M.Sc. Previous Year
Physics, MP-03

CLASSICAL ELECTRODYNAMICS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

MADHYA PRADESH BHOJ (OPEN) UNIVERSITY - BHOPAL

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SYLLABI-BOOK MAPPING TABLE

Classical Electrodynamics

Syllabi	Mapping in Book
UNIT 1 Electrostatics Uniqueness Theorem, Solution of Laplace and Poisson Equation in Rectangular, Cartesian and Spherical Polar Coordinates, Methods of Electric Images, Greens Function for Potential Problem, Solutions of Conducting and Dielectric Sphere in Uniform Electric Field. Laws of Magnetostatics, Magnetic Scalar and Vector Potential, Magnetisation, Vector Magnetic Susceptibility and Permeability, Magneto-Static Energy, Uniformly Magnetised Sphere in Magnetic Field, Classical Theories of Para, Dia and Ferro Magnetism, Magnetic Circuits and Their Comparison with Electric Circuits. Radiation from an Accelerated Charge at Low Velocity (Larmor's Formula), Radiation from an Oscillating Electric Dipole, Linear Antenna, Radiation from a Charged Particle Moving in a Circular Orbit, Electric Quadrupole Radiation, Radiation Damping.	**Unit-1:** Electrostatics and Magnetostatics **(Pages 3-87)**
UNIT 2 Electromagnetics Time Varying Fields, Maxwell's Electromagnetic Field Equations in Stationary and Moving Media, Electromagnetic Scalar Wave Equations and Their Solution, Hertz Vector, Plane Wave Propagation in Conducting and Ionised Media, Radiation Pressure and Momentum, Reflection, Refraction, Total Internal Reflection, Polarisation, Scattering (Rayleigh and Thomson) and Dispersion of Plane E. M. Waves, Elements of Wave Guides.	**Unit-2:** Electromagnetics **(Pages 89-176)**
UNIT 3 Electromagnetic Radiation Retarded Potential, Lienard-Wiechert Potentials due to Uniformly Moving and Accelerated Charges, Lorentz Formula, Bremsstrahlung. Radiation from an Accelerated Charge at Low Velocity (Larmor's Formula), Radiation from an Oscillating Electric Dipole, Linear Antenna, Radiation from a Charged Particle Moving in a Circular Orbit, Electric Quadrupole Radiation, Radiation Damping.	**Unit-3:** Electromagnetic Radiation **(Pages 177-210)**
UNIT 4 Plasma Physics Concept of Plasma, Plasma Oscillation, Debye Shielding, Plasma Parameters, Magnetoplasma, Plasma Confinement, Hydrodynamical Desorption of Plasma, Fundamental Equations, Hydromagnetic Waves, Magnetosonic Wave and Alfvén Wave, Wave Phenomenon in Magnetoplasma, Phase and Group Velocity Cut Offs, Resonance for Electromagnetic Wave Propagating Parallel and Perpendicular to the Magnetic Field, Appleton–Hartree Formula, Propagation Through Ionosphere and Magneto-Sphere Helicon, Whistle, Faraday Rotation.	**Unit-4:** Plasma Physics **(Pages 211-237)**

UNIT 5 Relativistic Formulations Covariant Formulation of Electrodynamics, Continuity Equation, Lorentz Force, Potentials, Operators, Electromagnetic Field Tensor, Transformation of Fields, Transformation of Field due to a Point Charge in Uniform Motion, Lagrangian and Hamiltonian Formulation of the Motion of a Charged Particle in an Electromagnetic Field, Radiation from Relativistically Moving Particles.	**Unit-5:** Relativistic Formulations **(Pages 239-265)**

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INTRODUCTION

NOTES

Classical electrodynamics or classical electromagnetism is a branch of theoretical physics that studies the interactions between electric charges and currents using an extension of the classical Newtonian model. The theory provides a description of electromagnetic phenomena whenever the relevant length scales and field strengths are large enough that quantum mechanical effects are negligible. For small distances and low field strengths, such interactions are better described by quantum electrodynamics.

Electromagnetism is a branch of physics involving the study of the electromagnetic force, a type of physical interaction that occurs between electrically charged particles. The electromagnetic force is carried by electromagnetic fields composed of electric fields and magnetic fields, is responsible for electromagnetic radiation, such as light, and is one of the four fundamental interactions, commonly termed as the forces in nature. The other three fundamental interactions are the strong interaction, the weak interaction, and gravitation. At high energy the weak force and electromagnetic force are unified as a single electroweak force. Electromagnetic phenomena are defined in terms of the electromagnetic force, sometimes called the Lorentz force, which includes both electricity and magnetism as different manifestations of the same phenomenon. The electromagnetic force plays a major role in determining the internal properties of most objects encountered in daily life and are responsible for the chemical bonds between atoms which create molecules, and intermolecular forces.

There are numerous mathematical descriptions of the electromagnetic field. In classical electrodynamics, electric fields are described as electric potential and electric current. In Faraday's law, magnetic fields are associated with electromagnetic induction and magnetism, and Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents.

This book, *Classical Electrodynamics*, is divided into five units which will help to understand the basic concepts of classical thermodynamics, electrostatics and electromagnetic, such as electrostatics, laws of magnetostatics, magnetic circuits, electric circuits, electromagnetics, polarisation, scattering (Rayleigh and Thomson), electromagnetic radiation, Lorentz formula, Bremsstrahlung, plasma physics, and relativistic formulations. The book follows the Self-Instruction Mode or the SIM format wherein each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The content is presented in a simple and structured form interspersed with Answers to 'Check Your Progress' for better understanding. A list of 'Summary' along with a 'Key Terms' and a set of 'Self-Assessment Questions and Exercises' is provided at the end of each unit for effective recapitulation.

UNIT 1 ELECTROSTATICS AND MAGNETOSTATICS

NOTES

Structure

1.0 INTRODUCTION

Electrostatics is a branch of physics that studies electric charges at rest (static electricity). The term static means a situation where the field does not vary with time. Static electric field also referred as electrostatics is created by the fixed charges in space. Electrostatic phenomena arise from the forces that electric charges exert on each other. Such forces are described by Coulomb's law. Even though electrostatically induced forces seem to be rather weak, some electrostatic forces, such as the one between an electron and a proton, that together make up a hydrogen atom, is about 36 orders of magnitude stronger than the gravitational force acting between them.

Coulomb's law states that, 'The magnitude of the electrostatic force of attraction or repulsion between two point charges is directly proportional to the product of the magnitudes of charges and inversely proportional to the square of the distance between them.'

The force is along the straight line joining them. If the two charges have the same sign, the electrostatic force between them is repulsive; if they have different signs, the force between them is attractive.

NOTES

Magnetostatics is the study of magnetic fields in systems where the currents are steady (not changing with time). It is the magnetic analogue of electrostatics, where the charges are stationary. The magnetization need not be static; the equations of magnetostatics can be used to predict fast magnetic switching events that occur on time scales of nanoseconds or less. Magnetostatics is even a good approximation when the currents are not static — as long as the currents do not alternate rapidly. Magnetostatic focussing can be achieved either by a permanent magnet or by passing current through a coil of wire whose axis coincides with the beam axis.

In this unit, you will study about the electrostatics, uniqueness theorem, solution of Laplace and Poisson equation in rectangular, Cartesian and spherical polar coordinates, methods of electric images, Greens function for potential problem, solutions of conducting and dielectric sphere in uniform electric field, laws of magnetostatics, magnetisation, uniformly magnetised sphere in magnetic field, classical theories of para, dia and ferro magnetism, magnetic circuits and their comparison with electric circuits.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the basic concept of electrostatics
- Understand the importance of uniqueness theorem
- Solve Laplace and Poisson equation in rectangular, Cartesian and spherical polar coordinates
- Explain the methods of electric images
- Discuss Greens function for potential problem
- Solve conducting and dielectric sphere in uniform electric field
- State the laws of magnetostatics and magnetisation
- Describe uniformly magnetised sphere in magnetic field
- Elaborate on the classical theories of para, dia and ferro magnetism
- Understand magnetic circuits and compare it with electric circuits

1.2 ELECTROSTATICS

The term static means a situation where the field does not vary with time. Static electric field also referred as electrostatics is created by the fixed charges in space.

There are many examples of electrostatic phenomena such as the attraction of the plastic wrap to your hand after you remove it from a package, and the attraction of paper to a charged scale, to the apparently spontaneous explosion of grain silos, the damage of electronic components during manufacturing, and photocopier and laser printer operation. Electrostatics involves the build-up of charge on the surface of objects due to contact with other surfaces. Although charge exchange happens whenever any two surfaces contact and separate, the effects of charge exchange are usually only noticed when at least one of the surfaces has a high resistance to electrical flow. This is because the charges that transfer are trapped there for a time long enough for their effects to be observed. These charges remains on the object until they either bleed off to ground or are quickly neutralized by a discharge: for example, the familiar phenomenon of a static 'Shock' is caused by the neutralization of charge built up in the body from contact with insulated surfaces.

Determination of the electrostatic field components, such as electric field, electric force, and electric flux density are explained by two important laws namely, Coulomb's law and Gauss law.

Coulomb's Law

Coulomb's law provides the relation between forces experienced by the charges when they are separated by a distance. This theory was first proposed by Coulomb in 1785. This law states that,

Force, *F* exerted between two point charges Q_1 and Q_2 as shown in Figure (1.1) is,

⇒ Directly proportional to the product of the two charges and

⇒ Inversely proportional to the square of the distance between the two charges.

⇒ The direction of the force will be in the same direction along the line joining the two charges.

Mathematically, Coulomb's law may be expressed as,

$$\vec{F} \propto \frac{Q_1 Q_2}{R^2}\,\overrightarrow{a_R}$$

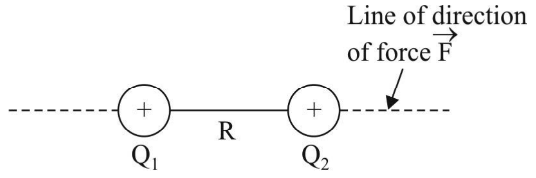

Fig. 1.1 *Coulomb's Force*

Removing the proportionality,

$$\vec{F} = k\frac{Q_1 Q_2}{R^2}\,\overrightarrow{a_R} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\,\overrightarrow{a_R}$$

Where,

$\overrightarrow{a_R}$ = Unit vector in the line of direction of force, *F*

$Q_1 Q_2$ = Charges

R = Distance seprating the charges (m^2)

$$k = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 m/F$$

Where, ϵ_0 = Permittivity in free space = $8.854 \times 10^{-12} = \frac{10^{-9}}{36\pi}\ F/m.$

NOTES

Now, assume two charges Q_1 and Q_2 at a distance of r_1 and r_2, respectively, from an observing point as shown in Figure (1.2). The force exerted by charge on is given by,

$$\overrightarrow{F_{12}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\overrightarrow{a_{R_{12}}}$$

Where

$$\overrightarrow{a_{R_{12}}} = \frac{\overrightarrow{R_{12}}}{|\overrightarrow{R_{12}}|}$$

And

$$\overrightarrow{R_{12}} = r_2 - r_1$$

Therefore,

$$\overrightarrow{a_{R_{12}}} = \frac{r_2 - r_1}{|r_2 - r_1|};$$

$$\overrightarrow{F_{12}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\frac{\overrightarrow{R_{12}}}{R_{12}^3}$$

$$\therefore \qquad \text{Force, } \vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0}\left\{\frac{(x_2 - x_1)\overrightarrow{a_x} + (y_2 - y_1)\overrightarrow{a_y}}{\left[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]^3}\right\}$$

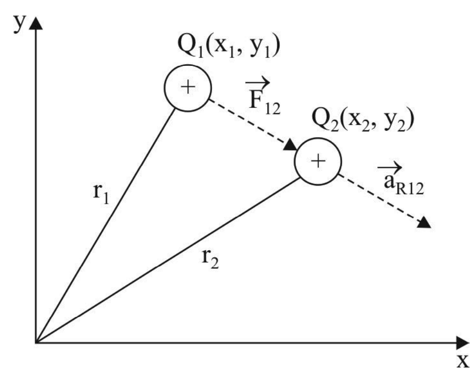

Fig. 1.2 *Coulomb's Force on Charges at a Distance*

Similarly, for force exerted by charge Q_2 on Q_1 is given by,

$$\overrightarrow{F_{21}} = -\overrightarrow{F_{12}} \qquad \left[\because\ \overrightarrow{a_{R_{12}}} = -\overrightarrow{a_{R_{21}}}\right]$$

For Many Charges: Generalising, the above expression when many charges are present,

$$\vec{F} = \frac{Q}{4\pi\epsilon_0}\sum_{i=1}^{N} Q_i\left[\frac{\vec{r} - \vec{r_i}}{|\vec{r} - \vec{r_i}|^3}\right]$$

NOTES

Electric Field Intensity $(\vec{E})$

Electric field intensity is defined as the strength of electric field at any point. It is equal to force per unit charge as experienced by test charge kept at that point. Therefore, it is expressed as,

$$\vec{E} = \frac{\vec{F}}{Q}$$

Also,

$$\vec{E} = \frac{1}{Q}\left[\frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\right]\overrightarrow{a_R}$$

Assuming that $Q = Q_1 = Q_2$,

$$\vec{E} = \left[\frac{Q}{4\pi\epsilon_0 R^2}\right]\overrightarrow{a_R}$$

In general,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} Q_i\left[\frac{\vec{r}-\vec{r_i}}{|\vec{r}-\vec{r_i}|^3}\right]$$

Charge Distribution

The presence of charge Q ensures the existence of electric field $\vec{E}$. The charges may be distributed on a line conductor, on a surface or inside a volume. Hence, based on the charge distribution,

Along a line, charge, $Q = \int_L \rho_L \,.\overrightarrow{dl}$

On a surface, charge, $Q = \int_S \rho_S .\overrightarrow{ds}$

Inside a volume, charge, $Q = \int_V \rho_V .\, dv$

Where, $\rho_L =$ line charge density (C/m)

$\rho_S =$ surface charge density (C/m^2)

$\rho_V =$ volume charge density (C/m^3)

Based on the above distribution of the charges on line, surface and volume, the electric field intensity, $\vec{E}$ can be given as,

Along a line, $\vec{E} = \frac{1}{4\pi\epsilon_0 R^2}\int_L \rho_L \,.\overrightarrow{dl}\,.\overrightarrow{a_R}$

On a surface, $\vec{E} = \frac{1}{4\pi\epsilon_0 R^2}\int_S \rho_S .\overrightarrow{ds}\,.\overrightarrow{a_R}$

Inside a volume, $\vec{E} = \frac{1}{4\pi\epsilon_0 R^2}\int_V \rho_V .\, dv\,.\overrightarrow{a_R}$

NOTES

Electric Field Intensity Due to a Line Charge

In this section, let us derive the electric field intensity, $\vec{E}$ due to a line charge. Cartesian coordinate system is considered for the analysis. Consider a uniformly charged line of length '*L*' with line charge density, $\rho_L(C/m)$. An incremental elemental length '*dl*' is considred for the analysis from an observing point '*P*' at a distance '*r*'. The arrangement is depicted in Figure (1.3).

The electric field along the line is given as,

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^2}\int_L \rho_L\,.\vec{dl}\,.\vec{a_r}$$

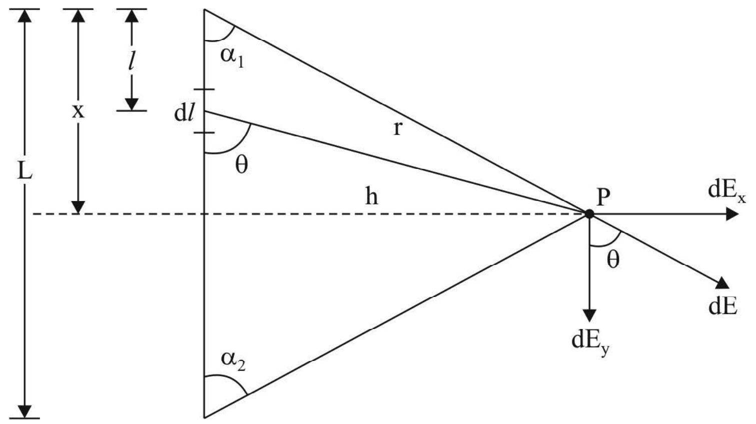

Fig. 1.3 *Charge Distribution Due to Line Charge*

Electric field intensity due to a small elemental length '*dl*' is given as,

$$\vec{dE} = \frac{1}{4\pi\epsilon_0 r^2}\,\rho_L\,.\vec{dl}.\vec{a_r}$$

The electric field at point '*P*'will be at an angle θ with respect to the normal axis. Hence, can be resolved in to *x*-component and *y*-component.

Therefore, $dE_x = dE \sin\theta$

And, $dE_y = dE \cos\theta$

Substituting the magnitude of *dE* in the above expressions,

$$dE_x = \frac{\rho_L\,.dl}{4\pi\epsilon_0 r^2}\sin\theta \qquad ...(1.1)$$

To obtain E_x, then the above expression needs to be integrated over length '*L*' and hence '*r*' must be determined.

From Figure (1.3), $\sin\theta = h/r$?

Therefore, $r = \frac{h}{\sin\theta} = h\ cosec\ \theta$

Also, from Figure (1.3),

$$\tan\theta = \frac{h}{x-l}$$

NOTES

$$x - l = \frac{h}{\tan\theta}$$

$$x - l = h\cot\theta$$

$$-dl = -h\, cosec^2\theta\, d\theta$$

Substituting *dl* and *r* in the Equation (1.1),

$$dE_x = \frac{-\rho_L \,.\sin\theta}{4\pi\epsilon_0 (h^2\, cosec^2\theta)}\ (h\, cosec^2\theta\, d\theta)$$

$$dE_x = \frac{-\rho_L \,.\sin\theta}{4\pi\epsilon_0 h}\ d\theta$$

Integrating from α_1 to $\pi - \alpha_2$ for the entire length of the wire,

$$E_x = \int_{\alpha_1}^{\pi-\alpha_2} \frac{-\rho_L \,.\sin\theta}{4\pi\epsilon_0 h} d\theta$$

$$= \frac{-\rho_L}{4\pi\epsilon_0 h}\ [-\cos\theta]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_x = \frac{\rho_L}{4\pi\epsilon_0 h}[\cos\alpha_1 + \cos\alpha_2]$$

Similarly from dE_y,

$$dE_y = \frac{\rho_L \,.\cos\theta}{4\pi\epsilon_0 h}\ d\theta$$

$$\therefore E_y = \int_{\alpha_1}^{\pi-\alpha_2} \frac{\rho_L \,.\cos\theta}{4\pi\epsilon_0 h}\ d\theta$$

$$E_y = \frac{\rho_L}{4\pi\epsilon_0 h}\int_{\alpha_1}^{\pi-\alpha_2} \cos\theta\ d\theta$$

$$= \frac{\rho_L}{4\pi\epsilon_0 h}[\sin\theta]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_y = \frac{\rho_L}{4\pi\epsilon_0 h}[\sin\alpha_2 - \sin\alpha_1]$$

There are two conditions associated with α. They are,

Case (i) If $\alpha = 0$, then $E_y = 0$ and $E_x = E = \frac{\rho_L}{2\pi\epsilon_0 h}$

Case (ii) If $\alpha = \alpha_1 = \alpha_2$, then $E_y = 0$ and $E_x = E = \frac{\rho_L}{2\pi\epsilon_0 h}\cos\alpha$

Electric Field Intensity Due to a Ring of Charge

NOTES

Consider a ring as shown in Figure (1.4), filled with charge Q. The x-axis is perpendicular to the ring and is at the center of the ring. The objective is to find the electric field at P due to the ring of radius 'R'.

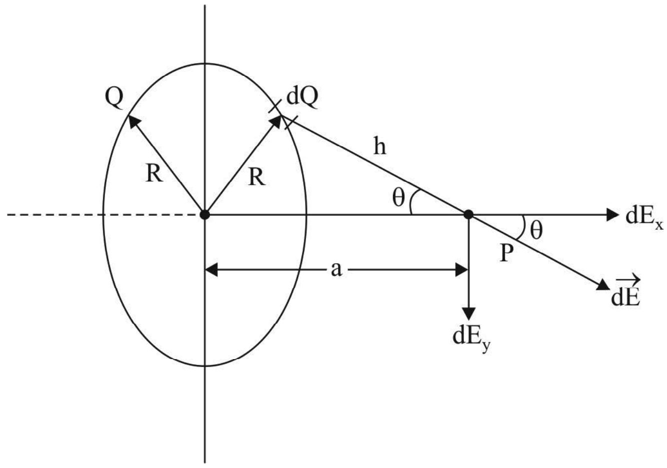

Fig. 1.4 *Ring of Charge*

Consider a small elemental charge, dQ on the ring. The electric field $\overrightarrow{dE}$ at point P is given as,

$$dE = k\frac{dQ}{h^2} = k\frac{dQ}{(R^2 + a^2)}$$

The x-component of dE is dE_x and is given as,

$$dE_x = dE\cos\theta = k\frac{dQ}{(R^2 + a^2)}\cos\theta$$

But, from Figure (1.4),

$$\cos\theta = \frac{a}{\sqrt{R^2 + a^2}}$$

$$dE_x = k\frac{dQ}{(R^2 + a^2)}\frac{a}{(R^2 + a^2)^{\frac{1}{2}}}$$

$$dE_x = k\frac{dQ\ (a)}{(R^2 + a^2)^{3/2}}$$

Referring to the Figure (1.4), neither k, R or a changes. Hence,

$$E_x = \int dE_x = \int k\frac{dQ\ (a)}{(R^2 + a^2)^{\frac{3}{2}}}$$

$$E_x = k\frac{(a)}{(R^2 + a^2)^{\frac{3}{2}}}\int dQ$$

$$E_x = k\frac{Q\ (a)}{(R^2 + a^2)^{3/2}}$$

When $R \rightarrow 0$, ring represents a point charge, therefore,

$$E \approx \frac{kQ(a)}{a^2} \approx \frac{kQ}{a}$$

Where $$k = \frac{1}{4\pi\epsilon_0}$$

NOTES

Electric Field Intensity Due to a Circularly Charged Disc

Unlike the previous structure of a ring, consider a disc of radius, R. The disc consists of a uniformly charged surface charge density of ρ_s C/m^2. Consider an elemental ring of radius dr at a distance 'r' from the center. The electric field at a point P is given as,

$$dE = k\,\frac{\rho_S.ds}{h^2}$$

The horizontal and vertical components of dE are dE_x and dE_y. The horizontal component dE_x is zero and the vertical component is given as,

$$dE_y = dE \cos\theta$$

$$dE_y = k\,\frac{\rho_S.ds}{h^2}\cos\theta$$

We know that for the differential surface element ds,

$$ds = 2\pi r\,dr$$

$$dE_y = k\,\frac{\rho_S.(2\pi r\,dr)}{h^2}\cos\theta$$

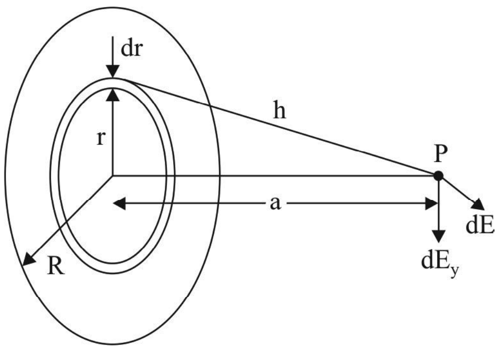

Fig. 1.5 *Electric Field Due to a Circularly Charged Disc*

From Figure (1.5),

$$\tan\theta = \frac{r}{a}$$

$$r = a\tan\theta$$

$$dr = a\sec^2\theta\;d\theta$$

NOTES

$$h = \frac{r}{\sin\theta}$$

Therefore,

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{\rho_S.(2\pi r)\ (a\sec^2\theta\ \ d\theta)}{\left(\frac{r}{\sin\theta}\right)^2}\cos\theta$$

$$dE_y = \frac{\rho_S(2\pi r)\ (a\,sec\,\theta\sin^2\theta\ \ d\theta)}{2\epsilon_0\tan\theta} \qquad [\because \tan\theta = r/a]$$

$$dE_y = \frac{\rho_S}{2\epsilon_0}\sin\theta\ \ d\theta \qquad [\tan\theta = \sec\theta\sin\theta]$$

Total electric field is given as,

$$E = \int_{\theta=0}^{\alpha} dE_y = \frac{\rho_S}{2\epsilon_0}\int_{\theta=0}^{\alpha}\sin\theta\ \ d\theta$$

$$E = \frac{\rho_S}{2\epsilon}(1-\cos\alpha)$$

$$E = \frac{\rho_S}{2\epsilon}\left[1 - \frac{a}{\sqrt{a^2+R^2}}\right]$$

Electric Flux Density

Electric flux density is an **imaginary field lines** that do not exist unlike magnetic field lines. Electric flux density do not exist practically and generally considered for theoretical reasoning only. Electric flux density is related to electric field by the following reason,

$$\vec{D} = \epsilon_0\vec{E}$$

Electric flux density $\vec{D}$ is independent of the medium and may also be defined in terms of electric flux ψ as,

$$\psi = \int \vec{D}.\vec{ds}$$

All the electric field expressions derived earlier can be substituted in the electric flux density expressions. Therefore, electric flux density due to a long conductor of charges is given as,

$$\vec{D} = \epsilon_0\vec{E}$$

Electric flux density due to a ring of charges is given by,

$$\vec{D} = \frac{aQ}{4\pi(R^2+a^2)^{3/2}}$$

Electric flux density due to a circularly charged disc is given by,

$$\vec{D} = \frac{\rho_S}{2}\left[1 - \frac{a}{\sqrt{a^2+R^2}}\right]$$

Gauss's Law and Applications

Gauss' law is a powerful tool for the calculation of electric fields. The applications of Gauss law includes determination of electric field due to a point charge, sheet of charge, line charge on surface of conductor and sphere of charges.

Gauss law states that total flux through a closed surface is equal to the charge enclosed by that surface. Mathematically, it is given as,

Electric flux, $\psi = Q$ (Charge enclosed).

Maxwell's Equation - I

From Gauss law, we know that, $\psi = Q$. Also, from the basic definition for electric flux ψ and charge Q on a volume,

$$\psi = \oint d\psi = \oint \vec{D}.\overrightarrow{ds} \qquad ...(1.2)$$

and

$$Q = \int_v \rho_v.dv \qquad ...(1.3)$$

Therefore, equating Equation (1.2) and Equation (1.3),

$$\oint_S \vec{D}.\overrightarrow{ds} = \int_v \rho_v.dv \qquad ...(1.4)$$

Applying divergence theorem on the LHS of the above expression in Equation (1.4),

$$\int_v \vec{\nabla}.\vec{D}\, dv = \int_v \rho_v.dv$$

Therefore,

$$\vec{\nabla}.\vec{D} = \rho_v \qquad ...(1.5)$$

Relating the units of the above expression in Equation (1.5),

$$\rho_v(C/m^3) = \vec{\nabla}.\{\vec{D}(C/m^2)\} = \{\vec{\nabla}.\vec{D}\}\,(C/m^3)$$

Equation (1.5) is called as **Maxwell's first equation** expressed in **differential form** and Equations (1.4) is called **Maxwell's first equation** expressed in **integral form**.

Gaussian Surfaces - Gauss's Law Application

A mathematically closed surface is called as a Gaussian surface. These surfaces are assumed to have a uniform symmetric charge distribution which are ideal for determining the electric field vector, $\vec{E}$ by applying Gauss law. Also, the electric flux density vector, $\vec{D}$ is assumed to act tangentially or normally on the Gaussian surface. Therefore, accordingly, when $\vec{D}$ is normal, then

$$\vec{D}.\overrightarrow{dS} = DdS$$

And when $\vec{D}$ is acting tangential,

$$\vec{D}.\overrightarrow{dS} = 0$$

NOTES

(a) Determining $\vec{D}$ Due to a Point Charge

Consider a point charge, Q located at point P as shown in Figure (1.6).

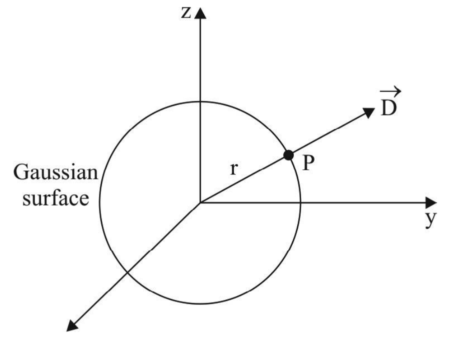

Fig. 1.6 *Electric Flux Density $\vec{D}$ Due to a Point Charge*

According to Gauss law,

$\psi = Q$

And,

$$Q = \oint \vec{D} \,.\, \overrightarrow{dS}$$

Assuming that $\vec{D}$ is normal to the Gaussian surface,

$$Q = \oint D.\, dS = D \oint dS$$

$$Q = D \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$Q = D(4\pi r^2)$$

$$\therefore \quad D = \frac{Q}{4\pi r^2}$$

$$\vec{D} = \frac{Q}{4\pi r^2} \overrightarrow{a_n}$$

(b) Determining $\vec{D}$ Due to Infinite Line Charge

The infinite line conductor is a cylindrical surface and hence, $\vec{D}$ needs to be operated in cylindrical coordinate system, and hence assuming $\vec{D}$ to be normal to the Gaussian surface as shown in Figure (1.7),

$$\vec{D} = D_\rho \overrightarrow{a_\rho}$$

Also, we know that,

$$Q = \rho_L . dl$$

Since the length of the conductor is assumed to be infinite with length '*l*',

$$Q = \rho_L . l = \oint \vec{D} \,.\, \overrightarrow{dS} = \oint D_\rho . \overrightarrow{a_\rho}\, \overrightarrow{dS}$$

$$\Rightarrow\ Q = \rho_L . l = D_\rho (2\pi\rho) . l\, \overrightarrow{a_\rho} \qquad \left[\because \oint \overrightarrow{dS} = 2\pi\rho l\right]$$

NOTES

Therefore,

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\overrightarrow{a_\rho}$$

Or

$$D = \frac{\rho_L}{2\pi\rho}$$

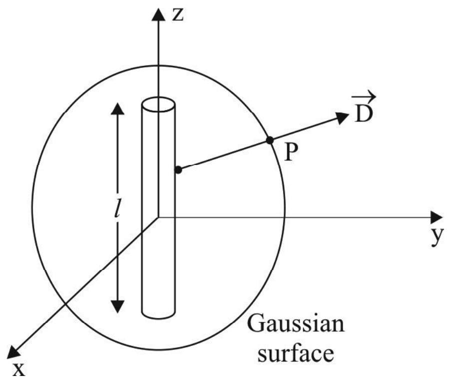

Fig. 1.7 *Electric Flux Density $\vec{D}$ Due to Infinite Line Charge*

(c) Determining $\vec{D}$ due to Charged Sphere

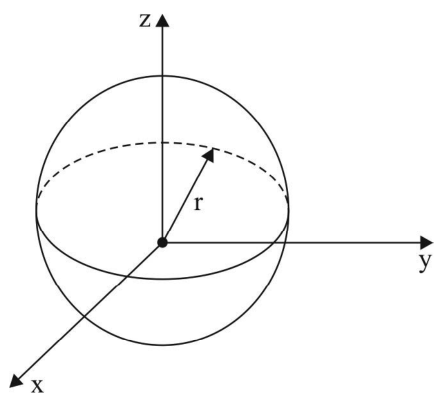

Fig. 1.8 *Electric Flux Density $\vec{D}$ Due to Charged Sphere*

Consider a sphere of radius, *r*. Electric flux density, $\vec{D}$ may either be inside the sphere ($R < r$) or outside the sphere ($R > r$). Hence accordingly, we have two cases to analysis as follows:

Case (i) When $R < a$

We know that,

$\psi = Q$

RHS:

$$Q = \int_V \rho_V . dV = \rho_V \int_V dV = \rho_V \int_{\rho=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$$

NOTES

$$Q = \rho_V \left[\frac{4}{3}\pi R^3\right] \quad ...(1.6)$$

LHS:

$$\psi = \oint_S \vec{D}\,.\,\overrightarrow{dS} = D_\rho \oint_S dS = D_\rho \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\theta \, d\phi$$

$$\psi = D_\rho \,[4\pi R^2] \quad ...(1.7)$$

Equating Equation (1.6) and Equation (1.7),

$$D_\rho = \frac{\rho_V\left(\frac{4}{3}\pi R^3\right)}{4\pi R^2}$$

$$D_\rho = \frac{\rho_V (R)}{3}$$

$$\vec{D} = \frac{R}{3}\rho_V . \overrightarrow{a_\rho}$$

Case (ii) When *R* > *a*

RHS:

$$Q = \int_V \rho_V . dV == \rho_V \left[\frac{4}{3}\pi R^3\right]$$

LHS:

$$\psi = D_\rho \,[4\pi R^2]$$

$$D_\rho = \frac{r^3}{3R^2}\rho_V \qquad or \qquad \vec{D} = \frac{r^3}{3R^2}\rho_V \,\overrightarrow{a_\rho}$$

$$\vec{D} = \begin{cases} \frac{R}{3}\rho_V \overrightarrow{a_\rho} & 0 < r < a \\ \frac{r^3}{3R^2}\rho_V \overrightarrow{a_\rho} & r > a \end{cases}$$

Electric Potential (V)

Electric field, $\vec{E}$ can be obtained by the following three ways.

(1) By using Coulomb's Law

$$-\vec{E} = \vec{F}/Q$$

(2) By using Gauss's Law

$$-\vec{E} = \frac{\vec{D}}{\epsilon_0}$$

(3) By a Scalar Potential Function, *V*

It is always simpler to determine the electric field, $\vec{E}$ by using the vector fields $\vec{F}$ and $\vec{D}$. Hence it is imperative to determine *V*.

NOTES

The scalar potential V is defined as the amount of work done in moving a charge Q. Hence V is expressed as,

$V = W/Q$

When the work done is to move the charge from A to B, then the potential is renamed as '**Potential Difference**'. Consider moving a charge Q from A to B subjected to an electric field $\vec{E}$. From Coulomb's law, the force experienced by the charge, Q is given as,

$$\vec{F} = Q\vec{E}$$

Therefore, small work done in moving the charge over a small distance $\overrightarrow{dl}$ is given as,

$$\overrightarrow{dW} = -\vec{F}.\overrightarrow{dl} = -Q\,\vec{E}.\overrightarrow{dl}$$

The negative sign indicates that the work done will be opposite to the force developed. Integrating $\overrightarrow{dw}$ to obtain $\vec{w}$ in moving the longer distance from A to B,

$$\vec{W} = \int_A^B \overrightarrow{dW} = -\int_A^B Q\,\vec{E}.\overrightarrow{dl}$$

We know that,

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\overrightarrow{a_n}$$

$$\vec{W} = -Q\int_A^B \frac{Q}{4\pi\epsilon r^2}\overrightarrow{a_n}.\overrightarrow{dl}$$

Potential difference, V_{AB} is

$$V_{AB} = \frac{W}{Q} = \frac{1}{Q}\int_A^B -Q\left(\frac{Q}{4\pi\epsilon r^2}\right)\overrightarrow{dl}$$

$$V_{AB} = -\frac{Q}{4\pi\epsilon}\left(\frac{1}{r^2}\right)_A^B$$

If A is at a distance of r_A from origin and B is at a distance of r_B from origin, then,

$$V_{AB} = -\frac{Q}{4\pi\epsilon}\left[\frac{1}{r^2}\right]_{r_A}^{r_B}$$

$$V_{AB} = -\frac{Q}{4\pi\epsilon}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$= \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

$$V_{AB} = \frac{Q}{4\pi\epsilon r_A} - \frac{Q}{4\pi\epsilon r_B}$$

$$V_{AB} = V_A - V_B$$

NOTES

If r_B is moved to infinity, then $V_B \rightarrow 0$, then

$$V_{AB} = V_A - 0 = V_A$$

In general, the potential in moving a charge from infinity to zero,

$$V = \frac{Q}{4\pi\epsilon r} \text{ volts}$$

Maxwell's Second Equation

We know that potential difference between two points *A* and *B* is negative if the potential difference between *B* and *A* is also negative. In other words, creating a loop between *A* and *B* must satisfy Kirchhoff's voltage law and hence,

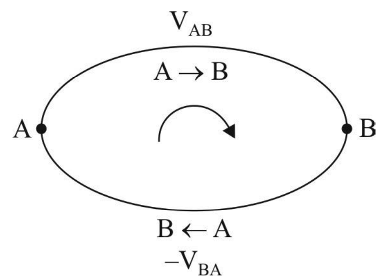

Fig. 1.9 *Voltage Around a Loop*

$$V_{AB} = -V_{BA}$$

Or $$V_{AB} + V_{BA} = 0$$

⇒ $$V = \oint_L \vec{E}.\overrightarrow{dl}$$

Therefore, applying Stokes's theorem to above closed line integral expression,

$$\oint_L \vec{E}.\overrightarrow{dl} = 0 = \oint_S (\vec{\nabla} \times \vec{E}).\overrightarrow{dS}$$

Hence,

$$\vec{\nabla} \times \vec{E} = 0$$

The above equation is called **Maxwell's second equation** in **differential form**. The equation can be briefed as 'Differential circulation of an electric field vector $\vec{E}$ is always zero or electric field vector, $\vec{E}$ vanishes when curled'. Such types of fields are **Conservative Fields.**

Relation between $\vec{E}$ and *V*

From the previous section on electric potential (*V*), we know that, work done is given as,

$$\vec{W} = -\int_A^B Q\,\vec{E}.\overrightarrow{dl}$$

$$V = \frac{W}{Q} = -\int_A^B \vec{E}.\overrightarrow{dl}$$

NOTES

Differentiating the above expression,

$$dV = -\vec{E}.\vec{dl}$$

In the Cartesian coordinate system,

$$dV = -[E_x\overrightarrow{a_x} + E_y\overrightarrow{a_y} + E_z\overrightarrow{a_z}].[dx\,\overrightarrow{a_x} + dy\,\overrightarrow{a_y} + dz\,\overrightarrow{a_z}]$$

$$dV = -[E_x dx + E_y dy + E_z dz]$$

But, LHS can be equated as,

$$dV = dx.\frac{\partial V}{\partial x} + dy.\frac{\partial V}{\partial y} + dz.\frac{\partial V}{\partial z}$$

Therefore,

$$dx.\frac{\partial V}{\partial x} + dy.\frac{\partial V}{\partial y} + dz.\frac{\partial V}{\partial z} = -[E_x dx + E_y dy + E_z dz]$$

Comparing LHS and RHS of the above expression, we get,

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad and \qquad E_z = -\frac{\partial V}{\partial z}$$

In general,

$$\vec{E} = -\vec{\nabla}V$$

Relating the above expression with Maxwell's second equation, i.e.,

$$\vec{E} = -\vec{\nabla}V \qquad and \qquad \vec{\nabla}\times\vec{E} = 0$$

Substituting for $\vec{E}$ in Maxwell's second equations,

$$\vec{\nabla}\times(-\vec{\nabla}V) = 0$$

The above is true from the property of curl which defines that curl of gradient is zero. The negative sign in $\vec{E} = -\vec{\nabla}V$ represents that electric potential 'V' is always opposite to the one created it, i.e., $\vec{E}$ according to Lenz's Law.

Electric Dipole

Similar to the poles of the magnet, when equal and opposite electric charges are separated by a short distance, they form an electric dipole as shown in Figure (1.10).

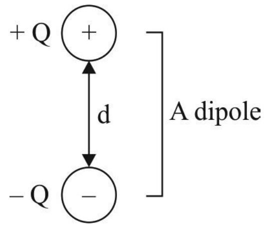

Fig. 1.10 *Electric Dipole*

NOTES

The objective of this section is to determine the scalar electric potential '*V*' due to the dipole. When two equal and opposite charges are separated by a distance '*d*' an electric dipole moment is formed equivalent to,

$m = Qd$

Consider the Figure (1.11) to determine the scalar potential '*V*' at a point '*P*' due to dipole.

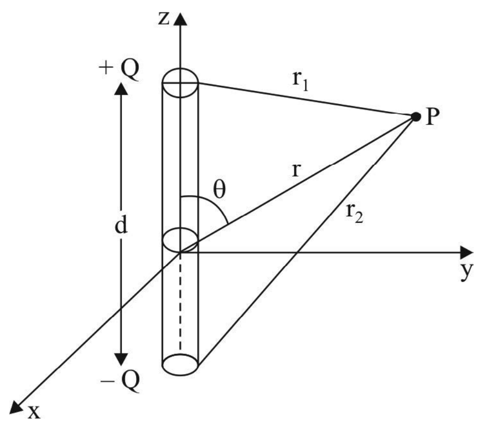

Fig. 1.11 *Potential Due to an Electric Dipole*

Scalar potential due to + Q is,

$$V_1 = \frac{Q}{4\pi\epsilon_0 r_1}$$

Scalar potential due to –Q is,

$$V_1 = \frac{-Q}{4\pi\epsilon_0 r_2}$$

Total potential is,

$$V = V_1 + V_2$$

$$= \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

For a far-field analysis, i.e., assuming that point *P* is at a far away distance, then,

$$r_1 \simeq r - \frac{d}{2}\cos\theta$$

$$r_2 \simeq r + \frac{d}{2}\cos\theta$$

Therefore,

NOTES

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{\left(r + \frac{d}{2}\cos\theta\right) - \left(r - \frac{d}{2}\cos\theta\right)}{r^2 - \frac{d^2}{4}\cos^2\theta}\right]$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{d\cos\theta}{r^2 - \frac{d^2}{4}\cos^2\theta}\right]$$

For a far field analysis and in general, $d << r$,

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{d\cos\theta}{r^2}\right]$$

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{m\cos\theta}{4\pi\epsilon_0 r^2}$$

$$V = \frac{m\cos\theta}{4\pi\epsilon_0 r^2}$$

The scalar potential is inversely proportional to the square of the distance between the observing point and the electric dipole.

1.3 UNIQUENESS THEOREM

The 'Cauchy Problem' is exactly the Initial Value Problem or IVP and is used to solve $x'(t)=f(t,x)$ with the condition $x(t_0)=x_0$. Picard's theorem is explained for given any point in the plane, (x_0, y_0) and a function $f(x, y)$, continuous on some neighborhood of (x_0, y_0) and Lipschitz in y on that neighborhood, then there exist a unique function $y(x)$ satisfying $y'=f(x, y)$ and $y(x_0) = y_0$. A 'Neighborhood' of a point is an open set containing that point. A function, $f(x)$, is 'Lipschitz' on a set if and only if there exist a positive number C such that for any x, y in that set, $|f(x)-f(y)| < C|x-y|$.

If f(x) is Lipschitz on a set then it is continuous at every point of that set. The mean value theorem can be used to show that if a function is differentiable at every point of a set, then it is Lipschitz on the set while 'Continuous' and 'Differentiable' are defined at points. If $f(x, y)$ is continuous but not Lipschitz on a set, then there may be many functions satisfying the differential equation and 'Initial Condition'. The Picard's method for solving an initial value problem is considered as the basis for his proof.

NOTES

Uniqueness

The system is equivalent to the integral equation. If we have a Lipschitz condition, then we can use the Picard iterates method on the integral equation to get a unique solution. We define,

$$y_0(x) = y_0$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

As we commented above, this converges to a unique solution if *f* is Lipschitz in *y*.

Alternately, we could use Gronwall's Inequality, as defined below.

Gronwall's Inequality

Let u, v be nonnegative continuous functions $[a, b]$ such that,

$$v(t)C + \int_a^t v(s)u(s)ds,\ a \le t \le b,$$

Then

$$v(t) \le Ce^{\int_a^t u(s)ds}$$

In particular, if $C = 0$, then $v = 0$.

Proof. Let $h(t) := C + \int_a^t v(s)u(s)ds$ Therefore,

$$h'(t) = v(t)u(t) \le h(t)u(t)$$

This reduces to the differential inequality,

$$h' - uh \le 0$$

Multiplying the LHS by,

$$e^{-\int_a^t u(s)ds},$$

We get

$$\left(h(t)e^{-\int_a^t u(s)ds}\right)' \le 0$$

And integrate from 0 to *x* to get,

$$h(x)e^{-\int_a^x u(s)ds} - h(a) \le 0$$

$$h(x) \le h(a)e^{\int_a^x u ds}$$

NOTES

Finally,

$$v(x) \le h(x) \le Ce^{\int_a^x u(s)ds}$$

This allows us to state a new uniqueness theorem.

Theorem: Uniqueness of Solutions to IVPs

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on,

$$Q := \{(x, y) : |x - x_0| \le a, |y - y_0| \le a\}$$

and satisfies,

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|.$$

Then the solution to the IVP exists on $[x_0 - \alpha, x_0 + \alpha]$, where $\alpha := \frac{a}{M}$, and the solution is unique.

Proof. Existence follows.

If there exists two solutions $\phi_1(t)$ and $\phi_2(t)$ then define,

$$w(t) := \phi_1(t) - \phi_2(t)$$

Then, $w'(t) = \phi_1'(t) - \phi_2'(t)$, and

$$\int_{x_0}^{x} w'(t)dt = w(x) - w(x_0) = \int_{x_0}^{x} [f(t, \phi_1(t)) - f(t, \phi_2(t))]dt$$

$$w(x_0) = \phi_1(x_0) - \phi_2(x_0) = 0$$

So, we get the following for w:

$$w(x) = \int_{x_0}^{x} [f(t, \phi_1(t) - f(t, \phi_2(t))]dt$$

Therefore,

$$|w(x)| \le \left| \int_{x_0}^{x} f(t, \phi_1) - f(t, \phi_2)dt \right|$$

$$\le \int_{x_0}^{x} |f(t, \phi_1) - f(t, \phi_2)|dt \le K \int_{x_0}^{x} |\phi_1(t) - \phi_2(t)|dt$$

$$= K \int_{x_0}^{x} |w(t)|dt$$

Thus, from Gronwell's Inequality with $u(t) := K$, $v(t) := |w(t)|$, and $C = 0$, we get $|w(t)| = 0$. Consequently, $\phi_1 = \phi_2$, and the uniqueness is shown.

NOTES

1.4 POISSON'S EQUATION AND LAPLACE'S EQUATION

We know that if ϕ is the scalar potential associated with the electric field $\vec{E}$, $\vec{E} = -\vec{\nabla}\phi$. Again from the differential form of Gauss' law, we can write $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$. If we couple these two equations, we get

$$\vec{\nabla}.(-\vec{\nabla}\phi) = \frac{\rho}{\epsilon_0}, \quad \text{i.e., } \vec{\nabla}.(\vec{\nabla}\phi) = -\frac{\rho}{\epsilon_0}$$

$$\therefore \qquad \nabla^2\phi = -\frac{\rho}{\epsilon_0} \qquad ...(1.8)$$

This Eqation (1.8) is known as **Poisson's equation** in electrostatics.

If we consider a particular region where there is no free charge, *i.e.*, $\rho = 0$, then from Equation (1.8), we get

$$\nabla^2\phi = 0 \qquad ...(1.8(a))$$

Equation (1.8(a)) is known as **Laplace's equation** in electrostatics and is of much importance for evaluating electrostatic potential in charge-free regions.

1.4.1 Solution of Laplace and Poisson Equation in Rectangular, Cartesian and Spherical Polar Coordinates

Laplace's equation can be used to evaluate potentials for charge-free regions. But for that purpose we have to solve that equation in an appropriate coordinate system. Then we have to impose boundary conditions on that solution to obtain the electrostatic potential. Here, we will discuss the various forms of Laplace's equation in different coordinate systems and their solutions. We shall not go through the details of solving Laplace's equation because all the techniques are beyond the scope of this book.

Cartesian System

Laplace's equation in Cartesian coordinate system takes the form,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad ...(1.8(b))$$

The solution of Equation 1.8(b) can be written as,

$$\phi(x, y, z) = (A \cos Kx + B \sin Kx) \times (\cos my + D \sin my)$$
$$\times \left(E.e^{(\sqrt{K^2+m^2})z} + F.e^{-(\sqrt{K^2+m^2})z}\right) \qquad ...(1.9)$$

Where, A, B, C, D, E, F, K and m all are constants.

Cylindrical System

Laplace's equation in cylindrical coordinate system takes the form,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad ...(1.10)$$

The solution of Equation (1.10) is of the form,

$$\phi(r, \theta, z) = [AJ_n(Kr) + BY_n(Kr)] \times [C \cos m\theta + D \sin m\theta] \times [Ee^{Kz} + Fe^{-Kz}]$$

Where A, B, C, D, E, F, K and m all are constants and $J_n(Kr)$ and $Y_n(Kr)$ are Bessel's functions of order n.

Spherical System

Laplace's equation in spherical polar coordinate system can be written as,

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

The solution of the above equation is of the form,

$$\Phi(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[Ar^n + Br^{-(n+1)}] \times [C \cos m\phi + D \sin m\phi] \times [EP_n^m(\cos\theta) + FQ_n^m(\cos\theta)]$$

Where, A, B, C, D, E and F all are constants and $P_n^m(\cos\theta)$, $Q_n^m(\cos\theta)$ are associated Legendre polynomials.

1.5 METHODS OF ELECTRIC IMAGES

At this stage, a question may arise that, "How does a static charged particle present in an electrostatic field interact with the field?" The answer can be realised by simply considering the idea of electric lines of force. We already know that electric lines of force are imaginary lines that generate from a positive charge and terminate on a negative charge. Let us consider an electrostatic field produced by a single, static positive charge $+Q$. The lines of force associated with $+Q$ are shown in the Figure 1.12. In the Figure 1.12 we can see that the density of lines is greater at the very surrounding of the charge, and this density decreases as we go farther from $+Q$. Density of field lines gives the measure for the strength of the electric field. Thus, the force of repulsion between $+Q$ and a positive test charge $+q$ placed at a large distance from $+Q$ is weaker in comparison to that when $+q$ is placed at the very surrounding of $+Q$. This can be verified from the expression of the electric field also, because electric field decreases inversely with the square of the separating distance. As discussed above, the density of lines of force is a measure for the strength of the electric field, electric flux also gives the measure for the density of lines of force. *Electric flux is generally defined as the number of lines of force passing through a unit area held normal to the direction of the lines of force.* If the electric flux is greater, the electric field is greater and vice versa.

If ϕ is the electric flux corresponding to the electric field $\vec{E}$, $\phi = \iint_S \vec{E}.d\vec{s}$, where $d\vec{s}$ is an elementary portion of the surface S.

NOTES

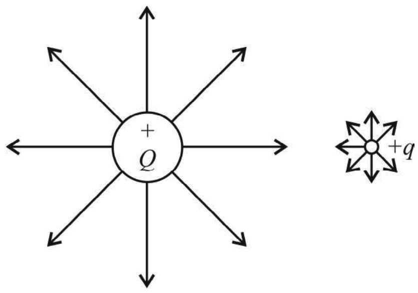

Fig. 1.12 *Lines of Force Associated with +Q*

Now consider a closed spherical surface S with radius r and centre at O. If we place a positive charge $+q$ at O, the lines of force over the surface S will be identical as shown in the Figure 1.13. Since, we have placed the charge at the centre, the electric field at the outer surface of S is uniform throughout and is $\vec{E} = \frac{q}{4\pi \in_0 r^2}\hat{r}$. So, the electric flux at the outer surface of S is given by,

$$\phi = \oiint_S \vec{E}.d\vec{s} = \oiint_S \frac{q}{4\pi \in_0 r^2}\hat{r}.d\vec{s}$$

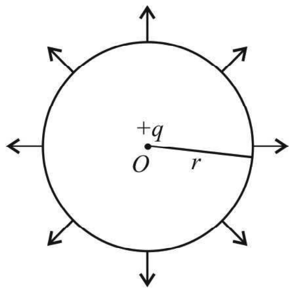

Fig. 1.13 *Electric Flux*

Here, $d\vec{s}$ is nothing but $(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$. The direction of $d\vec{s}$ is along a unit outward normal, drawn over s, which is identical to $\hat{r}$, because radius to a surface point of a sphere is always perpendicular to the tangent of the sphere at that point. Hence,

$$\phi = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{q}{4\pi \in_0 r^2}\hat{r}.(r^2 \sin\theta \, d\theta \, d\phi\hat{r}) = \frac{q}{\in_0} \quad ...(1.11)$$

Evidently if the charge was placed outside the surface S, the total flux over S would be zero. This result is valid for any shape of the closed surface S.

Electrostatic Potential

The electrostatic field is of the form,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi \in_0} \cdot \frac{q}{r^2}\hat{r}$$

Where, q is the source charge and r is the separation between the source and test charges. Now take the curl operation on the above equation.

$$\vec{\nabla} \times \vec{E} = \frac{q}{4\pi \in_0}\left(\vec{\nabla} \times \frac{\hat{r}}{r^2}\right) = \frac{q}{4\pi \in_0}\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right) = 0 \quad ...(1.12)$$

This establishes that the electric field $\vec{E}$ is a *conservative* one. From the idea of conservative field, we know that such a field is always associated with a scalar potential ϕ by the relation $\vec{E} = -\vec{\nabla}\phi$ (the negative sign comes to obey the principle of conservation of energy). Here, the **scalar potential** ϕ is known as the **electrostatic potential** associated with the **electrostatic field** $\vec{E}$.

Potentials are always the consequences of various interactions. Electrostatic potential ϕ is itself the result of the interaction between a charge and the electric field $\vec{E}$. We can define $\phi(\vec{r})$ as *the work done in bringing a unit positive charge from infinity to the point* $\vec{r}$. Using this definition, let us develop an expression for the potential.

$$\phi(\vec{r}) = -\int_{\infty}^{r} \vec{E}.d\vec{r} = \text{Work done by } \vec{E} \text{ to bring a}$$

positive unit charge from infinity to $\vec{r}$

$$= \int_{r}^{\infty} \frac{q}{4\pi \in_0 r^2}\hat{r}.d\vec{r}$$

$$= \frac{q}{4\pi \in_0}\int_{r}^{\infty} \frac{dr}{r^2} = \frac{q}{4\pi \in_0 r}$$

i.e.,
$$\phi(\vec{r}) = \frac{q}{4\pi \in_0 r} \quad ...(1.13)$$

which is the required expression for the electrostatic potential $\phi(\vec{r})$.

1.6 GREENS FUNCTION FOR POTENTIAL PROBLEM

In the field of electrostatics, the **Green's function** is precisely defined as the impulse or signal response of an inhomogeneous or nonuniform linear differential operator which is characteristically defined on a domain including the certain specific initial conditions or the boundary conditions. The term signal or impulse is uniquely used in context with the signal processing defining the input signal or impulse response or precisely the Impulse Response Function (IRF).

In mathematics, the notion Green's functions are precisely named after the British mathematician George Green, who initially or originally developed and established this concept in the year 1820s. In the contemporary analysis and evaluation of linear partial differential equations, the Green's functions are analysed principally on the basis of fundamental solutions.

The Green's function implies or indicates that if L is considered as the linear differential operator, then:

- Essentially, the Green's function G is precisely defined as the solution of the equation of the form L$G = \delta$, where δ is referred as the Dirac's delta function.

NOTES

- Fundamentally, the solution of the Initial Value Problem (IVP) L$y = f$ is referred as the convolution ($G * f$), where G is referred as the Green's function.

The principle of superposition is used for a given linear Ordinary Differential Equation (ODE) to solve considering that,

L (Solution) = Source

L (Green) = δ_s, for each s

Because the source is a sum of delta functions, and the solution is a sum of Green's functions through the linearity of L.

Characteristically, the Green's function, $G(x,s)$, precisely of a linear differential operator L = L(x) distinctively acting on distributions over a subset of the Euclidean space $\mathbb{R}^n$, at a specific point s, is any solution of,

$$\mathbf{L}\ G\ (x,\ s) = \delta\ (s - x)$$

Where δ is referred as the Dirac delta function. This specific property of a Green's function can be manipulated or exploited for solving the differential equations of the form,

$$\mathbf{L}\ u(x) = f(x)$$

Considering that the kernel of 'L' is non-trivial, then precisely the Green's function is not considered as unique. Consequently, in fact, certain specific combinations of symmetry, boundary conditions and/or other superficially or externally imposed and required criteria may provide the notion of a unique Green's function. Subsequently, the Green's functions may possibly be categorized or classified by means of the kind of boundary conditions that are precisely satisfied through a Green's function number. Additionally, the Green's functions are generally considered as the distributions and not essentially as the functions of a real variable.

Fundamentally, the Green's functions are referred as exceptionally effective and efficient methods for solving the wave equations and the diffusion equations. In the field of quantum mechanics, the Green's function of the Hamiltonian form is considered as a key model with significant concepts of density of states.

In physics, the Green's function is generally defined with the opposite sign, i.e.,

$$\mathbf{L}\ G\ (x,\ s) = \delta\ (x - s)$$

This definition of the Green's function does not substantially or considerably change or modify any of the properties of the Green's function because of the uniformity and consistency of the Dirac delta function.

If the operator is translation invariant, i.e., when **L** holds constant coefficients with respect to x, then the Green's function is typically considered as a convolution kernel, that is,

$$G\ (x,\ s) = G\ (x - s)$$

In this instance, the Green's function is considered equivalent with the impulse or signal response of linear time-invariant system theory.

NOTES

1.7 SOLUTIONS OF CONDUCTING DIELECTRIC SPHERE IN UNIFORM ELECTRIC FIELD

The term **dielectric sphere** in a uniform electric field can be uniquely defined with the help of the condition that define a conducting sphere in an electric field. At far distances from the sphere, the field is considered uniform and is equal or equivalent to $\vec{E} = E_0\hat{k}$ which corresponds to a potential of $\varphi = -E_0 r \cos\theta$.

The potential can be extended inside and outside with reference to Legendre polynomials. Since the distant potential only has Legendre polynomial of Order 1, therefore, outside the sphere, the field is given by the potential,

$$\varphi_{out} = A_1 r \cos\theta + \frac{B_1}{r^2}\cos\theta$$

Inside the sphere, the origin is included. Consequently, inside the sphere, the potential cannot have a singularity at the origin and is therefore given by,

$$\varphi_{in}(r,\theta) = A_2 r \cos\theta$$

Comparing with the asymptotic limit at long distances, $A_1 = -E_0$.

The potential itself is continuous at $r = R$,

$$-E_0 R + \frac{B_1}{R^2} = A_2 R$$

Since there are no free charges on the surface, therefore, the normal component of the displacement vector is continuous across the surface. The normal component being the radial direction, we have,

$$-\epsilon \frac{\partial \varphi_{in}}{\partial r}\bigg|_{r=R} = -\epsilon_0 \frac{\partial \varphi_{out}}{\partial r}\bigg|_{r=R}$$

$$-\epsilon A_2 = \epsilon_0 E_0 + \epsilon_0 \frac{2B_1}{R^3}$$

Solving, we get,

$$B_1 = E_0 R^3 \frac{\epsilon_0 - \epsilon}{\epsilon + 2\epsilon_0} = E_0 R^3 \frac{\kappa - 1}{\kappa + 2}$$

$$A_2 = -\frac{3E_0}{\kappa + 2}$$

So that,

$$\varphi_{in} = -\frac{3E_0}{\kappa + 2} r\cos\theta$$

And the electric field inside the dielectric is,

$$\vec{E}_{in} = \frac{3E_0}{\kappa + 2}\hat{k}$$

NOTES

It is specified by the analysis that because of the presence of dielectric, the electric field inside is reduced by,

$$\vec{E}_0 - \vec{E}_{in} = \frac{\kappa - 1}{\kappa + 2}E_0\hat{k}$$

If a dipole is placed at the origin of a sphere of radius R, then the radius of the sphere can be considered extremely large in comparison to the dimensions of the dipole, and the field is precisely given by,

$$\frac{\vec{p}}{4\pi\epsilon_0 R^3}$$

Therefore, the effect of dielectric is considered equivalent as that of replacing the dielectric with a dipole of moment,

$$\vec{p} = 4\pi\epsilon_0 R^3 \frac{\kappa - 1}{\kappa + 2}E_0\hat{k}$$

The polarization of the sphere is then obtained by dividing this by the volume of the sphere, i.e., by $\dfrac{4\pi R^3}{3}$

$$\vec{P} = 3\epsilon_0 \frac{\kappa - 1}{\kappa + 2}E_0\hat{k}$$

Consequently, the field due to the dielectric can be written as,

$$E_{dielectric} = -\frac{\kappa - 1}{\kappa + 2}E_0\hat{k} = -\frac{\vec{P}}{3\epsilon_0}$$

The polarization in the medium is considered as uniform and is precisely directed along the direction of the external field. The field lines approach the sphere from the left (negative z-direction) and leave from the right, making the left face of the sphere negatively charged (Refer Figure 1.14).

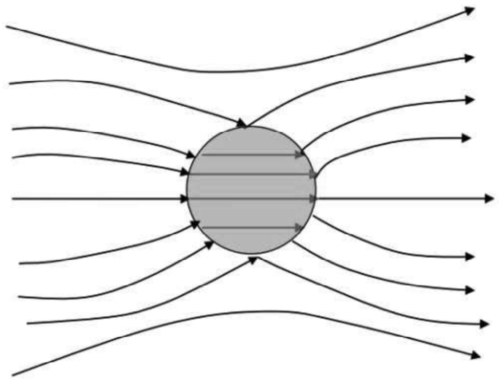

Fig. 1.14 *Polarization*

Microscopic Theory

The term dielectric is precisely considered as a collection of molecules. If the molecules are properly or appropriately separated, as in the case of a gas, then each molecule may experience or encounter an average macroscopic field represented as $\vec{E}$ at its position. This comprises of any external field and an average field because of all other existing molecules. The polarization induced is precisely written in terms of electric susceptibility,

$$\vec{P} = \chi\epsilon_0\vec{E}..$$

In the instance of gases, the electric field is primarily due to the external field.

