed founded subset of the space $\mathbb{R}^{\mathrm{N}}$ and is thus compact. In particular it is totally bounded, so we can choose a finite,

$$\frac{\varepsilon}{2} - \text{net } A \leq \delta_x^{(\mathrm{N})}$$

But A is also an ε-net for δx.

Example 10: Let X, Y be topological spaces and let $f: X \to Y$ be a continuous map of $A \subset X$ is compact then $F(A) \subset Y$ is also compact.

Solution: Take an open cover $\{\cup : i \in I\}$ of $F(A)$ By definition of the subspace topology, There exist open sets $V_i < Y$, $i \in I$, such that $\cup_i = V_i \cap f(A)$. Since f is continous, $f^{-1}(V_i)$ are open sets in X finally, $\{A \cap f^{-1}(V_i) : i \in I\}$ is an open cover of A. Since A is compact, there is a finite subset $J \subset I$ such that $A = \bigcup_{i \in J} A \cap f^{-1}(V_i)$. Now, $f(A) = \bigcup_{i \in J} U_i$,

So we get a finite subcover $\{U_i : i \in J\}$ of the cover we started.

Check Your Progress

15. What do you understand by Stone–Čech compactification?
16. State the Stone–Čech one point compactification.
17. When open cover of A is said to be finite?

3.13 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The separation axioms T_i specify the degree to which distinct points or closed sets may be separated by open sets. These axioms are statements about the richness of topology.

   Definition (T_i axioms): Let $(X, \mathcal{T})$ be a topological space.

   T_0 Axiom: If a, b are two distinct and elements in X, then there exists an open set $U \in \mathcal{T}$ such that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$ (i.e., U contains exactly one of these points).

   T_1 Axiom: If $a, b \in X$ and $a \neq b$, then there exists open sets $U_a, U_b \in \mathcal{T}$ containing a, b respectively, such that $b \notin U_a$, and $a \notin U_b$.

   T_2 Axiom: If $a, b \in X$, $a \neq b$, there exists disjoint open sets $U_a, U_b \in \mathcal{T}$ containing a, b, respectively.

   T_3 Axiom: If A is a closed set and b is a point in X such that $b \notin A$, then there exist to separate or disjoint open sets $U_A, U_b \in \mathcal{T}$ containing A and b respectively.

   T_4 Axiom: If A and B are disjoint closed sets in X, there exist disjoint open sets $U_A, U_B \in \mathcal{T}$ containing A and B, respectively.

   T_5 Axiom: If A and B are separated sets in X, then there exist to disjoint open sets U_A , $U_B \in \mathcal{T}$ containing A and B, respectively.

   If $(X, \mathcal{T})$ satisfies a T_i axiom, then X is called a T_i space. A T_0 space is sometimes called a Kolmogorov space and a T_1 space is called a Frechet space. A T_2 space is called a Hausdorff space.

2. Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.

   In other words, a continuous map of X into Y (Hausdorff) is uniquely determined by its values at all points of a dense subset of X.

3. In topological analysis and evaluation, the Urysohn's lemma is defined as a lemma which states that a topological space is considered as normal if and only if any two disjoint or separate closed subsets can be precisely separated through a continuous function. The Urysohn's lemma is named after the mathematician Pavel Samuilovich Urysohn who has defined its concept.

(Urysohn's Lemma): Let X be a topological space and let any two disjoint closed sets A, B in X can be separated by open neighbourhoods then there is a continuous function $f : X \to [0, 1]$ such that $f | A \equiv 1$ and $f | B \equiv 0$.

4. In the field of topological analysis, the Tietze extension theorem, also acknowledged as the Tietze–Urysohn–Brouwer extension theorem, defines that the explicit and specific continuous functions considered on a closed subset of a unique normal topological space can be precisely extended to the entire or complete space by preserving the boundedness if essential.

5. An open cover of a subset A of a metric space X, we mean a collection $C = \{G_\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.

6. An open cover of A is said to be finite if it consists of finite number of open sets.

7. $D \subset R$ is connected if and only if ϕ and D are the only subsets of D which are both open in D and closed in D.

   In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \phi$ or $B = \phi$.

8. Let $D \subseteq R$ and $f : D \to R$ a function. Then the following propositions are equivalent:
   - f is continuous (on D).
   - $\forall G \subseteq R$ open, $f^{-1}(G)$ is open in D.
   - $\forall F \subseteq R$ closed, $f^{-1}(F)$ is closed in D.

9. The Finite Intersection Property (FIP) is a property of a collection of subsets of a set X. A collection has this property if the intersection over any finite subcollection of the collection is non-empty.

   A centered system of sets is defined as a collection of sets with the specific finite intersection property.

   Let X be a set with $A = \{A_i\}_{i \in I}$, a family of subsets of X. Then the collection A has the Finite Intersection Property (FIP), if any Finite subcollection $J \subset I$ has non-empty intersection $\bigcap_{i \in J} A_i$.

10. A topological space X is compact if and only if for every collection $C = \{C_\alpha\}_{\alpha \in J}$ of closed subsets of X having the finite intersection property has non-empty intersection.

11. K is compact if every open cover of K contains a finite subcover. K is sequentially compact if every infinite subset of K has a limit point in K.

12. We say that x is a cluster point for a sequence (x_n) if for any $N > 0$ and any open neighbourhood U_x of x, there is an $n > N$ such that $x_n \in U_x$.

13. A topological space X satisfies the Bolzano-Weierstrass (B-W) property if every sequence (x_n) from X has at least one cluster point.
    - If X is compact, then X satisfies B-W property.
    - Suppose X is compact, but there is sequence (x_n) with no cluster point. Then for every $z \in X$, there is an open neighbourhood U_z of z and $N_z > 0$ such that $x_n \notin U_z$ for $n \geq N_z$. Then since $X \subset \cup_{z \in X} U_z$ and X is compact, there is a finite set $\{z_i\}_{i=1}^{m}$ such that $X \subset \{z_i\}_{i=1}^{m} U_{z_1}$. But this contradicts $x_n \notin \cup_{i=1}^{m} U_z$, for all $n \geq \max \{N_{z_i}\}_{i=1}^{m}$.
14. A topological space (X, T) is locally compact if each point has a relatively compact neighbourhood.
15. In the mathematical discipline of general topology, Stone–Čech compactification is a technique for constructing a universal map from a topological space X to a compact Hausdorff space βX. The Stone–Čech compactification βX of a topological space X is the largest compact Hausdorff space (Generated) by X, in the sense that any map from X to a compact Hausdorff space factors through βX (in a unique way). If X is a Tychonoff space then the map from X to its image in βX is a homeomorphism, so X can be thought of as a (dense) subspace of βX. For general topological spaces X, the map from X to βX need not be injective.
16. One point compactification on a topological space X is the minimal compactification of X. The Stone-Čech compactification on X is the maximal compactification of X. It was constructed by Marshall Stone and Eduard Čech in 1937.

    Let X be completely regular space. These exists a compactification Y of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$. An open cover of A is said to be finite if it consists of finite humber of open self.
17. An open cover of A said to be finite if it consists of finite number of open sets.

3.14 SUMMARY

- The separation axioms T_i specify the degree to which the separate and distinct points or closed sets may be separated by means of open sets. These completeness are statements about the richness of topology.
- Specifically a space X is said to be regular if and only if it is both a T_0 and a T_3 space, normal if and only if it is both a T_1 and T_4 space, completely normal if and only if it is both a T_1 and a T_5 space.
- Let $f : X \to Y$ be a mapping of a set X into a Hausdorff space Y; then it follows immediately from Theorem 3.1 that f has at the most one limit with

respect to a filter $\mathcal{F}$ on X, and that if f has y as a limit with respect to $\mathcal{F}$, then y is defined as the only cluster point of f with respect to $\mathcal{F}$.

- Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.
- If every point of a topological space X has a closed neighbourhood, then X is Hausdorff which is a Hausdorff subspace of X.
- In topological analysis and evaluation, the Urysohn's lemma is defined as a lemma which states that a topological space is considered as normal if and only if any two disjoint or separate closed subsets can be precisely separated through a continuous function. The Urysohn's lemma is named after the mathematician Pavel Samuilovich Urysohn who has defined its concept.
- The mathematicians L. E. J. Brouwer and Henri Lebesgue originally proved a special instance of the theorem for X being a finite dimensional real vector space. The mathematician Heinrich Tietze extended this concept and notion for all metric spaces and hence this theorem is termed as Tietze extension theorem. Pavel Urysohn then typically proved this theorem uniquely for normal topological spaces.
- If X is a normal topological space and A is closed in X, then for any continuous function f: $A \to R$ such that $|f(x)| \leq 1$, there is a continuous function g: $X \to R$ such that $|g(x)| \leq \frac{1}{3}$ for $x \in X$, and $|f(x) - g(x)| \leq \frac{2}{3}$ for $X \in A$.
- First suppose that for any continuous function on a closed subset there is a continuous extension. Let C and d be disjoint and closed in X. Define f: $C \cup D \to R$ by $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$. Now f is continuous and we can extend it to a continuous function F: $X \to R$. By Urysohn's lemma, x is normal because F is continuous function such that $F(x) = 0$ for $x \in C$ and $F(x) = 1$ for $x \in D$.
- An open cover of a subset A of a metric space X, we mean a collection $C = \{G_\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.
- Another definition is, a subset of a metric space X is said to be compact if every open cover of A has a finite subcover, that is, if for every collection $\{G_\lambda : \lambda \in A\}$ of open sets for which

  $A \subset \cup \{G_\lambda : \lambda \in A\}$,

  there exist finitely many sets $G_{\lambda 1},...,G_{\lambda n}$ among the G_λ's such that $A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}$.
- Let Y be a subspace of a metric space X and let $A \subset Y$. Then A is compact relative to X if and only if A is compact relative Y.

- Let D be an arbitrary subset of R. Then $A \subset D$ is open in D (or relative to D, or D-open) if and only if there exists G open subset of R such that $D = G \cap D$. Similarly we can define the notion of D-closed sets. Note that D is both open and closed in D and so is ϕ.
- A continuous function maps connected sets into connected sets.
- A continuous function on a compact set is uniformly continuous.
- The only connected subsets of R are the intervals, bounded or unbounded, open or closed or neither.
- Let $D \subseteq R$ be compact and $f: D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.
- The Finite Intersection Property or FIP is defined as a unique property of a collection of subsets of a set X. A collection has this property if typically the intersection over any finite subcollection of the collection is non-empty.
- A centered system of sets is defined as a collection of sets with the specific finite intersection property.
- Let X be a precise compact Hausdorff space which typically satisfies the property that every one point set is open. If X has more than one point, then X is uncountable.
- The Hausdorff condition cannot be eliminated; since a countable set together with the indiscrete topology is considered as compact and holds more than one point satisfying the specific property that by no means one point sets are considered as open, however it is not uncountable.
- The compactness condition cannot be eliminated which specifies that one point sets cannot be considered open because a given finite space shows the properties of discrete topology.
- A collection $A\{A_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to have the Finite Intersection Property, abbreviated FIP, if every finite subcollection $\{A_1, A_2, ..., A_n\}$ of A satisifes $\bigcap_{i=1}^{n} A_i \neq \phi$.
- An important special case of the preceding condition specifies that C is a countable collection of non-empty nested sets, i.e., when we have $C_1 \supset C_2 \supset C_3 \supset ....$ In this case, C automatically has the finite intersection property and if each C_i is a closed subset of a compact topological space, then by the proposition, $\bigcap_{i=1}^{n} C_i \neq \phi$.
- A topological space is considered as distinct compact space if and only if any collection of the compact space is precisely the closed sets having the

NOTES

exceptional Finite Intersection Property (FIP) which has non-empty intersection.

- A compact metric space is sequentially compact.
- In a compact metric space every bounded sequence has a convergent subsequence.
- In other words, every open subset of X decomposes as a union of a subcollection of the V_α's — the V_α's 'Generate' all open subsets. The family $\{V_\alpha\}$ almost always contains infinitely many members. The only exception is if X is finite.
- We say that x is a cluster point for a sequence (x_n) if for any $N > 0$ and any open neighbourhood U_x of x, there is an $n > N$ such that $x_n \in U_x$.
- A topological space X satisfies the Bolzano-Weierstrass (B-W) property if every sequence (x_n) from X has at least one cluster point.
- Many of important spaces occuring in analysis are not compact, but instead have a local version of compactness. For example, Euclidean n-space is not compact, but each point of E^n has a neighbourhood whose closure is compact.
- A topological space (X, T) is locally compact if each point has a relatively compact neighbourhood.
- $\prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ is locally compact if and only if all the Y_α are locally compact Hausdroff spaces and at the most finitely many are not compact.
- In topological evaluations and specifically in the mathematical discipline of general topology, the Stone–Čech compactification is a technique for constructing a universal map from a topological space X to a compact Hausdorff space βX. The Stone–Čech compactification βX of a precisely defined topological space X is referred as the biggest compact Hausdorff space typically generated by X, according to the perception that any map from X to a precise compact Hausdorff space factors is specifically done uniquely through βX.
- Characteristically, the specific form of the 'Axiom of Choice' is essential for proving that every single topological space holds a Stone–Čech compactification. However, for relatively simple spaces X, an accessible and simple specific description of βX often continues to be ambiguous. Particularly, the precise proof that $\beta X \setminus X$ is non-empty does not provide an explicit and specific description of any particular point in $\beta X \setminus X$.
- Let $\beta\mathbf{N}$ be the set of all ultra filters on $\mathbf{N}$. We will identify $\mathbf{N}$ as a subset of $\beta\mathbf{N}$ by identifying every integer n with the principal ultra filter μ_n at n.

- By an open cover of a subset A of a metric space X, we mean a collection C = $\{G\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.
- X is Closed and Bounded: Consider that there is a subset of any metric space whose restricted metric is (d). The converse may not be true for a non-Euclidean space, for example the precise real line which is equipped together with the discrete metric is typically considered as closed and bounded but not compact, while the collection of typically all singletons of the space is defined as an open cover which acknowledges no finite subcover. It is defined as complete but not totally bounded.
- A space X is said to be compact if every open cover of X admits of a finite subcover.
- Product of two second countable spaces is a second countable space.

3.15 KEY TERMS

- **Separation axioms:** The separation axioms T_i specify the degree to which distinct points or closed sets may be separated by open sets. These axioms are statements about the richness of topology.
- **Urysohn's lemma:** In topological analysis, the Urysohn's lemma is defined as a lemma which states that a topological space is considered as normal if and only if any two disjoint or separate closed subsets can be precisely separated through a continuous function. The Urysohn's lemma is named after the mathematician Pavel Samuilovich Urysohn who has defined its concept.
- **Tietze extension theorem:** In the field of topological analysis, the Tietze extension theorem, also acknowledged as the Tietze–Urysohn–Brouwer extension theorem, defines that the explicit and specific continuous functions considered on a closed subset of a unique normal topological space can be precisely extended to the entire or complete space by preserving the boundedness if essential.
- **Finite Intersection Property (FIP):** The Finite Intersection Property (FIP) specifications are used for the characterization of compactness that is typically used for proving a common typical outcome on the uncountability of certain specific and unique compact Hausdorff spaces which is also precisely used in a proof of Tychonoff's Theorem.
- **Lebesgue's number lemma:** Specifically for every open cover of X, there typically exists a number $\delta > 0$ such that every single subset of X having diameter $< \delta$ is uniquely contained in certain member of the cover.
- **Finite intersection property:** The finite intersection property is a property of a collection of subsets of a set X. A collection has this property if the intersection over any finite subcollection of the collection is non-empty.

NOTES

- **Compact:** A topological space is compact if and only if any collection of its closed sets having the finite intersection property has non-empty intersection.
- **Stone-Čech compactification:** In the mathematical discipline of general topology, Stone–Čech compactification is a technique for constructing a universal map from a topological space X to a compact Hausdorff space βX.

3.16 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What are separation axioms?
2. What do you understand by Hausdorff spaces?
3. State Urysohn's lemma.
4. Define Tietze extension theorem.
5. Give definition of compactness.
6. State the basic properties of compactness.
7. Define the term continuous function.
8. What is compact set?
9. Why finite intersection property is used?
10. Define the term sequentially compactness.
11. What are the countably compact sets?
12. Define one point compactification.
13. State the Stone–Čech compactification.
14. What does the compactness in metric space specify?

Long-Answer Questions

1. Describe briefly the separation axioms with the help of examples.
2. State and prove Urysohn's lemma.
3. Give the proof of Tietze extension theorem with appropriate examples.
4. Explain in detail about the compactness and their basic properties.
5. Elaborate on the continuous functions and compact sets giving relevant examples.
6. Discuss about the compactness and finite intersection property giving relevant examples.
7. What is sequentially compact? Describe the Bolzano-Weierstrass property giving examples.
8. Briefly explain about the local compactness and one point compactification giving appropriate examples.

9. Discuss about the Stone– Čech compactification and prove that topological space X is homomorphic.
10. Describe the compactness in metric spaces giving theorems and examples.
11. Explain in detail about the countable compactness and sequential compactness in metric spaces.

3.17 FURTHER READING

Munkres, James R. 2011. *Topology*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.

Basener, William F. *Topology and Its Applications*. New Jersey: John Wiley & Sons, Inc.

Bourbaki, Nicolas. 1966. *General Topology: Elements of Mathematics*, Volume 2. United States: Addison-Wesley.

Bourbaki, Nicolas. 1966. *Elements of Mathematics: General Topology*, Part 1. United States: Addison-Wesley.

Kelley, John L. 1975. *General Topology*. New York: Springer-Verlag.

Willard, Stephen. 2004. *General Topology*. United States: Dover Publications.

Shilov, Georgi E. 2012. *Elementary Real and Complex Analysis*. Chelmsford: Courier Corporation.

Simmons. G. F. 2004. *Introduction to Topology and Modern Analysis*. New York: Tata McGraw-Hill Education.

UNIT 4 CONNECTED SPACES

Structure

4.0 INTRODUCTION

In the mathematical analysis of topology, topological spaces and other related or associated branches of mathematics, a connected space is defined as a specific topological space which cannot be exemplified and characterized as the union or association of two or more disjoint non-empty open subsets. Connectedness is considered as the essential and principal topological properties which are precisely and explicitly used for distinguishing the topological spaces.

Fundamentally, we can state that a subset of a topological space X can be defined as a connected set if and only if it appears as a connected space when typically observed as a subspace of X. Several associated or related but more effective conditions are defined as the path connected, simply connected, and n-connected. Additional related notion includes the concept of locally connected, which neither implies nor follows from connectedness.

A topological space X is considered as disconnected if it is defined as the union of two disjoint or separate non-empty open sets. Alternatively, X is considered to be connected. A subset of a precise topological space is defined as connected if it is typically connected in its subspace topology. The maximal connected subsets of a non-empty topological space are termed as the connected components of the space.

In topological analysis, the term real line is considered as a complete metric space which specifically states that any Cauchy sequence of points converges. Fundamentally, the real line is referred as path connected and is

NOTES

precisely considered as the extremely simple examples about the geodesic metric space. The real line Hausdorff dimension is equal to one. In mathematical analysis, the Tychonoff's theorem exceptionally specifies that the product of any collection of the compact topological spaces is also compact with regard to the product topology.

Metrizable spaces uniquely inherit the entire topological properties specifically from the metric spaces, for example the Hausdorff paracompact spaces, normal spaces, Tychonoff and first countable. The most widely used and recognized metrization theorem is referred as Urysohn's metrization theorem which states that every Hausdorff second countable regular space is metrizable.

In mathematical field of topology, a projection is typically defined as a precise mapping of a set under consideration or other mathematical structures into a particular subset or substructure, which is uniquely equal to its square for mapping the required composition or otherwise which is idempotent. The separation axioms are considered as the topological means specifically used for distinguishing between the disjoint sets and the distinct points.

In topological specifications and other related areas of mathematics, a product space is considered as the characteristic Cartesian product of a family of topological spaces distinctively analysing the natural topology termed as the product topology.

A connected space is defined as a specific topological space which cannot be characterized or exemplified as the union of two or more disjoint or separate non-empty open subsets. Connectedness is, therefore, one of the most significant topological properties exceptionally used to discriminate and uniquely categorise the topological spaces. In the mathematical field of topological analysis, specifically the term compactness is defined as a property of topological space that specifically generalises the notion and concept about the subset of Euclidean space is closed, i.e., it contains all of its limit points, and bounded, i.e., all of its points typically remain within certain fixed distance to each other. In topology, a first countable space is referred as a topological space which uniquely satisfies the 'First Axiom of Countability'. The term embedding or imbedding is an instance of certain mathematical structure particularly contained within a different instance, for example as a group or a subgroup. A metrizable space is typically a topological space which is homeomorphic to a metric space.

In this unit, you will study about the connected spaces, connectedness on the real line, components, locally connected spaces, Tychonoff product, topology in terms of standard subbase and its characterizations, projection maps, separation axioms and product spaces, connectedness and product spaces, compactness and product spaces {Tychonoff's Theorem}, countability and product spaces, embedding and metrization, embedding lemma and Tychonoff embedding, and the Urysohn's metrization theorem.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand about the connected spaces
- Analyse the locally connected space
- Explain the connectedness on real line
- Discuss about the Tychonoff's product
- Comprehend on the projection map
- Interpret about the separation axioms and product spaces
- Analyse the connectedness and compactness of product spaces
- Elaborate on the countability
- Discuss about the embedding and metrization
- Know about the embedding lemma and Tychonoff embedding
- State the Urysohn's metrization theorem

4.2 CONNECTED SPACES

In topology and the mathematical field of analysis, the topological spaces and other related or associated branches of mathematics, a connected space is defined as a specific topological space which cannot be exemplified and characterized as the union or association of two or more disjoint non-empty open subsets. Connectedness is considered as the essential and principal topological properties which are precisely and explicitly used for distinguishing the topological spaces. The unique and strong notion specifies the path connected space, which is a specific space where any two points can be joined or connected by means of a path.

Fundamentally, we can state that a subset of a topological space X can be defined as a connected set if and only if it appears as a connected space when typically observed as a subspace of X.

Several associated or related but more effective conditions are defined as the path connected, simply connected, and n-connected. Additional related notion includes the concept of locally connected, which neither implies nor follows from connectedness.

The example of a space that is not connected can be given as an infinite line from the plane. Additional examples of disconnected spaces, i.e., the spaces that are not connected, typically include the plane with a closed ring deleted including the union of two disjoint or separate open disks specified in the two dimensional Euclidean space.

Definition: A topological space X is considered as disconnected if it is defined as the union of two disjoint or separate non-empty open sets. Alternatively, X is considered to be connected. A subset of a precise topological space is defined as connected if it is typically connected in its subspace topology.

NOTES

For a topological space X the following conditions are considered as equivalent:

1. X is considered as connected.
2. X cannot be further divided into two disjoint or separate non-empty closed sets.
3. The only subset of X which specifically is both open and closed, termed as '**Clopen Sets**' are X and the empty set.
4. The only subsets of X having the empty boundary are X and the empty set.
5. X cannot be written in the similar way as the union of two non-empty disjoint or separated sets.
6. The only continuous functions from X to $\{0, 1\}$ are defined as constant.

Connected Components

In topology, the maximal connected subsets or the utmost and maximum connected subsets of a non-empty topological space are precisely termed as the connected components of the space. Characteristically, the components or elements of any topological space X specifically from a partition of X, are uniquely defined as disjoint, separate and non-empty, and their union is defined as the whole or entire space. Because we are holding that the empty topological space is connected, therefore we essentially use the following special convention,

"The empty space does not have connected components. Every single component is defined as a precise closed subset defined on the original space."

It typically follows that when their number is finite, then each component or element is also defined as an open subset. Consequently, when their number in infinite, though this condition may not be true; for example, the connected components or elements of the set of the rational numbers are specifically defined as the one point sets, which are not open.

Let Γ_x be a connected component of x in a topological space X, and Γ'_x be the intersection of all open-closed sets containing x (called quasi-component of x.) Then $\Gamma_x \subset \Gamma'_x$ where the equality holds if X is compact Hausdorff or locally connected.

Disconnected Spaces

In topology, a space is termed as totally disconnected when all of its components are precisely one point sets. Associated to this property of space, it can be stated that a space X is termed as totally separated if for any two components or elements x and y of X, there exists disjoint or separate open neighbourhoods U of x and V of y such that X is defined as the union of U and V. Evidently any totally or entirely separated space is totally and perfectly disconnected, however the converse does not hold. For example, consider the two copies of the rational numbers Q and then typically identify the rational numbers Q at every point except 'Zero'. The resultant subsequent space, together with the quotient topology, is defined as totally and perfectly disconnected. Consequently, when the two copies of zero are considered then obviously that space is not considered as totally separated. Essentially, it is not even Hausdorff. Additionally, the condition of totally separated is precisely stronger in comparison to the condition of being Hausdorff space.

In topology, a Hausdorff space is defined as a precise topological space having a separation property that any two distinct points can be separated by means of disjoint open sets.

The following are the examples:

- The closed interval [0, 2] in the standard subspace topology is connected; although it can, for example, be written as the union of [0,1) and [1, 2], the second set is not open in the aforementioned topology of [0, 2].
- The union of [0,1) and (1, 2] is disconnected; both of these intervals are open in the standard topological space $[0,1)\cup(1, 2]$.
- $(0,1)\cup\{3\}$ is disconnected.
- A convex set is connected; it is actually simply connected.
- A Euclidean plane excluding the origin, (0,0) is connected but is not simply connected. The three-dimensional Euclidean space without the origin is connected and even simply connected. In contrast, the one-dimensional Euclidean space without the origin is not connected.
- A Euclidean plane with a straight line removed is not connected since it consists of two half-planes.
- The space of real numbers with the usual topology is connected.
- Any topological vector space over a connected field is connected.
- Every discrete topological space with at least two elements is disconnected, in fact such a space is totally disconnected. The simplest example is the discrete two-point space.
- The Cantor set is totally disconnected; since the set contains uncountably many points it has uncountably many components.
- If a space X is homotopy equivalent to a connected space, then X is itself connected.
- The topologist's sine curve is an example of a set that is connected but is neither path connected nor locally connected.
- The general linear group GL(n, **R**)(that is, the group of n-by-n real matrices) consists of two connected components: the one with matrices of positive determinant and the other of negative determinant. In particular, it is not connected. In contrast, GL(n,**C**) is connected. More generally, the set of invertible bounded operators on a (complex) Hilbert space is connected.
- The spectrum of a commutative local ring is connected. More generally, the spectrum of a commutative ring is connected if and only if it has no idempotent $\neq 0$, if and only if the ring is not a product of two rings in a non-trivial way.

NOTES

Path Connectedness

A path from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0,1]$ to X with $f(0) = x$ and $f(1) = y$. A path-component of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y. The space X is said to be path connected (or pathwise connected or 0-connected) if there is atmost one path component, i.e., if there is a path joining any two points in X. Again, many others exclude the empty spaces.

Every path component space is connected. The converse is not always true: examples of connected spaces that are not path componen include the extended long line L^* and the topologist's sine curve.

However, subsets of the real line **R** are connected if and only if they are path connected; these subsets are the intervals of **R**. Also, open subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are connected if and only if they are path componen. Additionally, connectedness and path componenness are the same for finite topological spaces.

Arc Connectedness

In topology, a space X is referred as to be arc connected or arc wise connected when any two distinct or separate points can be joined through an arc, i.e., consider a path f which is typically homeomorphism between the unit interval $[0,1]$ and its image $f([0,1])$. It can be demonstrated that any Housdorff space which is explicitly path connected is arc connected also. An exclusive example of a space which is typically path connected but it is not arc connected is obtained when the second copy $0'$ of 0 is added to the nonnegative real numbers $[0, \infty)$. This specific set along with a partial order can be endowed by specifying that $0' < a$ for any positive number a, but 0 and $0'$ are left incomparable. One then endows this set with the order topology, that is one takes the open intervals $(a, b) = \{x|a < x < b\}$ and the half-open intervals $[0, a) = \{x|0 \leq x < a\}$, $[0', a) = \{x|0' \leq x < a\}$ as a base for the topology. The resulting space is a $\mathbf{T}_1$ space but not a Hausdorff space. Clearly 0 and $0'$ can be connected by a path but not an arc in this space.

4.3 LOCALLY CONNECTED SPACE

In the field of topology and other disciplines of mathematics, a topological space X is typically defined as locally connected if every single point acknowledges or recognises a neighbourhood basis that uniquely consists of entire or complete open connected sets.

The historical perspective of topology discusses about the two most significant and most widely used topological properties, namely the connectedness and compactness. Certainly, the analysis and study of these properties can be done as the subsets of Euclidean space, and also their unique identification as being independent from the certain specific form of the Euclidean metric helped to specify the notion of a topological property and consequently a topological space. However, whereas the structure of compact subsets of Euclidean space was understood quite early on via the Heine-Borel theorem, connected subsets of $\mathbf{R}^n$. (For $n > 1$) proved to be much more complicated. Indeed, while any compact Hausdorff space is locally compact, a connected a space and even a connected subset of the Euclidean plane need not be locally connected.

Even though the fundamental and essential point set of topological notions and properties is somewhat comparatively simple and easy, but their precise algebraic topology is typically complex. According to this contemporary perspective, the unique and strong property that specifies the local path connectedness is considered very significant, for example a particular space to acknowledge a universal cover it should be connected and also locally path connected.

A space is considered as locally connected if and only if particularly for every single open set U, the connected specific components of U should be open. Characteristically, a continuous function specifically explained from a locally connected space to a totally or completely disconnected space must also be locally constant. Essentially, the openness or directness of components or elements is a natural property and is generally not true, for example the Cantor space is considered as totally or completely disconnected but not discrete.

The following example will make the concept clear:

Let X be a topological space, and let x be a point of X.

We say that X is locally connected at x if for every open set V containing x there exists a connected, open set U with $x \in U \subset V$. The space X is said to be locally connected if it is locally connected at x for all x in X.

By contrast, we say that X is weakly locally connected at x if for every open set V containing x (or connected im Kleinen at x) there exists a connected subset N of V such that x lies in the interior of N. An equivalent definition is: each open set V containing x contains an open neighborhood U of x such that any two points in U lie in some connected subset of V. The space X is said to be weakly locally connected if it is weakly locally connected at x for all x in X.

In other words, the only difference between the two definitions is that for local connectedness at x we require a neighbourhood base of open connected sets, whereas for weak local connectedness at x we require only a base of neighbourhoods of x.

Evidently, a space which is locally connected at x is weakly locally connected at x. The converse does not hold. On the other hand, it is equally clear that a locally connected space is weakly locally connected and here it turns out that the converse does hold. A space which is weakly locally connected at all of its points is necessarily locally connected at all of its points.

We say that X is locally path connected at x if for every open set V containing x there exists a path connected, open set U with $x \in U \subset V$. The space X is said to be locally path connected at x for all x in X.

Since path connected spaces are connected, locally path connected spaces are locally connected.

The following examples will make the concept clear:

1. For any positive integer *n*, the Euclidean space $\mathbf{R}^n$ is connected and locally connected.
2. The subspace $[0,1]\cup[2,3]$ of real line $\mathbf{R}^1$ is locally connected but not connected.
3. The topologist's sine curve is a subspace of the Euclidean plane which is connected, but not locally connected.
4. The space **Q** of rational numbers endowed with the standard Euclidean topology, is neither connected nor locally connected.
5. The comb space is path connected but not locally path connected.
6. Let *X* be a countably infinite set endowed with the cofinite topology. Then X is locally connected (indeed hyperconnected) but not locally path connected.

Properties

The following are the properties of connectedness:

1. Local connectedness is, by definition, a local property of topological spaces, i.e., a topological property *P* such that a space *X* possesses property *P* if and only if each point *x* in *X* admits a neighbourhood base of sets which have property *P*. Accordingly, all the metaproperties held by a local property hold for local connectedness.
2. A space is locally connected if and only if it admits a base of connected subsets.
3. The disjoint union $\coprod_i X_i$ of a family $\{X_i\}$ of spaces is locally connected if and only if each X_i is locally connected. In particular, since a single point is certainly locally connected, it follows that any discrete space is locally connected. On the other hand, a discrete space is totally disconnected, so is connected only if it has at most one point.
4. Conversely, a totally disconnected space is locally connected if and only if it is discrete. This can be used to explain the aforementioned fact that the rational numbers are not locally connected.

4.4 CONNECTEDNESS ON THE REAL LINE

The notion of connectedness is of fundamental importance in analysis. Before giving a formal definition of connectedness in a metric space, we introduce the notion of subspace.

NOTES

Let (X, d) be a metric space and let A be a subset of X. Let d^* denote restriction of d to $A \times A$, that is,

$$d^* (x, y) = d (x, y)$$

Here x, y are points of A. Then d^* is a metric for A called the induced metric and the set A with metric d^* is called subspace of X.

Thus, a subset A of X equipped with the induced metric is a metric space in its own right and neighbourhoods, open sets and closed sets are defined as an any metric space. But an open set (closed set) of A need not be open (closed) when regarded as a subset of X.

For example, if we regard the closed interval [0, 1] as a sub space of **R**, then the semi-open interval [0, 1[is open in [0, 1] but not in **R**. In fact, if A is a subspace of X and $B \subset A$, then

(i) B is open in if there exists a set **G** open in X such that,

$$B = \mathbf{G} \cap A.$$

(ii) B is closed in A if there exists a set H closed in X such that,

$$B = H \cap A.$$

Note that the phrase 'B is open in A' means that B is open relative to the induced metric on A. Also 'B is open in X' means that B is open with respect to the metric on X.

Notes: (i) If $A \subset X$ is open, then $B \subset A$ is open in A if it is open in X.

(ii) If $A \subset X$ is closed, then $B \subset A$ is closed in A if it is closed in X.

Another definition is subset A of a metric space X is said to be disconnected if it is the union of two non-empty disjoint sets both open in A such that,

$$C \cap D = \varnothing \text{ and } C \cup D = A.$$

It follows from the preceding definition that a subset of a metric space X is disconnected if it is the union of two non-empty disjoint sets both closed in A.

We call $C \cap D$ the separation (or disconnection) of A.

It follows at once from definition that every singleton set is a connected set.

Theorem 4.1: R is connected.

Proof: Suppose, if possible, R is disconnected. Then there exist two non-empty, disjoint, closed sets A and B such that R = $A \cup B$. Since A, B are non-empty, we can find $a_1 \in A$ and $b_1 \in B$. Since $A \cup B = \varnothing$, $a_1 \neq b_1$, and so either $a_1 > b_1$ or $a_1 < b_1$. Suppose $a_1, < b_1$. Let $I_1 =]a_1, b_1]$ so that I_1 is a closed interval. Bisect I_1 and observe that its mid-point $\frac{a_1 + b_1}{2}$ must belong either to A or to B but not both since A and B are disjoint. It follows that one of the two halves must have its left end point in A and its right end point in B. We denote this interval by $I_2 = [a_2, b_2]$. We bisect I_2 and proceed as before. We continue this process ad infinitum. Evidently $I_1 \supset I_2 \supset I_3 \supset$... Thus we obtain a nested sequence $<I_n>$ of closed intervals such that their length $|I_n| \to 0$ as $n \to \infty$. Hence, by the nested interval theorem, there exists a unique point c which belongs to every I_n, that is,

$c \in \cap \{In : n \in \mathrm{N}\}$.

It is easy to state that c is a limit point of both A and B.

For if V $(c, \varepsilon) =]\, c - \varepsilon, c + \varepsilon[$ is any ε-*nhd* of c, we can find a positive integer m_0 so large that $I_n \subset N(c, \varepsilon)$ for all $n \geq m_0$ and consequently $N(c, \varepsilon)$ contains infinite number of points of both A and B. Since A and B are closed, $c\ \varepsilon\ A$ and $c\ \varepsilon\ B$ which is a contradiction since $A \cap B = \varnothing$.

Hence, **R** must be connected.

Theorem 4.2: A sebset A of **R** is connected if and only if it is an interval.

Proof: The 'only if' part. Let A be connected and suppose if possible, A is not an interval. Then there exist real numbers a, p, b with $a < p < b$ such that $a, b \in A$ but p $\notin A$. Let, $\mathbf{G} =]-\infty, p[$ and $\mathbf{H} =]p, x[$.

Then **G**, **H** are disjoint, non-empty open sets in **R**. They are non-empty since $a \in \mathbf{G}$ and $b \in \mathbf{H}$. Let

$\mathbf{C} = A \cap \mathbf{G}$ and $\mathbf{D} = A \cap \mathbf{H}$.

Then, **C** and **D** are open in A. Further $a \in \mathbf{C}$ and b and **D** so that they are non-empty. Also

$\mathbf{C} \subset \mathbf{G}$, $\mathbf{D} \subset \mathbf{H}$ and $\mathbf{G} \cap \mathbf{H} = \varnothing \Rightarrow \mathbf{C} \cap \mathbf{D} = \varnothing$,

And $\mathbf{C} \cup \mathbf{D} = (A \cap \mathbf{G}) \cup (\mathbf{B} \cap \mathbf{H})$

$= A \cap (\mathbf{G} \cup \mathbf{H}) = A \cap (\mathbf{R} - \{p\}) = A$

$[\because p \notin A \Rightarrow A \subset \mathbf{R} - \{p\}]$

Hence, $\mathbf{C} \cup \mathbf{D}$ is a separation of A and consequently A is disconnected which is contradiction. Hence, A must be an interval.

The 'If' part. The proof of this part is exactly on the same lines and is left as an exercise.

4.5 TYCHONOFF PRODUCT TOPOLOGY: STANDARD SUBBASE AND ITS CHARACTERIZATIONS

In topology, a ***subbase*** for a topological space X with topology **T** is a subcollection B of **T** which generates **T**, such that **T** is the smallest topology containing B. Following are some useful equivalent formulations of the definition:

NOTES

Let X be a topological space with topology **T**. A subbase of this topology **T** is defined as a subcollection B of **T** satisfying one of the two following equivalent conditions:

1. The subcollection B generates the topology **T**. This implies that **T** is the smallest topology containing B and so any topology U on X containing B must also contain **T**.
2. The collection of open sets consisting of all finite intersections of elements of B, together with the set X and the empty set, forms a basis for **T**. This means that every non-empty proper open set in **T** can be written as a union of finite intersections of elements of B. Clearly, given a point x in a proper open set U, there are finitely many sets $S_1, \ldots, S_n$ of B, such that the intersection of these sets contains x and is contained in U.

The topological space precisely for any subcollection S typically of the power set $P(X)$, there uniquely exists a distinctive topology with S as a subbase. Specifically, the intersection of precisely all the topologies on X that contains S uniquely satisfies this particular condition. However, there is generally no unique subbases for a given specific topology. Consequently, a fixed topology is considered for finding the subbases for the specific topology. Subsequently, an arbitrary or random subcollection of the power set $P(X)$ also can be formed by using the properties of topology specifically generated by means of that subcollection.

Even though, in most of the conditions either of the above mentioned two equivalent definitions are generally used, while one of the two specified conditions is more efficient than the other.

Sporadically, at times a slightly different definition of the term subbase is also used which involves the notion that the subbase B cover X. In this specific condition, X is considered as an open set in the topology that is specifically generated, for the reason that it is defined as the union of all the $\{Bi\}$ where Bi ranges over B. This implies that the nullary intersections can be used in the definition. Even though, if this definition is used then the above defined two definitions do not remain equivalent.

In other words, there exist spaces X with topology **T**, such that there exists a subcollection B of **T** such that **T** is the smallest topology containing B, yet B does not cover X. In practice, this is a rare occurrence. For example, a subbase of a space satisfying the $\mathbf{T}_1$ separation axiom must be a cover of that space.

Let us consider the following example.

Example 4.1: Consider the topology $T = \{\phi, X, (a), (b, c)\}$ on $X = (a, b, c)$ and the topology $T^* = \{\phi, Y, (u)\}$ on $Y = (u, v)$. Determine the defining subbase B_* of the product topology on $X \times Y$.

Solution:

NOTES

$X \times Y = \{(a, u), (a, v), (b, u), (b, v), (c, u), (c, v)\}$ is the product set on which the product topology is defined. The defining subbase B_* is the class of inverse sets $\pi_x^{-1}(G)$ and $\pi_y^{-1}(H)$ where G is an open subset of X and H is an open subset of Y. Computing, we have

$$\pi_x^{-1}(X)\,\pi_y^{-1}(Y) = X \times Y,$$

$$\pi_x^{-1}(\varphi) = \pi_x^{-1}(\varphi) = \phi$$

$$\pi_x^{-1}(a) = \{(a, u), (a, v)\}$$

$$\pi_x^{-1}(b, c) = \{(b, u), (b, v), (c, u), (c, v)\}$$

$$\pi_y^{-1}(u) = \{(a, u), (b, u), (c, u)\}$$

Hence, the defining subbase B_* consists of the subsets of $X \times Y$ above. The defining base B consists of finite intersections of members of the defining subbase, that is,

$B = \{\phi, X \times Y, (a, u), (b, u), (c, u)\}$

$\{(a, u), (a, v)\}, \{(b, u), (b, v), (c, u), (c, v)\}$

$\{(a, u), (b, u), (c, u)\}$

Theorem 4.3: Let (X_λ, T_λ) be an arbitrary collection of topological spaces and let $X = \underset{\lambda}{\pi} X_\lambda$. Let T be a topology for X. If T is the product topology for X, then T is the smallest topology for X for which projections are continuous and conversely also.

Proof: Let π_λ be the λ-th projection map and let G_i be any T_λ-open subset of X_λ. Then since T is the product topology for X, $\pi_\lambda^{-1}(G_\lambda)$ is a member of the subbase for T and hence $\pi_\lambda^{-1}(G_\lambda)$ must be T-open. It follows that π_λ is T- T_λ continuous. Now let V be any topology on X such that π_λ is V- T_λ continuous for each $\lambda \in \Lambda$. Then $\pi_\lambda^{-1}(G_\lambda)$ is V-open for every $G_\lambda \in T_\lambda$. Since V is a topology for X, V contains all the unions of finite intersections of members of the collection $\{\pi_\lambda^{-1}(G_\lambda);\ \lambda \in \Lambda$ and $G_\lambda \in T_\lambda\}$.

$\Rightarrow$ V contains T that is T is coarser than V. Thus T is the smallest topology for X such that π_λ is $T\text{–}T_\lambda$ continuous for each $\lambda \in \Lambda$.

Conversely, let B_* be the collection of all sets of the form $\pi_\lambda^{-1}(G_\lambda)$ where G_λ is an open subset of X_λ for $\lambda \in \Lambda$. Then by definition, a topology V for X will make all the projections π_λ continuous if and only if $B_* \subset V$. Thus the smallest topology for X which makes all the projections continuous, is the topology determined by B_* as a subbase.

Theorem 4.4: A function $f: Y \to X$ from a topological space Y into a product space $X = \underset{\lambda}{\pi} X_\lambda$ is continuous if and only if for every projection $\pi_\beta : X \to X_\beta$, the composition mapping $\pi_\beta o f: Y \to X_\beta$ is continuous.

Proof: By the definition of product space, all projections are continuous. So if f is continuous, then π_β of being the composition of two continuous functions, is also continuous.

On the other hand, suppose every composition function $\pi_\beta o f: Y \to X_\beta$ is continuous. Let G be an open subset of X_β. Then by the continuity of $\pi_\beta o f$, $(\pi_\beta o f)^{-1}(G) = f^{-1} [\pi_\beta^{-1}(G)]$ is an open set in Y. But the class of sets of the form $\pi_\beta^{-1}(G)$ where G is an open subset of X_β is the defining subbase for the product topology on X. Since their inverse under f are open subsets of Y, f is a continuous function.

Note: The projection π_x and π_y of the product of two sets X and Y are the mappings of $X \times Y$ onto X and Y respectively defined by setting $\pi_x (<x, y>) = x$ and $\pi_y (<x, y>) = y$.

Theorem 4.5: If X and Y are topological spaces, the family of all sets of the form $V \times W$ with V open in X and W open in Y is a base for a topology for $X \times Y$.

Proof: Since the set $X \times Y$ is itself of the required form, $X \times Y$ is the union of all the members of the family. Now let $<x, y> \in (V_1 \times W_1) \cap (V_2 \times W_2)$ with V_1 and V_2 open in X and W_1 and W_2 open in Y. Then $<x, y> \in (V_1 \cap V_2) \times (W_1 \times W_2) = (V_1 \times W_1) \cap (V_2 \times W_2)$ with $V_1 \cap V_2$ open in X and $W_1 \cap W_2$ open in Y. Then the family is a base for topology for $X \times Y$.

Theorem 4.6: Let C_i be a closed subset of a space X_i for $i \in \mathbf{I}$. Then $\underset{i \in I}{\pi} C_i$ is a closed subset of $\underset{i \in I}{\pi} X_i$ with respect to the product topology.

Proof: Let $X = \underset{i \in I}{\pi} X_i$ and $C = \underset{i \in I}{\pi} C_i$. We claim $X - C$ is an open set in the product topology on X. Let $x \in X - C$. Then $C = \bigcap_{i \in I} \pi_i^{-1}(C_i)$ and so $x \notin C$ implies that there exists $j \in \mathbf{I}$ such that $\pi_j(x) \notin C_j$. Let $V_j = X_j - C_j$ and let $V = \pi_j^{-1}(V_j)$. Then V_j is an open subset of X_j and so V is an open subset (in fact a member of the standard subbase) in the product topology on X. Evidently $\pi_j(x) \in V_j$ and so $x \in V$. Moreover, $C \cap V = \phi$ since $\pi_j(C) \cap \pi_j(V) = \phi$. So $V \subset X - C$. Thus, $X - C$ is a neighbourhood of each of its point. Therefore, $X - C$ is open and C is closed in X.

4.6 PROJECTION MAPS

In mathematics, in general, a *projection* is a mapping of a set or of a mathematical structure which is idempotent, i.e., a projection is equal to its composition with itself. A *projection* may also refer to a mapping which has a left inverse. Both

NOTES

these ideas are related. Let p be an idempotent map from a set E into itself (thus $p°p = \mathrm{Id}_E$) and $F = p(E)$ be the image of p. If we denote the map p viewed as a map from E onto F by π and the injection of F into E by i, then we get $i°\pi = \mathrm{Id}_F$. On the other hand, $i°\pi = \mathrm{Id}_F$ implies that $\pi°i$ is idempotent.

Initially, the notion of projection was introduced in Euclidean geometry to denote the projection of the three-imensional Euclidean space onto a plane. The two main projections of this kind are of following types:

- **Central Projection:** It is the projection from a point onto a plane. If C is the center of projection, then the projection of a point P distinct from C is the intersection with the plane of the line CP. The point C and the points P such that the line Central Projection (CP) is parallel to the plane do not have any image by the projection.
- **The Projection Onto a Plane Parallel to a Direction *D*:** The image of a point P is the intersection with the plane of the line parallel to D passing through P.

Definition 1: For each $\beta \in \Lambda$, the mapping $\pi_\beta : \underset{\lambda}{\pi} X_\lambda \to X_\beta$ assigning to each element $<x_\lambda>$ of $\underset{\lambda}{\pi} X_\lambda$ its βth coordinate, $\underset{\beta}{\pi}(< x_\lambda >) = x_\beta$ is known as the projection mapping associated with the index β.

Consider the set $\underset{\beta}{\pi}^{-1}(G_\beta)$ where G_β is an open subset of X_β. It consists of all points $p = \{a_\lambda ; \lambda \in \Lambda\}$ in $\underset{\lambda}{\pi} X_\lambda$ such that $\underset{\beta}{\pi}(p) \in G_\beta$. In other words, $\underset{\beta}{\pi}^{-1}(G_\beta)$ $= \underset{\lambda}{\pi} Y_\lambda$ where $Y_\beta = G_\beta$ and $Y_\lambda = X_\lambda$ whenever $\lambda \neq \beta$ that is $\pi_\beta^{-1}(G_\beta) = X_1 \times X_2 \times \ldots \times X_{\beta-1} \times G_\beta \times X_{\beta+1} \ldots . \times \ldots$

Definition 2: For each λ in an arbitrary index set Λ, let (X_λ, T_λ) be a topological space and let $X = \underset{\lambda}{\pi} X_\lambda$. Then the topology T for X, which has a subbase the collection $B_* = \{\pi_\beta^{-1}(G_\lambda); \lambda \in \wedge \text{ and } G_\lambda \in T_\lambda\}$ is called the **product topology** or **Tichonov topology** for X and (X, T) is called the product space of the given spaces.

The collection B_* is called the defining subbase for T. The collection B of all finite intersection of elements of B_* would then form the base for T.

Note: The projection mappings are continuous for G_β in T_β–open in $X_\beta \Rightarrow \underset{\beta}{\pi}^{-1}(G_\beta) \in B_*$ which is a subbase for T and therefore $\underset{\beta}{\pi}^{-1}(G_\beta)$ is T-open in $X = \underset{\lambda}{\pi} X_\lambda$.

Theorem 4.7: Let A be a member of the defining base for a product space $X = \underset{\lambda}{\pi} X_\lambda$. Then the projection of A into any coordinate space is open.

NOTES

Proof: Since A belongs to the defining base for X.

$A = \underset{\lambda}{\pi}\{X_\lambda ; \lambda \neq \alpha_1, \alpha_{2,.....}\alpha_m\} \times G_{\alpha_1} \times ... \times G_{\alpha_m}$ where G_{α_i} is an open subset of X_{α_i}. So for any projection π_β: $X \to X_\beta$,

$$\pi_\beta(A) = \begin{cases} X_\beta \text{ if } \beta \neq \{\alpha_1, \alpha_2, ..., \alpha_m\} \\ G_\beta \ \ if \ \ \beta \in \{\alpha_1, \alpha_2, ..., \alpha_m\} \end{cases}$$

In either case $\pi_\beta(A)$ is an open set.

Theorem 4.8: Every projection $\pi_\beta : X \to X_\beta$ on a product space $X = \underset{\lambda}{\pi} X_\lambda$ is open.

Let G be an open subset of X. For every point $p \in G$, there is a member A of the defining base of the product topology such that $p \in A \subset G$. Thus for any projection $\pi_\beta : X \to X_\beta$, $p \in G \Rightarrow \pi_\beta(P) \in \pi_\beta(A) \subset \pi_\beta(G)$.

But $\pi_\beta(A)$ is an open set. Therefore, every point $\pi_\beta(P)$ in $\pi_\beta(G)$ belongs to an open set $\pi_\beta(A)$ which is contained in $\pi_\beta(G)$ is an open set.

Note: As each projection is continuous and open, but projections are not closed maps, e.g., consider the space $\mathbf{R} \times \mathbf{R}$ with product topology.

Let $H = \{(x, y); x, y \in \mathbf{R}$ and $xy = 2\}$

Here H is closed in $\mathbf{R} \times \mathbf{R}$ but $\pi_1(H) = \mathbf{R} \sim (0)$

This is not closed with respect to the usual topology for $\mathbf{R}$ where π_1 is the projection in the first coordinate space $\mathbf{R}$.

Check Your Progress

1. What is connected space?
2. When a subset of a topological space X is connected?
3. Define connected components.
4. When topological space termed as totally disconnected?
5. What do you understand by locally connected space?
6. When d^* is called subspace of X?
7. Give the definition of a subbase.
8. Define the term projection map.
9. What is central projection?

4.7 SEPARATION AXIOMS AND PRODUCT SPACES

NOTES

In topology and related fields of mathematics, there are many limitations that we often make on the kinds of topological spaces that we wish to consider. Some of these restrictions are given by the ***separation axioms***. These separation axioms are often termed as ***Tychonoff separation axioms***, after Andrey Tychonoff. The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The separation axioms are denoted with the letter '*T*' after the German *Trennungsaxiom*, meaning separation axiom.

The separation axioms are used to distinguish disjoint sets and distinct points. It is not enough for elements of a topological space to be distinct. They have to be *topologically distinguishable*. In the same way, it is not enough for subsets of a topological space to be disjoint. They have to be separated in some way. Suppose X is a topological space. Consequently, the two specific points x and y in X are typically defined as topologically distinguishable or distinct if they do not precisely possess the similar neighbourhoods or alternatively, we can specify that at least one of the points has a neighbourhood which is precisely not a neighbourhood of the other. Therefore, when x and y are topologically distinguishable or distinct points, then the singleton sets $\{x\}$ and $\{y\}$ must also be disjoint or separate.

Therefore, two points x and y are considered as ***separated*** if each of them holds a specific neighbourhood that is not at all the neighbourhood of the other. Additionally, the two subsets A and B of X are considered as ***separated*** if each of them is disjoint or separate from the closure of other; typically, the closures themselves are not disjoint or separate. Evidently, the points x and y are considered as separated if and only if their singleton sets $\{x\}$ and $\{y\}$ are also separated. Characteristically, remaining all conditions for sets can precisely be applied to the points or to a specific point and also a set by using the singleton sets.

Additionally, subsets A and B of X are separated by neighbourhoods if they have disjoint neighbourhoods. They are separated by closed neighbourhoods if they have disjoint closed neighbourhoods. They are separated by a function if there exists a continuous function f from the space X to the real line **R** such that the image $f(A)$ equals $\{0\}$ and $f(B)$ equals $\{1\}$. Lastly, they are precisely separated by a function if there exists a continuous function f from X to **R** such that the preimage $f^{-1}(\{0\})$ equals A and $f^{-1}(\{1\})$ equals B. Any two topologically distinguishable points must be distinct and any two separated points must be topologically distinguishable. Moreover, any two separated sets must be disjoint and any two sets separated by neighbourhoods must be separated, and so on.

Theorem 4.9: A topological space X is Hausdorff if and only if the diagonal is closed in $X \times X$ with the product topology.

Proof: Let D denote the diagonal $\{(x,x) \mid x \in X\}$ in $X \times X$.

Suppose D is closed. Then the complement of D is open. We want to show that X is Hausdorff. Let $x \neq y$. The point (x, y) lies in an open set disjoint from D. In particular, there is a basis open set about (x, y) that does not intersect D. Let $U \times V$ be such a basis open set (so U and V are both open in X). Clearly, if $y \in U \cap V$, then $(y, y) \in U \times V$. But $U \times V$ does not intersect D, and hence U and V are disjoint open sets.

Conversely, suppose X is Hausdorff. Then, we want to show that D is closed.

We know that given $x \neq y$, there are disjoint open sets U and V containing x and y respectively. Thus, any (x, y) lies inside an open set $U \times V$. Further, as $U \cap V$ is empty, the set $U \times V$ does not intersect D. Consequently, for every single point outside D, there is precisely a neighbourhood of the point outside D. The union of all these neighbourhoods are taken for concluding that the complement of D is open and therefore D is closed.

Even though, precisely the two directions of proof appear to be practically identical to one another, and hence there is a subtle or slight difference. To prove that precisely the diagonal is closed from Hausdorffness, then basically it can be said that products of open sets are open. Consequently, the identical or similar proof will be considered that more open sets can be added. Alternatively, the proof of Hausdorffness from the specified diagonal being closed significantly defines that the so called open rectangles form a basis, i.e., the products of open sets.

Theorem 4.10: The product of Hausdorff spaces is Hausdorff in the product topology.

Proof: Let X and Y be the two Hausdorff spaces. Then, the product space is $X \times Y$.

Select the points (x_1, y_1) and (x_2, y_2). Now either $x_1 \neq x_2$ or $y_1 \neq y_2$.

If $x_1 \neq x_2$, first separate x_1 and x_2 in X. That is, choose disjoint open sets in $X: U_1$ containing x_1 and U_2 containing x_2. Clearly $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2) respectively.

If $y_1 \neq y_2$, then separate y_1 and y_2 in Y, say, by V_1 and V_2. Then $X \times V_1$ and $X \times V_2$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2).

Theorem 4.11: Every subspace of a regular space is regular.

Proof: Let X be the regular topological space and A be a subset. Choose $x \in A$ and C closed in A. For $x \in X$, choose D, a closed subset of X such that $D \cap A = C$.

Such a D exists from the way the subspace topology is defined. Clearly, whatever D is picked up for the purpose, $x \notin D$ because the only points in $D \cap A$ are in a set not containing x.

NOTES

Since X is regular, we can find open sets U and V in X such that $x \in U$, $C \subseteq V$, and U and V are disjoint. Now, $U \cap A$ and $V \cap A$ are disjoint open subsets of A, with $x \in U \cap A$ and $C \subseteq V \cap A$.

The two important facts that we chose are that x being a point in the subspace, remained a point in the whole space and every closed set in the subspace was an intersection with the subspace of a closed set in the whole space.

4.8 CONNECTEDNESS AND COMPACTNESS OF PRODUCT SPACES

In topological analysis, the terms 'Connectedness' and 'Compactness' are the two most widely studied and used topological properties. Definitely, these properties are studied for analysing the subsets of the Euclidean space and the identification and acknowledgement of their individuality from the specific and unique form of the Euclidean metric notion of a topological property and consequently a topological space. However, the basic structure and unique notion of compact subsets of Euclidean space was recognized by the Heine-Borel theorem, who specified that connected subsets of $\mathbf{R}^n$ (for $n > 1$) proved to be very complicated and complex. Certainly, while any precise and compact Hausdorff space is considered locally compact, therefore a connected space – and even a connected subset of the Euclidean plane – essentially should not be locally connected.

A connected space is defined as a specific topological space which cannot be characterized or exemplified as the union of two or more disjoint or separate non-empty open subsets. Connectedness is, therefore, one of the most significant topological properties exceptionally used to discriminate and uniquely categorise the topological spaces. In the mathematical field of topological analysis, specifically the term compactness is defined as a property of topological space that specifically generalises the notion and concept about the subset of Euclidean space is closed, i.e., it contains all of its limit points, and bounded, i.e., all of its points typically remain within certain fixed distance to each other. In topology, a first countable space is referred as a topological space which uniquely satisfies the 'First Axiom of Countability'. The term embedding or imbedding is an instance of certain mathematical structure particularly contained within a different instance, for example as a group or a subgroup. A metrizable space is typically a topological space which is homeomorphic to a metric space.

In the end of the twentieth century, the locally homeomorphic properties to Euclidean space were considered more complicated and complex. This implies that even though the basic point set topology is relatively simple because

NOTES

essential metrizable spaces were studied and according to most of the definitions the concept of the algebraic topology is extremely complex. From this contemporary perspective, the strong property of local path connectedness is considered more significant, for example for a space to acknowledge a universal cover it must be connected and also locally path connected. Characteristically, a space is termed to be locally connected iff for every open set U, the connected components of U in the subspace topology are open. A continuous function from a locally connected space to a totally disconnected space must be locally constant.

Theorem 4.12: $\prod_\lambda X_\lambda$ is locally connected if and only if each space X_λ is locally connected and all but a finite number are connected.

Proof: Suppose $\prod_\lambda X_\lambda$ is locally connected, and let $x_\beta \in X_\beta$ be contained in some open set Y_β. Choose some point $z = <z_\lambda>$ with $z_\beta = X_\beta$ and we have z belonging to the open set $\pi_\beta^{-1}(Y_\beta)$. By local connectedness, there must exist a connected open set G containing z and contained in $\pi_\beta^{-1}(Y_\beta)$.Taking the β-th projection, we see that $z_\beta = x_\beta$ is contained in the connected open set $\pi_\beta(G)$ which is itself contained in Y_β and so X_β is locally connected. Further, if z is any point of the product space, it must be contained in some connected open set G. By definition, $z \in \pi_\lambda Y_\lambda \subseteq G$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for all but a certain finite number of values of λ. But then the projections of G are connected and are equal to X_λ, except for that finite number of values of λ.

Now suppose that X_λ is locally connected for old λ and connected for $\lambda \neq \beta_1, \beta_2, ..., \beta_n$. Let $X = <x_\lambda>_\lambda$ be an arbitrary point of $\pi_\lambda Y_\lambda$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for $\lambda \neq \beta_1{*}, \beta_2{*}, ..., \beta_k{*}$. Since $x_\lambda \in Y_\lambda$ for all λ and Y_λ is locally connected, there is a connected open set G_λ in X_λ such that $x_\lambda \in G_\lambda \subseteq Y_\lambda$. Consider the subset $\pi_\lambda Z_\lambda$ where $Z_\lambda = G_\lambda$ if $\lambda = \beta_1, \beta_2, ..., \beta_n, \beta_1{*}, ..., \beta_k{*}$ and $Z_\lambda = X_\lambda$ otherwise. But by the result, $\prod_\lambda X_\lambda$ is connected if and only if each X_λ is connected. This set is connected. Hence we have formed a connected open set containing X and contained in $\prod_\lambda Y_\lambda$.

4.9 COUNTABILITY

Following are the axioms of countability specifically used for topological spaces.

First Axiom of Countability

A **topological space** X is said to satisfy the **First Axiom of Countability** if, for every $x \in X$ there exists a countable collection $\mathcal{U}$ of neighbourhoods of x, such that if N is any neighbourhood of x, then there exists $U \in \mathcal{U}$ with $U \subseteq N$.

NOTES

A **topological space** that satisfies the **first axiom of countability** is said to be **First Countable**.

All **metric spaces** satisfy the first axiom of countability because for any neighbourhood N of a point x, there is an open ball $B_r(x)$ within N, and the countable collection of neighbourhoods of x that are $B_{1/k}(x)$ where $k \in \mathbb{N}$, has the neighbourhood $B_{1/n}(x)$ where $\frac{1}{n} < r$.

Theorem 4.13. If a topological space satisfies the first axiom of countability, then for any point x of closure of a set S, there is a sequence $\{a_i\}$ of points within S which converges to x.

Proof: Let $\{A_i\}$ be a countable collection of neighbourhoods of x such that for any neighbourhood N of x, there is an A_i such that $A_i \subset N$. Define,

$$B_n = \bigcap_{i=1}^{n} A_n.$$

Then form a sequence $\{a_i\}$ such that $A_i \subset B_i$. Consequently, $\{a_i\}$ converges to x.

Theorem 4.14. Let X be a topological space satisfying the first axiom of countability. Then, a subset A of X is closed if and only if all convergent sequences $\{x_n\} \subset A$ which converge to an element of X converge to an element of A.

Proof: Suppose that $\{x_n\}$ converges to x within X. The point x is a limit point of $\{x_n\}$ and thus is a limit point of A, and since A is closed, it is contained within A. Conversely, suppose that all convergent sequences within A converge to an element within A, and let x be any point of contact for A. Then by the theorem above, there is a sequence $\{x_n\}$ which converges to x, and so x is within A. Thus, A is closed.