Specifically, in a dense medium, a particular distinct molecule will be subject to a local field, which in addition to the external field, is because of interactions with the dipoles in its immediate neighbourhood in addition to the polarization of the other molecules in the medium. This local field induces a dipole moment $\vec{p}$ on the molecule and is precisely defined as the 'Atomic Polarizability α' by means of the relation,

$$\vec{p} = \alpha\vec{E}_{local}$$

If there are *n* atoms per unit volume, then the polarization is precisely given by,

$$\vec{P} = \alpha n\,\vec{E}_{local}$$

However, according to the theory of gases, because an atom cannot exert a force on itself, therefore in the computation of the local field we must subtract the contribution or influence on the polarization by means of the atom under consideration. The dielectric has two parts, a spherical volume of radius *r* assigned to the atom in question which contains other atoms in the immediate neighbourhood of the atom and the rest of the dielectric outside this volume. The 'Rest of the Dielectric' will be considered in a macroscopic approach which gives rise to an average field $\vec{E}_0$, taken along the **z** direction. To this field, during the computation the field must be added or included which occurs because of the spherical volume discussed above.

Check Your Progress

1. Define electric field intensity.
2. What is electric flux density?
3. State Gauss' law.
4. When is a function *f*(*x*) Lipschitz on a set?
5. Give the equation for Laplace in the cylindrical coordinate system.
6. What is electric flux?
7. Define the term Green's function.
8. What does Green's function imply?
9. What is dielectric sphere in a uniform electric field?

NOTES

1.8 LAWS OF MAGNETOSTATICS

Magnetic Flux (ϕ)

Magnetic flux lines are imaginary lines that flow from the north to south poles. Magnetic flux lines constitutes the magnetic field. Magnetic field is denoted by $\vec{H}$. The unit of magnetic flux is Weber and Denoted as ϕ.

The unit Weber is named after the German Physicist Wilhelm Eduard Weber and the symbol used ins *Wb*.

Magnetic Flux Density $(\vec{B})$

Magnetic flux density is defined as the magnetic flux lines passing through a unit surface area. It is denoted as $\vec{B}$ and the unit is weber/metre^2 or Tesla.

Magnetic flux density, $\vec{B}$ is given as,

$$B = \phi/A \tag{1.14}$$

Also in terms of magnetic field intensity. $\vec{H}$,

$$\vec{B} = \mu\vec{H} \tag{1.15}$$

Where

$\mu = \mu_o\mu_r$ = Permeability

$\mu_o = 4\pi * 10^{-7}\ H/m$ = Free Space Permeability

μ_r = Relative Permeability = 1 (for Air).

In electrostatics, it is possible to have an isolated charge and the isolated charge has electric field. Whereas there is nothing known as magnetic charge and magnetic field exists only if there are two equal and opposite poles.

The magnetic flux, ϕ is given as,

$$\phi = \oint \vec{B} \,.\overrightarrow{ds} \tag{1.16}$$

In electrostatics, from Gauss's law,

$$Q = \oint_s \vec{D}\,.\,\overrightarrow{ds} = \Psi$$

The above electrostatic equation states that electric flux Ψ is created due to the source '*Q*'. But, in magnetostatics, there is no magnetic source to create a magnetic flux, ϕ, hence Equation (1.16) is given as,

$$\oint_s \vec{B}\,.\,\overrightarrow{ds} = 0 \tag{1.17}$$

Applying divergence theorem to Equation (1.17), we obtain ,

$$\oint_s \vec{B} \cdot \vec{ds} = \int_v \vec{\nabla} \cdot \vec{B} \cdot dv = 0$$

Therefore,

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (1.18)$$

Equation (1.18) is Maxwell's third equation in differential form and reveals that, divergence of magnetic field is zero. Equation (1.17) is called the integral form of Maxwell's Equation.

Steady Electric Currents and Current Density

The electric currents are produced either by conduction or convection. Current is defined as the rate of movement of charge across a plane in a given time. They are expressed as,

$$I = \frac{dQ}{dt}$$

Current Density

Current density is defined as the amount of current flowing through a given area of a material. It is a vector component with magnitude equivalent to the electric current per cross sectional area. Current I is related to current density $\vec{J}$ as,

$$I = \int \vec{J} . \vec{ds}$$

Different current density are produced based on the nature of movement of charges. They are classified as,

(a) Convection Current Density

(b) Conduction Current Density

(c) Displacement Current Density

Based on the above classification, currents are classified as ***Conduction Currents*** and ***Convection Currents***.

(a) Convection Current

The flow of charges through convection constitutes convection current. Beam of electrons inside Cathode Ray Tube (CRT) or in vacuum tubes are due to convection currents. Convection currents neither they obey Ohm's law nor they involve any conductor for the flow of current. Convection current density, $\vec{J}$ is given as,

$$\vec{J} = \rho_V \vec{u}$$

Where $\vec{u}$ is the directional vector indicating the flow of charges. Convection currents are depicted in Figure (1.15).

NOTES

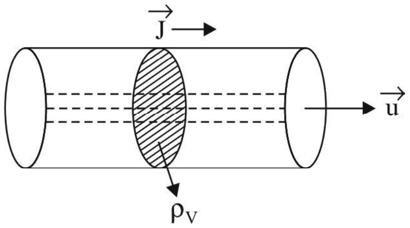

Fig. 1.15 *Convection Current*

(b) Conduction Current

Conduction current requires conductor to flow. The flow of charges from one end of conductor to the other is facilitated by application of electric field between the conductors. When an electric field is applied, the electrons experience a force given as,

$$\vec{F} = -e\vec{E}$$

The force on the electrons make them move constituting the flow of current. conduction current density also depends on the conductivity of the conducting medium. Hence the conduction current density is given as,

$$\vec{J} = \sigma\vec{E}$$

Where σ is the conductivity of the conductor and $\vec{E}$ is the applied electric field in (V/m). Unlike convection current, conduction current obeys Ohm's law and depends on the medium or the conductor.

Conductivity of the conductor in turn depends on the resistivity of the medium and given as,

$$\sigma = \frac{1}{\rho}$$

Where $\rho = \frac{RA}{l}$. Resistance of the conductor is *R*, the length of the conductor is $'l'$ in (m) and the area is A in m^2.

Ohm's Law

Ohm's law states that electric current is proportional to voltage and inversely proportional to resistance.

$I = V/R$

The term *Ohm's law* is also used to refer to various generalizations of the law originally formulated by Ohm. The current density and the electric field are related as: $\vec{J} = \sigma\vec{E}$

The above expression will be useful in the electric circuit analysis when the above expression is expressed in terms of potential and current rather than in terms of electric field and current density. The above expression may be expressed as,

$$\vec{E} = \frac{1}{\sigma}\vec{J} = \rho\,\vec{J}$$

NOTES

We know that, in a closed conductor, the scalar electric potential is given as,

$$\Delta V = -\int_A^B \vec{E}\,.\,\overrightarrow{dl}$$

Hence,

$$V = E.\,l_{AB}$$

Where l_{AB} is the length of the conductor with terminals as AB. Therefore,

$$E = \frac{V}{l_{AB}}$$

The current density J is defined as the current passing through a unit area cross section and is given as,

$$J = \frac{I}{a}$$

Where a is the area of cross section of the conductor. Substituting for J and E in the expression $\vec{E} = \rho\,\vec{J}$, we get,

$$\frac{V}{l_{AB}} = \rho\,\frac{I}{a}$$

$$\therefore V = \rho\, l_{AB}\frac{I}{a}$$

$$V = I\,R$$

Where, $R = \frac{\rho l}{a} = Resistance\ of\ the\ conductor.$

Boundary Conditions of Current Density

From the properties of conductor, it is well known for conductors that the electric field inside a conductor must be zero. If it were not, it would cause current to flow, and propagation of current involves the dissipation of energy, and this cannot occur without any external sources of energy. Hence, it follows that any charges in the conductor must be located on its surface.

Since the mean magnetic field is assumed to be zero, therefore the derivative,

$$\frac{\partial\vec{H}}{\partial t} = 0$$

And since the electric field inside the vacuum is satisfied as,

$$\nabla\cdot\vec{E} = 0\ and\ \nabla\times\vec{E} = 0$$

The boundary conditions on the field $\vec{E}$ at the surface of the conductor follow from the equation $\nabla\times E = 0$, and is both valid outside and inside the body. Take the z-axis in the direction of the normal n to the surface at some point in the conductor. If the surface is homogeneous, the derivatives, $\partial Ez/\partial x$ and $\partial Ez/\partial y$ along the surface remain finite even though E_z may be quite large.

NOTES

Hence, since $\nabla x \times E = \partial Ez/\partial y - \partial Ey/\partial z = 0$, we find that $\partial Ey/\partial z$ is finite. This means that E_y is continuous at the surface since a discontinuity in E_y would mean an infinity of the derivative $\partial Ey/\partial z$. The same applies to E_x and since $E = 0$ inside the conductor, this implies that the tangential components of the electric field at the surface must be zero,

$Et = 0$

For the case of two conductors under static field conditions (i.e., $\partial E/\partial t = 0$ and $\partial B/\partial t = 0$,) there can be no charge build up at the interface and hence,

$Jn1 = Jn2$

- The normal components must be equal, $J1n = J2n$, because otherwise there would be an accumulation of charge into the surface, which cannot be sustained in the steady state. This is because, for a normal from 1 into 2, the flow of charge that exits Medium 1 in a small area dA is $J_{1n}\,dA$ and the charge entering Medium 2 is $J_{2n}\,dA$.
- The tangential components on the other hand, can be different, because charge is just flowing past the boundary at different speeds. On the other hand, the electric field's tangential components must be continuous across the boundary, because otherwise there would be a nonzero circulation in the loop. In a metal where Ohm's law holds, the current density is proportional to electric field,

  $\vec{J} = \sigma\vec{E}$

  And therefore the current density's tangential components will not in general be constant across the boundary.

Equation of Continuity and Kirchhoff's Law

Continuity equation on the basis of law of conservation of charge states that in a given volume, the total current coming out of the volume is equal to the rate of decrease of charge inside the volume. It is expressed as,

$$I_{out} = -\frac{dQ}{dt} \quad (1.19)$$

We know that,

$$I = \oint \vec{J}.\overrightarrow{ds}$$

And $Q = \int_V \rho_V . dV$

Substituting I and Q in Equations (1.19), we have,

$$\oint \vec{J}.\overrightarrow{ds} = -\frac{d}{dt}\int_V \rho_V . dV \quad (1.20)$$

Equation (1.20) is called the ***integral from of continuity equation***. Also, invoking divergence theorem on LHS of Equation (1.20), we have,

NOTES

$$\oint \vec{J}.\overrightarrow{ds} = \int_V \vec{\nabla}.\vec{J}\, dV \quad (1.21)$$

Now RHS of Equations (1.20) is given as,

$$-\frac{d}{dt}\int_V \rho_V.dV = -\int_V -\frac{\partial \rho_V}{\partial t}\, dV \quad (1.22)$$

Equating Equations (1.21) and (1.22), we have,

$$\int_V \vec{\nabla}.\vec{J}\, dV = -\int_V -\frac{\partial \rho_V}{\partial t}\, dV$$

Therefore,

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho_V}{\partial t} \quad (1.23)$$

When steady current flows out of the volume, i.e., $\frac{\partial \rho_V}{\partial t} = 0,$

$$\vec{\nabla}.\vec{J} = 0$$

Equation (1.23) is known as the continuity equation which states that there is no accumulation of charges at any position.

Kirchhoff's Law

In general, Kirchhoff Law relates the potential in a closed circuit or current in the nodes of a circuit. Hence Kirchhoff's Voltage Law, states that the sum of voltage drop and voltage rises in a closed loop of an electric circuit is zero and is given as,

$$\sum_{i=1}^{N} V_i = 0$$

Where i indicates the number of circuit elements in the loop. V_i is positive for voltage rise across an element or negative if voltage drops across an element. Similarly, at any node of an electric circuit, Kirchhoff's current law states that,

$$\sum_{i=1}^{N} I_i = 0$$

Where i indicates the number of circuit branches connected to a node. I_i is positive, if the current enters the node and negative if the current leaves the node.

Postulates of Magnetostatics: Biot–Savart's Law

Similar to the postulates of electrostatics governed by Coulombs law and Gauss law, Magnetostatics postulates are governed by the Biot-Savart law and Ampere circuital law. These laws typically relates the magnetic field with the current flowing in the circuit.

Biot–Savart's Law

Assume a current carrying conductor with current magnitude, *I* amperes. The objective is to find the effect of magnetic field intensity at an observing point, *p* due to a small length of the conductor carrying current.

NOTES

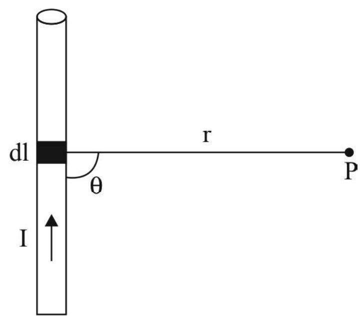

Fig. 1.16 *Biot–Savart's Law*

Biot–Savart's law states that, the magnetic field intensity at a point *p* away by a distance of *r* from a current carrying conductor due to a differential element '*dl*' is proportional to the product of current flowing through the differential element and sine of angle, θ between the current carrying conductor and the line joining the current element with *p*, and inversely proportional to the square of the distance between them. Mathematically,

$$dH \propto \frac{(Idl)}{r^2} \sin\theta \quad (1.24)$$

Removing the proportionality constant,

$$dH = \frac{1}{4\pi} \frac{(Idl)}{r^2} \sin\theta \quad (1.25)$$

We know that,

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

$$\vec{A} \times \overrightarrow{a_n} = |\vec{A}|(1) \sin\theta$$

Similarly,

$$\overrightarrow{dH} = \frac{1}{4\pi} \left[\frac{I(\overrightarrow{dl} \times \overrightarrow{a_r})}{r^2}\right] \quad (1.26)$$

Where, $\overrightarrow{dl} \times \overrightarrow{a_r} = |\overrightarrow{dl}|(1) \sin\theta$

But $\overrightarrow{a_r} = \frac{\vec{r}}{|\vec{r}|}$

$$\overrightarrow{dH} = \frac{I\overrightarrow{dl}}{4\pi r^2} \times \frac{\vec{r}}{r}$$

$$\overrightarrow{dH} = \frac{I}{4\pi r^2}(\overrightarrow{dl} \times \overrightarrow{a_r}) \quad (1.27)$$

To obtain magnetic field intensity, integrate Equation (1.26),

$$\vec{H} = \int \overrightarrow{dH} = \int \frac{I}{4\pi r^2} (\overrightarrow{dl} \times \overrightarrow{a_r})$$

For line current, $\vec{H} = \frac{1}{4\pi r^2} \int I(\overrightarrow{dl} \times \overrightarrow{a_r})$

For surface current, $\vec{H} = \frac{1}{4\pi r^2} \int I'(\overrightarrow{ds} \times \overrightarrow{a_r})$

For volume current, $\vec{H} = \frac{1}{4\pi r^2} \int I''(dv \times \overrightarrow{a_r})$

Where $I' = \vec{K}$ = Surface current density and $I'' = \vec{J}$ = Volume current density.

Magnetic Field Intensity $\vec{H}$ and Magnetic Field $\vec{B}$ Due to Line Current: On an Infinite Conductor

NOTES

Consider an infinite long conductor carrying a line current *I*. The objective of this section is to determine $\vec{H}$ and $\vec{B}$ at an observing point $P(x, y, z)$ due to a differential current element $I\,dl$. The observing point *P* is located at a distance '*r*'. The arrangement is depicted in Figure (1.17).

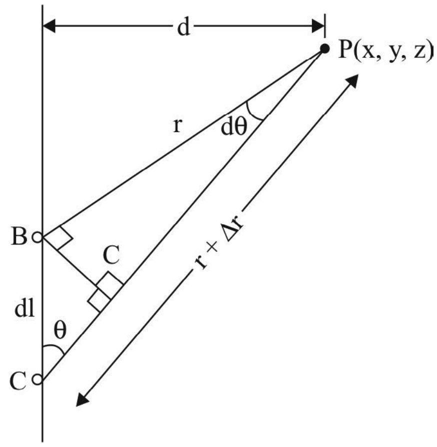

Fig. 1.17 *Infinte Current Carrying Conductor*

From Equation (1.25), we have,

$$\int dH = H = \frac{1}{4\pi r^2}\int I\,dl\sin\theta$$

$$B = \mu H = \frac{\mu I}{4\pi r^2}\int I\,dl\sin\theta \tag{1.28}$$

From Figure (1.17). in $\Delta\,ABC$,

$$\sin\theta = \frac{BC}{BA} \quad\Rightarrow\quad BC = AB\sin\theta$$

$$BC = dl\sin\theta$$

From ΔPBC

$$BC = rd\theta$$

$$rd\theta = dl\sin\theta$$

$$d\theta = \frac{dl\sin\theta}{r} \tag{1.29}$$

Substituting Equation (1.29) in Equation (1.28),

$$B = \frac{\mu I}{4\pi r}\int d\theta \tag{1.30}$$

From Figure (1.17),

$$\sin\theta = \frac{d}{r}$$

$$\frac{1}{r} = \frac{\sin\theta}{d} \tag{1.31}$$

Substituting Equation (1.31) in Equation (1.30), we get,

$$B = \frac{\mu I}{4\pi d}\int_0^{\lambda}\sin\theta\,d\theta$$

NOTES

$$B = \frac{\mu I}{4\pi d}[-\cos\theta]_0^{\pi}$$

$$B = \frac{\mu I}{4\pi d}[2]$$

$$B = \frac{\mu I}{4\pi d} \quad wb/m^2$$

$$H = \frac{B}{\mu} = \frac{I}{2\pi d} \quad A/m$$

$$H = \frac{I}{2\pi d} \quad A/m$$

$\vec{B}$ and $\vec{H}$ Due to Finite Conductor

Consider a finite conductor carrying a current I with a small differential current element Idl at a distance of r from the observing point $p(x, y, z)$ at which the magnetic field, $\vec{B}$ and magnetic field intensity $\vec{H}$ is to be determined.

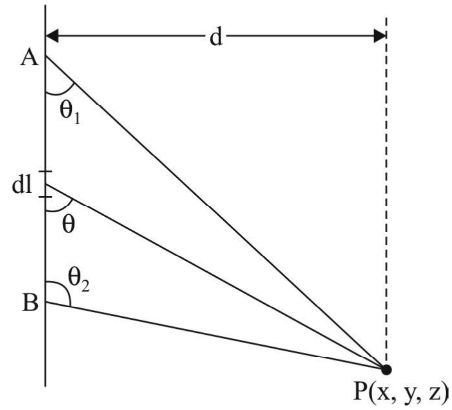

Fig. 1.18 *Finite Length Conductor*

The procedure is similar to the infinite conductor case discussed in the previous section till Equation (1.31). Hence, repeat the derivation.

According to Biot-Savart's law,

$$\int dH = H = \frac{1}{4\pi r^2}\int I\, dl \sin\theta$$

$$B = \mu H = \frac{\mu I}{4\pi r^2}\int dl \sin\theta$$

Also,

$$dl \sin\theta = r d\theta$$

$$d\theta = \frac{dl \sin\theta}{r}$$

$$B = \frac{\mu I}{4\pi r}\int d\theta$$

From Figure (1.18),

$$\sin\theta = \frac{d}{r}$$

$$\frac{1}{r} = \frac{\sin\theta}{d}$$

Therefore,

$$B = \frac{\mu I}{4\pi d}\int \sin\theta \, d\theta$$

Integrating from limits $\theta_1 \; to \; \pi - \theta_2$

$$B = \frac{\mu I}{4\pi d}\left[-\cos\theta\right]_{\theta_1}^{\pi-\theta_2}$$

$$B = \frac{\mu I}{4\pi d}\left[\cos\theta_1 + \cos\theta_2\right]$$

When the conductors are infinitely long, i.e., $\theta_1 = \theta_2 = 0$

$$B = \frac{\mu I}{4\pi d}\left[\cos 0 + \cos 0\right]$$

$$B = \frac{\mu I}{4\pi d}(2) = \frac{\mu I}{2\pi d}$$

The magnetic field intensity *H* is given as,

$$H = \frac{B}{\mu}$$

$$H = \frac{I}{4\pi d}\left[\cos\theta_1 + \cos\theta_2\right]$$

NOTES

B and *H* Due to a Circular Loop

Consider a circular loop as shown in Figure (1.19). The observing point is at a height, *h* on the z-axis. Since the structure under consideration is a circular loop, let us solve this analysis using cylindrical coordinate system with coordinates (δ, ϕ, z).

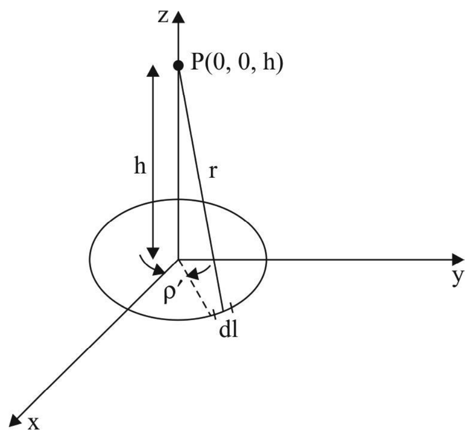

Fig. 1.19 *Circular Loop*

From Biot-Savart's law, given in Equation (1.27),

$$\overrightarrow{dH} = \frac{I}{4\pi r^3}\left(\overrightarrow{dl} \times \vec{r}\right)$$

$\overrightarrow{dl}$ in cylindrical coordinate system is given as, $\overrightarrow{dl} = \rho d\phi$ and $\vec{r} = (0,0,h) - (x,y,0)$

$$\vec{r} = -\rho\overrightarrow{a_\rho} + 0\overrightarrow{a_\phi} + h\overrightarrow{a_z}$$

NOTES

$$|\vec{r}| = \sqrt{\rho^2 + h^2}$$

$$\vec{dl} \times \vec{r} = \begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h d\phi \vec{a_\rho} + \rho^2 d\phi \vec{a_z}$$

$$\vec{dH} = \frac{I}{4\pi r^3}\left[\rho h d\phi \vec{a_\rho} + \rho^2 d\phi \vec{a_z}\right]$$

$$\vec{dH} = \frac{I}{4\pi[\rho^2+h^2]^{3/2}}\left[\rho h d\phi \vec{a_\rho} + \rho^2 d\phi \vec{a_z}\right] \qquad (1.32)$$

At a point p, $\vec{a_\rho} = 0$ and only $\vec{a_z}$ component exists. Therefore,

$$\vec{dH} = \vec{H} = \int_0^{2\pi} \frac{I\rho^2 d\phi}{4\pi[\rho^2+h^2]^{\frac{3}{2}}}\vec{a_z}$$

$$= \frac{I\rho^2}{4\pi[\rho^2+h^2]^{\frac{3}{2}}}\vec{a_z} \quad [2\pi - 0]$$

$$\vec{H} = \frac{I\rho^2}{2[\rho^2+h^2]^{\frac{3}{2}}}\vec{a_z}$$

The magnetic field, $\vec{B}$ is given as,

$$\vec{B} = \mu\vec{H}$$

$$\vec{B} = \frac{\mu I\rho^2}{2[\rho^2+h^2]^{\frac{3}{2}}}\vec{a_z}$$

Magnetic Potential

(a) Scalar Magnetic Potential

Similar to electrostatics, scalar potential V, the scalar potential is denoted as V_m.

In electrostatics, $\vec{E} = -\vec{\nabla}V$

In magnetostatics, applying the equivalence,

$$\vec{H} = -\vec{\nabla}V_m \qquad (1.33)$$

From Maxwell's fourth equation,

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{\nabla} \times (-\vec{\nabla}V_m) = \vec{J} \qquad (1.34)$$

But, from a vector identity, curl of gradient is zero, given by,

$$\vec{\nabla} \times (-\vec{\nabla}V) = 0$$

Equation (1.34) becomes,

$$\vec{\nabla} \times (-\vec{\nabla}V_m) = 0$$

From the above, the equation is valid only if,

$$\vec{J} = 0$$

NOTES

From Maxwell's third equation,

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \mu\vec{H} = 0$$

From Equation (1.33),

$$\vec{\nabla} \cdot \mu\left(-\vec{\nabla}V_m\right) = 0$$

$$\mu\left[\vec{\nabla}\left(-\vec{\nabla}V_m\right)\right] = 0$$

$$\nabla^2 V_m = 0$$

The above equation is the Laplace's equation. Hence scalar magnetic potential V_m satisfies Laplace's equation.

(b) Vector Magnetic Potential

Vector magnetic potential is denoted as $\vec{A}$. When $\vec{J} \neq 0$, vector magnetic potential exists.

From Maxwell's third equation,

$$\vec{\nabla} \cdot \vec{B} = 0$$

From a vector identity, divergence of a curl of a zero is given as,

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Comparing the above two equations,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Where $\vec{A}$ is the magnetic vector potential. In electrostatics, scalar potential v is given as,

$$V = \frac{1}{4\pi\epsilon_o r}\int dQ$$

Similarly $\vec{A}$ can be defined as,

$\vec{A} = \frac{\mu_o}{4\pi\epsilon_o R}\int_L \vec{I} \cdot \overrightarrow{dl}$ For line current

$\vec{A} = \frac{\mu_o}{4\pi\epsilon_o R}\int_S \vec{I'} \cdot \overrightarrow{ds}$ For surface current

$\vec{A} = \frac{\mu_o}{4\pi\epsilon_o R}\int_v \vec{I''} \cdot dv$ For volume current

We may also define,

$$\vec{I'} = \vec{K} \quad and \;\; \vec{I''} = J^{\rightarrow}$$

The magnetic flux in terms of magnetic flux density is defined as,

$$\psi = \oint_s \vec{B} \times \overrightarrow{ds}$$

$$\psi = \int_s \left(\vec{\nabla} \times \vec{A}\right) \cdot \overrightarrow{ds}$$

NOTES

Applying Stoke's Theorem,

$$\psi = \int_s (\vec{\nabla} \times \vec{A}) \cdot \overrightarrow{ds} = \oint_L \vec{A} \cdot \overrightarrow{dl}$$

Therefore,

$$\psi = \oint_L \vec{A} \cdot \overrightarrow{dl}$$

From vector triple product identity,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \quad (1.35)$$

Adding Equation $\vec{\nabla} \times \vec{A} = \vec{B}$ in Equation (1.35)

$$\vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \quad (1.36)$$

For a static magnetic field, there will be no current and in turn the current density, i.e., $\vec{J} = 0$.

$$\vec{\nabla} \times \vec{A} = 0 \quad (1.37)$$

We know that,

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \vec{J}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} \quad (1.38)$$

Substituting Equations (1.37) and (1.38) in Equation (1.36), we get,

$$-\nabla^2 \vec{A} = \mu \vec{J}$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

Which means,

$$\nabla^2 [A_x \overrightarrow{a_x} + A_y \overrightarrow{a_y} + A_z \overrightarrow{a_z}] = \mu [J_x \overrightarrow{a_x} + J_y \overrightarrow{a_y} + J_z \overrightarrow{a_z}]$$

$$\therefore \ \nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

Forces Due to Magnetostatics

For the force to be developed, there must be at least two fields with a phase difference. Hence force due to magnetic field can be experienced by either of the following,

(a) A unit charge, *Q* travelling in a magnetic field experiences a force.

(b) A current element placed in a magnetic field experiences a force.

(c) Two current carrying conductor when placed close to each other experience force between them.

(a) Force Due to a Unit Charge, *Q* or Lorentz Force

In electrostatics, recall the force on a charge, given as,

$\vec{F} = Q\vec{E} = \vec{F_e}$

The suffix 'e' represents the electrostatics. Similarly, force is experienced in magnetic field only when charge is moving. Hence force in magnetic field is given as,

$\vec{F_m} = Q\,[\vec{V} \times \vec{B}]$

Where $\vec{V}$ is the velocity at which charge Q is moving in a magnetic field $\vec{B}$.

The total force exerted on a charge when influenced by electromagnetic fields is given as,

$\vec{F} = \vec{F_e} + \vec{F_m}$

$\vec{F} = Q\vec{E} + \left[Q\,[\vec{V} \times \vec{B}]\right]$

$\vec{F} = Q[\vec{E} + \vec{V} \times \vec{B}]$

This is called as **Lorentz-Force Equation**.

(b) Force Due to a Current Element

Let the current element be $I\,\vec{dl}$ for which the force is to be determined. The differential current element $\vec{dl}$ induces a differential force, $\vec{dF}$, hence,

$$\vec{dF} = I\,\vec{dl} \times \vec{B} \qquad (1.39)$$

Total force, $\vec{F}$ is obtained by integrating Equation (1.39) on both sides,

$\int \vec{dF} = \int I\,\vec{dl} \times \vec{B}$

$\vec{F} = \int I\,\vec{dl} \times \vec{B}$ (For line current)

$= I\,B\,l\,\sin\theta = B\,I\,l\sin\theta$.

Similarly,

$\vec{F} = \int \vec{K}\,\vec{dS} \times \vec{B}$ (For surface current density)

$\vec{F} = \int \vec{J}\,\vec{dV} \times \vec{B}$ (For volume current density)

(c) Force between Two Current Elements (Wires)

Consider two current elements dl_1 and dl_2 with currents I_1 and I_2, respectively, current element 1, $I_1\vec{dl_1}$ produced a magnetic field $\vec{dB_1}$ that links with current element 2, $I_2\vec{dl_2}$ and similarly vice versa. Therefore force on element *I* due to field from element 2 is given as,

$$d_{F_{12}} = I_1\vec{dl_1} \times dB_2 \qquad (1.40)$$

And force on element 2 due to field from element 1 is given as,

$$d_{F_{21}} = I_2\vec{dl_2} \times dB_1 \qquad (1.41)$$

It is dB_1 and dB_2 in Equations (1.41) and (1.40), respectively, as only differential field links and other fields fringes away.

NOTES

But, from Biot–Savart's law (Equation 1.27), we get,

$$\overrightarrow{dH} = \frac{I}{4\pi r^3}(\overrightarrow{dl} \times \vec{r})$$

And we know that,

$$\vec{B} = \mu_0 \vec{H}$$

$$\overrightarrow{dB} = \mu_0 \overrightarrow{dH} = \frac{\mu_0 I}{4\pi r^2}(\overrightarrow{dl} \times \vec{r}) \quad (1.42)$$

Substituting Equations (1.42) in (1.40) and Equation (1.41), we get,

$$d_{F_{12}} = I_1 dl_1 \times \frac{\mu_0 I_2}{4\pi r^2}(\overrightarrow{dl_2} \times \vec{r}) \quad (1.43)$$

Integrating Equation (1.43), We obtain,

$$\int d_{F_{12}} = F_{12} = \frac{I_1 I_2}{4\pi r^2} \int_{L_1} \int_{L_2} \overrightarrow{dl_1} \times (\overrightarrow{dl_2} \times \vec{r})$$

Similarly,

$$F_{21} = \frac{I_1 I_2}{4\pi r^2} \int_{L_1} \int_{L_2} \overrightarrow{dl_2} \times (\overrightarrow{dl_1} \times \vec{r})$$

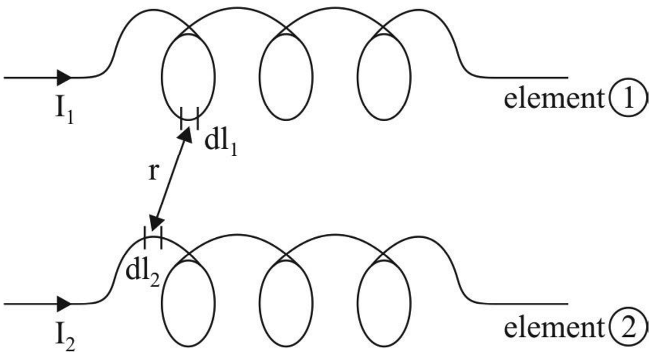

Fig. 1.20 *Force Due to Two Current Elements.*

Ampere's Circuit Law

Ampere's circuit law states that the closed line integral of magnetic field intensity is equal to the current circulating in the closed path.

Mathematically,

$$\oint \vec{H} \cdot \overrightarrow{dl} = I \quad (1.44)$$

Similar to Gauss law, from which we applied Stoke's theorem to arrive at Maxwell's second equation.

From Equation (1.44), applying Stoke's theorem,

$$I = \oint_L \vec{H} \cdot \overrightarrow{dl} = \int_s \vec{\nabla} \times \vec{H} \cdot \overrightarrow{ds} \quad (1.45)$$

We also know that,

$$I = \int_s \vec{J} \cdot \overrightarrow{ds} \quad (1.46)$$

Equating Equations (1.45) and (1.46), we get

$$\int_s \vec{J} \cdot \overrightarrow{ds} = \int_s \vec{\nabla} \times \vec{H} \cdot \overrightarrow{ds}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

This is Maxwell's fourth equation which states that curl of magnetic field intensity is equal to the current density.

Check Your Progress

10. What are magnetic flux lines?
11. Define the terms steady electric currents and current density.
12. What are convection currents?
13. State Ohm's Law.
14. Define Kirchhoff Law.

NOTES

1.9 MAGNETIC SCALAR AND VECTOR POTENTIAL

The magnetic Scalar and Vector potential can be explained using the precise equations on the scalar electic potential and Vector magnetic potential.

In this section, the general form of wave equations despite medium properties will be covered. During the process of deriving we will observe the conditions for relations between the scalar electric potential, *'V'* and vector magnetic potential $'A'$. This condition is called Lorentz condition. The procedure to obtain the wave equations include, starting from Maxwell's equation and applying the vector identities and simple manipulations of the obtained equations will result in the wave equations.

From Maxwell's Second Equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{dt}$$

We know that,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Therefore,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times \vec{E} + \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = 0$$

$$\vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0 \qquad (1.47)$$

We know that the vector identity and the curl of a gradient is zero. Therefore,

$$\vec{\nabla} \times (-\vec{\nabla} V) = 0 \qquad (1.48)$$

Comparing Equations (1.47) and (1.48), we get,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

NOTES

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\left[\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t}\right] \quad (1.49)$$

Taking divergence of Equation (1.49), we get,

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left[-\left(\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t}\right)\right]$$

$$\vec{\nabla}.\vec{E} = -\left[\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A})\right] \quad (1.50)$$

But from Maxwell's First Equation,

$$\vec{\nabla}.\vec{E} = \frac{\rho_V}{\epsilon}$$

Therefore, equating the above equation with Equation (1.50), we have,

$$\frac{\rho_V}{\epsilon} = -\left[\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A})\right]$$

Or

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho_V}{\epsilon} \quad (1.51)$$

So far, we have used Maxwell's equation for electric field and now let us use Maxwell's equation for magnetic field to couple the two fields to define the electromagnetic waves. Considering, Maxwell's Fourth Equation,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{dt}$$

But, $\vec{B} = \mu\vec{H}$. Therefore,

$$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \vec{J} + \frac{\partial \vec{D}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\frac{\partial \vec{D}}{dt}$$

Also, $\vec{D} = \epsilon\vec{E}$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial \vec{E}}{dt} \quad (1.52)$$

Substituting for $\vec{E}$, from Equation (1.49), we get,

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial}{dt}\left[-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right]$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

But $\vec{B} = \vec{\nabla} \times \vec{A}$, Hence,

NOTES

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} \quad (1.53)$$

Applying vector identity to the LHS of Equation (1.53), we get,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$$

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} \quad (1.54)$$

Observing Equation (1.54), we find that scalar potential V and vector potential $\vec{A}$ can be separated and decoupled. Therefore for vector potential, $\vec{A}$, Equation (1.54) becomes,

$$-\nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

Or

$$\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} = \mu\vec{J} \quad (1.55)$$

1.10 MAGNETISATION

In 1831, Michael Faraday reported on a series of ground-breaking experiments which showed that,

(*i*) Whenever there is any change in the magnetic flux over the surface of a closed circuit, there is an ElectroMotive Force (EMF) induced in the same.

(*ii*) The induced EMF in the closed circuit is directly proportional to the time rate of change of magnetic flux over the surface of the circuit.

Lenz noticed that this induced EMF opposes the very cause (rate of change of magnetic flux) of its creation. Thus, if ϕ is assumed to be the magnetic flux linked with a closed circuit, the induced EMF acting in the circuit must be

$$\varepsilon = -\frac{\partial\phi}{\partial t} \quad ...(1.56)$$

Also we know that,

$$\phi = \iint_S \vec{B}.d\vec{S} \quad ...(1.57)$$

Where $\vec{B}$ is the magnetic induction vector associated with the changing magnetic flux.

Thus, $$\varepsilon = -\frac{\partial}{\partial t}\iint_S \vec{B}.d\vec{S} \quad ...(1.58)$$

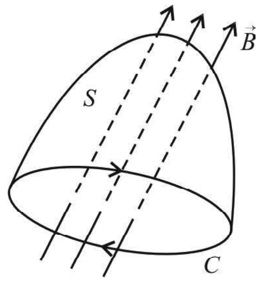

Fig. 1.21 *Mognetic Flux*

NOTES

Here, S is an open surface such that the closed circuit acts as its boundary (C) [Refer Figure 1.21]. If $\vec{E}$ is the electric field generated within the circuit due to the time varying magnetic flux, then the emf acting in the circuit is,

$$\varepsilon = \oint_C \vec{E}.d\vec{l} \quad ...(1.59)$$

Thus, combining Equations (1.58) and (1.59), we get

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}.d\vec{S} \quad ...(1.60)$$

Equation (1.60) is known as the **integral form of Faraday's law**.

If we use Stokes' theorem on the left hand side of Equation (1.60), we get,

$$\oint_C \vec{E}.d\vec{l} = \iint_S (\vec{\nabla} \times \vec{E}).d\vec{S} \quad ...(1.61)$$

So, from Equations (1.60) and (1.61), we can write,

$$\iint_S (\vec{\nabla} \times \vec{E}).d\vec{S} = -\frac{\partial}{\partial t}\iint_S \vec{B}.d\vec{S}$$

Or,
$$\iint_S (\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}).d\vec{S} = 0$$

As S is an arbitrary open surface, the above equation is true when the integrand vanishes, i.e.,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ...(1.62)$$

Equation (1.62) represents the **differential form of Faraday's law.**

Thus, from the Faraday's law of electromagnetism, it is evident that a changing magnetic field induces an electric field. This fact may somehow seems to violate the principle of conservation of energy. But, according to Lenz's idea, the induced emf always opposes the rate of change of magnetic flux, i.e., the work done by the magnetomotive force compensates in inducing the electromotive force on the closed circuit. So, the principle of conservation of energy also holds good in this case.

In the context of electromagnetic induction, we get an idea of **motional EMF**. If a conducting rod (or wire) moves in a magnetic field, an EMF is induced between the ends of the rod. Such an emf induced due to the motion of a conducting elements is known as motional EMF. From the Faraday's law, this phenomenon can be easily explained. When the conducting rod moves in the magnetic field, the magnetic flux linked with the rod changes in time. As a result, EMF develops within the rod. Now, let us try to explain this phenomenon of electromagnetic induction from the basic electronic theory.

Suppose, AB is a conducting rod of length L parallel to X-axis. If it moves with a velocity $\vec{v}$ along Y-axis in the presence of a magnetic induction $\vec{B}$ along Z-axis, the magnetic Lorentz force acting on each electron within the conductor is,

$$\vec{F} = -e(\vec{v} \times \vec{B}) \quad ...(1.63)$$

This force acts along negative X-axis. Due to the action of this force some electrons will be drawn towards A end resulting B end as positively charged. This fact, in turn, produces an electric field $\vec{E}$ and hence an electrostatic force,

$$\vec{F}_e = -e\vec{E} \quad ...(1.64)$$

acting on each electron. At equilibrium, the two forces $\vec{F}_m$ and $\vec{F}_e$ balance one another giving a steady condition. Thus, in steady state,

NOTES

$$\vec{F}_m = \vec{F}_e$$

i.e., $$-e(\vec{v} \times \vec{B}) = -e\vec{E}$$

i.e., $$\vec{E} = (\vec{v} \times \vec{B}) \quad ...(1.65)$$

Hence, the emf induced between the two ends of the rod under steady state is,

$$\varepsilon = \int_0^L \vec{E}.d\vec{l} = \int_0^L (\vec{v} \times \vec{B}).d\vec{l} = vBL \quad ...(1.66)$$

This is a direct consequence of electromagnetic induction.

1.10.1 Magnetic Circuits

The source of magnetic flux is either a permanent magnet or a current carrying coil. The lines of the magnetic flux always form a closed path. The closed path followed by the lines of magnetic flux is called a magnetic circuit. Thus, a magnetic circuit provides a closed path for the magnetic flux and is similar to an electric circuit which provides a closed path for the flow of electric current. In this section, definitions about the various magnetic quantities and simple analysis of magnetic circuits are explained.

First Law of Magnetics: Like poles of magnets repel each other whereas unlike poles attract each other.

Second/Coulomb's Law: Second law of magnetics known as Coulomb's law accounts for the force exerted between two magnetic poles.

Coulomb's magnetic law states that the force F between two magnetic poles P_1 and P_2 is directly proportional to the product of pole strengths and inversely proportional to square of their disance apart d.

$$F \propto \frac{P_1 P_2}{d^2}$$

The constant of proportionality depends on the relative premeability of the medium through which the lines of magnetic field passes.

$$F = \frac{\mu_o \mu_r}{4\pi} \frac{P_1 P_2}{d^2} \quad \text{N}$$

F – Force between the poles in N (Newton)

P_1, P_2 – Pole strength in A/m

d – Distance between poles in m

μ_o – Permeability of free spaces ($4\pi \times 10^{-7}$ H/m)

μ_r – Relative permeability of the medium

$\mu_r = 1$ for air

From the above equation for force, an unit magnetic pole is defined as one which when situated 1 m distance in vacuum from an equal pole experiences a force of $\mu_o/4\pi$ Newton.

1.10.2 Vector Magnetic Susceptibility and Permeability

NOTES

Magneto Motive Force (MMF)

MMF or Magneto Motive Force is the source of producing flux in a magnetic circuit. For a current *I* flowing through a coil of *T* turns, the magnetic flux is obtained as a product of *I* and *T*. Its variable symbol is F_m and its unit is Ampere-Turn (AT).

$$F_m = IT \quad \text{AT} \tag{1.67}$$

Magnetising Force

The Magnetising Force MF, otherwise called as Magnetic Field Intensity (MFI), is defined as the magneto motive force per unit length of the magnetic flux path.

Magnetising force is a measure of the ability of a magnetised body to produce magnetic induction in other magnetic substances. H is the variable symbol to denote magnetising force and Ampere-Turns/metre (AT/m) is the unit.

$$H = \frac{F_m}{l} = \frac{IT}{l} \quad \text{AT/m} \tag{1.68}$$

Magnetic Flux

The magnetic lines of force or the amount of lines of a magnetic field provided by a magnet is called the magnetic flux. It is represented by the variable symbol ϕ and its unit is Weber (Wb).

$$1 \text{ Wb} = 10^8 \text{ Magnetic Lines} = 10^8 \text{ Maxwells} \tag{1.69}$$

Magnetic Flux Density

Magnetic flux density or Magnetic Induction (MT) is defined as the magnetic flux per unit area at right angles to the direction of the flux. *B* is the variable symbol used to denote the magnetic flux density and its unit is Weber per square meter (Wb/m^2) or Tesla (T).

$$B = \frac{f}{A} \text{ T} \tag{1.70}$$

Permeability

This is the ability of the medium to set up a magnetic flux density (*B*) by the magnetising force (*H*).

Permeability of Free Space

The flux density established in a vacuum changes linearly with respect to the magnetising force and the proportionality constant is called the permeability of free space. It is denoted by μ_0 and has the unit of Wb/AT-m or Henry per metre (H/m).

$$\mu_0 = \frac{B}{H} = 4\pi \times 10^{-7} \quad \text{H/m} \tag{1.71}$$

Relative Permeability

In ferromagnetic materials, like steel, by virtue of their inherent property, a given magnetising force sets up much more magnetic flux density compared with that in a vacuum.

The ratio of flux density produced in a medium or material to the flux density produced in a vacuum by the same magnetising force is called as the relative permeability, denoted by μ_r.

$$\mu_r = \frac{\text{Flux density in the medium}}{\text{Flux density in the vacuum}} \qquad (1.72)$$

$$\mu_r = \frac{\text{Flux density in the medium (B)}}{\mu_0 H}$$

or $$B = \mu_r \mu_0 H \qquad (1.73)$$

For many magnetic materials, the value of μ_r itself changes with different values of the magnetising force. The value of relative permeability in different media are

μ_r = 1000 – 10000 for magnetic materials

μ_r = 1 for non-magnetic materials.

Absolute Permeability

The product of relative permeability and permeability of free space is called the absolute permeability and is denoted by μ.

$$\mu = \mu_r \mu_0 \qquad (1.74)$$

And $$B = \mu H \qquad (1.75)$$

Reluctance and Permeance

The reluctance and permeance properties of the magnetic plux and circuits are explained in this section.

Reluctance

The opposition offered by a magnetic circuit to the establishment of a magnetic flux is called as reluctance of the magnetic circuit. R_m is the variable symbol used to denote the reluctance of the magnetic circuit and its unit is Ampere-Turn/Weber (AT/Wb).

The reluctance in a magnetic circuit is directly proportional to the length of the field path l, and inversely proportional to the area of a cross-section A, of the magnetic field path.

$$R_m \propto \frac{l}{A}$$

$$R_m = \frac{l}{\mu A} \quad \text{AT/Wb} \qquad (1.76)$$

Further from Equations (1.68), (1.70) and (1.75),

$$\begin{aligned} F_m &= Hl \\ &= \frac{B}{\mu} l = B \times \frac{A}{A}\frac{l}{\mu} \\ &= \phi R_m \end{aligned}$$

Thus $$R_m = \frac{F_m}{\phi} \qquad (1.77)$$

NOTES

Permeance

The reciprocal of reluctance is called permeance. Therefore, the permeance of the magnetic circuit is the readiness with which a magnetic flux is developed. P_m is the variable symbol and its unit is either Weber/Ampere-Turn (Wb/AT) or Henry (H).

$$P_m = \frac{1}{R_m} \quad \text{Wb/AT} \tag{1.78}$$

Example

The flux produced in the air-gap between two electromagnetic pole faces is 6×10^{-2} Wb. Length of air-gap is 1.4 cm and cross-sectional area of the gap is 0.3 m^2. Find (i) The flux density (ii) Magnetic field intensity, (iii) Reluctance, (iv) Permeance and (v) MMF dropped.

Data $\phi = 6 \times 10^{-2}$ Wb; $l_g = 1.4 \times 10^{-2}$ m; $A = 0.3$ m^2; $\mu_r = 1$

Aim B?, H? R_m? P_m? MMF?

Solution:

(i) Flux density $B = \dfrac{\phi}{A} = \dfrac{6\times10^{-2}}{0.3}$

$$B = 0.2 \text{ T}$$

(ii) Magnetic field intensity $H = \dfrac{B}{\mu_0} (\because \mu_r = 1 \text{ for air})$

$$= \frac{0.2}{(4\pi\times10^{-7})}$$

$$H = 159.155\times10^3 \text{ AT/m}$$

(iii) Reluctance $R_m = \dfrac{l_g}{\mu_0 \text{A}}$

$$= \frac{1.4\times10^{-2}}{(4\pi\times10^{-7})\times0.3}$$

$$R_m = 37.136\times10^3 \text{ AT/Wb}$$

(iv) Permeance $P_m = \dfrac{1}{R_m} = \dfrac{1}{37.136\times10^3}$

$$P_m = 26.928\times10^{-6} \text{ Wb/AT}$$

(v) MMF for Air-Gap $F_m = \varphi \text{R}_\text{m}$

$$= (6\times10^{-2})\times(37.136\times10^3)$$

$$F_m = 2228.16 \text{ AT}$$

1.10.3 Magneto-Static Energy

The analysis of electric and magnetic fields are generally facilitated by the use of auxiliary functions those are termed as electromagnetic potentials.

NOTES

(*i*) **Electromagnetic Vector Potential**

From Maxwell's second equation, we have

$$\vec{\nabla}.\vec{B} = 0 \qquad ...(1.79)$$

But, we know that div curl of a vector is always zero, i.e.,

$$\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0 \qquad ...(1.80)$$

Thus, from Equations (1.79) and (1.80), we can write,

$$\vec{\nabla} \times \vec{A} = \vec{\beta} \qquad ...(1.81)$$

So, $\vec{B}$ is the curl of a vector $\vec{A}$ and $\vec{A}$ is defined by,

$$\vec{A} = \vec{A}(x, y, z; t)$$

The vector $\vec{A}$ is called electromagnetic "Vector Potential".

(*ii*) **Electromagnetic Scalar Potential**

From Maxwell' third equation, we have,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right)$$ (Putting the value of $\vec{B}$ in terms of electromagnetic vector potential from Equation (1.81)

$$= -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \qquad \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \qquad ...(1.82)$$

From the above Equation (1.82) we can say that the term $\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)$ is a irrotational vector and should be the negative sign of gradient of a scalar function.

Therefore, $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$, where ϕ is a scalar function and often called electromagnetic "Scalar Potential".

E.M. Wave in a Charge Free Conducting Media

For a charge free conducting media $\rho = 0$. Thus, the Maxwell's equations are,

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad ...(1.83)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \qquad ...(1.84) \qquad [\because \vec{B} = \mu \vec{H}]$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad ...(1.85) \qquad [\because \vec{B} = \mu \vec{H}]$$

$$\vec{\nabla} \times \vec{H} = \delta \vec{E} + \in \frac{\partial \vec{E}}{\partial t} \qquad ...(1.86) \qquad [\because \vec{B} = \mu \vec{H}]$$

NOTES

Taking curl on both sides of Equation (1.85), we get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \qquad \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\delta\vec{E} + \in\frac{\partial\vec{E}}{\partial t}\right)$$

[Putting the value of $\vec{\nabla}\times\vec{H}$ from Equations (1.86)]

$$\Rightarrow \qquad \nabla^2\vec{E} - \mu\delta\frac{\partial\vec{E}}{\partial t} - \mu\in\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad ...(1.87)$$

[Putting the value of $\vec{\nabla}\cdot\vec{E}$ from Equations (1.88)]

Similarly, taking curl on both sides of Equations (1.86), we can get,

$$\nabla^2\vec{H} - \mu\delta\frac{\partial\vec{H}}{\partial t} - \mu\in\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad ...(1.87(a))$$

Let us consider the solution of eqn. (1.87) is of the form

$$\vec{E} = \vec{E}_0 e^{i(\vec{K}\cdot\vec{r} - \omega t)} \qquad ...(1.88)$$

Thus, we can get the value of $\nabla^2\vec{E}, \frac{\partial\vec{E}}{\partial t}$ and $\frac{\partial^2\vec{E}}{\partial t^2}$ from Equations (1.88)

$$\left.\begin{aligned} \nabla^2\vec{E} &= -K^2\vec{E} \\ \frac{\partial\vec{E}}{\partial t} &= -i\omega\vec{E} \\ \frac{\partial^2\vec{E}}{\partial t^2} &= -\omega^2\vec{E} \end{aligned}\right\} \qquad ...(1.89)$$

Now, putting the value of $\nabla^2\vec{E}, \frac{\partial\vec{E}}{\partial t}$ and $\frac{\partial^2\vec{E}}{\partial t^2}$ from Equations (1.89) to Equations (1.87), we get,

$$(-K^2 + i\mu\delta\omega + \mu\in\omega^2)\,\vec{E} = 0$$

$$\because \qquad -K^2 + i\mu\delta\omega + \mu\in\omega^2 = 0 \qquad [\because \vec{E} \neq 0]$$

$$\Rightarrow \qquad K^2 = \mu\in\omega^2\left(1 + \frac{i\delta}{\in\omega}\right) \qquad ...(1.89(a))$$

The above equation is known as "Dispersion Equation" in E.M. Theory. The first term of the equation corresponds to the displacement current and the second to the conduction current contribution.

Let us consider $K = \alpha + i\beta$. So,

$$K^2 = \alpha^2 - \beta^2 + 2i\alpha\beta \qquad ...(1.90)$$

Now, comparing Equations (1.89(a)) and (1.90), we have,

$$\alpha^2 - \beta^2 = \mu\in\omega^2$$

And $\quad 2\alpha\beta = \mu\delta\omega$

Thus,

$$\alpha = \omega\sqrt{\frac{\mu\in}{2}}\left[1+\left\{1+\left(\frac{\delta}{\in\omega}\right)^2\right\}^{1/2}\right]^{1/2}$$

...(1.91)

NOTES

And $$\beta = \omega\sqrt{\frac{\mu\in}{2}}\left[-1+\left\{1+\left(\frac{\delta}{\in\omega}\right)^2\right\}^{1/2}\right]^{1/2}$$

Now, putting the value of K in Equations (1.88), we have,

$$\vec{E} = \vec{E}_0 e^{-\beta\vec{r}} e^{i(\alpha\vec{r} - \omega t)} \quad ...(1.92)$$

Following the same procedure, solution of Equations (1.87(a)) will be,

$$\vec{H} = \vec{H}_0 e^{-\beta\vec{r}} e^{i(\alpha\vec{r} - \omega t)} \quad ...(1.93)$$

Let us consider the field vectors are propagated in the conducting media with speed $v\left(=\frac{\omega}{K}\right)$, thus,

$$v = \frac{\omega}{K} = \frac{1}{\sqrt{\mu\in}}\left[\frac{\sqrt{\left\{1+\left(\frac{\delta}{\in\omega}\right)^2\right\}}+1}{2}\right]^{-1/2} \quad ...(1.94)$$

Now, we will consider the different conductor types.

Case 1: For a poor conductor, $\frac{\delta}{\in\omega} << 1;$ thus, the values of α and β reduces to,

$$\alpha = \omega\sqrt{\mu\in} \quad \text{and} \quad \beta = \frac{\delta}{2}\sqrt{\frac{\mu}{\in}}$$

So, $$K = \alpha + i\beta = \omega\sqrt{\mu\in} + i\frac{\delta}{2}\sqrt{\frac{\mu}{\in}} \quad ...(1.95)$$

Case 2: For a good conductor, $\frac{\delta}{\in\omega} >> 1;$ thus the values of α and β reduces to,

$$\alpha = \beta = \sqrt{\frac{\mu\delta\omega}{2}}$$

So, $$K = \alpha + i\beta = (1 + i)\sqrt{\frac{\mu\delta\omega}{2}} \quad ...(1.96)$$

1.10.4 Uniformly Magnetised Sphere in Magnetic Field

To explain the magnetised sphere in a magnetic field, consider that there is a uniformly magnetised sphere having the radius $r = a$. The sphere holds a uniform permanent magnetisation which is denoted as $\mathbf{M} = M_0\, \mathbf{e}_z$, which is precisely in the direction '**z**' and is typically surrounded or enveloped by vacuum. This problem can be solved using the scalar magnetic potential, denoted as Φ_m.

The magnetic scalar potential satisfies the Laplace equation of the form:

$$\nabla^2 \phi_m = e_m = 0 \quad ....(1.97)$$

NOTES

Fundamentally, this is due to the reason that there is precisely a zero net current in the sphere and also the magnetic volume charge density in a uniformly magnetized magnetic medium or vacuum is zero.

Even though, considering that there is a net magnetic surface charge density on the surface of the sphere, we can write the equation as:

$$\sigma_m = \mathbf{M}.\hat{r} = M_0 \cos\theta$$

In the above equation, the terms r and θ are referred as the spherical polar coordinates. Consequently, the boundary conditions determine and establish the notion that the tangential component of the magnetic field '**H**' must be continuous on the surface of the sphere. Additionally, it can also be stated that the scalar magnetic potential should also be continuous at $r = a$, because,

$$\nabla.\mathbf{B} = \mu_0 \nabla.(\mathbf{H} + \mathbf{M}) = 0$$

Therefore, we can write,

$$\phi_m(r = a+, \theta) = \phi_m(r = a-, \theta)$$

Furthermore, we know that,

$$\mathbf{e}_m = -\nabla.\mathbf{M} \quad \text{.... (1.98)}$$

Applying the Equations (1.97) and (1.98) and integrating the Equation (1.98) over a Gaussian pill-box typically spanning or overlapping the surface of the sphere gives us the equation of the form:

$$\left[\frac{\partial\phi_m}{\partial r}\right]_{r=a-}^{r=a+} = -\sigma_m = -M_0 \cos\theta \quad \text{....(1.99)}$$

This implies that the magnetic charge sheet on the surface of the sphere eventually causes a discontinuity in the magnetic scalar potential, which is radially at the surface, $r = a$.

The extremely common axisymmetric solution to Equation (1.99) given above is that it uniquely satisfies the given boundary conditions at $r = a$ and $r = \infty$ and also involve the Legendre polynomials. The equation can then be written as:

$\phi_m(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$, for $r < a$ (Inside the Sphere)

And for outside the sphere, we have ($r \geq a$):

$$\phi_m(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

Applying the boundary condition given in Equation (1.99) for all values of l, we can state taht,

$$B_l = A_l a^{2l+1}$$

Substituting the expression for the given boundary condition and solving we get:

NOTES

$$-\frac{(l+1)B_l}{a^{l+2}} - lA_l a^{l-1} = - M_0 \delta_{l1}$$

Now, $P_l(\cos\theta) = \cos\ \theta$, we get (for $l \neq 1$)

$$A_l = B_l = 0$$

And (for $l = 1$),

$$A_l = \frac{M_0}{3}\ and\ B_l = \frac{M_0 a^3}{3}$$

Therefore, we can write, (for $r < a$):

$$\phi_m(r,\theta) = \frac{M_0 a^2}{3}\frac{r}{a^2}\cos\ \theta$$

And for ≥ , we have:

$$\phi_m(r,\theta) = \frac{M_0 a^2}{3}\frac{a}{r^2}\cos\ \theta$$

Since it is obvious and evident from the uniqueness theorem of the Poisson's equation that this extremely common axisymmetric potential is the only solution that uniquely satisfies the given boundary conditions at 'Origin' and at '∞'.

Consequently, because outside the sphere there is vacuum, therefore we can write the following equation:

$$\mathbf{B} = \mu_0 \mathbf{H} = \mu_0 \nabla \phi_m$$

In addition, we have,

$$\mathbf{B}(r > a) = \frac{\mu_0}{4\pi}\left[-\frac{\boldsymbol{m}}{r^3} + \frac{3(\boldsymbol{m}.\boldsymbol{r})\boldsymbol{r}}{r^5}\right]$$

Where, $\boldsymbol{m} = \frac{4}{3}\pi a^3 \mathbf{M}$

Basically, this is considered as the essential magnetic field of a magnetic dipole having the moment '***m***'. Consequently, we have the net dipole moment of the sphere which is considered as equal to the integral of the magnetisation '**M**' and essentially which too is exactly considered as the dipole moment per unit volume.

Inside the sphere, we can write the equation of the form:

$$\mathbf{H} = -\boldsymbol{\nabla}\phi_m$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

Solving the above given two equations, we have:

$$\mathbf{H} = -\frac{\mathbf{M}}{3}$$

And,

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M}$$

From the above evaluations it is obvious that both the '**H**' and '**B**' fields are uniform inside the sphere, even though the magnetic intensity of the field is considered as opposite with respect to the magnetisation of the sphere. This implies and signifies that the external field uniquely acts in a manner to demagnetise the sphere.

1.11 CLASSICAL THEORIES OF PARA, DIA AND FERRO MAGNETISM

Magnetism is a phenomenon by which a material exerts either a positive or a repulsive force on another material. The source of 'Magnetism' of any material can be traced back to its electrical behaviour, which in turn is dictated by the structure of its constituents, i.e., atoms and molecules.

All magnetic fields, including the ones inside a material, are due to tiny currents or electric charges in motion. If we examine a small piece of any magnetic material we will find electrons orbiting around the nuclei or electron spinning on its axis, each of which contributes towards generating the magnetic field inside a material. We, generally, treat these tiny current loops as magnetic dipoles for macroscopic purposes. In general, these dipoles cancel each other on an average because of the random orientation of atoms in a material. However, when an external magnetic field is applied, these tiny current loops which form magnetic moments act as tiny magnets and reorient themselves thereby making the material 'Magnetically Polarised' or 'Magnetised'. This orientation of these magnetic dipoles in a material can be either be parallel or opposite to the external magnetic field, depending on its magnetic nature.

All materials, whether they are solid, liquid or gaseous, can be divided broadly into three categories of magnetism, depending on how they respond to an external magnetic field. This is measured quantitatively by a material's magnetic susceptibility, its magnitude as well as its sign.

Here are some important relations involving magnetic quantities, such as: B, M, H, X_m and u (u_0 and u_r) that must be understood and remembered.

1. **H: Strength of the External Magnetic Field.**
2. **M: Magnetisation of the material in an external magnetic:** It is defined as the total magnetic moments per unit volume. M = N * pm; where pm is the dipole magnetic moment and N is the number of dipoles of per unit volume.
3. **Magnetic Susceptibility χ_m:** Magnetic susceptibility of a material is a measure of how weakly or strongly it is affected by the presence of an external magnetic field. It measures the ratio of M by B. Magnetic susceptibility is a dimensionless quantity.

$$\chi_m = M/H$$

Thus, $$M = \chi_m H$$

4. **B = Total Magnetic Field Intensity in the Material**

$$B = u_0 H + u_0 M = u_0 u_r M$$

NOTES

Where u_0 is the magnetic permeability of the material in vacuum and u_r is the relative magnetic permeability.