Second Axiom of Countability

A **topological space** is said to satisfy the **Second Axiom of Countability** if it has a **countable base**. Consequently, the topological space that satisfies the second axiom of countability is said to be **Second Countable**.

Fundamentally, the topological space that satisfies the second axiom of countable is first countable, since the countable collection of neighbourhoods of a point can be all neighbourhoods of the point within the countable base, so that any neighbourhood N of that point must contain at least one neighbourhood A within the collection, and A must be a subset of N.

Theorem 4.15. If a topological space X satisfies the second axiom of countability, then all open covers of X have a countable subcover.

Proof: Let $\mathcal{G}$ be an open cover of X, and let B be a countable base for X. B covers X. For all points x, select an element of $\mathcal{G}$, C_x which contains x, and an element of the base B_x, which contains x and is a subset of C_x (which is possible because B is a base). $\{B_x\}$ forms a countable open cover for X. For each B_x, select an element of $\mathcal{G}$ which contains B_x, and this is a countable subcover of $\mathcal{G}$.

Check Your Progress

10. What do you understand by separation axioms?
11. What are the uses of separation axioms?
12. Define connectedness and compactness?
13. When does a topological space satisfy the first axiom of countability?
14. What do you mean by a first axiom countability in topological space?

4.10 EMBEDDING AND METRIZATION

Determining if two given spaces are homeomorphic is one of the fundamental problems in topology.

Definition 1: A one-one and onto (bijection) continuous map $f : X \to Y$ is a homeomorphism if its inverse is continuous.

A bijection $f : X \to Y$ induces a bijection between subsets of X and subsets of Y and it is a homeomorphism iff this bijection restricts to a bijection,

$$\{\text{Open (or closed) subsets of } X\} \underset{f^{-1}(V) \leftarrow V}{\overset{U \to f(U)}{\rule{3em}{0.4pt}}} \{\text{Open (or closed) subsets of } Y\}$$

between open (or closed) subsets of X and open (or closed) subsets of Y.

Definition 2: Suppose X is a set, Y a topological space and $f : X \to Y$ an injective map. The embedding topology on X (for the map f) is the collection,

$f^{-1}(\mathcal{T}_Y) = \{ f^{-1}(V) | V \subset Y \text{ open}\}$ of subsets of X.

The subspace topology for $A \subset X$ is the embedding topology for the inclusion map $A \to X$.

Theorem 4.16. (Characterization of the Embedding Topology): Let X has the embedding topology for the map $f : X \to Y$. Then,

1. $X \to Y$ is continuous.
2. For any map $A \to X$ into X,

   $A \to X$ is continuous iff $A \to X \xrightarrow{f} Y$ is continuous.

The embedding topology is the only topology on X with these two properties. The embedding topology is the most common topology on X such that $f : X \to Y$ is continuous.

Proof: The reason is that $A \xrightarrow{g} X$ is continuous.

$$\Leftrightarrow g^{-1}(\mathcal{T}_X) \subset \mathcal{T}_A \Leftrightarrow g^{-1}(f^{-1}\mathcal{T}_Y) \subset \mathcal{T}_A \Leftrightarrow (fg)^{-1} (\mathcal{T}_Y) \subset \mathcal{T}_A \Leftrightarrow A \xrightarrow{g} X \xrightarrow{f} Y$$

is continuous by definition of the embedding topology. The identity map of X is a homeomorphism whenever X is equipped with a topology with these two properties.

Definition: An injective continuous map $f: X \to Y$ is an embedding if the topology on X is the embedding topology for f, i.e., $\mathcal{T}_X = f^{-1}\mathcal{T}_Y$.

Any injective map $f: X \to Y$ induces a bijection between subsets of X and subsets of $f(X)$ and it is an embedding iff this bijection restricts to a bijection,

$$\{\text{Open (or closed) subsets of } X\} \frac{U \to f(U)}{f^{-1}(V) \leftarrow V} \{\text{Open (or closed) subsets of } f(X)\}$$

between open (or closed) subsets of X and open (or closed) subsets of $f(X)$.

Alternatively, the injective map $f: X \to Y$ is an embedding iff the bijective corestriction $f(X)| f: X \to f(X)$ is a homeomorphism. An embedding is a homeomorphism followed by an inclusion. The inclusion $A \to X$ of a subspace is an embedding. Any open (or closed) continuous injective map is an embedding.

For example, the map $f(x) = 3x + 1$ is a homeomorphism from $\mathbf{R} \to \mathbf{R}$.

Lemma 1: If $f: X \to Y$ is a homeomorphism (embedding) then the corestriction of the restriction $f(A)| f \setminus A: A \to f(A)$ $(B| f \setminus A: A \to B)$ is a homeomorphism (embedding) for any subset A of X (and any subset B of Y containing $f(A)$). If the maps $f_j: X_j \to Y_j$ are homeomorphisms (embeddings) then the product map $\prod f_j : \prod X_j \to Y_j$ is a homeomorphism (embedding).

Proof: In case of homeomorphisms employ that there is a continuous inverse in both cases. In case of embeddings, employ that an embedding is a homeomorphism followed by an inclusion map.

Lemma 2: (Composition of Embeddings): Let $X \overset{f}{\to} Y \overset{g}{\to} Z$ be continuous maps. Then f and g are embeddings implies that $g \circ f$ is an embedding which in turn implies that f is an embedding.

Proof: For proving the second implication, first note that f is an injective continuous map. Let $U \subset X$ be open. Since $g \circ f$ is an embedding, $U = (g \circ f)^{-1} W$ for some open $W \subset Z$. But $(g \circ f)^{-1} = f^{-1} g^{-1} W$ where $g^{-1} W$ is open in Y since g is contimuous. This shows that f is an embedding.

Theorem 4.17: (Characterization of the Product Topology): Given the product topology $\prod Y_j$. Then,

1. The projections $\pi_j : \prod Y_j \to Y_j$ are continuous, and
2. For any map $f: X \to \prod_{j \in J} Y_j$ into the product space we have,

$$X \overset{f}{\to} \prod_{j \in J} Y_j \text{ is continuous} \Leftrightarrow \forall\, j \in J: X \overset{f}{\to} \prod_{j \in J} Y_j \overset{\pi_j}{\to} Y_j \text{ is continuous.}$$

The product topology is the only topology on the product set with these two properties.

Proof: Let $\mathbf{T}_X$ be the topology on X and $\mathbf{T}_j$ the topology on Y_j. Then $S_\Pi = \bigcup_{j\in J} \pi_j^{-1}(\mathcal{T}_j)$ is a subbasis for the product topology on $\prod_{j\in J} Y_j$. Therefore,

$$f : X \to \prod_{j\in J} Y_j \text{ is continuous} \Leftrightarrow f^{-1}(\bigcup_{j\in J} \pi_j^{-1}(\mathcal{T}_j)) \subset \mathcal{T}_X$$

$$\Leftrightarrow (\bigcup_{j\in J} f^{-1}(\pi_j^{-1}(\mathcal{T}_j))) \subset \mathrm{T}_X$$

$$\Leftrightarrow \forall\, j \in J\text{: } (\pi_j \circ f)^{-1}(\mathcal{T}_j) \subset \mathcal{T}_X$$

$$\Leftrightarrow \forall\, j \in J\text{: } \pi_j \circ f \text{ is continuous by definition of continuity}$$

Now, we have to show that the product topology is the unique topology with these properties. Take two copies of the product set $\prod_{j\in J} X_j$. Provide one copy with the product topology and the other copy with some topology that has the two properties of the above theorem. Then the identity map between these two copies is a homeomorphism.

Theorem 4.18: Let $(X_j)_{j\in J}$ be an indexed family of topological spaces with subspaces $A_j \subset X_j$. Then $\prod_{j\in J} A_j$ is a subspace of $\prod_{j\in J} X_j$.

1. $\overline{\prod A_j} = \prod \overline{A_j}$
2. $(\prod A_j)^\circ \subset \prod A_j^\circ$ and equality holds if $A_j = X_j$ for all but finitely many $j \in J$.

Proof: (1) Let (x_j) be a point of $\prod X_j$. Since $S_\Pi = \bigcup_{j\in J} \pi_j^{-1}(\mathcal{T}_j)$ is a subbasis for the product topology on $\prod X_j$, we have

$(x_j) \in \overline{\prod A_j} \Leftrightarrow \forall\, k \in J\text{: } \pi_k^{-1}(U_k) \cap \prod A_j \neq \varphi$ for all neighbourhoods U_k of x_k.

$$\Leftrightarrow \forall\, k \in J\text{: } U_k \cap A_k \neq \phi \text{ for all neighbourhoods } U_k \text{ of } x_k$$

$$\Leftrightarrow \forall\, k \in J\text{: } x_k \in \overline{A_k}$$

$$\Leftrightarrow (x_j) \prod \overline{A_j}$$

(2) $(\prod A_j)^\circ \subset \prod A_j^\circ$ because π_j is an open map so that $\pi_j((\prod A_j)^\circ) \subset A_j^\circ$ for all $j \in J$. If $A_j = X_j$ for all but finitely many $j \in J$ then $\prod A_j^\circ \subset (\prod A_j)^\circ$ because $\prod A_j^\circ$ is open and contained in $\prod A_j$.

NOTES

It follows that a product of closed sets is closed.

Note: A product of open sets need not be open in the product topology.

4.10.1 Embedding Lemma and Tychonoff Embedding

Theorem 4.19 (Embedding Lemma): Let $\mathcal{F}$ be a family of mappings where each member $f \in \mathcal{F}$ maps $X \to Y_f$. Then,

1. The evaluation mapping $e: X \to \prod_{f \in \mathcal{F}} Y_f$ defined by $\pi_f oe(x) = f(x)$, for all $x \in X$, is continuous.
2. The mapping e is an open mapping onto $e(X)$ if $\mathcal{F}$ distinguishes points and closed sets.
3. The mapping e is one-to-one if and only if $\mathcal{F}$ distinguishes points.
4. The mappping e is an embedding if $\mathcal{F}$ distinguished points and closed sets.

Proof: (1) Let $\pi_g: \prod_{f \in \mathcal{F}} Y_f \to Y_g$ be the projection map to the space Y_g. Then $\pi_g oe = g$ so that $\pi_g oe$ is continuous. Therefore e must be continuous as g is continuous.

(2) Suppose that U is open in X and $x \in U$. Choose $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(X \setminus U)}$. The set $B = \{z \in e(X) | \pi_f(z) \notin \overline{f(X \setminus U)}\}$ is a neighbourhood of $e(x)$ as the set is open (it is defined for components not being in the closed set $\overline{f(X \setminus U)}$ and clearly $e(x) \in B$. Moreover $\pi_f(B) \subset f(U)$ by construction. It is now claimed that $f(U) \subset \pi_f(B)$. This follows trivially from the definition of a family of functions distinguishing points and closed sets. Therefore $f(U) = \pi_f(B)$ and subsequently $f(U)$ is an open subset of $\pi_g oe(X)$. Therefore the evaluation map is an open mapping.

(3) The definition of distinguishing points implies injectivity.

(4) Combining a, b and c, we see that $X \cong e(X)$ as e is a continuous, open, injective, surjective (as a continuous map is always surjective onto its image) map.

Definition: If X is a space and A, a set then by the power X^A we mean the product space $\Pi_\alpha X_\alpha$, where $X_\alpha = X$, for each $\alpha \in A$. Any power of [0, 1] is called a cube. A map e: $X \to Y$ is an embedding iff the map e: $X \to e(X)$ is a homeomorphism. If there is an embedding e: $X \to Y$ then we say that X can be embedded in Y.

Theorem 4.20 (Tychonoff's Embedding Theorem): A space is Tychonoff iff it can be embedded in a cube.

Proof: $\Rightarrow$ Let X be a Tychonoff space and let $A = \{f : X \to [0, 1] \,/\, f$ is continuous$\}$. Define e: $X \to [0, 1]^A$ by $e(x)(f) = f(x)$.

(*i*) e is injective: If $x, y \in X$ with $x \neq y$, then there is $f \in A$ so that $f(x) = 0$ and $f(y) = 1$. Then $e(x)(f) \neq e(y)(f)$, so $e(x) \neq e(y)$.

(*ii*) e is continuous: This is immediate since $\pi_f e = f$.

(*iii*) e: $X \to e(X)$ carries open sets of X to open subsets of $e(X)$: For let U be open in X and let $x \in U$. Then there is $f \in A$ so that $f(x) = 0$ and $f(X - U) = 1$. Let $V = \pi_f^{-1}([0, 1))$, an open subset of $[0, 1]^A$. Then $e(x) \in V$ and if $y \in X$ is such that $e(y) \in V$, then $e(y)(f) \in [0, 1)$, so $f(y) < 1$ and $y \in U$. Thus $e(x) \in V \cap e(X) \subset e(U)$.

(*i*), (*ii*) and (*iii*) together imply that e is an embedding.

$\Leftarrow$: [0, 1] is clearly so $[0, 1]^A$ is Tychonoff for any A. Any subspace of a Tychonoff space is Tychonoff. Thus if X can be embedded in a cube, then X is homeomorphic to a Tychonoff space and so is itself Tychonoff.

Theorem 4.21: Let $(\mathbf{T}, \mathcal{T})$ be the 3-point topological space defined by $\mathbf{T} = \{0, 1, 2\}$ and $\mathcal{T} = \{\varphi, \{0\}, \mathbf{T}\}$. Let $(X, \mathcal{U})$ be any topogical space and suppose that $\mathcal{U} \cap X = \varphi$. Then there is an embedding e: $X \to \mathcal{T}^{\mathcal{U} \cup X}$.

Proof: For each $U \in \mathcal{U}$, define f_U: $X \to \mathbf{T}$ by $f_U(y) = 0$ if $y \in U$ and $f_U(y) = 1$ if $y \notin U$. Then f_U is continuous. For each $x \notin X$, define f_x: $X \to \mathbf{T}$ by $f_x(y) = 2$ if $y = x$ and $f_x(y) = 1$ if $y \neq x$. Then f_x is also continuous.

Define e by $e_i(y) = f_i(y)$ for each $i \in \mathcal{U} \cup X$. Then

(*i*) e is injective, for if $x, y \in X$ with $x \neq y$ then $e_x(y) = 1$ but $e_x(x) = 2$, so $e_x(x) \neq e_x(y)$ and hence $e(x) \neq e(y)$.

(*ii*) e is continuous because each f_i is continuous.

(*iii*) e is open into $e(X)$, for if $U \in \mathcal{U}$ and $x \in U$ then $V = \pi_U^{-1}(0)$ is open in $\mathbf{T}^{\mathcal{U} \cup X}$. Furthermore, so $\pi_U e(x) = 0$, so $e(x) \in V$ while if $y \in X$ is such that $e(y) \in V$ then $\pi_U e(y) = 0$ and hence $y \in U$. Thus $V \cap e(X) \subset e(\mathcal{U})$.

NOTES

4.10.2 Urysohn's Metrization Theorem

Theorem 4.22 (Urysohn's Metrization Theorem): Suppose $(X, \mathcal{T})$ is a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.

Proof: Let $(X, \mathcal{T})$ be a regular metrizable space with countable basis $\mathcal{B}$. For this proof, we will first create a countable collection of functions $\{f_n\}_{n\in\mathbf{N}}$, where $f_m : X \to \mathbf{R}$ for all $m \in \mathbf{N}$, such that given any $x \in X$ and any open neighbourhood U of x there is an index N such that $f_N(x) > 0$ and zero outside of U. We will then use these functions to imbed X in $\mathbf{R}^w$.

Let $x \in X$ and let U be any open neighbourhood of x. There exists $B_m \in \mathcal{B}$ such that $x \in B_m$. Now, since X has a countable basis and is regular, we know that X is normal. Next, as B_m is open, there exists some $B_n \in \mathcal{B}$ such that $\overline{B_n} \subset B_m$. Thus we now have two closed sets $\overline{B_n}$ and $X \setminus B_m$, and so we can apply Urysohn's lemma to give us a continuous function $g_{n,m}: X \to \mathbf{R}$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X \setminus B_m) = \{0\}$. Notice here that this function satisfies requisite: $g_{n,m}(y) = 0$ for $y \in X \setminus B_m$ and $g_{n,m}(x) > 0$. Here, g was indexed purposely, as it shows us that $\{g_{n,m}\}$ is indexed by $\mathbf{N}\times\mathbf{N}$, which is countable (since the cross product of two countable sets is countable). Considering this, relable the functions $\{g_{n,m}\}_{n,m\in\mathbf{N}}$ as $\{f_n\}_{n\in\mathbf{N}}$.

We now imbed X in the metrizable space $\mathbf{R}^w$. Let $F: X \to \mathbf{R}^w$ where $F(x) = (f_1(x), f_2(x), f_3(x), \ldots)$, where f_n are the functions constructed above. We claim that F is an imbedding of X into $\mathbf{R}^w$.

For F to be an imbedding it is required of F to be homeomorphic onto its image. First, this needs that F should be a continuous bijection onto its image. We know that F is continuous as each of its component functions f_N are continuous by construction. Now we show that F is an injection.

Let $x, y \in X$ be distinct. From the Hausdorff condition there exist open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ with $U_x \cap U_y = \phi$. By the construction of our maps f there exists an index $N \in \mathbf{N}$ such that $f_N(U_x) > 0$ and $F_N(X \setminus U_x) = 0$. It follows that $f_N(x) \neq f_N(y)$ and so $F(x) \neq F(y)$. Hence, F is injective.

Now, as it is clear that F is surjective onto its image $F(X)$, all that is left to show is that F is an embedding. We will show that for any open set $U \in X$, $F(U)$ is open in $\mathbf{R}^w$. Let $U \subset X$ be open and let $x \in U$. Pick an index N such that $f_N(x) > 0$ and $f_N(X \setminus U) = 0$.

Let $F(x) = z \in F(U)$. Let $V = \pi_N^{-1}((0, \infty))$, i.e., all elements of $\mathbf{R}^w$ with a positive Nth coordinate. Now let $W = F(X) \cap V$. We claim that

$z \in W \subset F(U)$ showing that $F(U)$ can be written as a union of open sets, hence making it open.

First we show that W is open in $F(X)$. We know that V is an open set in $\mathbf{R}^w$. $W = F(X) \cap V$, and W is open by the definition of the subspace topology.

Thereafter, we will first show that $f(x) = z \in W$ and then $W \subset F(U)$.

To prove our first claim,

$F(x) = z \Rightarrow (F(x)) = f_N(x) > 0 \Rightarrow - \pi_N(z) = \pi_N(F(x)) = f_N(x) > 0$

$\Rightarrow \pi_N(z) > 0$ which means that $z \in \pi_N^{-1}(V)$ and also $z \in F(X) \Rightarrow z \in F(X) \cap V = W$. Now we show that $W \subset F(U)$. Let $y \in W$. This means $y \in F(X) \cap V = W$.

Now we show that $W \subset F(U)$. Let $y \in W$. This means $y \in F(X) \cap V$. This means there exists some $w \in X$ such that $F(w) = y$. But, since $y \in V$ we have that:

$\pi_N(y) = \pi_N(F(w)) = f_N(w) > 0$ since $y \in V$, but $f_N(w) = 0$ for all $w \in X \backslash U$ and so $y \in F(U)$.

In conclusion, as we have shown that $F: X \to \mathbf{R}^w$ is a map that preserves open sets in both directions and bijective onto its image, we have shown that F is an embedding of the space X into the metrizable space $\mathbf{R}^w$ and X is therefore metrizable, the metric being given by the induced metric from $\mathbf{R}^w$.

Example 4.2: The topology generated by the dictionary ordering on $\mathbf{R}^2$ is metrizable.

Proof: From previous Theorem 4.22, all we have to do for showing that $\mathbf{R}^2$ is metrizable in the dictionary ordering is to prove that this space is regular with a countable basis.

Now, since the set $\{(a, b), (c, d)| \ a \leq c, \ b < d; \ a, b, c, d \in \mathbf{R}\}$ is a basis for the dictionary ordering on $\mathbf{R}^2$ and the set of intervals with rational end-points are a basis for the usual topology on $\mathbf{R}$, it follows that the set $\{(a, b), (c, d)|a \leq c, \ b < d; \ a, b, c, d \in \mathbf{Q} \}$ is a countable basis for the dictionary ordering.

Now we will show that the dictionary ordering is regular. Let $a \in \mathbf{R}^2$ and $B \subseteq \mathbf{R}^2$ such that B is closed in the dictionary ordering and $a \notin B$. Let $\varepsilon = \inf \{d(a, b)| \ b \in B\}$. We know that ε is greater than 0, for otherwise a would be an accumulation point of B, which is a contradiction. It follows that the open sets $((a, a–\varepsilon/2), (a, a+\varepsilon/2))$ and $\bigcup_{b \in B} \ ((b, b–\varepsilon/2), (b, b+\varepsilon/2))$ are disjoint open sets containing a and B, respectively. Hence, the dictionary ordering over $\mathbf{R}^2$ is metrizable, since it is regular and has a countable basis.

Note: In this proof we have shown that a sequence of functions $\{f_n\}_{n \in \mathbf{N}}$ with the property that for each $x \in X$ and each neighbourhood U of x there is some $n \in \mathbf{N}$ such that $f_n(x) > 0$ and $f_n(y) > 0$ for all $y \in X \backslash U$, gives us an imbedding $F: X \to \mathbf{R}^w$. Notice that we have the very similar result if we have a sequence of functions $\{f_j\}_{j \in J}$ with same properties as above: given any $x \in X$ and any neighbourhood U of x there exists $j \in J$ such that $f_j(x) > 0$ and $f_j(y) = 0$ for all $y \in X \backslash U$, then we have an imbedding from $X \to \mathbf{R}^j$ given by $F(x) = (f_j(x))_{j \in J}$. This is known as the **imbedding theorem** and is a generalization of Urysohn's metrization theorem.

NOTES

Check Your Progress

15. Define the term embedding.
16. State the Urysohn's metrization theorem.

4.11 SOLVED EXAMPLES

Example 1: The continuous image of a connected space is connected.

Solution: Let $f:X\to Y$ is a continuous surjection suppose Y has a proper simultaneously open and closed subset then $f^{1}(\cup)$ is a proper simultaneously open and closed subset of X. Thus disconnectedness of Y implies disconnectedness of X.

Example 2: Let p denote an analytic polynomial in the complex variables $X_1,\ldots X_n$, and $X(p)$ denote the zero set of p. Then $\mathbb{C}^n\backslash X(p)$ is path connected.

Solution: Let $X,\omega\in\mathbb{C}^n X(p)$.

Consider the straight line path,

$r(t)=(1-t)x+t\omega(t\in\mathbb{C})$

Note that $\{t\in\mathbb{C}: r(t)\in X(p)$ is precisely the zero set $X(por):=X$. Since por is polynomial in one variable, X is a finite subset of $\mathbb{C}$ thus r maps the path connected set $\mathbb{C}\backslash X$ continuously into $\mathbb{C}^n\backslash X(p)$. In particular, X and ω belong to the path-connected subset $r(\mathbb{C}\backslash X)$ of $\mathbb{C}^n\backslash X(P)$.

Example 3: A metric space (X, d) is second countable if any only if the following holds true:

(1) X has a countable dense subset.

(2) X is compact.

Solution:

1. Let $\{x_n\}$ be a countable dense subset of X we see that the countable collection $\{B_d(x_n;\frac{1}{m}):n\geq 1, m\geq 1\}$ forms a basics:

   (i) Let $x\in X$ then there exists an integer $n\geq 1$ such that $X_n\in B_d(x,\frac{1}{2})$, then $x\in B_d(x_n,\frac{1}{2})$

   (ii) If $x\in\cup := B_d(x_n,\frac{1}{m_1})\cap B_d(x_{n_2},\frac{1}{m_2})$ then

   There exists integer $r\geq 1$ such that $B_d(x,\frac{1}{r})\subseteq U$

   By density of $\{x_n\}$, there exists $x_l\in$ $B_d(x,\frac{1}{4r})$

   Then $x\in B_d(x_l,\frac{1}{2r})\subseteq B_d(x,\frac{1}{r})\subseteq U$.

2. Suppose X is compact for each integer $n \geq 1$ consider the open cover $\{\mathbb{B}(x, \frac{1}{n})\}$ of X. Since X is compact, X has a finite subcover $\{\mathbb{B}(x_n, \frac{1}{n})\} : i = 1, ...k_n\}$ of X.

$\bigcup_{n=1}^{\infty} \{\mathbb{B}(x_n, \frac{1}{n}) : i = 1, ...k_n$ Forms a countable basis for X.

Example 4: Every compact Hausdorff space is normal.

Solution: Let A and B be two closed subset of a compact Hausdorff space X. It is already known that any closed set and a point in X can be separated by open sets. For each $a \in A$, there exists disjoint open sets for $\cup a$ and W_a such that $a \in \cup a$ and $B \subseteq Wa$

Clearly, $A \subseteq U\, a \in A\; Ua$. Since A is compact there exist $a_1 .... a_K \in A$ such that

$U := U_{a_1} \cap ...... \cap U_{a_k}$ is an open set containing A.

Then $\text{W} := \text{W}_{a_k} \cap ...... \cap \text{W}_{a_k}$ is an open set containing B and $U \cap W = \phi$.

Example 5: Prove that there exists a maximal family $F \supseteq C$ with the finite intersection property.

Solution: $F \supseteq C$, here consider the collection $\mathbb{F}$ of families $F \supseteq C$ of subsets of X with the finite intersection property with strictly partial order given by

$f_1 > f_2$ iff $f_1 \subseteq f_2$

Let $\mathbb{F}_0$ be a simply ordered subcollection of $\mathbb{F}$, i.e, $f_1 \neq f_2 \in \mathbb{F}_0$ then either $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$ has the upper bound $U = U_{f \in \mathbb{F}_0}$ in $\mathbb{F}$.

To see this, we must check that U belongs to F that in U has finite intersection property. If $A, ...A_n \in u$ Then $A_i \in f_i$ for some i, and hence $A, ... A_n \in F_j$ for some J.

Since F_j has finite intersection property $A_1 \cap ... \cap A_n$ in non-empty. Thus Zorn's Lamma can be applied which ensures existance of maximal family $F \supseteq C$ with the finite intersection property.

Example 6: Let X be a connected normal space, which is also Hausdorff and which contains at least two points then there exists a continous surjection $f:X \rightarrow [0,1]$. In particular, X is uncountable.

Solution: By **Urysohn's Lemma**, There exists a continous function $f:X \rightarrow [0,1]$ such that $f(a)=0$ and $f(b)=1$.

Where a,b are fixed distinct point in X suppose there is $r \in [0,1]$ such that r has no preimage. Then is $0 \in f(X) \cap [0, r]$ is open and closed subset of $f(X)$ which close not contain 1.

This is not possible since $f(X)$ is a connected set containing 1, so that f is surjective.

NOTES

Example 7: Prove that no pair of the following subspaces of $\mathbb{R}$ are homeomorphic: [0,1], [0,1], [0,1]

Solution: The distinguishing property is the minimal number of connected components when we take out two point such a procedure will always separate (0,1) into 3 components. If we choose one of them to be a boundry point it is 2 for [0,1] but not less if we take of two boundary points from [0,1] it will remain connected, hence this number is 1 for it.

Example 8: Given a space *X*, we define an equivalence relation on the element of *X* as follows for all *x,y*∈*X*. *x*~*y* there is a connected subset *A*⊂*X* with *x,y*∈*A* compute the connected components of *Q*.

Solution: Let $\phi = A \subset Q$ be connected of *A* contains more than one elements say $x<y \in Q$, then for a given irrational at *r* between *x* and *y* $(-\infty, r)$ and (r, ∞) separate *A* into two disjoint, non-empty open subset. Thus it cannot be connected, so it should contain at most one element, so the connected components are $\{x\}$, for $x \in Q$

Example 9: Suppose *X* admits a family $\{U_r\} r \in Q$ of nested neighbourhoods. Then $f{:}X \to [0, 1]$, given by $f(x) = \inf\ Q(x)$ is continous.

Solution: Let (c,d) be an open interval containing $f(x_0)$ where $x_0 \in X$. Choose rational number *p* and *q* such that,

$$c<p<f(x_0)<q<d,$$

x_0 belongs to U_q since $f(x_0)<q$ and $x_0 \notin \bar{U}_p$ since $f(x_0)>p$.

Thus $x_0 \in u := U_q \setminus \bar{U}_p \subseteq \bar{U}_q \setminus U_p$. We check that $f(U) \subseteq [p,q] \subseteq (c,d)$

In fact, if $x \in U$ then $x \in \overline{U}_q$, so that $f(x) \leq q$ and $x \notin U_p$, so that $f(x) \geq p$.

Example 10: A space *X* is locally path-connected if and only if the path components of open subsets of *X* are open.

Solution: Suppose that *X* is locally path connected and let $W \subset X$ be open and *P* a path component of *W* if $x \in P$, then there exists a path connected neighbourhood *V* of *x* with $V<W$. As $x \in P \cap V$.

$P \cup W$ is path connected and contained in *W*. Since *P* is a maximal path connected subset of *W* we have $P \cup V = P$.

Which implies that $V \subset P$,

Thus *P* is a neighbourhood of *X* and *P* is open conversely let *W* be an open set in *X* let $x \in W$ and let *V* be the path component of *x* in *W*. By hypothesis *V* is open therefore *X* is locally path connected.

4.12 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In topology and the mathematical field of analysis, the topological spaces and other related or associated branches of mathematics, a connected space is defined as a specific topological space which cannot

be exemplified and characterized as the union or association of two or more disjoint non-empty open subsets. Connectedness is considered as the essential and principal topological properties which are precisely and explicitly used for distinguishing the topological spaces. The unique and strong notion specifies the path connected space, which is a specific space where any two points can be joined or connected by means of a path.

2. A topological space X is considered as disconnected if it is defined as the union of two disjoint or separate non-empty open sets. Alternatively, X is considered to be connected. A subset of a precise topological space is defined as connected if it is typically connected in its subspace topology.
3. In topology, the maximal connected subsets or the utmost and maximum connected subsets of a non-empty topological space are precisely termed as the connected components of the space. Characteristically, the components or elements of any topological space X specifically from a partition of X, are uniquely defined as disjoint, separate and non-empty, and their union is defined as the whole or entire space.
4. In topology, a space is termed as totally disconnected when all of its components are precisely one point sets. Associated to this property of space, it can be stated that a space X is termed as totally separated if for any two components or elements x and y of X, there exists disjoint or separate open neighbourhoods U of x and V of y such that X is defined as the union of U and V. Evidently any totally or entirely separated space is totally and perfectly disconnected, however the converse does not hold.
5. A space is considered as locally connected if and only if particularly for every single open set U, the connected specific components of U should be open. Characteristically, a continuous function specifically explained from a locally connected space to a totally or completely disconnected space must also be locally constant. Essentially, the openness or directness of components or elements is a natural property and is generally not true, for example the Cantor space is considered as totally or completely disconnected but not discrete.
6. Let (X, d) be a metric space and let A be a subset of X. Let d^* denote restriction of d *to* $A \times A$, that is,

   $$d^* (x, y) = d (x, y)$$

   Here x, y are points of A. Then d^* is a metric for A called the induced metric and the set A with metric d^* is called subspace of X.
7. Let X be a topological space with topology **T**. A subbase of this topology **T** is defined as a subcollection B of **T** satisfying one of the two following equivalent conditions:
   - The subcollection B generates the topology **T**. This implies that **T** is the smallest topology containing B and so any topology U on X containing B must also contain **T**.