5. **Magnetic Permeability:** It is a measure of the degree of penetration of magnetic field in a substance. It is defined as the ratio of total magnetic flux density and external magnetic field intensity.

   We can thus write:

   $B = uH$, where u is the magnetic permeability in the material.

   $$B = u_0 H + u_0 M = u_0 u_r M$$

   Where u_0 is the magnetic permeability of the material in vacuum and u_r is the relative magnetic permeability.

   $$B = u_0H + u_0X_mH = (1+X_m)u_0H$$

   And,

   $$X_m = u_r - 1$$

   $$u = u_0 (1+X_m)$$

1.11.1 Properties of Magnetic Materials: Salient Features

Paramagnetism

1. Paramagnetic materials are those which are weakly attracted in an external magnetic field. When exposed to external magnetic field, they acquire feeble magnetisation in the same direction as the external field.
2. In the absence of an external field, the orientations of atomic magnetic moments are random, leading to no net magnetisation of the material.
3. If placed in a random orientation in an external field, a paramagnetic material will rotate and its magnetic dipoles will line up with the field until their longest axis becomes parallel to the external field. This results in positive magnetisation in the material. However, because the dipoles do not interact, extremely large magnetic fields are required to align all dipoles. Further, as soon as the external field is removed, the magnetisation in the material becomes zero.
4. The magnetisation in a material under the influence of an external field decreases with increase in temperature because the thermal agitation randomises the orientation direction of magnetic dipoles.
5. Platinum, aluminium and manganese are some examples of elements which are paramagnetic in nature. Other examples of such substances include alloys of copper, solutions of salts of iron, nickel and copper sulphate.
6. The relative permeability of a paramagnetic substance is slightly greater than 1, for example platinum has a relative permeability of 1.00036. manganese, which is the most known paramagnetic material, has a relative permeability of 1.00013.
7. The magnetic susceptibility of a paramagnetic substance is small ($\sim10^{-5}$ to 10^{-2}) and does not change with increase in external magnetic field.

8. The magnetic susceptibility of a paramagnetic substance decreases with increase in temperature, i.e., it varies inversely with absolute temperature. As a result, above a certain temperature a paramagnetic material acquires negative susceptibility and becomes diamagnetic.
9. Thus, in paramagnetic substances, the magnetism is weak and exists only in the presence of an external magnetic field.

Diamagnetism

1. These are the materials which are weaky repelled by an external magnetic field as opposed to paramagnets. When placed in an external magnetic field they acquire a feeble magnetisation in the direction opposite to that of external field.
2. If placed in a random orientation in an external field, a diamagnetic material will rotate until their longest axis becomes antiparallel to the external field. The magnetic dipoles in atoms act so as to oppose the applied field. The effect produces small negative magnetisation.
3. Gold, bismuth and copper are some examples of elements which are diamagnetic at room temperature. Other examples of such substances include alumina, alcohol, water and hydrogen gas.
4. The relative permeability of diamagnetic materials is slightly less than unity.
5. The magnetic susceptibility of a diamagnetic material is small and negative ($\sim -10^{-5}$). It does not vary with change in external field or temperature unlike that of paramagnetic material.
7. Just like in paramagnetics, in diamagnetic material the magnetism is weak and exists only in the presence of an external magnetic field, though the sense of direction is opposite to that in paramagnetic material.

Ferromagnetism

1. Both paramagnetic and diamagnetic materials are considered as non-magnetic materials as they exhibit magnetism only in the presence of an external magnetic field.
2. There are certain materials which are 'Magnetic' even in the absence of an external field. They possess permanent magnetic moments due to unpaired dipoles formed as a result of unpaired energy levels. Further, these materials are strongly magnetised even in the presence of a weak magnetic field. Such materials are called ferromagnetic materials.
2. The ferromagnets retain a substantial magnetisation even after the external field has been removed, hence for these materials the magnetisation is not just determined by the present field but by the complete magnetic history of the object.
3. Iron is unarguably the most well-known ferromagnetic material, in fact the term 'Ferromagnetism' can be traced to have its origin in the Latin name of Iron, the 'Ferrum'. Other ferromagnetic materials include nickel, cobalt and steel.

4. In their behaviour in an external field, the ferromagnetic materials resemble paramagnetic materials and bear some more properties.
5. Their magnetic susceptibility is positive and very large (~10^6 in value). Their relative permeability is also very large ~100s or 1000s in value.
6. The intensity of magnetisation in a ferromagnetic material increases with increase of external magnetic field. It is directly proportional for low values of H, but increases very rapidly at high value of external magnetic field, H. For very high H, the magnetisation of a ferromagnetic material approaches a constant value M_0.
7. The magnetic susceptibility remains nearly constant for low H, but increases rapidly for high H, and then decreases for H values.
8. The magnetic induction of the material, B essentially traces the behaviour of M vs H. However, its value remains lower than that of M at high H.
9. The magnetic permeability also follows the behaviour of susceptibility except at very high H, where permeability varies slowly as compared to susceptibility.
10. Magnetic susceptibility of a ferromagnetic material decreases with an increase in temperature; and above a certain temperature the material becomes paramagnetic. This temperature is known as 'CURIE TEMPERATURE' of a material. T_c is 1000^0 for Iron, 360^0 for Nickel and 1150^0 for Cobalt.

NOTES

1.11.2 Classical Theories of Magnetism

Although for a better knowledge of magnetic materials we may have to use quantum theory, but for basic understanding the classical picture which assumes that materials are made up of atoms wherein there is a central positive nucleus surrounded by electrons in various circular orbits will be sufficient.

Classical Theory of Diamagnetism

A diamagnet is a substance that exhibits negative magnetism under the influence of an external field. The origin of diamagnetism can be explained using the simple model of orbital motion of electron around a nucleus, which can be thought of as a small current and hence a tiny magnetic dipole with net magnetic moment. Even though it is composed of atoms which have no net magnetic moment over the whole material, it reacts in a particular way to an applied field.

The classical (non-quantum-mechanical) theory of this effect was first given by the French physicist Paul Langevin (1872– 1946) in his famous paper published in 1905. He had further refined and made quantitative some ideas which had been earlier advanced by Ampére and by the German physicist Wilhelm Weber (1804 – 1891).

According to this theory, the effect of an applied external magnetic field on a single electron orbit is to reduce the effective current of the orbit. In the process, it produces a magnetic moment opposing the applied field. This effect is then summed over all the electrons in the atom, and each atom is regarded as acting independently of the others. The values of diamagnetic susceptibility calculated in this way generally agrees with experimental values to an accuracy better than a factor of 10, which suggests that the model is at least qualitatively correct. The model also does not point to strong temperature dependence of susceptibility, which is also shown by the experiments.

Electrons which constitute a closed shell in an atom usually have their spin and orbital moments oriented so that the atom as a whole has no net moment. Thus, the monoatomic rare gases He, Ne, A, etc., which have closed-shell electronic structures, are all diamagnetic. So are most polyatomic gases, such as H_2, N_2, etc., because the process of molecule formation usually leads to filled electron shells and no net magnetic moment per molecule. The same argument can be used to explain the diamagnetism of ionic solids like NaCl. The process of bonding in this substance involves the transfer of an electron from each Na atom to each Cl atom; the resulting ions of sodium and chlorine, then have closed shells and are both diamagnetic. Covalent bonding by the sharing of electrons also leads to closed shells, and elements like C (diamond), Si, and Ge are diamagnetic. Most organic compounds are diamagnetic, and magnetic measurements have furnished much useful information about the size and shape of organic molecules.

But not all gases are diamagnetic and nor are all ionic or covalent solids. An interesting class of materials in this field is 'Superconductor' which under some conditions behave as perfect diamagnets.

Classical Theory of Paramagnetism

The classical theory of paramagnetism is quite similar to that of alignment of polar molecules placed in an electric field that we study in electrostatics. Langevin gave the theory of magnetic susceptibility for paramagnetic materials. It is characterised by a positive but small value of magnetic susceptibility. Langevin's theory describes the behaviour fairly accurately only for gaseous substances where there is negligible interaction between molecules. The molecules of such a substance have a permanent dipole magnetic moment and they are free to orient themselves in an external magnetic field. This dipole moment has contributions from both spin and orbital magnetic moment. Only the partially filled shells in an atom or molecule contribute towards its moment. However, due to small interaction, these dipoles are randomly oriented and net magnetic moment of material cancels out. When an external magnetic field is applied, it exerts torque on the individual dipoles and tends to align them along the field direction. This alignment is however resisted by collisions among molecules due to their thermal motion. Over a period of time an equilibrium is reached and the material gets weakly magnetised depending on the strength of external magnetic field as well as temperature.

The first systematic measurements of the susceptibility of a large number of substances over an extended range of temperature were made by Pierre Curie 1 and reported by him in 1895. He found that the mass susceptibility was independent of temperature for diamagnetics, but that it varied inversely with the absolute temperature for paramagnetics. This relation is called the Curie law, and C is the Curie constant per gram. It was later shown that the Curie law is only a special case of a more general law, called the Curie–Weiss law.

Curie's measurements on paramagnetics went without theoretical explanation for 10 years, until Langevin in 1905 took up the problem in the same paper in which he presented his theory of diamagnetism. Qualitatively, his theory of paramagnetism is simple. He assumed a paramagnetic to consist of atoms, or molecules, each of which has the same net magnetic moment **m**, because all the spin and orbital moments of the electrons do not cancel out. In the absence of an applied field, these atomic moments point at random and cancel one another, so that the magnetization of the specimen is zero. When a field is applied, there is a tendency for each atomic moment to turn toward the direction of the field; if no opposing force acts, complete alignment of the atomic moments would be produced and the specimen as a whole would acquire a very large moment in the direction of the field. But thermal agitation of the atoms opposes this tendency and tends to keep the atomic moments pointed at random. The result is only partial alignment in the field direction, and therefore a small positive susceptibility. The effect of an increase in temperature is to increase the randomizing effect of thermal agitation and therefore to decrease the susceptibility.

Diamagnetism vs Paramagnetism

The magnetic susceptibility of paramagnetic materials is small and positive as opposed to small but negative susceptibility of diamagnetic materials. The Diamagnetic material is not much affected by temperature unlike a paramagnetic substance. Though a small amount of diamagnetism is present in all substances, its effect is masked in substances where paramagnetic behaviour dominates. diamagnetism is prominent in materials with closed electron shells wherein magnetic moment due to unpaired electron is not present and the remaining paramagnetic moments cancel out. Whether a substance will behave as paramagnet or diamagnet depends on external conditions, such as temperature. Since the susceptibility of a paramagnetic substance decreases with an increase of temperature, at high enough temperature a paramagnet becomes a diamagnet.

Theory of Ferromagnetism

In paramagnetic and diamagnetic materials, which require an external magnetic field to exhibit 'magnetism', the alignment of atomic dipoles is maintained by that field. Being non-linear in nature, unlike paramagnets and diamagnets, ferromagnetic materials do not need external fields to sustain magnetisation. In the paramagnetism, the 'Alignment' is 'Frozen In'. The ferromagnetic phenomenon involves the magnetic dipoles associated with the spins of unpaired electrons, like paramagnetism. Besides that, ferromagnets have one additional feature which makes them so different than paramagnets. This is the interaction between neighbouring dipoles. In a ferromagnet, each dipole likes to point in the same direction as its neighbour. This correlation is so strong that nearly all unpaired electron spins are aligned in same direction.

This alignment, however, occurs in small batches known as domains. Further, these domains are randomly oriented in a material. This makes the net magnetisation in the material zero and as a result not all ferromagnets are 'Magnets' in the broad sense of terms.

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In certain materials, there is a preference for antiparallel alignment between neighbouring dipoles. Such materials exhibit 'Anti-ferromagnetic' behaviour.

Example 1.1: Let magnetization for a ferromagnet is **M**(**r**) is given and its free current density is denoted as $\mathbf{j_f} = \mathbf{0}$.

(a) Write down the Maxwell equations for the magnetic induction **B** and the magnetic field **H** in the absence of any time-varying electric field for this case.

(b) Give the relation between **B**, **H** and **M**, for this ferromagnet.

(c) Define the magnetic scalar potential Φ_M.

(d) Show that $\nabla^2\Phi_M$ can be expressed in terms of **M** by a Poisson equation and give its formal solution in the absence of boundaries by comparing with the solution of the electrostatic Poisson equation.

Solution:

(a) $\nabla\mathbf{B} = 0,\ \nabla\times\mathbf{B} = \mu_0\mathbf{j},$

Here, $\mathbf{j} = \mathbf{j}_f + \mathbf{j}_m$

And, $\mathbf{j}_m = \nabla\times\mathbf{M}.$

$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M},$

$\nabla\times\mathbf{H} = \mathbf{j}_f.$

Now, since $\mathbf{j}_f = 0$

We have Maxwell's equations as:

$\nabla\mathbf{B} = 0,$

$\nabla\times\mathbf{H} = 0.$

(b) $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}.$

(c) The curl of **H** is zero everywhere, Hence, we can write $\mathbf{H} = -\nabla\Phi_M$.

(d) $\nabla\mathbf{H} = \nabla\mathbf{B}/\mu_0 - \nabla\mathbf{M}.$
Thus we have, $\nabla^2\Phi_M = \nabla\mathbf{M}.$

In electrostatics, the solution to the Poisson equation $\nabla^2\Phi = -\rho/\varepsilon_0$ in free space is given as,

$$\Phi(\mathbf{r}) = [1/(4\pi\varepsilon_0)] \int\rho(\mathbf{r}')\ dV'/|\mathbf{r} - \mathbf{r}'|.$$

Consequently, we can write the equation:

$$\Phi_M(\mathbf{r}) = [-1/(4\pi)] \int\mathbf{M}(\mathbf{r}')\ dV'/|\mathbf{r} - \mathbf{r}'|.$$

1.12 MAGNETIC CIRCUITS AND THEIR COMPARISON WITH ELECTRIC CIRCUITS

The analysis of magnetic circuits is somewhat similar to the analysis of DC electric circuits. This is due to the existence of a close analogy between the magnetic circuits and DC electric circuit.

NOTES

MMF (Magneto Motive Force), F_m is analogous to EMF (Electro Motive Force), E; Flux ϕ is analogous to current I; Reluctance R_m is analogous to resistance R, etc.

The circuit laws, Ohm's law and Kirchhoff's laws discussed in relation to electric circuits hold good for magnetic circuits also.

Ohm's Law for magnetic circuit is,

$$\text{Flux} = \frac{\text{MMF}}{\text{Reluctance}}$$

$$\phi = \frac{F_m}{R_m} \qquad (1.100)$$

Equation (1.100) is just a simple rearrangement.

Kirchhoff's Voltage Law (KVL) for magnetic circuit is,

MMF set up in a loop = MMF expended in various parts of the loop.

$$F_m = H_1 l_1 + H_2 l_2 + H_3 l_3 + ... + H_n l_n$$

Kirchhoff's Current Law (KCL) is applied to parallel magnetic circuits in which the total flux set up by a MMF divides between the different parallel paths as,

$$\phi = \phi_1 + \phi_2 + \phi_3 + ...\phi_n$$

Further, in series magnetic circuits, the equivalent reluctance of a number of reluctances in series is given as,

$$R_{meqs} = [R_{m1} + R_{m2} + R_{m3} + ...R_{mn}]$$

In parallel magnetic circuits, the equivalent reluctance can be calculated as

$$\frac{1}{R_{meqp}} = \left[\frac{1}{R_{m1}} + \frac{1}{R_{m2}} + \frac{1}{R_{m3}} + ...\frac{1}{R_{mn}}\right]$$

Series Circuits

The series circuits can be easily understood with the help of following example.

Example 1.2:

A magnetic core in the form of a closed ring has a mean length of 30 cm and a cross-sectional area of 1.2 cm^2. The relative permeability of iron is 2500. What current will be required to pass on through a coil of 2000 turns uniformly wound around the ring to create a flux of 0.5 mWb in the iron?

Data $l = 30$ cm = 0.3 m; $A = 1.2$ cm^2 = 1.2×10^{-4} m^2

= 0.5×10^{-3} Wb; $\mu_r = 2500$; $T = 2000$; $\phi = 0.5$ mWb

Aim I?

Solution:

Refer Figure (1.22) given below:

Reluctance

$$R_m = \frac{l}{\mu A} = \frac{l}{\mu_r \mu_0 A}$$

$$= \frac{0.3}{2500 \times (4\pi \times 10^{-7}) \times 1.2 \times 10^{-4}}$$

$$= 7.96 \times 10^5 \text{ AT/Wb}$$

NOTES

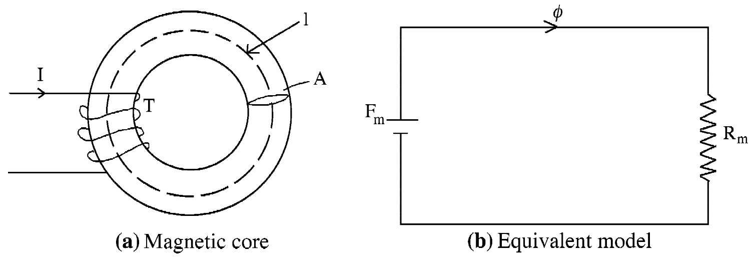


Fig. 1.22 *Diagram for Example 1.2*

MMF required,

$$F_m = \phi R_m$$

$$= (0.5 \times 10^{-3}) \times (7.96 \times 10^5) = 398 \text{ AT}$$

But

$$F_m = IT$$

$$I = \frac{F_m}{T} = \frac{398}{2000}$$

$$= 199 \times 10^{-3} \text{ A} = 199 \text{ mA.}$$

Example 1.3:

An iron rod of 1.8 cm diameter is bent to form a ring of mean diameter 25 cm and wound with 250 turns of wire. A gap of 1 mm exists in between the end faces. Calculate the current required to produce a flux of 0.6 mWb. Take relative permeability of iron as 1200.

Data d = 1.8 cm = 1.8 × 10^{-2} m; D = 25 cm =0.25 cm; T = 250

l_g=1 mm =1 × 10^{-3} m = 0.001 m; ϕ = 0.6 mWb = 0.6 × 10^{-3} Wb

Aim I?

Solution:

Refer Figure given below.

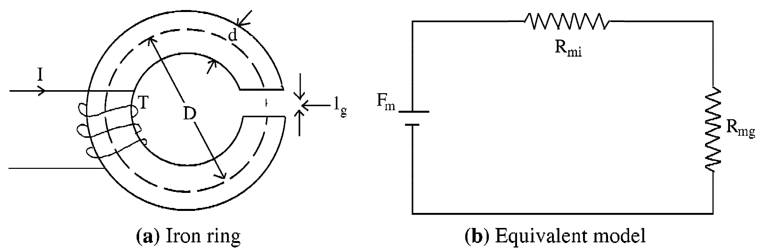


Fig. 1.23 *Diagram for Example 1.3*

Reluctance for iron path,

$$R_{mi} = \frac{l_i}{\mu_r \mu_0 A}$$

Total circumferencial length of the ring,

$$l = \pi D = \pi \times 0.25 = 0.785 \text{ m}$$

NOTES

Length of iron,

$$l_i = l - l_g = 0.785 - 0.001$$
$$= 0.784 \text{ m}$$
$$A = \frac{\pi d^2}{4} = \frac{\pi \times (1.8 \times 10^{-2})^2}{4}$$
$$= 2.54 \times 10^{-4} \text{ m}^2$$
$$R_{mi} = \frac{0.784}{1200 \times (4\pi \times 10^{-7}) \times (2.54 \times 10^{-4})}$$
$$= 2.047 \times 10^6 \text{ AT/Wb}$$

Reluctance for air-gap,

$$R_{mg} = \frac{l_g}{\mu_0 \times A} = \frac{0.001}{(4\pi \times 10^{-7})(2.54 \times 10^{-4})}$$
$$= 3.133 \times 10^6 \text{ AT/Wb}$$

Total reluctance of the magnetic circuit,

$$R_{mt} = R_{mi} + R_{mg}$$
$$= (2.047 \times 10^6) + (3.133 \times 10^6)$$
$$= 5.18 \times 10^6 \text{ AT/Wb}$$

MMF required,

$$F_m = \phi R_{mt}$$
$$= (0.6 \times 10^{-3}) \times (5.18 \times 10^6)$$
$$= 3108 \text{ AT}$$

Current required,

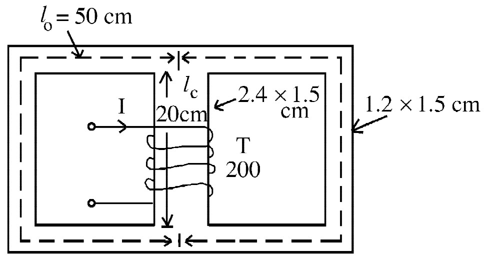
$$I = \frac{F_m}{T}$$
$$= \frac{3108}{250} = 12.432 \text{ A}$$

Parallel Circuits

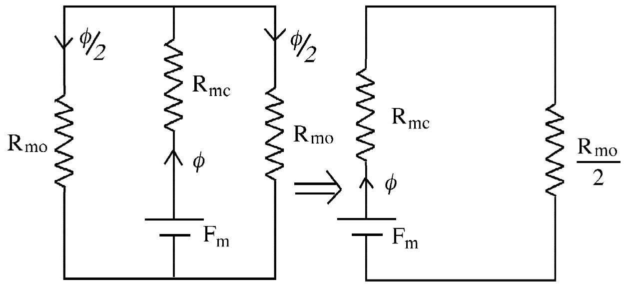
Example 1.4:

The magnetic circuit shown in Figure. 1.24 (a) has a cast steel core of relative permeability 1300. The dimensions are as marked in the figure. The centre limb has 200 turns of wire closely wound around. Find the current required to produce a flux of 1.2 mWb in the middle limb.

NOTES



(**a**) Magnetic circuit



(**b**) Equivalent circuits

Fig. 1.24 *Diagram for Example 1.4*

Data Dimensions as marked in Figure. (1.24 (a))

$T = 200$; $\phi = 1.2$ mWb; $\mu_r = 1300$

Aim I?

Solution:

For the centre limb,

$$R_{mc} = \frac{l_c}{\mu_r \mu_0 A_c}$$

$$l_c = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_c = 2.4 \times 1.5 \text{ cm}^2$$

$$= 3.6 \times 10^{-4} \text{ m}^2$$

$$R_{mc} = \frac{0.2}{1300 \times (4\pi \times 10^{-7}) \times (3.6 \times 10^{-4})}$$

$$= 340.075 \times 10^3 \text{ AT/Wb}$$

For each outer limb,

$$R_{m0} = \frac{l_0}{\mu_r \mu_0 A_0}$$

$$l_0 = 50 \text{ cm} = 0.5 \text{ m}$$

$$A_0 = 1.2 \times 1.5 = 1.8 \text{ cm}^2 = 1.8 \times 10^{-4} \text{ m}^2$$

$$R_{m0} = \frac{0.5}{1300 \times (4\pi \times 10^{-7}) \times (1.8 \times 10^{-4})}$$

$$= 1.7 \times 10^6 \text{ AT/Wb}$$

NOTES

Total reluctance,

$$R_{mt} = R_{mc} + \frac{R_{m0}}{2}$$
$$= 340.075\times10^{3} + \frac{1.7\times10^{6}}{2}$$
$$= 1.19\times10^{6} \text{ AT/Wb}$$
$$F_m = \phi R_{mt} = (1.2\times10^{-3}) * (1.19\times10^{6})$$
$$= 1428 \text{ AT}$$
$$I = \frac{F_m}{T} = \frac{1428}{200}$$
$$I = 7.14 \text{ A}$$

Example 1.5:

An inductor has a magnetic core built up of stampings of the shape as shown in Figure. 1.25. A coil of 600 turns being provided in the centre limb. There is a 1 mm air-gap in the centre limb which has a cross-sectional area of 3 cm^2. All the other paths in the core have a cross-sectional area of 2 cm^2. The mean magnetic path lengths in each portion of the core are as shown in the Figure 1.25. If the relative permeability of the steel core is 1100, find the current needed in the coil to produce a flux of 1 mWb in the centre limb.

Data $T = 600$; $l_g = 1\text{mm} = 1\times10^{-3}$ m; $A_c = 3$ cm^2 $=3\times10^{-4}$ m^2;

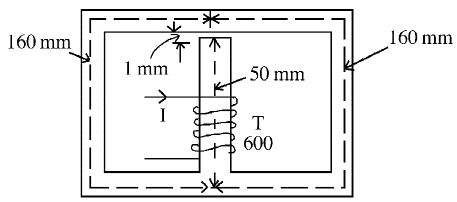
$A_0 = 2\text{cm}^2 = 2\times10^{-4}$ m^2; $\mu_r = 1100$; $\phi = 1$ mWb $= 1.1\times10^{-3}$ Wb

Aim *I*?

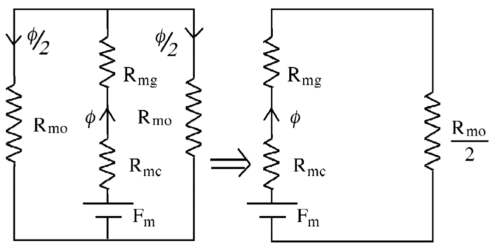
Solution:

For each outer limb,

$$R_{m0} = \frac{l_0}{\mu_r \mu_0 A_0}$$
$$= \frac{160\times10^{-3}}{1100\times(4\pi\times10^{-7})(2\times10^{-4})}$$
$$= 578.75\times10^{3} \quad \text{AT/Wb}$$



(a) Magnetic circuit



(b) Equivalent circuits

Fig. 1.25 *Diagram for Example 1.5*

NOTES

For the centre limb,

$$R_{mc} = \frac{l_c}{\mu_r \mu_0 A_c}$$

$$= \frac{50 \times 10^{-3}}{1100 \times (4\pi \times 10^{-7})(3 \times 10^{-4})}$$

$$= 120.572 \times 10^3 \text{ AT/Wb}$$

For the air-gap,

$$R_{mg} = \frac{l_g}{\mu_0 A_g}$$

$$= \frac{1 \times 10^{-3}}{(4\pi \times 10^{-7})\,(3 \times 10^{-4})}$$

$$= 2652.582 \times 10^3 \text{ AT/Wb}$$

Total Reluctance,

$$R_{mt} = R_{mc} + R_{mg} + \frac{R_{m0}}{2}$$

$$= \left(120.572 + 2652.582 + \frac{578.75}{2}\right)10^3 = 3.063 \times 10^6 \text{ AT/Wb}$$

$$\text{F}_\text{m} = \phi \times R_{mt}\,(1.1 \times 10^{-3}) \times (3.063 \times 10^6)$$

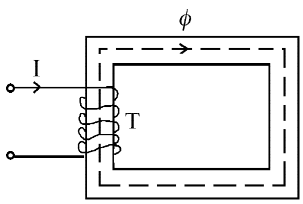
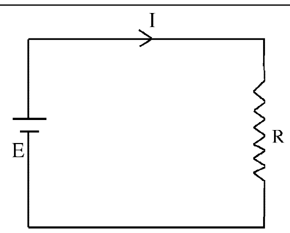
$$= 3369.3 \text{ AT}$$

$$\text{I} = \frac{\text{F}_\text{m}}{\text{T}} = \frac{3369.3}{600}$$

$$= 5.62 \text{ A}$$

Comparison between Magnetic and Electric Circuits

Table 1.1 *Comparison between Magnetic and Electric Circuits*

Magnetic Circuit	**Electric Circuit**
 *Magnetic Circuit*	 *Electric Circuit*
1. Closed path for magnetic flux *f* is known as a magnetic circuit.	Closed path for electric current *I* is known as electric circuit.
2. Magneto Motive Force, MMF in AT	Electro Motive Force, EMF in V
3. Flux $\phi = \dfrac{MMF}{\text{Reluctance}} = \dfrac{F_m}{R_m}$	Current $I = \dfrac{\text{EMF}}{\text{Resistance}} = \dfrac{E}{R}$
4. Reluctance $R_m = \dfrac{l}{\mu_r \mu_0 A}$	Resistance $R = \dfrac{\rho l}{a}$

NOTES

5. Reluctivity $= \frac{1}{\mu_r \mu_0}$	Resistivity $= \rho$
6. Permeance $P_m = \frac{1}{R_m}$	Conductance $G = \frac{1}{R}$
7. Magnetic Field Intensity $H = \frac{F_m}{l}$	Electric Field Intensity $E = \frac{V}{d}$
8. Flux Density $B = \frac{\phi}{A}$	Current Density $J = \frac{I}{a}$
9. The reluctance of a magnetic circuit is not constant and it depends upon flux density in the material.	The resistance of an electric circuit is constant at constant temperature.
10. There is no magnetic insulator.	There are many electric insulators.

Check Your Progress

15. Define the magnetic flux as stated by Michael Faraday and Lenz.
16. What does the Coulomb's magnetic law state?
17. State about the uniformly magnetised sphere in a magnetic field.
18. What is magnetism?
19. Define the terms paramagnetic, diamagnetic and ferromagnetic materials.

1.13 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Electric field intensity is defined as the strength of electric field at any point. It is equal to force per unit charge as experienced by test charge kept at that point.

   Therefore, it is expressed as,

   $$\vec{E} = \frac{\vec{F}}{Q}$$

2. Electric flux density is an imaginary field lines that do not exist unlike magnetic field lines. Electric flux density do not exist practically and generally considered for theoretical reasoning only.

   Electric flux density is related to electric field by the following reason,

   $$\vec{D} = \epsilon_0 \vec{E}$$

   Electric flux density $\vec{D}$ is independent of the medium and may also be defined in terms of electric flux ψ as,

   $$\psi = \int \vec{D}.\vec{ds}$$

   All the electric field expressions can be substituted in the electric flux density expressions. Therefore, electric flux density due to a long conductor of charges is given as,

   $$\vec{D} = \epsilon_0 \vec{E}$$

NOTES

Electric flux density due to a ring of charges is given by,

$$\vec{D} = \frac{aQ}{4\pi(R^2 + a^2)^{3/2}}$$

Electric flux density due to a circularly charged disc is given by,

$$\vec{D} = \frac{\rho_S}{2}\left[1 - \frac{a}{\sqrt{a^2 + R^2}}\right]$$

3. Gauss' law is a powerful tool for the calculation of electric fields. The applications of Gauss law include determination of electric field due to a point charge, sheet of charge, line charge on surface of conductor and sphere of charges. Gauss law states that total flux through a closed surface is equal to the charge enclosed by that surface.

   Mathematically, it is given as,

   Electric flux, $\psi = Q$ (Charge enclosed).
4. A function, $f(x)$, is 'Lipschitz' on a set if and only if there exist a positive number C such that for any x, y in that set, $|f(x)–f(y)|< C|x–y|$.
5. Laplace's equation in cylindrical coordinate system takes the form,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

6. Electric flux is generally defined as the number of lines of force passing through a unit area held normal to the direction of the lines of force. If the electric flux is greater, the electric field is greater and vice versa.
7. The Green's function is precisely defined as the impulse or signal response of an inhomogeneous or nonuniform linear differential operator which is characteristically defined on a domain including the certain specific initial conditions or the boundary conditions. The term signal or impulse is uniquely used in context with the signal processing defining the input signal or impulse response or precisely the Impulse Response Function (IRF).

   In mathematics, the notion Green's functions are precisely named after the British mathematician George Green, who initially or originally developed and established this concept in the year 1820s. In the contemporary analysis and evaluation of linear partial differential equations, the Green's functions are analysed principally on the basis of fundamental solutions.
8. The Green's function implies or indicates that if L is considered as the linear differential operator, then:
   - Essentially, the Green's function G is precisely defined as the solution of the equation of the form $LG = \delta$, where δ is referred as the Dirac's delta function.
   - Fundamentally, the solution of the Initial Value Problem (IVP) $Ly = f$ is referred as the convolution (G N f), where G is referred as the Green's function.

NOTES

The principle of superposition is used for a given linear Ordinary Differential Equation (ODE) to solve considering that,

L (Solution) = Source

L (Green) = δ_s, for each s

Because the source is a sum of delta functions, and the solution is a sum of Green's functions through the linearity of L.

9. The term **dielectric sphere** in a uniform electric field can be uniquely defined with the help of the condition that define a conducting sphere in an electric field. At far distances from the sphere, the field is considered uniform and is equal or equivalent to $\vec{E} = E_0\hat{k}$ which corresponds to a potential of $\varphi = -E_0 r \cos 6$.
10. Magnetic flux lines are imaginary lines that flow from the north to south poles.
11. Steady electric currents and current density can be defined based on the electric currents that are produced either by conduction or convection. Current is defined as the rate of movement of charge across a plane in a given time.

    Current density is defined as the amount of current flowing through a given area of a material. It is a vector component with magnitude equivalent to the electric current per cross sectional area.
12. The flow of charges through convection constitutes convection current. Beam of electrons inside Cathode Ray Tube (CRT) or in vacuum tubes are due to convection currents. Convection currents neither they obey Ohm's law nor they involve any conductor for the flow of current.
13. Ohm's law states that electric current is proportional to voltage and inversely proportional to resistance.

    $I = V/R$

    The term *Ohm's law* is also used to refer to various generalizations of the law originally formulated by Ohm. The current density and the electric field are related as: $\vec{J} = \sigma\vec{E}$
14. Kirchhoff Law relates the potential in a closed circuit or current in the nodes of a circuit. Hence Kirchhoff's Voltage Law, states that the sum of voltage drop and voltage rises in a closed loop of an electric circuit is zero.
15. In 1831, Michael Faraday reported on a series of ground-breaking experiments which showed that,
    (*i*) Whenever there is any change in the magnetic flux over the surface of a closed circuit, there is an ElectroMotive Force (EMF) induced in the same.
    (*ii*) The induced EMF in the closed circuit is directly proportional to the time rate of change of magnetic flux over the surface of the circuit.

Lenz noticed that this induced EMF opposes the very cause (rate of change of magnetic flux) of its creation. Thus, if ϕ is assumed to be the magnetic

NOTES

flux linked with a closed circuit, the induced EMF acting in the circuit must be

$$\varepsilon = -\frac{\partial \phi}{\partial t}$$

Also we know that,

$$\phi = \iint_S \vec{B}.d\vec{S}$$

Where $\vec{B}$ is the magnetic induction vector associated with the changing magnetic flux.

16. Coulomb's magnetic law states that the force F between two magnetic poles P_1 and P_2 is directly proportional to the product of pole strengths and inversely proportional to square of their disance apart d.

$$F \propto \frac{P_1 P_2}{d^2}$$

The constant of proportionality depends on the relative premeability of the medium through which the lines of magnetic field passes.

$$F = \frac{\mu_o \mu_r}{4\pi} \frac{P_1 P_2}{d^2} \quad \text{N}$$

F – Force between the poles in N (Newton)

P_1, P_2 – Pole strength in A/m

d – Distance between poles in m

μ_o – Permeability of free spaces ($4\pi \times 10^{-7}$ H/m)

μ_r – Relative permeability of the medium

$\mu_r = 1$ for air

From the above equation for force, an unit magnetic pole is defined as one which when situated 1 m distance in vacuum from an equal pole experiences a force of $\mu_o / 4\pi$ Newton.

17. To explain the magnetised sphere in a magnetic field, consider that there is a uniformly magnetised sphere having the radius $r = a$. The sphere holds a uniform permanent magnetisation which is denoted as $\mathbf{M} = M_0 \mathbf{e_z}$, which is precisely in the direction '**z**' and is typically surrounded or enveloped by vacuum. This problem can be solved using the scalar magnetic potential, denoted as Φ_m.

The magnetic scalar potential satisfies the Laplace equation of the form:

$$\nabla^2 \phi_m = e_m = 0$$

Fundamentally, this is due to the reason that there is precisely a zero net current in the sphere and also the magnetic volume charge density in a uniformly magnetized magnetic medium or vacuum is zero.

18. Magnetism is a phenomenon by which a material exerts either a positive or a repulsive force on another material. The source of 'Magnetism' of any material can be traced back to its electrical behaviour, which in turn is dictated by the structure of its constituents, i.e., atoms and molecules.

NOTES

19. Paramagnetic materials are those which are weakly attracted in an external magnetic field. When exposed to external magnetic field, they acquire feeble magnetisation in the same direction as the external field.

Diamagnetic materials are the materials which are weaky repelled by an external magnetic field as opposed to paramagnets. When placed in an external magnetic field they acquire a feeble magnetisation in the direction opposite to that of external field.

Both paramagnetic and diamagnetic materials are considered as non-magnetic materials as they exhibit magnetism only in the presence of an external magnetic field.

There are certain materials which are 'Magnetic' even in the absence of an external field. They possess permanent magnetic moments due to unpaired dipoles formed as a result of unpaired energy levels. Further, these materials are strongly magnetised even in the presence of a weak magnetic field. Such materials are called ferromagnetic materials.

1.14 SUMMARY

- The term static means a situation where the field does not vary with time. Static electric field also referred as electrostatics is created by the fixed charges in space.
- Determination of the electrostatic field components, such as electric field, electric force, and electric flux density are explained by two important laws namely, Coulomb's law and Gauss law.
- Coulomb's law provides the relation between forces experienced by the charges when they are separated by a distance. This theory was first proposed by Coulomb in 1785. This law states that,

  Force, F exerted between two point charges Q_1 and Q_2

  ⇒ Directly proportional to the product of the two charges and

  ⇒ Inversely proportional to the square of the distance between the two charges.

  ⇒ The direction of the force will be in the same direction along the line joining the two charges.

  Mathematically, Coulomb's law may be expressed as,

  $$\vec{F} \propto \frac{Q_1 Q_2}{R^2} \overrightarrow{a_R}$$

- Electric field intensity is defined as the strength of electric field at any point. It is equal to force per unit charge as experienced by test charge kept at that point. Therefore, it is expressed as,

  $$\vec{E} = \frac{\vec{F}}{Q}$$

NOTES

- The presence of charge Q ensures the existence of electric field $\vec{E}$. The charges may be distributed on a line conductor, on a surface or inside a volume. Hence, based on the charge distribution,

Along a line, charge, $Q = \int_L \rho_L \,.\, \overrightarrow{dl}$

On a surface, charge, $Q = \int_S \rho_S . \overrightarrow{ds}$

Inside a volume, charge, $Q = \int_V \rho_V . dv$

Where, $\rho_L =$ line charge density (C/m)

$\rho_S =$ surface charge density (C/m^2)

$\rho_V =$ volume charge density (C/m^3)

- The electric field along the line is given as,

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^2} \int_L \rho_L \,.\, \overrightarrow{dl} \,.\, \overrightarrow{a_r}$$

- Electric field intensity due to a small elemental length '*dl*' is given as,

$$\overrightarrow{dE} = \frac{1}{4\pi\epsilon_0 r^2} \rho_L \,.\, \overrightarrow{dl} . \overrightarrow{a_r}$$

- **Electric flux density** is an **imaginary field lines** that do not exist unlike magnetic field lines. Electric flux density do not exist practically and generally considered for theoretical reasoning only. Electric flux density is related to electric field by the following reason,

$\vec{D} = \epsilon_0 \vec{E}$

- Electric flux density $\vec{D}$ is independent of the medium and may also be defined in terms of electric flux ψ as,

$\psi = \int \vec{D} . \overrightarrow{ds}$

- Electric flux density due to a ring of charges is given by,

$$\vec{D} = \frac{aQ}{4\pi(R^2 + a^2)^{3/2}}$$

- Electric flux density due to a circularly charged disc is given by,

$$\vec{D} = \frac{\rho_S}{2}\left[1 - \frac{a}{\sqrt{a^2 + R^2}}\right]$$

- Gauss' law is a powerful tool for the calculation of electric fields. The applications of Gauss law includes determination of electric field due to a point charge, sheet of charge, line charge on surface of conductor and sphere of charges.
Gauss law states that total flux through a closed surface is equal to the charge enclosed by that surface. Mathematically, it is given as,
Electric flux, $\psi = Q$ (Charge enclosed).

- A mathematically closed surface is called as a Gaussian surface. These surfaces are assumed to have a uniform symmetric charge distribution which are ideal for determining the electric field vector, $\vec{E}$ by applying Gauss law. Also, the electric flux density vector, $\vec{D}$ is assumed to act tangentially or normally on the Gaussian surface. Therefore, accordingly, when $\vec{D}$ is normal, then

  $\vec{D}.\overrightarrow{dS} = DdS$

  And when $\vec{D}$ is acting tangential,

  $\vec{D}.\overrightarrow{dS} = 0$

- Electric field, $\vec{E}$ can be obtained by the following three ways.

  (1) By using Coulomb's Law

  $$-\vec{E} = \vec{F}/Q$$

  (2) By using Gauss's Law

  $$-\vec{E} = \frac{\vec{D}}{\epsilon_0}$$

  (3) By a Scalar Potential Function, V

  It is always simpler to determine the electric field, $\vec{E}$ by using the vector fields $\vec{F}$ and $\vec{D}$. Hence it is imperative to determine V.

  The scalar potential V is defined as the amount of work done in moving a charge Q. Hence V is expressed as,

  $$V = W/Q$$

- Potential difference between two points A and B is negative if the potential difference between B and A is also negative. In other words, creating a loop between A and B must satisfy Kirchhoff's voltage law and hence,

  $V_{AB} = -V_{BA}$

- Similar to the poles of the magnet, when equal and opposite electric charges are separated by a short distance, they form an electric dipole
- When two equal and opposite charges are separated by a distance 'd' an electric dipole moment is formed equivalent to,

  $m = Qd$

- A function, $f(x)$, is 'Lipschitz' on a set if and only if there exist a positive number C such that for any x, y in that set, $|f(x)-f(y)| < C|x-y|$.
- If f(x) is Lipschitz on a set then it is continuous at every point of that set. The mean value theorem can be used to show that if a function is differentiable at every point of a set, then it is Lipschitz on the set while 'Continuous' and 'Differentiable' are defined at points.
- Laplace's equation in Cartesian coordinate system takes the form,

  $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

NOTES

- Laplace's equation in cylindrical coordinate system takes the form,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- Laplace's equation in spherical polar coordinate system can be written as,

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

- Electric flux is generally defined as the number of lines of force passing through a unit area held normal to the direction of the lines of force. If the electric flux is greater, the electric field is greater and vice versa.
- If ϕ is the electric flux corresponding to the electric field $\vec{E}$, $\phi = \iint_S \vec{E}.d\vec{s}$, where $d\vec{s}$ is an elementary portion of the surface *S*.
- Potentials are always the consequences of various interactions. Electrostatic potential ϕ is itself the result of the interaction between a charge and the electric field $\vec{E}$. We can define $\phi(\vec{r})$ as the work done in bringing a unit positive charge from infinity to the point $\vec{r}$.
- In the field of electrostatics, the Green's function is precisely defined as the impulse or signal response of an inhomogeneous or nonuniform linear differential operator which is characteristically defined on a domain including the certain specific initial conditions or the boundary conditions.
- The term signal or impulse is uniquely used in context with the signal processing defining the input signal or impulse response or precisely the Impulse Response Function (IRF).
- In mathematics, the notion Green's functions are precisely named after the British mathematician George Green, who initially or originally developed and established this concept in the year 1820s.
- Essentially, the Green's function *G* is precisely defined as the solution of the equation of the form L*G* = δ, where δ is referred as the Dirac's delta function.
- Fundamentally, the solution of the Initial Value Problem (IVP) L$y = f$ is referred as the convolution (G N f), where *G* is referred as the Green's function.
- Characteristically, the Green's function, $G(x,s)$, precisely of a linear differential operator L = L(x) distinctively acting on distributions over a subset of the Euclidean space $\mathbb{R}^n$, at a specific point *s*, is any solution of,

  L $G(x, s) = \delta(s - x)$

  Where δ is referred as the Dirac delta function.
- This specific property of a Green's function can be manipulated or exploited for solving the differential equations of the form,

  L $u(x) = f(x)$

- In physics, the Green's function is generally defined with the opposite sign, i.e.,

  $$\mathbf{L}\, G(x, s) = \delta (x - s)$$

- If the operator is translation invariant, i.e., when **L** holds constant coefficients with respect to x, then the Green's function is typically considered as a convolution kernel, that is,

  $$G(x, s) = G(x - s)$$

- In this instance, the Green's function is considered equivalent with the impulse or signal response of linear time-invariant system theory.
- The term **dielectric sphere** in a uniform electric field can be uniquely defined with the help of the condition that define a conducting sphere in an electric field. At far distances from the sphere, the field is considered uniform and is equal or equivalent to $\vec{E} = E_0\hat{k}$ which corresponds to a potential of $\varphi = -E_0 r \cos 6$.
- The potential itself is continuous at $r = R$,

  $$-E_0 R + \frac{B_1}{R^2} = A_2 R$$

- Magnetic flux lines are imaginary lines that flow from the north to south poles. Magnetic flux lies constitutes the magnetic field. Magnetic field is denoted by $\vec{H}$. The unit of magnetic flux is Weber and Denoted as ϕ.

  The unit Weber is named after the German Physicist Wilhelm Eduard Weber and the symbol used in *Wb*.
- Magnetic flux density is defined as the magnetic flux lines passing through a unit surface area. It is denoted as $\vec{B}$ and the unit is weber/metre^2 or Tesla.

  Magnetic flux density, $\vec{B}$ is given as,

  $$B = \phi / A$$

- The electric currents are produced either by conduction or convection. Current is defined as the rate of movement of charge across a plane in a given time. They are expressed as,

  $$I = \frac{dQ}{dt}$$

- Current density is defined as the amount of current flowing through a given area of a material. It is a vector component with magnitude equivalent to the electric current per cross sectional area. Current I is related to current density $\vec{J}$ as,

  $$I = \int \vec{J}.\overrightarrow{ds}$$

NOTES

- The flow of charges through convection constitutes convection current. Beam of electrons inside Cathode Ray Tube (CRT) or in vacuum tubes are due to convection currents.
- Ohm's law states that electric current is proportional to voltage and inversely proportional to resistance.
  $I = V/R$
  The term *Ohm's law* is also used to refer to various generalizations of the law originally formulated by Ohm. The current density and the electric field are related as: $\vec{J} = \sigma\vec{E}$
- Continuity equation on the basis of law of conservation of charge states that in a given volume, the total current coming out of the volume is equal to the rate of decrease of charge inside the volume. It is expressed as,
  $$I_{out} = -\frac{dQ}{dt}$$
- In general, Kirchhoff Law relates the potential in a closed circuit or current in the nodes of a circuit. Hence Kirchhoff's Voltage Law, states that the sum of voltage drop and voltage rises in a closed loop of an electric circuit is zero and is given as,
  $$\sum_{i=1}^{N} V_i = 0$$
  Where i indicates the number of circuit elements in the loop. V_i is positive for voltage rise across an element or negative if voltage drops across an element.
- Similarly, at any node of an electric circuit, Kirchhoff's current law states that,
  $$\sum_{i=1}^{N} I_i = 0$$
  Where i indicates the number of circuit branches connected to a node. I_i is positive, if the current enters the node and negative if the current leaves the node.
- For the force to be developed, there must be at least two fields with a phase difference. Hence force due to magnetic field can be experienced by either of the following,

  (a) A unit charge, Q travelling in a magnetic field experiences a force.

  (b) A current element placed in a magnetic field experiences a force.

  (c) Two current carrying conductor when placed close to each other experience force between them.
- Ampere's circuit law states that the closed line integral of magnetic field intensity is equal to the current circulating in the closed path.
  Mathematically,
  $$\oint \vec{H} \cdot \vec{dl} = I$$

NOTES

- In 1831, Michael Faraday reported on a series of ground-breaking experiments which showed that,
    (*i*) Whenever there is any change in the magnetic flux over the surface of a closed circuit, there is an ElectroMotive Force (EMF) induced in the same.
    (*ii*) The induced EMF in the closed circuit is directly proportional to the time rate of change of magnetic flux over the surface of the circuit.
- Lenz noticed that this induced EMF opposes the very cause (rate of change of magnetic flux) of its creation. Thus, if ϕ is assumed to be the magnetic flux linked with a closed circuit, the induced EMF acting in the circuit must be

$$\varepsilon = -\frac{\partial \phi}{\partial t}$$

- The source of magnetic flux is either a permanent magnet or a current carrying coil. The lines of the magnetic flux always form a closed path. The closed path followed by the lines of magnetic flux is called a magnetic circuit. Thus, a magnetic circuit provides a closed path for the magnetic flux and is similar to an electric circuit which provides a closed path for the flow of electric current.
- First Law of Magnetics: Like poles of magnets repel each other whereas unlike poles attract each other.
  Second/Coulomb's Law: Second law of magnetics known as Coulomb's law accounts for the force exerted between two magnetic poles.
- MMF or Magneto Motive Force is the source of producing flux in a magnetic circuit. For a current I flowing through a coil of T turns, the magnetic flux is obtained as a product of I and T. Its variable symbol is F_m and its unit is Ampere-Turn (AT).

$$F_m = IT \quad \text{AT} \qquad (1.67)$$

- The Magnetising Force MF, otherwise called as Magnetic Field Intensity (MFI), is defined as the magneto motive force per unit length of the magnetic flux path.
- Magnetising force is a measure of the ability of a magnetised body to produce magnetic induction in other magnetic substances. H is the variable symbol to denote magnetising force and Ampere-Turns/metre (AT/m) is the unit.

$$H = \frac{F_m}{l} = \frac{IT}{l} \quad \text{AT/m}$$

- To explain the magnetised sphere in a magnetic field, consider that there is a uniformly magnetised sphere having the radius $r = a$.
- The sphere holds a uniform permanent magnetisation which is denoted as $\mathbf{M} = M_0\,\mathbf{e_z}$, which is precisely in the direction '**z**' and is typically surrounded or enveloped by vacuum. This problem can be solved using the scalar magnetic potential, denoted as Φ_m.
- The magnetic scalar potential satisfies the Laplace equation of the form:

$$\nabla^2 \phi_m = e_m = 0$$

NOTES

- Magnetism is a phenomenon by which a material exerts either a positive or a repulsive force on another material. The source of 'Magnetism' of any material can be traced back to its electrical behaviour, which in turn is dictated by the structure of its constituents, i.e., atoms and molecules.
- Magnetic susceptibility of a material is a measure of how weakly or strongly it is affected by the presence of an external magnetic field.
- Magnetic permeability is a measure of the degree of penetration of magnetic field in a substance. It is defined as the ratio of total magnetic flux density and external magnetic field intensity.
- Paramagnetic materials are those which are weakly attracted in an external magnetic field. When exposed to external magnetic field, they acquire feeble magnetisation in the same direction as the external field.
- Diamagnets are the materials which are weaky repelled by an external magnetic field as opposed to paramagnets. When placed in an external magnetic field they acquire a feeble magnetisation in the direction opposite to that of external field.
- A diamagnet is a substance that exhibits negative magnetism under the influence of an external field. The origin of diamagnetism can be explained using the simple model of orbital motion of electron around a nucleus, which can be thought of as a small current and hence a tiny magnetic dipole with net magnetic moment.
- Both paramagnetic and diamagnetic materials are considered as non-magnetic materials as they exhibit magnetism only in the presence of an external magnetic field.
- There are certain materials which are 'Magnetic' even in the absence of an external field. They possess permanent magnetic moments due to unpaired dipoles formed as a result of unpaired energy levels. Further, these materials are strongly magnetised even in the presence of a weak magnetic field. Such materials are called ferromagnetic materials.
- Closed path for magnetic flux f is known as a magnetic circuit.
- Closed path for electric current I is known as electric circuit.
- In paramagnetic and diamagnetic materials, which require an external magnetic field to exhibit 'Magnetism', the alignment of atomic dipoles is maintained by that field.
- Being non-linear in nature, unlike paramagnets and diamagnets, ferromagnetic materials do not need external fields to sustain magnetisation.
- In the paramagnetic materials, the 'Alignment' is 'Frozen In'.
- The ferromagnetic phenomenon involves the magnetic dipoles associated with the spins of unpaired electrons, like paramagnetism.
- The ferromagnets have one additional feature which is the interaction between neighbouring dipoles.
- In a ferromagnet, each dipole likes to point in the same direction as its neighbour. This correlation is so strong that nearly all unpaired electron spins are aligned in same direction.

1.15 KEY TERMS

NOTES

- **Coulomb's law:** Coulomb's law provides the relation between forces experienced by the charges when they are separated by a distance. This theory was first proposed by Coulomb in 1785.
- **Electric field intensity:** Electric field intensity is defined as the strength of electric field at any point. It is equal to force per unit charge as experienced by test charge kept at that point.
- **Gaussian surface:** A mathematically closed surface is called as a Gaussian surface. These surfaces are assumed to have a uniform symmetric charge distribution which are ideal for determining the electric field vector, by applying Gauss law.
- **Electric flux:** Electric flux is generally defined as the number of lines of force passing through a unit area held normal to the direction of the lines of force. If the electric flux is greater, the electric field is greater and vice versa.
- **Green's function:** A Green's function is defined as the impulse response of an inhomogeneous linear differential operator typically defined on a domain with particular initial conditions or boundary conditions. Green's functions are named after the British mathematician George Green, who first developed this concept in the year 1820s.
- **Current:** Current is defined as the rate of movement of charge across a plane in a given time.
- **Current density:** Current density is defined as the amount of current flowing through a given area of a material. It is a vector component with magnitude equivalent to the electric current per cross sectional area.
- **Conduction current:** Conduction current requires conductor to flow. The flow of charges from one end of conductor to the other is facilitated by application of electric field between the conductors.
- **First law of magnetics:** First law of magnetics states that like poles of magnets repel each other whereas unlike poles attract each other.
- **Second/Coulomb's law of magnetics:** Second/Coulomb's law of magnetics known as Coulomb's law accounts for the force exerted between two magnetic poles.

1.16 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the term electrostatics.
2. State Coulomb's law.
3. Define electric flux density.

NOTES

4. What does Gauss law state?
5. What is Gaussian surface?
6. State about Green's function.
7. Define steady electric currents and current density.
8. What is current density?
9. Differentiate between convection currents and conduction currents.
10. What is Kirchhoff Law?
11. State the first law of magnetics.
12. Define magnetism.
13. Differentiate between paramagnetic, diamagnetic and ferromagnetic materials.

Long-Answer Questions

1. Briefly discuss the concept of electrostatics and magnetostatics giving appropriate examples.
2. Explain uniqueness theorem with the help of examples.
3. Discuss how the solution of Laplace and Poisson equation in rectangular, Cartesian and spherical polar coordinates are obtained.
4. Explain the methods of electric images giving examples.
5. Discuss the concept of Greens function for potential problems giving appropriate examples.
6. Briefly explain about the dielectric sphere in uniform electric field giving examples.
7. Discuss the laws of magnetostatics.
8. Explain the basic theory and types of magnetisations giving appropriate examples.
9. Analyse the characteristic features of uniformly magnetised sphere in magnetic field giving examples.
10. Discuss the significant features of magnetism phenomenon with the help of examples.
11. Discuss the properties of paramagnetic, diamagnetic and ferromagnetic materials giving relevant examples.
12. Explain the classical theories of paramagnetic, diamagnetic and ferromagnetic magnetisation.
13. Explain magnetic circuits and differentiate it from electric circuits.

1.17 FURTHER READING

Prakash, Satya. 2007. *Electromagnetic Theory and Electrodynamics*: *Including Electrostatics and Magnetostatics*. Meerut: Kedar Nath Ram Nath.

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UNIT 2 ELECTROMAGNETICS

NOTES

Structure

2.0 INTRODUCTION

Electromagnetism is a branch of physics involving the study of the electromagnetic force, a type of physical interaction that occurs between electrically charged particles. The electromagnetic force is carried by electromagnetic fields composed of electric fields and magnetic fields, and it is responsible for electromagnetic radiation, such as light. It is one of the four fundamental interactions (commonly called forces) in nature, together with the strong interaction, the weak interaction, and gravitation. At high energy, the weak force and electromagnetic force are unified as a single electroweak force.

Electromagnetic phenomena are defined in terms of the electromagnetic force, sometimes called the Lorentz force, which includes both electricity and magnetism as different manifestations of the similar phenomenon. The electromagnetic attraction between atomic nuclei and their orbital electrons holds atoms together. Electromagnetic forces are responsible for the chemical bonds between atoms which create molecules, and intermolecular forces.

The theoretical implications of electromagnetism, particularly the establishment of the speed of light based on properties of the 'Medium' of propagation (permeability and permittivity), led to the development of special relativity by Albert Einstein in 1905.

In this unit, you will study about the time varying fields, Maxwell's electromagnetic field equations in stationary and moving media, electromagnetic scalar wave equations and their solution, Hertz vector, plane wave propagation in conducting and ionised media, radiation pressure and momentum, reflection, refraction, total internal reflection, polarisation, scattering (Rayleigh and Thomson) and dispersion of plane E. M. waves, and elements of wave guides.

NOTES

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss about the time varying fields
- Understand Maxwell's electromagnetic field equations in stationary and moving media
- Explain the electromagnetic scalar wave equations and their solution
- State the concept of Hertz vector
- Describe plane wave propagation in conducting and ionised media
- Elaborate on radiation pressure and momentum
- Understand reflection and refraction
- Discuss the significance of polarisation
- Explain the importance of total internal reflection
- Know about the scattering (Rayleigh and Thomson) and dispersion of plane E. M. waves
- Define the elements of wave guides

2.2 TIME VARYING FIELDS

Static electric fields are **constant fields**, which do not change in intensity or direction over time, in contrast to low and high frequency alternating fields. The strength of a static electric field is expressed in Volts per meter (V/m).

Time varying is a system in which certain quantities governing the system's behaviour change with time, so that the system will respond differently to the same input at different times.

When an electrically conducting structure is exposed to a time varying magnetic field, then **an electrical potential difference is induced across the structure**. The generation of electric potential by a time varying magnetic flux is very well described by Faraday's Law. This is a form of electromagnetic induction. According to Faradays law, **when magnetic flux changes in the region surrounded by conductor**, it produces electric field (induced Electro Magnetic Force EMF) in conductor.

It is known that a time varying electric field is produced **by a time varying magnetic field** and a **time varying magnetic field** is produced **by a time varying electric field**. The first concept was experimentally introduced by Michael Faraday and the second was theoretically introduced by James Clerk Maxwell difference between static fields and time varying fields.

Electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are produced due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time varying fields or waves are usually due to **accelerated charges** or **time varying current**.

NOTES

A time varying electric field is typically produced through a time varying magnetic field and the concept was experimentally introduced by Michael Faraday while a time varying magnetic field is specifically produced by means of a time varying electric field and the concept was theoretically introduced by James Clerk Maxwell.

Maxwell's equations are defined as a set of coupled partial differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism. The term "Maxwell's Equations" is often also used for equivalent alternative formulations. Versions of Maxwell's equations based on the electric and magnetic scalar potentials are preferred for explicitly solving the equations as a boundary value problem, analytical mechanics, or for use in quantum mechanics.

Faraday's Law

The Maxwell–Faraday version of Faraday's law of induction describes how a time varying magnetic field creates or induces an electric field. In integral form, it states that the work per unit charge required to move a charge around a closed loop equals the rate of change of the magnetic flux through the enclosed surface.

The electromagnetic induction is the operating principle behind many electric generators, for example, a rotating bar magnet creates a changing magnetic field, which in turn generates an electric field in a nearby wire.

Maxwell's equations are a set of the following four complex equations that describe electromagnetics. These equations describe how electric and magnetic fields propagate, interact, and how they are influenced by objects.

1. $\nabla \cdot \mathbf{D} = \rho_V$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

Maxwell was the first to determine the speed of propagation of Electro Magnetic (EM) waves and found that it was the same as the speed of light, hence he concluded that EM waves and visible light can be studied on the same basis. Maxwell's equations are critical in understanding Antennas and Electromagnetics.

2.3 MAXWELL'S ELECTROMAGNETIC FIELD EQUATIONS IN STATIONARY AND MOVING MEDIA

Maxwell's equations state the fundamentals of electricity and magnetism. The working relationships in the field of electricity and magnetism can be derived using these equations. As a consequence of their brief statement, they symbolize a high level of mathematical sophistication, and hence are typically defined as unifying equations for studying of electrical and magnetic phenomena.

Principally, the Maxwell's equations are a set of partial differential equations that, together with the 'Lorentz Force Law', form the foundation of classical electromagnetism, classical optics, and electric circuits. The equations provide a mathematical model for electric, optical, and radio technologies, such as power generation, electric motors, wireless communication, lenses, radar, etc. Maxwell's equations specifically describe how electric and magnetic fields are generated by charges, currents, and changes of the fields. One significant consequence of the Maxwell's equations is that they demonstrate how fluctuating electric and magnetic fields propagate at the speed of light. Acknowledged as electromagnetic radiation, the Maxwell's waves may occur at various wavelengths to produce a spectrum from radio waves to γ-rays. The equations are named after the physicist and mathematician James Clerk Maxwell, who between 1861 and 1862 published an early or initial form of the equations that included the Lorentz force law. Maxwell also was the first to use the equations to recommend that light is an electromagnetic phenomenon.

The Maxwell equations have two major variations/variants. Though the microscopic Maxwell equations have universal applicability, but these are cumbersome for common calculations. They relate the electric and magnetic fields to total charge and total current, including the complicated charges and currents in materials at the atomic scale.