NOTES

- The collection of open sets consisting of all finite intersections of elements of B, together with the set X and the empty set, forms a basis for **T**. This means that every non-empty proper open set in **T** can be written as a union of finite intersections of elements of B. Clearly, given a point x in a proper open set U, there are finitely many sets $S_1, \ldots, S_n$ of B, such that the intersection of these sets contains x and is contained in U.

8. In mathematics, in general, a *projection* is a mapping of a set or of a mathematical structure which is idempotent, i.e., a projection is equal to its composition with itself. A *projection* may also refer to a mapping which has a left inverse. Both these ideas are related. Let p be an idempotent map from a set E into itself (thus $p°p = \text{Id}_E$) and $F = p(E)$ be the image of p. If we denote the map p viewed as a map from E onto F by π and the injection of F into E by i, then we get $i°\pi = \text{Id}_F$. On the other hand, $i°\pi = \text{Id}_F$ implies that $\pi°i$ is idempotent.
9. Central Projection: It is the projection from a point onto a plane. If C is the center of projection, then the projection of a point P distinct from C is the intersection with the plane of the line CP. The point C and the points P such that the line Central Projection (CP) is parallel to the plane do not have any image by the projection.
10. In topology and related fields of mathematics, there are many limitations that we often make on the kinds of topological spaces that we wish to consider. Some of these restrictions are given by the separation axioms. These separation axioms are often termed as Tychonoff separation axioms, after Andrey Tychonoff. The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The separation axioms are denoted with the letter 'T' after the German *Trennungsaxiom*, meaning separation axiom.
11. The separation axioms are used to distinguish disjoint sets and distinct points. It is not enough for elements of a topological space to be distinct. They have to be *topologically distinguishable*. In the same way, it is not enough for subsets of a topological space to be disjoint. They have to be separated in some way.
12. Connectedness is, therefore, one of the most significant topological properties exceptionally used to discriminate and uniquely categorise the topological spaces. In the mathematical field of topological analysis, specifically the term compactness is defined as a property of topological space that specifically generalises the notion and concept about the subset of Euclidean space is closed, i.e., it contains all of its limit points, and bounded, i.e., all of its points typically remain within certain fixed distance to each other.
13. In topology, a first countable space is referred as a topological space which uniquely satisfies the 'First Axiom of Countability'. The term embedding or imbedding is an instance of certain mathematical structure particularly

contained within a different instance, for example as a group or a subgroup. A metrizable space is typically a topological space which is homeomorphic to a metric space.

14. A topological space X is said to satisfy the First Axiom of Countability if, for every $x \in X$ there exists a countable collection $\mathcal{U}$ of neighbourhoods of x, such that if N is any neighbourhood of x, then there exists $U \in \mathcal{U}$ with $U \subseteq N$.

    A topological space that satisfies the first axiom of countability is said to be First Countable.

15. Suppose X is a set, Y a topological space and $f : X \rightarrow Y$ an injective map. The embedding topology on X (for the map f) is the collection,

    $f^{-1}(\mathcal{T}_Y) = \{f^{-1}(V) | V \subset Y \text{ open}\}$ of subsets of X.

    The subspace topology for $A \subset X$ is the embedding topology for the inclusion map $A \rightarrow X$.

16. Urysohn's Metrization Theorem: Suppose $(X, \mathcal{T})$ is a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.

4.13 SUMMARY

- Fundamentally, we can state that a subset of a topological space X can be defined as a connected set if and only if it appears as a connected space when typically observed as a subspace of X.
- In topology, the maximal connected subsets or the utmost and maximum connected subsets of a non-empty topological space are precisely termed as the connected components of the space. Characteristically, the components or elements of any topological space X specifically from a partition of X, are uniquely defined as disjoint, separate and non-empty, and their union is defined as the whole or entire space.
- "The empty space does not have connected components. Every single component is defined as a precise closed subset defined on the original space."

  Let Γ_x be a connected component of x in a topological space X, and Γ'_x be the intersection of all open-closed sets containing x (called quasi-component of x.) Then $\Gamma_x \subset \Gamma'_x$ where the equality holds if X is compact Hausdorff or locally connected.
- In topology, a Hausdorff space is defined as a precise topological space having a separation property that any two distinct points can be separated by means of disjoint open sets.
- A Euclidean plane excluding the origin, (0,0) is connected but is not simply connected. The three-dimensional Euclidean space without the origin is connected and even simply connected. In contrast, the one-dimensional Euclidean space without the origin is not connected.

NOTES

- Every discrete topological space with at least two elements is disconnected, in fact such a space is totally disconnected. The simplest example is the discrete two-point space.
- The topologist's sine curve is an example of a set that is connected but is neither path connected nor locally connected.
- The spectrum of a commutative local ring is connected. More generally, the spectrum of a commutative ring is connected if and only if it has no idempotent $\neq 0$, if and only if the ring is not a product of two rings in a non-trivial way.
- A path from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0,1]$ to X with $f(0)=x$ and $f(1)=y$. A path-component of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y.
- However, subsets of the real line **R** are connected if and only if they are path connected; these subsets are the intervals of **R**. Also, open subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are connected if and only if they are path componen. Additionally, connectedness and path componenness are the same for finite topological spaces.
- In topology, a space X is referred as to be arc connected or arc wise connected when any two distinct or separate points can be joined through an arc, i.e., consider a path f which is typically homeomorphism between the unit interval $[0,1]$ and its image $f([0,1])$.
- The historical perspective of topology discusses about the two most significant and most widely used topological properties, namely the connectedness and compactness. Certainly, the analysis and study of these properties can be done as the subsets of Euclidean space, and also their unique identification as being independent from the certain specific form of the Euclidean metric helped to specify the notion of a topological property and consequently a topological space.
- In other words, the only difference between the two definitions is that for local connectedness at x we require a neighbourhood base of open connected sets, whereas for weak local connectedness at x we require only a base of neighbourhoods of x.
- Evidently, a space which is locally connected at x is weakly locally connected at x. The converse does not hold. On the other hand, it is equally clear that a locally connected space is weakly locally connected and here it turns out that the converse does hold. A space which is weakly locally connected at all of its points is necessarily locally connected at all of its points.
- Local connectedness is, by definition, a local property of topological spaces, i.e., a topological property P such that a space X possesses property P if and only if each point x in X admits a neighbourhood base of sets which

have property P. Accordingly, all the metaproperties held by a local property hold for local connectedness.

- A space is locally connected if and only if it admits a base of connected subsets.

NOTES

- Thus, a subset A of X equipped with the induced metric is a metric space in its own right and neighbourhoods, open sets and closed sets are defined as an any metric space. But an open set (closed set) of A need not be open (closed) when regarded as a subset of X.
- Note that the phrase 'B is open in A' means that B is open relative to the induced metric on A. Also 'B is open in X' means that B is open with respect to the metric on X.
- A sebset A of $\mathbf{R}$ is connected if and only if it is an interval.
- The topological space precisely for any subcollection S typically of the power set $P(X)$, there uniquely exists a distinctive topology with S as a subbase. Specifically, the intersection of precisely all the topologies on X that contains S uniquely satisfies this particular condition. However, there is generally no unique subbases for a given specific topology. Consequently, a fixed topology is considered for finding the subbases for the specific topology. Subsequently, an arbitrary or random subcollection of the power set $P(X)$ also can be formed by using the properties of topology specifically generated by means of that subcollection.
- Let (X_λ, T_λ) be an arbitrary collection of topological spaces and let $X = \underset{\lambda}{\pi} X_\lambda$. Let T be a topology for X. If T is the product topology for X, then T is the smallest topology for X for which projections are continuous and conversely also.
- Let C_i be a closed subset of a space X_i for $i \in \mathbf{I}$. Then $\underset{i \in I}{\pi} C_i$ is a closed subset of $\underset{i \in I}{\pi} X_i$ with respect to the product topology.
- Initially, the notion of projection was introduced in Euclidean geometry to denote the projection of the three-imensional Euclidean space onto a plane.
- The image of a point P is the intersection with the plane of the line parallel to D passing through P.
- Let A be a member of the defining base for a product space $X = \underset{\lambda}{\pi} X_\lambda$. Then the projection of A into any coordinate space is open.
- Consequently, the two specific points x and y in X are typically defined as topologically distinguishable or distinct if they do not precisely possess the similar neighbourhoods or alternatively, we can specify that at least one of

NOTES

the points has a neighbourhood which is precisely not a neighbourhood of the other. Therefore, when x and y are topologically distinguishable or distinct points, then the singleton sets $\{x\}$ and $\{y\}$ must also be disjoint or separate.

- Therefore, two points x and y are considered as separated if each of them holds a specific neighbourhood that is not at all the neighbourhood of the other. Additionally, the two subsets A and B of X are considered as separated if each of them is disjoint or separate from the closure of other; typically, the closures themselves are not disjoint or separate.
- A topological space X is Hausdorff if and only if the diagonal is closed in $X \times X$ with the product topology.
- The two important facts that we chose are that x being a point in the subspace, remained a point in the whole space and every closed set in the subspace was an intersection with the subspace of a closed set in the whole space.
- In topological analysis, the terms 'Connectedness' and 'Compactness' are the two most widely studied and used topological properties. Definitely, these properties are studied for analysing the subsets of the Euclidean space and the identification and acknowledgement of their individuality from the specific and unique form of the Euclidean metric notion of a topological property and consequently a topological space.
- In the end of the twentieth century, the locally homeomorphic properties to Euclidean space were considered more complicated and complex. This implies that even though the basic point set topology is relatively simple because essential metrizable spaces were studied and according to most of the definitions the concept of the algebraic topology is extremely complex.
- $\prod_{\lambda} X_{\lambda}$ is locally connected if and only if each space X_{λ} is locally connected and all but a finite number are connected.
- Fundamentally, the topological space that satisfies the second axiom of countable is first countable, since the countable collection of neighbourhoods of a point can be all neighbourhoods of the point within the countable base, so that any neighbourhood N of that point must contain at least one neighbourhood A within the collection, and A must be a subset of N.
- A one-one and onto (bijection) continuous map $f : X \to Y$ is a homeomorphism if its inverse is continuous.
- Alternatively, the injective map $f : X \to Y$ is an embedding iff the bijective corestriction $f(X)| f : X \to f(X)$ is a homeomorphism. An embedding is a homeomorphism followed by an inclusion. The inclusion $A \to X$ of a subspace is an embedding. Any open (or closed) continuous injective map is an embedding.
- If $f : X \to Y$ is a homeomorphism (embedding) then the corestriction of the restriction $f(A)| f \backslash A : A \to f(A)$ $(B| f \backslash A : A \to B)$ is a homeomorphism

(embedding) for any subset A of X (and any subset B of Y containing $f(A)$). If the maps $f_j: X_j \to Y_j$ are homeomorphisms (embeddings) then the product map $\prod f_j : \prod X_j \to Y_j$ is a homeomorphism (embedding).

- The mappping e is an embedding if $\mathcal{F}$ distinguished points and closed sets.
- If X is a space and A, a set then by the power X^A we mean the product space $\Pi_\alpha X_\alpha$, where $X_\alpha = X$, for each $\alpha \in A$. Any power of [0, 1] is called a cube. A map $e: X \to Y$ is an embedding iff the map $e: X \to e(X)$ is a homeomorphism. If there is an embedding $e: X \to Y$ then we say that X can be embedded in Y.
- A space is Tychonoff iff it can be embedded in a cube.

4.14 KEY TERMS

- **Connected Components:** The empty space does not have connected components. Every single component is defined as a precise closed subset defined on the original space.
- **Path connectedness:** A path from a point x to a point y in a topological space X is a continuous function f from the unit interval [0,1] to X with $f(0) = x$ and $f(1) = y$. A path-component of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y.
- **Locally connected:** In the field of topology and other disciplines of mathematics, a topological space X is typically defined as locally connected if every single point acknowledges or recognises a neighbourhood basis that uniquely consists of entire or complete open connected sets.
- **Subbase:** In topology, a subbase for a topological space X with topology **T** is a subcollection B of **T** which generates **T**, such that **T** is the smallest topology containing B.
- **Central projection:** It is the projection from a point onto a plane. If C is the center of projection, then the projection of a point P distinct from C is the intersection with the plane of the line CP. The point C and the points P such that the line Central Projection (CP) is parallel to the plane do not have any image by the projection.
- **Second axiom of countability:** All metric spaces satisfy the first axiom of countability because for any neighbourhood N of a point x, there is an open ball $B_r(x)$ within N, and the countable collection of neighbourhoods of x that are $B_{1/k}(x)$ where $k \in \mathbb{N}$, has the neighbourhood $B_{1/n}(x)$ where $\frac{1}{n} < r$.

4.15 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. What is connected space?
2. Define the term disconnected spaces.
3. What do you mean by arc connectedness?
4. Define the term connectedness.
5. What is connectedness on the real line?
6. Define the term subspace.
7. State Tychonoff's theorem.
8. What do you understand by projection map?
9. Give the types of projection map.
10. When projections is continuous?
11. State the Tychonoff's separation axioms.
12. Why connectedness is used?
13. State the compactness of product space.
14. When does a metric space satisfy the first axiom of countability?
15. What do you mean by second axioms of countability?
16. Define the term embedding.
17. State the Tychonoff's embedding theorem.

Long-Answer Questions

1. Explain in detail about the connected spaces giving appropriate properties and examples.
2. Describe the term connectedness on the real line with the help of theorems and examples.
3. State and prove the Tychonoff product topology in terms of standard subbase with the help of theorem and examples
4. Describe the projection mappings with respect to topology giving examples.
5. Describe briefly about separation axioms and product space giving relevant examples.
6. Explain in detail about the connectedness and compactness of product space with the help of theorems and examples.
7. Analyse the first and second countable spaces giving appropriate examples.

8. State and prove embedding lemma and Urysohn's metrization theorem with the help of relevant examples.

NOTES

4.15 FURTHER READING

Munkres, James R. 2011. *Topology*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.

Basener, William F. *Topology and Its Applications*. New Jersey: John Wiley & Sons, Inc.

Bourbaki, Nicolas. 1966. *General Topology: Elements of Mathematics*, Volume 2. United States: Addison-Wesley.

Bourbaki, Nicolas. 1966. *Elements of Mathematics: General Topology*, Part 1. United States: Addison-Wesley.

Kelley, John L. 1975. *General Topology*. New York: Springer-Verlag.

Willard, Stephen. 2004. *General Topology*. United States: Dover Publications.

Shilov, Georgi E. 2012. *Elementary Real and Complex Analysis*. Chelmsford: Courier Corporation.

Simmons. G. F. 2004. *Introduction to Topology and Modern Analysis*. New York: Tata McGraw-Hill Education.

UNIT 5 NETS AND FILTERS

NOTES

Structure

5.0 INTRODUCTION

In mathematical analysis and more precisely in the context of general topology and other concerned fields the concept of 'Net' also known as 'Moore–Smith Sequence' is defined as the simplification or generality which specifies the notion of a sequence. The expression 'Net' is termed as 'Moore–Smith Sequence' because the unique concept of net was originally introduced by E. H. Moore and Herman L. Smith in the year 1922, while the term or name 'Net' was first coined by John L. Kelley. Basically, a sequence is described as a specific function whose domain holds the natural numbers. Generally, the codomain of this specific function mostly includes certain topological spaces.

Nets are referred as the specialized tools in topology and are typically used for generalizing specific and particular concepts, notations and theories that are usually sufficient within the context or perspective of metric spaces.

The term 'Filter', an associated notion was developed and established by Henri Cartan in the year 1937. In mathematical analysis, the term 'Filter' or 'Order Filter' is defined as a unique and particular subset of a partially ordered set. In the

NOTES

field of topology, filters are defined as a subfield of mathematics which are used for studying and analysing the topological spaces, identifying and describing all the basic topological concepts and notions, for instance compactness, convergence, continuity, limits, etc. The term 'Ultrafilters' are typically used for defining specific types of filters which possess numerous effective technical and mathematical properties due to which they are frequently used and preferred in place of arbitrary filters.

Characteristically, topology when used in mathematical analysis then it specifically studies the properties of a geometric object which are uniquely defined in continuous deformations, for instance crumpling, stretching, bending and twisting, i.e., without closing holes, opening holes, tearing, gluing, or passing through itself.

In topology and their related branches of mathematical analysis, a Hausdorff space, separated space or T_2 space specifically defined as a topological space in which for any two distinct points there exists neighbourhoods of each that are disjoint from each other.

Characteristically, in the field of general topology, the compactness is defined as a property that generalises and simplifies the concept, theory and notations of a subset of Euclidean space being 'Closed', i.e., it contains all its limit points and 'Bounded', i.e., it has all its points lying within certain fixed or static distance of each other.

Fundamentally, every single net induces or generates a canonical filter and then dually every filter induces or generates a canonical net, wherein this induced net or induced filter converges to a point if and only if the same is true of the original filter. An ultrafilter is a powerful tool both in set theory and in topology.

The term local finiteness, as per the mathematical field of topology, is defined as a specific property of collections or groups of subsets of a topological space. It is essential and significant, especially for analysing the paracompactness and topological dimensions. A paracompact space is referred as a topological space where every single open cover holds an open refinement which is locally finite. Consequently, every compact space is considered as paracompact.

The Nagata–Smirnov metrization theorem is distinctively named after Junichi Nagata and Yuri- Mikha-lovich Smirnov, who independently published their proofs in 1950 and 1951, respectively. In topology, the Nagata–Smirnov metrization theorem distinguishes and exemplifies when can a topological space is metrizable. The Nagata–Smirnov metrization theorem states that, "A topological space X is metrizable iff and only iff it is regular, Hausdorff and holds a countably locally finite, i.e., σ-locally finite basis".

The algebraic topology includes a specific branch termed as homotopy theory which exceptionally and distinctively studies the 'Paths' and 'Loops' considered as the essential subjects in topology. The homotopy of paths considers the precise and accurate notions about the continuously/constantly deforming a path while its endpoints are kept fixed. Additionally, as per the mathematical field of algebraic topology, the term fundamental group of a topological space specifically defines the characteristic group of the equivalence classes typically considered under the field of homotopy of the loops that are contained or enclosed in the space. It documents the specific information and evidence about the basic or fundamental essential shapes or holes of the concerned or related topological spaces. Covering spaces has significant key role in the homotopy theory, differential topology, harmonic analysis and Riemannian geometry.

In this unit, you will study about the nets and filters, topology and convergence of nets, Hausdorffness and nets, compactness and nets, filters and their convergence, canonical way of converting nets to filters and vice-versa, ultrafilters and compactness, metrization theorem and paracompactness, local finiteness, the Nagata-Smirnov metrization theorem, paracompactness, the Smirnov metrization theorem, the fundamental group and covering spaces, homotopy of paths, the fundamental group, covering spaces, the fundamental group of the circle and the fundamental theorem of algebra.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the significance of nets and filters
- Explain about the topology related to convergence of nets
- Discuss about the Hausdorffness and nets
- Comprehend on the compactness and nets
- Interpret about the filters and their convergence
- Understand the canonical way of converting nets to filters and vice versa
- Comprehend on the ultrafilters and compactness
- Elaborate on the local finiteness
- State the Nagata-Smirnov metrization theorem
- Define the term paracompactness
- Analyse the homotopy of paths
- Discuss about on the fundamental group
- Explain about the covering spaces
- Describe the fundamental group of the circle and the fundamental theorem of algebra.

5.2 NETS AND FILTERS

In general topology and the related branches, a net or Moore-Smith sequence is a generalization or simplification of the concept and notion of a sequence. A sequence is defined as a specific function with domain of the defined set of natural numbers and also the range which is normally defined for any topological space. Even though, as per the perspective of topology, the sequences cannot completely encode the entire information regarding a function that is defined between the said topological spaces. In particular, for a specific definite map f between topological spaces X and Y, the two conditions that are given below are not equivalent:

1. The map f is stated as continuous.
2. Given any point $x \in X$ and any sequence in X which converges to x, then specifically the composition of f with this unique sequence converges to $f(x)$.

The Condition 1 stated above implies Condition 2, in the context of all spaces. However in general, Condition 2 does not implies Condition 1 as topological spaces are, generally, not first-countable. On imposing the first-countability axiom to the topological spaces to be analysed, it can be stated that the above mentioned two conditions can be equivalent. Specifically, these two given conditions are then considered as equivalent for the metric spaces being studied.

The principle and key concept of the term 'Net' was originally proposed by E. H. Moore and H. L. Smith in the year 1922. The term net was specifically used for generalizing the notion of any sequence in order to validate the equivalence of the specified conditions through the sequence that are replaced/changed by means of net in the specified Condition 2. Particularly, instead of defining the net on a countable linearly ordered set, the net can be typically defined on an arbitrary and subjective directed form of set. Consequently, this permits to use theorems that are analogous to the type in order to assert the equivalence of both the specified Condition 1 and Condition 2 for holding the framework of topological spaces which essentially do not have either a countable or a linearly ordered neighbourhood origin about a point. Subsequently, because the sequences cannot encode the sufficient and essential information about the functions between the topological spaces, therefore the nets are typically used for this analysis since the collections of open sets in the specified topological spaces are similar to directed sets.

John L. Kelley finally coined term 'Net' and stated that nets are the distinctive tools that are exceptionally used in topology for generalizing specific concepts, notations, and theories that are sufficiently adequate in the perspective of metric spaces. Similarly, the concept of the filter was defined and established by Henri Cartan in the year 1937.

Since every single non-empty totally ordered set can be directed, therefore, it can be stated that every single function that is precisely defined on such a set is uniquely a 'Net'. Specifically, the natural numbers having the standard and common order typically forms or generates precise set and sequence which can be identified and designated as a function on the natural numbers such that every single sequence can be considered as a net.

Another significant example is given below.

Given a point x in a topological space, let N_x denote the set of all neighbourhoods containing x. Then N_x is defined as a directed set, where the direction is specified through reverse inclusion, such that the form $S \geq \mathbf{T}$ iff S is contained in $\mathbf{T}$. For S in N_x, let x_S be a point in S. Then (x_S) is uniquely a net.

When the S increases or expands with respect to $\geq$, then the points x_S defined in context of the net are precisely constrained or restricted to be in decreasing or reducing neighbourhoods of x, consequently this gives the notion that x_S must precisely tend towards x in certain perception.

NOTES

Nets: In mathematics, more specifically in general topology and related branches, a net or Moore-Smith sequence is a generalization of the notion of a sequence. Given any point x in X, and any sequence in X converging to x, the composition of f with this sequence converges to $f(x)$.

Definition 1: A directed set is defined as a set $\mathcal{D}$ with a relation $\leq$ on it satisfying the following:

1. $\leq$ is Reflexive.
2. $\leq$ is Transitive.
3. $\leq$ is Directed: for all $a, b \in \mathcal{D}$ there exists $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$.

Definition 2: A net in defined a set X which is a map $\lambda: \mathcal{D} \to X$. If X is a topological space, then we state that the net λ converges to $x \in X$ and write $\lambda \to X$ if and only if for every neighbourhood U of x certain specific tail,

$$\Lambda_d = \{\lambda(c) : d \leq c \in \mathcal{D}\} \subseteq U.$$

Now let $\lambda: \mathcal{D} \to \mathrm{X}$ be a net in X with directed set $\mathcal{D}$.

Definition 3: Eventually a net λ is in $A \subseteq X$ if and only if A contains certain tail of λ. We state that a net λ is frequently in A if and only if for every $d \in \mathcal{D}$ there exists $c \geq d$, such that $\lambda(c) \in A$.

Definition 4: If x is a cluster point or an accumulation point of a net λ in X if and only if λ is continuously in every neighbourhood of x.

Definition 5: A net is termed as maximal iff and only iff for every $A \subseteq X$, it is ultimately defined in either A or $X - A$. Maximal form of nets are precisely termed as ultra nets.

Nets are not as completely straightforward a generalization of sequences as one might wish. Consider the obvious naïve definition of a subnet λ' of a net $\lambda: \mathcal{D} \to X$ as the restriction of λ a subset $\mathcal{D}'$ of $\mathcal{D}$.

Definition 6: A subset $\mathcal{D}'$ of a directed set $\mathcal{D}$ is called co final in $\mathcal{D}$ if and only if for every $d \in \mathcal{D}$ there exists $d' \in \mathcal{D}'$ with $d \leq d'$. Given such a co final subset, we may refer to the co final subnet $\lambda' = \lambda/\mathcal{D}'$ of any net $\lambda: \mathcal{D} \to X$.

Sequence in a Topological Space

A sequence $(a_1, a_2, ...)$ in a topological space V can be considered a net in V defined on $\mathbf{N}$. The net is eventually in a subset Y of V if there exists an N in $\mathbf{N}$ such that for every $n \geq N$, the point a_n is in Y.

We have $\lim_{x \to c} a_n = L$ if and only if for every neighbourhood Y of L, the net is eventually in Y. The net is frequently in a subset Y of V iff for every N in $\mathbf{N}$ there exists some $n \geq N$ such that a_n is in Y, i.e., iff infinitely many elements of the sequence are in Y. Thus a point y in V is a cluster point of the net iff every neighbourhood Y of y contains infinitely many elements of the sequence.

Function from a Metric Space to a Topological Space

NOTES

Consider a function from a metric space M to a topological space V and a point c of M. We direct the set $M\setminus\{c\}$ reversely according to distance from c, with the relation 'has at least the same distance to c as', such that 'large enough' with respect to the relation means 'close enough to c'. The function f is a net in V defined on $M\setminus\{c\}$. The net f is eventually in a subset Y of V if there exists $a \in M\setminus\{c\}$ such that for every $x \in M\setminus\{c\}$ with $d(x,c) \le d(a,c)$, the point $f(x) \in Y$. We have $\lim_{x\to c} f(x) = L$ iff for every neighbourhood Y of L, f is eventually in Y. The net f is frequently in a subset Y of V iff for every $a \in M\setminus\{c\}$ there exists some $x \in M\setminus\{c\}$ with $d(x, c) \le d(a, c)$ such that $f(x) \in Y$. A point $y \in V$ is a cluster point of the net f iff for every neighbourhood Y of y, the net is frequently in Y.

Function from a Well-Ordered Set to a Topological Space

Consider a well-ordered set $[0, c]$ with limit point c and a function f from $[0, c)$ to a topological space V. This function is a net on $[0, c)$. It lies finally in a subset Y of V if there exists $a \in [0, c)$ such that for every $x \ge a$, the point $f(x) \in Y$. We have $\lim_{x\to c} f(x) = L$ iff for every neighbourhood Y of L, f is eventually in Y. The net f is frequently present in a subset Y of V iff for every $a \in [0, c)$ there exists some $x \in [a, c)$ such that $f(x) \in Y$. A point $y \in V$ is a cluster point of the net f iff for every neighbourhood Y of y, the net is frequently in Y.

Filters: Filter is a subfield of mathematics and can be used to study topological spaces and define all basic topological notion such a convergence, continuity, compactness, etc.

Definition 1: A filter is non-empty collection $\mathcal{F}$ of subsets of a topological space X such that:

1. $\phi \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A maximal filter is also called an ultra filter.

Definition 2: A filter $\mathcal{F}$ is said to converge to $x \in X$, denoted by $\mathcal{F} \to x$, if and only if every neighbourhood of x belongs to $\mathcal{F}$.

$\mathcal{N}_x$ is a sub filter of $\mathcal{F}$.

Definition 3: If $x \in \overline{F}$ for every $F \in \mathcal{F}$, we call x a cluster point or an accumulation point of $\mathcal{F}$.

Theorem 5.1: Any collection of sets $\mathcal{F}'$ satisfying the finite intersection property is contained in an ultra filter $\mathcal{U}$.

NOTES

Proof: Partially order, by inclusion, the set of all collections satisfying finite intersection property, which contain $\mathcal{F}$. Then each chain has its union as an upper bound, so there exists a maximal element $\mathcal{U}$. We claim that $\mathcal{U}$ is a filter; hence, an ultra filter as it is maximal. Indeed, the first and third properties of a filter are true by definition, so we need to consider only the second. But this follows from maximality.

Theorem 5.2: A filter $\mathcal{U}$ is maximal if and only if every set A that intersects every member of $\mathcal{U}$ (non emptily) belongs to $\mathcal{U}$.

Theorem 5.3: If $\mathcal{F} \to x$, then x is a cluster point of $\mathcal{F}$. Conversely, if x is a cluster point of an ultra filter $\mathcal{U}$, then $\mathcal{U} \to x$.

Proof: If $\mathcal{F} \to x$, then $\mathcal{N}_x \subseteq \mathcal{F}$. If $A \in \mathcal{F}$, then $A \cap \mathbf{N} \neq \phi$ for every $\mathbf{N} \in \mathcal{N}_x$. Thus $x \in \overline{A}$.

Conversely, if x is a cluster point of an ultra filter $\mathcal{U}$, then it follows from definition that each neighbourhood $\mathbf{N}$ of x intersects (non-emptily) every member of $\mathcal{U}$. By the previous Theorem 5.2, $\mathbf{N} \in \mathcal{U}$. Therefore $\mathcal{N}_x \subseteq \mathcal{U}$ and $\mathcal{U} \to x$.

Definition 1: A filter base of a filter $\mathcal{F}$ is a sub collection $\mathcal{B}$ such that for every $F \in \mathcal{F}$ there exists $B \in \mathcal{B}$ with $B \subseteq F$.

Definition 2: A filter base $\mathcal{B}$ in X is said to converge to $x \in X$ iff for every neighbourhood $\mathbf{N} \in \mathcal{N}_x$, there exists $B \in \mathcal{B}$ with $B \subseteq \mathbf{N}$ and we write $\mathrm{B} \to x$.

In mathematics, a filter is defined to be a special subset of a partially ordered set. A frequently used special case is the situation that the ordered set under consideration is just the power set of some set and ordered by set inclusion. Filters emerge in order and lattice theory, but can also be found in topology from where they originate. The dual notion of a filter is an ideal.

A non-empty subset F of a partially ordered set $(P, \leq)$ is a *filter* if the following conditions hold:

1. For every x, y in F, there is some element z in F, such that $z \leq x$ and $z \leq y$. (F is a filter base).
2. For every x in F and y in P, $x \leq y$ implies that y is in F (F is an *upper set*).
3. A filter is proper if it is not equal to the whole set P. This condition is sometimes absent from the definition of a filter.

Even though the definition given above is considered as the most common and conventional method for defining a filter for arbitrary or random posets, but originally it was precisely defined only for lattices. In this case, the above described definition can be characterized and distinguished by means of the equivalent statement given below:

The lattice $(P, \leq)$ having a non-empty subset F is defined as a filter iff and only iff it is an upper set which is typically closed under finite meets termed as infima, i.e., for all x, y in F, we can state that $x \wedge y$ is also in F. Additionally, the

NOTES

smallest filter which comprises of a given element p is termed as the principal filter and also p is defined as the principal element in this state. The principal filter for p is simply given by the set $\{x \in P \mid p \leq x\}$ and is precisely denoted by prefixing p with an upward arrows denoted as $\uparrow p$. The dual notion of a filter, i.e., the concept and notation obtained by reversing all $\leq$ and then exchanging $\wedge$ with $\vee$ is considered as *ideal*.

Consequently, in topological analysis the term filters are precisely used for defining the convergence in a precise manner analogous to the function and specifications of sequences on the basis of a metric space. Furthermore, in topology and other concerned fields of mathematical analysis, a filter is uniquely defined as a generalization or interpretation of a net. Additionally, both nets and filters provide extremely general perspectives and framework for unifying the different notions of limit precisely for the topological spaces that are arbitrary in type.

Subsequently, the term sequence is typically indexed by means of the natural numbers which are precisely referred as a totally ordered set. Therefore, in the first-countable spaces the limits can be specifically described by means of sequences. Although, if the said space does not have properties of first-countable, then in this case specifically the nets or filters should be used. Nets characteristically generalize the notion and concept of a sequence simply involving the index set to be a directed set. Filters can be further considered as the specific types of sets precisely developed from the multiple nets. Therefore, the notion of limit of a filter and also the notion of limit of a net both are conceptually and theoretically defined same as the notion limit of a sequence.

5.3 TOPOLOGY AND CONVERGENCE OF NETS

Let $\mathcal{F}$ be a filter and let $\mathcal{D}$ be a set that is bijective with $\mathcal{F}$. We shall call $\mathcal{D}$ an index set for $\mathcal{F}$ and denote the bijective correspondence by subscript labeling: $\mathcal{F} = \{F_d: d \in \mathcal{D}\}$.

Definition 1: Let $\mathcal{F}$ be an indexed filter in X with index set $\mathcal{D}$. Any net $\lambda: \mathcal{D} \to X$ with $\lambda(d) \in F_d$ is called a **derived net** of $\mathcal{F}$.

Definition 2: Let λ be a net in X with directed set $\mathcal{D}$. Then $\mathcal{F} = \{F \subseteq X : \lambda$ is eventually in $F\}$ and is called the derived filter of λ.

Theorem 5.4: A filter $\mathcal{F}$ in X converges to $x \in X$ if and only if every derived net λ does.

Proof: Assume $\mathcal{F} \to X$ and index $\mathcal{F}$ with an index set $\mathcal{D}$. If N is any neighbourhood of x, then $N = F_d \in \mathcal{F}$. Now, if $c \geq d$ then $F_c \subseteq F_d$, so $\lambda(c) \in F_c \subseteq N$ and λ is eventually in N. Thus, $\lambda \to x$.

Conversely, if $\mathcal{F} \to x$ then there exists some neighbourhood N of x such that $F_d \neq N$ for every $F_d \in \mathcal{F}$. Choose any net λ with $\lambda(d) \in F_d - N$ for each $d \in \mathcal{D}$. Then λ is a derived net of $\mathcal{F}$ and does not converge to x.

Theorem 5.5: A net $\lambda: \mathcal{D} \to X$ converges to $x \in X$ iff and only iff the derived filter $\mathcal{F}$ exists.

Proof: If $\lambda \to X$, then λ ultimately exist in each neighbourhood N of x. Consequently, each N belongs to the derived filter $\mathcal{F}$, therefore $\mathcal{N}_x \subseteq \mathcal{F}$ and $F \to X$.

Conversely, if the derived filter $\mathcal{F} \to X$ then each neighbourhood N of x belongs to $\mathcal{F}$, so λ is ultimately in N. Therefore, $\lambda \to X$.

Finally, we show that closure in any topological space is completely determined by convergence of nets or filters.

Theorem 5.6: Given a topological space X and a subset A of X, a point x is in $\overline{A}$ if and only if there is a net (equivalently, a filter) in A converging to x.

Proof: According to the two preceding results, sufficiently and essentially the nets are considered. Considering $V = X - \overline{A}$, it will be sufficient to prove that V is open iff and only iff there is no limit of a convergent net λ in $X - V = \overline{A}$ that lies in V.

When V is open, then typically V is defined as a neighbourhood of each $x \in V$. Therefore, no tail of λ is contained in V, and hence λ cannot converge to any $x \in V$.

Conversely, consider that for every neighbourhood U of certain $x \in V$ there typically exists a point of the net which is precisely denoted by $\lambda_U \in U - V$. Then $\lambda \to X$ but λ was assumed to be a net in $X - V$, a contradiction. Therefore, there exists some neighbourhood U of x with $U - V = \phi$, whence $U \subseteq V$ so V is a neighbourhood of x for every $x \in V$ and V is open.

5.4 HAUSDORFFNESS AND NETS

Theorem 5.7: Let X be a Hausdorff topological space. Then no net in X can have two different limits.

Proof: Let X be a Hausdorff topological space and $(x_\lambda)_{\lambda \in \wedge}$ be a net in X. Suppose $a, b \in X$ and we have both $x_\lambda \to a$ and $x_\lambda \to b$. We claim $a = b$ must hold.

If not, by the Hausdorff property of X, there are $U, V \subset X$ open, disjoint with $a \in U$ and $b \in V$. By the definition of $x_n \to a$, there is $\lambda_a \in \Lambda$ so that $x_\lambda \in U \ \forall \lambda \geq \lambda_a$. By the definition of $x_\lambda \to b$, there is $\lambda_b \in \Lambda$ so that $x_\lambda \in B \ \forall \lambda \geq \lambda_b$. Since Λ is directed, there is $\lambda \in \Lambda$ with $\lambda \geq \lambda_a$ and $\lambda \geq \lambda_b$. Then both $x_\lambda \in U$ and $x_\lambda \in V$ hold. But this leads to the contradiction $x_\lambda \in U \cap V = \theta$.

Theorem 5.8: Let X be a topological space in which there is no net with two different limits. Then X is Hausdorff.

Proof: Take $a, b \in X$ with $a \neq b$. Our claim is that there exist U and V open with $a \in U$, $b \in V$ and $U \cap V \neq \phi$. If not, we have $U \cap V \neq \phi$ for any open $U, V \subseteq X$ where $a \in U$, $b \in V$.

NOTES

Now find a net with two limits (which will be a and b). We take for the index set $\Lambda = \{(U,V): U,V \subseteq X \text{ open}, a \in U, b \in V\}$ where we define the order $(U_1,V_1) \le (U_2,V_2)$ to mean that both $U_1 \supseteq U_2$ and $V_1 \supseteq V_2$ hold. It is straightforward to check that Λ is then a directed set. The least trivial part is to show that $(U_1,V_1),(U_2,V_2) \in \Lambda$ implies there is $(U_3,V_3) \in \Lambda$ with $(U_1,V_1) \le (U_3,V_3)$ and $(U_2,V_2) \le (U_3,V_3)$. We just take $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$ and check U_3 and V_3 open, $a \in U_3$, $b \in V_3$ and the four containments we need.

For each $\lambda = (U,V) \in \Lambda$ choose one $x_\lambda \in U \cap V$. Then $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X and we claim both $x_\lambda \to a$ and $x_\lambda \to b$.

To verify $x_\lambda \to a$, take a neighbourhood $N_a \in \mathcal{U}_a$. Then $\lambda_0 = ((N_a)^\circ, X) \in \Lambda$. If $\lambda = (U,V) \in \Lambda$ satisfies $\lambda \ge \lambda_0$, then $x_\lambda \in U \cap V \subseteq U \subseteq (N_a)^\circ \subseteq N_a$. To verify $x_\lambda \to b$ we argue in a similar way but take $\lambda_0 = (X,(N_b)^\circ)$.

5.5 COMPACTNESS AND NETS

Definition: We say that a net $\{x_\lambda\}$ has $x \in X$ as a cluster point if and only if for each neighbourhood U of x and for each $\lambda_0 \in \Lambda$ there exist some $\lambda \ge \lambda_0$ such that $x_\lambda \in U$. In this case we say that $\{x_\lambda\}$ is cofinally (or frequently) in each neighbourhood of x.

Theorem 5.9: A net $\{x_\lambda\}$ has $y \in X$ as a cluster point if and only if it has a sub net, which converges to y.

Proof: Let y be a **cluster point** of $\{x_\lambda\}$. Define $M := \{(\lambda, U): \lambda \in \Lambda, U$ a neighbourhood of y such that $x_\lambda \in U\}$ and order M as follows: $(\lambda_1,U_1) \le (\lambda_2,U_2)$ if and only if $\lambda_1 \le \lambda_2$ and $U_2 \subseteq U_1$. This is easily verified to be a direction on M. Define $\varphi: M \to \Lambda$ by $\varphi(\lambda,U) = \lambda$. Then φ is increasing and cofinal in Λ, so φ defines a subnet of $\{x_\lambda\}$. Let U_0 be any neighbourhood of y and find $\lambda_0 \in \Lambda$ such that $x_{\lambda 0} \in U_0$. Then $(\lambda_0,U_0) \in M$, and moreover, $(\lambda,U) \ge (\lambda_0,U_0)$ implies $U \subseteq U_0$, so that $x_\lambda \in U \subseteq U_0$. It follows that the subnet defined by φ converges to y.

Suppose $\varphi: M \to \Lambda$ defines a subnet of $\{x_\lambda\}$ which converges to y. Then for each neighbourhood U of y, there is some u_U in M such that $u \ge u_U$ implies $x_{\varphi(u)} \in U$. Suppose a neighbourhood U of y and a point $\lambda_0 \in \Lambda$ are given. Since $\varphi(M)$ is cofinal in Λ, there is some $u_0 \in M$ such that $\varphi(u_0) \ge \lambda_0$. But there is also some $u_U \in M$ such that $u \ge u_U$ implies $x_{\varphi(u)} \in U$. Pick $u^* \ge u_0$ and $u^* \ge u_U$. Then $\varphi(u^*) = \lambda^* \ge \lambda_0$, since $\varphi(u^*) \ge \varphi(u_0)$, and $x_{\lambda^*} = x_\varphi(u^*) \in U$, since $u^* \ge u_U$. Thus for any neighbourhood U of y and any $\lambda_0 \in \Lambda$, there is some $\lambda^* \ge \lambda$ with $x_{\lambda^*} \in U$. It follows that y is a cluster point of $\{x_\lambda\}$.

Theorem 5.10: A topological space X is compact if and only if every net on X has a convergent subnet on X.

Proof: Assume that X is compact and suppose that we have a net $\{x_\lambda\}$ that does not has any convergent subnet. Hence, using the previous Theorem 5.9, the net $\{x_\lambda\}$ does not has cluster points. This means that for each $x \in X$ we can find a neighbourhood U_x of x and an index λ_x such that $x_\lambda \notin U_x$ for every $\lambda \geq \lambda_x$. Since X is compact then there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1} U_{xi}$. Take any $\lambda \geq \lambda_{x1}, \lambda_{x2}, \ldots, \lambda_{xn}$. Then $x_\lambda \notin X$ which is contradiction.

NOTES

Assume that every net on X has a convergent subnet on X. We will show that X is compact. To this end take a family $\mathcal{F} = \{F_i : i \in \mathbf{I}\}$ of closed subnets of X with the finite intersection property, that is $F_{i1} \cap F_{i2} \cap \ldots \cap F_{in} \neq \phi$ for every $\{i_1, i_2, \ldots, i_n\} \subseteq \mathbf{I}$. We will prove that $\bigcap_{i \in I} F_i \neq \phi$. Define a net as follows: Let $\Lambda = \{\{i_1, i_2, \ldots, i_n\} : i_1, i_2, \ldots, i_n \in \mathbf{I} \text{ and } n \in \mathbf{N}\}$.

And order Λ as follows: $\lambda_1 = \{i_1, i_2, \ldots, i_k\} \leq \lambda_2 = \{j_1, j_2, \ldots, j_n\}$ if and only if $\{i_1, i_2, \ldots, i_k\} \subseteq \{j_1, j_2, \ldots, j_n\}$. This is easily verified to be a direction on Λ. Since the family F has the finite intersection property then for every $\lambda = \{i_1, i_2, \ldots, i_n\} \in \Lambda$ we can find $x_\lambda \in F_{i1} \cap F_{i2} \cap, \ldots, \cap F_{i_n}$. Using our hypothesis, the net $\{x_\lambda\}$ has a convergent subnet, let say $\{x_{\lambda m}\}$. That is, there exists $x \in X$ such that $x_{\lambda m} \to x$. We will show prove $x \in F_i$ for all $i \in \mathbf{I}$. Fix some F_i. Hence, there exists m_0 such that $\lambda_{m0} \geq \{i\}$. Thus, for every $\lambda_m = \{i_1, i_2, \ldots, i_n, i\} \geq \lambda_{m0} \geq \{i\}$ we have that $x_{\lambda m} \in F_{i1} \cap F_{i2} \cap \ldots \cap F_{in} \cap F_i \subseteq F_i$. Since $x_{\lambda m} \to x$ and F_i is closed then $x \in F_i$. This completes the proof of the theorem.

Check Your Progress

1. What is net?
2. Define topological space.
3. Define the term ultra net.
4. What do you understand by filter?
5. What are the uses of filter?
6. What is drive filter of λ?
7. When X is Hausdorff space?
8. When a net can have a cluster point?

5.6 FILTERS AND THEIR CONVERGENCE

Let X be a topological space and consider a point $x \in X$. Recall that we define a set V to be a neighbourhood of x if there is an open set U such that $x \in U \subseteq V$.

Let N_x be the set of all neighbourhoods of point x. It is trivial to verify the following properties:

1. $X \in N_x$
2. If $V \in N_x$ and $V \subseteq W$ then $W \in N_x$

NOTES

3. If U, $V \in N_x$ then $U \cap V \in N_x$
4. $\phi \notin N_x$

If F is a collection of subsets of X which precisely satisfy the properties defined above, then we consider it a filter. In addition, N_x is called the neighbourhood filter of x. Remember that filters are closed under finite intersection as well as pairwise (by induction).

The significant example of a filter is the cofinite filter. Let X be an infinite set; then $F = \{U \subseteq X : X \backslash U \text{ is finite}\}$.

Now, we will consider the methods for generating filters. It can be easily checked that if C is a collection of filters on X then $\bigcap C$ is also a filter on X. Therefore, if there is a filter which continues a collection of subsets of X, G then there exists atleast such filter F. We will then state that filter F is generated by G.

Clearly if there exist $U_1, \ldots, U_n \in G$ with $\bigcap_{i=1}^{n} U_i = \varphi$ then G cannot generate a filter, as the U_i would be in any filter containing G, so the empty set would be in that filter, which is a contradiction.

Theorem 5.11: Let G be a collection of subsets of X such that for all $n \in N$ and $U_1, \ldots, U_n \in G$ we have $\bigcap_{i=1}^{n} U_i \neq \varphi$. Then $F = \{U \subseteq X: \exists\ U_1, \ldots, U_n \in G$ $\bigcap_{i=1}^{n} U_i \subseteq U\}$. F is a filter containing G. Indeed F is the filter generated by G.

Thus, a set G generates a filter iff it satisfies this condition. Such a G is called a subbase for F. If G is closed under finite (pair wise) intersection then it is called a base for F and F takes the simpler form $F = \{U \subseteq X: \exists\ V \in G, V \subseteq U\}$.

Definition: Let F be a filter on X. F is said to be an ultra filter if for all $A \subseteq X$ either $A \in F$ or $A^c \in F$.

For example, if $x \in X$ and $F = \{U \subseteq X: x \in U\}$ then F is an ultrafilter.

Theorem 5.12 (The Ultrafilter Theorem): Let F be a filter on X. Then there is an ultra filter **U** such that $F \subseteq$ **U**.

Proof: The proof is very easy. Consider that the set of all filters on X precisely contains F together with the partial ordering $\subseteq$.

Let C be a chain in this set. $\bigcup C$ is closed under pairwise intersection, and therefore forms a base for a filter. This filter is then defined as an upper bound for C. Hence, every chain has an upper bound, therefore the Zorn's Lemma provides us a maximal element of the set **U**. Consequently it can be easily checked that a maximal filter must be an ultra filter. Therefore, **U** is an ultra filter which contains F.

This does not contradict our original assertion that ultra filters other than the one generated by a single point cannot be described – the use of the axiom of choice means that the ultra filter theorem is highly non-constructive. However, it does prove the existence of such an ultra filter – let F be an ultra filter containing the cofinite filter. Then F cannot be generated by $\{a\}$, as $X\backslash\{a\} \in F$. Such an ultra filter is called free. Note here that any ultra filter which is not free is generated by a singleton, and if there exists an a with $X\backslash\{a\} \notin F$ then $\{a\} \in F$. The complements of singletons generate the cofinite filter, so if F is not generated by a singleton then it contains the cofinite filter and is thus free.

Definition: Let F be a filter and $x \in X$. We say that F converges to x, or that x is a limit of F if $N_x \subseteq F$. We shall write $F \to x$ to mean F converges to x.

Theorem 5.13: Let X be a topological space. X is Hausdorff iff every filter has at most one limit.

Proof: Suppose, X is Hausdorff and let $x \neq y$. Then there are neighbourhood U and V of x and y respectively with $U \cap V = \phi$. Thus no filter contains both U and V, and so no filter can converge to both x and y. Hence, all filters have at most one limit.

Conversely, suppose that x and y do not have disjoint neighbourhoods. Then $N_x \cup N_y$ forms a subbase for a filter which converges to both x and y. Therefore, if every filter has at most one limit then X is Hausdorff.

Therefore, considering X to be Hausdorff is equivalent to consider unique limits, which is a natural condition to impose when we assume that filters are an accurate method of describing limits. In a Hausdorff space we will write lim $F = x$ which means that x is unique limit of F. However remember that all the filters do not have limits.

Let X, Y be sets, F a filter on X and g: $X \to Y$. In general the set $\{g(U)$: $U \in F\}$ will not be a filter on Y. However because $g(U \cap V) \subseteq g(U) \cap g(V)$, it will be a subbase generating some filter. We shall denote the filter it generates by $g(F)$. Clearly, $V \in g(F)$ iff there exists $U \in F$ such that $g(U) \subseteq V$.

Theorem 5.14: Let X, Y be topological spaces with $x \in X$ and g: $X \to Y$. g is continuous at x iff whenever F is a filter such that $F \to x$ we have $g(F) \to g(x)$.

Proof: Suppose g is continuous at x and let $F \to x$. Let V be a neighbourhood of $g(x)$. By continuity there is a neighbourhood U of x such that $g(U) \subseteq V$. But $U \in F$, so $g(U) \in g(F)$. Thus, as $g(F)$ is a filter, $V \in g(F)$. Hence $g(F) \to g(x)$. Then $g(N_x) \to g(x)$ by hypothesis. So for every V, a neighbourhood of $g(x)$, we have $V \in g(F)$. Thus by our observation above there exists $U \in N_x$ such that $g(U) \subseteq V$. As N_x is the set of neighbourhoods of x, g is continuous at x.

5.7 CANONICAL WAY OF CONVERTING NETS TO FILTERS AND VICE VERSA

Theorem 5.15: Let $P : \Lambda \to X$ be a net and **H** the filter associated with it. Let $x \in X$. Then P converges to x as a net iff **H** converges to x as filter. Also x is a cluster point of the net P iff x is a cluster point of the filter **H**.

NOTES

Proof: Suppose P converges to x. Let U be a neighbourhood of x in X. Then $\exists$ $\lambda_0 \in \Lambda$ such that $B_{\lambda_0} = \{P(\lambda); \lambda \in \wedge; \lambda \geq \lambda_0\}$. But this means $U \in$ **H** by the definition of **H**. So every neighbourhood of x belongs to **H**. Hence, **H** converges to x.

Conversely suppose that **H** converges to x. Let U be as open neighbourhood of x. Then $U \in$**H**. Recalling how **H** was generated, there exists $\lambda_0 \in \Lambda$ such that $B_{\lambda_0} \subset U$ where B_{λ_0} is defined as above. This means that $P(\lambda) \in U$ for all $\lambda \in \Lambda$; $\lambda \geq \lambda_0$. Thus P converges to x in X.

Suppose now that the net P clusters at x. Therefore for each neighbourhood U of x and for each $\lambda_0 \in \Lambda$, there is some $\lambda \geq \lambda_0$ such that $P(\lambda) \in U$. But the sets $\{P(\lambda); \lambda \in \wedge; \lambda \geq \lambda_0\}$ form a base for **H**. Let $F \in$ **H**. Then $F \supset \{P(\lambda); \lambda \in \wedge; \lambda \geq \lambda_0\}$, i.e., $P(\lambda) \in F$. Thus $P(\lambda) \in F \cap U$.

Hence, every neighbourhood of x intersects every member of **H**. Hence x is a cluster point of **H**. Conversely suppose that x is a cluster point of **H**, i.e., every neighbourhood U of x intersects every member F of **H**. Thus $F \cap U \neq \phi$. But $F \supset \{P(\lambda); \lambda \in \wedge; \lambda \geq \lambda_0\}$ for some λ_0. Being a superset of $\{P(\lambda); \lambda \in \wedge; \lambda \geq \lambda_0\}$ let F belongs to **H**. Therefore $P(\lambda) \in U$ for $\lambda \geq \lambda_0$.

Theorem 5.16: Let **H** be filter on a space X and P be the associated net in X. Let $x \in X$. Then **H** converges to x as a filter iff P converges to x as a net. Moreover, x is a cluster point of the filter **H** iff it is a cluster point of the net P.

Proof: Suppose $\mathbf{H} \rightarrow x$. If U is a neighbourhood of x, then $U \in$**H**. Pick $p \in U$. Then $(p, U) \in \Lambda_{\mathbf{H}}$ and if $(q, F) \geq (p, U)$, then $q \in F \subset U$. Thus for each neighbourhood U of x, there exists $(p, U) \in \Lambda_{\mathbf{H}}$ such that $(q, F) \geq (p, U)$ implies $P(q, F) = q \in U$. Hence, the net P converges to x.

Conversely, suppose that the net based on **H** converges to x. Let U be a neighbourhood of x. Then for some $(p_0, F_0) \in \Lambda_{\mathbf{H}}$ we have $(p, F) \geq (p_0, F_0)$ implies $P(p, F) = p \in U$. But then $F_0 \subset U$, otherwise there is some $q \in F_0 - U$ and then $(q, F_0) \geq (p_0, F_0)$ but $q \notin U$. Hence U being a super set of the members (F_0) of **H**, belongs to **H**. Hence every neighbourhood U of x belongs to **H**. Hence $\mathbf{H} \rightarrow x$.

Now we come to the result concerning cluster points.

Suppose first **H** has x as a cluster point. Recall that the associated net $P : \Lambda \rightarrow X$ is defined by taking $\Lambda = \{(y, F); F \in \mathbf{H}, y \in F\}$ and putting $P(y, F) = y$. Let an open neighbourhood U of x and an element (y, F) of Λ be given. Then $F \cap U \neq \phi$ by definition of the cluster point of a filter. Let $z \in F \cap U$. Then $(z, F) \in \Lambda$, $(z, F) \geq (y, F)$ and $P(z, F) = z \in U$.

Thus to each neighbourhood U of x and $(y, F) \in \Lambda$, there is $(z, F) \in \Lambda$ such that $(z, F) \geq (y, F)$ implies $P(z, F) \in U$. Hence x is a cluster point of P. Let U be

NOTES

any open neighbourhood of x and let $F \in$ **H**. We have to show that $F \cap U \neq \phi$. Let z be any point of F. Then $(z, F) \in \Lambda$. Since x is a cluster point of P, there exists $(y, G) \in \Lambda$ such that $(y, G) \geq (z, F)$ and $P(y, G) \in U$. But then $y \in G$, $G \subset F$ and $y \in U$ ($\therefore$ $P(y, G) = y$) showing that $y \in F \cap U$ and so $F \cap U \neq \phi$. Hence every neighbourhood of x intersects every member of **H**, i.e., x is a cluster point of **H**.

Theorem 5.17: A topological space is Hausdorff iff no filter can converge to more than one point in it.

Proof: Suppose X is a Hausdorff space and a filter **H** converges to x as well as y. This means $\mu_x \subset$ **H** and $\mu_y \subset$ **H**. Now if $x \neq y$, then there exist $U \in \mu_x$ and $V \in \mu_y$ such that $U \cap V \neq \phi$ which contradicts the fact that **H** has the finite intersection property. So $x = y$. Thus limits of convergent filters in X are unique.

Conversely, suppose that no filter in X has more than one limit in X. If X is not Hausdorff, there exists $x, y \in X$, $x \neq y$ such that every neighbourhood of x intersects every neighbourhood of y. From this it follows that the family $\mu_x \cap \mu_y$ has finite intersection property. Evidently **H** converges both to x and y contradicting the hypothesis. So X is Hausdorff.

Theorem 5.18: For a topological space X, the following statements are equivalent:

(i) X is compact.

(ii) Every filter on X has a cluster point in X.

(iii) Every filter on x has a convergent sub filter.

Proof: (ii) $\Leftrightarrow$ (iii) has been shown in the definition of sub filter.

Moreover, if a filter **H** has a cluster point x in X, then the net based on **H** also has cluster point x in X. Therefore (i) $\Leftrightarrow$ (ii) follows from the result already proved for nets.

Theorem 5.19: A topological space is compact iff and only iff every ultrafilter in it is of convergent type.

Proof: When a space is defined as compact, then every filter contained in it holds a cluster point. Specifically, every ultrafilter holds a cluster point and consequently it is termed convergent by a consequence or result, i.e., an ultrafilter typically converges precisely to a point iff that point is exactly a cluster point of it.

Conversely, assume that X is a space having the property that every ultrafilter on this space is precisely convergent. It can be appropriately shown that every filter on X holds a cluster point because only then the result will follow as per the above mentioned Theorem 5.19. Suppose **H** is a filter on X. Therefore $\exists$ an ultra filter G containing **H**. By hypothesis, y converges to a point, say x on X. Then x is a cluster point of G. So every neighbourhood of x meets every member of G and in particular every member of **H**, therefore, since **H** $\subset G$. Consequently, x is also a cluster point of **H**. Thus every filter on X has a cluster point in X. Hence X is compact.

Now we are in a position to prove the following theorem which is typically the characterization of compact sets.

NOTES

Theorem 5.20: For a topological space X, the following are equivalent:

(*i*) X is compact.

(*ii*) Each family C of closed sets in X with the finite intersection property has non-empty intersection.

(*iii*) Each filter in X has a cluster point.

(*iv*) Each net in X has a cluster point.

(*v*) Each ultra net in X converges.

(*vi*) Each ultra filter in X converges.

Proof: (*i*) $\Rightarrow$ (*ii*). Let X be compact. Suppose on the contrary that $\{F_\lambda\}$ is a family of closed sets in X having empty intersection, i.e., $\bigcap_\lambda F_\lambda = \varphi$

$$\Rightarrow \left(\bigcap_\lambda F_\lambda\right)^C = \varphi^C$$

$$\Rightarrow \bigcup_\lambda F_\lambda^C = X$$

Thus $\{F_\lambda^C\}$ is a converging of X. Then by compactness of X, there must be a finite sub covering of X, i.e.,

$$X = \bigcup_{i=1}^{n} F_i^C$$

But the $\phi = X^c = (\bigcup_{i=1}^{n} F_i^C)^C = \bigcap_{i=1}^{n} F_i$

so that the family cannot have the finite intersection property. This contradiction proves that (*ii*) holds.

(*ii*) $\Rightarrow$ (*iii*). If $\mathbf{H}$ is a filter on X, then $\{\overline{F}\,;\, F \in H\}$ is a family of closed sets with the finite intersection property, so (*ii*) implies that there is a point x in $\cap$ $\{\overline{F}\,;\, F \in H\}$, i.e., each neighbourhood of x intersects every member of $\mathbf{H}$. Hence, x is a cluster point of the filter $\mathbf{H}$.

(*iii*) $\Rightarrow$ (*iv*). Suppose that the filter $\mathbf{H}$ has x as a cluster point. We know that net $P : \Lambda \to X$ based on $\mathbf{H}$ is defined by taking $\Lambda = \{(y, F); F \in \mathbf{H}; y \in F\}$ and putting $P(y, F) = y$. Let an open neighbourhood U of x and an element (y, F) of Λ be given. Then $F \cap U \neq \phi$ because x is a cluster point iff every neighbourhood U of x meets every member F of $\mathbf{H}$.

$Z \in F \cap U$. Then $(z, F) \in \Lambda$;

$(z, F) \geq (y, F)$ and $P(z, F) = z \in U$.

Thus to each neighbourhood U of x and $(y, F) \in \Lambda$ there is $(z, F) \in \Lambda$ such that $(z, F) \geq (y, F)$ implies $P(z, F) \in U$. Hence, x is a cluster point of the net P.

$(iv) \Rightarrow (iii)$. If an ultra net has a cluster point then it converges to that point.

$(v) \Rightarrow (vi)$. Let **H** be an ultra filter on X. Then the neighbourhood on **H** is an ultra net which converges by (v). But the limits are preserved in satiating over from ultra nets to ultra filters. Hence **H** is convergent.

$(vi) \Rightarrow (i)$. Suppose μ is an open cover of X with no finite sub cover. Then $X - (U_1 \cup U_2 \cup, \ldots, \cup U_n) \neq \phi$ for each finite collection $\{ U_1, \ldots, U_n\}$ from μ the sets of the form $X - (U_1 \cup U_2 \cup, \ldots, \cup U_n)$. Then form a filter base (since the intersection of two such sets has again the same form) generating a filter **H**. Now since each filter is contained in an ultra filter. It follows that **H** is contained in some ultra filter **H***. But **H*** converges to x by (f). Now $x \in U$ for some $U \in \mu$. Since U is a neighbourhood of x, by the definition of convergence ultra filter, we have $U \in$ **H***. By the construction, $X - U \in$ **H** $\subset$ **H***. Since it is impossible for both U and $X - U$ to belong to an ultra filter, we have a contradiction. Thus, μ must have a finite sub cover and hence X is compact.

5.8 ULTRAFILTERS AND COMPACTNESS

Definition 1: A filter on a set X is a collection $\mathcal{F}$ of subsets of X satisfying:

1) $X \in \mathcal{F}$, but $\phi \neq \mathcal{F}$.
2) If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.
3) A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$: if $A_{12} \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

Definition 2: Let X be a topological space, $\mathcal{F}$ a filter on X, and x a point in X. We say that $\mathcal{F}$ converges to x and write $\mathcal{F} \to x$ if every neighbourhood of x is in $\mathcal{F}$. If $\mathcal{F}$ converges to exactly one point x of X, then we will call that point the limit of $\mathcal{F}$ and write $x = \lim \mathcal{F}$.

Theorem 5.21: A filter on a Hausdorff space X may converge to at most one point in X.

Definition: Let X and Y be sets, $f : X \to Y$ be any function, and let $\mathcal{F}$ be a filter on X. The collection $f_*\mathcal{F} = \{A \subset Y: f^{-1}(A) \in \mathcal{F}\}$ is a filter on Y, called the push forward of the filter $\mathcal{F}$ via the map f.

Theorem 5.22: Let X and Y be topological spaces. A function $f : X \to Y$ is continuous iff whenever a filter $\mathcal{F}$ on X converges to a point $x \in X$, the filter $f_*\mathcal{F}$ on Y converges to $f(x)$.

Definition: An ultra filter on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion, i.e., it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F}' \supset \mathcal{F}$ actually satisfies $\mathcal{F}' = \mathcal{F}$.

Theorem 5.23: Every filter is contained in some ultra filter.

NOTES

Theorem 5.24: The following are equivalent for a filter $\mathcal{F}$ on a set X:

1) $\mathcal{F}$ is an ultra filter.
2) For every set $A \subset X$ either $A \in \mathcal{F}$ or $A^c = X - A \in \mathcal{F}$.
3) For every finite cover $\{A\}_{i=1}^{n}$ of a set $A \in \mathcal{F}$, $A_i \in \mathcal{F}$ for some i.