In addition, the term 'Maxwell's Equations' is also frequently used for equivalent alternative formulations. Versions of Maxwell's equations that are based on the electric and magnetic potentials are ideal for explicitly solving the equations, such as a boundary value problem, analytical mechanics, and in quantum mechanics. In the spacetime formulations, i.e., on spacetime rather than space and time separately, the Maxwell's equations are commonly used in high energy and gravitational physics because they make the compatibility of the equations with special and general dependence evident. Essentially, Einstein developed special and general relativity/ dependence to accommodate the invariant speed of light that drops out of the Maxwell equations with the principle that only relative movement has physical consequences. Principally, the Maxwell's equations are not exact, but a classical limit of the fundamental theory of quantum electrodynamics. Maxwell's four equations describe the electric and magnetic fields arising from distributions of electric charges and currents, and how those fields change in time. The second Maxwell equation is the analogous one for the magnetic field, which has no sources or sinks, i.e., no magnetic monopoles, the field lines just flow around in closed curves.

Maxwell's Equation for Static Fields and Magnetic Dipole

Summarizing all the Maxwell's equation from electrostatics and magnetostatics, we get the following four Maxwell's equation for static fields.

Sl. No	Differential form	Integral form
1	$\vec{\nabla}\cdot\vec{D} = \delta v$	$\oint_s \vec{D}\cdot\vec{ds} = \int_v \delta v \cdot dv$
2	$\vec{\nabla}\cdot\vec{E} = 0$	$\oint_L \vec{E}\cdot\vec{dl} = 0$
3	$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_s \vec{B}\cdot\vec{ds} = 0$
4	$\vec{\nabla}\cdot\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot\vec{dl} = \int_s \vec{J}\cdot\vec{ds}$

NOTES

Magnetic Dipole

The 'Magnetic Dipole Moment', is equal to the product of the current flowing through the loop and area of the loop with the moment acting normal to the loop (Refer Figure 2.1). Mathematically,

$$\vec{m} = I\,A\,\vec{a_n} \quad (2.1)$$

Where, I is the current in the loop with area $A(m^2)$

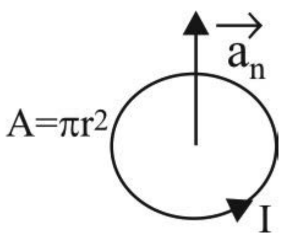

***Fig. 2.1** Magnetic Moment*

Magnetic dipole naturally exists on permanent magnets as 'North' and 'South' poles or in current carrying coils.

Unlike electrostatics, in magnetostatics monopole does not exist. When a magnet is broken into two pieces, North and South poles exists within the broken pieces (Refer Figure 2.2).

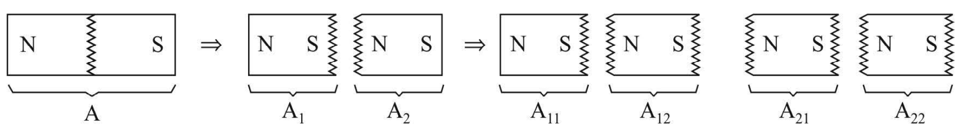

***Fig. 2.2** Magnetic Dipoles*

Recalling the $\vec{E}$ and V for electric dipoles, and comparing for magnetic dipoles, there exists an equivalence.

NOTES

Electrostatics	Magnetostatics
$V = Q\cos\dfrac{\theta}{4\pi\epsilon_o r^2}$ Rewritten as, $V = \dfrac{1}{4\pi r^2}\left(\dfrac{1}{\epsilon_o}\right)Q\cos\theta$ $\vec{E} = \dfrac{1}{4\pi r^3}\left(\dfrac{1}{\epsilon_o}\right)Q_d[2\cos\theta\overrightarrow{a_\delta} + \sin\theta\overrightarrow{a_\theta}]$	$\vec{A} = \frac{1}{4\pi r^2}(\mu_0)m\sin\theta\overrightarrow{a_\phi}$ $\vec{B} = \dfrac{1}{4\pi r^3}(\mu_0)m[2\cos\theta\overrightarrow{a_\delta} + \sin\theta\overrightarrow{a_\theta}]$

Magnetization

The equivalence of magnetization in electrostatics is polarisation, $\vec{P}$. Consider a single magnetic moment shown in Figure 2.3. When a magnetic material is non magnetised, i.e., when $\vec{B} = 0$, the magnetisation $\vec{M} = 0$. Also in a non-magnetised materials, the different dipole moments that exists within the atoms of the material are not polarized, or in other words, their unit vectors of the moment, $\overrightarrow{m}$ points in random direction as shown in Figure 2.3(a).

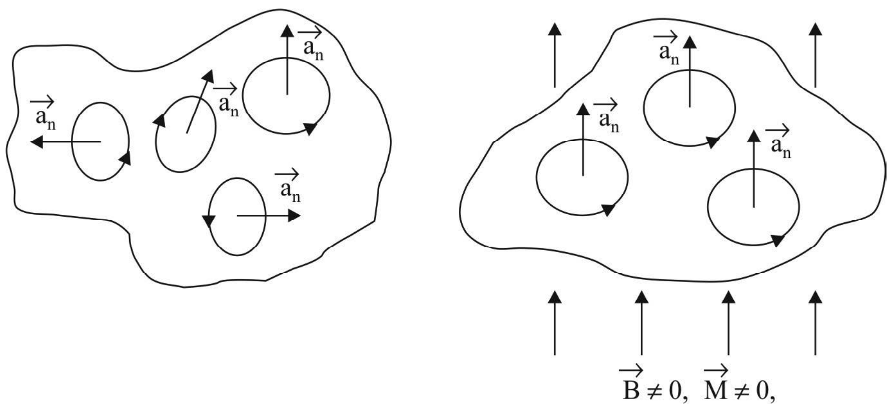

Fig. 2.3 (a) $\vec{B} = 0$ $\vec{M} = 0$ ***(b) Magnetization***

When a magnetic field is applied to the magnetic material, then the magnetic moments align in a particular direction. Hence, magnetisation is defined as the net magnetic dipole moment in a given volume. For a single magnetic moment,

$$Magnetisation, M = \frac{\vec{m}}{Volume}$$

For N magnetic moments,

$$\vec{M} = [\overrightarrow{m_1} + \overrightarrow{m_2} + \overrightarrow{m_3} + \cdots \ldots\ldots\ldots\ldots + \overrightarrow{m_N}]/volume$$

$$\vec{M} = \textstyle\sum_{i=1}^{N}\overrightarrow{m_i}/volume$$

Magnetic Susceptibility (x_m)

According to Maxwell's fourth equation, in free space, with magnetisation, $\vec{M} = 0$,

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad \Rightarrow \quad \vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{J} \qquad [\because \vec{B} = \mu_0 \vec{H}]$$

But for a magnet, with magnetisation, $\vec{M} \neq 0$,

$$\vec{B} = \mu_0[\vec{H} + \vec{M}]$$

$$\vec{B} = \mu_0 \vec{H}\left[1 + \frac{\vec{M}}{\vec{H}}\right]$$

But $\vec{B} = \mu\vec{H}$. Therefore,

$$\mu\vec{H} = \mu_0\vec{H}\left[1 + \frac{\vec{M}}{\vec{H}}\right]$$

$$\mu = \mu_0[1 + x_m]$$

$$\mu = \mu_0\mu_r$$

Therefore, $\mu_r = 1 + x_m$. Also

$$x_m = \frac{\vec{M}}{\vec{H}}$$

Hence, magnetic susceptibility is defined as the ratio of magnetisation to magnetic field intensity.

Therefore,

$$\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}$$

$$\mu_r = 1 + x_m = \frac{\mu}{\mu_0}$$

Maxwell's Equations in Time Varying Fields

Maxwell's equations in time varying fields are the final form of equations that interlinks the electric and magnetic fields. This section summarizes the interlinked electric and magnetic fields for time varying fields with modified form.

1. Maxwell's First Equation

We know from electrostatics,

$$\vec{\nabla}.\vec{D} = \rho_V$$

As, $\vec{D} = \epsilon\vec{E}$,

$$\vec{\nabla}.\epsilon\vec{E} = \rho_V$$

$$\vec{\nabla}.\vec{E} = \frac{\rho_V}{\epsilon}$$

NOTES

2. Maxwell's Second Equation

From electrostatics and from Faraday's law for electromagnetic fields,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{dt}$$

As, $\vec{B} = \mu\vec{H}$,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial(\mu\vec{H})}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\mu\frac{\partial\vec{H}}{dt}$$

3. Maxwell's Third Equation

From Gauss's law,

$$\vec{\nabla}.\vec{B} = 0$$

As, $\vec{B} = \mu\vec{H}$,

$$\vec{\nabla}.(\mu\vec{H}) = 0$$

$$\vec{\nabla}.\vec{H} = 0$$

4. Maxwell's Fourth Equation

Applying Ampere's law, the modified Maxwell's equation is given as,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial\vec{D}}{dt}$$

Also, $\vec{J} = \sigma\vec{E}$ and $\vec{D} = \epsilon\vec{E}$. Therefore,

$$\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \frac{\partial(\epsilon\vec{E})}{dt}$$

$$\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{dt}$$

Maxwell's Equation in Final Form

Table 2.1 Maxwell's Equation in Differential and Integral Form

Sl. No	Differential Form	Integral Form
1	$\vec{\nabla}.\vec{D} = \rho_v$ or $\vec{\nabla}.\vec{E} = \frac{\rho_V}{\epsilon}$	$\oint_S \vec{D}.\vec{ds} = \int_v \rho_v.dv$
2	$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{dt}$ or $\vec{\nabla} \times \vec{E} = -\mu\frac{\partial\vec{H}}{dt}$	$\oint_L \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{dS}$ or $\oint_L \vec{E}.\vec{dl} = -\mu\int_S \frac{\partial\vec{H}}{\partial t}.\vec{dS}$
3	$\vec{\nabla}.\vec{B} = 0$ or $\vec{\nabla}.\vec{H} = 0$	$\oint_S \vec{B}.\vec{dS} = 0$
4	$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial\vec{D}}{dt}$ or $\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{dt}$	$\oint_L \vec{H}.\vec{dl} = \int_S \left(\vec{J} + \frac{\partial\vec{D}}{dt}\right).\vec{dS}$

NOTES

Maxwell's Equation in Free Space

In free space, $\rho_V = 0$ and $\sigma = \vec{J} = 0$, therefore, the Maxwell's equation in free space are written in the Table 2.2 given below:

NOTES

Table 2.2 Maxwell's Equation in Free Space

Sl. No	Differential Form	Integral Form
1	$\vec{\nabla}.\vec{D} = 0$ or $\vec{\nabla}.\vec{E} = 0$	$\oint_S \vec{D}.\overrightarrow{ds} = 0$
2	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{dt}$	$\oint_L \vec{E}.\overrightarrow{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}.\overrightarrow{dS}$
3	$\vec{\nabla}.\vec{B} = 0$ or $\vec{\nabla}.\vec{H} = 0$	$\oint_S \vec{B}.\overrightarrow{dS} = 0$
4	$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{dt}$	$\oint_L \vec{H}.\overrightarrow{dl} = \int_S \frac{\partial \vec{D}}{dt}.\overrightarrow{dS}$

2.4 ELECTROMAGNETIC SCALAR WAVE EQUATIONS AND THEIR SOLUTION

The wave equation is a second-order linear partial differential equation for the description of waves—as they occur in classical physics, such as mechanical waves (e.g., water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics. Due to the fact that the second order wave equation describes the superposition of an incoming wave and an outgoing wave (i.e., rather a standing wave field) it is also called 'Two-Way Wave Equation', in contrast, the 'First Order One-Way Wave Equation' describes a single wave with predefined wave propagation direction and is much easier to solve due to the first order derivatives.

The (two-way) wave equation is a second order partial differential equation explaining waves. The **scalar wave equation** describe waves in scalars by **scalar functions** $u = u\ (x_1, x_2, ..., x_n; t)$ of a time variable t (a variable representing time) and one or more spatial variables $x_1, x_2, ..., x_n$ (variables representing a position in a space) while there are vector wave equations describing waves in vectors, such as waves for electrical field, magnetic field, and magnetic vector potential and elastic waves. By comparison with vector wave equations, the scalar wave equation can be seen as a special case of the vector wave equations; in the Cartesian coordinate system, the scalar wave equation is the equation to be satisfied by each component (for each coordinate axis, such as the x-component for the x-axis) of a vector wave without sources of waves in the considered domain (i.e., a space and time).

NOTES

The scalar wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right)$$

Where c is a fixed non-negative real coefficient.

The **electromagnetic wave equation** is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. It is a three-dimensional form of the wave equation. The homogeneous form of the equation, written in terms of either the electric field **E** or the magnetic field **B**, takes the form:

$$\left(v_{\mathrm{ph}}^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = \mathbf{0}$$

$$\left(v_{\mathrm{ph}}^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \mathbf{B} = \mathbf{0}$$

Where,

$$v_{\mathrm{ph}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

This is referred as the speed of light (i.e., phase velocity) in a medium with permeability ¼, and permittivity μ, and ∇^2 is the Laplace operator. In a vacuum, $v_{ph} = c_0 = 299792458$ m/s, a fundamental physical constant. The electromagnetic wave equation is derived from Maxwell's equations. Basically, **B** is termed as the *magnetic flux density* or *magnetic induction*.

2.4.1 Hertz Vector

Hertz vector is also known as **polarization potentials**, which are useful auxiliary fields that permit the calculation of the fundamental electromagnetic fields in many cases of practical importance. This provides a new light on the physical meaning of a Hertz potential.

The **Hertz vector potentials** are an alternative formulation of the electromagnetic potentials.

Magnetic vector potential, **A**, is the vector quantity in classical electromagnetism defined so that its curl is equal to the magnetic field.

Hertz vectors can be advantageous when solving for the electric and magnetic fields in certain scenarios, as they provide an alternative way to define the scalar potential ϕ and the vector potential **A** which are used to find the fields.

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Considering cases of electric and magnetic polarization separately for simplicity, each can be defined in terms of the scalar and vector potentials which then allows for the electric and magnetic fields to be found. For cases of just electric polarization the following relations are used.

$$\phi = -\nabla \cdot \mathbf{\Pi}_e$$

NOTES

$$\mathbf{A} = \mu\epsilon \frac{\partial \mathbf{\Pi}_e}{\partial t}$$

And for cases of solely magnetic polarization they are defined as:

$$\phi = 0$$
$$\mathbf{A} = \nabla \times \mathbf{\Pi}_m$$

Together with the electric potential Æ, the magnetic vector potential can be used to specify the electric field **E** as well. Therefore, many equations of electromagnetism can be written either in terms of the fields **E** and **B**, or equivalently in terms of the potentials ϕ and **A**. In more advanced theories, such as quantum mechanics, most equations use potentials rather than fields.

2.5 PLANE WAVE PROPAGATION IN CONDUCTING AND IONISED MEDIA

In physics, a plane wave is a special case of wave or field - a physical quantity whose value, at any moment, is constant over any plane that is perpendicular to a fixed direction in space. Principally, the electromagnetic wave equation is a second order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. It is a three dimensional form of the wave equation.

Electromagnetic waves are created by the vibration of an electric charge. This vibration creates a wave which has both an electric and a magnetic component. An electromagnetic wave transports its energy through a vacuum at a speed of 3.00×10^8 m/s. The propagation of an electromagnetic wave through a material medium occurs at a net speed which is less than 3.00×10^8 m/s. An electromagnetic wave consists of an electric field, typically defined in terms of the force per charge on a stationary charge, and a magnetic field, defined in terms of the force per charge on a moving charge.

As already discussed in the previous section, the mechanical waves travel through a medium, such as a string, water or air. Possibly the most significant prediction of Maxwell's equations is the existence of electromagnetic fields, i.e., combined electric and magnetic fields that propagate through space as electromagnetic waves. Because Maxwell's equations hold in free space, therefore the predicted electromagnetic waves, unlike mechanical waves, do not require a medium for their propagation.

Electromagnetic Waves in One Direction

An electromagnetic wave consists of an electric field, typically defined in terms of the force per charge on a stationary charge, and a magnetic field defined in terms of the force per charge on a moving charge.

The one-dimensional scalar equation is given below,

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{dt^2} = 0$$

NOTES

For a wave traveling in free space, $\mu = \mu_0$ and $\epsilon = \epsilon_0$. Therefore, generalizing equation for all three coordinates is given as,

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad (2.2)$$

The general solution of the above second order equation is given by,

$$\vec{E} = f_1(z - ut) + f_2(z + ut)$$

For wave traveling in z-direction,

$$\vec{E} = f_1(z - ut) + f_2(z + ut) \quad (2.3)$$

Some examples of the above function include $e^{j(z \pm ut)}, \sin(z \pm ut)$ ....

In the above general solution given in Equation (2.3), $f_1(z - ut)$ represents the wave traveling in positive z – direction and function $f_2(z + ut)$ represents wave traveling in the negative z – direction.

Wave Propagation in a Lossy Dielectric

Conduction based on the conductivity property of the material can be classified as lossy dielectric with $\sigma \neq 0$, lossless dielectric with $\sigma = 0$ and good conductors with $\sigma = \infty$. Considering a lossy, charge free medium, Maxwell's equation becomes,

$$\vec{\nabla}.\vec{E} = 0 \quad (2.4a)$$

$$\vec{\nabla}.\vec{H} = 0 \quad (2.4a)$$

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H} \quad (2.4c)$$

$$\vec{\nabla} \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} \quad (2.4d)$$

Taking a curl of Equation (2.4c),

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -j\omega\mu(\vec{\nabla} \times \vec{H})$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -j\omega\mu(\sigma + j\omega\epsilon)\vec{E} \quad (2.5)$$

Applying vector identity to LHS of Equation (2.5),

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$$

Equation (2.5) implies,

$$\vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = -j\omega\mu(\sigma + j\omega\epsilon)\vec{E} \quad (2.6)$$

From Equation (2.4a), $\vec{\nabla}.\vec{E} = 0$. Therefore Equation (2.6) becomes,

$$-\nabla^2\vec{E} = -j\omega\mu(\sigma + j\omega\epsilon)\vec{E}$$

$$\nabla^2\vec{E} + j\omega\mu(\sigma + j\omega\epsilon)\vec{E} = 0$$

$$\nabla^2\vec{E} + \mathrm{v}^2\vec{E} = 0 \quad (2.7)$$

Where,

$$\mathrm{v}^2 = j\omega\mu(\sigma + j\omega\epsilon) = -\omega^2\mu\epsilon + j\omega\mu\sigma \quad (2.8)$$

Similar to Equation (2.7), for magnetic field,

$$\nabla^2\vec{H} + \nu^2\vec{H} = 0 \quad (2.9)$$

Equation (2.8) implies,

$$\nu^2 = -\omega^2\mu\epsilon + j\omega\mu\sigma = \alpha + j\beta$$

NOTES

$$Re(\nu^2) = \alpha = -\omega^2\mu\epsilon \quad (2.10(a))$$

Where,

$$Im(\nu^2) = \beta = \omega\mu\sigma \quad (2.10(b))$$

In Equation (2.10) α is called as ***attenuation constant*** and β is called the ***propagation constant***.

$$|\nu^2| = \alpha^2 + \beta^2 = \sqrt{(-\omega^2\mu\epsilon)^2 + (\omega\mu\sigma)^2}$$

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\omega\epsilon)^2 + (\sigma)^2} \quad (2.11)$$

$$\nu^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$$

From Equation (2.10(a)),

$$Re(\nu^2) = \alpha^2 - \beta^2 = -\omega^2\mu\epsilon \quad (2.12)$$

Adding Equation (2.11) and Equation (2.12)

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon$$

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$-----------------$$

$$2\alpha^2 = -\omega^2\mu\epsilon + \omega\mu\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$\alpha^2 = -\frac{\omega^2\mu\epsilon}{2} + \frac{\omega\mu}{2}\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$= -\frac{\omega^2\mu\epsilon}{2} + \frac{\omega\mu}{2}\sqrt{(\omega\epsilon)^2\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]}$$

$$= -\frac{\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]}$$

$$\alpha^2 = \omega^2\left[\frac{\mu\epsilon}{2}\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - \frac{\mu\epsilon}{2}\right]$$

$$\alpha = \sqrt{\omega^2\left[\frac{\mu\epsilon}{2}\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - \frac{\mu\epsilon}{2}\right]}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]} \quad (2.13)$$

Subtracting Equations (2.11) and (2.12):

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$-\alpha^2 + \beta^2 = \omega^2\mu\epsilon$$

NOTES

$$2\beta^2 = \omega^2\mu\epsilon + \omega\mu\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$\beta^2 = \frac{\omega^2\mu\epsilon}{2} + \frac{\omega\mu}{2}\sqrt{(\omega\epsilon)^2 + (\sigma)^2}$$

$$= \frac{\omega^2\mu\epsilon}{2} + \frac{\omega\mu}{2}\sqrt{(\omega\epsilon)^2\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]}$$

$$= \frac{\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]}$$

$$\beta^2 = \omega^2\left[\frac{\mu\epsilon}{2}\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + \frac{\mu\epsilon}{2}\right]$$

$$\beta = \sqrt{\omega^2\left[\frac{\mu\epsilon}{2}\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + \frac{\mu\epsilon}{2}\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]} \qquad (2.14)$$

Intrinsic Impedance

Intrinsic impedance is also called as the wave impedance in free space. It is denoted as η_0. Intrinsic impedance relates the electric and magnetic field. Intrinsic impedance is the ratio of electric to magnetic field given as,

$$\eta_0 = \frac{E}{H}$$

Proof

Let the electric field, $\vec{E}$ is given as,

$$\vec{E} = E_x\overrightarrow{a_x} + E_y\overrightarrow{a_y} + E_z\overrightarrow{a_z}$$

If the electric field component has only x–direction component, then

$$\vec{E} = E_x\overrightarrow{a_x} = \overrightarrow{E_x} \qquad 2.15)$$

Therefore, if the wave is assumed to propagate in the positive z–direction, with only x–component,

$$\left[\nabla^2\overrightarrow{E_x} - \gamma^2\overrightarrow{E_x}\right] = 0$$

Expanding the above equation with the coordinate components,

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right][E_x\overrightarrow{a_x}]-\gamma^2[E_x\overrightarrow{a_x}]=0$$

NOTES

$$\overrightarrow{a_x}\left\{\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}-\gamma^2 E_x\right\}=0$$

Since the wave with *x*–component alone is travelling in the +*z* direction, $\frac{\partial^2 E_x}{\partial x^2}=\frac{\partial^2 E_x}{\partial y^2}=0$. Hence,

$$\frac{\partial^2 E_x}{\partial z^2}-\gamma^2 E_x=0 \qquad (2.16)$$

The solution of the Equation (2.16) is similar to the solution of Equation (2.3). Therefore,

$$E_x=E_{x_1}e^{-\gamma z}+E_{x_2}e^{\gamma z} \qquad (2.17)$$

Since we assume the wave to travel in the +*z* direction, Equation (2.17) becomes,

$$E_x=E_{x_1}e^{-\gamma z}$$

From Equation (2.15),

$$\vec{E}=E_{x_1}e^{-\gamma z}\overrightarrow{a_x}$$

$$=Re\left[E_{x_1}e^{-\gamma z}.e^{j\omega t}.\overrightarrow{a_x}\right]$$

$$=Re\left[E_{x_1}e^{-(\alpha+j\beta)z}.e^{j\omega t}.\overrightarrow{a_x}\right]$$

$$=Re\left[E_{x_1}e^{-\alpha z}e^{j(\omega t-\beta z)}.\overrightarrow{a_x}\right]$$

$$\vec{E}=E_{x_1}e^{-\alpha z}\cos(\omega t-\beta z)\overrightarrow{a_x}$$

Similarly,

$$\vec{H}=H_{x_1}e^{-\alpha z}\cos(\omega t-\beta z)\overrightarrow{a_y}$$

$$H_{x_1}=\frac{E_{x_1}}{\eta} \qquad (2.18)$$

Where, η = Intrinsic Impedance (Ω)

From Maxwell's third equation,

$$\vec{\nabla}\times\vec{E}=-j\omega\mu\vec{H}$$

$$E_{x_1}e^{-\gamma z}.e^{j\omega t}(-\gamma)=-j\omega\mu\left(H_{x_1}e^{-\gamma z}.e^{j\omega t}\right)$$

$$\Rightarrow E_{x_1}(-\gamma)=-j\omega\mu H_{x_1}$$

$$\Rightarrow H_{x_1}=\frac{\gamma}{j\omega\mu}E_{x_1} \qquad (2.19)$$

Comparing Equation (2.18) and Equation (2.19),

$$\eta = \frac{j\omega\mu}{\gamma}$$

$$= \frac{j\omega\mu}{\alpha + j\beta} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}}$$

NOTES

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \quad (2.20)$$

$$\eta = \sqrt{\frac{j\omega\mu(\sigma - j\omega\epsilon)}{\sigma^2 + \omega^2\epsilon^2}}$$

$$\eta^2 = \frac{j\omega\mu\sigma}{\sigma^2 + \omega^2\epsilon^2} + \frac{\omega^2\mu\epsilon}{\sigma^2 + \omega^2\epsilon^2}$$

$$\eta^2 = \frac{\omega^2\mu\epsilon}{\sigma^2 + \omega^2\epsilon^2} + j\frac{\omega\mu\sigma}{\sigma^2 + \omega^2\epsilon^2}$$

$$|\eta^2| = \sqrt{\frac{(\omega^2\mu\epsilon)^2 + (\omega\mu\sigma)^2}{[\sigma + (\omega\epsilon)^2]^2}} = \sqrt{\frac{(\omega\mu)^2(\sigma^2 + (\omega\epsilon)^2)}{[\sigma + (\omega\epsilon)^2]^2}}$$

$$|\eta^2| = \frac{\omega\mu}{\sqrt{\sigma^2 + (\omega\epsilon)^2}}$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sqrt{\sigma^2 + (\omega\epsilon)^2}}} = \sqrt{\frac{\omega\mu}{\sqrt{(\omega\epsilon)^2\left(\frac{\sigma^2}{(\omega\epsilon)^2} + 1\right)}}}$$

$$= \sqrt{\frac{\omega\mu}{\omega\epsilon\sqrt{\left(\frac{\sigma^2}{(\omega\epsilon)^2} + 1\right)}}} = \sqrt{\frac{\omega\mu}{\omega\epsilon\left(\frac{\sigma^2}{(\omega\epsilon)^2} + 1\right)^{\frac{1}{2}}}}$$

$$= \frac{\sqrt{\omega\mu}}{\sqrt{\omega\epsilon}\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}}$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}}$$

Let,

$$\theta_\eta = \tan^{-1}\left(\frac{\alpha}{\beta}\right) \quad (2.21)$$

NOTES

$$\theta_\eta = \tan^{-1}\left\{\frac{\omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]}}{\omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]}}\right\}$$

$$\tan\theta_\eta = \frac{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}}{\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1}}$$

$$\tan^2\theta_\eta = \frac{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1} \qquad (2.22)$$

We know that, $tan\ 2\theta = \frac{\tan^2\theta+1}{\tan^2\theta-1} = \sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon}\ \left[\because\ \frac{\sigma}{\omega\epsilon} \gg 1\right] \qquad (2.23)$$

Therefore,

$$\vec{H} = H_{x_1} e^{-\alpha z}\cos(\omega t-\beta z)\,\overrightarrow{a_y} = \frac{E_{x_1}}{|\eta|\angle\theta_\eta} e^{-\alpha z}\cos(\omega t-\beta z)\,\overrightarrow{a_y}$$

$$\vec{H} = \frac{E_{x_1}}{|\eta|} e^{-\alpha z}\cos(\omega t-\beta z-\theta_\eta)\,\overrightarrow{a_y}$$

Wave Equation for Conducting Medium

Wave propagation parameters gets altered when they travel across different medium like free space, dielectric and conductors. In the below section, wave parameters for different medium will be discussed.

(a) Plane Waves Traveling in Lossless Dielectric

In lossless dielectric, the material has the following properties.

$$\sigma \simeq 0$$

$$\epsilon = \epsilon_0\epsilon_r$$

$$\mu = \mu_0\mu_r$$

Recalling α and β from Equation (2.13) and Equation (2.14),

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]}\Bigg|_{\sigma=0} = 0$$

Substituting the material properties, we get α=0. Similarly,

$$= \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]} = \omega\sqrt{\mu\epsilon} \quad (2.24)$$

NOTES

$$|\eta| = \left.\frac{\sqrt{\mu\epsilon}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}}\right|_{\sigma=0} = \sqrt{\frac{\mu}{\epsilon}}$$

Also, $\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 0$

$$2\theta_\eta = 0$$

$$\theta_\eta = 0^\circ$$

$$\therefore \eta = \sqrt{\frac{\mu}{\epsilon}}\angle 0^\circ$$

For lossless dielectric, electromagnetic waves does not undergo phase change as $\theta_\eta = 0^\circ$.

(b) Plane Waves in Free Space

When plane waves travel in free space with the properties, $\sigma = 0, \epsilon = \epsilon_0$ and $\mu = \mu_0$. Substituting in the α and β expressions,

$$\alpha = 0$$

$$\beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0} = \omega/c \quad (2.25)$$

Where, c = Velocity of light.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi = 377\Omega$$

$$\eta = \eta_0 = 377\Omega$$

(c) Plane Waves in Good Conductors

Good conductors have the following material properties with regard to conductivity,

$$\sigma \cong \infty$$

$$\epsilon = \epsilon_0\epsilon_r$$

$$\mu = \mu_0\mu_r$$

NOTES

From the expressions for α and β.

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\frac{\sigma}{\omega\epsilon}\right)} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\frac{\sigma}{\omega\epsilon}\right)} = \sqrt{\frac{\omega\mu\sigma}{2}} \quad (2.26)$$

$$\Rightarrow \alpha = \beta$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}} \approx \frac{\sqrt{\frac{\mu}{\epsilon}}}{\sqrt{\frac{\sigma}{\omega\epsilon}}} = \sqrt{\frac{\omega\mu}{\sigma}} \quad (2.27)$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = \infty$$

$$\theta_\eta = 45^\circ$$

Therefore,

$$\therefore \eta = \sqrt{\frac{\omega\mu}{\sigma}}\ \angle 45^\circ \quad (2.28)$$

The electric field leads magnetic field by an angle of 45° in good conductors. Therefore $\vec{E}$ and $\vec{H}$ may be rewritten as,

$$\vec{E} = E_{x_1} e^{-\alpha z}\cos(\omega t - \beta z)\,\overrightarrow{a_x}$$

$$\vec{H} = \frac{E_{x_1}}{|\eta|} e^{-\alpha z}\cos(\omega t - \beta z - 45^\circ)\,\overrightarrow{a_y}$$

Depth of Penetration or Skin Depth (δ)

When an electromagnetic wave travels, they tend to attenuate. This attenuation depends on the frequency of the wave travelling. Attenuation is larger, when the frequency is larger. This implies that wave die out faster for larger frequencies and travel a very short distance. This distance of travel for a wave till they are attenuated to a value of 36.8% of the original value is called as ***depth of penetration*** or ***skin depth*** (Refer Figure 2.4). They are represented as δ.

$$\delta \propto \frac{1}{\alpha}$$

We know that,

$$e^{\alpha z} = \frac{1}{e}$$

If is the distance travelled as shown in Figure (2.4).

$$-\alpha\delta = -1$$

$$\alpha\delta = 1$$

$$\alpha = \frac{1}{\delta} \quad (2.29)$$

Where α is the attenuation factor. Therefore,

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]}}$$

For good conductors, $\alpha \approx \infty$. Hence, from Equation (2.26), we have,

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (2.30)$$

From Equation (2.30), it is observed that, as the frequency, ω increases, α increases and δ decreases.

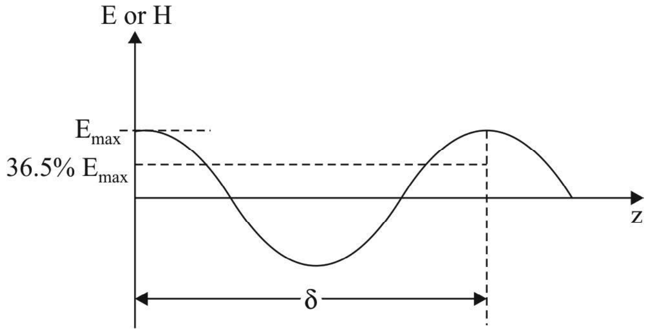

Fig. 2.4 *Skin Depth*

2.6 RADIATION PRESSURE AND MOMENTUM

The mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field is the 'Radiation Pressure'. The associated force is called the radiation pressure force, or sometimes just the force of light.

Radiation pressure is the mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field. This includes the momentum of light or electromagnetic radiation of any wavelength that is absorbed, reflected, or otherwise emitted by the matter on any scale (ranging from macroscopic objects to dust particles to gas molecules) is also known as black-body radiation. The associated force is called the radiation pressure force, or sometimes also referred as just the force of light.

The forces generated by radiation pressure are generally too small to be noticed under everyday circumstances; however, they are important in some physical processes and technologies. This particularly includes objects in outer space, where it is usually the main force acting on objects besides gravity, and where the net effect of a tiny force may have a large cumulative effect over long periods of time. For example, if the effects of the Sun's radiation pressure on the spacecraft of

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the Viking program had been ignored, the spacecraft would have missed Mars' orbit by about 15,000 km (9,300 mi). The significance of radiation pressure increases rapidly at extremely high temperatures and can sometimes dwarf the usual gas pressure, for instance, in stellar interiors and thermonuclear weapons.

Radiation pressure can equally well be accounted for by considering the momentum of a classical electromagnetic field or in terms of the momenta of photons, particles of light. The interaction of electromagnetic waves or photons with matter may involve an exchange of momentum. Due to the law of conservation of momentum, any change in the total momentum of the waves or photons must involve an equal and opposite change in the momentum of the matter it interacted with (Newton's third law of motion), as is illustrated in the following Figure for the case of light being perfectly reflected by a surface. This transfer of momentum is the general explanation for what we term radiation pressure.

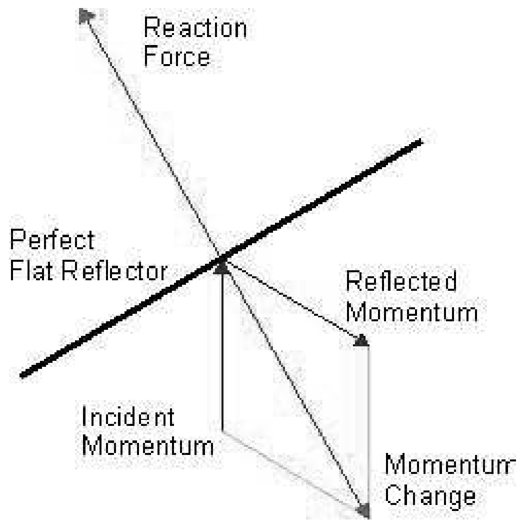

Fig. 2.5 *Force on a Reflector Results from Reflecting the Photon Flux Johannes Kepler put forward the concept of radiation pressure in 1619 to explain the observation that a tail of a comet always points away from the Sun.*

The assertion that light, as electromagnetic radiation, has the property of momentum and thus exerts a pressure upon any surface that is exposed to it was published by James Clerk Maxwell in 1862, and proven experimentally by Russian physicist Pyotr Lebedev in 1900 and by Ernest Fox Nichols and Gordon Ferrie Hull in 1901. The pressure is very small, but can be detected by allowing the radiation to fall upon a delicately poised vane of reflective metal in a Nichols radiometer.

Theory

Radiation pressure can be viewed as a consequence of the conservation of momentum given the momentum attributed to electromagnetic radiation. That momentum can be equally well calculated based on electromagnetic theory or from the combined momenta of a stream of photons, giving identical results.

Radiation Pressure from Momentum of An Electromagnetic Wave

According to Maxwell's theory of electromagnetism, an electromagnetic wave carries momentum, which will be transferred to an opaque surface it strikes.

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The energy flux (irradiance) of a plane wave is calculated using the Poynting vector,

S = **E** × **H**, whose magnitude is denoted by **S**. Characteristically S divided by the speed of light is the density of the linear momentum per unit area (pressure) of the electromagnetic field. So, dimensionally, the Poynting vector is,

$$S = \frac{\text{power}}{\text{area}} = \frac{\text{rate of doing work}}{\text{area}} = \frac{\frac{\Delta F}{\Delta t}\Delta x}{\text{area}},$$

Which is the speed of light, c=Δ*x*/Δ*t*, times pressure, Δ*F*/area. That pressure is experienced as radiation pressure on the surface:

$$P_{\text{incident}} = \frac{\langle S \rangle}{c} = \frac{I_f}{c}$$

Where *P* is pressure (usually in Pascals), I_f is the incident irradiance (usually in W/m^2) and *c* is the speed of light in vacuum.

2.7 REFLECTION AND REFRACTION

Reflection basically occurs when a wave is incident on a boundary between two media in which the wave speed is different, and then remains in the original medium rather than passing into the second medium. While reflection occurs at any boundary, often only a small proportion of the wave is reflected. Refraction is the change of the direction of propagation of waves when they pass into a medium where they have a different speed. It is observed whenever the waves are incident to the surface at an angle different to the normal to the surface.

When an electromagnetic field faces an abrupt change in the permittivity and permeability, then certain conditions on electric and magnetic fields on the interface are to be respected for the continuity. These conditions of continuity are termed as the boundary conditions for the electromagnetic field.

Similar to electrostatics boundary conditions, magnetostatics boundary conditions are formulated using the two parameters $\vec{B}\ and\ \vec{H}$. The magnetic field, $\vec{B}$ is defined using Gauss's law and given as,

In integral form, $\oint \vec{B}.\overrightarrow{ds} = 0$ (2.31)

The magnetic field intensity, $\vec{H}$ is defined as Ampere's circuit law,

In integral form, $\oint \vec{H}.\overrightarrow{dl} = I$ (2.32)

(a) Magnetic Material Boundary

Consider two magnetic materials as shown in Figure (2.6) to identify the magnetic boundary conditions using $\vec{B}\ and\ \vec{H}$.

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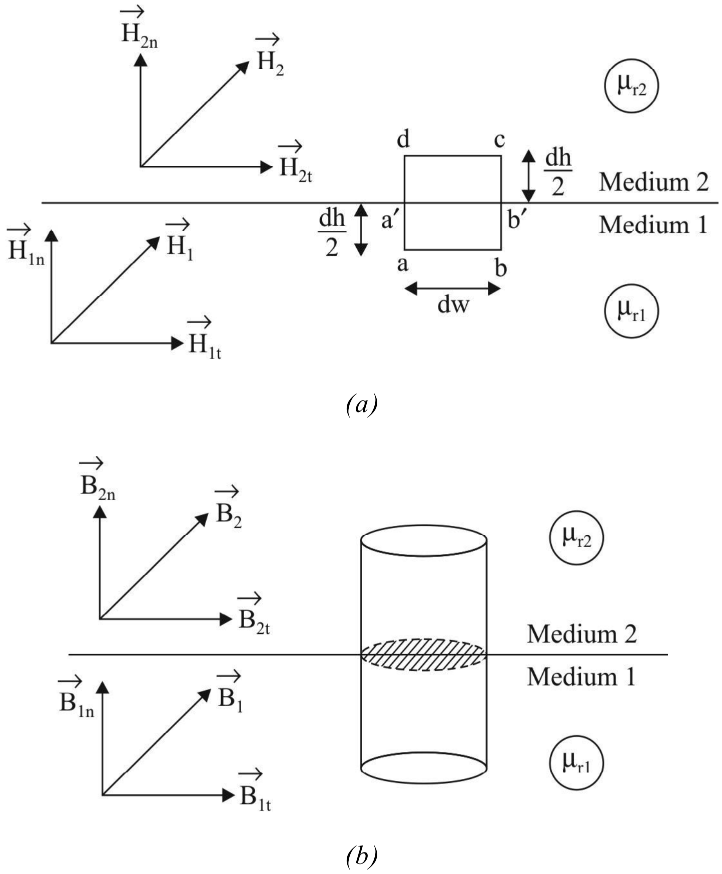

Fig. 2.6 *(a) Line Integral for $\vec{H}$ (b) Surface Integral for $\vec{B}$*

Using Equation (2.31) for the boundary condition as shown in Figure (2.6), and then integrating the pill box from Medium 1 to Medium 2,

$$\oint_s \vec{B}.\overrightarrow{ds} = \oint_{s1} \overrightarrow{B_{1n}}.\overrightarrow{ds} - \oint_{s2} \overrightarrow{B_{2n}}.\overrightarrow{ds} = 0$$

$$\Rightarrow \overrightarrow{B_{1n}}.\overrightarrow{ds} - \overrightarrow{B_{2n}}.\overrightarrow{ds} = 0$$

$$\Rightarrow \overrightarrow{B_{1n}} = \overrightarrow{B_{2n}} \quad (2.33)$$

Since $\vec{B} = \mu\vec{H},$ Equation (2.33) becomes,

$$\overrightarrow{H_{1n}}\,\mu_{r_1} = \overrightarrow{H_{2n}}\,\mu_{r_2} \quad (2.34)$$

From Equations (2.33) and (2.34), the normal component of $\vec{B}$ is continuous and the normal component of $\vec{H}$ is discontinuous as undergoes a change by across the boundary.

Similarly applying the line integral Equation (2.32) along the boundary path abb′cda′a in Figure (2.6),

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$$\oint \vec{H}.\vec{dl} = I$$

$$\Rightarrow \int_a^b \vec{H}_{1t}\,\vec{dl} + \int_b^{b'} \vec{H}_{1n}\,\vec{dl} + \int_{b'}^c \vec{H}_{2n}\,\vec{dl} + \int_c^d \left(-\vec{H}_{2t}\right)\vec{dl} + \int_d^{a'} \left(-\vec{H}_{2n}\right)\vec{dl}$$

$$+\int_{a'}^a \left(-\vec{H}_{1n}\right)\vec{dl} = \text{I.}\ \vec{dl}$$

$$\vec{H}_{1t}dw + \vec{H}_{1n}\frac{dh}{2} + \vec{H}_{2n}\frac{dh}{2} - \vec{H}_{2t}dw - \vec{H}_{2n}\frac{dn}{2} - \vec{H}_{1n}\frac{dh}{2} = I.dw$$

As $dh/2$=0,

$$\vec{H}_{1t}dw - \vec{H}_{2t}dw = I.dw$$

$$\Rightarrow \vec{H}_{1t} - \vec{H}_{2t} = I$$

If I=0 at the boundary, the,

$$\vec{H}_{1t} = \vec{H}_{2t} \qquad (2.35)$$

As $\vec{B} = \mu\vec{H}$,

$$\frac{\vec{B}_{1t}}{\mu_{r_1}} = \frac{\vec{B}_{2t}}{\mu_{r_2}} \qquad (2.36)$$

From Equations (2.35) and (2.36), the tangential component of magnetic field intensity, $\vec{H}$ is continuous across boundary and tangential component of flux density, $\vec{B}$ is a discontinuous by a factor of μ_r across the boundary.

Additional relation can be obtained by considering the detailed representation of Figure (2.6(a)) in Figure (2.7).

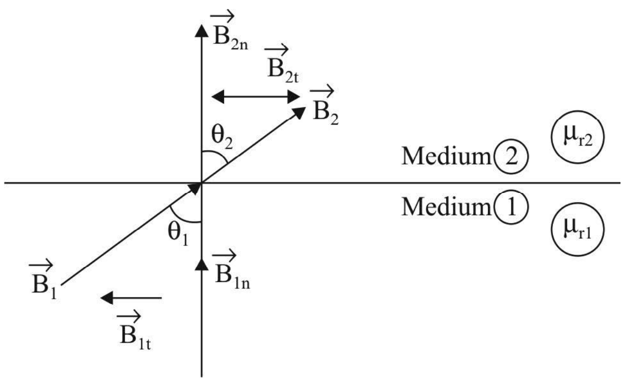

Fig. 2.7 *Boundary Condition*

From Figure (2.7),

$$B_{n_1} = B_1\cos\theta_1$$

$$and\ B_{n_2} = B_2\cos\theta_2.$$

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But from Equation (2.33),

$$B_{n_1} = B_{n_2}$$

$$\Rightarrow B_1 \cos\theta_1 = B_2 \cos\theta_2 \quad (2.37)$$

Also, from Equation (2.36), we have

$$\frac{B_{1t}}{\mu_{r_1}} = \frac{B_{2t}}{\mu_{r_2}}$$

$$\Rightarrow \mu_{r_2} B_{1t} = \mu_{r_1} B_{2t} \quad (2.38)$$

But from Figure (2.7),

$$B_{1t} = B_1 \sin\theta_1$$

$$B_{2t} = B_2 \sin\theta_2$$

Equation (2.38) implies,

$$\mu_{r_2}(B_1 \sin\theta_1) = \mu_{r_1}(B_2 \sin\theta_2) \quad (2.39)$$

Dividing Equation (2.39) by Equation (2.37),

$$\mu_{r_2} \frac{(B_1 \sin\theta_1)}{B_1 \cos\theta_1} = \mu_{r_1} \frac{(B_2 \sin\theta_2)}{B_2 \cos\theta_2}$$

$$\mu_{r_2} \tan\theta_1 = \mu_{r_1} \tan\theta_2$$

Therefore, $\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_{r_1}}{\mu_{r_2}} = \frac{\mu_0\mu_{r_1}}{\mu_0\mu_{r_2}} = \frac{\mu_1}{\mu_2}$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

(b) Magnetic Materials

A material is considered to be magnetic based on the magnetic susceptibility x_m.

If $x_m = 1$, the material is a magnet.

If $x_m = 0$, the material is a non-magnet.

The magnetic materials are generally classified depending on the relative permeability, μ_r. Hence there are mainly three different types of magnetic materials. They are:

(i) Diamagnets
(ii) Paramagnets
(iii) Ferromagnets

(i) Diamagnets

Diamagnets are weakly affected by magnetic field. They have $\mu_r < 1$ and $x_m < 0$. The magnetic moment of each atom in diamagnet is ***m*=0**.

Examples: Diamond, Silicon, Copper, etc.,

(ii) Paramagnets

Paramagnets are temperature dependent with a positive magnetic moment, ***m***. They have $\mu_r > 1 \;\; and\; x_m > 0.$

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Examples: Tungsten, Potassium and Platinum.

(iii) Ferromagnets

Ferromagnets are largely affected by magnetic field with a strong magnetic moment ***m***. They have $\mu_r \gg 1 \;\; and \;\; x_m \gg 0.$ Ferromagnets loose their magnetic property when temperature is raised above a certain level.

Ferromagnets exhibit a non-linear characteristics between the magnetic field, $\vec{B}$ and magnetic field intensity $\vec{H}.$ That is,

$$\vec{B} \neq \mu_0 \mu_r \vec{H}$$

Hence μ_r is nonlinear. The nonlinear relationship between $\vec{B}\; and\; \vec{H}$ are well understood by a 'Magnetisation Curve' or a 'Hysterisis Curve'.

Examples: Iron, Cobalt, etc.

These ferromagnets act as a short circuit path for the magnetic field. Hence they are used in magnetic screening applications. For example, they are used in transformers that links or act as short circuit path for the magnetic field produced by the primary winding.

Table 2.3 illustrates the properties of different magnetic materials.

Table 2.3 *Summary of Magnetic Materials*

Sl. No	Magnetic Material	μ_r	x_m	m
1	Diamagnets	$\lesssim 1$	-10^{-5}	0
2	Paramagnets	$\gtrsim 1$	10^{-5}	>0
3	Ferromagnets	$\gg 1$	$\gg 0$	$\gg 0$

(c) Hysteresis Curve

Hysteresis curve provides the nonlinear relationship between $\vec{B}\; and\; \vec{H}.$ To obtain a hysterisis curve or a magnetisation curve, assume a Ferromagnet, initially magnetised. This initial magnetisation is represented in the hysteresis curve from O to A as shown in Figure (2.8).

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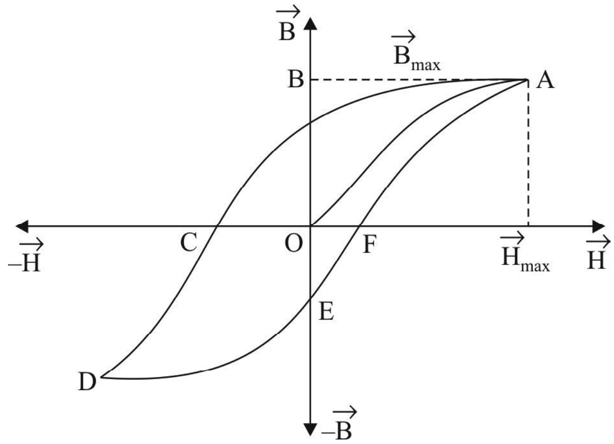

Fig. 2.8. *-B-H Curve or Hysteresis Curve*

Once the Ferromagnet is saturated with $\vec{B}_{max}$ at $\vec{H}_{max}$, it is demagnetised. When demagnetised, reducing the $\vec{H}$ reduces $\vec{B}$, but not on the same path of magnetisation OA. It follows AB while demagnetisation in Figure (2.8) on further reducing the $\vec{H}$ to negative, $\vec{B}$ reaches zero only at C. On further reducing , achieves negative saturation, following the path CD. When magnetised again with increase in , increases to attain saturation of through the path DEFA. The magnetisation and demagnetisation of the ferromagnetic material creates a Hysteresis Loop. This area of the closed loop indicates the energy loss of the material during magnetisation and demagnetisation. The larger the area of the hysteresis, the higher will be the losses and hence choice of the ferromagnetic material will be towards a narrower 'Hysteresis Curve'.

Reflection and Refraction of Electromagnetic Waves at the Interface of Non-Conducting Media

In physics, the term reflection refers to the change in direction of a wavefront at an interface between two different media so that the wavefront returns into the medium from which it originated. Common examples include the reflection of light, sound and water waves. The law of reflection says that for specular reflection the angle at which the wave is incident on the surface equals the angle at which it is reflected. Mirrors exhibit specular reflection.

When a plane wave travels across a medium they may be reflected at the boundary of the medium and refracted after they cross the boundary. The amount of reflection and refraction depends on the following factors:

1. The type of medium in which the wave travels.
2. The angle of incidence of the wave.

Based on the angle of incidence, the incidence may be classified as:

1. Normal Incidence
2. Oblique Incidence

Reflection and Refraction of Plane Waves with Normal Incidence

When a plane wave is incident normally on a plane, or in other words, if a wave is incident on a plane perpendicular to the traveling wave, then they are said to represent normal incidence as shown in Figure (2.9).

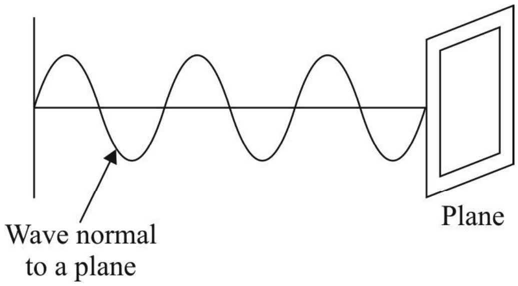

Fig. 2.9 *Normal Incidence*

NOTES

Let the wave travel from Medium 1 to Medium 2. Medium 1 has the following properties $\sigma_1, \epsilon_1\ and \mu_1$, and the Medium 2 has the properties $\sigma_2, \epsilon_2\ and \mu_2$. The wave that is travelling has three stages and needs ti be represented uniquely. Hence the following subscripts will be used.

Wave	*Subscript*
Incident wave	i
Reflected wave	r
Refracted or transmitted wave	t

The EM or Electro Magnetic wave is assumed to travel in the $+\overrightarrow{a_z}$ direction. Since the TEM (Transverse Electric and Magnetic) waves are considered electric field, the magnetic field and direction of travel are mutually orthogonal. Following vectors identities should be recalled:

$$\vec{E} \times \vec{H} = \vec{k} \qquad [\vec{E} \perp \vec{H} \perp \vec{k}]$$

$$\vec{H} \times \vec{k} = \vec{E}$$

$$\vec{k} \times \vec{E} = \vec{H}$$

(a) Incident Wave

Let $\overrightarrow{E_i}$ and $\overrightarrow{H_i}$ travel in $+\overrightarrow{a_z}$ direction

$$\overrightarrow{E_i}(z) = E_{0_i} e^{-\gamma_1 z}\, \overrightarrow{a_x} = E_{0_i} e^{-(-\alpha_1 - j\beta_1)z}\, \overrightarrow{a_x}$$

$$\overrightarrow{E_i}(z) = E_{0_i} e^{-j\beta_1 z}\, \overrightarrow{a_x} \qquad [let\ \alpha_1 = 0] \tag{2.40a}$$

$$\overrightarrow{H_i}(z) = H_{0_i} e^{-j\beta_1 z}\, \overrightarrow{a_y} \tag{2.40b}$$

In Equations (2.40a) and (2.40b), the time harmonic term $e^{j\omega t}$ is removed for the sake of convenience and will be reinstated at the end of the derivation.

(b) Reflected Wave

The reflected wave consisting of $\overrightarrow{E_r}$ and $\overrightarrow{H_r}$ get reflected and travel in the $-\overrightarrow{a_z}$ direction.

$$\overrightarrow{E_r}(z) = E_{0_r} e^{j\beta_1 z}\, \overrightarrow{a_x} \quad (2.41a)$$

$$\overrightarrow{H_r}(z) = H_{0_r} e^{j\beta_1 z}(-\overrightarrow{a_y}) \qquad [-\vec{k} \times \vec{E} = -\vec{H}]$$

NOTES

$$\overrightarrow{H_r}(z) = -H_{0_r} e^{j\beta_1 z}\, \overrightarrow{a_y} \quad (2.41b)$$

(c) Transmitted Wave

The transmitted wave, of $\overrightarrow{E_t}$ and $\overrightarrow{H_t}$ travel in the same direction as that of the incident wave, i.e., in the $+\overrightarrow{a_z}$ direction.

$$\overrightarrow{E_t}(z) = E_{0_t} e^{-j\beta_2 z}\, \overrightarrow{a_x} \quad (2.42)$$

$$\overrightarrow{H_t}(z) = H_{0_t} e^{-j\beta_2 z}\, \overrightarrow{a_y}$$

The incident, reflected and transmitted waves are shown in Figure (2.10).

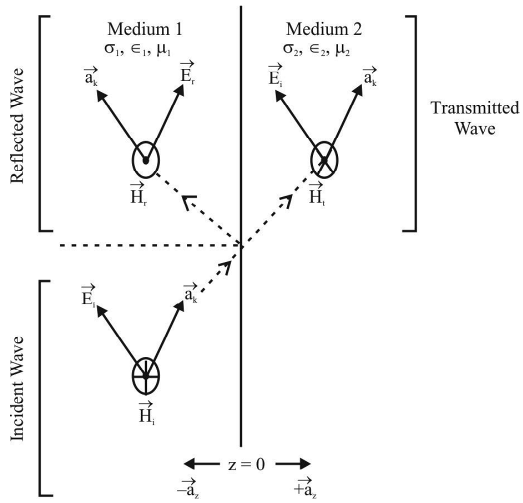

Fig. 2.10 *Reflection of Plane Waves at Normal Incidence*

The total field in Medium 1 includes the incident field and the reflected field. Similarly, total field in Medium 2 includes the transmitted field. Therefore,

In Medium 1,

$$\overrightarrow{E_1} = \overrightarrow{E_i} + \overrightarrow{E_r} \quad (2.43a)$$

$$\overrightarrow{H_1} = \overrightarrow{H_i} + \overrightarrow{H_r} \quad (2.43b)$$

In Medium 2,

$$\overrightarrow{E_2} = \overrightarrow{E_t} \quad (2.44a)$$

$$\overrightarrow{H_2} = \overrightarrow{H_t} \quad (2.44b)$$

Substituting Equations (2.40) and (2.41) in Equation (2.43), we have,

$$\overrightarrow{E_1} = E_{0_i} e^{-j\beta_1 z}\, \overrightarrow{a_x} + E_{0_r} e^{j\beta_1 z}\, \overrightarrow{a_x}$$

And

$$\overrightarrow{H_1} = H_{0_i} e^{-j\beta_1 z}\, \overrightarrow{a_y} - H_{0_r} e^{j\beta_1 z}\, \overrightarrow{a_y}$$

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At the boundary, i.e., at $z = 0$,

$$\overrightarrow{E_1} = E_{0_i} + E_{0_r} \quad (2.45a)$$

$$\overrightarrow{H_1} = H_{0_i} - H_{0_r}$$

$$= \frac{E_{0_i}}{\eta_1} - \frac{E_{0_r}}{\eta_1} \quad (2.45b)$$

Similarly, substituting Equation (2.42) in Equation (2.49), we have,

$$\overrightarrow{E_t} = E_{0_t} e^{-j\beta_2 z} \overrightarrow{a_x}$$

$$\overrightarrow{H_t} = H_{0_t} e^{-j\beta_2 z} \overrightarrow{a_y}$$

At the boundary at $z = 0$,

$$\overrightarrow{E_2} = E_{0_t} \quad (2.46a)$$

$$\overrightarrow{H_2} = H_{0_t} \quad (2.46b)$$

$$= \frac{E_{0_t}}{\eta_2}$$

Across the boundary, the fields are continuous and hence,

$\overrightarrow{E_1} = \overrightarrow{E_2}$ and $\overrightarrow{H_1} = \overrightarrow{H_2}$

$$\therefore E_{0_i} + E_{0_r} = E_{0_t} \quad (2.47a)$$

And

$$\frac{E_{0_i}}{\eta_1} - \frac{E_{0_r}}{\eta_1} = \frac{E_{0_t}}{\eta_2} \quad (2.47b)$$

Multiplying Equation (2.47a) by $\frac{1}{\eta_1}$ and adding with eqn. Equation (2.47b),

we have $\left[\frac{1}{n_2}\text{Equation (2.47a)} + \text{Equation (2.47b)}\right]$

$$\left\{\left[\frac{E_{0_i}}{\eta_1} + \frac{E_{0_r}}{\eta_1}\right] + \frac{E_{0_i}}{\eta_1} - \frac{E_{0_r}}{\eta_1}\right\} = \frac{E_{0_t}}{\eta_1} + \frac{E_{0_t}}{\eta_2}$$

$$\frac{2}{\eta_1} E_{0_i} = E_{0_t}\left[\frac{1}{\eta_1} + \frac{1}{\eta_2}\right]$$

$$E_{0_t} = E_{0_i} \frac{2}{\eta_1}\left[\frac{\eta_1 \eta_2}{\eta_1 + \eta_2}\right]$$

$$E_{0_t} = E_{0_i}\left[\frac{2\eta_2}{\eta_1 + \eta_2}\right] \quad (2.48)$$

Similarly, $\left[\frac{1}{n_2}\text{Equation (2.47a)} - \text{Equation (2.47b)}\right]$ implies,

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$$\left\{\left[\frac{E_{0_i}}{\eta_1} + \frac{E_{0_r}}{\eta_1}\right] - \left[\frac{E_{0_i}}{\eta_1} - \frac{E_{0_r}}{\eta_1}\right]\right\} = \frac{E_{0_t}}{\eta_1} - \frac{E_{0_t}}{\eta_2}$$

$$\frac{2}{\eta_1}E_{0_r} = E_{0_t}\left[\frac{1}{\eta_1} - \frac{1}{\eta_2}\right]$$

$$E_{0_r} = E_{0_t}\left[\frac{\eta_2 - \eta_1}{2\eta_2}\right] \quad (2.49)$$

Substituting Equation (2.48) in Equation (2.49),

$$E_{0_r} = E_{0_i}\left[\frac{2\eta_2}{\eta_1 + \eta_2}\right]\left[\frac{\eta_2 - \eta_1}{2\eta_2}\right]$$

$$E_{0_r} = E_{0_i}\left[\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\right]$$

Check Your Progress

1. Define the term time varying.
2. What are Maxwell's equations?
3. What does electromagnetic scalar wave equations describe?
4. Give the equation for scalar wave.
5. State about the Hertz vector.
6. What are electromagnetic waves?
7. What is radiation pressure?
8. Define the term reflection

2.8 POLARISATION

Polarization or Polarisation is a property applied to transverse waves that specifies the geometrical orientation of the oscillations. In a transverse wave, the direction of the oscillation is perpendicular to the direction of motion of the wave. A simple example of a polarized transverse wave is vibrations traveling along a taut string, for example, in a musical instrument like a guitar string. Depending on how the string is plucked, the vibrations can be in a vertical direction, horizontal direction, or at any angle perpendicular to the string. In contrast, in longitudinal waves, such as sound waves in a liquid or gas, the displacement of the particles in the oscillation is always in the direction of propagation, so these waves do not exhibit polarization. Transverse waves that exhibit polarization include electromagnetic waves, such as light and radio waves, gravitational waves, and transverse sound waves (shear waves) in solids. In some types of transverse waves, the wave displacement is limited to a single direction, so these also do not exhibit polarization; for example, in surface waves in liquids (gravity waves), the wave displacement of the particles is always in a vertical plane.

NOTES

An electromagnetic wave, such as light consists of a coupled oscillating electric field and magnetic field which are always perpendicular; by convention, the 'Polarization' of electromagnetic waves refers to the direction of the electric field. In linear polarization, the fields oscillate in a single direction. In circular or elliptical polarization, the fields rotate at a constant rate in a plane as the wave travels. The rotation can have two possible directions; if the fields rotate in a right hand sense with respect to the direction of wave travel, it is called right circular polarization, or, if the fields rotate in a left hand sense, it is called left circular polarization.

Light or other electromagnetic radiation from many sources, such as the sun, flames, and incandescent lamps, consists of short wave trains with an equal mixture of polarizations; this is called unpolarized light. Polarized light can be produced by passing unpolarized light through a polarizer, which allows waves of only one polarization to pass through. The most common optical materials (such as, glass) are isotropic and do not affect the polarization of light passing through them; however, some materials—those that exhibit birefringence, dichroism, or optical activity—can change the polarization of light. Some of these are used to make polarizing filters. Light is also partially polarized when it reflects from a surface.