Theorem 5.25: A topological space X is compact iff every ultra filter on X is convergent.

Theorem 5.26: If $\mathcal{F}$ is an ultra filter on a set X and $f: X \to Y$ is a function, then $f_*\mathcal{F}$ is also an ultra filter.

Theorem 5.27: There exists a functional l: $l^\infty \to \mathbf{R}$ (called a generalized limit) satisfying:

1) l is defined on all bounded sequences.
2) If (x_n) is a sequence whose limit exists in the usual sense, then $l((x_n)) = \lim_{n\to\infty} x_n$.

l is linear and multiplicative, whenever (x_n) and (y_n) are bounded sequences, and a and b are real numbers, $l((ax_n + by_n)) = al\ ((\ ax_n)) + bl\ ((y_n))$ and l (($x_n y_n$))) = $l((x_n))\ l((y_n))$.

Theorem 5.28: Non-standard models of first order arithmetic (models containing infinite integer and like creatures) exist.

Theorem 5.29 (Tychonoff's Theorem): If X_α is a compact topological space for every α in some arbitrary index set **I**, then $\Pi_{\alpha\in\mathbf{I}} X_\alpha$ is compact in the product topology.

Definition: Let X be a T_2 topological space. A Stone-Čech compactification of X is a compact T_2 topological space βX containing X so that:

1) The topology induced on X as a subset of βX is the original topology of X.
2) Whenever $f: X \to Y$ is a continuous map of X into some compact T_2 space Y, there exists a unique continuous map $\tilde{f}: \beta X \to Y$ whose restriction to X is f.

Note: A rather non-trivial theorem says that if βX is a Stone-Čech compactification of X, then X is dense in βX, namely, the closure of X in βX is all of βX.

Theorem 5.30: Any two Stone-Čech compactifications of the same topological space X are homomorphic.

For simplicity, we will work below only with the space $X = \mathbf{N}$ the natural numbers with the discrete topology. The results in this section all have analogues for an arbitrary completely regular topological space, and in particular, for an arbitrary metric space.

NOTES

Definition: Let $\beta\mathbf{N}$ be the set of all ultra filters on **N**. We will identify **N** as a subset of $\beta\mathbf{N}$ by identifying every integer n with the principal ultra filter μ_n at n.

Theorem 5.31: There is a topology on $\beta\mathbf{N}$ for which it is a Stone-Cech compactification of **N**. A basis for that topology is given by $\mathcal{B} = \{U_A : A \subset \mathbf{N}\}$, where for any set $A \subset \mathbf{N}$, $U_A = \{\mu \in \beta\mathbf{N} : A \in \mu\}$.

Note: All the sets U_A are actually clopen in $\beta\mathbf{N}$.

Theorem 5.32: **N** is dense in $\beta\mathbf{N}$.

Check Your Progress

9. What is an ultrafilter?
10. State the ultrafilter theorem.
11. Give the equivalent statements for a topological space X.
12. Define the limit of a filter.

5.9 LOCAL FINITENESS

According to the mathematical analysis in the field of topology, the term local finiteness can be defined as a unique property which satisfies the property of collections of subsets precisely for a topological space. Fundamentally these properties are used for studying the paracompactness and topological dimension. A collection of subsets that are contained in a topological space X are precisely defined as locally finite, if each point in the space uniquely has a neighbourhood which only intersects finitely several sets in the collection. Consequently, for a topological space a finite collection of subsets is termed as locally finite. Further, the infinite collections of sets can also be locally finite. For example, the collection of all subsets of **R** of the form $(n, n + 2)$ with integer n is locally finite. A countable collection of subsets need not be locally finite, as shown by the collection of all subsets of **R** of the form $(-n, n)$ with integer n. If a collection of sets is locally finite, then the collection of all closures of these sets is also locally finite. The converse of this, however, can fail if the closures of the sets are not distinct. For example, in the finite complement topology on **R**, the collection of all open sets is not locally finite, but the collection of all closures of these sets is locally finite as the only closures are **R** and the empty set.

No infinite collection of a compact space can be locally finite. Consider $\{G_a\}$ to be an infinite family of subsets of a space and suppose this collection is locally finite. For each point x of this space, choose a neighbourhood U_x that intersects the collection $\{G_a\}$ at only finitely many values of a. Clearly, $\bigcup_x$ for each $x \in X$ is an open covering in X and hence has a finite subcover, $U_{a1} \cup *, ... \cup *, U_{an}$. Since each U_{ai} intersects $\{G_a\}$ for only finitely many values of a, the union of all such U_{ai} intersects the collection $\{G_a\}$ for only finitely many values of a. It therefore follows that X intersects the collection $\{G_a\}$ at only finitely many values of a thus contradicting the infinite cardinality of the collection $\{G_a\}$.

NOTES

A topological space in which every open cover admits a locally finite open refinement is called paracompact. Every locally finite collection of subsets of a topological space X is also point-finite. A topological space in which every open cover admits a point-finite open refinement is termed as metacompact. A collection in a space is *countably locally finite* (or σ-locally finite) if it is the union of a countable family of locally finite collections of subsets of X. Countable local finiteness is a significant hypothesis in the Nagata-Smirnov metrization theorem, which states that a topological space is metrizable iff it is regular Hausdorff and has a countably locally finite basis.

Definition 1: A collection μ of subsets of X if called locally finite or (neighbourhood finite) if and only if for every $x \in X$ has a neighbourhood meeting only finite $\cup \in \mu$. We call μ point finite if and only if each $x \in X$ belongs to only finitely many $\cup \in \mu$. Clearly every locally finite collection is point finite.

Notes:

1. Since every point-finite may not be locally finite, e.g., consider the covering of the set **R**. $\{\{X\}; X \in \mathbf{R}\}$ which is a covering of **R**. It is a defined as point finite covering of **R** but it is not locally finite. Because in general topology of **R**, every point has neighbourhood (an open interval) which contains uncountable number of points of **R**. Therefore, every point do not has neighbourhood which intersects finite number of members of the covering. However when the discrete topology of **R** is considered, then above covering is also locally finite. Thus for any space X, $\{\{X\}; x \in X\}$ there is a pointfinite cover, which is locally finite only under stringent conditions on X.
2. The covering of **R** by the sets $[n, n+1]$, as n ranges through all integers is point finite.

Definition 2: A collection μ of subsets of X is discrete if and only if each $x \in X$ has a neighbourhood meeting on at most one element of μ. Clearly every discrete collection of sets is locally finite.

Lastly, for any property of collection of sets in X, there is a corresponding σ-property which we illustrate with an example.

A collection v of subsets of X is called σ-locally finite iff $v = \bigcup_{n=1}^{\infty} V_n$ where each V_n is a locally finite collection in X, i.e., iff v is the countable union of locally finite collections family in X.

In the same way, a collection v of subsets of X is σ-discrete iff v is the countable union of discrete collection in X. If v is a σ-locally finite cover of X, then the sub collections V_n, which are locally finite and make up v will not usually be covers.

Theorem 5.33: If $\{A_\lambda\,;\lambda \in \wedge\}$ is locally finite system of sets in X, then so is $\{\overline{A_\lambda}\,;\lambda \in \wedge\}$.

NOTES

Proof: Since $\{A_\lambda ; \lambda \in \wedge\}$ is a locally finite system of sets, given $p \in X$, find an open neighbourhood $\cup$ of p such that $\cup \cap A_\lambda = \phi$ except for finitely many λ.

$\Rightarrow U \cap \overline{A_\lambda} = \phi$ except for some λ

$U \cap \overline{A_\lambda} \neq \phi$ for some λ

$\Rightarrow \{\overline{A_\lambda} ; \lambda \in \wedge\}$ is locally finite. Hence the theorem is proved.

Theorem 5.34: If $\{A_\lambda ; \lambda \in \wedge\}$ is a locally finite system of sets, then $\cup \overline{A_\lambda} = \overline{\cup A_\lambda}$. In particular, the union of a locally finite collection of closed sets is closed.

Proof: $\Rightarrow$: $\cup \overline{A_\lambda} = \overline{\cup A_\lambda}$ (trivial).

Conversely, suppose $p \in \overline{\cup A_\lambda}$. Now some neighbourhood of p meets only finitely many of the sets A_λ, say $A_{\lambda_1}, ..., A_{\lambda_n}$. Since every neighbourhood of p meets $\cup A_\lambda$, every neighbourhood of p must then meet $A_{\lambda_1}, \cup .... \cup A_{\lambda_n}$.

Hence $p \in A_{\lambda_1} \cup A_{\lambda_2} \cup .... \cup A_{\lambda_n}$ so that for some k, $p \in \overline{A_{\lambda k}}$. Thus $\overline{\cup A_\lambda} \subset \cup \overline{A_\lambda}$ proving the lemma.

Theorem 5.35: For each $\lambda \in \wedge$ $\{\cup \overline{A_\lambda} ; \lambda \in \wedge\}$ is closed.

Proof: Let $B = \cup \{\overline{A_\lambda} ; \lambda \in \wedge\}$. We will show $\overline{B} = B$

$\Rightarrow B$ is closed.

Suppose $x \notin B$, we will show x cannot be a limit point of B.

$x \notin B \Rightarrow x \notin \cup \overline{A_\lambda} \Rightarrow x \notin \overline{A_\lambda} \ \forall \lambda \in \wedge$.

If there is a neighbourhood $\cup$ which is not disjoint for finite number of $\overline{A_\lambda}$'s,

i.e., $U \cap \overline{A_{\lambda_i}}$ for $i = 1, 2, ..., n$, then $X - \overline{A_{\lambda_i}}$ is open neighbourhood of x and so $\bigcap_1^n \{X - \overline{A_\lambda}\}$ is open neighbourhood of x.

Thus $U \cap \{\bigcap_1^n (X - \overline{A_{\lambda_i}})\}$ is neighbourhood of x disjoint with each $\overline{A_\lambda}$ and $U \cap (\bigcap_1^n \{X - \overline{A_{\lambda_i}}\})$ is disjoint with $\cup \overline{A_\lambda} = B$. Thus, x is not a limit point of B.

5.10 THE NAGATA-SMIRNOV METRIZATION THEOREM

NOTES

The Nagata-Smirnov Metrization theorem gives a full characterization of metrizable topological spaces. In other words, the theorem describes the necessary and sufficient conditions for a topology on a space to be defined using some metric. As a motivational example, consider the discrete topology on some space (every subset of the space is open). Though it might not be apparent to the untrained observer, this topology is actually defined by the following metric:

$$d(x, y) = \begin{cases} 1 \text{ when } x \neq y \\ 0 \text{ when } x = y \end{cases}$$

The open balls of radius 1/2 under this metric each contain only a single point (the point around which the ball is centered); using these open balls as a basis, we define the discrete topology. Hidden in the discrete topology is the underlying metric defined above. The Nagata-Smirnov Metrization Theorem lists the exact condition that any topology must have in order for there to be such an underlying metric. Before proving the full metrization theorem, we will start with a more specific result: the characterization of compact metric spaces.

Part 1: We will prove that a topological space X is a compact metric space if and only if X is compact Hausdorff with a countable basis.

We will begin with some relatively simple preliminary results that occur often in the lemmas and theorems to follow. When used, these results will not be cited by name.

Result 1: In a topological space X, suppose A is a compact set and $C \subset A$ is closed. Then C is compact.

Proof: Take any open cover of C. This cover and the open set $X \setminus C$ form an open cover of A. Because A is compact, there is a finite sub cover, which must also cover C, because $C \subset A$. Thus, any open cover of C can be reduced to a finite sub cover, so C is compact.

Result 2: Suppose $f: X \rightarrow Y$ is continuous, and $A \subset X$ is compact. Then $f(A)$ is compact.

Proof: Take an open cover $C = \{U\}$ of $f(A)$. Take $x \in A$. Then $f(x) \in f(A)$, so $f(x) \in U$ for some $U \in C$, so $x \in f^{-1}(U)$. Thus the pre-image of the sets in C, which themselves are open because f is continuous, cover A. Because A is compact, some finite sub cover $f^{-1}(U_1), ..., f^{-1}(U_n)$ covers A. Take $f(x) \in f(A)$. Then $x \in A$, so $x \in f^{-1}(U_i)$ for some i, $1 \leq i \leq n$. Therefore $f(x) \in U_i$, and the open sets $U_1, \ldots, U_n$ cover $f(A)$. Thus any open cover of $f(A)$ can be reduced to a finite sub cover, so $f(A)$ is compact.

Result 3: Suppose X is Hausdorff and $A \subset X$ is compact. Then A is closed.

Proof: To prove that A is closed, we will prove that $X\backslash A$ is open. Take some point $x \in X\backslash A$. For every point $y \in A$, we know that $x \neq y$ because $X\backslash A$ is by definition disjoint from A, so by Hausdorffness there exist disjoint open sets $U(x, y)$ and $V(x, y)$ with $x \in U(x, y)$ and $y \in V(x, y)$. Then $\bigcup_{y \in A} V(x, y)$ is an open cover of A, so because A is compact there is a finite sub cover, $V_1(x),....,V_n(x)$. Each $V_i(x)$ is disjoint from an open set $U_i(x)$ containing x, so $U(x)$ is also disjoint from C, hence $U(x) \subset X\backslash C$. Taking the union of all $U(x)$, for all $x \in X$, must therefore also be contained in $X\backslash C$, but also cover $X\backslash C$; therefore $X\backslash C$ is the union of open sets and hence is open.

NOTES

Result 4: The function $f : X \rightarrow Y$ is continuous if and only if for each $x \in X$ and open set $U \subset Y$ containing $f(x)$, there exists an open set $V \subset X$ such that $x \in V$ and $f(V) \subset U$.

Proof: Suppose first that f is continuous. Take $x \in X$ and an open set U containing $f(x)$. Then $f^{-1}(U)$ is open by continuity, $x \in f^{-1}(U)$, and $f(f^{-1}(U)) = U \subset U$.

Now we will prove the converse. Take an open set $U \subset Y$, and $x \in f^{-1}(U)$. So $f(x) \in U$, therefore there is an open set $V(x) \in X$ containing x with $f(V) \subset U$. Take $y \in V(x)$; then $f(y) \in f(V(x)) \subset U$, so $y \in f^{-1}(U)$. Therefore, $V(x) \subset f^{-1}(U)$. Therefore $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V(x)$, which is open because it is the union of open sets. So when $U \subset Y$ is open, then $f^{-1}(U)$ is open, proving that f is continuous.

Part 2:

Now we will prove the Nagata-Smirnov metrization theorem, if a topological space X is a metric space if and only if X is regular with a countably locally finite basis.

Lemma: Suppose A is a locally finite collection of subsets of a topological space X. Let $Y = \bigcup_{A \in \mathrm{A}} A$. Then $\overline{Y} = \bigcup_{A \in \mathrm{A}} \overline{A}$.

Proof: First, we will show that $\bigcup_{A \in \mathrm{A}} \overline{A} \subset \overline{Y}$, which is generally true. For each $A \in$ A, it is true that $A \subset Y \subset \overline{Y}$. $\overline{A}$ is the intersection of all closed sets containing A and $\overline{Y}$ is a closed set containing A; thus if $x \in \overline{A}$, then $x \in \overline{Y}$, so $\overline{A} \subset \overline{Y}$; thus $\bigcup_{A \in \mathrm{A}} \overline{A} \subset \overline{Y}$ as desired.

Now we will prove that $\overline{Y} \subset \bigcup_{A \in \mathrm{A}} \overline{A}$. Take $x \in \overline{Y}$. By local finiteness, there exists an open neighbourhood U containing x that intersects only a finite subset of elements of A ; Denote these elements as $A_1,\ldots,A_k$. Suppose that x was not contained in any of $\overline{A_1},....,\overline{A_k}$, i.e., $x \notin \bigcup_{j=1}^{k} \overline{A_j}$, which is a closed set. Then $x \in U\backslash\left(\bigcup_{j=1}^{k} \overline{A_j}\right)$, which is an open neighbourhood of x that is disjoint from every element of A. Thus x itself must be disjoint from every element of A, contradicting the fact that $x \in \overline{Y}$. Therefore x must be contained in some $\overline{A}_j$, $1 \le j \le k$, and $\overline{Y} \subset \bigcup_{A \in \mathrm{A}} \overline{A}$, implying that $\overline{Y} = \bigcup_{A \in \mathrm{A}} \overline{A}$.

NOTES

Lemma: Suppose X is a regular space with a countably locally finite basis B. Then X is normal.

Proof: We will prove this in two steps.

Step 1: Suppose $W \subset X$ is open. Then there is a countable collection of open sets $\{U_n\}$ such that $W = \bigcup_{n \in N} U_n = \bigcup_{n \in N} \overline{U_n}$.

Proof: Because B is countably locally finite, $\mathrm{B} = \bigcup_{n \in N} B_n$ where each B_n is a locally finite collection of subsets of X. For each $n \in N$, let $\mathrm{C}_n = \{B \in \mathrm{B}_n \mid \overline{B} \subset W\}$. Then $\mathrm{C}_n \subset \mathrm{B}_n$, so C_n must also be locally finite. Let $U_n = \bigcup_{B \in C_n} B$. Because each B is open, U_n is also open. Furthermore, by above lemma, $\overline{U_n} = \bigcup_{B \in C_n} \overline{B}$, because C_n is locally finite. Each $\overline{B} \subset W$, so $\overline{U_n} = \bigcup_{B \in C_n} \overline{B} \subset W$; therefore, $\bigcup_{n \in N} U_n \subset \bigcup_{n \in N} \overline{U_n} \subset W$.

Now we need to show that $W \subset \bigcup_{n \in N} U_n$. Take $x \in W$. Then $\{x\}$ is disjoint from $X \backslash W$ and both sets are closed ($\{x\}$ is closed by definition of regularity), so by regularity there exist disjoint open sets U and V such that $\{x\} \subset U$ and $X \backslash W \subset V$. Then $x \in \{x\} \subset U \subset X \backslash W \subset V$. Then $x\{x\} \subset U \subset X \backslash W \subset W$. For some $n \in N$, there exists a basis element $B \in \mathrm{B}_n$ such that $x \in B \subset U \subset X \backslash V$; because $X \backslash V$ is closed, $\overline{B} \subset X \backslash V \subset W$. Therefore $B \in \mathrm{C}_n$. This means that $x \in B \subset \bigcup_{n \in N} U_n$. Hence $W \subset \bigcup_{n \in N} U_n$, as desired.

Step 2: X is normal.

Proof: Take disjoint closed subsets $C, D \subset X$. Then $X \backslash D$ is open, so by Step 1, there exists a countable collection of open sets $\{U_n\}$ such that $X \backslash D = \bigcup_{n \in N} U_n = \bigcup_{n \in N} \overline{U_n}$. Of course, every $\overline{U_n}$ is disjoint from D, $C \subset \bigcup_{n \in N} U_n$. By the exact same reasoning, there exists a collection of open sets $\{V_n\}$ that cover D such that each $\overline{V_n}$ is disjoint from C.

$\bigcup_{n \in N} U_n$ and $\bigcup_{n \in N} V_n$ are open covers of C and D, respectively. But we cannot guarantee they are disjoint. For each $n \in N$, let $U'_n = U_n \setminus \left(\bigcup_{j=1}^{n} \overline{V_j}\right)$ and let $V'_n = V_n \setminus \left(\bigcup_{j=1}^{n} \overline{U_j}\right)$. These sets are open. Then let $U' = \bigcup_{n \in N} U'_n$ and let $V' = \bigcup_{n \in N} V'_n$. U' and V' are clearly open, because they are the union of open sets. Our claim is that they are disjoint covers of C and D.

NOTES

Take $x \in C$. Then $x \in U_n$ for some n, because $C \subset \bigcup_{n \in N} U_n$. Furthermore, $x \notin \overline{V_n}$ for all n. Therefore $x \in U_n \setminus \left(\bigcup_{j=1}^{n} \overline{V_j} \right) = U'_n \subset U'$, so $C \subset U'$. Similarly, $D \subset V'$.

Now suppose that U' and V' are not disjoint. Then there exists some $x \in X$ such that $x \in U' = \bigcup_{n \in N} U'_n$ and $x \in V' = \bigcup_{n \in N} V'_n$. This implies that for some $m, n, x \in U'_m = U_m \setminus \left(\bigcup_{j=1}^{m} \overline{V_j} \right)$β and $x \in V'_n = V_n \setminus \left(\bigcup_{j=1}^{n} \overline{U_j} \right)$, i.e., $x \in U_m, V_n$, but $x \notin \overline{V_1}, \ldots, \overline{V_m}, \overline{U_1}, \ldots, \overline{U_n}$. Suppose $m \leq n$. Then $x \in U_m$. But $x \notin \overline{U_1}, \ldots, \overline{U_m}, \ldots, \overline{U_n}$, which is a contradiction. Similarly, we get a contradiction if $n \leq m$. Therefore, U' and V' are disjoint open covers of C and D, proving that X is normal.

Theorem 5.36: Suppose X is a regular space with a countably locally finite basis B. Then X is metric space.

Proof: We will show that X can be embedded in the metric space $[0,1]$B, with the following metric:

Suppose $p, q \in [0,1]^{\mathrm{B}}$. Then let $d(p,q) = \sup_{B \subseteq \mathrm{B}} \{| p(B) - q(B) |\}$.

We will prove this is a metric.

1. Each term $| p(B) - q(B) | \geq 0$, so $d(p,q) = \sup_{B \subseteq \mathrm{B}} \{| p(B) - q(B) |\} \geq 0$ as well.

   If $d(p,q) = \sup_{B = \mathrm{B}} \{| p(B) - q(B) |\} = 0$, then for every $B \in$ B we have $0 \leq | p(B) - q(B) | \leq 0$, so $p(B) = q(B)$, hence $p = q$. Conversely, if $p = q$, then $| p(B) - q(B) | = 0$ for all $B \in$ B, so $d(p, q) = \sup_{B \subseteq \mathrm{B}} \{| p(B) - q(B) |\} = 0$.

2. $d(p,q) = \sup_{B \subseteq \mathrm{B}} \{| p(B) - q(B) |\} = \sup_{B \subseteq \mathrm{B}} \{| q(B) - p(B) |\} = d(q,p)$.

3. It is generally true that if W and Y are sets of real numbers, then sup (W) + sup (Y) = sup $(W+Y)$, where $W+Y = \{ W+Y \mid w \in W, y \in Y \}$. It is also true that if $K \subset L$, then sup $(K) \leq$ sup (L). Therefore, $d(p,r) = \sup_{B \subseteq \mathrm{B}} \{| p(B) - r(B) |\} \leq \sup_{B \subseteq \mathrm{B}} \{| p(B) - q(B) |$

$$+|q(B)-r(B)|\} \leq \sup_{B \subseteq \mathcal{B}}\{|p(B)-q(B)|\} + \sup_{B \subseteq \mathcal{B}}\{|q(B)-r(B)|\} = d(p,q) + d(q,r)$$

NOTES

Now we can proceed.

Fact 1: If $W \subset X$ is open, then there exists a continuous function f such that $f|W > 0$ and $f|(X \backslash W) \equiv 0$.

Proof: By Step 1 of Lemma 2, $W = \bigcup_{n \in N} A_n$, where each A_n is closed. Each A_n is disjoint from the closed set $X \backslash W$. By Lemma 2, X is normal, so we may apply Urysohn's lemma: for each A_n, there exists a continuous function $f_n: X \to [0, 1]$ such that $f_n|A_n \equiv 1$ and $f_n|(X \backslash W) \equiv 0$. For $x \in X$, let $f(x) = \sum_{n=1}^{\infty} f_n(x)/2^n$. This is well defined, because $0 \leq f_n(x) \leq 1$, so $0 \leq f_n(x)/2^n \leq 1/2^n$. $\sum_{n=1}^{\infty} 1/2^n$ converges, so by the comparison test $\sum_{n=1}^{\infty} f_n(x)/2^n$ also converges. In fact, by the Weierstrass M-test it converges uniformly. Therefore, because each term $f_n/2^n$ is a continuous function (f_n is continuous and 2^n is just a constant), f is also continuous. Each f_n is uniformly zero on $X \backslash W$, hence $f|(X \backslash W) \equiv 0$ and for $x \in W$, we know that $x \in A_n$ for some n. Therefore $f_n(x) = 1$, so at least one term in the infinite sum $\sum_{n=1}^{\infty} f_n(x)/2^n$ is greater than zero. Because all the other terms cannot be less than zero, this guarantees that $f(x) > 0$; so $f|W > 0$, as desired.

Now we are ready to define the embedding $g: X \to [0,1]^{\mathcal{B}}$. First, it should be briefly noted that while Fact 1 proved the existence of a continuous function from any open set to the closed interval [0,1], we could just as easily have replaced [0,1] by any closed interval $[a,b]$, $a, b \in \mathbf{R}$, by defining a continuous function between [0, 1] and $[a, b]$ and considering the composition of the function we did define and this new function, using the fact that the composition of continuous functions is continuous.

Because $\mathcal{B}$ is countably locally finite, it is true that $\mathcal{B} = \bigcup_{n=N} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite. We might as well assume that for each $m, n \in \mathbf{N}$, $\mathcal{B}_m \cap \mathcal{B}_n = \phi$, because if there were some basis element $B \in \mathcal{B}_m, \mathcal{B}_n$, we could define $\mathcal{B}'_m$ that contained all the same element as $\mathcal{B}_m$ except for B. $\mathcal{B}'_m$ would still be locally finite and $\mathcal{B}$ would still be a basis, because B is still contained in $\mathcal{B}_n$.

If we take any basis element B, we therefore know that $B \in \mathcal{B}_n$ for exactly one n. Because B is open, we can, by Fact 1, define a continuous function f_B: X $\to [0,1/n]$ such that $f_B|(X \backslash B) \equiv 0$. Now we will define the embedding g as follows: let $g(x) = \{(B, f_B(x)) | B \in \mathcal{B}\}$. We must prove that this is an embedding by showing three things: g is injective, g is continuous and g^{-1} is continuous.

Fact 2: g is injective.

Proof: Suppose $x \neq y$. By definition of regularity, the single point sets $\{x\}$ and $\{y\}$ are closed, and they are obviously disjoint. By Lemma 2, X is normal. Therefore, there are disjoint open sets U, V such that $x \in \{x\} \subset U$ and $y \in \{y\} \subset V$. There must then exist a basis element B such that $x \in B \subset U$. Then $f_B(x) > 0$, but $f_B(y) = 0$ because $y \notin B$. Therefore, the function $g(x)$ contains the ordered pair (B, ε) where $\varepsilon > 0$. But the function $g(y)$ contains the ordered pair $(B, 0)$. Therefore, $g(x) \neq g(y)$.

Fact 3: g is continuous.

Proof: To prove that g is continuous, we must show that for any open set V around a point $g(x)$, there is an open set $U \subset X$ such that $x \in U$ and $g(U) \subset V$. In particular, if we can find an open set U containing x such that $g(U) \subset B\,(g(x), \varepsilon) \subset V$, then we will be done.

We know that $\mathsf{B} = \bigcup_{n=N} \mathsf{B}_n$, where each $\mathcal{B}_n$ is locally finite. Take some $\mathcal{B}_n$ and find an open neighbourhood U_n of x that intersects only finitely many basis elements in $\mathcal{B}_n$. Suppose that $B \in \mathcal{B}_n$ does not intersect U. Then it certainly does not contain x, so $f_B(x) = 0$. Furthermore, $f_B(y) = 0$ for all $y \in U_n$. Therefore, $|f_B(x) - f_B(y)| = 0$.

Now suppose $B \cap U_n \neq \varphi$. We know that $f_B: X \to [0, 1/n]$ is continuous, so given the open set $(f_B(x) - \varepsilon/2, f_B(x) + \varepsilon/2) \cap [0,1/n] \subset [0,1/n]$, there will be an open set $W_n \subset X$ containing x such that $f_B(W_n) \subset (f_B(x) - \varepsilon/2,\ f_B(x) + \varepsilon/2) \cap [0,1/n]$. Let $V_n = W_n \cap U_n$, which is open because it is the finite intersection of open sets, and is not empty because both W_n and U_n contain x. Then for $Y \in V_n \subset W_n$, $f_B(y) \in (f_B(x) - \varepsilon/2, f_B(x) + \varepsilon/2) \cap [0,1/n]$, so $|f_B(x) - f_B(y)| < \varepsilon/2$. So for $y \in V_n$, $|f_B(x) - f_B(y)| < \varepsilon/2$ for all $B \in \mathcal{B}_n$.

Take N large enough so that $1/N < \varepsilon/2$. Let $V = V_1 \cap \ldots \cap V_N$, which is open as it is the finite intersection of open sets. If $y \in V$, then $y \in V_1, \ldots, V_N$, so for $n \leq N$ and $B \in \mathcal{B}_n$, $|f_B(x) - f_B(y)| < \varepsilon/2$. Furthermore, for $n > N$ and $B \in \mathrm{B}_n$, we know that $|f_B(x) - f_B(y)| < 1/n < 1/N < \varepsilon/2$, because the maximum value of f_B is $1/n$ and the minimum value is 0. So the maximum difference between any two values is $1/n$. So for $y \in V$, $|f_B(x) - f_B(y)| < \varepsilon/2$ for all $B \in \mathcal{B}$. Therefore, $d(g(x), g(y)) = \sup_{B = \mathsf{B}} \{|f_B(x) - f_B(y)|\} \leq \varepsilon/2 < \varepsilon$, implying that $g(y) \in B\,(g(x), \varepsilon)$, proving that g is continuous.

Fact 4: g^{-1} is continuous.

Proof: To prove that g^{-1} is continuous, we need to show that if $U \subset X$ is open, then $g(U) = (g^{-1})^{-1}(U)$ is open in $g(x)$. It suffices to show that for any $z \in g(U)$ and g is injective by Fact 2, there exists a unique $x \in U$ such that $g(x) = z$. Because U

is open, there exists a basis element β such that $x \in \beta \subset U$. Therefore $f_\beta(x) > 0$ and $f_\beta|(X\backslash\beta) \equiv 0$.

NOTES

Let $V = \{h: \mathcal{B} \rightarrow [0,1] | h(\beta) > 0\} \subset [0,1]$B, for β chosen above. Our claim is that V is open. To show this, we must show that any function in V has an open neighbourhood contained entirely within V. Take some function $h \in V$ and consider the open ball $B(h, h(\beta))$. For any function $h' \in B(h, h(\beta))$, $\sup_{B \subseteq \mathrm{B}}\{|h(B) - h'(B)|\} = d(h, h') < h(\beta)$, so $|h(B) - h'(B)| < h(\beta)$ for all $B \in \mathcal{B}$. In particular, $|h(\beta) - h'(\beta)| < h(\beta)$, so $h'(\beta) > 0$. Thus, $h' \in V$ and $B(h, h(\beta)) \subset V$, so V is open.

Let $W = V \bigcap g(x)$. Then W is open in $g(x)$. By our choice of β, $f_\beta(x) > 0$. So $g(x)$ contains the ordered pair (β,δ) for $\delta = f_\beta(x) > 0$. Thus $g(x) \in V$. It is also true that $g(x) \in g(X)$. Therefore, $g(x) \in V \bigcap g(x) = W$. Now we need to show that $W \subset g(U)$. Take any function $p \in W$. Then $p = g(y)$ for some $y \in X$. Because $p \in W = V \bigcap g(x)$, $p \in V$, so $f_\beta(y) = p(\beta) > 0$. This means that $y \in \beta \subset U$, so $p = g(y) \in g(U)$. Therefore $W \subset g(U)$, implying that $g(U)$ is open. This shows that $(g^{-1})^{-1}(U)$ is open whenever U is open. Hence g^{-1} is continuous.

5.11 PARACOMPACTNESS

Let X be a space and Y a subspace of X. A subspace Y is said to be normal (respectively, strongly normal) in X if for each disjoint closed subsets F_0, F_1 of X (respectively, of Y), there exist disjoint open subsets G_0, G_1 of X such that $F_i \cap Y \subset G_i$ for $i = 0, 1$. A subspace Y is said to be 1- (respectively, 2-) paracompact in X if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. Here, $\mathcal{V}$ is said to be a partial refinement of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V. The term 2- paracompact is often shortened to paracompact.

Theorem 5.37: Every paracompact space is normal.

Proof: We shall show first that a paracompact space is regular. Suppose A is closed set in a paracompact space X and $x \notin A$. For each $y \in A$, $\exists$ open set V_y containing y such that $x \notin \overline{V_y}$. Then the sets $V_y; y \in A$ together with the set $X - A$, form an open cover of X. Let w be a locally finite refinement and $V = U\{W \in w; W \cap A \neq \phi\}$.

Then V is open and contains A, and $\overline{V} = U\{\overline{W}; W \cap A \neq \phi\}$

But each set W is contained in some V_y since W is refinement and hence, $\overline{W}$ is contained in $\overline{V_y}$. Hence $x \notin \overline{W}$ (since $x \notin \overline{V_y}$). Thus $x \notin \overline{V}$. But $V \supseteq A$. Thus

x and A are separated by open sets in X, i.e., the space is regular. Now we will prove that the space is normal. Suppose A and B are disjoint closed sets in X. Since the space is regular, to each $y \in A$, $\exists$ an open set V_y containing y and $\overline{V_y} \cap B = \phi$. Then the sets V_y together with $X - A$ from an open cover of X. Let w be an open locally finite refinement and $V = \{ W \in w; W \cap A \neq \phi\}$. Then V is open and contains A and $\overline{V} = U\{\overline{W}; W \cap A \neq \phi\}$. But each such W is contained in some V_y (since w is refinement) and hence each $\overline{W}$ is contained in $\overline{V_y}$. Thus there is an open set V such that $A \subset V$ and $\overline{V} \cap B = \phi$. Thus X is normal.

Note: Every normal space need not be paracompact, e.g., space of ordinals which are less than the first uncountable rationals with respect to order topology is a normal space but not paracompact.

Theorem 5.38: Consider F_σ= set in a paracompact space X, where each F_n is closed in X. Let $\{U_\alpha; \alpha \in A\}$ be an open covering of F and each $U_\alpha = F \cap V_\alpha$ where V_α is open in X. For each fixed n, $\{X - F_n\} \cup \{V_\alpha; \alpha \in A\}$ is an open covering of X and so has an open locally finite refinement W_n. Let $\beta_n = \{ W \cap F ; W \in W_n\}$.

Then each β_n is locally finite collection of open subsets of F and $\bigcup_{n=1}^{\infty} \beta_n$ refines $\{U_\alpha; \alpha \in A\}$. Thus $\{U_\alpha; \alpha \in A\}$ has an open σ = locally finite refinement. Thus by Michael theorem, F is paracompact.

5.12 HOMOTOPY OF PATHS

In mathematics, a *path* in a topological space X is a continuous map f from the unit interval $I = [0, 1]$ to X given as,

$$f : I \to X$$

The *initial point* of the path is $f(0)$ and the *terminal point* is $f(1)$. In a path from x to y, the points x and y are respectively, the initial point and terminal point of the path. Remember that a path is not simply a subset of X resembles a curve, but it also includes a parameterization. For example, the maps $f(x) = x$ and $g(x) = x^2$ precisely represent two different paths from 0 to 1 on the real line.

A *loop* in a space X based at $x \in X$ is a path from x to x. A loop may be appropriately defined as a map $f : I \to X$ with $f(0) = f(1)$ or as a continuous map from the unit circle S^1 to X, given as,

$$f : S^1 \to X$$

This is because S^1 may be regarded as a quotient of I under the identification $0 \sim 1$. The set of all loops in X forms a space called the loop space of X. A topological space for which there exists a path connecting any two points is termed as path-connected. Any space may be broken up into a set of path-connected components. The set of path-connected components of a space X is denoted by $\pi_0(X)$.

NOTES

We can also define paths and loops in pointed spaces that are important in homotopy theory. If X is a topological space with basepoint x_0, then a path in X is one whose initial point is x_0. In the same way, a loop in X is one that is based at x_0. Paths and loops are central subjects of study in the branch of algebraic topology known as homotopy theory. Specifically, a homotopy of paths or path-homotopy, in X is a family of paths $f_t : I \rightarrow X$ indexed by I such that

- $f_t(0) = x_0$ and $f_t(1) = x_1$ are fixed.
- The map $F : I \times I \rightarrow X$ given by $F(s, t) = f_t(s)$ is continuous.

The paths f_0 and f_1 connected by a homotopy are said to homotopic. Similarly, a homotopy of loops can be defined keeping the base point fixed. The relation of being homotopic is an equivalence relation on paths in a topological space. The equivalence class of a path f under this relation is called the homotopy class of f and is precisely denoted by $[f]$. Assume that f is a path from x to y and g is a path from y to z. Than the path fg is defined as the path obtained by first traversing f and then traversing g:

$$fg(s) = \begin{cases} f(2s) & 0 \leq s \leq \dfrac{1}{2} \\ g(2s-1) & \dfrac{1}{2} \leq s \leq 1. \end{cases}$$

Evidently, path composition is only defined when the terminal point of f and the initial point of g coincide. If we consider all loops based at a point x_0, then path composition is termed as a binary operation. Path composition, when precisely defined, is not associative in general due to the difference in parametrization. However, it is associative up to path-homotopy, i.e., $[(fg)h] = [f(gh)]$. Path composition defines a group structure on the set of homotopy classes of loops based at a point x_0 in X. The resultant group is termed as the fundamental group of X based at x_0 and is typically denoted by $\pi_1(X, x_0)$.

In the case of associativity of path composition, a path f in X can be defined as a continuous map from an interval $[0, a]$ to X for any real $a \geq 0$. This path f has a length $|f|$ defined as a. Path composition can be defined using the following modification:

$$fg(s) = \begin{cases} f(s) & 0 \leq s \leq |f| \\ g(s-|f|) & |f| \leq s \leq |f| + |g| \end{cases}$$

Wherein, with the previous definition, f, g and fg all have length 1, this definition makes $|fg| = |f| + |g|$. The previous definition do not follow or support associativity properly because although $(fg)h$ and $f(gh)$ have the same length 1, and the midpoint of $(fg)h$ occurred between g and h, whereas the midpoint of $f(gh)$

occurred between f and g. In this modified definition $(fg)h$ and $f(gh)$ holds the same length, namely $|f|+|g|+|h|$, and the same midpoint found at $(|f|+|g|+|h|)/2$ in both $(fg)h$ and $f(gh)$. Hence $(fg)h$ and $f(gh)$ have the same parametrization throughout.

NOTES

Definition: Let X and Y be topological spaces. Let $X' \subset X$ and $f_0, f_1 : X \to Y$ be continuous and agree on X'. f_0 is homotopic to f_1 relative to X' if there exists a continuous map $F : X \times \mathbf{I} \to Y$ such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ for $x \in X$, and $F(x,t) = f_0(x)$ for $x \in X$ and $t \in \mathbf{I}$.

If f_0 and f_1 are homotopic relative to $X,'$ we write $f_0 \cong f_1$ rel X'. If $X' = \phi$ we omit writing rel X'.

Theorem 5.39: Homotopy relative to X' is an equivalence relation in the set of continuous maps $X \to Y$.

Proof: Reflexivity: $f \cong f$ because let $F(x, t) = f(x)$.

Symmetry: Let F be the homotopy for $f_0 \cong f_1$ rel X'. Then the homotopy F' for $f_1 \cong f_0$ rel X' is

$F'(x, t) = F(x, 1-t)$.

Transitivity: Let F_1 be the homotopy for $f_0 \cong f_1$ rel X' and F_2 be the homotopy for $f_1 \cong f_2$ rel X'. Then, F_3 defined as

$$F_3(x, t) = \begin{cases} F_1(x, 2t) & 0 \le t \le 1/2 \\ F_2(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

is the homotopy for $f_0 \cong f_2$ rel X'. F_3 is continuous because both its restrictions are continuous and equal for $t = 1/2$ (i.e., $F_1(x, 1) = f_1(x) = F_2(x, 0)$).

Theorem 5.40: Composites of homotopic maps are homotopic.

Definition 1: A topological space X is said to be pointed if X is non empty and it has a base point $x_0 \in X$. We denote such a space by (X, x_0).

Definition 2: A topological space X is said to be contractible if its identity map is homotopic to some constant map of X to itself, i.e., $f(x) = x$ and $f \cong c$.

Theorem 5.41: Any two maps of an arbitrary space to a contractible space are homotopic.

Proof: Let Y be a contractible space and let its identity map be homotopic to the map c (constant in Y). Let $f_0, f_1 : X \to Y$ be arbitrary. By Theorem 5.2, $f_0 \cong cf_0$ and $f_1 \cong cf_1$. Because $cf_0 = cf_1$ therefore by Theorem 5.1, we have $f_0 \cong f_1$.

Let (X, b) and (Y, c) be pointed topological spaces. A map $f : (Y, c) \to (X, b)$ is a continuous function f that satisfies $f(c) = b$. Then the map $f_* : \pi_1(Y, c) \to \pi_1(X, b)$ is defined by making the path γ in Y correspond to the path $f \circ \gamma$ in X.

5.13 THE FUNDAMENTAL GROUP

NOTES

According to the mathematical analysis in the field of topology, the notion and concept of fundamental group is defined by Henri Poincaré. The fundamental group is referred as a group which is typically associated with any given pointed topological space which provides methodology to determine that when the two paths, namely starting path and ending path at a fixed base point, can be continuously or constantly deformed into each other. Characteristically, it documents the information which significantly defines the basic shapes or holes of the topological space. The concept of fundamental group is the initial and simplest form of the homotopy groups. Basically, the fundamental groups can be studied on the basis of the theory of covering spaces, because a fundamental group precisely coincides with the group of surface transformations of the related or associated universal covering space. The abelianization of the fundamental groups can be identified and defined with the help of first homology group of the space. The concept and notion of fundamental group was first defined in the theory of Riemann surfaces, which precisely was based on the work of Bernhard Riemann, Henri Poincaré and Felix Klein, who typically described the monodromy properties of complex functions and also provided the comprehensive topological classification for the closed surfaces.

First consider a space, for example a surface and certain point in it, and also all the starting and ending loops of this point. When the two loops are combined together in a transparent manner then they move along the first loop and then along the second loop. Any two loops can only be considered similar or equivalent if one loop can be precisely deformed into the other loop without breaking. Eventually, with the help of this method the set of all such loops can be combined and this typical equivalence between them is termed as the fundamental group.

Definition: Suppose σ and τ are two paths in X with the same end points ($\sigma(0) = \tau(0) = x_0$ and $\sigma(1) = \tau(1) = x_1$). We say that σ and τ are homotopic relative to the end points ($\sigma \cong \tau$ rel $\{0,1\}$) if there exists a map F: $\mathbf{I} \times \mathbf{I} \to X$ such that $F(t, 0) = \sigma(t)$, $F(t, 1) = \tau(t)$, $F(1, t') = x_1 t$, $t' \in \mathbf{I}$.

We can see that the above definition is the general definition of homotopy applied to paths and the set $\{0,1\} \subset \mathbf{I}$. Therefore the homotopy of paths will be an equivalence relation in the set of paths $\mathbf{I} \to X$ for a given space x.

We define the multiplication of two paths as,

$$\sigma\tau(t) = \begin{cases} \sigma(2t)\, 0 \le t \le 1/2 \\ \tau(2t-1)\, 1/2 \le t \le 1 \end{cases}$$

Because homotopy is an equivalence relation, we can speak of multiply the equivalence class of one path with the equivalence class of another path presuming that their ends are the same.

Theorem 5.42: Let $\pi_1(X, x_0)$ be the set of classes of homotopy of loops in X at the point x_0. From the above definition of multiplication, $\pi_1(X, x_0)$ is a group. The identity element being the constant loop in x_0 and the inverse of $[\sigma]$ being $[\sigma^{-1}]$ defined $\sigma^{-1}(t) = \sigma(1-t)$, $0 \leq t \leq 1$.

Theorem 5.43: If X is path connected space then for all points $x_0 \in X$ the groups $\pi_1(X, x_0)$ are isomorphic.

In the above case (X is path connected) we call $\pi_1(X, x_0)$ the fundamental groups of X and write it as $\pi_1(X)$.

We can now look at contractible spaces as being a special case of a more general class of spaces.

Definition: A topological space (X, A) is defined as just a connected space if it is typically path connected and the fundamental group of X is trivial.

Therefore, a simply connected space is referred as a path connected space in which all paths between two points can be continuously transformed into each other. Examples of simply connected spaces are $\mathbf{R}^2$ and S^n $(n \geq 2)$.

Theorem 5.44: A contractible space is a simply connected space.

Check Your Progress

13. State about local finiteness.
14. Give the definition of local finiteness?
15. State the Nagata-Smirnov metrization theorem.
16. Give the proof that every paracompact space is regular.
17. What is path composition?
18. Define fundamental group.

5.14 COVERING SPACES

In algebraic topology, a *covering map* is a continuous surjective function p from a topological space, C, to a topological space, X, such that each point in X has a neighbourhood which is evenly covered by p. This implies that for each point $x \in X$, there is associated an ordered pair, (K, U), where U is a neighbourhood of x and where K is a collection of disjoint open sets in C, each of which gets mapped homeomorphically, via p to U. Particularly, this means that every covering map is necessarily a local homeomorphism. Under this definition, C is called a covering space of X.

Covering spaces are considered significant in the analysis and evaluation of homotopy theory, harmonic analysis, Riemannian geometry, and differential topology. In the Riemannian geometry, the ramification is defined as a generalization of the concept of covering maps. Covering spaces are also considered interconnected which was proved by the study of homotopy groups and specifically the fundamental group. The most significant application is obtained from the result which specifies that if X is a sufficiently appropriate topological space, then there is typically a bijection from the collection of all the isomorphism forms of connected coverings of X and the subgroups of the fundamental group of X.

NOTES

Let X be a topological space. A covering space of X is a space C together with a continuous surjective map,

$p: C \to X$

such that for every $x \in X$, there exists an open neighbourhood U of x, such that the inverse image of U under p, $p^{-1}(U)$, is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p. Fundamentally, the map p is termed as the covering map. The space X is generally characterized as the base space of the covering while the space C is characterized as the total space of the covering. Consider any point x in the base, then the inverse image of x in C is essentially and certainly a discrete space termed as the fiber over x.

The special and unique open neighbourhoods U of x as specified in the given definition are termed as evenly-covered neighbourhoods. These evenly-covered neighbourhoods form an open cover of the space X. The homeomorphic versions in C of an evenly-covered neighbourhood U are typically termed as the sheets over U. In particular, the covering maps are referred as the locally trivial which specifies that locally each and every covering map is isomorphic to a projection such that there is a unique homeomorphism from the pre-image of an evenly covered neighbourhood U to $U \times F$, where F is termed as the fiber which essentially satisfies the local trivialization condition. Consequently, if this homeomorphism is projected onto U, then the derived structure or resultant composition is equal to p.

Consider the unit circle $S^1 \subseteq \mathbf{R}^2$. The map p: $\mathbf{R} \to S^1$ with $p(t) = (\cos t, \sin t)$ is a cover where each point of S^1 is covered infinitely often. For example, take the complex plane with the origin removed (denoted by $\mathbf{C}^*$) and choose a non-zero integer n. Then $q_n: \mathbf{C}^* \to \mathbf{C}^*$ given by, $q_n(z) = z^n$ is a cover. Here, every fiber has n elements. The map q_n leaves the unit circle S^1 invariant and the restriction of this map to S^1 is an n-fold cover of the circle by itself.

In fact, S^1 and $\mathbf{R}$ are the only connected covering spaces of the circle. For proving this, first note that the fundamental group of the circle is isomorphic to the additive group of integers $\mathbf{Z}$. As follows from the correspondence between equivalence classes of connected coverings and conjugacy classes of subgroups of the fundamental group of the base space discussed below, a connected covering f: $C \to S^1$ is determined by a subgroup $f_\# (\pi_1(C))$ of $\pi_1(S^1) = \mathbf{Z}$, where $f_\#$ is the induced homomorphism. The group $\mathbf{Z}$ is abelian and it only has two kinds of subgroups, namely the trivial subgroup that has infinite subgroup index in $\mathbf{Z}$ and the subgroups $H_n = \{kn | \, k \in \mathbf{Z}\}$ for $n = 1, 2, 3, \ldots$, where H_n has index n in $\mathbf{Z}$. Each of the subgroups H_n *of* $\mathbf{Z}$ is realized by the covering q_n: $S^1 \to S^1$ since one can check that $(q_n)_\#$: $\mathbf{Z} \to \mathbf{Z}$ maps an integer k to kn and hence, $(q_n)_\#(\mathbf{Z}) = H_n$. The trivial subgroup of $\mathbf{Z}$ is realized by the covering p: $\mathbf{R} \to S^1$, since $\mathbf{R}$ is simply connected and has trivial fundamental group and hence $p_\#(\pi_1(\mathbf{R})) = \{0\}$, the trivial subgroup of $\mathbf{Z}$. Since the total space of the coverings q_n is S^1 and the total space of the covering p is **R**, this shows that every connected cover of S^1 is either S^1 or **R**.

Suppose X and E are topological spaces and $p : E \to X$ is a continuous map.

Definition 1: An open subset U of X is said to be evenly covered by p if the inverse image $p^{-1}(U)$ is a union of disjoint open subsets of E, each of which is mapped homeomorphically by p onto U.

Definition 2: The map p is said to be a covering map if p continuously maps E onto X such that each $x \in X$ has an open neighbourhood, which is evenly covered by p.

In the above case, E is called a covering space over X. A covering map is also known as a cover. The open cover elementary topology, is a special case of a covering space, when E is a disjoint union of a collection of open sets X_i with union X.

If $x \in X$, then the set $p^{-1}(x)$ is called the fiber over x. From the definition of a covering map it follows that it is a discrete subspace of E and each $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(x) \times U$. The subsets of $p^{-1}(U)$ mapped homeomorphically onto U are called the sheets of $p^{-1}(U)$. Of course, if U is connected, then the sheets of $p^{-1}(U)$ coincide with the connected components of $p^{-1}(U)$ (as connectedness is preserved by homeomorphism). Every cover p: $E \to X$ is a local homeomorphism which implies that X and its covering space E always share the same local topological properties. However, if X is simply connected, topological properties of X and E are held globally as well, with p being a homeomorphism of these two spaces.

Theorem 5.45: Product of covering spaces is covering space.

Example 5.1: $\mathbf{R}^n$ is the covering space over the n-dimensional torus $T^n = S^1 \times ... \times S^1$ with the covering map, p: $\mathbf{R}^n \to T^n$ given by:

$$p(x_1, \ldots, x_2) = (e^{2\pi i x_1}, \ldots.., e^{2\pi i x_n}).$$

Suppose that E, X and Y are topological spaces and p: $E \to X$ is a covering map. Let f: $Y \to X$ be a map. We need to determine whether there exists a map g: $Y \to E$ such that $pog = f$. In this case, g is called a lift of f as shown in Figure.

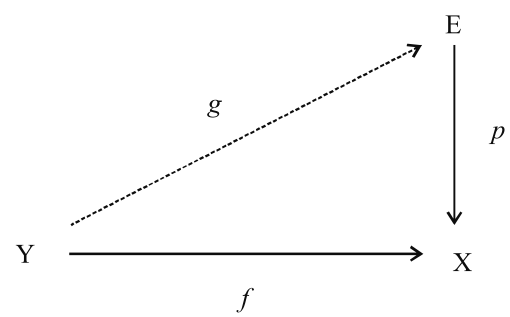


Fig. 1 *A Diagram of the Lift g*

NOTES

When $pog = f$, then the Figure commutes. The following lemma explains about the uniqueness of lifts.

Lemma: Let p: $E \rightarrow X$ be a covering map and let Y be a connected topological space. Let f: $Y \rightarrow X$ be a map and let g, h: $Y \rightarrow E$ be two lifts of f. If $g(y) = h(y)$ for some point $y \in Y$, then $g = h$.

5.15 THE FUNDAMENTAL GROUP OF THE CIRCLE AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Theorem 5.46: The map ϕ: $Z \rightarrow \pi_1(S^1)$ sending an integer n to the homotopy class of the loop $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ based at $(0,1)$ is an isomorphism.

Proof: The idea is to compare paths in S^1 with paths in **R** via the map p : $\mathbf{R} \rightarrow S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. This map can be visualized geometrically by embedding **R** in $\mathbf{R}^3$ as the helix parametrized by $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$, and then p is the restriction to the helix of the projection of $\mathbf{R}^3$ onto $\mathbf{R}^2$, $(x, y, z) \mapsto (x, y)$, as shown in the Figure 5.1.

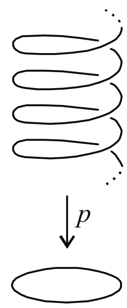


Fig. 5.1 *Helix*

Observe that the loop ω_n is the composition $p\,\tilde{\omega}_n$ where $\tilde{\omega}_n$:$\mathbf{I} \rightarrow \mathbf{R}$ is the path $\tilde{\omega}_n(s) = ns$, starting at 0 and ending at n, winding around the helix $|n|$ times, upward if $n > 0$ and downward if $n < 0$. The relation $\omega(s) = p\,\tilde{\omega}_n$ is expressed by saying that $\tilde{\omega}_n$ is a lift of ω_n.

The definition of ϕ can be reformulated by setting $\phi(n)$ equal to the homotopy class of the loop $p\,\tilde{f}$ for $\tilde{f}$ any path in **R** from 0 to n. Such an $\tilde{f}$ is homotopic to $\tilde{\omega}_n$ via the linear homotopy $(1-t)\,\tilde{f} + t\,\tilde{\omega}_n$, hence $p\,\tilde{f}$ is homotopic to $p\,\tilde{\omega}_n = \omega_n$ and the new definition of $\phi(n)$ agrees with the old one.

To verify that ϕ is a homomorphism, let τ_m: $\mathbf{R} \rightarrow \mathbf{R}$ be the translation $\tau_m(x) = x + m$. Then $\tilde{\omega}_m .(\tau_m \tilde{\omega}_n)$ is a path in **R** from $m+n$, so $\phi(m+n)$ is the homotopy class of the loop in S^1 that is the image of this path under p. This image is just $\omega_m . \omega_n$, so $\phi(m+n) = \phi(m).\phi(n)$.

To show that ϕ is an isomorphism we will use following two facts:

NOTES

(a) For each path f: $\mathbf{I} \to S^1$ starting at a point $x_0 \in S^1$ and each $\tilde{x}_0 \in p^{-1}(x)$ there is a unique lift $\tilde{f} : \mathbf{I} \to \mathbf{R}$ starting at $\tilde{x}_0$.

(b) For each homotopy f_t: $\mathbf{I} \to S^1$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x)$, there is a unique lifted homotopy $\tilde{f}_t : \mathbf{I} \to \mathbf{R}$ of paths starting at $\tilde{x}_0$.

To show that ϕ is surjective, let f: $\mathbf{I} \to S^1$ be a loop at the base point (1, 0), representing a given element of $\pi_1(S^1)$. By (a) there is a lift $\tilde{f}$ starting at 0. This path $\tilde{f}$ ends at some integer n since $p\tilde{f}(1) = f(1) = (1, 0)$ and $p^{-1}(1, 0) = Z \subset \mathbf{R}$. By the extended definition of ϕ we then have $\phi(n) = [p\tilde{f}] = [f]$. Hence, ϕ is surjective. To show that ϕ is injective, suppose $\phi(m) = \phi(n)$, which means $\omega_m \cong \omega_n$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By (b) this homotopy lifts to a homotopy $\tilde{f}_t$ of paths starting at 0. The uniqueness part of (a) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths, the endpoints $\tilde{f}_t(1)$ are independent of t. For $t = 0$ this endpoint is m and for $t = 1$ it is n, so $m = n$.

It remains to prove (a) and (b). Both statements can be deduced from the following assertion:

(c) Given a map F: $Y \times \mathbf{I} \to S^1$ and a map $\tilde{F} : Y \times \{0\} \to \mathbf{R}$ lifting $F| Y \times \{0\}$, there is a unique map $\tilde{F} : Y \times \mathbf{I} \to \mathbf{R}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$. Statement (a) is the special case that Y is a point and (b) is obtained by applying (c) with $Y = 1$ in the following way: The homotopy f_t in (b) gives a map F: $\mathbf{I} \times \mathbf{I} \to S^1$ by setting $F(s, t) = f_t(s)$ as usual. A unique lift $\tilde{F} : \mathbf{I} \times \{0\} \to \mathbf{R}$ is obtained by an application of (a). Then (c) gives a unique lift $\tilde{F} : \mathbf{I} \times \mathbf{I} \to \mathbf{R}$. The restrictions $\tilde{F} |\{0\} \times \mathbf{I}$ and $\tilde{F} |\{1\} \times \mathbf{I}$ are paths lifting constant paths, hence they must also be constant by the uniqueness part of (a). So $\tilde{f}_t(s) = \tilde{F}(s, t)$ is a homotopy of paths and $\tilde{f}_t$ lifts f_t since $p\tilde{F} = F$.

We shall prove (c) using just one special property of the projection p: $\mathbf{R} \to S^1$, namely:

There is an open cover $\{U_\alpha\}$ of S^1 such that for each α, $p^{-1}(U_\alpha)$ can be (*) decomposed as a disjoint union of open sets each of which is mapped homeomorphically onto U_α by p.

NOTES

For example, we could take the cover $\{U_\alpha\}$ to consist of any two open arcs in S^1 whose union is S^1.

There is an open cover $\{U_\alpha\}$ of S^1 such that for each α, $p^{-1}(U_\alpha)$ can be decomposed as a disjoint union of open sets each of which is mapped homeomorphically onto U_α by p.

For example, we could take the cover $\{U_\alpha\}$ to consist of any two open arcs in S^1 whose union is S^1.

To prove (c) we will first construct a lift $\tilde{F} : \mathbf{N} \times \mathbf{I} \to \mathbf{R}$ for $\mathbf{N}$ some neighbourhood in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times \mathbf{I}$ has a product neighbourhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some α. By compactness of $\{y_0\} \times \mathbf{I}$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times \mathbf{I}$. This implies that we can choose a single neighbourhood $\mathbf{N}$ of Y_0 and a partition $0 = t_0 < t_1 < ...... < t_m = 1$ of $\mathbf{I}$ so that for each i, $F(\mathbf{N} \times [t_i, t_{i+1}])$ is contained in some U_α, which we denote by U_i. Assume inductively that $\tilde{F}$ has been constructed on $\mathbf{N} \times [0, t_i]$, starting with the given $\tilde{F}$ on $\mathbf{N} \times \{0\}$. We have $F(\mathbf{N} \times [t_i, t_{i+1}]) \subset U_i$, so by statement (*) there is an open set $\tilde{U}_i \subset \mathbf{R}$ projecting homeomorphically onto U_i by p and containing the point $\tilde{F}(y_0, t_i)$. After replacing $\mathbf{N}$ by a smaller neighbourhood of y_0 we may assume that $\tilde{F}(\mathbf{N} \times \{t_i\})$ is contained in $\tilde{U}_i$, namely, replace $\mathbf{N} \times \{t_i\}$ by its intersection with $(\tilde{F} \mid \mathbf{N} \times \{t_i\})^{-1}(\tilde{U}_i)$. Now we can define $\tilde{F}$ on $\mathbf{N} \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $p^{-1}: U_i \to \tilde{U}_i$. After a finite number of steps we eventually get a lift $\tilde{F} : \mathbf{N} \times \mathbf{I} \to \mathbf{R}$ for some neighbourhood $\mathbf{N}$ of y_0.

Next, we show the uniqueness part of (c) in the special case that Y is a point. In this case we can omit Y from the notation. So suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F: \mathbf{I} \to S^1$ such that $\tilde{F}(0) = \tilde{F}'(0)$. As before, choose a partition $0 = t_0 < t_1 < ...... < t_m = 1$ of $\mathbf{I}$ so that for each $\mathbf{I}$, $F([t_i, t_{i+1}])$ is contained in some U_i. Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, so is $\tilde{F}[t_i, t_{i+1}]$, which must therefore lie in a single one of the disjoint open sets $\tilde{U}_i$ projecting homeomorphically to U_i as in the statement (*). By the same token, $\tilde{F}'([t_i, t_{i+1}])$ lies in a single $\tilde{U}_i$, in fact in the same one that contains $\tilde{F}([t_i, t_{i+1}])$ since $\tilde{F}'(t_i) = \tilde{F}(t_i)$. Because p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$, and the induction step is finished.

NOTES

The last step in the proof of (c) is to observe that since the $\tilde{F}$'s constructed above on sets of the form $\mathbf{N} \times \mathbf{I}$ are unique when restricted to each segment $\{y\} \times \mathbf{I}$, they must agree whenever two such sets $\mathbf{N} \times \mathbf{I}$ overlap. So we obtain a well-defined lift $\tilde{F}$ on all of $Y \times \mathbf{I}$. This $\tilde{F}$ is continuous since it is continuous on each $\mathbf{N} \times \mathbf{I}$, and $\tilde{F}$ is unique since it is unique on each segment $\{y\} \times \mathbf{I}$.

Theorem 5.47: Every nonconstant polynomial with coefficients in **C** has a root in **C**.

Proof: We may assume the polynomial is of the form $p(z) = z^n + a_1 z^{n-1} + ... + a_n$. If $p(z)$ has no roots in **C**, then for each real number $r \geq 0$ the formula,

$$f_r(s) = \frac{p\left(re^{2\pi is}\right)/\,p(r)}{|\,p\left(re^{2\pi is}\right)/\,p(r)|}$$

defines a loop in the unit circle $S^1 \subset \mathbf{C}$ based at 1. As r varies, f_r is a homotopy of loops based at 1. Since f_0 is the trivial loop, we deduce that the class $[f_r] \in \pi_1(S^1)$ is zero for all r. Now fix a large value of r, bigger than $|a_1| + ... + |a_n|$ and bigger than 1. Then for $|z| = r$ we have $|z^n| = r^n = r.\, r^{n-1} > (|a_1| + ... + |a_n|)\, |z^{n-1}| \geq |a_1 z^{n-1} + ... + a_n|$.

From the inequality $|z^n| > |a_1 z^{n-1} + ... + a_n|$ it follows that the polynomial

$p_t(z) = z^n + t(a_1 z^{n-1} + ... + a_n)$ has no roots on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p by p_t in the formula for f_r above and letting t go from 1 to 0, we obtain a homotopy from the loop f_r to the loop $\omega_n(s) = e^{2\pi ins}$. By Theorem 5.46, ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1)$. Since we have shown that $[\omega_n] = [f_r] = 0$, we conclude that $n = 0$. Thus the only polynomials without roots in **C** are consonants.