Basically, the electric and magnetic vibrations of an electromagnetic wave occur in numerous planes. A light wave that is vibrating in more than one plane is referred to as unpolarized light. It is possible to transform unpolarized light into polarized light. Polarized light waves are light waves in which the vibrations occur in a single plane. The process of transforming unpolarized light into polarized light is known as polarization.

Wave Polarization

Electromagnetic waves travel through any medium. These wave are produced by the vibration of the electron charges. These wave are traverse waves that has both the electric and magnetic components.

Polarization, also called wave polarization, is an expression of the orientation of the lines of electric flux in an ElectroMagnetic field (EM field). Polarization can be constant — that is, existing in a particular orientation at all times, or it can rotate with each wave cycle.

Polarization is important in wireless communications systems. The physical orientation of a wireless antenna corresponds to the polarization of the radio waves received or transmitted by that antenna. Thus, a vertical antenna receives and emits vertically polarized waves, and a horizontal antenna receives or emits horizontally polarized waves. The best short-range communications is obtained when the transmitting and receiving (source and destination) antennas have the same polarization. The least efficient short-range communications usually takes place when the two antennas are at right angles (for example, one horizontal and one vertical). Over long distances, the atmosphere can cause the polarization of a radio wave to fluctuate, so the distinction between horizontal and vertical becomes less significant.

NOTES

Some wireless antennas transmit and receive EM waves whose polarization rotates 360 degrees with each complete wave cycle. This type of polarization, called elliptical or circular polarization, can be either clockwise (right handed) or counter clockwise (left handed). The best communications results are obtained when the transmitting and receiving antennas have the same sense of polarization (both clockwise or both counter clockwise). The worst communications usually takes place when the two antennas radiate and receive in the opposite sense (one clockwise and the other counter clockwise).

Polarisation may be defined as the orientation of the field in a particular direction. Considering a uniform plane wave when traveling in +z direction, assuming the y component of the electric field to be zero, i.e., $E_y = 0$, then only E_m component exists pointing towards x-direction. Then the electric field is said to be oriented towards *x*-direction or in other words, the wave is said to be polarised in x-direction. Similarly when $E_x = 0$, wave is polarised to be in y-direction.

Polarisation of waves may be:

(1) Linear Polarisation

(2) Circular Polarisation

(3) Elliptical Polarisation

1. Linear Polarisation

If the electric field components $E_x \; and \; E_y$ are in phase with each other, the resultant electric field, $\vec{E}$ given as,

$$\vec{E} = E_x \overrightarrow{a_x} + E_y \overrightarrow{a_y}$$

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2}$$

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right)$$

$|\vec{E}|$ is said to be linear. The resultant vector is said to be linear, if the magnitude and the phase of the vector is constant with time, then the wave is said to be linearly polarized.

Fig. 2.11 *Linear Polarisation*

2. Circular Polarisation

When the field components E_x and E_y are out of phase by 90° with each other. The variation of these field components makes the resultant vector rotate in a circular path. Such polarisation is called circular polarisation.

We know that,

$$\vec{E} = E_x \overrightarrow{a_x} + E_y \overrightarrow{a_y}$$

NOTES

Since the fields vary sinusoidally, each component of electric field is represented with an instantaneous value with magnitude E_m.

$\therefore \quad E_x = E_m \sin \omega t$

And $E_y = E_y \sin(90 - \omega t)$

$\therefore \vec{E} = E_m \sin \omega t \overrightarrow{a_x} + E_m \sin(90 - \omega t)\overrightarrow{a_y}$

$= E_m \sin \omega t \overrightarrow{a_x} + E_m \cos \omega t \overrightarrow{a_y}$

$|\vec{E}| = \sqrt{(E_m \sin \omega t)^2 + (E_m \cos \omega t)^2} = \sqrt{E_x^2 + E_y^2}$

$= E_m$

$or\ E_m^2 = E_x^2 + E_y^2$

The above expression is a locus of a circle and hence said to be circularly polarised.

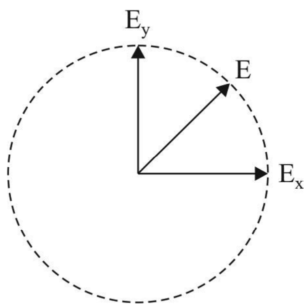

Fig. 2.12 *Circularly Polarized*

3. Elliptical Polarisation

When the field components E_x and E_y have different magnitude but have 90° phase difference, the resultant field envelopes results in elliptical polarisation.

We know that,

$\vec{E} = E_x \overrightarrow{a_x} + E_y \overrightarrow{a_y}$

Let the magnitudes of E_x and E_y be E_{m_1} and , respectively. Therefore,

$$E_x = E_{m_1} \sin \omega t \quad (2.50)$$

$$E_y = E_{m_2} \sin(90 - \omega t) = E_{m_2} \cos \omega t \quad (2.51)$$

Therefore,

$\vec{E} == E_{m_1} \sin \omega t \overrightarrow{a_x} + E_{m_2} \cos \omega t \overrightarrow{a_y}$

$|\vec{E}| = \sqrt{\left(E_{m_1} \sin \omega t\right)^2 + \left(E_{m_2} \cos \omega t\right)^2}$

$= E_x^2 + E_y^2$

From Equations (2.50) and (2.51), we have

$\frac{E_x}{E_{m_1}} = \sin \omega t \qquad and \qquad \frac{E_y}{E_{m_2}} = \cos \omega t$

$\sin^2 \omega t + \cos^2 \omega t = 1 = \frac{E_x^2}{E_{m_1}^2} + \frac{E_y^2}{E_{m_2}^2}$

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The locus of the above expression is ellipse and hence said to be elliptically polarised.

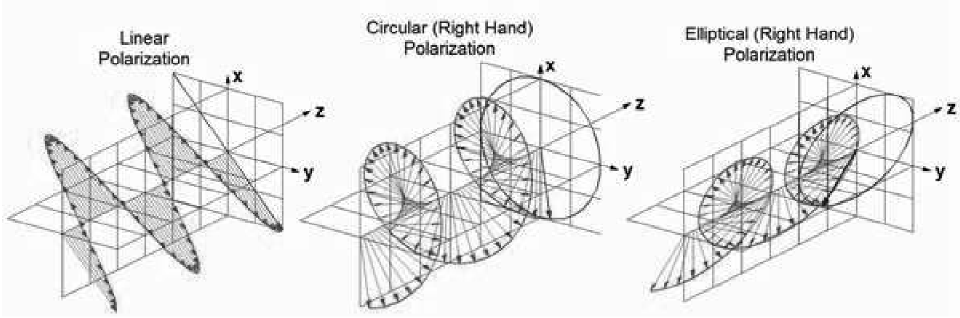

Fig. 2.13 *Elliptical Polarisation*

The following are four different methods of polarisation:

- Polarisation by Transmission
- Polarization by Reflection
- Polarization by Refraction
- Polarization by Scattering

Perpendicular Polarization

In perpendicular polarization, electric field is perpendicular to the plane of incidence, across the boundary, electric field must be continuous and hence,

$$E_t = E_i + E_r$$

$$1 + \frac{E_r}{E_i} = \frac{E_t}{E_i}$$

We how that,

$$\frac{E_r^2}{E_i^2} = 1 - \frac{E_t^2 \cos\theta_t}{E_i^2 \cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

$$= 1 - \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_i}{\cos\theta_t}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

Following the similar procedure as that of parallel polarization

$$\left(1 - \frac{E_r}{E_i}\right)\left(1 + \frac{E_r}{E_i}\right) = \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

$$1 - \frac{E_r}{E_i} = \left(1 + \frac{E_r}{E_i}\right)\frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

$$\left(1 - \frac{E_r}{E_i}\right) = \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}} + \frac{E_r}{E_i}\frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

$$\frac{E_r}{E_i}\left[1 + \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right] = 1 - \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$$

$$\frac{E_r}{E_i} = \frac{1 - \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}}{1 + \frac{\cos\theta_t}{\cos\theta_i}\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}}$$

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$$\Delta_E = \frac{E_r}{E_i} = \frac{\sqrt{\epsilon_1}\,\cos\theta_i - \sqrt{\epsilon_2}\cos\theta_t}{\sqrt{\epsilon_1}\cos\theta_i + \sqrt{\epsilon_2}\,\cos\theta_t}$$

Similarly,

$$\tau_E = \frac{2\,\sqrt{\epsilon_2}\cos\theta_i}{\sqrt{\epsilon_2}\cos\theta_i + \sqrt{\epsilon_1}\,\cos\theta_t}$$

$$1 + \Delta_E = 1 + \frac{\sqrt{\epsilon_1}\,\cos\theta_i - \sqrt{\epsilon_2}\cos\theta_t}{\sqrt{\epsilon_1}\cos\theta_i + \sqrt{\epsilon_2}\,\cos\theta_t}$$

$$= \frac{2\sqrt{\epsilon_1}\,\cos\theta_i}{\sqrt{\epsilon_1}\cos\theta_i + \sqrt{\epsilon_2}\,\cos\theta_t} = \tau_E$$

$$1 + \Delta_E = \tau_E$$

2.9 TOTAL INTERNAL REFLECTION

According to electromagnetic theory of light, electric field, magnetic field and the propagation vector of light travel along three mutually perpendicular directions. It is the electric field of light that creates optical sensation in our eyes, in photographic cameras and in all other optical instruments. That is why electric field is known as **light vector**.

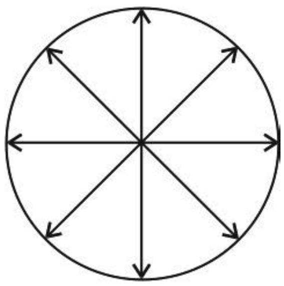

Fig. 2.14 *Direction of Propagation*

In case of propagation of light, electric field, being always perpendicular to the direction of propagation, vibrates in its own plane with all possible orientations. If we consider a circular cross-section perpendicular to the direction of propagation, the oscillating profile of the electric field in all possible directions will be similar to that in Figure (2.14). Here the propagation vector of light is perpendicular to the plane of the paper. If, by some means, all the oscillating directions except a single one of the electric field are cut off, the resultant light will be said to be **polarised light** and this phenomenon of eliminating all directions of electric field and retaining a single preferred direction of light vector perpendicular to the propagation vector is known as **polarisation**.

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Production of Polarised Light

Suppose, S is some source of unpolarised light (natural light). If we place a tourmaline crystal in the path of the unpolarised light in such a way that the optic axis of the crystal is perpendicular to the direction of propagation of the unpolarised light, then those vibrations of the electric field of unpolarised light which are parallel to the optic axis of the crystal, will pass through the crystal and the resultant light is polarised (containing only one preferred direction of electric field). If a second tourmaline crystal is placed in the path of polarised light in such a manner that both the crystals have their optic axes parallel, then polarised light will be visible beyond the second crystal also.

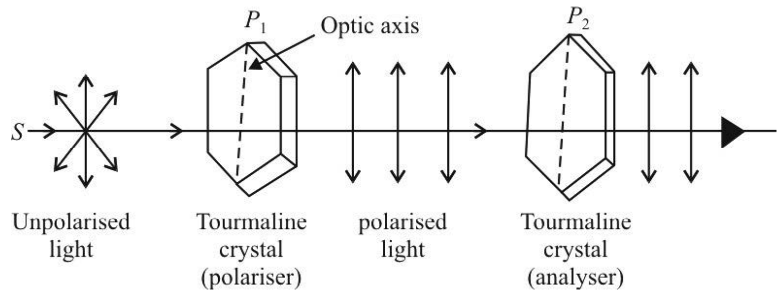

Fig. 2.15 *Propagation of Polarized Light*

Now, if the optic axis of the second crystal is rotated around the direction of propagation of polarised light coming from the first crystal, then the intensity of palarised light beyond the second crystal goes on decreasing and ultimately becomes zero when the optic axis of the 2nd crystal is perpendicular to that of the first crystal. The first tourmaline crystal is called polariser and the 2nd one is called analyser.

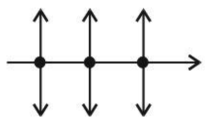

Fig. 2.16 *Unpolarised Light*

Fig. 2.17 *Polarised Light (Vibrations are Perpendicular to the Plane of Paper)*

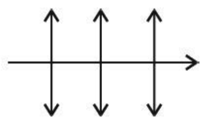

Fig. 2.18 *Polarised Light Vibrations are in the Plane of Paper)*

Planes of Vibration and Polarisation

The imaginary plane which contains the vibrations of electric field of a polarised light is called plane of vibration. An imaginary plane perpendicular to the plane of vibration is called plane of polarisation.

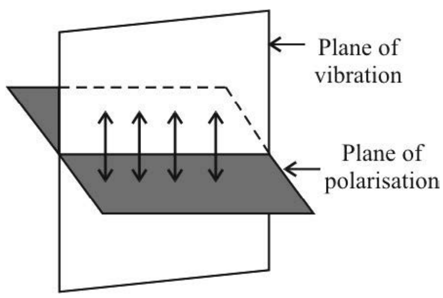

Fig. 2.19 *Plane of Vibration and Polarization*

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Qualitative Discussion on Types of Polarised Light

In general, there are three types of polarised light. These are:

Plane Polarised Light: The direction of electric field remains fixed, but its magnitude changes during vibration.

Circularly Polarised Light: The magnitude of electric field remains fixed, but its direction changes during vibration.

Elliptically Polarised Light: Both the magnitude and direction change continuously during vibration.

Let us consider two mutually orthogonal light vectors given by,

$$E_x = E_1 \sin \omega t \quad ...(2.52)$$

$$E_y = E_2 \sin (\omega t + \phi) \quad ...(2.53)$$

[$\phi \rightarrow$ Phase Difference between them]

$$\Rightarrow \quad \frac{E_x}{E_1} = \sin \omega t \quad ...(2.54)$$

$$\Rightarrow \quad \frac{E_y}{E_2} = \sin \omega t \cos \phi + \cos \omega t \sin \phi \quad ...(2.55)$$

Using Equations (2.54) in (2.55), we get

$$\frac{E_y}{E_2} = \frac{E_x}{E_1}\cos\phi + \sqrt{1-\frac{E_x^2}{E_1^2}}\sin\phi$$

$$\text{or,} \quad \left(\frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\phi\right)^2 = \left(1-\frac{E_x^2}{E_1^2}\right)\sin^2\phi$$

$$\text{or} \quad \left(\frac{E_y}{E_2}\right)^2 + \left(\frac{E_x}{E_1}\right)^2\cos^2\phi - 2\frac{E_y}{E_2}.\frac{E_x}{E_1}\cos\phi = \left(1-\frac{E_x^2}{E_1^2}\right)\sin^2\phi$$

$$\text{or,} \quad \left(\frac{E_y}{E_2}\right)^2 + \left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_y}{E_2}\right)\left(\frac{E_x}{E_1}\right)\cos\phi = \sin^2\phi \quad ...(2.56)$$

This represents the general equation for an ellipse. If Equation (2.56) is held, the associated state of polarisation is called **Elliptical polarisation in general**.

For $\phi = n\pi$, $n = 0, 1, 2, ...$

$$\text{Equation (2.56)} \Rightarrow \left(\frac{E_y}{E_2}\right)^2 + \left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_y}{E_1}\right)\left(\frac{E_x}{E_1}\right) = 0$$

[$\because$ $\sin(n\pi) = 0$, $\cos(n\pi) = 1$]

$$\Rightarrow \quad \left(\frac{E_y}{E_2} - \frac{E_x}{E_1}\right)^2 = 0$$

$$\Rightarrow \quad E_y = \left(\frac{E_2}{E_1}\right)E_x \quad ...(2.57)$$

Equation (2.57) represents a straight line with slope $\left(\frac{E_2}{E_1}\right)$. If Equation (2.57) is held, the associated state of polarisation is called **plane polarisation**.

For $\phi = (2n + 1)\frac{\pi}{2}$, $n = 0, 1, 2, ...$

$$\text{Equation (2.56)} \Rightarrow \left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1 \quad \text{...(2.58)}$$

$\Rightarrow$ This represents **Elliptical state of polarisation**.

For $E_1 = E_2 = E$, and $\phi = (2n + 1)\frac{\pi}{2}$, $n = 0, 1, 2, ...$

$$(E_x)^2 + (E_y)^2 = E^2 \quad \text{...(2.59)}$$

$\Rightarrow$ This represents **Circular state of polarisation**.

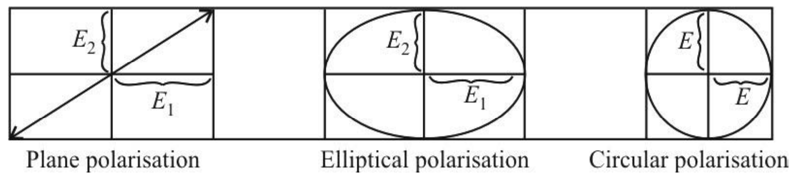

Fig. 2.20 *State of Polarisation*

Thus, it is clear that elliptical polarisation is the most general type of polarisation. Other types of polarisation are merely special cases of elliptical polarisation.

Polarisation through Reflection and Brewster's Law

Brewster's law states that the relationship for light waves as the maximum polarization (vibration in one plane only) of a ray of light may be achieved by letting the ray fall on a surface of a transparent medium in such a way that the refracted ray makes an angle of 90° with the reflected ray. The law is named after a Scottish physicist, Sir David Brewster, who first proposed it in 1811. Brewster's angle, also known as the polarization angle, is an angle of incidence at which light with a particular polarization is perfectly transmitted through a transparent dielectric surface, with no reflection. When unpolarized light is incident at this angle, the light that is reflected from the surface is therefore perfectly polarized.

Suppose *AB* is the interface between two dielectric media having refractive indices μ_1 and μ_2, respectively. It is found that if an ordinary (unpolarised) ray of light is incident on the interface at a particular angle θ_B such that the reflected and refracted rays are mutually perpendicular, then the reflected ray will be plane polarised with plane of vibration perpendicular to the plane of incidence.

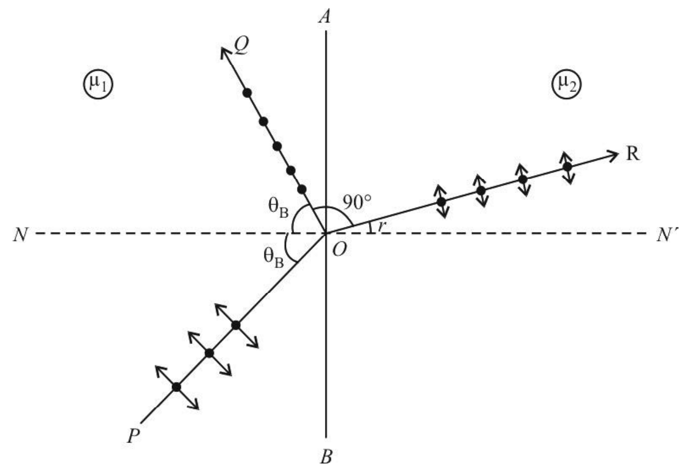

Fig. 2.21 *Polarisation through Reflection*

Here the angle of incidence is called Brewster's angle after the name of physicist Sir David Brewster who first studied this phenomenon.

From the Figure (2.21), it is clear that

$$\mu_1 \sin\theta_B = \mu_2 \sin r \qquad \text{(Snell's Law)}$$

or,
$$\frac{\mu_2}{\mu_1} = \frac{\sin\theta_B}{\sin r} = \mu \text{ (say)}$$

Again, $\theta_B + 90° - r = 180° \Rightarrow r = 90° - \theta_B$

$$\Rightarrow \frac{\sin\theta_B}{\sin(90° - \theta_B)} = \mu = \frac{\sin\theta_B}{\cos\theta_B} = \tan\theta_B$$

$$\Rightarrow \tan\theta_B = \mu \qquad \text{...(2.60)}$$

This is Brewster's Law.

Brewster's Law

When unpolarised light is incident on the interface between two dielectric media at the Brewster's angle θ_B ($= \tan^{-1}\mu$), the reflected light is plane polarised with the plane of vibration perpendicular to the plane of incidence and the angle between reflected and refracted rays is 90°.

Malus's Law

When light (unpolarised) is incident on a polariser, the transmitted light is plane polarised with its plane of polarisation perpendicular to the optic axis of the polariser. If this transmitted plane polarised light is allowed to pass through an analyser, the intensity of the transmitted ray through the analyser varies with the angle between the plane of polariser and that of the analyser. Malus studied this problem and stated that the variation of intensity in terms of a law known as Malus's law.

It states that the intensity of the polarised light transmitted through the analyser varies as the square of cosine of the angle between the plane of transmission of the analyser and the plane of polariser.

Mathematically, intensity of the polarised light transmitted through the analyser

$$I = A^2\cos^2\theta$$

or,
$$I = I_o\cos^2\theta \qquad \text{...(2.61)}$$

$A \rightarrow$ Amplitude of the light vector of plane polarised light

$\theta \rightarrow$ Angle between the transmission plane of analyser and the plane of polariser

If $\theta = \frac{\pi}{2}$, $I = 0$

This shows that when the two planes are at right angle to each other, the intensity of the transmitted light is zero.

Double Refraction

If a beam of unpolarised light is allowed to pass through an anisotropic crystal (Calcite or Quartz), it splits up into two refracted beams instead of one. This phenomenon is called **Double Refraction** or **Birefringence**. If a ray of light *SA* from a point source is incident on a calcite crystal making an angle of incidence '*i*', it is refracted along two paths *AB* and *AC* making angles of refraction r_1 and r_2, respectively. These rays emerge out as *BO* and *CE* parallel to each other as shown

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in Figure (2.22). If the calcite crystal is rotated about the incident beam as axis, *O*-ray remains fixed but *E*-ray rotates round *O*-ray. Here the *O*-ray obeys ordinary laws of refraction and hence it is called **Ordinary ray.** The *E*-ray does not obey the ordinary laws of refraction. It is called **Extra-ordinary ray**. It is noteworthy that both *O*-ray and *E*-ray are plane polarised with the vibrations of *O*-ray are perpendicular and those of the *E*-ray are parallel to the principal section of the calcite crystal. For *O*-ray, $\left(\frac{\sin i}{\sin r_1}\right) \Rightarrow$ Constant, for *E*-ray $\left(\frac{\sin i}{\sin r_2}\right) \Rightarrow$ Function of '*i*'.

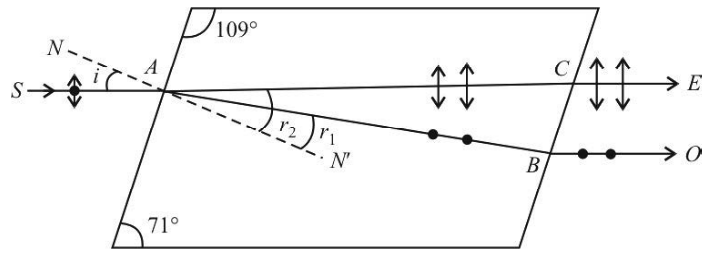


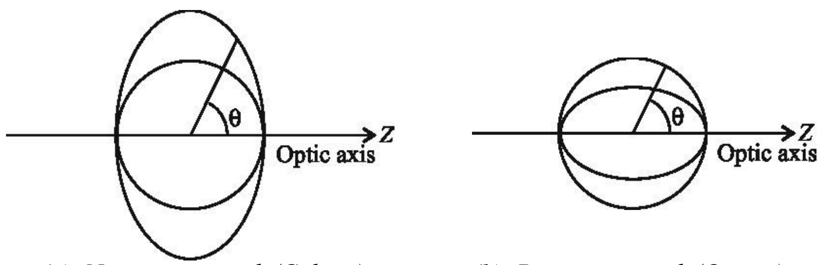
Fig. 2.22 *Principal Section of Calcite ($CaCO_3$)*

If we place a **calcite** crystal in front of a point marked on a paper, we will, in general, see two images of the point. The appearence of the two images is due to the phenomenon of **double refraction**. When a light ray entering such a crystal will split up into two rays and for a crystal like calcite, one of the rays will follow the Snell's law of refraction and the other will not. The former is termed as **ordinary ray** (*O*-ray) and the latter as **extra-ordinary ray** (*E*-ray). The velocity of the *O*-ray is same in all directions whereas the velocity of *E*-ray is different in different directions. But, along a particular direction the velocity of *O*-ray and *E*-ray is same; this direction is known as the **optic axis** of the crystal.

Indeed, the wavefront due to the ordinary ray is spherical whereas the wavefront due to the extra-ordinary ray is an ellipsoid in nature. If the ellipsoid of revolution lies outside the sphere (i.e., the velocity of *E*-ray is greater than the velocity of *O*-ray everywhere except optic axis), then the crystal is known as a **negative crystal** (For example: Calcite).

On the other hand, if the ellipsoid of revolution lies inside the sphere (i.e., the velocity of *E*-ray is less than the *O*-ray except optic axis), then the crystal is known as a **positive crystal** (For example: Quartz).

The images formed by the *O*-ray and *E*-ray are given in Figure (2.23(a)) and Figure (2.23(b)) for negative and positive crystal, respectively.



(a) *Negative crystal (Calcite)* ***(b)*** *Positive crystal (Quartz)*

Fig. 2.23 *Images by O-Ray and E-Ray*

NOTES

Nicol Prism

It is an optical device made from a calcite crystal and is used to produce and analyse the plane polarised light of extra-ordinary nature.

When unpolarised light is incident on a Nicol prism, two plane polarised lights are produced by double refraction, the *O*-ray, thus produced, is eliminated by total internal reflection and the *E*-ray is transmitted through the crystal.

Construction: The Nicol prism is constructed from a calcite crystal whose length is nearly three times its width. The crystal is cut through *AC* and the cut faces are joined together by some adhesive material canadabalsam. The angles of principal section are 112° and 68°. In calcite medium, the refractive indices for *O*-ray and *E*-ray are $\mu_o = 1.66$, $\mu_e = 1.49$, respectively. The refractive index of canadabalsam is $\mu_{cd} = 1.55$.

Polarising Action: Nicol prism can be used as a polariser to produced plane polarised (*E*-ray) light from unpolarised light. The unpolarised light after entering at principal section of a Nicol prism is split up into *E*-ray and *O*-ray by the technique of double refraction. Since $\mu_o > \mu_{cd}$, the *O*-ray suffers total internal reflection at Calcite-Canadabalsam Interface and thus it is eliminated. For $\mu_e < \mu_{cd}$, the extra-ordinary *E*-ray is transmitted through the end face *CD* of the Nicol prism. Thus by using a Nicol prism we can get plane polarised *E*-rays from unpolarised light.

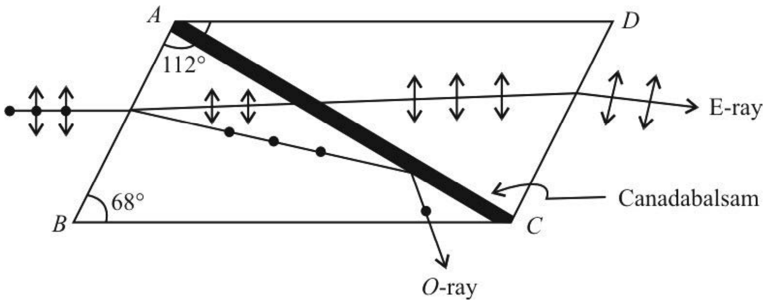


Fig. 2.24 *Polarising Action*

Analysing Action: Nicol prism can be used to analyse a polarised light. If two Nicol prisms are placed one-by-one such that their principal sections are parallel to one another, then the first prism is used as polariser and second one as analyser.

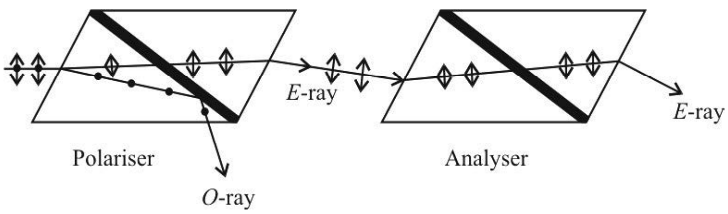


Fig. 2.25 *Analysing Action*

If we rotate the principal section of the analysing Nicol prism with respect to that of the polariser, the intensity of the *E*-ray emitted from the analyser decreases and it becomes zero when the principal planes of the two Nicol prisms are perpendicular to one another.

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Polaroids

Due to the non-availability of larger sizes of polarising crystals (Tourmaline, Nicol prisms, etc.) it is not possible to obtain larger cross-section of plane polarised light. To obtain plane polarised light with larger cross-sections, polaroids are used.

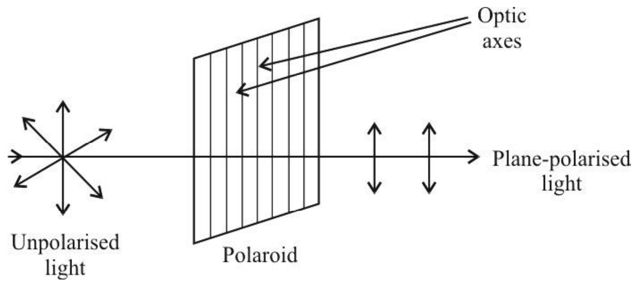

Fig. 2.26 *Polarids*

The materials used to produce polaroids is basically Iodoquinine Sulphate which is also known as Herapathite. These look like tiny needles. This is mixed in Nitrocellulose solution. The solution is placed between two glass plates to produce polaroids. If a polaroid is placed in the path of unpolarised light, plane polarised light is produced. In this case, the tiny needles behave as parallel optic axes.

Polaroids are two types: (*i*) H-Polaroid and (*ii*) K-Polaroid.

H-Polaroid: It is prepared by using PolyVinyl Alcohol (PVA) with Iodine dopant.

K-Polaroid: It is prepared by using the same material PVA heated in the presence of dehydrating agent, such as HCl.

Applications of Polaroid

Polaroids are used in Liquid Crystal Display (LCD), sunglasses, optical microscopes, etc., but the major use of these is in production of polarised light of large cross-section.

Polarisation by Absorption

A number of crystalline materials absorb more light in one incident plane than another, so that light progressing through the material become more and more polarised as they proceed. This anisotropy in absorption is called *dichroism*. There are several naturally occurring dichroic materials, and the commercial material polaroid also polarises by selective absorption.

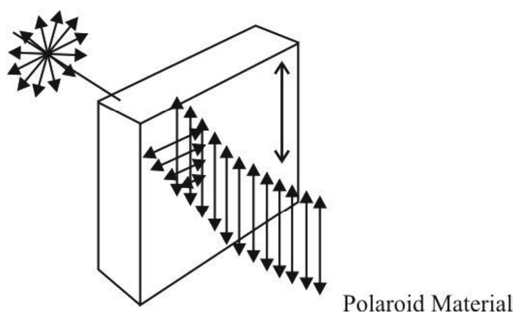

Fig. 2.27 *Absorption*

Polariod is the trade name for the most commonly used dichroic material. It selectively absorbs light from one plane, typically transmitting less than 1% through a sheet of polariod. It may transmit more than 80% of light in the perpendicular

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plane. The word 'polaroid' usually refers to polaroid H-sheet, which is a sheet of iodine-impregnated polyvinyl alcohol. A sheet of polyvinyl alcohol is heated and stretched in one direction while softened, which has the effect of aligning the long polymeric molecules in the direction of stretch. When dipped in iodine, the iodine atoms attach themselves to the aligned chains. The iodine atoms provide electrons which can move easily along the aligned chains, but not perpendicular to them. Light waves with electric fields parallel to these chains are strongly absorbed because of the dissipative effects of the electron motion in the chains. The direction perpendicular to the polyvinyl alcohol chains is the 'pass' direction since the electrons cannot move freely to absorb energy.

Polaroid Sunglasses

The polaroid material used in sunglasses makes use of dichroism, or selective absorption, to achieve polarisation.

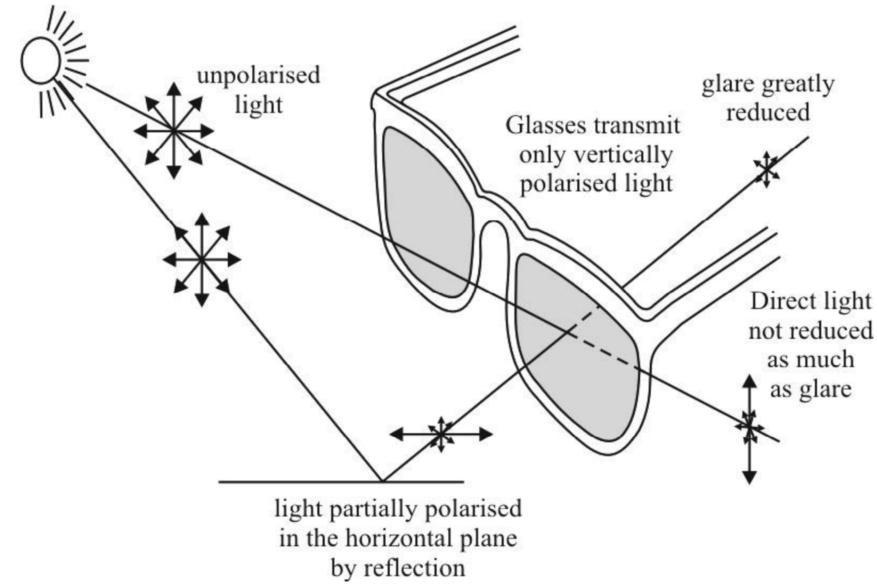

Fig. 2.28 *Polaroid Sunglasses*

Retardation Plates (Half and Quarter Wave Plates)

A double refracting crystalline plate cut parallel to its optic axis with the refracting faces of particular phase and path difference between ordinary ray (*O*-ray) and extraordinary ray (*E*-ray) is termed as **retardation plate**. If '*d*' be the thickness of the crystalline plate, μ_o and μ_e be the refractive indices of the *O*-ray and *E*-ray respectively, then the path difference by the plate is given by,

$$(\mu_o - \mu_e) \bullet d \qquad ...(2.62)$$

And the corresponding phase difference is given by,

$$\phi = \frac{2\pi}{\lambda}(\mu_o - \mu_e) \bullet d \qquad ...(2.63)$$

If the value of thickness '*d*' of the plate is such that the path difference is $\frac{\lambda}{4}$ or a phase difference is $\frac{\pi}{2}$, then the plate is called **Quarter Wave Plate**. Thus the thickness of a quarter wave plate is given by,

$$(\mu_o - \mu_e)\bullet d = \frac{\lambda}{4}$$

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$$\Rightarrow \quad d = \frac{\lambda}{4(\mu_o - \mu_d)} \quad ...(2.64)$$

If the value of thickness '*d*' of the plate is such that the path difference is $\frac{\lambda}{2}$ or a phase difference is π, then the plate is called **Half Wave Plate**. Thus the thickness of a half wave plate is given by,

$$(\mu_o - \mu_e)\bullet d = \frac{\lambda}{2}$$

$$\Rightarrow \quad d = \frac{\lambda}{2(\mu_o - \mu_e)} \quad ...(2.65)$$

Production of Elliptically and Circularly Polarised Light

For Elliptical Polarisation

An elliptically polarised light can be produced by the superimposing of two perpendicular coherent linear vibrations of light ray having different amplitudes with phase difference $\frac{\pi}{2}$. Let us consider two mutually perpendicular light rays given by (at $z = 0$),

$$E_x = E_1 \sin \omega t \quad ...(2.66)$$

And $$E_y = E_2 \sin\left(\omega t + \frac{\pi}{2}\right) \quad ...(2.67)$$

Hence, an elliptical vibration is given by,

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1 \quad \text{(by Equation 2.58)} \quad ...(2.68)$$

Such two linear vibrations [given in Equations (2.66) and (2.67)] can be produced by allowing a light ray of plane polarised beam to be incident perpendicularly on a quarter-wave plate with the direction of vibration making an angle α except $\frac{\pi}{4}$ radian (about $\frac{\pi}{6}$ radian) with the optic axis of the plate. The incident ray of amplitude '*E*' can be resolved into two components having amplitude $E_1 = E \cos\theta$ along the optic axis formed *E*-ray and another having amplitude $E_2 = E \sin\theta$ normal to the optic axis formed *O*-ray. Thus, these two vibrations (at $z = 0$) can be represented by,

$$E_x = E \cos\theta \sin \omega t + ...$$

And $$E_y = E \sin\theta \cos \omega t + ...$$

On passing through the quarter wave plate having path difference $\frac{\lambda}{4}$, relative phase difference $\frac{\pi}{2}$ will be maintained between the two vibrations. Thus, as a result an elliptically polarised emergent light will be produced as we discussed above.

Circular Polarisation

NOTES

A circularly polarised light can be produced by the superimposing of two perpendicular coherent linear vibrations of light ray having same amplitudes with phase difference $\frac{\pi}{2}$. Let us consider two mutually perpendicular light rays given by (at $z = 0$),

$$E_x = E \sin \omega t \qquad ...(2.69)$$

And $$E_y = E \sin\left(\omega t + \frac{\pi}{2}\right) \qquad ...(2.70)$$

Hence, a circular vibration is given by

$$E_x^2 + E_y^2 = E^2 \qquad \text{(By Equation 2.59)} \qquad ...(2.71)$$

Such two linear vibrations [given in Equations (2.70) and (2.71)] can be produced by allowing a light ray of plane polarised beam to be incident perpendicularly on a quarter wave plate with the direction of vibration making an angle $\frac{\pi}{4}$ radian with the optic axis of the plate. The incident ray of amplitude 'E' can be resolved into two components having amplitude $E \cos \frac{\pi}{4} = \frac{E}{\sqrt{2}}$ along the optic axis formed E-ray and another having amplitude $E \sin \frac{\pi}{4} = \frac{E}{\sqrt{2}}$ normal to the optic axis formed O-ray. Thus, these two light vibrations (at $z = 0$) can be represented by,

$$E_x = \frac{E}{\sqrt{2}} \sin \omega t$$

And $$E_y = \frac{E}{\sqrt{2}} \cos \omega t$$

On passing through the quarter wave plate having path difference $\frac{\lambda}{4}$, relative phase difference $\frac{\pi}{2}$ will be maintained between the two vibrations. Thus, as a result a circularly polarised emergent light will be produced as we discussed above.

Degree of Polarization

Principally, the Degree Of Polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized. A perfectly polarized wave has a DOP of 100%, whereas an unpolarized wave has a DOP of 0%. A wave which is partially polarized, and therefore can be represented by a superposition of a polarized and unpolarized component, will have a DOP somewhere in between 0 and 100%. DOP is calculated as the fraction of the total power that is carried by the polarized component of the wave.

DOP can be used to map the strain field in materials when considering the DOP of the photoluminescence. The polarization of the photoluminescence is related to the strain in a material by way of the given material's photo-elasticity tensor.

DOP is also visualized using the Poincaré sphere representation of a polarized beam. In this representation, DOP is equal to the length of the vector measured from the center of the sphere.

2.10 SCATTERING: RAYLEIGH AND DISPERSION OF PLANE E. M. WAVES

NOTES

Electromagnetic waves are one of the best known and most commonly encountered forms of radiation that undergo scattering. Scattering of light and radio waves (especially in radar) is particularly important. Major forms of elastic light scattering including the negligible energy transfer are Rayleigh scattering and Mie scattering. Light scattering is one of the two major physical processes that contribute to the visible appearance of most objects, the other being absorption. Surfaces described as white owe their appearance to multiple scattering of light by internal or surface inhomogeneities in the object, for example by the boundaries of transparent microscopic crystals that make up a stone or by the microscopic fibers in a sheet of paper. More generally, the gloss of the surface is determined by scattering. Highly scattering surfaces are described as being dull or having a matte finish, while the absence of surface scattering leads to a glossy appearance, as with polished metal or stone.

Spectral absorption, the selective absorption of certain colours, determines the colour of most objects with some modification by elastic scattering. The apparent blue colour of veins in skin is a common example where both spectral absorption and scattering play important and complex roles in the colouration. Light scattering can also create colour without absorption, often shades of blue, as with the sky (Rayleigh scattering), the human blue iris, etc.

Models of light scattering can be divided into three domains based on a dimensionless size parameter, α which is defined as:

$$\alpha = \pi D_p/\lambda,$$

Where πD_p is the circumference of a particle and λ is the wavelength of incident radiation. Based on the value of α, these domains are defined as:

$\alpha << 1$: Rayleigh scattering, small particle compared to wavelength of light.

$\alpha \approx 1$: Mie scattering, particle about the same size as wavelength of light, valid only for spheres.

$\alpha >> 1$: Geometric scattering, particle much larger than wavelength of light.

Rayleigh scattering is a process in which electromagnetic radiation (including light) is scattered by a small spherical volume of variant refractive indexes, such as a particle, bubble, droplet, or even a density fluctuation. This effect was first modeled successfully by Lord Rayleigh, from whom it gets its name. In order for Rayleigh's model to apply, the sphere must be much smaller in diameter than the wavelength (λ) of the scattered wave; typically the upper limit is taken to be about 1/10 the wavelength. In this size regime, the exact shape of the scattering center is usually not very significant and can often be treated as a sphere of equivalent volume. The inherent scattering that radiation undergoes passing through a pure gas is due to microscopic density fluctuations as the gas molecules move around, which are normally small enough in scale for

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Rayleigh's model to apply. This scattering mechanism is the primary cause of the blue colour of the Earth's sky on a clear day, as the shorter blue wavelengths of sunlight passing overhead are more strongly scattered than the longer red wavelengths according to Rayleigh's famous $1/\lambda^4$ relation. Along with absorption, such scattering is a major cause of the attenuation of radiation by the atmosphere. The degree of scattering varies as a function of the ratio of the particle diameter to the wavelength of the radiation, along with many other factors including polarization, angle, and coherence.

For larger diameters, the problem of electromagnetic scattering by spheres was first solved by Gustav Mie, and scattering by spheres larger than the Rayleigh range is therefore usually known as Mie scattering. In the Mie regime, the shape of the scattering center becomes much more significant and the theory only applies well to spheres and, with some modification, spheroids and ellipsoids. Closed-form solutions for scattering by certain other simple shapes exist, but no general closed-form solution is known for arbitrary shapes.

Both Mie and Rayleigh scattering are considered elastic scattering processes, in which the energy (and thus wavelength and frequency) of the light is not substantially changed. However, electromagnetic radiation scattered by moving scattering centers does undergo a Doppler shift, which can be detected and used to measure the velocity of the scattering center/s in forms of techniques, such as lidar and radar. This shift involves a slight change in energy.

At values of the ratio of particle diameter to wavelength more than about 10, the laws of geometric optics are mostly sufficient to describe the interaction of light with the particle, and at this point, the interaction is not usually described as scattering.

For modeling of scattering in cases where the Rayleigh and Mie models do not apply, such as irregularly shaped particles, there are many numerical methods that can be used. The most common are finite-element methods which solve Maxwell's equations to find the distribution of the scattered electromagnetic field.

Scattering of Electromagnetic Waves by Particles

Interaction between electromagnetic waves and particles produce unique scattering patterns that are wavelength and particle size dependent. As electromagnetic waves propagate through matter they interact with particles and locally perturb the local electron distribution. This variation produces periodic charge separation within the particle causing oscillation of the induced local dipole moment, this periodic acceleration acts as a source of electromagnetic radiation thus causing scattering. The majority of the scattered wave oscillates at the same frequency as the incident wave and is termed elastic scattering. Interaction with the incident beam may also lead to absorption in the form of thermal energy. The combination of scattering and absorption attenuate the incident beam leading to extinction.

Scattering of electromagnetic waves by particles can be explained using the following two theoretical frameworks:

1. Rayleigh scattering that is applicable to small, dielectric, non-absorbing spherical particles.

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2. Mie scattering that provides a general solution to scattering independent of particle size. Mie scattering theory provide a generalized approach, has no particle size limitations and converges to the limit of geometric optics at large particle sizes.

Consequently Mie scattering theory can be used to describe most scattering by spherical particles, including Rayleigh scattering, but due to the complexity of implementation, Rayleigh scattering theory is often preferred.

Rayleigh scattering is strongly dependent upon the size of the particle and the wavelength of the illuminating radiation. The intensity of the Rayleigh scattered radiation increases rapidly as the ratio of particle size to wavelength increases and is identical in the forward and reverse directions. The Rayleigh scattering model breaks down when the particle size becomes larger than approximately 10% of the wavelength of the incident radiation at which point Mie theory must be applied. The Mie solution is obtained through an analytical solution of Maxwell's equations for the scattering of electromagnetic radiation by spherical particles in terms of infinite series rather than a simple mathematical expression.

Mie scattering differs from Rayleigh scattering in several respects. It is roughly independent of wavelength and it is larger in the forward direction than in the reverse direction, as shown in Figure (2.29). The greater the particle size, the more of the light is scattered in the forward direction. In addition to explaining many atmospheric effects of light scattering, applications of Mie scattering include environmental areas, such as dust particles in the atmosphere and oil droplet in water, etc. Figure (2.29) illustrates the electric field due to Mie scattering of incident wave in x direction showing enhanced scattering in forward direction.

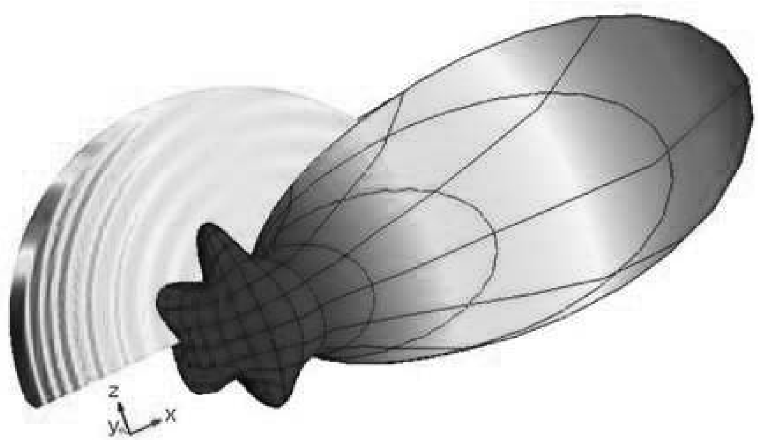

Fig. 2.29 *Electric Field due to Mie Scattering*

Analysis of Mie Scattering

It is a complex process to explain the Mie scattering by a particle or object and requires solution of Maxwell's equations to represent the incident, scattered and internal fields. These are not simple mathematical expressions and take the form of infinite series expansion of vector spherical harmonics that permits the cross sections, efficiency factors and distributions of intensity to be predicted. Additionally, the influence of particle geometry, incident of the incident wave and the particle's material properties can be examined.

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In **electromagnetic wave** scattering problems, the total wave decomposes into the incident and scattered wave components:

$$\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{sca}$$

$$\mathbf{H} = \mathbf{H}_{inc} + \mathbf{H}_{sca}$$

Maxwell's wave equation can be solved with respect to scattered electric field as:

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \mathbf{E}_{sca} \right) - k_0^2 \left(\varepsilon_r - j \frac{\sigma}{\omega \varepsilon_0} \right) \mathbf{E}_{sca} = 0$$

The scattered magnetic field is typically calculated from Faraday's law as:

$$\mathbf{H}_{sca} = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}_{sca}$$

The time-average Poynting vector for time-harmonic fields gives the energy flux as:

$$\mathcal{P} = \frac{1}{2} \mathrm{Re}\left[\mathbf{E} \times \mathbf{H}^*\right], \quad \left[W/m^2\right]$$

For an incident plane wave, the magnetic field is related to the electric field and is represented by:

$$\mathbf{H}_{inc} = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}_{inc}$$

Where $\hat{\mathbf{k}}$ is direction of the incident wave propagation, $\eta = (\mu/\varepsilon)^{1/2}$ is the characteristic impedance, ε is permittivity and μ is permeability of ambient medium. Hence, incident energy flux is calculated as,

$$\mathcal{P}_{inc} = \frac{1}{2\eta} \left| \mathbf{E}_{inc} \right|^2 \hat{\mathbf{k}}$$

Significant physical quantities can be obtained from the scattered fields. One of these is the cross section, which can be defined as the net rate at which electromagnetic energy (*W*) crosses the surface of an imaginary sphere centered at the particle divided by the incident irradiation $(\mathcal{P}_{inc})$. To quantify the rate of the electromagnetic energy that is absorbed (W_{abs}) and scattered (W_{sca}) by the particle, the absorption (σ_{abs}), scattering (σ_{sca}) and extinction cross sections are defined as:

$$\sigma_{abs} = \frac{W_{abs}}{\mathcal{P}_{inc}}, \quad \sigma_{sca} = \frac{W_{sca}}{\mathcal{P}_{inc}}, \quad \sigma_{ext} = \sigma_{abs} + \sigma_{sca}$$

The total absorbed energy is derived by integrating the energy loss over the volume of the particle:

$$W_{abs} = \frac{1}{2} \iiint_{V_p} \text{Re}\left[(\sigma \mathbf{E} + j\omega \mathbf{D}) \cdot \mathbf{E}^* + j\omega \mathbf{B} \cdot \mathbf{H}^*\right] dV, \quad [W]$$

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The scattered energy is derived by integrating the Poynting vector over an imaginary sphere around the particle:

$$W_{sca} = \oiint_S \mathcal{P}_{sca} \cdot \mathbf{n} dS = \frac{1}{2} \oiint_S \text{Re}\left[\mathbf{E}_{sca} \times \mathbf{H}^*_{sca}\right] \cdot \mathbf{n} dS, \quad [W]$$

Where **n** is unit vector is considered normal to the imaginary surface *S*.

Due to the particulate nature of electromagnetic waves they also carry momentum $\mathcal{P}/c$ and exert a force on the particle, termed as the *radiation pressure* which can be obtained by integrating the Maxwell stress tensor over the surface of the sphere:

$$\sigma_{pr} = \sigma_{ext} - \langle \cos\theta \rangle \sigma_{sca}, \quad [m^2]$$

Where σ_{pr} is the *pressure cross-section*, and $\langle \cos \theta \rangle$ is the *asymmetry parameter*.

The radiation pressure cross section can be used to calculate force which the particle experiences in the incident direction:

$$\mathbf{F} = \frac{1}{c} \sigma_{pr} \mathcal{P}_{inc}, \quad [N]$$

The total time-averaged force **F** acting on a particle illuminated with light can also be calculated using surface integral of the time-averaged Maxwell's stress tensor $\hat{\mathbf{T}}$:

$$\mathbf{F} = \oiint_{S_p} \hat{\mathbf{T}} \cdot \mathbf{n} dS_p, \quad [N]$$

Where S_p is surface enclosing particle volume V_p and **n** is unit normal vector to surface S_p.

Scattering Parameters

Scattering parameters or S-parameters are the elements of a scattering matrix or S-matrix which define the electrical behaviour of linear electrical networks when undergoing various steady state stimuli by electrical signals. The parameters are used in different branches of electrical engineering, including electronics, communication systems design, and especially for microwave engineering.

The S-parameters are members of a family of similar parameters, other examples include Y-parameters, Z-parameters, H-parameters, T-parameters or ABCD-parameters. They differ from these, in the sense that S-parameters do not use open or short circuit conditions to characterize a linear electrical network; instead, matched loads are used. These terminations are easily used at high signal frequencies as compared to the open-circuit and short-circuit terminations. Moreover, the quantities are measured in terms of power.

Many electrical properties of networks of components (inductors, capacitors, resistors) may be expressed using S-parameters, such as gain, return loss, Voltage Standing Wave Ratio (VSWR), reflection coefficient and amplifier stability. The term 'Scattering' refers to the effect observed when a plane electromagnetic wave is incident on an obstruction or passes across dissimilar dielectric media. In the context of S-parameters, scattering refers to the way in which the traveling currents and voltages in a transmission line are affected when they meet a discontinuity caused by the insertion of a network into the transmission line. This is equivalent to the wave meeting an impedance differing from the line's characteristic impedance.

Although applicable at any frequency, S-parameters are mostly used for networks operating at Radio Frequency (RF) and microwave frequencies where signal power and energy considerations are more easily quantified than currents and voltages. S-parameters change with the measurement frequency, so frequency must be specified for any S-parameter measurements stated, in addition to the characteristic impedance or system impedance. S-parameters are readily represented in matrix form and obey the rules of matrix algebra.

Types of S-Parameters

The S-parameters are of following types.

Small Signal S-Parameters: By small signal, we mean that the signals have only linear effects on the network, small enough so that gain compression or other non-linear effects do not take place. For passive networks, the small signal act linearly at any power level.

Large Signal S-Parameters: In this case, the S-matrix may vary depending upon the input signal strength.

Mixed-Mode S-Parameters: It refers to a special case of analysing balanced circuits.

Pulsed S-Parameters: These are measured on power devices so that an accurate representation is captured before the device heats up.

Cold S-Parameters: By cold, we refer to active devices that are not powered up. This can be an individual device, or an amplifier, or module, or anything active that is operated passively.

How S-Parameters Function

The scattering matrix is a mathematical construct that quantifies how Radio Frequency (RF) energy propagates through a multi-port network. The S-matrix is what allows us to accurately describe the properties of incredibly complicated networks as simple 'Black Boxes'. For an RF signal incident on one port, some fraction of that signal gets reflected back out of the incident port, some of it enters *into* the incident port and then exits at or ***scatters*** to some or all of the other ports, perhaps being amplified or attenuated. What is left of that incident power disappears as heat or even electromagnetic radiation. The S-matrix for an N-port contains N^2 coefficients (S-parameters), each one representing a possible input-output path.

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S-parameters are complex numbers, having real and imaginary parts or magnitude and phase parts, because both the magnitude and phase of the incident signal are changed by the network. S-parameters are defined for a given frequency and system impedance, and vary as a function of frequency for any non-ideal network.

Additionally, the S-parameters are usually displayed in a matrix format with the number of rows and columns equal to the number of ports. For the S-parameter, S_{ij} the 'j' subscript stands for the port that is the input port and the 'i' subscript is for the output port. Thus S_{11} refers to the ratio of the amplitude of the signal that reflects from port one to the amplitude of the signal incident on port one. Parameters along the diagonal of the S-matrix are referred to as reflection coefficients because they only refer to what happens at a single port, while off-diagonal S-parameters are referred to as transmission coefficients, because they refer to what happens at one port when it is excited by a signal incident at another port. Following are the examples of S-matrices for one, two and three-port networks:

$$\begin{pmatrix} S_{11} \end{pmatrix} \quad (one-port)$$

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (two-port)$$

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \quad (three-port)$$

Etc.

Remember that each S-parameter is a complex number, so if actual data has to be presented in matrix format, then a magnitude and phase angle has to be presented for each S_{ij}. The input and output reflection coefficients of networks, such as S_{11} and S_{22} can be plotted on the Smith chart while the transmission coefficients (S_{21} and S_{12}) are generally not plotted on the Smith chart.

S-Parameters for Network Ports

S-parameters describe the response of an N-port network to signal(s) incident to any or all of the ports. The first number in the subscript refers to the responding port, while the second number refers to the incident port. Thus S_{21} means the response at port 2 due to a signal at port 1. The most common 'N-port' networks in microwaves are one-port and two-port networks.

Consider a two-port network. The signal at a port, say port 1, can be thought of as the superposition to two waves traveling in opposite directions. By convention each port is represented as two nodes so as to give a name and value to these opposite direction waves. The variable a_i represents a wave incident to port i and the variable b_j represent a wave reflected from port j. The magnitude of the a_i and b_j variables can be thought of as voltage-like variables, normalized using a specified reference impedance. This is very convenient since the square of these magnitudes are then equal to the power level of the waves. Remember, S-parameters can be used if the value of the reference impedance (frequently called Z_0) is known (Refer Figure (2.30)).

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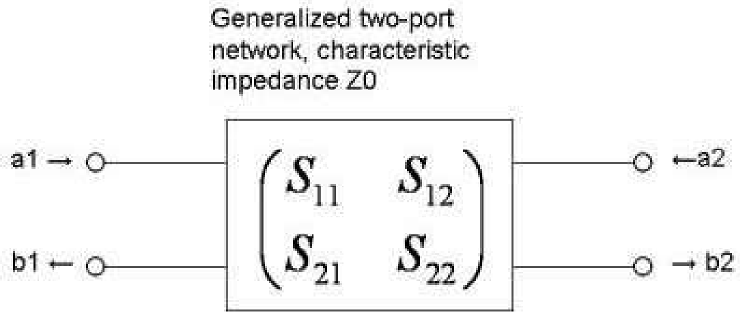

Fig. 2.30 *S-Parameters for Two-Port Network*

If we assume that each port is terminated in the reference impedance Z_0, we can define the four S-parameters of the 2-port as follows:

$$S_{11} = b1/a1$$

$$S_{12} = b1/a2$$

$$S_{21} = b2/a1$$

$$S_{22} = b2/a2$$

The above equations for S_{11} and S_{21} are derived from network analysis or measurements by setting the value of the incident signal $a_2 = 0$ and solving for the above S-parameter ratios as a function of a_1. Similarly, S_{12} and S_{22} are derived by setting the value of $a_1 = 0$ and solving for the other ratios.

The subscript precisely follows the parameters in the ratio, $S_{11}=b_1/a_1$, etc. The matrix algebraic representation of 2-port S-parameters is:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

In order to measure S_{11}, a signal at port one is inserted to measure its reflected signal. In this case, no signal is injected into Port 2, so a2 = 0; for almost all laboratory S-parameter measurements, only one signal is inserted at a time. To measure S_{21}, a signal at Port 1 is inserted to measure the resulting signal power exiting Port 2. For S_{12} a signal is inserted into Port 2 to measure the signal power leaving Port 1, and for S_{22} a signal is inserted at Port 2 to measure its reflected signal.

All the ‘a’ and ‘b’ measurements can be complex numbers, hence for complex S-parameters these complex numbers are sometimes called vectors, therefore termed as the Vector Network Analyzers (VNA).

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Polarization of Scattered Light

Polarization also occurs when light is scattered while traveling through a medium. When light strikes the atoms of a material, it will often set the electrons of those atoms into vibration. The vibrating electrons then produce their own electromagnetic wave that is radiated outward in all directions. This newly generated wave strikes neighboring atoms, forcing their electrons into vibrations at the same original frequency. These vibrating electrons produce another electromagnetic wave that is once more radiated outward in all directions. This absorption and reemission of light waves causes the light to be scattered about the medium. This scattered light is partially polarized. Polarization by scattering is observed as light passes through our atmosphere. The scattered light often produces a glare in the skies. Photographers know that this partial polarization of scattered light leads to photographs characterized by a *washed-out* sky. The problem can easily be corrected by the use of a Polaroid filter. As the filter is rotated, the partially polarized light is blocked and the glare is reduced.

How it Works

For example, when the unpolarized white light from a slide projector enters a fish tank of very slightly milky water. Some of the electromagnetic waves impinge on the colloidal particles and molecules in the water, are absorbed and re-radiated. The horizontal component of the polarization decreases as $\cos^2\theta$, where θ is the scattering angle (Refer Figure (2.31)) The maximum scattered intensity is perpendicular to the plane of oscillation of the molecule, where it is also totally plane polarized, $\theta = 90°$. At other angles the light is partially plane polarized.

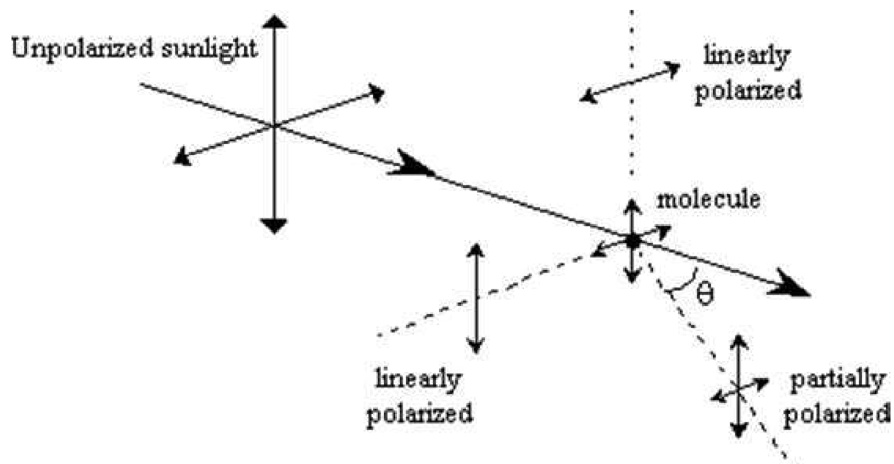

Fig. 2.31 *Polarization by Scattering*

This can be observed on the tank at right angles to the initial direction of propagation of the light. A mirror angled over the tank allows to view the phenomenon of scattered light emerging from two surfaces perpendicular to each other (Refer Figure (2.32)). By placing a Polaroid sheet between the projector and the tank with its polarizing axis horizontally, the scattered light from the side of the fish tank is blocked, whereas that from the top of the tank remains unaffected. Rotating the Polaroid 90° blocks the light from the top of the tank, but now the scattered light from the side of the tank reappears. Alternatively, let the scattering process polarize an unpolarized beam from the slide projector and let the Polaroid sheet be the analyzer, as shown in Figure (2.32).

Rayleigh scattering has a wavelength dependence of $1/(\lambda^4)$, so it affects blue light much more strongly as compared to red. By adding milk to the tank, the scattering can be increased because the milky water begins to develop a bluish tint and the un-scattered beam reddish.

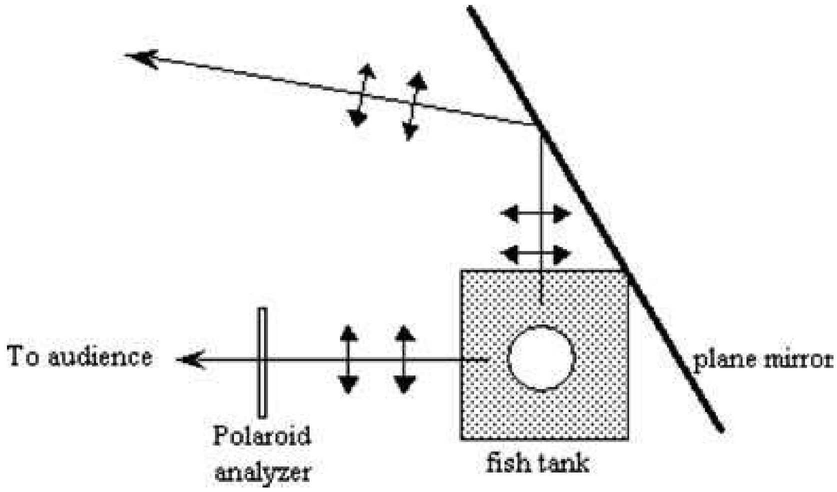

Fig. 2.32 *Fish Tank and Mirror Arrangement*

2.11 SCATTERING: THOMSON AND DISPERSION OF PLANE E. M. WAVES

Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle, as described by classical electromagnetism. It is the low-energy limit of Compton scattering: the particle's kinetic energy and photon frequency do not change as a result of the scattering. This limit is valid as long as the photon energy is much smaller than the mass energy of the particle: $\nu \ll mc^2/h$, or equivalently, if the wavelength of the light is much greater than the Compton wavelength of the particle (e.g., for electrons, longer wavelengths than hard x-rays).

In the low-energy limit, the electric field of the incident wave (photon) accelerates the charged particle, causing it, in turn, to emit radiation at the same frequency as the incident wave, and thus the wave is scattered. Thomson scattering is an important phenomenon in plasma physics and was first explained by the physicist J. J. Thomson. So long as, the motion of the particle is non-relativistic (i.e., its speed is much less than the speed of light), the main cause of the acceleration of the particle will be due to the electric field component of the incident wave. In a first approximation, the influence of the magnetic field can be neglected. The particle will move in the direction of the oscillating electric field, resulting in electromagnetic dipole radiation. The moving particle radiates most strongly in a direction perpendicular to its acceleration and that radiation will be polarized along the direction of its motion. Therefore, depending on where an observer is located, the light scattered from a small volume element may appear to be more or less polarized.

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The electric fields of the incoming and observed wave (i.e., the outgoing wave) can be divided up into those components lying in the plane of observation (formed by the incoming and observed waves) and those components perpendicular to that plane. Those components lying in the plane are referred to as 'Radial' and those perpendicular to the plane are 'Tangential'.

The scattering is best described by an emission coefficient which is defined as ε where ε dt dV dΩ dλ is the energy scattered by a volume element dV in time dt into solid angle dΩ between wavelengths λ and λ + dλ. From the point of view of an observer, there are two emission coefficients, μ_r corresponding to radially polarized light and μ_t corresponding to tangentially polarized light. For unpolarized incident light, these are given by:

$$\varepsilon_t = \frac{3}{16\pi}\sigma_t In$$
$$\varepsilon_r = \frac{3}{16\pi}\sigma_t In\cos^2\chi$$

Where n is the density of charged particles at the scattering point, I is incident flux (i.e., energy/time/area/wavelength) and σ_t is the Thomson cross section for the charged particle, defined below. The total energy radiated by a volume element dV in time dt between wavelengths λ and λ+dΩ is found by integrating the sum of the emission coefficients over all directions (solid angle):

$$\int \varepsilon\, d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\chi(\varepsilon_t + \varepsilon_r)\sin\chi = I\frac{3\sigma_t}{16\pi}n2\pi(2 + 2/3) = \sigma_t In$$

The Thomson differential cross section, related to the sum of the emissivity coefficients, is given by

$$\frac{d\sigma_t}{d\Omega} = \left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right)^2\frac{1+\cos^2\chi}{2}$$

expressed in SI units; q is the charge per particle, m the mass of particle, and μ_0 a constant, the permittivity of free space. To obtain an expression in cgs units, drop the factor of $4\pi\varepsilon_0$. Integrating over the solid angle, we obtain the Thomson cross section

$$\sigma_t = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right)^2 \quad \text{in SI units.}$$

The important feature is that the cross section is independent of photon frequency. The cross section is proportional by a simple numerical factor to the square of the classical radius of a point particle of mass m and charge q, namely

$$\sigma_t = \frac{8\pi}{3}r_e^2$$

Alternatively, this can be expressed in terms of λ_c, the Compton wavelength, and the fine structure constant:

$$\sigma_t = \frac{8\pi}{3}\left(\frac{\alpha\lambda_c}{2\pi}\right)^2$$

For an electron, the Thomson cross-section is numerically given by:

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$$\sigma_t = \frac{8\pi}{3}\left(\frac{\alpha\hbar c}{mc^2}\right)^2 = 6.6524587158\ldots \times 10^{-29}\ \mathrm{m}^2 = 66.524587158\ldots\ (\mathrm{fm})^2$$

2.12 ELEMENTS OF WAVE GUIDES

Depending upon the purpose for which waveguide is to be used and the frequency of the wave to be transmitted, there are many different structures of waveguide that include *parallel plate waveguide*, *rectangular waveguide*, *circular waveguide*, *optical fiber waveguide*, and *dielectric slab waveguide*. These waveguide structures are shown in Figure (2.33) and discussed in next sections. The wave propagates through these guided structures whose propagating medium is not a free space and they are no more uniform plane waves.

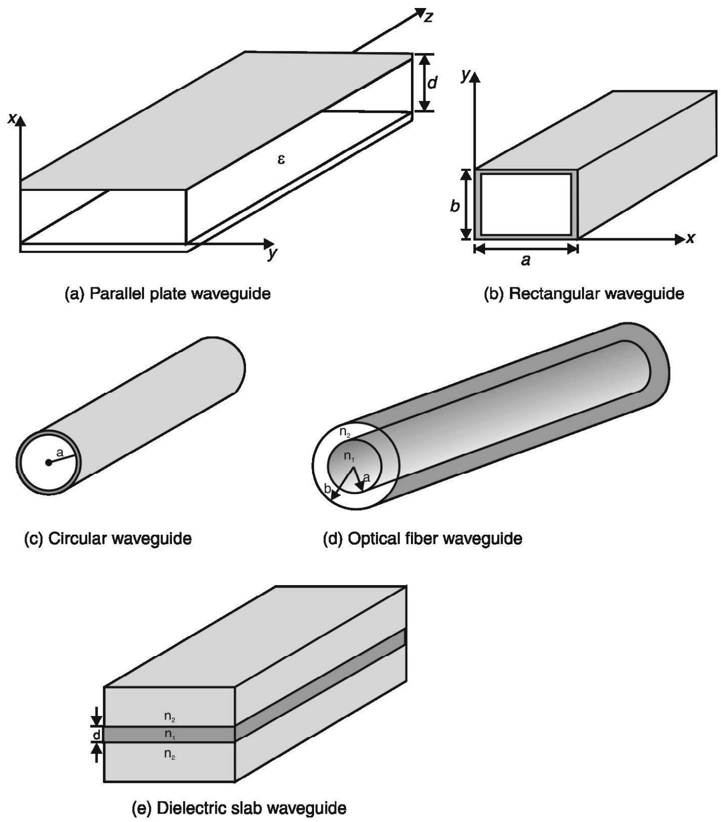

Fig. 2.33 *Different Forms of Waveguide*

Rectangular Waveguides

For the parallel plate waveguides the fields vary only in one transverse or orthogonal direction. Thus, the concept of parallel plate waveguide is simple, however, it is not of practical use due to infinite dimensions.

NOTES

Here, we will discuss about a rectangular waveguide which is the most commonly used among various waveguide structures. Consider a rectangular waveguide that is a hollow metallic device with its inner dimensions as *a* and *b* meters as shown in Figure (2.34). The walls of the waveguide are perfectly conducting having conductivity σ • and it is filled with a charge free lossless dielectric material having conductivity σ 0. The direction of wave propagation is assumed to be along *z*-direction.

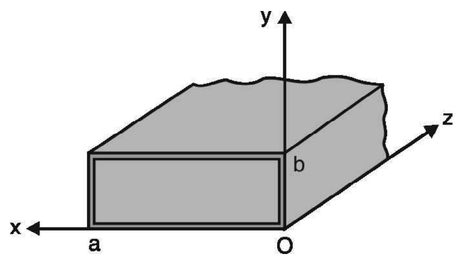

Fig. 2.34 *A Rectangular Waveguide*

The Maxwell's equation can be expressed in phasor form as:

$$\nabla \times \vec{H}_s = j\omega\varepsilon \vec{E}_s$$

Expanding the above equation, we get:

$$\nabla \times \vec{H}_s = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & H_{zs} \end{vmatrix} = \left(\frac{\partial H_{zs}}{\partial y} - \frac{\partial H_{ys}}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_{xs}}{\partial z} - \frac{\partial H_{zs}}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y}\right)\hat{a}_z$$

$$= j\omega\varepsilon E_{xs}\hat{a}_x + j\omega\varepsilon E_{ys}\hat{a}_y + j\omega\varepsilon E_{zs}\hat{a}_z$$

Equating the coefficients, we obtain:

$$\frac{\partial H_{zs}}{\partial y} - \frac{\partial H_{ys}}{\partial z} = j\omega\varepsilon E_{xs}$$

$$\frac{\partial H_{xs}}{\partial z} - \frac{\partial H_{zs}}{\partial x} = j\omega\varepsilon E_{ys}$$

$$\frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y} = j\omega\varepsilon E_{zs} \qquad \text{...(2.72)}$$

Similarly, using Maxwell's equation given as:

$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

And expanding and equating the coefficients, we get:

$$\frac{\partial E_{zs}}{\partial y} - \frac{\partial E_{ys}}{\partial z} = -j\omega\mu H_{xs}$$

NOTES

$$\frac{\partial E_{xs}}{\partial z} - \frac{\partial E_{zs}}{\partial x} = -j\omega\mu H_{ys}$$

$$\frac{\partial E_{ys}}{\partial x} - \frac{\partial E_{xs}}{\partial y} = -j\omega\mu H_{zs} \quad ...(2.73)$$

Now, at $z = 0$, the field components may be written as:

$$H_{ys} = H_{yo}e^{-\gamma_r z}$$

$$H_{xs} = H_{xo}e^{-\gamma_r z}$$

$$E_{ys} = E_{yo}e^{-\gamma_r z}$$

$$E_{xs} = E_{xo}e^{-\gamma_r z} \quad ...(2.74)$$

Here, subscript r is used to denote the propagation inside rectangular waveguide.

Substituting Equation (2.74) into Equations (2.72) and (2.73), we get:

$$\frac{\partial H_{zs}}{\partial y} + \gamma_r H_{ys} = j\omega\varepsilon E_{xs}$$

$$-\gamma_r H_{xs} - \frac{\partial H_{zs}}{\partial x} = j\omega\varepsilon E_{ys}$$

$$\frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y} = j\omega\varepsilon E_{zs} \quad ...(2.75)$$

And,

$$\frac{\partial E_{zs}}{\partial y} + \gamma_r E_{ys} = -j\omega\mu H_{xs}$$

$$-\gamma_r E_{xs} - \frac{\partial E_{zs}}{\partial x} = -j\omega\mu H_{ys}$$

$$\frac{\partial E_{ys}}{\partial x} - \frac{\partial E_{xs}}{\partial y} = -j\omega\mu H_{zs} \quad ...(2.76)$$

Rearranging and solving Equations (2.75) and (2.76) to get the field components as:

$$H_{ys} = -\frac{\gamma_r}{h_r^2}\frac{\partial H_{zs}}{\partial y} - \frac{j\omega\varepsilon}{h_r^2}\frac{\partial E_{zs}}{\partial x}$$

$$H_{xs} = -\frac{\gamma_r}{h_r^2}\frac{\partial H_{zs}}{\partial x} + \frac{j\omega\varepsilon}{h_r^2}\frac{\partial E_{zs}}{\partial y} \qquad ...(2.77)$$

NOTES

$$E_{xs} = -\frac{\gamma_r}{h_r^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h_r^2}\frac{\partial H_{zs}}{\partial y}$$

$$E_{ys} = -\frac{\gamma_r}{h_r^2}\frac{\partial E_{zs}}{\partial y} + j\omega\mu\frac{\partial H_{zs}}{\partial x}$$

Where $h_r^2 = \gamma_r^2 + \omega^2\mu\varepsilon$ ...(2.78)

From Equation (2.77), we can say that the - and y-components of electric and magnetic fields depend upon their respective z component. Thus, if E_{zs} and H_{zs} become zero, all the field components get vanished which implies that in a rectangular waveguide Transverse ElectroMagnetic (TEM) wave does not exist, it only supports transverse electric and transverse magnetic waves. To obtain the solutions for these two fields, let us consider the wave equations and substituting conductivity σ 0, we get:

$$^2\vec{E}_s = j\omega\mu\,(\sigma + j\omega\varepsilon)\,\vec{E}_s$$

$$^2\vec{E}_s + \omega^2\mu\varepsilon\,\vec{E}_s = 0$$

And, $^2\vec{H}_s = j\omega\mu\,(\sigma + j\omega\varepsilon)\,\vec{H}_s$

$$^2\vec{H}_s + \omega^2\mu\varepsilon\,\vec{H}_s = 0$$

Expanding these equations in Cartesian coordinates we get:

$$\frac{\partial^2\vec{E}_s}{\partial x^2} + \frac{\partial^2\vec{E}_s}{\partial y^2} + \frac{\partial^2\vec{E}_s}{\partial z^2} = -\omega^2\mu\varepsilon\vec{E}_s$$

$$\frac{\partial^2\vec{H}_s}{\partial x^2} + \frac{\partial^2\vec{H}_s}{\partial y^2} + \frac{\partial^2\vec{H}_s}{\partial z^2} = -\omega^2\mu\varepsilon\vec{H}_s$$

Here, the electric and magnetic fields can be split in their respective components. Thus, to obtain the $\vec{E}_s$ and $\vec{H}_s$ fields, we have to solve six equations. For example, for z-coordinate the above equations can be written as:

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} = -\omega^2\mu\varepsilon E_{zs}$$

$$\frac{\partial^2 H_{zs}}{\partial x^2} + \frac{\partial^2 H_{zs}}{\partial y^2} + \frac{\partial^2 H_{zs}}{\partial z^2} = -\omega^2\mu\varepsilon H_{zs} \qquad ...(2.79)$$

These equations are partial differential equations solved by using the method of product solution (or separation of variables). Let E_{zs} be written as:

$$E_{zs}(x, y, z) = X(x)\, Y(y)\, Z(z)$$

Where $X(x)$, $Y(y)$, and $Z(z)$ are the functions of x, y, and z, respectively. Substituting this in Equation (2.79), we get:

NOTES

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = -\omega^2\mu\varepsilon XYZ$$

Dividing the above equation by XYZ, we get:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -\omega^2\mu\varepsilon \qquad ...(2.80)$$

Here, each term on the left hand side is independent of each other, also their sum is a constant which implies that each of these terms must be a constant. Let, , and be the separation constants, then Equation (2.80) can be written as:

$$-k_x^2 - k_y^2 - k_z^2 = \omega^2\mu\varepsilon$$

Thus, using these separation constants, Equation (2.80) can be written as:

$$\frac{d^2X}{dx^2} = -Xk_x^2$$

$$\frac{d^2Y}{dy^2} = -Yk_y^2$$

$$\frac{d^2Z}{dz^2} = -Zk_z^2$$

The respective solutions of the above equations can be given by the relations:

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$Y(y) = C\sin(k_y y) + D\cos(k_y y)$$

$$Z(z) = E\sin(k_z z) + F\cos(k_z z)$$

Where A, B, C, D, E, and F are constants.

Now, using the above solution to find $\vec{E}_{zs}$ field component in general can be computed as:

$$E_{zs}(x, y, z) = [A\sin(k_x x) + B\cos(k_x x)]\,[C\sin(k_y y) + D\cos(k_y y)]\,[E\sin(k_z z) + F\cos(k_z z)]$$

Here, it is to be noted that h_r can be represented in terms of k_x and k_y as:

$$h_r^2 = k_x^2 + k_y^2 \qquad ...(2.81)$$

In the similar way, the general solution for magnetic field component H_{zs} can be written as:

$$H_{zs}(x, y, z) = \left[A' \sin(k_x x) + B' \cos(k_x x)\right]\left[C' \sin(k_y y) + D' \cos(k_y y)\right]$$

$$\left[E' \sin(k_z z) + F' \cos(k_z z)\right]$$

NOTES

Now, since the wave is propagating in positive z-direction, the solution of wave equation for z-axis can be written in terms of propagation constant as:

$$E_{zs}(x, y, z) = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]e^{-\gamma_r z} \quad ...(2.82)$$

$$H_{zs}(x, y, z) = [A'\sin(k_x x) + B'\cos(k_x x)][C'\sin(k_y y) + D'\cos(k_y y)]e^{-\gamma_r z} \quad ...(2.83)$$

Since we have derived these general solutions, let us derive the field expressions for two different waves, that is, Transverse Magnetic (TM) wave and Transverse Electric (TE) wave. It should be noted that rectangular waveguides does not support TEM waves as we have studied earlier.

Transverse Magnetic Waves

To determine the field expressions, let us apply the boundary conditions on the walls of the waveguide which are perfectly conducting. Since we know that for TM waves, $H_{zs} = 0$ and the tangential components of electric field $\vec{E}$ must be continuous, which implies:

$$E_{zs} = 0 \text{ at } x = 0 \text{ and } y = 0 \text{ [Refer to Figure (2.33)]} \quad ...(2.84)$$

Also,

$$E_{zs} = 0 \text{ at } x = a \text{ and } y = b \quad ...(2.85)$$

Substituting the boundary condition described in Equation (2.84) in Equation (2.82), we get:

$$B = 0 \text{ and } D = 0$$

Thus, Equation (2.82) becomes:

$$E_{zs} = AC\sin(k_x x)\sin(k_y y)e^{-\gamma_r z}$$

Or, it can be written as:

$$E_{zs} = E_o \sin(k_x x)\sin(k_y y)e^{-\gamma_r z} \quad ...(2.86)$$

Where, $E_o = AC$ is constant.

Now, applying boundary condition given in Equation (2.85), Equation (2.86) becomes:

$$\sin k_x a = 0$$

And, $\quad \sin k_y b = 0$

This implies that;

$$K_x a = m\pi, \text{ where } m = 1, 2, 3, ...$$

And, $K_y b = n\pi$, where, $n = 1, 2, 3, ...$

Here m and n denotes the number of half cycle variations in x- and y-directions, respectively. Hence, Equation (2.86) becomes:

$$E_{zs} = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

Now, substituting the value of E_{zs}, from Equation (2.76) the other field components can easily be obtained. Here, $H_{zs} = 0$, as it is a transverse magnetic wave. Thus, we get:

$$E_{xs} = -\frac{\gamma_r}{h_r^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$E_{ys} = -\frac{\gamma_r}{h_r^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$H_{xs} = \frac{j\omega\varepsilon}{h_r^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$E_{ys} = -\frac{j\omega\varepsilon}{h_r^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

Where $h_r^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ [Refer to Equation (2.81)]

Here, it is to be noted that the lowest mode that can be transmitted using a rectangular waveguide is TM_{11} mode as neither m nor n can be zero for TM wave. Since on substituting m or n zero, all the field components vanish. The field patterns of TM_{11} are shown in Figure 2.35.

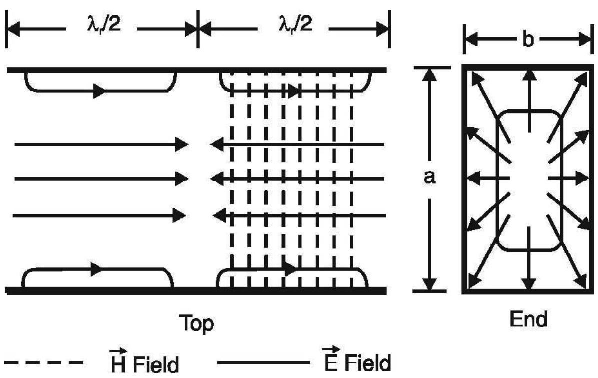

***Fig. 2.36** TM_{11} Wave inside a Rectangular Waveguide*

Transverse Electric Waves

Since in TE waves, $E_{zs} = 0$, and the tangential components of electric field $\vec{E}$ at the walls of the waveguide must be continuous, we have:

$$E_{xs} = 0 \text{ at } y = 0 \text{ and } y = \text{b}$$

Also,

$$E_{ys} = 0 \text{ at } x = 0 \text{ and } x = a$$

NOTES

Substituting these conditions in Equation (2.77), we get:

$$\frac{\partial H_{zs}}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b \qquad ...(2.87)$$

Also,

$$\frac{\partial H_{zs}}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a \qquad ...(2.88)$$

Now, using Equations (2.83), (2.87), and (2.88) and proceeding in similar way as we did for transverse magnetic waves, we get:

$$H_{zs} = H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

Where $H_o = B'D'$ and $m = 0,1,2,3,...$ and $n = 0,1,2,3,...$

Substituting the value of H_{zs}, from Equation (2.77), the other field components becomes:

$$E_{xs} = \frac{j\omega\mu}{h_r^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$E_{ys} = -\frac{j\omega\mu}{h_r^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$H_{xs} = \frac{\gamma_r}{h_r^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

$$H_{ys} = \frac{\gamma_r}{h_r^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\gamma_r z}$$

Notice that unlike TM waves, the TE waves can exist for zero value of m and n. However, if both are simultaneously zero it will result in zero field components. Thus, the lowest mode that can be transmitted by rectangular waveguide is TE_{01} mode or TE_{10} mode depending upon the dimensions of waveguide. The field variations for TE_{10} mode are shown in Figure (2.36).

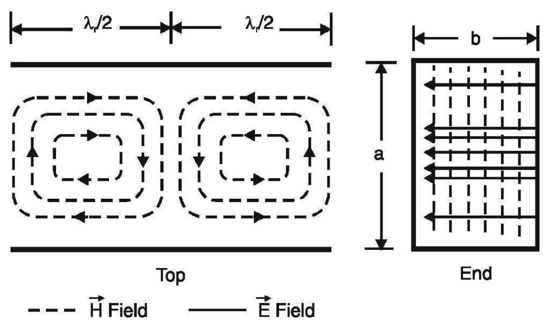

Fig. 2.35 *TE_{10} Wave inside a Rectangular Waveguide*

Propagation Characteristics of TE and TM Waves in Rectangular Waveguides

NOTES

Here we will discuss about various propagation characteristics of the rectangular waveguides. As we studied in the previous section that:

$$h_r^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Also, it can be expressed as:

$$h_r^2 = \gamma_r^2 + \omega^2\mu\varepsilon$$

Or, it can be written as:

$$\gamma_r^2 = h_r^2 - \omega^2\mu\varepsilon$$

$$\gamma_r = \sqrt{h_r^2 - \omega^2\mu\varepsilon}$$

Thus, we get:

$$\gamma_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

Now, depending upon the value of h_r and $\omega^2\mu\varepsilon$, there are three cases.

Case 1: If $\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] > \omega^2\mu\varepsilon$, then the value of propagation constant γ_r becomes purely real. Thus, we obtain only the attenuation constant α_r as:

$$\gamma_r = \alpha_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

Case 2: If $\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] < \omega^2\mu\varepsilon$, then the propagation constant γ_r will be purely imaginary, that is, the real part α_r is equal to zero.

Thus, the imaginary part of propagation constant, that is, the phase constant β_r is obtained as:

$$\gamma_r = j\beta_r$$

It is expressed as:

$$\beta_r = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad ...(2.89)$$

NOTES

Case 3: There exists one more case for rectangular waveguides where $\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] = \omega^2\mu\varepsilon$, then the value of propagation constant γ_r comes out to be zero which implies, both attenuation constant α_r and phase constant β_r are equal to zero. In addition, there will be no propagation of wave and this is considered as the critical condition for cut-off propagation. The value of ω in this case is known as **angular cut-off frequency**, denoted by ω_c, expressed as:

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad ...(2.90)$$

Thus, the cut-off frequency f_c is expressed as:

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad ...(2.91)$$

From Equations (2.89) and (2.91), the phase constant β_r can be written as:

$$\beta_r = \beta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad ...(2.92)$$

Where β is the free space phase constant.

Now, the cut-off wavelength λ_c can be obtained as:

$$\lambda_c = \frac{u}{f_c}$$

Where u is the velocity of the wave in lossless dielectric medium given as:

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Thus, the cut-off wavelength λ_c is obtained as:

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad ...(2.93)$$

Now, the guide wavelength λ_r is obtained as:

$$\lambda_r = \frac{2\pi}{\beta_r}$$

Substituting the value of β_r from Equation (2.89), we get:

$$\lambda_r = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

NOTES

Or, it can be written as:

$$\lambda_r = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}} \qquad \text{[Refer to Equation (2.90)]}$$

$$= \frac{2\pi}{\omega\sqrt{\mu\varepsilon\left(1 - \frac{\omega_c^2}{\omega^2}\right)}}$$

The guide wavelength λ_r in terms of cut-off wavelength λ_c is expressed as:

$$\lambda_r = \frac{\lambda}{\left[1 - \left(\frac{\lambda}{\lambda_c}\right)^2\right]^{1/2}}$$

The phase velocity u_{ph} and group velocity u_g for rectangular waveguides can be obtained by substituting the value of β_r in the respectively following equations:

We get

$$u_{ph} = \frac{\omega}{\beta}$$

$$u_g = \frac{\partial\omega}{\partial\beta}$$

$$u_{ph} = \frac{u}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$u_g = u\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

From above two expressions it is obtained that:

$$\sqrt{u_{ph}u_g} = u$$

Now the dominant mode for a rectangular waveguide is TE_{10} mode as it has the lowest cut-off frequency among all the modes. All the propagation characteristics expressed above are same for both transverse electric and transverse magnetic waves. However, as the field expressions for magnetic and electric fields are different for TE and TM waves the expressions for intrinsic impedance also differ. Let us now derive the expression of intrinsic impedance for both TE and TM waves in rectangular waveguides. The intrinsic impedance η_r is expressed as:

$$\eta_r = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

NOTES

For TM wave, intrinsic impedance η_{rTM} is obtained by substituting the values of E_x and H_y, we get:

$$\eta_{rTM} = \frac{\gamma_r}{j\omega\varepsilon}$$

Now, for a propagating wave, we have:

$$\gamma_r = j\beta_r$$

Thus, we obtain:

$$\eta_{r\,TM} = \frac{\beta_r}{\omega\varepsilon}$$

Substituting the value of β_r from Equation (2.89), we get:

$$\eta_{r\,TMm} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Or, it can be written as:

$$\eta_{r\,TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad ...(2.94)$$

Where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of lossless dielectric material.

In the similar way, the intrinsic impedance for TE wave $\eta_{r\,TE}$ is obtained by substituting the values of E_x and H_y we get:

$$\eta_{r\,TE} = \frac{\omega\mu}{\beta_r}$$

Or,

$$\eta_{r\,TE} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\eta_{r\,TE} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad ...(2.95)$$

From Equations (294) and (2.95), it can be observed that $\eta_{r\,TM}$ and η_{rTE} vary with frequency and are purely resistive in nature. Also, we have:

$$\sqrt{\eta_{rTM}\,\eta_{rTE}} = \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Power Transmission and Losses in Rectangular Waveguides

The power transmitted in a waveguide can be calculated using Poynting theorem. We know that the average Poynting vector is given as:

NOTES

$$\vec{S}_{av} = \frac{1}{2}\text{Re}(\vec{E}_s \times \vec{H}_s^*)$$

Here, the wave is propagating along z-direction and hence, the Poynting vector is also along z-direction. Thus, we have:

$$\vec{S}_{av} = \frac{1}{2}\text{Re}(E_{xs}H_{ys}^* - E_{ys}H_{xs}^*)\hat{a}_z$$

Or, it can be written as:

$$\vec{S}_{av} = \frac{1}{2\eta_r}|E_s|^2\ \hat{a}_z$$

where and $\eta_r = \eta_{r\,\text{TM}}$ for TM wave and $\eta_r = \eta_{r\,\text{TE}}$ for TE wave. Thus, the above equation becomes:

$$\vec{S}_{av} = \frac{1}{2\eta_r}[|E_{xs}|^2 + |E_{ys}|^2]\hat{a}_z \qquad ...(2.96)$$

Now, the total time-average power transmitted is given by the relation:

$$S_{av} = \int_s \vec{S}_{av}\,.d\vec{s}$$

$$S_{av} = \frac{1}{2\eta_r}\int_{x=0}^{a}\int_{y=0}^{b}|E_s|^2\ dxdy$$

Or, $$S_{av} = \frac{1}{2\eta_r}\int_0^a\int_0^b |E_{xs}|^2 + |E_{ys}|^2\ dxdy \qquad ...(2.97)$$

Thus, the power transmitted for TM mode is given as:

$$S_{av} = \frac{1}{2\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\int_0^a\int_0^b |E_s|^2\ dxdy$$ [Refer to Equation (2.94)]

For TE mode is given as:

$$S_{av} = \frac{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{2\eta}\int_0^a\int_0^b |E_s|^2\ dxdy$$ [Refer to Equation (2.95)]

Power Losses

So far for a waveguide, we have assumed that its walls to be perfectly conducting and the dielectric between them to be lossless. However practically, if the waveguide walls are not perfectly conducting and the dielectric medium is lossy, consequently, there incurs some power loss along the wave propagation. These losses can be classified into two types as:

NOTES

- Losses in the dielectric
- Losses in the walls of the guide

The power flow in the waveguide is expressed as:

$$S_{av} = S_o e^{-2\alpha z}$$

Where $\alpha = \alpha_w + \alpha_d$. Here, α_w denotes the losses occurring in the walls of the waveguide and α_d represents the losses due to dielectric. Let us determine the losses due to dielectric and then we will determine the losses due to guide walls. For lossy dielectrics, we know that $\gamma \pi j\beta$ (thus, only the value of propagation constant needs to be modified to obtain the results for propagation in lossy dielectrics and it is done by replacing ε with ε_c in Equation (2.78), where ε_c denotes the complex permittivity. Thus, Equation (2.78) becomes:

$$h_r^2 = \gamma_r^2 + \omega^2 \mu \varepsilon_c$$

Or, it can be written as:

$$\gamma_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon_c} = \alpha_d + j\beta_d \quad \text{...(2.98)}$$

where $\varepsilon_c = \varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right) = \varepsilon' - j\varepsilon''$

Substituting this relation in Equation (2.98), it becomes:

$$\gamma_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon + j\omega\mu\sigma} = \alpha_d + j\beta_d$$

Squaring both sides, we get:

$$\gamma_r^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon + j\omega\mu\sigma = \alpha_d^2 - \beta_d^2 + j2\alpha_d\beta_d$$

Equating real and imaginary parts on both sides, we get:

$$\alpha_d^2 - \beta_d^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon \quad \text{...(23.99)}$$

And, $2\alpha_d\beta_d = \omega\mu\sigma$ ...(2.100)

Assuming , thus Equation (2.99) becomes:

$$\alpha_d^2 - \beta_d^2 \simeq -\beta_d^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon$$

Or,

$$\beta_d = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\beta_d = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

NOTES

Now, from Equation (2.100), α_d can be written as:

$$\alpha_d = \frac{\omega\mu\sigma}{2\beta_d}$$

Substituting the value of β_d as obtained above, we get:

$$\alpha_d = \frac{\omega\mu\sigma}{2\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\sigma\eta}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad ...(2.101)$$

Where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$.

Now let us determine the losses due to waveguide walls, that is, α_w. We will determine the value of α_w for TE_{10} mode, as it is comparatively easy and less time consuming. Substituting the expressions of E_{xs} and E_{ys} for TE_{10} mode in Equation (2.97) where $E_{xs} = 0$ for $m = 1$ and $n = 0$ mode, we get:

$$S_{av} = \frac{1}{2\eta_r}\int_0^a\int_0^b |E_{xs}|^2 + |E_{ys}|^2\, dxdy = \frac{1}{2\eta_r}\int_0^a\int_0^b 0 + \frac{\omega^2\mu^2a^2H_o^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right)dxdy$$

Or, it can be written as:

$$S_{av} = \frac{\omega^2\mu^2a^2H_o^2}{2\pi^2\eta_r}\int_0^a\int_0^b \sin^2\left(\frac{\pi x}{a}\right)dxdy$$

Integrating the above equation, we get:

$$S_{av} = \frac{\omega^2\mu^2a^3H_o^2b}{4\pi^2\eta_r} \quad ...(2.102)$$

Now, the total power loss per unit length of the walls is given by the relation:

$$S_l = [S_l]_{x=0} + [S_l]_{x=a} + [S_l]_{y=0} + [S_l]_{y=b}$$

$$= 2\ \{[S_l]_{x=0} + [S_l]_{y=0}\} \quad ...(2.103)$$

As the same power is dissipated in the walls $x = 0$ and $x = a$ then $[S_l]_{x=0} = [S_l]_{x=a}$ or $y = 0$ and $y = b$ then $[S_l]_{y=0} = [S_l]_{y=b}$. For $y = 0$, we have:

$$[S_l]_{y=0} = \frac{1}{2}\text{Re}\left[\eta_{rw}\int(|H_{xs}|^2 + |H_{zs}|^2)dx\right]_{y=0}$$

NOTES

Here, subscript w denotes the intrinsic impedance for conducting walls. Let R_s be the real part of intrinsic impedance, given as:

$$R_s = \frac{1}{\sigma_w \delta} = \sqrt{\frac{\pi f \mu}{\sigma_w}}$$

Thus, we obtain:

$$[S_l]_{y=0} = \frac{1}{2}\text{R}_\text{s}\left[\int_{x=0}^{a}\frac{\beta^2 a^2}{\pi^2}H_o^2 \sin^2\left(\frac{\pi x}{a}\right)dx + \int_{x=0}^{a}H_o^2\cos^2\left(\frac{\pi x}{a}\right)dx\right]$$

$$= \frac{R_s a H_o^2}{4}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right) \quad ...(2.104)$$

Now for $x = 0$, we have:

$$[S_l]_{x=0} = \frac{1}{2}\text{Re}\left[\eta_{rw}\int|H_{zs}|^2\, dy\right]_{x=0}$$

$$= \frac{1}{2}\text{R}_\text{s}\int_{y=0}^{b}H_o^2 dy$$

$$= \frac{R_s b H_o^2}{2} \quad ...(2.105)$$

Thus, using Equations (2.104) and (2.105), Equation (2.103) becomes:

$$S_l = R_s H_o^2\left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right] \quad ...(2.106)$$

As per the law of conservation of energy, it must be conserved which implies that the power loss per unit length is equal to the rate of decrease of average power, that is,

$$S_l = -\frac{dS_{av}}{dz} = 2\alpha S_{av}$$

$$\alpha = \frac{S_l}{2S_{av}}$$

Substituting the values from Equations (2.102) and (2.106), we get:

$$\alpha_w = \frac{2R_s H_o^2 \pi^2 \eta_r\left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]}{\omega^2 \mu^2 a^3 H_o^2 b}$$

In terms of frequency, it can be written as:

$$\alpha_w = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

NOTES

This is the required expression of attenuation constant of walls of waveguide for TE_{10} mode. For TE_{mn} modes, it can be derived using the same procedure, and is given as:

$$\alpha_{w,\mathrm{TE}} = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\left(1+\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2+\frac{\frac{b}{a}\left(\frac{b}{a}m^2+n^2\right)}{\frac{b^2}{a^2}m^2+n^2}\left(1-\left(\frac{f_c}{f}\right)^2\right)\right] \quad ...(2.107)$$

And, for TM_{mn} modes, it is given as:

$$\alpha_{w,\mathrm{TM}} = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{(b/a)^3m^2+n^2}{(b/a)^2m^2+n^2}\right] \quad ...(2.108)$$

Thus, the total loss in the rectangular waveguide can be obtained using Equations (2.101), (2.107), and (2.108) as:

For TM wave:

$$\alpha = \alpha_w + \alpha_d$$

$$\alpha = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{(b/a)^3m^2+n^2}{(b/a)^2m^2+n^2}\right]+\frac{\sigma\eta}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

For TE wave:

$$\alpha = \alpha_w + \alpha_d$$

$$\alpha = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\left(1+\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2+\frac{\frac{b}{a}\left(\frac{b}{a}m^2+n^2\right)}{\frac{b^2}{a^2}m^2+n^2}\left(1-\left(\frac{f_c}{f}\right)^2\right)\right]+\frac{\sigma\eta}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Cylndrical or Circular Waveguides

A circular tubular conductor is considered as a circular waveguide, used to transmit EM waves from source to destination. However, they are different from that of rectangular waveguides as they do not have unique orientation due to its symmetricity around the axis as shown in Figure (2.37). Also, circular waveguides are easy to manufacture. Both TE and TM modes can propagate through a circular waveguide. The field expressions for circular waveguides are obtained using Bessel functions and are derived below.

NOTES

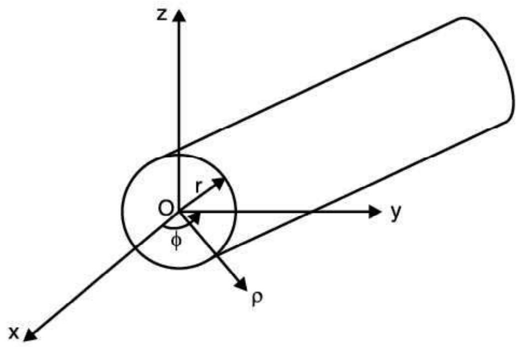

Fig. 2.37 *A Circular Waveguide*

Consider a circular waveguide with dimension of radius r as shown in Figure (2.37). Proceeding in similar way as we did for rectangular waveguides, the general solution of electric and magnetic field can be written in terms of Bessel functions as:

$$E_{zs}(\rho, \phi, z) = \left[A_n J_n(k_c \rho) + B_n N_n(k_c \rho)\right](C_n \cos n\phi + D_n \sin n\phi)e^{\pm j\beta_{cir} z} \quad ...(2.109)$$

$$H_{zs}(\rho, \phi, z) = \left[A'_n J_n(k_c \rho) + B'_n N_n(k_c \rho)\right](C'_n \cos n\phi + D'_n \sin n\phi)e^{\pm j\beta_{cir} z} \quad ...(2.110)$$

Where $J_n(k_c\rho)$ is the Bessel function of first kind and $N_n(k_c\rho)$ is the Bessel function of second kind. Also,

$$\beta_{cir} = \pm\sqrt{\omega^2\mu\varepsilon - k_c^2}$$

Here, subscript *cir* is used to denote the propagation inside circular waveguide. Now, at $\rho = 0$ the field must be finite which implies $B_n = 0$, thus Equations (2.109) and (2.110) becomes:

$$E_{zs}(\rho, \phi, z) = A_n J_n(k_c \rho)(C_n \cos n\phi + D_n \sin n\phi)e^{\pm j\beta_{cir} z} \quad ...(2.111)$$

$$H_{zs}(\rho, \phi, z) = A'_n J_n(k_c \rho)(C'_n \cos n\phi + D'_n \sin n\phi)e^{\pm j\beta_{cir} z} \quad ...(2.112)$$

Also, using trigonometric manipulations, we have:

$$C_n \cos n\phi + D_n \sin n\phi = \sqrt{C_n^2 + D_n^2}\cos\left[n\phi + \tan^{-1}\left(\frac{D_n}{C_n}\right)\right]$$

$$= K_n \cos(n\phi)$$

Where K_n is another constant.

Thus, Equations (2.111) and (2.112) becomes:

$$E_{zs}(\rho, \phi, z) = E_o J_n(k_c \rho)\cos(n\phi)e^{-j\beta_{cir} z} \quad ...(2.113)$$

$$H_{zs}(\rho, \phi, z) = H_o J_n(k_c \rho)\cos(n\phi)e^{-j\beta_{cir} z} \quad ...(2.114)$$

Where $E_o = A_n K_n$ and $H_o =$ $H_o = A'_n K_n$

Now, the solution of TE and TM waves can be obtained using Maxwell's equation given as:

$$\nabla \times \vec{H}_s = j\omega\varepsilon\vec{E}_s$$

NOTES

Expanding above equation, we get:

$$\nabla \times \vec{H}_s = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ H_{\rho s} & \rho H_{\phi s} & H_{zs} \end{vmatrix} = \left[\frac{1}{\rho}\frac{\partial H_{zs}}{\partial\phi} - \frac{\partial H_{\phi s}}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial H_{\rho s}}{\partial z} - \frac{\partial H_{zs}}{\partial\rho}\right]\hat{a}_\phi$$

$$+\frac{1}{\rho}\left[\frac{\partial\left(\rho H_{\phi s}\right)}{\partial\rho} - \frac{\partial H_{\rho s}}{\partial\phi}\right]\hat{a}_z$$

$$= \ j\omega\varepsilon E_{\rho s}\hat{a}_\rho + j\omega\varepsilon E_{\phi s}\hat{a}_\phi + j\omega\varepsilon E_{zs}\hat{a}_z$$

Equating the coefficients, we obtain:

$$\frac{1}{\rho}\frac{\partial H_{zs}}{\partial\phi} - \frac{\partial H_{\phi s}}{\partial z} = j\omega\varepsilon E_{\rho s}$$

$$-j\beta_{cir}H_{\rho s} - \frac{\partial H_{zs}}{\partial\rho} = j\omega\varepsilon E_{\phi s}$$

$$\frac{1}{\rho}\frac{\partial(\rho H_{\phi s})}{\partial\rho} - \frac{1}{\rho}\frac{\partial H_{\rho s}}{\partial\phi} = j\omega\varepsilon E_{zs} \qquad ...(2.115)$$

Similarly using Maxwell's equation given as:

$$\nabla \times \vec{E}_s = -\ j\omega\mu\vec{H}_s$$

And expanding and equating the coefficients, we get:

$$\frac{1}{\rho}\frac{\partial E_{zs}}{\partial\phi} - \frac{\partial E_{\phi s}}{\partial z} = -\ j\omega\mu H_{\rho s}$$

$$\frac{\partial E_{\rho s}}{\partial z} - \frac{\partial E_{zs}}{\partial\rho} = -\ j\omega\mu H_{\phi s}$$

$$\frac{1}{\rho}\frac{\partial(\rho E_{\phi s})}{\partial\rho} - \frac{1}{\rho}\frac{\partial E_{\rho s}}{\partial\phi} = -\ j\omega\mu H_{zs} \qquad ...(2.116)$$

Transverse Magnetic Waves

To determine the field expressions, let us apply the boundary conditions. We know that the tangential components of electric field $\vec{E}$ must be continuous, which implies:

$$E_{zs} = 0 \text{ at } \rho = r$$

Using this boundary condition, we get:

$$J_n(k_c r) = 0 \qquad ...(2.117)$$

NOTES

Also, the field equations by substituting $H_{zs} = 0$ and , Equations (2.115) and (2.116) becomes:

$$E_{\rho s} = \frac{-j\beta_{cir}}{k_c^2}\frac{\partial E_{zs}}{\partial \rho}$$

$$E_{\phi s} = \frac{-j\beta_{cir}}{k_c^2}\frac{1}{\rho}\frac{\partial H_{zs}}{\partial \phi}$$

$$E_{zs} = E_o J_n(k_c \rho)\cos(n\phi)e^{-j\beta_{cir}z} \quad ...(2.118)$$

$$H_{\rho s} = \frac{j\omega\varepsilon}{k_c^2}\frac{1}{\rho}\frac{\partial E_{zs}}{\partial \phi}$$

$$E_{\phi s} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{zs}}{\partial \rho}$$

$$H_{zs} = 0 \quad ...(2.119)$$

where $\quad k_c^2 = \omega^2\mu\varepsilon - \beta_{cir}^2$

Now, from Equations (2.117), (2.118), and (2.119) we get the field components of TM wave as:

$$E_{\rho s} = E_o J_n'\left(\frac{X_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

$$E_{\phi s} = E_o J_n\left(\frac{X_{np}\rho}{r}\right)\sin(n\phi)e^{-j\beta_{cir}z}$$

$$E_{zs} = E_o J_n\left(\frac{X_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

$$H_{\rho s} = H_o J_n\left(\frac{X_{np}\rho}{r}\right)\sin(n\phi)e^{-j\beta_{cir}z}$$

$$H_{\phi s} = H_o J_n'\left(\frac{X_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

Where $\frac{X_{np}}{r} = k_c$. Here, X_{np} are the roots of the Bessel function in which subscript n denotes the number of full cycles of field variation in one revolution and subscript p denotes the number of zeroes of E_ϕ, however, zero on the axis is excluded. The values of n and p are given as:

$$n = 0, 1, 2, 3, 4, ...$$

$$p = 1, 2, 3, 4, ...$$

Also, $J_n'\left(\frac{X_{np}\rho}{r}\right)$ indicates the derivative of $J_n\left(\frac{X_{np}\rho}{r}\right)$

Propagation Characteristics of Transverse Magnetic Wave

NOTES

The phase constantis β_{cir} given as:

$$\beta_{cir} = \sqrt{\omega^2\mu\varepsilon - k_c^2} = \sqrt{k^2 - k_c^2}$$

Or, the phase constant can be written as:

$$\beta_{cir} = \left[k^2 - \left(\frac{X_{np}}{r}\right)^2\right]^{1/2}$$

Also, the cut-off frequency is given by the relation:

$$f_c = \frac{uX_{np}}{2\pi r}$$

where u is the velocity of the wave given as:

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Now, the cut-off wavelength for circular waveguide can be given as:

$$\lambda_c = \frac{2\pi r}{X_{np}}$$

And the guide wavelength λ_{cir} is given as:

$$\lambda_{cir} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

The intrinsic impedance of the circular waveguide is similar to that of rectangular waveguide and thus, given by the relation:

$$\eta_{cir\,\mathrm{TM}} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Transverse Electric Waves

As the tangential components of electric field $\vec{E}$ must be continuous, we have:

$$E_{\phi s} = 0 \text{ at } \rho = r$$

Substituting this condition in Equation (2.117), we get:

$$\frac{\partial H_{zs}}{\partial \rho} = 0$$

Using the above equation, we get:

$$J_n'(k_c\, r) = 0 \qquad \text{...(2.120)}$$

Also, the field components on substituting $E_{zs} = 0$ and $\frac{\partial}{\partial z} = -j\beta_{cir}$ in Equations (2.115) and (2.116), we get:

NOTES

$$E_{\rho s} = \frac{-j\omega\mu}{k_c^2}\frac{1}{\rho}\frac{\partial H_{zs}}{\partial\phi}$$

$$E_{\phi s} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{zs}}{\partial\rho} \qquad ...(2.121)$$

$$E_{zs} = 0$$

$$H_{\rho s} = \frac{-j\beta_{cir}}{k_c^2}\frac{\partial H_{zs}}{\partial\rho}$$

$$H_{\phi s} = \frac{-j\beta_{cir}}{k_c^2}\frac{1}{\rho}\frac{\partial H_{zs}}{\partial\phi} \qquad ...(2.122)$$

$$H_{zs} = H_o J_n(k_c\,\rho)\cos(n\phi)e^{-j\beta_{cir}z}$$

Now, from Equations (2.120), (2.121), and (2.122), we get the field components of TE wave as:

$$E_{\rho s} = E_o J_n\left(\frac{X'_{np}\rho}{r}\right)\sin(n\phi)e^{-j\beta_{cir}z}$$

$$E_{\phi s} = E_o J'_n\left(\frac{X'_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

$$H_{\rho s} = -H_o J_n\left(\frac{X'_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

$$H_{\phi s} = H_o J_n\left(\frac{X'_{np}\rho}{r}\right)\sin(n\phi)e^{-j\beta_{cir}z}$$

$$H_{zs} = H_o J_n\left(\frac{X'_{np}\rho}{r}\right)\cos(n\phi)e^{-j\beta_{cir}z}$$

Where $\frac{X'_{np}}{r} = k_c$

Propagation Characteristics of Transverse Electric Wave

Let us now discuss the propagation parameters of transverse electric wave. Some of the characteristics are same to those of TM wave while some are different. However, we will represent them all here in summarized way.

$$\beta_{cir} = \left[k^2 - \left(\frac{X'_{np}}{r}\right)^2\right]^{1/2}$$

$$f_c = \frac{uX'_{np}}{2\pi r}$$

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda_c = \frac{2\pi r}{X'_{np}}$$

$$\lambda_{cir} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

And,

$$\eta_{cir\,\text{TE}} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

NOTES

Now, the dominant mode of a circular waveguide is TE_{11}.

Note: The major drawback of circular waveguide is that they occupy more space as compared to rectangular waveguide. Hence, to carry the same signal, their cross section is much larger.

Check Your Progress

9. Define the term polarisation.
10. What is light vector?
11. Define the term double refraction or birefringence.
12. What is Degree Of Polarization (DOP)?
13. State about the scattering of light and radio waves.
14. What are the elements of waveguides?

2.13 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Time varying is a system in which certain quantities governing the system's behaviour change with time, so that the system will respond differently to the same input at different times.

   When an electrically conducting structure is exposed to a time varying magnetic field, an electrical potential difference is induced across the structure.

   The generation of electric potential by a time varying magnetic flux is very well described by 'Faraday's Law'. This is a form of electromagnetic induction. According to Faradays law, when magnetic flux changes in the region surrounded by conductor, it produces electric field (induced Electo Motive Force or EMF) in conductor.
2. Maxwell's equations state the fundamentals of electricity and magnetism. The working relationships in the field of electricity and magnetism can be derived using these equations. As a consequence of their brief statement, they symbolize a high level of mathematical sophistication, and hence are typically defined as unifying equations for studying of electrical and magnetic phenomena.

   Principally, the Maxwell's equations are a set of partial differential equations that, together with the 'Lorentz Force Law', form the foundation of classical electromagnetism, classical optics, and electric circuits.
3. The (two-way) wave equation is a second order partial differential equation explaining waves. The scalar wave equation describes waves in scalars by scalar functions $u = u\,(x_1, x_2, ..., x_n; t)$ of a time variable t (a variable representing time) and one or more spatial variables $x_1, x_2, ..., x_n$ (variables representing a position in a space) while there are vector wave equations

describing waves in vectors, such as waves for electrical field, magnetic field, and magnetic vector potential and elastic waves.

4. The scalar wave equation is,

NOTES

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right)$$

Where c is a fixed non-negative real coefficient.

5. Hertz vector is also known as polarization potentials, which are useful auxiliary fields that permit the calculation of the fundamental electromagnetic fields. The Hertz vector potentials are an alternative formulation of the electromagnetic potentials.

6. Principally, the electromagnetic wave equation is a second order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. It is a three-dimensional form of the wave equation. Electromagnetic waves are created by the vibration of an electric charge. This vibration creates a wave which has both an electric and a magnetic component.

   An electromagnetic wave transports its energy through a vacuum at a speed of 3.00×10^8 m/s. The propagation of an electromagnetic wave through a material medium occurs at a net speed which is less than 3.00×10^8 m/s. An electromagnetic wave consists of an electric field, typically defined in terms of the force per charge on a stationary charge, and a magnetic field, defined in terms of the force per charge on a moving charge.

7. The mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field is the radiation pressure. The associated force is called the radiation pressure force, or sometimes just the force of light. Radiation pressure is the mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field.

8. Reflection basically occurs when a wave is incident on a boundary between two media in which the wave speed is different, and then remains in the original medium rather than passing into the second medium. While reflection occurs at any boundary, often only a small proportion of the wave is reflected. Refraction is the change of the direction of propagation of waves when they pass into a medium where they have a different speed. It is observed whenever the waves are incident to the surface at an angle different to the normal to the surface. When an electromagnetic field faces an abrupt change in the permittivity and permeability, then certain conditions on electric and magnetic fields on the interface are to be respected for the continuity.

9. Polarization or polarisation is a property applied to transverse waves that specifies the geometrical orientation of the oscillations. In a transverse wave, the direction of the oscillation is perpendicular to the direction of motion of the wave.

10. According to electromagnetic theory of light, electric field, magnetic field and the propagation vector of light travel along three mutually perpendicular

directions. It is the electric field of light that creates optical sensation in our eyes, in photographic cameras and in all other optical instruments. That is why electric field is known as light vector.

11. If a beam of unpolarised light is allowed to pass through an anisotropic crystal (Calcite or Quartz), then it splits up into two refracted beams instead of one. This phenomenon is called 'Double Refraction' or 'Birefringence'.
12. Principally, the Degree Of Polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized.
13. Electromagnetic waves are one of the best known and most commonly encountered forms of radiation that undergo scattering. Scattering of light and radio waves (especially in radar) is particularly important. Major forms of elastic light scattering including the negligible energy transfer are Rayleigh scattering and Mie scattering. Light scattering is one of the two major physical processes that contribute to the visible appearance of most objects, the other being absorption.
14. Depending upon the purpose for which waveguide is to be used and the frequency of the wave to be transmitted, there are many different structures of waveguide that include *parallel plate waveguide*, *rectangular waveguide*, *circular waveguide*, *optical fiber waveguide*, and *dielectric slab waveguide*. For the parallel plate waveguides the fields vary only in one transverse or orthogonal direction. Thus, the concept of parallel plate waveguide is simple, however, it is not of practical use due to infinite dimensions.

    A circular tubular conductor is considered as a circular waveguide, used to transmit EM waves from source to destination. However, they are different from that of rectangular waveguides as they do not have unique orientation due to its symmetricity around the axis. Also, circular waveguides are easy to manufacture. Both TE and TM modes can propagate through a circular waveguide.

2.14 SUMMARY

- Time varying is a system in which certain quantities governing the system's behaviour change with time, so that the system will respond differently to the same input at different times.
- When an electrically conducting structure is exposed to a time varying magnetic field, an electrical potential difference is induced across the structure.
- The generation of electric potential by a time varying magnetic flux is very well described by 'Faraday's Law'. This is a form of electromagnetic induction. According to Faradays law, when magnetic flux changes in the region surrounded by conductor, it produces electric field (induced Electro Motive Force or EMF) in conductor.
- Maxwell's equations are defined as a set of coupled partial differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism.

NOTES

- The Maxwell–Faraday version of Faraday's law of induction describes how a time varying magnetic field creates induces, an electric field. In integral form, it states that the work per unit charge required to move a charge around a closed loop equals the rate of change of the magnetic flux through the enclosed surface.
- Maxwell's equations state the fundamentals of electricity and magnetism. The working relationships in the field of electricity and magnetism can be derived using these equations. As a consequence of their brief statement, they symbolize a high level of mathematical sophistication, and hence are typically defined as unifying equations for studying of electrical and magnetic phenomena.
- Principally, the Maxwell's equations are a set of partial differential equations that, together with the 'Lorentz Force Law', form the foundation of classical electromagnetism, classical optics, and electric circuits.
- The magnetic dipole moment is equal to the product of the current flowing through the loop and area of the loop with the moment acting normal to the loop.
- Magnetic dipole naturally exists on permanent magnets as North and South poles or in current carrying coils.
- When a magnetic field is applied to the magnetic material, then the magnetic moments align in a particular direction. Hence, magnetisation is defined as the net magnetic dipole moment in a given volume.
- Hence, magnetic susceptibility is defined as the ratio of magnetisation to magnetic field intensity.
- Maxwell's equations in time varying fields are the final form of equations that interlinks the electric and magnetic fields.
- The (two-way) wave equation is a second order partial differential equation explaining waves.
- The scalar wave equation describes waves in scalars by scalar functions $u = u(x_1, x_2, ..., x_n; t)$ of a time variable t (a variable representing time) and one or more spatial variables $x_1, x_2, ..., x_n$ (variables representing a position in a space) while there are vector wave equations describing waves in vectors, such as waves for electrical field, magnetic field, and magnetic vector potential and elastic waves.
- The scalar wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right)$$

  Where c is a fixed non-negative real coefficient.
- Hertz vector is also known as polarization potentials, which are useful auxiliary fields that permit the calculation of the fundamental electromagnetic fields in many cases of practical importance. This provides a new light on the physical meaning of a Hertz potential.

NOTES

- The Hertz vector potentials are an alternative formulation of the electromagnetic potentials.
- A plane wave is a special case of wave or field - a physical quantity whose value, at any moment, is constant over any plane that is perpendicular to a fixed direction in space.
- Principally, the electromagnetic wave equation is a second order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. It is a three-dimensional form of the wave equation.
- Intrinsic impedance is also called as the wave impedance in free space. It is denoted as η_0. Intrinsic impedance relates the electric and magnetic field.
- Wave propagation parameters gets altered when they travel across different medium like free space, dielectric and conductors.
- When an electromagnetic wave travels, they tend to attenuate. This attenuation depends on the frequency of the wave travelling. Attenuation is larger, when the frequency is larger. This implies that wave die out faster for larger frequencies and travel a very short distance.
- The mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field is the radiation pressure. The associated force is called the radiation pressure force, or sometimes just the force of light.
- Radiation pressure is the mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field.
- Reflection basically occurs when a wave is incident on a boundary between two media in which the wave speed is different, and then remains in the original medium rather than passing into the second medium. While reflection occurs at any boundary, often only a small proportion of the wave is reflected.
- Refraction is the change of the direction of propagation of waves when they pass into a medium where they have a different speed. It is observed whenever the waves are incident to the surface at an angle different to the normal to the surface.
- When an electromagnetic field faces an abrupt change in the permittivity and permeability, then certain conditions on electric and magnetic fields on the interface are to be respected for the continuity.
- The law of reflection says that for specular reflection the angle at which the wave is incident on the surface equals the angle at which it is reflected. Mirrors exhibit specular reflection.
- Polarization or Polarisation is a property applied to transverse waves that specifies the geometrical orientation of the oscillations. In a transverse wave, the direction of the oscillation is perpendicular to the direction of motion of the wave.

NOTES

- An electromagnetic wave, such as light consists of a coupled oscillating electric field and magnetic field which are always perpendicular; by convention, the 'Polarization' of electromagnetic waves refers to the direction of the electric field.
- In linear polarization, the fields oscillate in a single direction. In circular or elliptical polarization, the fields rotate at a constant rate in a plane as the wave travels.
- The rotation can have two possible directions; if the fields rotate in a right hand sense with respect to the direction of wave travel, it is called right circular polarization, or, if the fields rotate in a left hand sense, it is called left circular polarization.
- According to electromagnetic theory of light, electric field, magnetic field and the propagation vector of light travel along three mutually perpendicular directions. It is the electric field of light that creates optical sensation in our eyes, in photographic cameras and in all other optical instruments. That is why electric field is known as light vector.
- The imaginary plane which contains the vibrations of electric field of a polarised light is called plane of vibration. An imaginary plane perpendicular to the plane of vibration is called plane of polarisation.
- If a beam of unpolarised light is allowed to pass through an anisotropic crystal (Calcite or Quartz), it splits up into two refracted beams instead of one. This phenomenon is called 'Double Refraction' or 'Birefringence'.
- A number of crystalline materials absorb more light in one incident plane than another, so that light progressing through the material become more and more polarised as they proceed. This anisotropy in absorption is called dichroism. There are several naturally occurring dichroic materials, and the commercial material polaroid also polarises by selective absorption.
- Principally, the Degree Of Polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized.
- Electromagnetic waves are one of the best known and most commonly encountered forms of radiation that undergo scattering.
- Scattering of light and radio waves (especially in radar) is particularly important. Major forms of elastic light scattering including the negligible energy transfer are Rayleigh scattering and Mie scattering.
- Light scattering is one of the two major physical processes that contribute to the visible appearance of most objects, the other being absorption.
- Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle, as described by classical electromagnetism. It is the low-energy limit of Compton scattering: the particle's kinetic energy and photon frequency do not change as a result of the scattering. This limit is valid as long as the photon energy is much smaller than the mass energy of the particle: $\nu \ll mc^2/h$, or equivalently, if the wavelength of the light is much greater than the Compton wavelength of the particle (e.g., for electrons, longer wavelengths than hard x-rays).

NOTES

- Depending upon the purpose for which waveguide is to be used and the frequency of the wave to be transmitted, there are many different structures of waveguide that include *parallel plate waveguide*, *rectangular waveguide*, *circular waveguide*, *optical fiber waveguide*, and *dielectric slab waveguide*.
- For the parallel plate waveguides the fields vary only in one transverse or orthogonal direction. Thus, the concept of parallel plate waveguide is simple, however, it is not of practical use due to infinite dimensions.
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2.15 KEY TERMS

- **Time varying:** Time varying is a system in which certain quantities governing the system's behaviour change with time, so that the system will respond differently to the same input at different times. When an electrically conducting structure is exposed to a time varying magnetic field, an electrical potential difference is induced across the structure.
- **Magnetic dipole moment:** The magnetic dipole moment is equal to the product of the current flowing through the loop and area of the loop with the moment acting normal to the loop.
- **Magnetic susceptibility:** Magnetic susceptibility is defined as the ratio of magnetisation to magnetic field intensity.
- **Intrinsic impedance:** Intrinsic impedance is also called as the wave impedance in free space. It is denoted as η_0. Intrinsic impedance relates the electric and magnetic field.
- **Polarization:** Polarization or polarisation is a property applied to transverse waves that specifies the geometrical orientation of the oscillations.
- **Plane of polarisation:** The imaginary plane which contains the vibrations of electric field of a polarised light is called plane of vibration. An imaginary plane perpendicular to the plane of vibration is called plane of polarisation.

2.16 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define time varying fields.
2. What is Maxwell's electromagnetic field equations?
3. State about the electromagnetic scalar wave equations.