According to the fundamental theorem of algebra, every non-constant single-variable polynomial with complex coefficients has at least one complex root. Equivalently, the field of complex numbers is algebraically closed. This theorem is also stated as: every non-zero single-variable polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity. In spite of its name, there is no purely algebraic proof of the theorem, since any proof must either use the completeness of the reals or some other equivalent formulation of completeness, which is not an algebraic concept. Moreover, it is not even fundamental for modern algebra; its name was given during the time when the study of algebra was mainly concerned with the solutions of polynomial equations with real or complex coefficients.

Theorem 5.48 (Fundamental Theorem of Algebra): Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point x with $h(x) = x$.

NOTES

Here we are using the standard notation D^n for the closed unit disk in $\mathbf{R}^n$, all vectors x of length $|x| \le 1$. Thus the boundary of D^n is the unit sphere S^{n-1}. Shown in Figure 5.2.

Proof:

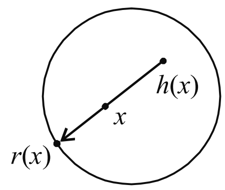


Fig. 5.2 *Closed Unit D*

Suppose on the contrary that $h(x) \neq x$ for all $x \in D^2$. Then we can define a map $r: D^2 \to S^1$ by letting $r(x)$ be the points of S^1 where the ray in $\mathbf{R}^2$ starting at $h(x)$ and passing through x leaves D^2. Continuity of r is clear since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points. The crucial property of r, besides continuity, is that $r(x) = x$ if $x \in S^1$. Thus, r is a retraction of D^2 onto S^1. We will show that no such retraction can exist.

Let f_0 be any loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop, for example the linear homotopy $f_t(s) = (1-t) f_0(s) + t(x_0)$ where x_0 is the base point of f_0. Since the retraction r is the identity on S^1, the composition rf_t is then a homotopy in S^1 from $rf_0 = f_0$ to the constant loop at x_0. However, this contradicts the fact that $\pi_1(S^1)$ is nonzero.

Check Your Progress

19. What is a covering space?
20. When map p is said to be a covering map?
21. Define fundamental theorem of algebra.

5.16 SOLVED EXAMPLES

Example 1: Let (X,T) be a topological space and let $A \subseteq X$. Then $x \in \bar{A}$ iff there is a net $w:D \to A$ such that $w \to x$.

Solution: Suppose $w:D \to A$ is a net such that $w \to x$ we want to show that $x \in \bar{A}$ so fix an open set P containing x. By definition of net convergence, there is a $d \in D$ such that $w(e) \in P$ for all $e \geq d$.

Since $w(e) \in A$ for all $e \in D$ in particular $P \cap A$ is non-empty.

Converse$\to$ Suppose $x \in \bar{A}$. We need to show that there is a net w that converges to x.

NOTES

Let $D_x = \{P \in T, x \in P\}$, this is a directed set $P \leq v$ iff $P \supseteq V$.

By definition of $\bar{A}$, for every $P \in D_x$ we can fix a point $x_p \in P \cap A$. Define $w: D_x \to A$ by $w(P) = x_p$ then w is a net and we claim that $w \to x$ indeed, fix an open set P containing x. Then $P \in D_x$ and so for all $V \geq P$ in Dx (for $V \subseteq P$) we have,

$$w(V) = x_v \in V \cap A \subseteq P \cap A \subseteq P$$

This shows that the tail $T_p = \{V \in D_x : P \leq V\} = \{V \in D_x : V \subseteq P\}$ of the net is contained in P.

Example 2: A subset A of a topological space (X,T) is closed if and only if the limit points of all convergent nets in A is again in A.

Solution: We know that the closure $\bar{A}$ of a set A is precisely the set of all limit points of nets in A.

The result then follows from the fact that A is closed if and only if $A = \bar{A}$.

Example 3: Let (X,T) be a topological space if F is a filter on X and $F \to x$, then x is an accumulation point of F. Conversely if x is an accumulation point of an ultra filter U on X, then $U \to x$.

Solution: First, suppose F is a filter and $F \to x$ then $F_x \subseteq F$, and so in particular for any open set $\mathbb{U} \in f_x$ and any $\mathbb{F} \in F$, $\mathbb{U} \cap \mathbb{F} \neq \phi$ since F is closed under finite intersection and $\phi \in F$.

Second suppose x is an accumulation point of an ultrafilter U on X.

We want to show that $F_x \subseteq U$, so fix $\mathbb{F} \in F_x$. Since x is an accumulation point of U and F contains an open set containing x, we have that $\text{U} \cap \mathbb{F} \neq \phi$ for every $\mathbb{U} \in \text{U}$ But then the collection $U \cup \{\mathbb{F}\}$ has the finite intersection property and so it must be contained in some filter that filter must be U itself since U is not properly contained in any other filter on X.

Example 3a: Assume that $X \neq \phi$ and $F_\alpha \in F(X)$, $\alpha \in I$ then $\cap F_\alpha \in F(X)$ is the finest filter on X, which is converse than each F_α, $\alpha \in I$

Solution: Note i) $\phi \in, F_\alpha, \alpha \in I$ implies that $\phi \notin \bigcap_{\alpha \in \int I} F_\alpha$ and also X belongs to each F_α, $\alpha \in I$ Thus $X \in \bigcap_{\alpha \in I} F_\alpha$ and it follows that $\bigcap_{\delta \in I} F_\alpha \neq \phi$.

(ii) Now let $A, B \in \bigcap_{\delta \in I} F_\alpha$; then A and B belongs to each F_α. Therefore $A \cap B$ belongs to each F_α, $\alpha \in I$ and thus $A \cap B \in \bigcap_{\alpha \in I} F_\alpha$.

(iii) Let $A \in \bigcap_{\alpha \in I} F_\alpha$ then $A \in$ each F_α, $\alpha \in I$. Let $B \supseteq A$ thus B belongs to each F_α since B is an over set of A. It follows that $B \in \bigcap_{\alpha \in I} F_\alpha$ since (i), (ii) and (iii) are satisfied, $\bigcap_{\alpha \varepsilon \text{I}} \text{F}_\alpha$ is a filter clearly $\bigcap_{\alpha \varepsilon \text{I}} \text{F}_\alpha \leq \text{F}_\alpha$ for each $\alpha \in I$.

Now let $G \subseteq F_\alpha$ for each $\alpha \in I$ and let us prove that $G \subseteq \bigcap_{\alpha \in I} F_\alpha$ Let $A \in G$, Thus A belongs to to each F_α, $\alpha \in I$, and then $A \in \bigcup_{\alpha \in I} F_x$. It follows that $G \subseteq \bigcup_{\alpha \in I} F_\alpha$

NOTES

Example 4: Let *f: (Xτ)→(Y,σ)* be a continuous function and onto. If *(X, τ)* is compact then *(Y,σ)* is compact.

Solution: Let U be an ultrafilter on Y, $F=f^{1}(U)$ and there exists an ultrafilter on X.

$G \supseteq F$. Then $G \xrightarrow{r} x$, for some $x \in X$ since (X, τ) is compact.

The contunuity assumption of f implies that $f(G)\to f(x)$ according to theorem.

Since $G \supseteq F$ then $f(G) \supseteq f(F)=f(f^{1}(U \geq U$. Also f is onto then $f(f^{1}(U))= U$.

Hence, $f(G) \supseteq U$ and since V is an ultrafilter $f(G)=U$ Consequently, $f(G)\to f(x)$ implies that $U\to f(x)$ therefore (Y,σ) is compact.

Example 5: Let *X, Y* be topological spaces with *x∈X* and *g: X→Y* then *g* is continuous at *x* if and only if whenever *F* is filter such that *F→ x, g(F)→g(x).*

Solution: Suppose g is continuous at x and $F\to x$. Let V be a neighborhood of $g(x)$. By continuity there is a neighborhood U of x such that g (U) $\subseteq V$.

Since $U \in F$, $g(\text{U}) \in g(F)$.

And thus $V \in g(F)$ Hence $g(F) \to g(x)$.

Conversely, suppose that whenever $F\to x$, $g(F)\to g(x)$.

Then $g(U(x)) \to g(x)$ by hypothesis

Then for each neighborhood V of $g(x)$, $V \in g(U(x))$. Then there exists a $V \in U(x)$ such that $g(U)V$ and thus g is a continous at x.

Example 6: Let *A* be a subset of a topological space *X* then for *x∈X, x∈Ā* if and only if there exists a filter on *X* which contains *A* and converse to *x*.

Solution: Assume that $x \in \bar{A}$ then any neighborhood of x has a non-empty intersection with A.

Now all the sets $A \cap U$, where U is a neighborhood of x form a filter base and the corresponding filter converges to x.

Conversely, assume that F is a filter containing A and converging to x. Choose any neighborhood U of x then $U \in F$. And thus $U \cap A \neq \phi$ since $A \in F$ This prove that $x \in \bar{A}$.

Example 7: Let (X, τ) be a topological space. Then the following statements are equivalent:

(a) (X, τ) is compact.

(b) Each ultrafilter on *X* converges.

(c) $\cap\{\bar{A}: A \in F\}$ For each filter on *X*.

Solution: Suppose (X,τ) is compact and that there exist an ultrafilter F that does not converge. Then for each $x \in X$ There exists $V_x \notin F$ and thus $C = \{V_x : x \in X\}$ is an open cover of X_n.

Hence $\bigcup_{i=1}^{n} V_{x_i} = X$ Since F is an ultrafilter, implies that $V_x^c \in F$ for $x \in X$ and therefore $X^c = \bigcup_{i=1}^{n} V_{x_i}^c = \phi \in F$, which is a contradiction.

Thus, each ultrafilter on X converges Hence (a) $\Rightarrow$(b)

(b) $\Rightarrow$(c): Given any filter F on X; Let G be an ultrafilter containing F then G converges to x in (X, τ) for some $x \in X$ given any neighborhood V of x and $A \in F$, then $A \in G$ and $V \in G$.

NOTES

Hence $A \cap V \in G$ and thus $A \cap V \neq \phi$. Therefore, $x \in \bar{A}$

Suppose (X, τ) is not compact Let $C = \{Q_\alpha : \alpha \in J\}$ be on open cover of X with no finite subcover.

Then $\bigcup_{i=1}^{n} Q_{\alpha_i} \neq X$, for each n. Let F be the filter on X whose base

$\left\{ \bigcup_{i=1}^{n} Q_{\alpha_i}^c : n \geq I, Q_\alpha \in C \right\}$ However $\Lambda\{\bar{A} : A \in F\} \subseteq \Lambda Q_\alpha = (\bigcup_{\alpha \in J} Q_\alpha)^c = X^c = \phi$

which is a contradiction. Hence (c) $\Rightarrow$ (a).

Example 8: (X, τ) is T_2 if and only if each filter converges to at most one point, $F \xrightarrow{r} x, y$ implies $x = y$.

Solution: Suppose that $(X, \tau)$is Hausdorff and space suppose $F \xrightarrow{r} x, y$ where $x \neq y$.

Then there exist $Q_x, Q_y \in \tau$ such that $x \in Q_x$, $y \in Q_y$ and $Q_x \cap Q_y = \phi$

However F $\xrightarrow{r} x, y$ implies that $Q_x \in F$, $Q_y \in F$ and $Q_x \cap Q_y = \phi = F$, which is a contradiction therefore each filter converges to at most one point conversely, suppose that $x \neq y$ and assume $Q_x \cap Q_y = \phi$ for each Q_x, $Q_y \in \tau$, $x \in Q_x$, $y \in Q_y$

We claim that $r = \{Q_x \cap Q_y : \forall Q_x \in \tau, Q_y \in \tau, x \in Q_x, y \in Q_y\}$ is a base for some filter F observe that $(Q_x \cap Q_y)\,(G_x \cap G_y) = (Q_x \cap G_x)(Q_y \cap G_y) \in r$.

Since $(Q_x \cap G_x)$ is an open set containing x and $(Q_y \cap G_y)$ is an open set containing y. Thus r is a base for some filter F since $Q_x \supseteq (Q_x \cap Q_y)$ implies that each $Q_x \in F$

F converges to x likewise $Q_y \supseteq (Q_x \cap Q_y)$ implies that $Q_y \in F$, and thus F converges to y, a contradiction therefore, there does not exist an Q_x and Q_y such that $Q_x \cap Q_y \neq \phi$ where $x \in Q_x$ and $y \in Q_y$ hence,(X, τ) is Hausdorff.

Example 9: Let $f: X \rightarrow Y$ be a function and U is an ultrafilter on X then $f(U)$ is an ultrafilter on Y.

Solution: Assume that $A \in f(U)$ then for each $U \in \bigcup$

$A^c \cap f(U) \neq \phi$

Hence $f^{-1}(A^c) \cap U \neq \phi$ for each U$\in u$ and $f^{-1}(A^c) \in U$

Therefore $f(f^{-1}(A^c) \in f(U)$ and thus $A^c \in f(U)$ since $f(f^{-1}(A^c) \subseteq A^c$ Hence $f(U)$ is an ultrafilter on Y.

NOTES

Example 10: Give some examples of simple directed sets.

Solution:

1. D $=\mathbb{N}$ with its usual ordinary relation $\leq$.
2. Let *(X,T)* be a topological space and let $x \in X$ then the set $D_x := \{U \in T : x \in U\}$ is a directed set when equipped with the either the subset relation $\subseteq$, or more usually the subset relation $\supseteq$.
3. D$=\{\{n, n+1, n+2, \ldots\} \subseteq \mathbb{N}\}$, with the subset relation $\subseteq$ or the super set relation $\supseteq$.
4. If $(D_1 \subseteq_1)$ and $(D_1 \subseteq_2)$ are directed sets then $(D_1 \times D_2, \subseteq)$ is a directed set where $\subseteq$ is defined by $(a,b) \subseteq (x,y)$ if and only if $a \subseteq_1 x$ and $b \subseteq_2 y$.

5.17 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In general topology and the related branches, a net or Moore-Smith sequence is a generalization or simplification of the concept and notion of a sequence. A sequence is defined as a specific function with domain of the defined set of natural numbers and also the range which is normally defined for any topological space. Even though, as per the perspective of topology, the sequences cannot completely encode the entire information regarding a function that is defined between the said topological spaces.
2. In particular, for a specific definite map *f* between topological spaces *X* and *Y*, the two conditions that are given below are not equivalent:
   (i) The map *f* is stated as continuous.
   (ii) Given any point $x \in X$ and any sequence in *X* which converges to *x*, then specifically the composition of *f* with this unique sequence converges to *f*(*x*).
3. A net is termed as maximal iff and only iff for every $A \subseteq X$, it is ultimately defined in either *A* or $X - A$. Maximal nets are precisely termed as ultra nets.
4. In mathematics, a filter is defined to be a special subset of a partially ordered set. Filters are used to study topological spaces and define all basic topological notion, such a convergence, continuity, compactness, etc. A frequently used special case is the situation that the ordered set under consideration is just the power set of some set and ordered by set inclusion. Filters emerge in order and lattice theory but can also be found in topology from where they originate. The dual notion of a filter is an ideal.
5. Consequently, in topological analysis the term filters are precisely used for defining the convergence in a precise manner analogous to the function and specifications of sequences on the basis of a metric space. Furthermore, in topology and other concerned fields of mathematical analysis, a filter is uniquely defined as a generalization or interpretation of a net. Additionally, both nets and filters provide extremely general perspectives and framework for unifying the different notions of limit precisely for the topological spaces that are arbitrary in type.

NOTES

6. Let λ be a net in X with directed set $\mathcal{D}$. Then $\mathcal{F} = \{F \subseteq X : \lambda$ is eventually in $F\}$ and is called the derived filter of λ.
7. Let X be a topological space in which there is no net with two different limits. Then X is Hausdorff.
8. A net $\{x_\lambda\}$ has $y \in X$ as a cluster point if and only if it has a sub net, which converges to y.
9. Let F be a filter on X. F is said to be an ultrafilter if for all $A \subseteq X$ either $A \in F$ or $A^c \in F$.
10. The Ultrafilter Theorem: Let F be a filter on X. Then there is an ultra filter **U** such that $F \subseteq$ **U**.
11. For a topological space X, the following are equivalent:
    (*i*) X is compact.
    (*ii*) Each family C of closed sets in X with the finite intersection property has non-empty intersection.
    (*iii*) Each filter in X has a cluster point.
    (*iv*) Each net in X has a cluster point.
    (*v*) Each ultra net in X converges.
    (*vi*) Each ultra filter in X converges.
12. Let X be a topological space, $\mathcal{F}$ a filter on X, and x a point in X. We say that $\mathcal{F}$ converges to x and write $\mathcal{F} \to x$ if every neighbourhood of x is in $\mathcal{F}$. If $\mathcal{F}$ converges to exactly one point x of X, then we will call that point the limit of $\mathcal{F}$ and write $x = \lim \mathcal{F}$.
13. According to the mathematical analysis in the field of topology, the term local finiteness can be defined as a unique property which satisfies the property of collections of subsets precisely for a topological space. Fundamentally these properties are used for studying the paracompactness and topological dimension. A collection of subsets that are contained in a topological space X are precisely defined as locally finite, if each point in the space uniquely has a neighbourhood which only intersects finitely several sets in the collection. Consequently, for a topological space a finite collection of subsets is termed as locally finite. Further, the infinite collections of sets can also be locally finite.
14. A collection μ of subsets of X if called locally finite or (neighbourhood finite) if and only if for every $x \in X$ has a neighbourhood meeting only finite $\cup \in \mu$. We call μ point-finite if and only if each $x \in X$ belongs to only finitely many $\cup \in \mu$. Clearly every locally finite collection is point-finite.
15. The Nagata-Smirnov Metrization theorem gives a full characterization of metrizable topological spaces. In other words, the theorem describes the necessary and sufficient conditions for a topology on a space to be defined using some metric. As a motivational example, consider the discrete topology on some space (every subset of the space is open).

This topology is actually defined by the following metric:

$$d(x, y) = \begin{cases} 1 \text{ when } x \neq y \\ 0 \text{ when } x = y \end{cases}$$

NOTES

The Nagata-Smirnov Metrization Theorem lists the exact condition that any topology must have in order for there to be such an underlying metric.

16. We shall show first that a paracompact space is regular. Suppose A is closed set in a paracompact space X and $x \notin A$. For each $y \in A$, $\exists$ open set V_y containing y such that $x \notin \overline{V_y}$. Then the sets $V_y; y \in A$ together with the set $X-A$, form an open cover of X. Let w be a locally finite refinement and $V = U\{ W \in w; W \cap A \neq \phi\}$.

    Then V is open and contains A, and $\overline{V} = U\{\overline{W}; W \cap A \neq \phi\}$

    But each set W is contained in some V_y since W is refinement and hence, $\overline{W}$ is contained in $\overline{V_y}$. Hence $x \notin \overline{W}$ (since $x \notin \overline{V_y}$). Thus $x \notin \overline{V}$. But $V \supseteq A$. Thus x and A are separated by open sets in X, i.e., the space is regular.

17. Path composition is only defined when the terminal point of f and the initial point of g coincide. If we consider all loops based at a point x_0, then path composition is termed as a binary operation. Path composition, when precisely defined, is not associative in general due to the difference in parametrization.

18. According to the mathematical analysis in the field of topology, the notion and concept of fundamental group is defined by Henri Poincaré. The fundamental group is referred as a group which is typically associated with any given pointed topological space which provides methodology to determine that when the two paths, namely starting path and ending path at a fixed base point, can be continuously or constantly deformed into each other. Characteristically, it documents the information which significantly defines the basic shapes or holes of the topological space. The concept of fundamental group is the initial and simplest form of the homotopy groups.

19. Covering spaces are considered significant in the analysis and evaluation of homotopy theory, harmonic analysis, Riemannian geometry, and differential topology. In the Riemannian geometry, the ramification is defined as a generalization of the concept of covering maps. Covering spaces are also considered interconnected which was proved by the study of homotopy groups and specifically the fundamental group.

20. The map p is said to be a covering map if p continuously maps E onto X such that each $x \in X$ has an open neighbourhood, which is evenly covered by p.

21. According to the fundamental theorem of algebra, every non-constant single-variable polynomial with complex coefficients has at least one complex root. Equivalently, the field of complex numbers is algebraically closed. This theorem is also stated as every non-zero single-variable polynomial with

complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity.

Fundamental Theorem of Algebra: Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point x with $h(x) = x$.

NOTES

5.18 SUMMARY

- In general topology and the related branches, a net or Moore-Smith sequence is a generalization or simplification of the concept and notion of a sequence.
- A sequence is defined as a specific function with domain of the defined set of natural numbers and also the range which is normally defined for any topological space. Even though, as per the perspective of topology, the sequences cannot completely encode the entire information regarding a function that is defined between the said topological spaces.
- The principle and key concept of the term 'Net' was originally proposed by E. H. Moore and H. L. Smith in the year 1922. The term net was specifically used for generalizing the notion of any sequence in order to validate the equivalence of the specified conditions through the sequence that are replaced/changed by means of net.
- Particularly, instead of defining the net on a countable linearly ordered set, the net can be typically defined on an arbitrary and subjective directed form of set.
- Because the sequences cannot encode the sufficient and essential information about the functions between the topological spaces, therefore the nets are typically used for this analysis since the collections of open sets in the specified topological spaces are similar to directed sets.
- John L. Kelley finally coined term 'Net' and stated that nets are the distinctive tools that are exceptionally used in topology for generalizing specific concepts, notations, and theories that are sufficiently adequate in the perspective of metric spaces. Similarly, the concept of the filter was defined and established by Henri Cartan in the year 1937.
- Since every single non-empty totally ordered set can be directed, therefore, it can be stated that every single function that is precisely defined on such a set is uniquely a 'Net'.
- Specifically, the natural numbers having the standard and common order typically forms or generates precise set and sequence which can be identified and designated as a function on the natural numbers such that every single sequence can be considered as a net.
- When the S increases or expands with respect to $\geq$, then the points x_S defined in context of the net are precisely constrained or restricted to be in decreasing or reducing neighbourhoods of x, consequently this gives the notion that x_S must precisely tend towards x in certain perception.

NOTES

- Eventually a net λ is in $A \subseteq X$ if and only if A contains certain tail of λ. We state that a net λ is frequently in A if and only if for every $d \in \mathcal{D}$ there exists $c \geq d$, such that $\lambda(c) \in A$.
- The net is frequently in a subset Y of V iff for every N in $\mathbf{N}$ there exists some $n \geq N$ such that a_n is in Y, i.e., iff infinitely many elements of the sequence are in Y. Thus a point y in V is a cluster point of the net iff every neighbourhood Y of y contains infinitely many elements of the sequence.
- A filter $\mathcal{F}$ is said to converge to $x \in X$, denoted by $\mathcal{F} \to x$, if and only if every neighbourhood of x belongs to $\mathcal{F}$.

  $\mathcal{N}_x$ is a sub filter of $\mathcal{F}$.
- Filter is a subfield of mathematics and can be used to study topological spaces and define all basic topological notion such a convergence, continuity, compactness, etc.
- The lattice $(P, \leq)$ having a non-empty subset F is defined as a filter iff and only iff it is an upper set which is typically closed under finite meets termed as infima, i.e., for all x, y in F, we can state that $x \wedge y$ is also in F. Additionally, the smallest filter which comprises of a given element p is termed as the principal filter and also p is defined as the principal element in this state.
- Consequently, in topological analysis the term filters are precisely used for defining the convergence in a precise manner analogous to the function and specifications of sequences on the basis of a metric space.
- In topology and other concerned fields of mathematical analysis, a filter is uniquely defined as a generalization or interpretation of a net. Additionally, both nets and filters provide extremely general perspectives and framework for unifying the different notions of limit precisely for the topological spaces that are arbitrary in type.
- The term sequence is typically indexed by means of the natural numbers which are precisely referred as a totally ordered set. Therefore, in the first-countable spaces the limits can be specifically described by means of sequences. Although, if the said space does not have properties of first-countable, then in this case specifically the nets or filters should be used.
- Nets characteristically generalize the notion and concept of a sequence simply involving the index set to be a directed set. Filters can be further considered as the specific types of sets precisely developed from the multiple nets. Therefore, the notion of limit of a filter and also the notion of limit of a net both are conceptually and theoretically defined same as the notion limit of a sequence.
- Conversely, consider that for every neighbourhood U of certain $x \in V$ there typically exists a point of the net which is precisely denoted by λ_U

$\in U - V$. Then $\lambda \to X$ but λ was assumed to be a net in $X - V$, a contradiction. Therefore, there exists some neighbourhood U of x with $U - V = \phi$, whence $U \subseteq V$ so V is a neighbourhood of x for every $x \in V$ and V is open.

NOTES

- Let X be a Hausdorff topological space. Then no net in X can have two different limits.
- If F is a collection of subsets of X which precisely satisfy the properties defined above, then we consider it a filter. In addition, N_x is called the neighbourhood filter of x. Remember that filters are closed under finite intersection as well as pairwise (by induction).
- The complements of singletons generate the cofinite filter, so if F is not generated by a singleton then it contains the cofinite filter and is thus free.
- Let F be a filter and $x \in X$. We say that F converges to x, or that x is a limit of F if $N_x \subseteq F$. We shall write $F \to x$ to mean F converges to x.

  Let X be a topological space. X is Hausdorff iff every filter has at most one limit.
- An ultra filter on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion, i.e., it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F}' \supset \mathcal{F}$ actually satisfies $\mathcal{F}' = \mathcal{F}$.
- According to the mathematical analysis in the field of topology, the term local finiteness can be defined as a unique property which satisfies the property of collections of subsets precisely for a topological space. Fundamentally these properties are used for studying the paracompactness and topological dimension.
- A collection of subsets that are contained in a topological space X are precisely defined as locally finite, if each point in the space uniquely has a neighbourhood which only intersects finitely several sets in the collection. Consequently, for a topological space a finite collection of subsets is termed as locally finite. Further, the infinite collections of sets can also be locally finite.
- A topological space in which every open cover admits a locally finite open refinement is called paracompact. Every locally finite collection of subsets of a topological space X is also point-finite.
- A topological space in which every open cover admits a point-finite open refinement is termed as metacompact.
- A collection in a space is countably locally finite (or σ-locally finite) if it is the union of a countable family of locally finite collections of subsets of X.
- Countable local finiteness is a significant hypothesis in the Nagata-Smirnov metrization theorem, which states that a topological space

NOTES

is metrizable iff it is regular Hausdorff and has a countably locally finite basis.

- The paths f_0 and f_1 connected by a homotopy are said to homotopic. Similarly, a homotopy of loops can be defined keeping the base point fixed. The relation of being homotopic is an equivalence relation on paths in a topological space.
- According to the mathematical analysis in the field of topology, the notion and concept of fundamental group is defined by Henri Poincaré.
- The fundamental group is referred as a group which is typically associated with any given pointed topological space which provides methodology to determine that when the two paths, namely starting path and ending path at a fixed base point, can be continuously or constantly deformed into each other. Characteristically, it documents the information which significantly defines the basic shapes or holes of the topological space.
- The concept of fundamental group is the initial and simplest form of the homotopy groups. Basically, the fundamental groups can be studied on the basis of the theory of covering spaces, because a fundamental group precisely coincides with the group of surface transformations of the related or associated universal covering space.
- The abelianization of the fundamental groups can be identified and defined with the help of first homology group of the space.
- Covering spaces are considered significant in the analysis and evaluation of homotopy theory, harmonic analysis, Riemannian geometry, and differential topology.
- In the Riemannian geometry, the ramification is defined as a generalization of the concept of covering maps.
- Covering spaces are also considered interconnected which was proved by the study of homotopy groups and specifically the fundamental group.
- In particular, the covering maps are referred as the locally trivial which specifies that locally each and every covering map is isomorphic to a projection such that there is a unique homeomorphism from the pre-image of an evenly covered neighbourhood U to $U \times F$, where F is termed as the fiber which essentially satisfies the local trivialization condition. Consequently, if this homeomorphism is projected onto U, then the derived structure or resultant composition is equal to p.
- According to the fundamental theorem of algebra, every non-constant single-variable polynomial with complex coefficients has at least one complex root. Equivalently, the field of complex numbers is algebraically closed. This theorem is also stated as: every non-zero single-variable polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted up to its multiplicity.

5.19 KEY TERMS

NOTES

- **Moore-Smith sequence:** In general topology and the related branches, a net or Moore-Smith sequence is a generalization or simplification of the concept and notion of a sequence.
- **Sequence:** A sequence is defined as a specific function with domain of the defined set of natural numbers and also the range which is normally defined for any topological space.
- **Filter:** In mathematics, a filter is defined to be a special subset of a partially ordered set.
- **Local finiteness:** According to the mathematical analysis in the field of topology, the term local finiteness can be defined as a unique property which satisfies the property of collections of subsets precisely for a topological space. Fundamentally these properties are used for studying the paracompactness and topological dimension.
- **Nagata-Smirnov metrization theorem:** The Nagata-Smirnov metrization theorem gives a full characterization of metrizable topological spaces.
- **Fundamental group:** According to the mathematical analysis in the field of topology, the notion and concept of fundamental group is defined by Henri Poincaré. The fundamental group is referred as a group which is typically associated with any given pointed topological space which provides methodology to determine that when the two paths, namely starting path and ending path at a fixed base point, can be continuously or constantly deformed into each other.

5.20 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is net?
2. Define subbase for a topology.
3. Give the uses of filter in topology.
4. Define convergence of net.
5. State the necessary and sufficient condition for a topological space to be Hausdorff.
6. Define the term compactness.
7. State about the filter and their convergence.
8. What is collection of subset?
9. Define the canonical way of converting nets to filters.
10. What do you understand by ultrafilter?

NOTES

11. Define local finiteness.
12. State the Nagata-Smirnov metrization theorem.
13. Define the term paracompactness.
14. Define the homotopy of paths.
15. What is the fundamental group?
16. What do you understand by covering space?
17. State the fundamental theorem of algebra.

Long-Answer Questions

1. Describe briefly about net and filter with the help of appropriate examples.
2. Discuss about the convergence of net and filter with reference to topology.
3. Describe Hausdorffness and compactness with reference to nets.
4. Explain in detail about the canonical way of converting nets to filters and vice versa giving relevant examples.
5. Elaborate on the ultrafilter and local finiteness giving appropriate examples.
6. State and prove Nagata-Smirnov metrization theorem.
7. Describe the paracompactness and homotopy of paths in topology with the help of relevant examples.
8. Explain in detail about the fundamental group giving examples.
9. Prove that the product of a covering space is a also covering space giving relevant examples.
10. State and prove the fundamental theorem of algebra giving examples.

5.21 FURTHER READING

Munkres, James R. 2011. *Topology*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.

Basener, William F. *Topology and Its Applications*. New Jersey: John Wiley & Sons, Inc.

Bourbaki, Nicolas. 1966. *General Topology: Elements of Mathematics*, Volume 2. United States: Addison-Wesley.

Bourbaki, Nicolas. 1966. *Elements of Mathematics: General Topology*, Part 1. United States: Addison-Wesley.

Kelley, John L. 1975. *General Topology*. New York: Springer-Verlag.

Willard, Stephen. 2004. *General Topology*. United States: Dover Publications.

Shilov, Georgi E. 2012. *Elementary Real and Complex Analysis*. Chelmsford: Courier Corporation.

Simmons. G. F. 2004. *Introduction to Topology and Modern Analysis*. New York: Tata McGraw-Hill Education.