4. Define Hertz vector.
5. What is plane wave propagation?
6. Define radiation pressure.

NOTES

7. Differentiate between reflection, refraction and total internal reflection.
8. What is polarisation?
9. Define scattering of plane E. M. waves.
10. What are the elements of wave guides?

Long-Answer Questions

1. Briefly discuss time varying fields with the help of examples.
2. Discuss the Maxwell's electromagnetic field equations in stationary and moving media giving appropriate examples.
3. Explain the characteristic features of electromagnetic scalar wave equations with the help of examples.
4. What is the importance of Hertz vector? Explain giving examples.
5. Describe the significant features of plane wave propagation in ionised media giving appropriate examples.
6. Interpret about radiation pressure and momentum giving examples.
7. Briefly discuss the concept of reflection, refraction, and total internal reflection with the help of general reaction mechanism and examples.
8. Brief a note on polarisation giving examples.
9. Discuss in detail the characteristic equations for scattering (Rayleigh and Thomson) and dispersion of plane E. M. waves with the help of relevant examples.
10. Explain the elements of wave guides giving examples.

2.17 FURTHER READING

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UNIT 3 ELECTROMAGNETIC RADIATION

NOTES

Structure

3.0 INTRODUCTION

ElectroMagnetic Radiation (EMR) typically consists of waves of the ElectroMagnetic (EM) field, that are propagating through the space, carrying electromagnetic radiant energy. All of the waves, such as radio waves, microwaves, InfraRed (IR) waves, white (visible) light, UltraViolet (UV) waves, X-rays, and gamma rays form the significant part of the electromagnetic spectrum. All these wave types are considered as the synchronized oscillations of electric field and magnetic field.

Electromagnetic radiation or electromagnetic waves are typically created as a result of periodic change of the electric field or the magnetic field. Depending on how this periodic change happens and the power is generated, different wavelengths of electromagnetic spectrum are produced. In a vacuum, electromagnetic waves travel at the speed of light.

The position of an electromagnetic wave within the electromagnetic spectrum, such as radio waves, microwaves, InfraRed (IR) radiation, visible light, UltraViolet (UV) radiation, X-rays and gamma rays. can be ascertained by either its frequency of oscillation or its wavelength. Electromagnetic waves of different frequency are called by different names since they have different sources and effects on matter. Characteristically, they are categorized in order of increasing frequency and decreasing wavelength.

NOTES

Electromagnetic waves are emitted by electrically charged particles undergoing acceleration, and these waves can subsequently interact with other charged particles, exerting force on them. EM waves carry energy, momentum and angular momentum away from their source particle and can impart those quantities to matter with which they interact. Electromagnetic radiation is associated with those EM waves that are free to propagate or radiate themselves without the continuing influence of the moving charges that produced them, because they have achieved sufficient distance from those charges. Consequently, ElectroMagnetic Radiation (EMR) is sometimes referred to as the far field.

In physics, the 'Retarded Electromagnetic Potentials' are typically derived from the Maxwell's equations and the Lorenz condition. The Maxwell's equations are given by the physicist James Clerk Maxwell. The key difference observed between these retarded electromagnetic potentials and the conventional Liénard–Wiechert potentials is precisely explained by ignoring the dependency of motion of the effective charge density. In addition, the subsequent retarded fields for a point-like charge specifically in the arbitrary or random motion are precisely compared and evaluated with the notions, formulae and equations given by the Oliver Heaviside, Richard Phillips Feynman, Oleg Dmitrovich Jefimenko and other authors. Electromagnetic radiation in the form of waves can be obtained from these potentials. The expressions for the Liénard–Wiechert potentials are named after the physicists Alfred-Marie Liénard who developed in part in the year 1898 and then by Emil Wiechert who independently developed in the year 1900.

Bremsstrahlung is also termed as the braking radiation. This radiation is typically produced because of the deacceleration or the negative acceleration of a charged particle, which contains the synchrotron radiation process in which the emission of photon takes place through a relativistic particle and the cyclotron radiation in which emission of photon takes place through a non-relativistic particle, in addition it also explains the electrons and positrons emission during the beta decay.

Bremsstrahlung radiation is specifically defined as the radiation that is released by means of a charged particle the 'Electron' owing to its acceleration that is caused by means of an electric field of another charged particle the 'Proton' or an atomic nucleus. The word "Bremsstrahlung" is a German word which means "Braking Radiation", and specifically refers to the approach in which the electrons are "Braked" when they typically hit a metal target. The incident electrons are considered as free, i.e., they are not bound to an atom or ion, both either before or after the braking.

In this unit, you will study bout the retarded potential, Liénard -Wiechert potentials due to uniformly moving and accelerated charges, Lorentz formula, Bremsstrahlung, radiation from on accelerated charged at low velocity (Larmor's formula), radiation from an oscillating electric dipole, linear antenna, radiation from a charged particle moving in a circular orbit, electric quadrupole radiation, and radiation damping.

3.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Understand what retarded potential is
- Explain the Liénard-Wiechert potentials due to uniformly moving and accelerated charges
- Analyse the Lorentz formula
- Interpret the Bremsstrahlung concept
- Define radiation due to accelerated charge at low velocity (Larmor's formula)
- Explain radiation due to an oscillating electric dipole
- Describe the significance of linear antenna
- Discuss about the radiation from a charged particle moving in a circular orbit
- Elucidate on electric quadrupole radiation
- Know what radiation damping is

3.2 RETARDED POTENTIAL

In physics, the 'Retarded Electromagnetic Potentials' are typically derived from the Maxwell's equations and the Lorenz condition. The Maxwell's equations are given by the physicist James Clerk Maxwell. Characteristically, the Liénard–Wiechert potentials precisely explain and evaluate the classical electromagnetic effect or consequence of a moving electric point charge with reference to a vector potential and a scalar potential in the Lorenz gauge. Reducing or decreasing directly from the Maxwell's equations, the Liénard–Wiechert potentials uniquely explain the comprehensive, relativistically appropriate, time-varying electromagnetic field for a point charge in arbitrary or random motion, but the Liénard–Wiechert potentials are not modified or amended for explaining the quantum mechanical effects.

Electromagnetic radiation in the form of waves can be obtained from these potentials. The expressions for the Liénard–Wiechert potentials are named after the physicists Alfred-Marie Liénard who developed in part in the year 1898 and then by Emil Wiechert who independently developed in the year 1900.

The key difference observed between these retarded electromagnetic potentials and the conventional Liénard–Wiechert potentials is precisely explained by ignoring the dependency of motion of the effective charge density. In addition, the subsequent retarded fields for a point-like charge specifically in the arbitrary or random motion are precisely compared and evaluated with the notions, formulae and equations given by the Oliver Heaviside, Richard Phillips Feynman, Oleg Dmitrovich Jefimenko and other authors. Consequently, the explanation and derivations given by Feynman about the fields of an accelerated charge are similar and identical to the explanation and derivations that are precisely derived from the Liénard–Wiechert potentials, although it is not similar to the explanations and derivations that are given and specified in the Jefimenko formulae. In the derivations and explanations given in the Jefimenko equations were not correct since a mathematical error or inaccuracy pertaining to partial space and time derivatives were observed and pointed out in the derivations.

Derivation of Retarded Electromagnetic Potentials from Inhomogeneous d'Alembert Equations

The retarded electromagnetic potentials may be derived from the Maxwell equations as follows:

$$\vec{\nabla} \cdot \vec{E} = 4\pi J_0, \tag{3.1}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J} \tag{3.2}$$

And the Lorenz condition,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial A_0}{\partial t} = 0 \tag{3.3}$$

Where the current density J is a 4-vector:

$$J(\vec{x}_J(t), t) = (J_0; \vec{J}) \equiv (\gamma_u \rho^*; \gamma_u \vec{\beta}_u \rho^*) = \frac{u\rho^*}{c}. \tag{3.4}$$

The system of source charges is assumed to be at rest in the frame S^*, where the charge density is ρ^*, and to move with velocity $\vec{u} = c\vec{\beta}_u$ relative to the frame S in which the potential is defined. The 4-vector velocity of the charge system in this last frame is:

$$u \equiv (c\gamma_u; c\gamma_u \vec{\beta}_u) \tag{3.5}$$

Where,

$$\beta_u \equiv \frac{u}{c}, \quad \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}}.$$

The first step of the calculation uses the Lorenz condition in Equation (3.3) to eliminate either $\vec{J}$ or J_0 from Equations (3.1) and (3.2) to obtain the following inhomogeneous d'Alembert equations:

$$\nabla^2 A_0 - \frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} = -4\pi J_0. \tag{3.6}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -4\pi \vec{J}. \tag{3.7}$$

The solutions give the retarded 4-vector potential of the form:

$$A_\mu^{ret}(\vec{x}_q, t) = \int dt' \int d^3x_J(t') \frac{J_\mu(\vec{x}_J(t'), t')}{|\vec{x}_q - \vec{x}_J(t')|}\delta(t' + \frac{|\vec{x}_q - \vec{x}_J(t')|}{c} - t). \tag{3.8}$$

In the special case of a single point-like source charge the current density in Equation (3.8) is given by the expression:

$$\mathrm{J}^Q(\vec{x}_J(t'), t') = \frac{Qu}{c}\delta(\vec{x}_J(t') - \vec{x}_Q(t')) \quad (3.9)$$

NOTES

Where $\vec{x}_Q(t')$ is the position of the charge at time t'. Inserting Equations (3.9) in (3.8), and integrating over $\vec{x}_J$., gives,

$$\mathrm{A}^{ret}_\mu(\vec{x}_q, t) = \frac{Qu_\mu}{c}\int dt' \frac{\delta(t' - t'_Q)}{r'} \quad (3.10)$$

Where,

$$r' \equiv |\vec{x}_q - \vec{x}_Q(t')|, \quad t'_Q \equiv t - \frac{|\vec{x}_q - \vec{x}_Q(t'_Q)|}{c} = t - \left.\frac{r'}{c}\right|_{t'=t'_Q} \quad (3.11)$$

The retarded 4-vector potential is therefore:

$$(\mathrm{A}^{ret}_0; \vec{\mathrm{A}}^{ret}) = \left(\left.\frac{Q\gamma_u}{r'}\right|_{t'=t'_Q}; \left.\frac{Q\gamma_u\vec{\beta}_u}{r'}\right|_{t'=t'_Q} \right) \quad (3.12)$$

3.3 LIÉNARD-WIECHERT POTENTIALS DUE TO UNIFORMLY MOVING AND ACCELERATED CHARGES

In physics, the theory and derivations of classical electrodynamics was initiated and structured by the physicist Albert Einstein's during the development of the 'Relativity Theory'. The significant analysis of the motion theory and propagation of electromagnetic waves formed the basis of the special and distinctive relativity description and derivations of space and time equations. The Liénard–Wiechert formulation is considered as an essential and substantial launchpad for the in depth analysis of relativistic moving particles. Typically, the Liénard–Wiechert potentials are used for describing the classical electromagnetic effect of a moving electric point charge precisely with regards to a vector potential and a scalar potential in the Lorenz gauge.

As already discussed, the Liénard–Wiechert potentials are directly constructed from the Maxwell's equations, therefore the Liénard–Wiechert potentials precisely explain the wide-ranging, relativistically precise, time-varying electromagnetic field for a point charge in arbitrary or random motion.

Characteristically, the Liénard–Wiechert description and derivation is considered accurate and exact for a substantial and significant independently moving particle, i.e., the analysis and derivation is 'Classical' and consequently the **acceleration of the charge** is because of a force that is precisely independent of the ElectroMagnetic Field or EMF.

NOTES

Principally, the Liénard–Wiechert formulation gives the following two sets of solutions:

1. Advanced Fields are Absorbed by the Charges.
2. Retarded Fields are Emitted.

Fundamentally, the Liénard–Wiechert potential formulation represents explicit and specific expressions for time-varying electromagnetic fields that are uniquely caused by means of charge in arbitrary or random motion. However, the Liénard–Wiechert potentials were distinctively derived from the retarded potentials, which in sequence are derived and explained from the Maxwell equations.

Equations of Liénard–Wiechert Potentials

The Liénard–Wiechert potentials φ (scalar potential field) and $\mathbf{A}$ (vector potential field) are for a source point charge q at position $\mathbf{r}_s$ traveling with velocity $\mathbf{v}_s$:

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(1 - \mathbf{n}\cdot\boldsymbol{\beta}_s)|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r}$$

And,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 c}{4\pi}\left(\frac{q\boldsymbol{\beta}_s}{(1 - \mathbf{n}\cdot\boldsymbol{\beta}_s)|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r} = \frac{\boldsymbol{\beta}_s(t_r)}{c}\varphi(\mathbf{r}, t)$$

Where,

- $\boldsymbol{\beta}_s(t) = \frac{\mathbf{v}_s(t)}{c}$ is the velocity of the source expressed as a fraction of the speed of light.
- $|\mathbf{r} - \mathbf{r}_s|$ is the distance from the source.
- $\mathbf{n} = \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|}$ is the unit vector pointing in the direction from the source.

Corresponding Values of Electric and Magnetic Fields

We can calculate the electric and magnetic fields directly from the potentials using the following equations:

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} \text{ and } \mathbf{B} = \nabla\times\mathbf{A}$$

The calculation is nontrivial and requires a number of steps. The electric and magnetic fields are (in non-covariant form):

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q(\mathbf{n} - \boldsymbol{\beta})}{\gamma^2(1 - \mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r} - \mathbf{r}_s|^2} + \frac{q\mathbf{n}\times\big((\mathbf{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big)}{c(1 - \mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r}$$

And,

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\left(\frac{qc(\boldsymbol{\beta}\times\mathbf{n})}{\gamma^2(1 - \mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r} - \mathbf{r}_s|^2} + \frac{q\mathbf{n}\times\Big(\mathbf{n}\times\big((\mathbf{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big)\Big)}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r} - \mathbf{r}_s|}\right)_{t_r} = \frac{\mathbf{n}(t_r)}{c}\times\mathbf{E}(\mathbf{r}, t)$$

Where,

$$\beta(t) = \frac{\mathbf{v}_s(t)}{c},\ \mathbf{n}(t) = \frac{\mathbf{r} - \mathbf{r}_s(t)}{|\mathbf{r} - \mathbf{r}_s(t)|} \text{ and } \gamma(t) = \frac{1}{\sqrt{1 - |\beta(t)|^2}} \text{ (the Lorentz factor).}$$

Note that the $\mathbf{n} - \beta$ part of the first term updates the direction of the field towards the instantaneous position of the charge, if it continues to move with constant velocity $c\beta$. This term is connected with the 'Static' part of the electromagnetic field of the charge.

The second term, which is connected with electromagnetic radiation by the moving charge, requires charge acceleration $\dot{\beta}$ and if this is zero, then the value of this term is zero, and the charge does not radiate, i.e., emit electromagnetic radiation. This term requires additionally that a component of the charge acceleration be in a direction transverse to the line which connects the charge q and the observer of the field $\mathbf{E}(\mathbf{r}, t)$. The direction of the field associated with this radiative term uniquely specifies the time-retarded position of the charge, i.e., where the charge was when it was accelerated.

3.4 LORENTZ FORMULA

In physics, principally in the field of electromagnetism the term Lorentz force also sometimes referred as the electromagnetic force is defined as the combination of electric force and magnetic force on a point charge that exclusively occurs as a result of electromagnetic fields. The Lorentz formula refers to the Lorentz force which was formulated by Hendrik Lorentz who precisely derived the contemporary or present form of the formula and its derivation typically for the electromagnetic force which includes the analysis and formulation of the total force obtained from both the electric fields and the magnetic fields.

The Lorentz force is defined as a force that is exerted by means of the electromagnetic field on the charged particle, i.e., it is precisely specified as the rate at which the linear momentum is transferred from the electromagnetic field to the particle. From a contemporary perspective it is possible that the Maxwell's 1865 formulation and equations can be identified as the unique form of the Lorentz force equation with respect to electric currents.

In this equation, the general form of wave equations despite medium properties will be covered. During the process of deriving we will observe the conditions for relations between the scalar electric potential, 'V' and vector magnetic potential '$\vec{A}$'. This condition is called Lorentz condition. The procedure to obtain the wave equations include, starting from Maxwell's equation and applying the vector identities and simple manipulations of the obtained equations will result in the wave equations.

NOTES

From Maxwell's second equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{dt}$$

We know that,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Therefore,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times \vec{E} + \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = 0$$

$$\vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0 \qquad (3.13)$$

We know the vector identity that curl of a gradient is zero. Therefore,

$$\vec{\nabla} \times \left(-\vec{\nabla} V\right) = 0 \qquad (3.14)$$

Comparing Equations (3.13) and (3.14), we get,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\left[\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right] \qquad (8.3)$$

Taking divergence of Equation (3.15), we get,

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left[-\left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t}\right)\right]$$

$$\vec{\nabla}.\vec{E} = -\left[\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A})\right] \qquad (3.16)$$

But from Maxwell's first equation,

$$\vec{\nabla}.\vec{E} = \frac{\rho_V}{\epsilon}$$

Therefore, equating the above equation with Equation (3.16),

$$\frac{\rho_V}{\epsilon} = -\left[\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A})\right]$$

Or

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho_V}{\epsilon} \quad (3.17)$$

So far, we have used Maxwell's equation for electric field and now let us use Maxwell's equation for magnetic field to couple the two fields to arrive at the electromagnetic waves. Considering, Maxwell's fourth equation,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{dt}$$

But, $\vec{B} = \mu\vec{H}$. Therefore,

$$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \vec{J} + \frac{\partial \vec{D}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\frac{\partial \vec{D}}{dt}$$

Also, $\vec{D} = \epsilon\vec{E}$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial \vec{E}}{dt} \quad (3.18)$$

Substituting for $\vec{E}$, from Equation (3.15) , we get,

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial}{dt}\left[-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right]$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

But $\vec{B} = \vec{\nabla} \times \vec{A}$, Hence,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} \quad (3.19)$$

Applying vector identity to the LHS of Equation (3.19), we get,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$$

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\vec{\nabla}\frac{\partial V}{dt} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} \quad (3.20)$$

Observing Equation (3.20), we find that scalar potential V and vector potential $\vec{A}$ can be separated and decoupled. Therefore for vector potential, $\vec{A}$, Equation (3.20) becomes,

$$-\nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

NOTES

Or

$$\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J} \quad (3.21)$$

Similarly, it is good to separate for scalar potential V in terms of $\vec{A}$ because we obtain a relation relating $\vec{A}$ and V known as ***Lorentz condition*** for potentials.

$$\vec{\nabla}(\vec{\nabla}.\vec{A}\,) = -\mu\epsilon\vec{\nabla}\frac{\partial V}{dt}$$

$$= \vec{\nabla}\left(-\mu\epsilon\frac{\partial V}{dt}\right)$$

$$\vec{\nabla}.\vec{A} = -\mu\epsilon\frac{\partial V}{dt} \quad (3.22)$$

The above relation is known as **Lorentz condition for potentials**.

Recalling Equation (3.17),

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\frac{\rho_V}{\epsilon}$$

Substituting for $\vec{\nabla}.\vec{A}$ from Equation (3.22) in Equation (3.17) above,

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\epsilon\frac{\partial V}{dt}\right) = -\frac{\rho_V}{\epsilon}$$

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{dt^2} = -\frac{\rho_V}{\epsilon} \quad (3.23)$$

Recalling Equation (3.21) as below,

$$\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J}$$

The above two equations are called as ***wave equations***. Equations (3.21) and (3.22) are decoupled equations of $\vec{A}$ and V, whereas Equation (3.17) and (3.20) are coupled equations, in which and are interlinked.

Free Space Wave Equations

In free space, $\rho_V = 0$ and $\vec{J} = 0$ and hence, the wave equations simplify to,

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{dt^2} = -\frac{\rho_V}{\epsilon} \quad (3.24(a))$$

NOTES

And

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = \mu\vec{J} \qquad (3.24(b))$$

From electrostatics and magnetostatics,

$$V = -\oint_L \vec{E}.\vec{dl}$$

And

$$\vec{\nabla} \times \mu\vec{H} = \vec{A}$$

Eqnation (3.24) can be rewritten as,

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{dt^2} = 0 \qquad (3.25(a))$$

And

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = \mu\vec{J} \qquad (3.25(b))$$

Uniform Plane Waves

Plane waves are waves with same phase at all points of existence. Uniform plane waves are plane waves with constant amplitude.

Consider an electric field wave equation from Equation (3.25(a))

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{dt^2} = 0$$

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{dt^2}$$

We know that, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and

$\vec{E} = E_x \overrightarrow{a_x} + E_y \overrightarrow{a_y} + E_z \overrightarrow{a_z}$. Therefore,

$$\frac{\partial^2 E_x}{\partial x^2}\overrightarrow{a_x} + \frac{\partial^2 E_y}{\partial y^2}\overrightarrow{a_y} + \frac{\partial^2 E_z}{\partial z^2}\overrightarrow{a_z} = \mu\epsilon \left[\frac{\partial^2 E_x}{\partial t^2}\overrightarrow{a_x} + \frac{\partial^2 E_y}{\partial t^2}\overrightarrow{a_y} + \frac{\partial^2 E_z}{\partial t^2}\overrightarrow{a_z}\right]$$

In the above equation, it should be noted that $\frac{\partial^2 E_y}{\partial y^2} = 0$ and likewise for unequal variables. Splitting the above three dimensional vector equation to a one-dimensional scalar equation,

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} \qquad (3.26(a))$$

$$\frac{\partial^2 E_y}{\partial y^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2} \quad (3.26(b))$$

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$$\frac{\partial^2 E_z}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_z}{\partial t^2} \quad (3.26(c))$$

Also, in free space, where the electromagnetic waves travel, (they can also travel through different medium), $\rho_V = 0$ and $\vec{J} = 0$. Therefore, from Maxwell's first equation,

$$\vec{\nabla}.\vec{D} = 0$$

$$\vec{\nabla}.\epsilon\vec{E} = 0$$

$$\vec{\nabla}.\vec{E} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial y}\overrightarrow{a_y} + \frac{\partial}{\partial z}\overrightarrow{a_z}\right).\left(E_x\overrightarrow{a_x} + E_y\overrightarrow{a_y} + E_z\overrightarrow{a_z}\right) = 0$$

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z = 0$$

Since, E_x is travelling in the x –direction, $\frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = 0$. Then, $\frac{\partial E_x}{\partial x}$ represents that no variation of E_x in x-direction and also $\vec{E}$ is independent of y and z. Differentiating with respect to x,

$$\frac{\partial^2 E_x}{\partial x^2} = 0$$

The solution of the above second order differential equation exists only if,

$$E_x = 0 \quad or \quad E_x = K(constant)$$

If $E_x = K$, then E_x is not a wave, but a constant dc line. Hence, a uniform plane wave travelling in x –direction do not have an component of $\vec{E}$. Similarly for the other two directions y and z.

For the magnetic field vector, following the similar approach, from Maxwell's third equation,

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}.\mu\vec{H} = 0$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial y}\overrightarrow{a_y} + \frac{\partial}{\partial z}\overrightarrow{a_z}\right).\left(H_x\overrightarrow{a_x} + H_y\overrightarrow{a_y} + H_z\overrightarrow{a_z}\right) = 0$$

$$\frac{\partial}{\partial x}H_x + \frac{\partial}{\partial y}H_y + \frac{\partial}{\partial z}H_z = 0$$

Since, H_x is travelling in the x –direction, $\frac{\partial H_y}{\partial y} = \frac{\partial H_z}{\partial z} = 0$. $\vec{H}$ is independent of y and z. Therefore,

$$\frac{\partial H_x}{\partial x} = 0 \Rightarrow \frac{\partial^2 H_x}{\partial x^2} = 0$$

Since H_x cannot be constant, to satisfy the above second order differential equation, $H_x = 0$ for uniform plane wave.

Properties of Electromagnetic Waves

Electromagnetic waves transport energy or information from one point to the other. Few examples of electromagnetic waves include the waves in the electromagnetic spectrum as shown in Figure (3.1). Few electromagnetic waves include X-rays, Gamma rays, microwave, TV signals, radar signals, light rays, etc. The electromagnetic waves are also called as Hertzian waves.

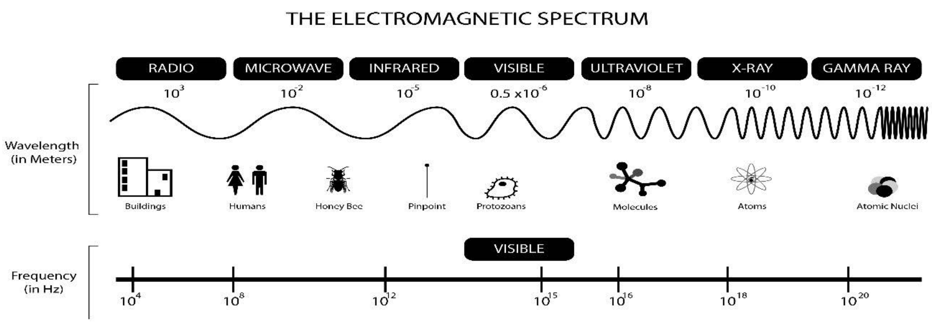

Fig. 3.1 *Electromagnetic Spectrum*

Characteristics of EM Waves

Following are the characteristics of EM waves.

1. They travel at a speed of light in vacuum.
2. They travel similar to the waves with the same their same property.
3. They radiate away from the source
4. They can travel across any medium
5. EM waves are generated by vibration of electrons resulting in energy emission called as electromagnetic radiation.
6. Electromagnetic waves have both electric and magnetic components.
7. Electric and magnetic components are orthogonal (perpendicular) to each other.
8. The direction of wave propagation will be orthogonal to the electric and magnetic waves. Such waves are called Transverse ElectroMagnetic waves (TEM waves).

Parameters of Wave

A simple wave is a sinusoidal signal as represented as in Figure (3.2).

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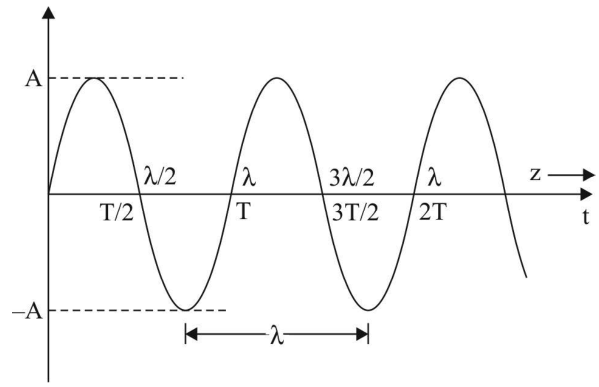

Fig. 3.2 *EM Wave*

Let $\lambda = wavelength\ (m)$

$T = Time\ period\ (S)$

$u = speed\ (m/s)$

$\therefore \quad u = \frac{\lambda}{T} = f\lambda \qquad [f = 1/T]$

Let $\omega = angular\ frequency\ (rad/s)$

$\beta = phase\ constant\ or\ wave\ number\ (rad/m)$

$$\omega = 2\pi f\ (rad/s)$$

$$\beta = \frac{\omega}{u} = \frac{2\pi f}{u}$$

$$\beta = \frac{2\pi f}{f\lambda}$$

$$\beta = \frac{2\pi}{\lambda}$$

3.5 BREMSSTRAHLUNG

Bremsstrahlung is also termed as the **braking radiation**. This radiation is typically produced because of the deacceleration or the negative acceleration of a charged particle, which contains the **synchrotron radiation** process in which the emission of photon takes place through a relativistic particle and the **cyclotron radiation** in which emission of photon takes place through a non-relativistic particle, in addition it also explains the electrons and positrons emission during beta decay.

Though, this term is commonly used in the perception of radiation from electrons and the Bremsstrahlung that is emitted from plasma is also occasionally described as **free–free radiation**. This implies that the radiation is created in this instance by the electrons which were free before and stay free after also when there is the emission of a photon. In the similar parlance, the **bound–bound** radiation describes the discrete spectral lines, An electron typically 'Jumps' between two bound states, even though the **free–bound** one to the radiative combination method, wherein a free electron recombines or reunites with an ion from whatsoever source decelerating in matter.

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The 'Bremsstrahlung' or 'Braking Radiation' is specifically identified as the radiation that is typically given off by means of the free electrons which are deflected, i.e., the electrons are accelerated in the electric fields of charged particles and the nuclei of atoms. Thermal bremsstrahlung refers to the emission typically given off by means of an ionized gas of plasma in thermal equilibrium at a specific or particular distinct temperature, wherein the unique distribution of electron velocities typically goes along with the recognised Maxwellian distribution. Additionally, the Relativistic electrons whose energy distribution frequently follows a power-law shape specifically in the astrophysical settings, which further produce relativistic Bremsstrahlung radiation which in turn is also of power-law shape having the equivalent spectral index as the emitting electrons.

If the quantum effects are considered negligible, then an accelerating charged particle radiates the power based on the theory as explained by the Larmor formula and its relativistic generalization.

Total Radiated Power

The total radiated power is given by the equation,

$$P = \frac{q^2\gamma^4}{6\pi\varepsilon_0 c}\left(\dot{\beta}^2 + \frac{\left(\beta \cdot \dot{\beta}\right)^2}{1-\beta^2}\right)$$

Where,

$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}$ is defined as the Velocity of the Particle divided by the Speed of Light

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is defined as the Lorentz Factor

$\dot{\boldsymbol{\beta}}$ Denotes a Unique Time Derivative of $\boldsymbol{\beta}$.

q is defined as the Charge of the Particle.

Bremsstrahlung Radiation

Bremsstrahlung radiation is specifically defined as the radiation that is released by means of a charged particle the '**Electron**' owing to its acceleration that is caused by means of an electric field of another charged particle the '**Proton**' or an atomic nucleus. The word "Bremsstrahlung" is a German word which means "Braking Radiation", and specifically refers to the approach in which the electrons are "Braked" when they typically hit a metal target. The incident electrons are considered as free, i.e., they are not bound to an atom or ion, both

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either before or after the braking. Consequently, this type of the radiation spectrum is considered continuous, dissimilar atomic spectra which comprises of sharp spectral lines, and occasionally referred to as "Free-Free" radiation. If the energy of the incident electrons is adequately high, then they specifically emit X-rays once they have been braked.

One of the greatest universally recognized and identified examples of Bremsstrahlung radiation in the **universe** is the one which comes from the hot intracultural gas of the **galaxy clusters**. In this specific situation, the electrons do not deflect or bounce off a metal target although they are deflected or bounced off by means of the electric field of **protons**. The gas holds X-ray **luminosities** of 10^{36} to 10^{38} W (roughly 10 billion to 1 trillion times the **luminosity** of the **sun**) and temperatures on the order of 10 million K. X-ray **telescopes** can detect this radiation as diffuse **light**, as witnessed in the Coma cluster.

Bremsstrahlung is electromagnetic radiation which is characteristically similar to x-radiation. It is emitted by means of a charged particle because it decelerates in a series of collisions with atomic particles. Due to this deflection a deceleration of the beta particle is obtained and consequently a reduction in its kinetic energy with the emission of energy as a photon of bremsstrahlung or 'Braking Radiation'. The phenomenon is explained and illustrated through Bremsstrahlung radiation imaging.

Bremsstrahlung is, therefore, a physical phenomenon typically used in the radiology apparatus. When an electron or a beta (β) particle passes through matter then it decelerates or slows down, and a fraction of its energy is directly converted into X-rays. The spectrum of X-ray emission is continuous, and its maximum energy is the initial energy of the electron. For example, a beta (β) emitter, such as ^{14}C can emit X-rays of up to 156 keV in any given sample. The similar phenomenon can be used to produce X-rays. The Bremsstrahlung yield is in fact 'Actually' proportional to the atomic number of the media and 'Roughly' proportional to the square of the energy. In the biological tissue, the atomic number is low (between 7 and 8), and the yield stays very low. For ^{90}Y (E_{max} = 2.2 MeV), only 1% of the energy is converted into X-rays, i.e., 20 keV per beta (β) particle, spread over a spectrum, the maximum energy of which is 2.2 MeV. In probability terms, less than 20% of beta (β) particles provide an X-ray that can contribute to form an image. Characteristically, the collimators used for scintigraphy have an efficiency of 100 cps/MBq, even for high-energy beta-emitting isotopes, the total efficiency of Bremsstrahlung imaging is certainly not more than 20 cps/MBq. For a lower-energy tracer, such as ^{14}C, the yield is at least 100-fold weaker and therefore not suitable. Bremsstrahlung radiation imaging is therefore mainly used in clinical imaging, where it is combined with high-energy tracers requiring visualization that could not be seen otherwise. A good example is ^{90}Y, a tracer widely used for radiotherapy. Bremsstrahlung scintigraphy allows imaging of the specific localization of the tracer to target tumor sites.

Thermal Bremsstrahlung: Emission and Absorption

Internal bremsstrahlung evolves in the process of radioactive disintegration of beta decay, which typically consists of the production and emission of electrons (or positrons, positive electrons) by means of unstable atomic nuclei or the capture by nuclei of one of their individual orbiting electrons. These electrons, deflected in the vicinity of their specific individual associated nuclei, emit internal bremsstrahlung.

In Figure 3.3, the dashed line represents bremsstrahlung emission coefficients included only ion-ion correlation effects, the solid-dashed line represents ion-ion correlation and free electron shielding effects, and solid line represents ion-ion correlation and total electron shielding effects. Bound electron shielding effects are dominant for reduction in bremsstrahlung emission.

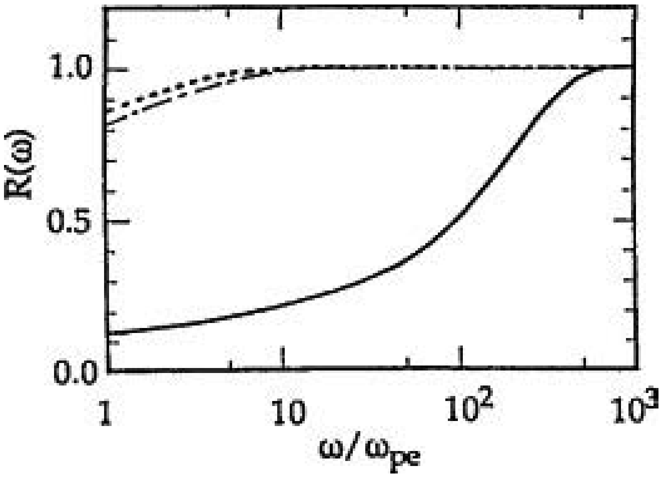

Fig. 3.3 *Bremsstrahlung Emission - Ion-Ion Correlation Effects*

The simulation in a fictitious plasma is performed, Z=3.97, Γ=0.553, T=1keV, for investigating the effective ionization state for Bremsstrahlung emission. Figure 3.4 shows the reduction factor in the fictitious plasma. The three types of lines characterize the identical as per the Bremsstrahlung emission. Therefore, if the plasma is free ionized, then the effective ionization state for Bremsstrahlung emission can be approximated by the plasma with Γ_{eff} and Z_{eff}, for two component plasma.

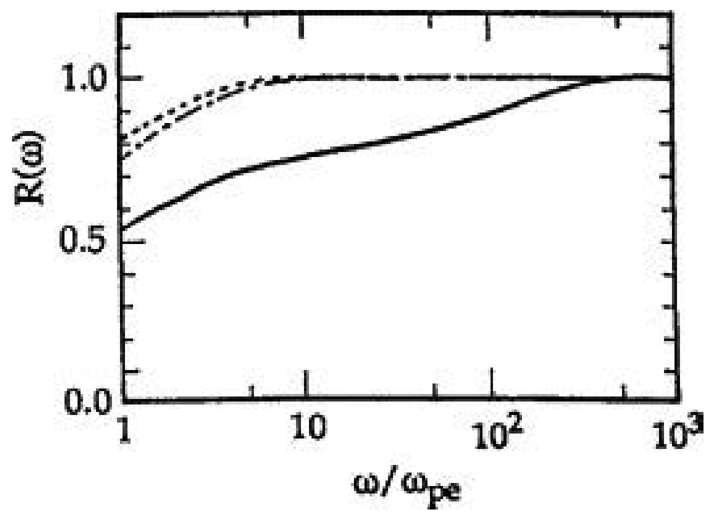

Fig. 3.4 *Ionization State for Bremsstrahlung Emission*

Bremsstrahlung radiation is, therefore, the standard radiation that is given off through a charged particle, typically an electron, due to its acceleration that is caused by means of an electric field of another charged particle, specifically a proton or an atomic nucleus.

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3.6 RADIATION FROM AN ACCELERATED CHARGE AT LOW VELOCITY (LARMOR'S FORMULA)

Accelerating charges produce EM radiation, because the changing electric fields yield magnetic fields (Ampere-Maxwell law) and the resulting changing magnetic fields yield electric fields (Faraday's Law). Consequently, these are the electromagnetic radiation, where the changing fields propagate.

Additionally, it can be stated as, "An accelerated point charge emits energy and impulses in the form of radiation". With regard to the conservation of energy, it is said that, "The radiated energy is directly extracted from the mechanical kinetic energy of the charged particle and not from the potential energy of the electromagnetic system".

The phenomenon in nuclear magnetic resonance is termed as the Larmor precession.

In electrodynamics, the **Larmor formula** is specifically used for calculating the total power radiated through a nonrelativistic point charge as it accelerates. It was originally derived by J. J. Larmor in 1897, in the perspective of the wave theory of light. When any charged particle, such as an electron, a proton or an ion accelerates, then it typically radiates away energy in the form of electromagnetic waves. For those velocities which are small comparative to the speed of light, the total power radiated is given by the Larmor formula:

$$P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c}\left(\frac{\dot{v}}{c}\right)^2 = \frac{2}{3}\frac{q^2 a^2}{4\pi\varepsilon_0 c^3} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \text{ (SI units)}$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} \text{ (cgs units)}$$

As it is known that charge particle is at rest emits electric field. Charge particle is in motion or moving with constant velocity emits **Electric Field + Magnetic Field**. Charge particle is in accelerated motion emits electric field, magnetic field as well electromagnetic radiation or EM wave.

From the Maxwell's electromagnetic field equations, it is evident that both the electric field and magnetic field are function of velocity.

E (Electric Field)- Function of Velocity.

B (Magnetic Field)- Function of Velocity.

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When v is constant means particle moving with constant velocity, then it has both the electric field and the magnetic field. While the velocity of the particle changes the means particle is in accelerated motion and both it's electric field and magnetic field will change. According to Faraday's law of ElectroMagnetic Induction (EMI), "Changing electric field can produce magnetic field and changing magnetic field can generate electric field".

Consequently, the changing of electric field is basically producing magnetic field and vice-versa. Therefore, electric field and magnetic field is being oscillated upon each other. Thus, the wave producing is the oscillation of the **Electric Field** and **Magnetic Field** is characteristically termed as EM wave or radiation which is capable of travelling even though the vacuum with the speed of 3×10^{8}m/s.

Therefore, EM wave is a wave of the oscillation of changing electricity and magnetism, and it will be produced when the particle is in acceleration.

The behaviour of an accelerated electric charge, regardless of the accelerating force, is a limit problem in classical physics. The radiation of a system of charges is typically described through Poynting's theorem, which is a logical consequence of Maxwell's equations. The key point is that the Poynting vector, which is defined in this theorem, is interpreted as a flow of radiant energy using the principle of the conservation of energy in an electromagnetic system. From this perspective, it seems that radiation, similar to potential energy, is a behaviour that is associated with a system of charges and not with individual charges. In this sense, the literature discusses dipole radiation, quadrupole radiation, etc. However, in classical theory, it is immediately clear that the radiation of a system of charges can be calculated if the motion of the charges is known because that is sufficient to determine the fields that define the Poynting vector. There is a direct relationship between the motion of a system of charges and the resulting radiation. H.A. Lorentz went further and extended the result for an isolated charge that is accelerated by any force (a magnetic field, gravity, etc., independently of the existence of electromagnetic potential energy. This author demonstrated that the field in the proximity of a charge with spherical symmetry becomes distorted by the combined effects of the acceleration of the charge and the finite propagation velocity of changes in the field. This distortion generates a net "Self-Force" of the field on the particle on its own source; the displacement of this force can represent, at least in specific situations, the radiated electromagnetic energy. Consequently, Lorentz did not ascribe radiation to the relative accelerations between the charges in the system as was expected if there was a relationship with the potential energy, but to the acceleration of a charge with respect to any inertial coordinate system. An accelerated point charge emits energy and impulses in the form of radiation. The reason for this attribution is the presence of emitted energy in the form of oscillations of the particle's electromagnetic field in regions relatively far from the particle; these make up the radiation field. An observer in the radiation field can relate this energy to an event that occurred at a point occupied by the charge at an earlier time. This delay time corresponds to the propagation velocity of an electromagnetic signal from the particle, and the event referred to is a change in the velocity of the charged particle. In regard to the conservation of energy, the radiated energy is directly extracted from the mechanical kinetic energy of the charged particle, not from the potential energy of the electromagnetic system.

3.7 RADIATION FROM AN OSCILLATING ELECTRIC DIPOLE

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In physics, a **dipole** is referred as a quantity which involves some form of polarity. The most significant resource of **electromagnetic radiation** is possibly the oscillating electric dipole. When the current density in a localized source oscillates harmonically with angular frequency, then it has an electric dipole moment of the form where there is the complex amplitude.

An electric dipole is concerned with the separation of the positive and negative charges observed in any electromagnetic system. A simple example of this system is a pair of electric charges of equal magnitude, although opposite sign separated by means of certain typically small distance. A permanent electric dipole is occasionally termed as an electret.

The electric dipole moment is defined as a measure of the separation of positive and negative electrical charges within a system, i.e., it is a measure of the system's overall polarity. The SI unit for electric dipole moment is the Coulomb meter (C·m). The Debye (D) is another unit of measurement used in atomic physics and chemistry.

Theoretically, an electric dipole is defined by the first-order term of the multipole expansion; it consists of two equal and opposite charges that are infinitesimally close together, although real dipoles have separated charge.

An object with an electric dipole moment is subject to a torque τ when placed in an external electric field. The torque tends to align the dipole with the field. A dipole aligned parallel to an electric field has lower potential energy than a dipole making some angle with it. For a spatially uniform electric field **E**, the energy U and the torque τ are given by,

$$U = -\mathbf{p} \cdot \mathbf{E}, \qquad \boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}.$$

where **p** is the dipole moment, and the symbol '×' refers to the vector cross product. The field vector and the dipole vector define a plane, and the torque is directed normal to that plane with the direction given by the right-hand rule.

A dipole oriented co- or anti-parallel to the direction in which a non-uniform electric field is increasing (gradient of the field) will experience a torque, as well as a force in the direction of its dipole moment. It can be shown that this force will always be parallel to the dipole moment regardless of co- or anti-parallel orientation of the dipole.

Magnetic dipole is the closed circulation of an electric current system. A simple example is a single loop of wire with constant current through it. A bar magnet is an example of a magnet with a permanent magnetic dipole moment.

Dipoles, whether electric or magnetic, can be characterized by their dipole moment, a vector quantity. For the simple electric dipole, the electric dipole moment points from the negative charge towards the positive charge and has a magnitude equal to the strength of each charge times the separation between the charges. To be specific, for the definition of the dipole moment, one should always consider the 'Dipole Limit', where, for example, the distance of the generating charges should *converge* to 0 while simultaneously, the charge strength should *diverge* to infinity in such a way that the product remains a positive constant.

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Electric Dipole Radiation in Free Space

The most significant source of **electromagnetic radiation** is possibly the oscillating electric dipole. When the current density in a localized source oscillates harmonically with angular frequency, it has an electric dipole moment of the form where is the complex amplitude.

Dipole Radiation

Instead of heading straight to dipole radiation, it would be beneficial to first define certain preliminary terms and understand the notion of radiation as a general phenomenon. In general, radiation can be thought of as an irreversible transport of energy, out to infinity, by fields, due to the acceleration of charges. In other words, accelerated charges create electromagnetic radiation fields, that carry energy from one point in space, to another, in an irreversible fashion. Irreversible simply means that there is no definable way to extract the energy back from the wave, i.e., the energy radiated outwards, does not come back in. The term 'To Infinity' in the definition of radiation, implies that it is important that the observer is at a very distant location from the source, or that the source is a point source.

It is known to us, that even static charges and steady currents produce electromagnetic fields. However, accelerated charges and changing currents produce a slightly difierent kind of electromagnetic fields, called electromagnetic radiation fields.

Calculating the potentials due to an electric dipole is possible about the radiation mathematically.

3.8 RADIATION FROM A CHARGED PARTICLE MOVING IN A CIRCULAR ORBIT

It is established that a moving charge produces ElectroMagnetic Waves (EMWs) as such charge moving in a circular orbit experiences centripetal acceleration will also produces electromagnetic wave.

An accelerated charge is referred as the source of ElectroMagnetic Waves (EMWs). When the charge is in a circular motion, the direction of its velocity continuously changes and thus it is in accelerated motion and produces EMWs. A charge falling in an electric field is accelerated by the electric force and thus produces ElectroMagnetic Waves (EMWs).

Radiation Energy produced by a charged particle circulating in a magnetic field does radiate energy, which it is called **synchrotron radiation**. All circular particle accelerators have energy losses due to this radiation. Similarly when a charged particle is projected in the plane perpendicular to a uniform magnetic field it executes uniform circular motion with radius $r = mv/qBr = mv/qB$.

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Synchrotron radiation is the electromagnetic radiation also known as **magneto bremsstrahlung radiation is** emitted when charged particles are accelerated radially, e.g., when they are subject to an acceleration perpendicular to their velocity (**a** ⊥ **v**). It is produced, for example, in synchrotrons using bending magnets, undulators and/or wigglers. If the particle is non-relativistic, the emission is called cyclotron emission. If the particles are relativistic, sometimes referred to as ultra-relativistic then the emission is called synchrotron emission.

Synchrotron radiation may be achieved artificially in synchrotrons or storage rings, or naturally by fast electrons moving through magnetic fields. The radiation produced by this manner has a characteristic polarization and the frequencies generated can range over the entire electromagnetic spectrum, which is also called continuum radiation.

In astrophysics it is stated that, synchrotron emission occurs, due to ultra-relativistic motion of a source around a black hole. When the source performs a circular geodesic around the **black hole**, the synchrotron radiation occurs for orbits close to the photosphere where the motion is in the ultra-relativistic regime.

Synchrotron radiation was named after it was discovered in Schenectady, New York from a General Electric synchrotron accelerator built in 1946 and announced in May 1947 by Frank Elder, Anatole Gurewitsch, Robert Langmuir and Herb Pollock in a letter entitled "Radiation from Electrons in a Synchrotron".

Emission Mechanism

Synchrotron radiation is produced when moving particles accelerate, e.g. when electrons move freely in a magnetic field. This is similar to a radio antenna, but with the difference that, in theory, the relativistic speed will change the observed frequency due to the Doppler effect by the Lorentz factor γ. Relativistic length contraction then bumps the frequency observed by another factor of γ, thus multiplying the Giga Hertz (GHz) frequency of the resonant cavity that accelerates the electrons into the X-ray range. The radiated power is given by the relativistic Larmor formula, while the force on the emitting electron is given by the Abraham–Lorentz–Dirac force.

The radiation pattern can be distorted from an isotropic dipole pattern into an extremely forward-pointing cone of radiation. Synchrotron radiation is the brightest artificial source of X-rays.

The planar acceleration geometry appears to make the radiation linearly polarized when observed in the orbital plane, and circularly polarized when observed at a small angle to that plane. Amplitude and frequency are, however, focused to the polar ecliptic.

Synchrotron radiation is also generated by astronomical objects, typically where relativistic electrons spiral (and hence change velocity) through magnetic fields. Two of its characteristics include non-thermal power-law spectra, and polarization. It is considered to be one of the most powerful tools in the study of extra-solar magnetic fields wherever relativistic charged particles are present. Most known cosmic radio sources emit synchrotron radiation. It is often used to estimate the strength of large cosmic magnetic fields as well as analyse the contents of the interstellar and intergalactic media.

Some of the characteristic properties of synchrotron radiation are stated below:

1. Having a Broad Spectrum (from Microwaves to Hard X-Rays): The users can select the wavelength required for their experiment.
2. High Flux: High-intensity photon beam allows rapid experiments or use of weakly scattering crystals.
3. High Brilliance: Highly collimated photon beam generated by a small divergence and small-size source (spatial coherence).
4. High Stability: Sub micrometre source stability.
5. Polarization: Both linear and circular.
6. Pulsed Time Structure: Pulsed duration down to tens of picoseconds allows the resolution of process on the same time scale.

3.8.1 Linear Antenna

As per the radio engineering, an **antenna** also known as **aerial** is the interface between radio waves propagating through space and electric currents moving in metal conductors, used with a transmitter or receiver.

In the process of transmission, a radio transmitter supplies an electric current to the antenna's terminals, and the antenna radiates the energy from the current as electromagnetic waves (radio waves). In reception, an antenna intercepts some of the power of a radio wave in order to produce an electric current at its terminals, that is applied to a receiver to be amplified. Antennas are essential components of all radio equipment.

An antenna is an array of conductors (elements), electrically connected to the receiver or transmitter. Antennas can be designed to transmit and receive radio waves in all horizontal directions equally (omnidirectional antennas), or preferentially in a particular direction (directional, or high-gain, or "beam" antennas). An antenna may include components not connected to the transmitter, parabolic reflectors, horns, or parasitic elements, which serve to direct the radio waves into a beam or other desired radiation pattern.

Antennas were first time built by German physicist Heinrich Hertz in 1888 during his experiments to prove the **existence of waves** predicted by the electromagnetic theory of James Clerk Maxwell. Hertz placed dipole antennas at the focal point of parabolic reflectors for both transmitting and receiving. Starting in 1895, Guglielmo Marconi began development of antennas practical for long-distance, wireless telegraphy, for which he received a Nobel Prize.

Any radio receiver or transmitter needs the antenna to couple its electrical connection to the electromagnetic field. Radio waves are electromagnetic waves which carry signals through the air (or through space) at the speed of light with almost no transmission loss.

There are numerous distinct types of antennas which can be broadly categorized into three broad types, namely **Omni-Directional**, **Directional**, and **Semi-Directional**. Further antennas can be classified by operating principles or by their application, some of which are listed below:

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- Radio Frequency Antenna Types
- Radio Frequency Propagation
- Cellular Repeater
- Dxing
- Electromagnetism
- Mobile Broadband Modem
- Numerical Electromagnetics Code
- Radial (Radio)
- Radio Masts and Towers
- RF Connector
- Smart Antenna
- TETRA
- Shortwave Broadband Antenna
- Personal RF Safety Monitor

Linear antennas are considered to be those antennas that imply the use of electrically thin conductors (wavelength conductor diameter). Electric current flows over the conductor surface in order to calculate radiated fields in these antennas. Conductors are modelled as if they were current lines with no diameter.

3.8.2 Electric Quadrupole Radiation

A distribution of charge or magnetization which produces an **electric or magnetic field equivalent** to that produced by two electric or magnetic dipoles whose dipole moments have the same magnitude but point in opposite directions, and which are separated from each other by a small distance.

Electric quadrupole is referred as a charge distribution that produces an electric field equivalent to that produced by two electric dipoles whose dipole moments have the same magnitude but point in opposite directions and which are separated from each other by a small distance.

A general distribution of electric charge may be characterized by its net charge, by its dipole moment, its quadrupole moment and higher order moments. An elementary quadrupole can be represented as two dipoles oriented antiparallel. Quadrupole and higher order multipoles are not important for the characterization of dielectric materials. Dipole fields are much smaller than the fields of isolated charges, but in dielectrics where there are no free charges, the dipole effects are dominant. There is no such circumstance favouring the quadrupole effects, since they must arise from the same number of molecules as the dipole effects.

The simplest example of an electric quadrupole consists of alternating positive and negative charges, arranged on the corners of a square. The monopole moment (just the total charge) of this arrangement is zero. Similarly, the dipole moment is zero, regardless of the coordinate origin that has been chosen. But the quadrupole moment of the arrangement in the diagram cannot be reduced to zero, regardless of where we place the coordinate origin.

The electric potential of an electric charge quadrupole is given by,

$$V_{\mathrm{q}}(\mathbf{R}) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{R}|^3}\sum_{i,j}\frac{1}{2}Q_{ij}\,\hat{R}_i\hat{R}_j$$

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where ε_0 is the electric permittivity, and Q_{ij} follows the definition above.

3.8.3 Radiation Damping

Radiation damping in accelerator physics is a way of reducing the beam emittance of a high-velocity charged particle beam by synchrotron radiation.

The two main ways of using radiation damping to reduce the emittance of a particle beam are the use of ***undulators*** and ***damping rings*** (often containing undulators), both relying on the same principle of inducing synchrotron radiation to reduce the particles' momentum, then replacing the momentum only in the desired direction of motion.

Damping Rings

As particles are moving in a closed orbit, the lateral acceleration causes them to emit synchrotron radiation, thereby reducing the size of their momentum vectors (relative to the design orbit) without changing their orientation (ignoring quantum effects for the moment). In longitudinal direction, the loss of particle impulse due to radiation is replaced by accelerating sections (RF cavities) that are installed in the beam path so that an equilibrium is reached at the design energy of the accelerator. Since this is not happening in transverse direction, where the emittance of the beam is only increased by the quantization of radiation losses (quantum effects), the transverse equilibrium emittance of the particle beam will be smaller with large radiation losses, compared to small radiation losses.

Because high orbit curvatures (low curvature radii) increase the emission of synchrotron radiation, damping rings are often small. If long beams with many particle bunches are needed to fill a larger storage ring, the damping ring may be extended with long straight sections.

Undulators and Wigglers

When faster damping is required than can be provided by the turns inherent in a damping ring, it is common to add undulator or wiggler magnets to induce more synchrotron radiation. These are devices with periodic magnetic fields that cause the particles to oscillate transversely, equivalent to many small tight turns. These operate using the same principle as damping rings and this oscillation causes the charged particles to emit synchrotron radiation.

The many small turns in an undulator have the advantage that the cone of synchrotron radiation is all in one direction, forward. This is easier to shield than the broad fan produced by a large turn.

The process of radiation damping is important in many areas of electron accelerator operation:

1. It can give rise to a stable (Gaussian) distribution of transverse and longitudinal beam dimensions due to an equilibrium between the competing forces of

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radiation damping and "Quantum Excitation" – the growth of oscillation amplitudes due to the discrete emission of radiation quanta;

2. It permits an efficient multi-cycle injection scheme to be employed in storage rings, by allowing the beam dimensions to damp in size between injection pulses;
3. It allows large beam dimensions, produced in a LINAC, for example, to be reduced in specially designed 'Damping Rings';
4. It helps to counteract beam growth due to various processes such as intra-beam scattering and collective instabilities.

Check Your Progress

1. From what are retarded electromagnetic potentials derived?
2. Who structured the relativity theory?
3. What does Liénard-Wiechert potentials describe?
4. Define the term Lorentz force.
5. Give the characteristics of EM waves.
6. State Bremsstrahlung radiation.
7. Why is Larmor formula used?
8. Define the term electric quadrupole.

3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In physics, the 'Retarded Electromagnetic Potentials' are typically derived from the Maxwell's equations and the Lorenz condition. The Maxwell's equations are given by the physicist James Clerk Maxwell.
2. The theory and derivations of classical electrodynamics was initiated and structured by the physicist Albert Einstein's during the development of the 'Relativity Theory'. The significant analysis of the motion theory and propagation of electromagnetic waves formed the basis of the special and distinctive relativity description and derivations of space and time equations.
3. The Liénard–Wiechert formulation is considered as an essential and substantial launchpad for the in depth analysis of relativistic moving particles. Typically, the Liénard–Wiechert potentials are used for describing the classical electromagnetic effect of a moving electric point charge precisely with regards to a vector potential and a scalar potential in the Lorenz gauge. The Liénard–Wiechert potentials are directly constructed from the Maxwell's equations, therefore the Liénard–Wiechert potentials precisely explain the wide-ranging, relativistically precise, time-varying electromagnetic field for a point charge in arbitrary or random motion.
4. In the field of electromagnetism, the term Lorentz force also sometimes referred as the electromagnetic force is defined as the combination of electric force and magnetic force on a point charge that exclusively occurs as a

result of electromagnetic fields. The Lorentz formula refers to the Lorentz force which was formulated by Hendrik Lorentz who precisely derived the contemporary or present form of the formula and its derivation typically for the electromagnetic force which includes the analysis and formulation of the total force obtained from both the electric fields and the magnetic fields.

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5. Following are the characteristics of EM waves.
   - They travel at a speed of light in vacuum.
   - They travel similar to the waves with the same their same property.
   - They radiate away from the source
   - They can travel across any medium
   - EM waves are generated by vibration of electrons resulting in energy emission called as electromagnetic radiation.
   - Electromagnetic waves have both electric and magnetic components.
   - Electric and magnetic components are orthogonal (perpendicular) to each other.
   - The direction of wave propagation will be orthogonal to the electric and magnetic waves. Such waves are called Transverse ElectroMagnetic waves (TEM waves).
6. Bremsstrahlung radiation is also termed as the braking radiation. This radiation is typically produced because of the deacceleration or the negative acceleration of a charged particle, which contains the synchrotron radiation process in which the emission of photon takes place through a relativistic particle and the cyclotron radiation in which emission of photon takes place through a non-relativistic particle, in addition it also explains the electrons and positrons emission during beta decay.
7. In electrodynamics, the Larmor formula is specifically used for calculating the total power radiated through a nonrelativistic point charge as it accelerates. It was originally derived by J. J. Larmor in 1897, in the perspective of the wave theory of light. When any charged particle, such as an electron, a proton or an ion accelerates, then it typically radiates away energy in the form of electromagnetic waves. For those velocities which are small comparative to the speed of light, the total power radiated is given by the Larmor formula:

$$P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c}\left(\frac{\dot{v}}{c}\right)^2 = \frac{2}{3}\frac{q^2 a^2}{4\pi\varepsilon_0 c^3} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \text{ (SI units)}$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3} \text{ (cgs units)}$$

8. Electric quadrupole is referred as a charge distribution that produces an electric field equivalent to that produced by two electric dipoles whose dipole moments have the same magnitude but point in opposite directions and which are separated from each other by a small distance.

3.10 SUMMARY

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- In physics, the 'Retarded Electromagnetic Potentials' are typically derived from the Maxwell's equations and the Lorenz condition. The Maxwell's equations are given by the physicist James Clerk Maxwell.
- Characteristically, the Liénard–Wiechert potentials precisely explain and evaluate the classical electromagnetic effect or consequence of a moving electric point charge with reference to a vector potential and a scalar potential in the Lorenz gauge.
- Reducing or decreasing directly from the Maxwell's equations, the Liénard–Wiechert potentials uniquely explain the comprehensive, relativistically appropriate, time-varying electromagnetic field for a point charge in arbitrary or random motion, but the Liénard–Wiechert potentials are not modified or amended for explaining the quantum mechanical effects.
- Electromagnetic radiation in the form of waves can be obtained from these potentials. The expressions for the Liénard–Wiechert potentials are named after the physicists Alfred-Marie Liénard who developed in part in the year 1898 and then by Emil Wiechert who independently developed in the year 1900.
- In physics, the theory and derivations of classical electrodynamics was initiated and structured by the physicist Albert Einstein's during the development of the 'Relativity Theory'. The significant analysis of the motion theory and propagation of electromagnetic waves formed the basis of the special and distinctive relativity description and derivations of space and time equations.
- The Liénard–Wiechert formulation is considered as an essential and substantial launchpad for the in depth analysis of relativistic moving particles.
- Typically, the Liénard–Wiechert potentials are used for describing the classical electromagnetic effect of a moving electric point charge precisely with regards to a vector potential and a scalar potential in the Lorenz gauge.
- Characteristically, the Liénard–Wiechert description and derivation is considered accurate and exact for a substantial and significant independently moving particle, i.e., the analysis and derivation is 'Classical' and consequently the acceleration of the charge is because of a force that is precisely independent of the ElectroMagnetic Field or EMF.
- Principally, the Liénard–Wiechert formulation gives the following two sets of solutions:
    1. Advanced Fields are Absorbed by the Charges.
    2. Retarded Fields are Emitted.
- Fundamentally, the Liénard–Wiechert potential formulation represents explicit and specific expressions for time-varying electromagnetic fields that are uniquely caused by means of charge in arbitrary or random motion. However,

the Liénard–Wiechert potentials were distinctively derived from the retarded potentials, which in sequence are derived and explained from the Maxwell equations.

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- In physics, principally in the field of electromagnetism the term Lorentz force also sometimes referred as the electromagnetic force is defined as the combination of electric force and magnetic force on a point charge that exclusively occurs as a result of electromagnetic fields.
- The Lorentz formula refers to the Lorentz force which was formulated by Hendrik Lorentz who precisely derived the contemporary or present form of the formula and its derivation typically for the electromagnetic force which includes the analysis and formulation of the total force obtained from both the electric fields and the magnetic fields.
- The Lorentz force is defined as a force that is exerted by means of the electromagnetic field on the charged particle, i.e., it is precisely specified as the rate at which the linear momentum is transferred from the electromagnetic field to the particle.
- Electromagnetic waves transport energy or information from one point to the other. Few examples of electromagnetic waves include the waves in the electromagnetic spectrum. Few electromagnetic waves include X-rays, Gamma rays, microwave, TV signals, radar signals, light rays, etc. The electromagnetic waves are also called as Hertzian waves.
- Bremsstrahlung is also termed as the braking radiation. This radiation is typically produced because of the deacceleration or the negative acceleration of a charged particle, which contains the synchrotron radiation process in which the emission of photon takes place through a relativistic particle and the cyclotron radiation in which emission of photon takes place through a non-relativistic particle, in addition it also explains the electrons and positrons emission during beta decay.
- The 'Bremsstrahlung' or 'Braking Radiation' is specifically identified as the radiation that is typically given off by means of the free electrons which are deflected, i.e., the electrons are accelerated in the electric fields of charged particles and the nuclei of atoms.
- Thermal bremsstrahlung refers to the emission typically given off by means of an ionized gas of plasma in thermal equilibrium at a specific or particular distinct temperature, wherein the unique distribution of electron velocities typically goes along with the recognised Maxwellian distribution.
- If the quantum effects are considered negligible, then an accelerating charged particle radiates the power based on the theory as explained by the Larmor formula and its relativistic generalization.
- Bremsstrahlung radiation is specifically defined as the radiation that is released by means of a charged particle the 'Electron' owing to its acceleration that is caused by means of an electric field of another charged particle the 'Proton' or an atomic nucleus.

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- The word "Bremsstrahlung" is a German word which means "Braking Radiation", and specifically refers to the approach in which the electrons are "Braked" when they typically hit a metal target.
- The incident electrons are considered as free, i.e., they are not bound to an atom or ion, both either before or after the braking. Consequently, this type of the radiation spectrum is considered continuous, dissimilar atomic spectra which comprises of sharp spectral lines, and occasionally referred to as "Free-Free" radiation.
- One of the greatest universally recognized and identified examples of Bremsstrahlung radiation in the universe is the one which comes from the hot intracultural gas of the galaxy clusters.
- Bremsstrahlung is, therefore, a physical phenomenon typically used in the radiology apparatus. When an electron or a beta (β) particle passes through matter then it decelerates or slows down, and a fraction of its energy is directly converted into X-rays.
- Internal Bremsstrahlung evolves in the process of radioactive disintegration of beta decay, which typically consists of the production and emission of electrons (or positrons, positive electrons) by means of unstable atomic nuclei or the capture by nuclei of one of their individual orbiting electrons. These electrons, deflected in the vicinity of their specific individual associated nuclei, emit internal Bremsstrahlung.
- Bremsstrahlung radiation is, therefore, the standard radiation that is given off through a charged particle, typically an electron, due to its acceleration that is caused by means of an electric field of another charged particle, specifically a proton or an atomic nucleus.
- Accelerating charges produce EM radiation, because the changing electric fields yield magnetic fields (Ampere-Maxwell law) and the resulting changing magnetic fields yield electric fields (Faraday's Law). Consequently, these are the electromagnetic radiation, where the changing fields propagate.
- Additionally, it can be stated as, "An accelerated point charge emits energy and impulses in the form of radiation". With regard to the conservation of energy, it is said that, "The radiated energy is directly extracted from the mechanical kinetic energy of the charged particle and not from the potential energy of the electromagnetic system".
- The phenomenon in nuclear magnetic resonance is termed as the Larmor precession.
- In electrodynamics, the Larmor formula is specifically used for calculating the total power radiated through a nonrelativistic point charge as it accelerates. It was originally derived by J. J. Larmor in 1897, in the perspective of the wave theory of light.
- When any charged particle, such as an electron, a proton or an ion accelerates, then it typically radiates away energy in the form of electromagnetic waves. For those velocities which are small comparative to the speed of light, the total power radiated is given by the Larmor formula:

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- The charge particle is at rest emits electric field. Charge particle is in motion or moving with constant velocity emits Electric Field + Magnetic Field. Charge particle is in accelerated motion emits electric field, magnetic field as well electromagnetic radiation or EM wave.
- EM wave is a wave of the oscillation of changing electricity and magnetism, and it will be produced when the particle is in acceleration.
- In physics, a dipole is referred as a quantity which involves some form of polarity. The most significant resource of electromagnetic radiation is possibly the oscillating electric dipole. When the current density in a localized source oscillates harmonically with angular frequency, then it has an electric dipole moment of the form where there is the complex amplitude.
- An electric dipole is concerned with the separation of the positive and negative charges observed in any electromagnetic system. A permanent electric dipole is occasionally termed as an electret.
- The electric dipole moment is defined as a measure of the separation of positive and negative electrical charges within a system, i.e., it is a measure of the system's overall polarity. The SI unit for electric dipole moment is the Coulomb meter (C·m). The Debye (D) is another unit of measurement used in atomic physics and chemistry.
- Theoretically, an electric dipole is defined by the first-order term of the multipole expansion; it consists of two equal and opposite charges that are infinitesimally close together, although real dipoles have separated charge.
- The most significant source of electromagnetic radiation is possibly the oscillating electric dipole. When the current density in a localized source oscillates harmonically with angular frequency, it has an electric dipole moment of the form where is the complex amplitude.
- A moving charge produces ElectroMagnetic Waves (EMWs) as such charge moving in a circular orbit experiences centripetal acceleration will also produces electromagnetic wave.
- An accelerated charge is referred as the source of ElectroMagnetic Waves (EMWs). When the charge is in a circular motion, the direction of its velocity continuously changes and thus it is in accelerated motion and produces EMWs. A charge falling in an electric field is accelerated by the electric force and thus produces ElectroMagnetic Waves (EMWs).
- Radiation Energy produced by a charged particle circulating in a magnetic field does radiate energy, which it is called synchrotron radiation.
- As per the radio engineering, an antenna also known as aerial is the interface between radio waves propagating through space and electric currents moving in metal conductors, used with a transmitter or receiver.
- In the process of transmission, a radio transmitter supplies an electric current to the antenna's terminals, and the antenna radiates the energy from the current as electromagnetic waves (radio waves).
- In reception, an antenna intercepts some of the power of a radio wave in order to produce an electric current at its terminals, that is applied to a

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receiver to be amplified. Antennas are essential components of all radio equipment.

- Linear antennas are considered to be those antennas that imply the use of electrically thin conductors (wavelength conductor diameter). Electric current flows over the conductor surface in order to calculate radiated fields in these antennas. Conductors are modelled as if they were current lines with no diameter.
- A distribution of charge or magnetization which produces an electric or magnetic field equivalent to that produced by two electric or magnetic dipoles whose dipole moments have the same magnitude but point in opposite directions, and which are separated from each other by a small distance.
- Electric quadrupole is referred as a charge distribution that produces an electric field equivalent to that produced by two electric dipoles whose dipole moments have the same magnitude but point in opposite directions and which are separated from each other by a small distance.
- Radiation damping in accelerator physics is a way of reducing the beam emittance of a high-velocity charged particle beam by synchrotron radiation.
- The two main ways of using radiation damping to reduce the emittance of a particle beam are the use of undulators and damping rings (often containing undulators), both relying on the same principle of inducing synchrotron radiation to reduce the particles' momentum, then replacing the momentum only in the desired direction of motion.

3.11 KEY TERMS

- **Retarded electromagnetic potentials:** In physics, the 'Retarded Electromagnetic Potentials' are typically derived from the Maxwell's equations and the Lorenz condition. The Maxwell's equations are given by the physicist James Clerk Maxwell.
- **Liénard–Wiechert potentials:** Characteristically, the Liénard–Wiechert potentials precisely explain and evaluate the classical electromagnetic effect or consequence of a moving electric point charge with reference to a vector potential and a scalar potential in the Lorenz gauge.
- **Lorentz force:** The term Lorentz force also sometimes referred as the electromagnetic force is defined as a force that is exerted by means of the electromagnetic field on the charged particle, i.e., it is precisely specified as the rate at which the linear momentum is transferred from the electromagnetic field to the particle. The Lorentz force formula was formulated by Hendrik Lorentz.
- **Bremsstrahlung radiation:** The Bremsstrahlung radiation is also termed as the braking radiation. This radiation is typically produced because of the deacceleration or the negative acceleration of a charged particle, which contains the synchrotron radiation process in which the emission of photon takes place through a relativistic particle and the cyclotron radiation in which

emission of photon takes place through a non-relativistic particle, in addition it also explains the electrons and positrons emission during beta decay.

- **Larmor precession:** The phenomenon in nuclear magnetic resonance is termed as the Larmor precession.
- **Dipole**: In physics, a dipole is referred as a quantity which involves some form of polarity.
- **Electric dipole:** An electric dipole is concerned with the separation of the positive and negative charges observed in any electromagnetic system.
- **Electric quadrupole:** Electric quadrupole is referred as a charge distribution that produces an electric field equivalent to that produced by two electric dipoles whose dipole moments have the same magnitude but point in opposite directions and which are separated from each other by a small distance.
- **Radiation damping:** Radiation damping in accelerator physics is a way of reducing the beam emittance of a high-velocity charged particle beam by means of synchrotron radiation.

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3.12 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is retarded potential?
2. Define Liénard-Wiechert potentials for moving and accelerated charges.
3. State the Lorentz formula.
4. What is Bremsstrahlung radiation?
5. Give the Larmor's formula.
6. What is an oscillating electric dipole?
7. Define the term linear antenna.
8. Write short notes on electric quadrupole radiation and radiation damping.

Long-Answer Questions

1. Briefly discuss the concept and significance of retarded potential giving appropriate examples.
2. Explain the connotation, notions, explanations, and derivations of Lienard-Wiechert potentials for uniformly moving and accelerated charges with the help of relevant examples.
3. State and prove Lorentz formula giving the derivations and explanations.
4. Explain the concept of Bremsstrahlung radiation typically arising from an accelerated and charged particle at low velocity giving proper examples.
5. Briefly discuss about the radiation typically occurring from an accelerated charged particle at low velocity with reference to the Larmor's formula. Give explanation and derivation to support your answer.

6. Discuss in detail the notions and derivations of radiation occurring precisely from an oscillating electric dipole.
7. What is the linear antenna? Explain the significance of linear antenna in the field of electromagnetic radiation.
8. Explain the concept of radiation from a charged particle moving in a circular orbit.
9. Discuss about electric quadrupole radiation and radiation damping giving relevant examples.

3.13 FURTHER READING

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UNIT 4 PLASMA PHYSICS

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Structure

4.0 INTRODUCTION

Plasma, in physics, is an electrically conducting medium in which there are roughly equal numbers of positively and negatively charged particles, produced when the atoms in a gas become ionized. Principally, the plasma is a state of matter. The three other common states of matter are solids, liquids and gases, so plasma is sometimes also called the fourth state of matter.

Plasma oscillations are rapid oscillations of the electron density in conducting media, such as plasmas or metals in the UltraViolet or UV region. These oscillations are also known as Langmuir waves (named after the Irving Langmuir), and precisely the oscillations are described as an instability in the dielectric function of a free electron gas. The frequency depends weakly on the wavelength of the oscillation. The quasiparticle resulting from the quantization of these oscillations is the plasmon.

American physicists Irving Langmuir and Lewi Tonks discovered 'Langmuir Waves' in 1920s. These waves are parallel in form to Jean's instability waves, which are caused by gravitational instabilities in a static medium. In plasma physics and electrolytes, the Debye length λ_D is also termed as the Debye radius; this is considered as a measure of a charge carrier's net electrostatic effect in a solution and that how much its electrostatic effect continues. With each of the Debye length the charges increase and electrically examined while the electric potential decreases

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in magnitude by 1/e. A Debye sphere is defined as a volume whose radius is the Debye length. Debye length is a significant and essential parameter in plasma physics, electrolytes, and colloids (DLVO theory). Plasma parameters are the various characteristics of a plasma, an electrically conductive collection of charged particles that responds collectively to electromagnetic forces. Plasma is the typically in the form of neutral gas-like clouds or charged beams of ions, but unlike gas it may also include dust and grains.

As per plasma physics, plasma confinement refers to the act of containment of a plasma by various forces at the extreme conditions in a discrete volume necessary for thermonuclear fusion reactions. Confinement of plasma is required in order to achieve fusion power. There are two major approaches to confinement: magnetic confinement and inertial confinement. Magnetic confinement fusion is an approach for generating thermonuclear fusion power using magnetic fields to confine fusion fuel in the form of a plasma. Magnetic confinement is one of two major branches of fusion energy research, along with inertial confinement fusion.

The advancement and expansion of Magnetic Fusion Energy (MFE) was defined in three distinct and significant phases. In the 1950s, it was assumed that Magnetic Fusion Energy (MFE) can be relatively simple to accomplish by just setting and building an appropriate machine. By the late 1950s, it was evident that plasma turbulence and instabilities were challenging and during the 1960s, 'The Doldrums', effort was considered as the best theoretical explanation to study and understand the concept of plasma physics.

The MagnetoHydroDynamic (MHD) waves lose their defining nature and get mixed properties in case of an inhomogeneous plasma, i.e., a plasma where at least one of the background quantities is not constant. Whereas in some cases, such as the axisymmetric waves in a straight cylinder with a circular basis which is one of the simplest models for a coronal loop, the three MagnetoHydroDynamic (MHD) waves can still be clearly distinguished. But generally, the pure Alfvén, fast and slow magnetosonic waves do not exist and the waves in the plasma are coupled to each other in intricate ways.

Alfvén wave, named after Hannes Alfvén, is a type of magnetohydrodynamic wave where in ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines. An Alfvén wave in a plasma is considered as low frequency in comparison to the ion cyclotron frequency naturally travelling oscillation of the ions and the magnetic field. The ion mass density gives the inertia, and the magnetic field line tension provided that the restoring force exists. The wave propagates in the direction of the magnetic field, on the contrary of it, waves exist at oblique incidence and smoothly change into the magnetosonic wave where the propagation is perpendicular to the magnetic field. If the motion of the ions and the perturbation of the magnetic field are in the same direction and transverse to the direction of propagation, then the wave is dispersionless.

In this unit, you will study about the concept of plasma, plasma oscillation, Debye shielding, plasma parameters, magnetoplasma, plasma confinement, hydrodynamical desorption of plasma, fundamental equations, hydromagnetic waves, magnetosonic wave and Alfvén wave, wave phenomenon in magnetoplasma, phase and group velocity cut offs, resonance for electromagnetic wave propagating parallel and perpendicular to the magnetic field, Appleton–Hartree formula, propagation through ionosphere and magnetosphere helicon, whistles, and Faraday rotation.

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4.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe the basic concept of plasma
- Explain the conditions for the plasma existence
- Define plasma oscillations
- Understand the Debye shielding and plasma parameters
- State the various theories for plasma confinement
- Elaborate on the fundamental equations and hydromagnetic waves
- Discuss about the magnetosonic wave and Alfvén wave
- State the wave phenomenon in magnetoplasma
- Conceptualize the laws and equations explaining plasma
- Explain the hydrodynamical desorption of plasma
- Describe the various parameters of plasma
- State the Appleton–Hartree formula
- Understand the propagation through ionosphere and magnetosphere helicon, whistles, and the Faraday rotation

4.2 CONCEPT OF PLASMA

In physics, the term 'Plasma' is referred as an electrically conducting medium in which there are approximately equal numbers of positively charged particles and negatively charged particles, typically produced when the atoms in a gas become ionized. Principally, the plasma is a state of matter. The three other common states of matter are the solids, the liquids, and the gases, and consequently plasma is sometimes also called the fourth state of matter.

In the plasma, the negative charge is generally carried by the electrons, each of which has precisely one unit of negative charge. Characteristically, the positive charge is carried by atoms or molecules that are specifically missing those identical electrons. In some rare but interesting cases, electrons missing from one type of atom or molecule become attached to another component, resulting in a plasma containing both positive and negative ions. The most extreme case of this type occurs when small but macroscopic dust particles become charged in a state referred to as a dusty plasma. The uniqueness of the plasma state is due to the significance of electric and magnetic forces that precisely act on a plasma in addition to such forces as gravity that affect all forms of matter. Since these electromagnetic forces can act at large distances, therefore a plasma will also act collectively much like a fluid even when the particles seldom collide with one another.

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Approximately all the visible matter in the universe exists in the plasma state, uniquely occurring predominantly in this form specifically in the Sun and stars, and also in interplanetary space and interstellar space. Auroras, lightning, and welding arcs are also termed as plasmas; basically, the plasmas exist in neon and fluorescent tubes, in the crystal structure of metallic solids, and in many other phenomena and objects. The Earth itself is immersed in a tenuous plasma called the solar wind and is distinctively surrounded by a dense plasma called the ionosphere.

A plasma may be produced in the laboratory by heating a gas to an extremely high temperature, which causes such vigorous collisions between its atoms and molecules that electrons are ripped free, yielding the requisite electrons and ions. A similar process occurs inside stars. In space the dominant plasma formation process is photoionization, wherein photons from sunlight or starlight are absorbed by an existing gas, causing electrons to be emitted. Since the Sun and stars shine continuously, virtually all the matter becomes ionized in such cases, and the plasma is said to be fully ionized. A completely ionized hydrogen plasma, consisting solely of electrons and protons (hydrogen nuclei), is the most elementary plasma.

Definition

Plasma is a state of matter in which an ionized gaseous substance becomes highly electrically conductive to the point that long-range electric and magnetic fields dominate the behaviour of the matter. The plasma state can be contrasted with the other states: solid, liquid, and gas.

Plasma is an electrically neutral medium of unbound positive and negative particles (i.e., the overall charge of a plasma is roughly zero). Although these particles are unbound, they are not 'Free' in the sense of not experiencing forces. Moving charged particles generate an electric current within a magnetic field, and any movement of a charged plasma particle affects and is affected by the fields created by the other charges. In turn this governs collective behaviour with many degrees of variation. The following three factors define a plasma:

1. **Plasma Approximation:** The plasma approximation applies when the plasma parameter, Λ, representing the number of charge carriers within a sphere (called the Debye sphere whose radius is the Debye screening length) surrounding a given charged particle, is sufficiently high as to shield the electrostatic influence of the particle outside of the sphere.
2. **Bulk Interactions:** The Debye screening length is compared to the physical size of the plasma. This criterion means that interactions in the bulk of the plasma are more important than those at its edges, where boundary effects may take place. When this criterion is satisfied, the plasma is quasi-neutral.
3. **Plasma Frequency:** The electron plasma frequency (measuring plasma oscillations of the electrons) is large compared to the electron-neutral collision frequency, i.e., the measuring frequency of collisions between electrons and neutral particles. When this condition is valid, electrostatic interactions dominate over the processes of ordinary gas kinetics.

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Conditions for Plasma Existence

Plasma temperature is commonly measured in Kelvin or electron volts and is, informally, a measure of the thermal kinetic energy per particle. High temperatures are usually needed to sustain ionisation, which is a defining feature of a plasma. The degree of plasma ionisation is determined by the electron temperature relative to the ionisation energy and more weakly by the density. At low temperatures, ions and electrons tend to recombine into bound states—atoms—and the plasma will eventually become a gas.

In most cases the electrons are close enough to thermal equilibrium that their temperature is relatively well-defined; this is true even when there is a significant deviation from a Maxwellian energy distribution function, for example, due to UltraVoilet or UV radiation, energetic particles, or strong electric fields. Because of the large difference in mass, the electrons come to thermodynamic equilibrium amongst themselves much faster than they come into equilibrium with the ions or neutral atoms. For this reason, the ion temperature may be very different from (usually lower than) the electron temperature. This is especially common in weakly ionized technological plasmas, where the ions are often near the ambient temperature.

Therefore, for plasma to exist, ionisation is necessary. The term 'Plasma Density' by itself usually refers to the 'Electron Density', that is, the number of free electrons per unit volume. The degree of ionisation of a plasma is the proportion of atoms that have lost or gained electrons, and is controlled by the electron and ion temperatures and electron-ion vs. electron-neutral collision frequencies. The degree of ionisation, α, is defined as $\alpha = \frac{n_i}{n_i + n_n}$, where n_i is the number density of ions and n_n is the number density of neutral atoms. The *electron density* is related to this by the average charge state $\langle Z \rangle$ of the ions through $n_e = \langle Z \rangle n_i$, where n_e is the number density of electrons.

In a plasma, the electron-ion collision frequency ν_{ei} is much greater than the electron-neutral collision frequency ν_{en}. Therefore, with a weak degree of ionization, α the electron-ion collision frequency can equal the electron-neutral collision frequency: $\nu_{ei} = \nu_{en}$ is the limit separating a plasma from being partially or fully ionized.

- The term *fully ionised gas* introduced by Lyman Spitzer does not mean the degree of ionisation is unity, but only that the plasma is in a *Coulomb-collision dominated regime*, i.e., when $\nu_{ei} > \nu_{en}$, which can correspond to a degree of ionisation as low as 0.01%.
- A *partially* or *weakly ionised gas* means the plasma is not dominated by Coulomb collisions, i.e., when $\nu_{ei} < \nu_{en}$.

Most of technological engineered plasmas are weakly ionised gases.

Occurrence of Plasma

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Plasma is created by adding energy to a gas so that some of its electrons leave its atoms. This is called ionization. It results in negatively charged electrons, and positively charged ions. Unlike the other states of matter, the charged particles in a plasma will react strongly to electric and magnetic fields (i.e., electromagnetic fields). If a plasma loses heat, the ions will reform into a gas, emitting the energy which had caused them to ionize.

Over 99% of the matter in the visible universe is believed to be plasma. When the atoms in a gas are broken up, the pieces are called electrons and ions. Because they have an electric charge, they are pulled together or pushed apart by electric fields and magnetic fields. This makes a plasma act differently than a gas. For example, magnetic fields can be used to hold a plasma, but not to hold a gas. Plasma is a better conductor of electricity than copper.

Plasma is usually very hot, because it takes very high temperatures to break the bonds between electrons and the nuclei of the atoms. Sometimes plasmas can have very high pressure, like in stars. Stars (including the Sun) are mostly made of plasma. Plasmas can also have very low pressure, like in outer space.

On Earth, lightning makes plasma. Artificial (man-made) uses of plasma include fluorescent light bulbs, neon signs, and plasma displays used for television or computer screens, as well as plasma lamps and globes which are a popular children's toy and room decoration.

Motion of Charged Particles in a Uniform Electric Field

When a particle of charge 'q' and mass 'm' is placed in an electric field 'E', then the electric force exerted on the charge is 'qE'. If this is the only force exerted on the particle, it must be the net force and so must cause the particle to accelerate. In this case, Newton's second law applied to the particle gives electric force on the charged particles in uniform electric field.

$$\mathbf{F}_e = q\mathbf{E} = m\mathbf{a}$$

The acceleration of the particle is therefore electric force on the charge particles,

$$\mathbf{a} = \frac{q\mathbf{E}}{m}$$

If E is uniform (that is, constant in magnitude and direction), then the acceleration is constant. If the particle has a positive charge, then its acceleration is in the direction of the electric field. If the particle has a negative charge, then its acceleration is in the direction opposite the electric field.

4.2.1 Plasma Oscillation

Plasma oscillations are rapid oscillations of the electron density in conducting media, such as plasmas or metals in the UltraViolet or UV region. These oscillations are also known as Langmuir waves (named after the Irving Langmuir), and precisely the oscillations are described as an instability in the dielectric function of a free electron gas. The frequency depends weakly on the wavelength of the oscillation. The quasiparticle resulting from the quantization of these oscillations is the plasmon.

American physicists Irving Langmuir and Lewi Tonks discovered 'Langmuir Waves' in 1920s. These waves are parallel in form to Jean's instability waves, which are caused by gravitational instabilities in a static medium.

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4.2.2 Debye Shielding

The quasi--neutrality leads are considered as the significant quantity termed as the **Debye Length** λ_D. Assume that we place a plane grid into a plasma, which is kept at a specific potential, φ_g.

In plasmas physics and electrolytes, the **Debye length** λ_D is also termed as the **Debye radius**; this is considered as a measure of a charge carrier's net electrostatic effect in a solution and that how much its electrostatic effect continues.

With each of the Debye length the charges increase and electrically examined while the electric potential decreases in magnitude by 1/e. A **Debye sphere** is defined as a volume whose radius is the Debye length. Debye length is a significant and essential parameter in plasma physics, electrolytes, and colloids (DLVO theory).

The corresponding Debye screening wave vector $k_D = 1/\lambda_D$ for particles of density *n*, charge *q* at a temperature *T* is given by $k_D^2 = 4\pi nq^2/(k_B T)$ in the Gaussian units.

The analogous quantities at very low temperatures $(T \to 0)$ are termed as the **Thomas–Fermi length** and the **Thomas–Fermi wave vector**. These are considered as the significant notation to describe the behaviour of electrons in metals at room temperature.

4.2.3 Plasma Parameters

Plasma parameters are the various characteristics of a plasma, an electrically conductive collection of charged particles that responds collectively to electromagnetic forces. Plasma is the typically in the form of neutral gas-like clouds or charged beams of ions, but unlike gas it may also include dust and grains.

Fundamental Plasma Parameters

All the quantities of plasma parameters are defined in Gaussian (cgs) units except energy and temperature which are typically expressed in eV and ion mass expressed in units of the proton mass, $\mu = m_i / m_p$; Z $\mu = m_i/m_p$; Z is the ion charge in units of the elementary charge '*e*' for the state of fully charged ion and Z is the respective atomic number; *k* is Boltzmann's constant; *c* is the speed of light; $\ln\Lambda$ is the Coulomb logarithm.

NOTES

Check Your Progress

1. What do you mean by plasma?
2. Define the term plasma physics.
3. What do you understand by ionization?
4. Define plasma oscillations.
5. What is the Debye length?

4.3 MAGNETOPLASMA

Magnetic fields are used to contain high-density, high-temperature plasmas because such fields exert pressures and tensile forces on the plasma. An equilibrium configuration is reached only when at all points in the plasma these pressures and tensions exactly balance the pressure from the motion of the particles. A well-known example of this is the pinch effect observed in specially designed equipment. If an external electric current is imposed on a cylindrically shaped plasma and flows parallel to the plasma axis, the magnetic forces act inward and cause the plasma to constrict, or pinch. An equilibrium condition is reached in which the temperature is proportional to the square of the electric current. This result suggests that any temperature may be achieved by making the electric current sufficiently large, the heating resulting from currents and compression.

The plasma can be defined by a magnetic field by measuring containment time (τ_c), or the average time for a charged particle to diffuse out of the plasma; this time is different for each type of configuration. Various types of instabilities can occur in plasma. These lead to a loss of plasma and a catastrophic decrease in containment time. The most important of these is called magnetohydrodynamic instability. Although an equilibrium state may exist, it may not correspond to the lowest possible energy. The plasma, therefore, seeks a state of lower potential energy, just as a ball at rest on top of a hill (representing an equilibrium state) rolls down to the bottom if perturbed; the lower energy state of the plasma corresponds to a ball at the bottom of a valley. In seeking the lower energy state, turbulence develops, leading to enhanced diffusion, increased electrical resistivity, and large heat losses. In toroidal geometry, circular plasma currents must be kept below a critical value called the Kruskal-Shafranov limit, otherwise a particularly violent instability consisting of a series of kinks may occur. Although a completely stable system appears to be virtually impossible, considerable progress has been made in devising systems that eliminate the major instabilities. Temperatures on the order of 10,000,000 K at densities of 10^{19} particles per cubic metre and containment times as high as 1/50 of a second have been achieved.

4.4 PLASMA CONFINEMENT

As per plasma physics, plasma confinement refers to the act of containment of a plasma by various forces at the extreme conditions in a discrete volume necessary for thermonuclear fusion reactions. Confinement of plasma is required in order to achieve fusion power. There are two major approaches to confinement: magnetic confinement and inertial confinement.

Magnetic confinement fusion is an approach for generating thermonuclear fusion power using magnetic fields to confine fusion fuel in the form of a plasma. Magnetic confinement is one of two major branches of fusion energy research, along with inertial confinement fusion. This magnetic approach was initiated in the 1940s and absorbed the majority of subsequent development.

Fusion reactions combine light atomic nuclei such as hydrogen to form heavier ones such as helium gas, producing energy. In order to overcome the electrostatic repulsion between the nuclei, they require a temperature in the range of tens of millions of degrees, creating a plasma. In addition, the plasma is required to be contained at a sufficient density for a sufficient time, as specified by the Lawson criterion (triple product).

Magnetic confinement fusion attempts to use the electrical conductivity of the plasma to contain it through interaction with magnetic fields. The magnetic pressure offsets the plasma pressure. Developing a suitable arrangement of fields that contain the fuel without excessive turbulence or leaking is the primary challenge of this technology.

The advancement and expansion of Magnetic Fusion Energy (MFE) was defined in three distinct and significant phases. In the 1950s, it was assumed that Magnetic Fusion Energy (MFE) can be relatively simple to accomplish by just setting and building an appropriate machine. By the late 1950s, it was evident that plasma turbulence and instabilities were challenging and during the 1960s, 'The Doldrums', effort was considered as the best theoretical explanation to study and understand the concept of plasma physics.

4.5 HYDRODYNAMICAL DESORPTION OF PLASMA

Ions are produced in a given sample through Plasma desorption mass spectrometry This technique is similar to those of MALDI-MS. Adequately large ion extraction voltage is required in particular to accelerate ions through the time of-flight analyzer and allow detection of slow (large) as well as fast (small) ions from an unfractionated sample or one of relatively wide polydispersity. Plasma desorption mass spectrometry has been used to analyze heavy distillation residues from direct coal liquefaction processes. The average molecular masses derived from plasma desorption were compared with those from gel permeation chromatography.

4.6 MAGNETOSONIC WAVE AND ALFVÉN WAVE

A linear MagnetoHydroDynamic (MHD) wave that is driven by thermal pressure, magnetic pressure, and magnetic tension is known as magnetosonic wave which is also called a magnetoacoustic wave. The magnetosonic waves have two types, the one is known as fast magnetosonic wave and the other is slow magnetosonic wave. Both fast and slow magnetosonic waves are present in the solar corona providing an observational foundation for the technique for coronal plasma diagnostics, coronal seismology.

NOTES

In the homogeneous plasma of infinite extent, and in the absence of gravity, the magnetosonic waves form, together with the Alfvén wave, the three basics linear MagnetoHydroDynamic (MHD) waves.

Underneath the assumption of normal modes, specifically that the linear perturbations of the physical quantities are of the form

$$f_1 = \tilde{f}_1 e^{i\left(k_x x + k_y y + k_z z - \omega t\right)}$$

with $\tilde{f}_1$ being the constant amplitude, a dispersion relation of the magnetosonic waves is typically derived using specifically the system of ideal MagnetoHydroDynamic (MHD) equations:

$$\omega^4 - k^2 \left(v_s^2 + v_A^2\right) \omega^2 + k_{||} k^2 v_s^2 v_A^2 = 0,$$

where v_A is the Alfvén speed, v_s is the sound speed, k is the magnitude of the wave vector and $k_{||}$" is the component of the wave vector along the background magnetic field which is straight and constant, because the plasma is assumed homogeneous.

Inhomogeneous Plasma

The MagnetoHydroDynamic (MHD) waves lose their defining nature and get mixed properties in case of an inhomogeneous plasma, i.e., a plasma where at least one of the background quantities is not constant. Whereas in some cases, such as the axisymmetric waves in a straight cylinder with a circular basis which is one of the simplest models for a coronal loop, the three MagnetoHydroDynamic (MHD) waves can still be clearly distinguished. But generally, the pure Alfvén, fast and slow magnetosonic waves do not exist and the waves in the plasma are coupled to each other in intricate ways.

Alfvén Wave

Alfvén wave, named after Hannes Alfvén, is a type of magnetohydrodynamic wave where in ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines.

An Alfvén wave in a plasma is considered as a low frequency in comparison to the ion cyclotron frequency naturally travelling oscillation of the ions and the magnetic field. The ion mass density gives the inertia, and the magnetic field line tension provided that the restoring force exists.

As per Physics the wave propagates in the direction of the magnetic field, on the contrary of it, waves exist at oblique incidence and smoothly change into the magnetosonic wave where the propagation is perpendicular to the magnetic field.

If the motion of the ions and the perturbation of the magnetic field are in the same direction and transverse to the direction of propagation then the wave is dispersionless.

4.7 WAVE PHENOMENON IN MAGNETOPLASMA

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As per plasma physics, the waves in plasmas are an interconnected set of particles and fields which propagate in a periodically repeating fashion. It is also known that plasma is a quasi-neutral, electrically conductive fluid. Hence in the simplest case, it is composed of electrons and a single species of positive ions, but it may also contain multiple ion species including negative ions as well as neutral particles. Due to its electrical conductivity, a plasma couples to electric and magnetic fields. This complex of particles and fields supports a wide variety of wave phenomena.

In a plasma, the electromagnetic fields are presumed to have two parts, the first is static/equilibrium part and the second is oscillating/perturbation part. Waves in plasmas are characteristically classified as electromagnetic or electrostatic according to whether or not there exists an oscillating magnetic field. When the Faraday's law of induction to plane waves is applied, then we find $\mathbf{k} \times \tilde{\mathbf{E}} = \omega\tilde{\mathbf{B}}$, implying that an electrostatic wave must be purely longitudinal. On the contrary, an electromagnetic wave must possess a transverse component, however it may also be partially longitudinal.

Waves can further be classified by the oscillating species. In most plasmas of interest, the electron temperature is comparable to or larger than the ion temperature. Consequently the fact that when coupled with the much smaller mass of the electron it indicates that the electrons move much faster than the ions. Mode of an electron depends on the mass of the electrons, but the ions may be assumed to be infinitely massive, i.e., stationary. Mode of an ion depends on the ion mass, but the electrons are assumed to be massless and to redistribute themselves instantaneously according to the Boltzmann relation. It is rarely, that in the lower hybrid oscillation, will a mode depend on both the electron and the ion mass.

The various modes can also be classified depending upon that whether their propagation is in an unmagnetized plasma or parallel, perpendicular, or oblique to the stationary magnetic field. Finally, for perpendicular electromagnetic electron waves, the perturbed electric field can be parallel or perpendicular to the stationary magnetic field.

Check Your Progress

6. What do you understand by plasma confinement?
7. Define magnetic confinement fusion.
8. What is a magnetostatic wave?
9. What do you understand by inhomogenous plasma?
10. Define Alfvén wave.

4.8 PHASE AND GROUP VELOCITY CUT OFFS

If the thermal motion of the electrons is ignored, it is possible to show that the charge density oscillates at the plasma frequency

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{m^* \varepsilon_0}}, [\mathrm{rad/s}]$$

NOTES

$$\omega_{pe} = \sqrt{\frac{4\pi n_e e^2}{m^*}}, [\mathrm{rad/s}]$$

Where, n_e is the number density of electrons, ***e*** is the effective mass of the electron, and m^* is the permittivity of free space. Remember that the above formula is specifically derived using the approximation that the mass of an ion is infinite. Generally, this is considered as a good approximation because the electrons are significantly lighter as compared to ions.

Proof using Maxwell Equations. Assume that the charge density oscillations is given as $\rho(\omega) = \rho_0 e^{-i\omega t}$ the continuity equation:

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t} = i\omega\rho(\omega)$$

the Gauss law

$$\nabla \cdot \mathbf{E}(\omega) = 4\pi\rho(\omega)$$

and the conductivity

$$\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$$

it remains:

$$i\omega\rho(\omega) = 4\pi\sigma(\omega)\rho(\omega)$$

which is always true only if

$$1 + \frac{4\pi i\sigma(\omega)}{\omega} = 0$$

This expression must be modified in the case of electron-positron plasmas, often encountered in astrophysics. Since the frequency is independent of the wavelength, these oscillations have an infinite phase velocity and zero group velocity.

4.9 RESONANCE FOR ELECTROMAGNETIC WAVE PROPAGATING PARALLEL AND PERPENDICULAR TO THE MAGNETIC FIELD

In the process by adding a weak magnetic undulator the propagation of electromagnetic waves in magnetized plasma near the electron cyclotron frequency can be modified. Such as, both right- and left-hand circularly polarized waves can propagate along the magnetic field without experiencing resonant absorption. This phenomenon of 'Eliminating Electron Cyclotron Heating' is referred to as the 'Undulator-Induced Transparency (UIT)' of the plasma and is the classical equivalent of the well-known quantum mechanical effect of electromagnetically induced transparency. As such UIT can dramatically slow down the waves group velocity, resulting in the extreme compression of the wave energy in the plasma.

Compressed waves are polarized along the propagation direction and can be used for synchronous electron or ion acceleration. Strong coupling between the two wave helicities is explored to impart the waves with high group velocities for vanishing wave numbers *k*.

For treating the propagation of electromagnetic waves in uniform, weakly interacting plasmas near equilibrium in the absence of external magnetic fields, "Green's Function Techniques" are used. The frequency and the damping of electromagnetic waves in a medium are related to the local complex conductivity tensor, which is calculated by the diagrammatic techniques of modern field theory. Physical quantities are calculated in terms of a consistent many-particle perturbation expansion in powers of a (weak) coupling parameter. To simplify the calculation of absorptive parts an 'Open-Diagram Technique' is introduced. For long-wavelength longitudinal waves such as electron plasma oscillations, it is found that the main absorption mechanism in the electron-ion plasma is the two-particle collision process appropriately corrected for collective effects and not the one-particle (or Landau) damping process. Electron-ion collisions produce a damping effect which remains finite for long wavelengths.

4.9.1 Appleton–Hartree Formula

The Appleton–Hartree equation, also sometimes referred to as the Appleton–Lassen equation is a specific mathematical expression used to explain the refractive index for electromagnetic wave propagation in a cold magnetized plasma. The Appleton–Hartree equation was independently developed by numerous different scientists, including Edward Victor Appleton, Douglas Hartree and German radio physicist H. K. Lassen. Lassen's work, completed two years prior to Appleton and five years prior to Hartree, included an additional comprehensive treatment of collisional plasma; but was only published in German, and hence it has not been commonly read and used in the English speaking world of radio physics. Further, regarding the derivation by Appleton, it was noted in the historical study by Gilmore that Wilhelm Altar (while working with Appleton) first calculated the dispersion relation in 1926.

The dispersion relation can be written as an expression for the frequency (squared), but it is also common to write it as an expression for the index of refraction:

$$n^2 = \left(\frac{ck}{\omega}\right)^2$$

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The Appleton-Hartree Equation

The equation is typically given as follows,

$$n^2 = 1 - \frac{X}{1 - iZ - \frac{\frac{1}{2}Y^2 \sin^2\theta}{1-X-iZ} \pm \frac{1}{1-X-iZ}\left(\frac{1}{4}Y^4 \sin^4\theta + Y^2\cos^2\theta\,(1-X-iZ)^2\right)^{1/2}}$$

Definition of Terms

n = Complex Refractive Index

$i = \sqrt{-1}$

$$X = \frac{\omega_0^2}{\omega^2}$$

$$Y = \frac{\omega_H}{\omega}$$

$$Z = \frac{\nu}{\omega}$$

ν = Electron Collision Frequency

$\omega = 2\pi f$

f = Wave Frequency

$\omega_0 = 2\pi f_0 = \sqrt{\frac{Ne^2}{\epsilon_0 m}}$ = Electron Plasma Frequency

$\omega_H = 2\pi f_H = \frac{B_0|e|}{m}$ = Electron Gyro Frequency

ε_0 = Permittivity of Free Space

μ_0 = Permeability of Free Space

B_0 = Ambient Magnetic Field Strength

e = Electron Charge

m = Electron Mass

θ = Angle Between the Ambient Magnetic Field Vector and the Wave Vector

Modes of Propagation: The presence or existence of the ± sign in the Appleton–Hartree equation provides two separate solutions for the refractive index. For propagation perpendicular to the magnetic field, i.e., the '+' sign represents the 'Ordinary Mode', and the '–' sign represents the 'Extraordinary Mode'. For propagation parallel to the magnetic field, i.e., $\mathbf{k} \perp \mathbf{B}_0$, the '+' sign represents a left-hand circularly polarized mode, and the '–' sign represents a right-hand circularly polarized mode.

k is the vector of the propagation plane.

4.10 PROPAGATION THROUGH IONOSPHERE AND MAGNETO-SPHERE HELICON

NOTES

Ionosphere and magnetosphere, regions of Earth's atmosphere in which the number of electrically charged particles—ions and electrons—are large enough to affect the propagation of radio waves. The charged particles are created by the action of extra-terrestrial radiation (mainly from the Sun) on neutral atoms and molecules of air. The ionosphere begins at a height of about 50 km (30 miles) above the surface, but it is most distinct and important above 80 km (50 miles). In the upper regions of the ionosphere, beginning several hundred kilometres above Earth's surface and extending tens of thousands of kilometres into space, is the magnetosphere, a region where the behaviour of charged particles is strongly affected by the magnetic fields of Earth and the Sun.

It is in the lower part of the magnetosphere that overlaps with the ionosphere that the spectacular displays of the aurora borealis and aurora australis take place. The magnetosphere also contains the Van Allen radiation belts, where highly energized protons and electrons travel back and forth between the poles of Earth's magnetic field.

4.10.1 Whistler

Whistler is Very Low Frequency or VLF electromagnetic (radio) wave which is generated during lightning discharges or thunderstorms and lightning flash. This wave propagates through the ionosphere, the portion of the atmosphere where the number of ions is large enough; it begins at a height of about 50 km above the Earth's surface, which is guided by ducts or region along the earth magnetic field. Frequencies of whistlers are usually much smaller than the electron cyclotron frequency ($\omega \ll \omega_{ce}$) in the earth ionosphere and is 100 Hz to 10 kHz, with a maximum amplitude usually at 3 kHz to 5 kHz. These waves are electromagnetic waves but they comprise audio frequencies hence can be detected by a sensitive audio amplifier or loudspeaker. Because these waves produce sound thus also called as whistling atmospheric radio wave. This wave generates gliding sound or descending pitch whistle from high-to-low-frequency. This is due to that these waves get dispersed in course of time in such a way that the higher frequencies wave moves faster than the lower ones. Thus, at point of detection, higher frequency wave arrives sooner than the lower ones. When the whistlers are detected at magnetic conjugate points, it is called as **short whistlers**. However, electromagnetic signal may be reflected at the earth surface and get back along the earth magnetic field to a point close to where it is originated. If whistler is detected at this point, it is called as **long whistler**. Initially, whistlers last about half a second, and they may be repeated at regular intervals of several seconds, growing progressively longer and faints with time.

The phenomenon of atmospheric whistler propagation can be explained in terms of very low frequency region of propagation of right circularly polarized wave.

4.10.2 Faraday Rotation

NOTES

The polarization rotator based on the Faraday effect, also known as 'Faraday rotator is a magneto-optic effect involving transmission of light through a material wherein a longitudinal static magnetic field is present. The state of polarization such as, the axis of linear polarization or the orientation of elliptical polarization is rotated as the wave traverses the device, which is explained by a slight difference in the phase velocity between the left and right circular polarizations. It can be considered as an example of circular birefringence, as is optical activity, but involves a material which has this property in the presence of a magnetic field. Circular birefringence, involving a difference in propagation between opposite circular polarizations, is distinct from linear birefringence which also transforms a wave's polarization but not through a simple rotation.

The polarization state is rotated in proportion to the applied longitudinal magnetic field according to:

$$\beta = VBd$$

where β is the angle of rotation (in radians), ***B*** is the magnetic flux density in the direction of propagation (in teslas), *d* is the length of the path (in metres) where the light and magnetic field interact, and V is the Verdet constant for the material. This empirical proportionality constant (in units of radians per tesla per metre, rad/(T·m)) varies with wavelength and temperature and is tabulated for various materials.

Faraday rotation is a rare example of non-reciprocal optical propagation. Although reciprocity is a basic tenet of electromagnetics, the apparent non-reciprocity in this case is a result of not considering the static magnetic field but only the resulting device. Unlike the rotation in an optically active medium such as a sugar solution, reflecting a polarized beam back through the same Faraday rotator does not undo the polarization change the beam underwent in its forward pass through the medium, but actually doubles it. Then by implementing a Faraday rotator with a rotation of 45°, inadvertent downstream reflections from a linearly polarized source will return with the polarization rotated by 90° and can be simply blocked by a polarizer; this is the basis of optical isolators used to prevent undesired reflections from disrupting an upstream optical system (particularly a laser).

Difference between Faraday rotation and other polarization rotation mechanisms can be stated as. The polarization direction in an optically active medium twists or rotates in the same sense such as like a right-handed screw for either direction, thus in the case of a plane reflection the original rotation is reversed, returning the incident beam to its original polarization. Whereas, in case of Faraday rotator, passage of light in opposite directions experience a magnetic field in opposite directions relative to the propagation direction, and since the rotation, relative to the direction of propagation is determined by the magnetic field, therefore the rotation is opposite between the two propagating directions.

NOTES

Check Your Progress

11. Define Appleton-Hartree formula.
12. What do you mean by whistler?
13. Define short whistlers.
14. Define long whistlers.
15. What do you understand by Faraday's rotation?

4.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In physics, the term 'Plasma' is referred as an electrically conducting medium in which there are approximately equal numbers of positively charged particles and negatively charged particles, typically produced when the atoms in a gas become ionized. Principally, the plasma is a state of matter. The three other common states of matter are the solids, the liquids, and the gases, and consequently plasma is sometimes also called the fourth state of matter. In the plasma, the negative charge is generally carried by the electrons, each of which has precisely one unit of negative charge.
2. Plasma physics is referred as the study of the state of matter of charged particles in which an ionized gaseous substance becomes highly electrically conductive to the point that long-range electric and magnetic fields dominate the behaviour of the matter. The plasma state can be contrasted with the other states: solid, liquid, and gas.
3. The degree of plasma ionisation is determined by the electron temperature relative to the ionisation energy and precisely more weakly by the density. At low temperatures, ions and electrons tend to recombine into bound states—atoms—and the plasma will eventually become a gas. Plasma is created by adding energy to a gas so that some of its electrons leave its atoms. This is called ionization. It results in negatively charged electrons, and positively charged ions. Unlike the other states of matter, the charged particles in a plasma will react strongly to electric and magnetic fields (i.e., electromagnetic fields). If a plasma loses heat, the ions will reform into a gas, emitting the energy which had caused them to ionize.
4. Plasma oscillations are rapid oscillations of the electron density in conducting media, such as plasmas or metals in the UltraViolet or UV region. These oscillations are also known as Langmuir waves (named after the Irving Langmuir), and precisely the oscillations are described as an instability in the dielectric function of a free electron gas. The frequency depends weakly on the wavelength of the oscillation. The quasiparticle resulting from the quantization of these oscillations is the plasmon.

   American physicists Irving Langmuir and Lewi Tonks discovered 'Langmuir Waves' in 1920s. These waves are parallel in form to Jean's instability waves, which are caused by gravitational instabilities in a static medium.

NOTES

5. The quasi--neutrality leads are considered as the significant quantity termed as the Debye Length λ_D. Assume that we place a plane grid into a plasma, which is kept at a specific potential, φ_g.

   In plasmas physics and electrolytes, the Debye length λ_D is also termed as the Debye radius; this is considered as a measure of a charge carrier's net electrostatic effect in a solution and that how much its electrostatic effect continues.

   With each of the Debye length the charges increase and electrically examined while the electric potential decreases in magnitude by 1/e. A Debye sphere is defined as a volume whose radius is the Debye length. Debye length is a significant and essential parameter in plasma physics, electrolytes, and colloids (DLVO theory).

6. As per plasma physics, plasma confinement refers to the act of containment of a plasma by various forces at the extreme conditions in a discrete volume necessary for thermonuclear fusion reactions. Confinement of plasma is required in order to achieve fusion power. There are two major approaches to confinement: magnetic confinement and inertial confinement.

7. Fusion reactions combine light atomic nuclei, such as hydrogen to form heavier ones, such as helium gas, producing energy. Magnetic confinement fusion attempts to use the electrical conductivity of the plasma to contain it through interaction with magnetic fields. The magnetic pressure offsets the plasma pressure. Developing a suitable arrangement of fields that contain the fuel without excessive turbulence or leaking is the primary challenge of this technology.

8. A linear MagnetoHydroDynamic (MHD) wave that is driven by thermal pressure, magnetic pressure, and magnetic tension is known as magnetosonic wave which is also called a magnetoacoustic wave. The magnetosonic waves have two types, the one is known as fast magnetosonic wave and the other is slow magnetosonic wave. Both fast and slow magnetosonic waves are present in the solar corona providing an observational foundation for the technique for coronal plasma diagnostics, coronal seismology. In the homogeneous plasma of infinite extent, and in the absence of gravity, the magnetosonic waves form, together with the Alfvén wave, the three basics linear MagnetoHydroDynamic (MHD) waves.

9. The MagnetoHydroDynamic (MHD) waves lose their defining nature and get mixed properties in case of an inhomogeneous plasma, i.e., a plasma where at least one of the background quantities is not constant. Whereas in some cases, such as the axisymmetric waves in a straight cylinder with a circular basis which is one of the simplest models for a coronal loop, the three MagnetoHydroDynamic (MHD) waves can still be clearly distinguished.

10. Alfvén wave, named after Hannes Alfvén, is a type of magnetohydrodynamic wave where in ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines. An Alfvén wave in a plasma is

considered as a low frequency in comparison to the ion cyclotron frequency naturally travelling oscillation of the ions and the magnetic field. The ion mass density gives the inertia, and the magnetic field line tension provided that the restoring force exists.

NOTES

11. The Appleton–Hartree equation, also sometimes referred to as the Appleton–Lassen equation is a specific mathematical expression used to explain the refractive index for electromagnetic wave propagation in a cold magnetized plasma. The Appleton–Hartree equation was independently developed by numerous different scientists, including Edward Victor Appleton, Douglas Hartree and German radio physicist H. K. Lassen. Further, regarding the derivation by Appleton, it was noted in the historical study by Gilmore that Wilhelm Altar (while working with Appleton) first calculated the dispersion relation in 1926.

    The Appleton-Hartree Equation: The equation is typically given as follows,

$$n^2 = 1 - \frac{X}{1 - iZ - \frac{\frac{1}{2}Y^2 \sin^2\theta}{1-X-iZ} \pm \frac{1}{1-X-iZ}\left(\frac{1}{4}Y^4 \sin^4\theta + Y^2\cos^2\theta\,(1-X-iZ)^2\right)^{1/2}}$$

12. Whistler is Very Low Frequency or VLF electromagnetic (radio) wave which is generated during lightning discharges or thunderstorms and lightning flash. This wave propagates through the ionosphere, the portion of the atmosphere where the number of ions is large enough; it begins at a height of about 50 km above the Earth's surface, which is guided by ducts or region along the earth magnetic field. Frequencies of whistlers are usually much smaller than the electron cyclotron frequency ($\omega \ll \omega_{ce}$) in the earth ionosphere and is 100 Hz to 10 kHz, with a maximum amplitude usually at 3 kHz to 5 kHz. These waves are electromagnetic waves but they comprise audio frequencies hence can be detected by a sensitive audio amplifier or loudspeaker.
13. At point of detection, higher frequency wave arrives sooner than the lower ones. When the whistlers are detected at magnetic conjugate points, it is called as short whistlers.
14. If whistler is detected at this point, it is called as long whistler. Initially, whistlers last about half a second, and they may be repeated at regular intervals of several seconds, growing progressively longer and faints with time.
15. The polarization rotator based on the Faraday effect, also known as 'Faraday rotator is a magneto-optic effect involving transmission of light through a material wherein a longitudinal static magnetic field is present. The state of polarization such as, the axis of linear polarization or the orientation of elliptical polarization is rotated as the wave traverses the device, which is explained by a slight difference in the phase velocity between the left and right circular polarizations. Faraday rotation is a rare example of non-reciprocal optical propagation.

4.12 SUMMARY

NOTES

- In physics, the term 'Plasma' is referred as an electrically conducting medium in which there are approximately equal numbers of positively charged particles and negatively charged particles, typically produced when the atoms in a gas become ionized.
- Principally, the plasma is a state of matter. The three other common states of matter are the solids, the liquids, and the gases, and consequently plasma is sometimes also called the fourth state of matter.
- In the plasma, the negative charge is generally carried by the electrons, each of which has precisely one unit of negative charge.
- Characteristically, the positive charge is carried by atoms or molecules that are specifically missing those identical electrons. In some rare but interesting cases, electrons missing from one type of atom or molecule become attached to another component, resulting in a plasma containing both positive and negative ions.
- The uniqueness of the plasma state is due to the significance of electric and magnetic forces that precisely act on a plasma in addition to such forces as gravity that affect all forms of matter. Since these electromagnetic forces can act at large distances, therefore a plasma will also act collectively much like a fluid even when the particles seldom collide with one another.
- Approximately all the visible matter in the universe exists in the plasma state, uniquely occurring predominantly in this form specifically in the Sun and stars, and also in interplanetary space and interstellar space. Auroras, lightning, and welding arcs are also termed as plasmas; basically, the plasmas exist in neon and fluorescent tubes, in the crystal structure of metallic solids, and in many other phenomena and objects.
- The Earth itself is immersed in a tenuous plasma called the solar wind and is distinctively surrounded by a dense plasma called the ionosphere.
- Plasma is a state of matter in which an ionized gaseous substance becomes highly electrically conductive to the point that long-range electric and magnetic fields dominate the behaviour of the matter. The plasma state can be contrasted with the other states: solid, liquid, and gas.
- Plasma is an electrically neutral medium of unbound positive and negative particles (i.e., the overall charge of a plasma is roughly zero). Although these particles are unbound, they are not 'Free' in the sense of not experiencing forces.
- Moving charged particles generate an electric current within a magnetic field, and any movement of a charged plasma particle affects and is affected by the fields created by the other charges. In turn this governs collective behaviour with many degrees of variation.
- The electron plasma frequency (measuring plasma oscillations of the electrons) is large compared to the electron-neutral collision frequency (measuring frequency of collisions between electrons and neutral particles).

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When this condition is valid, electrostatic interactions dominate over the processes of ordinary gas kinetics.

- Plasma temperature is commonly measured in Kelvin or electron volts and is, informally, a measure of the thermal kinetic energy per particle. High temperatures are usually needed to sustain ionisation, which is a defining feature of a plasma.
- The degree of plasma ionisation is determined by the electron temperature relative to the ionisation energy and more weakly by the density. At low temperatures, ions and electrons tend to recombine into bound states—atoms—and the plasma will eventually become a gas.
- Plasma is created by adding energy to a gas so that some of its electrons leave its atoms. This is called ionization. It results in negatively charged electrons, and positively charged ions.
- Unlike the other states of matter, the charged particles in a plasma will react strongly to electric and magnetic fields (i.e., electromagnetic fields). If a plasma loses heat, the ions will reform into a gas, emitting the energy which had caused them to ionize.
- When a particle of charge 'q' and mass 'm' is placed in an electric field 'E', then the electric force exerted on the charge is 'qE'. If this is the only force exerted on the particle, it must be the net force and so must cause the particle to accelerate.
- Plasma oscillations are rapid oscillations of the electron density in conducting media, such as plasmas or metals in the UltraViolet or UV region. These oscillations are also known as Langmuir waves (named after the Irving Langmuir), and precisely the oscillations are described as an instability in the dielectric function of a free electron gas.
- The frequency depends weakly on the wavelength of the oscillation. The quasiparticle resulting from the quantization of these oscillations is the plasmon.
- American physicists Irving Langmuir and Lewi Tonks discovered 'Langmuir Waves' in 1920s. These waves are parallel in form to Jean's instability waves, which are caused by gravitational instabilities in a static medium.
- The quasi--neutrality leads are considered as the significant quantity termed as the Debye Length λ_D. Assume that we place a plane grid into a plasma, which is kept at a specific potential, φ_g.
- In plasmas physics and electrolytes, the Debye length λ_D is also termed as the Debye radius; this is considered as a measure of a charge carrier's net electrostatic effect in a solution and that how much its electrostatic effect continues.
- With each of the Debye length the charges increase and electrically examined while the electric potential decreases in magnitude by 1/e. A Debye sphere is defined as a volume whose radius is the Debye length. Debye length is a significant and essential parameter in plasma physics, electrolytes, and colloids (DLVO theory).

NOTES

- Plasma parameters are the various characteristics of a plasma, an electrically conductive collection of charged particles that responds collectively to electromagnetic forces. Plasma is the typically in the form of neutral gas-like clouds or charged beams of ions, but unlike gas it may also include dust and grains.
- All the quantities of plasma parameters are defined in Gaussian (cgs) units except energy and temperature which are typically expressed in eV and ion mass expressed in units of the proton mass, $\mu = m_i / m_p$; Z is the ion charge in units of the elementary charge '*e*' for the state of fully charged ion and Z is the respective atomic number; *k* is Boltzmann's constant; *c* is the speed of light; $\ln\Lambda$ is the Coulomb logarithm.
- Magnetic fields are used to contain high-density, high-temperature plasmas because such fields exert pressures and tensile forces on the plasma. An equilibrium configuration is reached only when at all points in the plasma these pressures and tensions exactly balance the pressure from the motion of the particles.
- As per plasma physics, plasma confinement refers to the act of containment of a plasma by various forces at the extreme conditions in a discrete volume necessary for thermonuclear fusion reactions. Confinement of plasma is required in order to achieve fusion power. There are two major approaches to confinement: magnetic confinement and inertial confinement.
- Magnetic confinement fusion is an approach for generating thermonuclear fusion power using magnetic fields to confine fusion fuel in the form of a plasma. Magnetic confinement is one of two major branches of fusion energy research, along with inertial confinement fusion. This magnetic approach was initiated in the 1940s and absorbed the majority of subsequent development.
- Fusion reactions combine light atomic nuclei such as hydrogen to form heavier ones such as helium gas, producing energy. In order to overcome the electrostatic repulsion between the nuclei, they require a temperature in the range of tens of millions of degrees, creating a plasma. In addition, the plasma is required to be contained at a sufficient density for a sufficient time, as specified by the Lawson criterion (triple product).
- Magnetic confinement fusion attempts to use the electrical conductivity of the plasma to contain it through interaction with magnetic fields. The magnetic pressure offsets the plasma pressure. Developing a suitable arrangement of fields that contain the fuel without excessive turbulence or leaking is the primary challenge of this technology.
- A linear MagnetoHydroDynamic (MHD) wave that is driven by thermal pressure, magnetic pressure, and magnetic tension is known as magnetosonic wave which is also called a magnetoacoustic wave.
- The magnetosonic waves have two types, the one is known as fast magnetosonic wave and the other is slow magnetosonic wave. Both fast and slow magnetosonic waves are present in the solar corona providing an observational foundation for the technique for coronal plasma diagnostics, coronal seismology.

NOTES

- In the homogeneous plasma of infinite extent, and in the absence of gravity, the magnetosonic waves form, together with the Alfvén wave, the three basics linear MagnetoHydroDynamic (MHD) waves.
- Alfvén wave, named after Hannes Alfvén, is a type of magnetohydrodynamic wave where in ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines.
- An Alfvén wave in a plasma is considered as a low frequency in comparison to the ion cyclotron frequency naturally travelling oscillation of the ions and the magnetic field. The ion mass density gives the inertia, and the magnetic field line tension provided that the restoring force exists.
- As per Physics the wave propagates in the direction of the magnetic field, on the contrary of it, waves exist at oblique incidence and smoothly change into the magnetosonic wave where the propagation is perpendicular to the magnetic field.
- The MagnetoHydroDynamic (MHD) waves lose their defining nature and get mixed properties in case of an inhomogeneous plasma, i.e., a plasma where at least one of the background quantities is not constant. Whereas in some cases, such as the axisymmetric waves in a straight cylinder with a circular basis which is one of the simplest models for a coronal loop, the three MagnetoHydroDynamic (MHD) waves can still be clearly distinguished.
- As per plasma physics, the waves in plasmas are an interconnected set of particles and fields which propagate in a periodically repeating fashion. It is also known that plasma is a quasi-neutral, electrically conductive fluid. Hence in the simplest case, it is composed of electrons and a single species of positive ions, but it may also contain multiple ion species including negative ions as well as neutral particles.
- Due to its electrical conductivity, a plasma couples to electric and magnetic fields. This complex of particles and fields supports a wide variety of wave phenomena.
- If the thermal motion of the electrons is ignored, it is possible to show that the charge density oscillates at the plasma frequency

$$\omega_{\rm pe} = \sqrt{\frac{n_{\rm e} e^2}{m^* \varepsilon_0}}, \, [\text{rad/s}]$$

$$\omega_{\rm pe} = \sqrt{\frac{4\pi n_{\rm e} e^2}{m^*}}, \, [\text{rad/s}]$$

Where, n_e is the number density of electrons, ***e*** is the effective mass of the electron, and m^* is the permittivity of free space. Remember that the above formula is specifically derived using the approximation that the mass of an ion is infinite.

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- In the process by adding a weak magnetic undulator the propagation of electromagnetic waves in magnetized plasma near the electron cyclotron frequency can be modified.
- The phenomenon of 'Eliminating Electron Cyclotron Heating' is referred to as the 'Undulator-Induced Transparency (UIT)' of the plasma and is the classical equivalent of the well-known quantum mechanical effect of electromagnetically induced transparency. As such UIT can dramatically slow down the waves group velocity, resulting in the extreme compression of the wave energy in the plasma.
- Compressed waves are polarized along the propagation direction and can be used for synchronous electron or ion acceleration. Strong coupling between the two wave helicities is explored to impart the waves with high group velocities for vanishing wave numbers *k*.
- The Appleton–Hartree equation, also sometimes referred to as the Appleton–Lassen equation is a specific mathematical expression used to explain the refractive index for electromagnetic wave propagation in a cold magnetized plasma.
- The Appleton–Hartree equation was independently developed by numerous different scientists, including Edward Victor Appleton, Douglas Hartree and German radio physicist H. K. Lassen.
- Further, regarding the derivation by Appleton, it was noted in the historical study by Gilmore that Wilhelm Altar (while working with Appleton) first calculated the dispersion relation in 1926.
- The dispersion relation can be written as an expression for the frequency (squared), but it is also common to write it as an expression for the index of refraction:

$$n^2 = \left(\frac{ck}{\omega}\right)^2$$

- The Appleton-Hartree Equation: The equation is typically given as follows,

$$n^2 = 1 - \frac{X}{1 - iZ - \frac{\frac{1}{2}Y^2 \sin^2\theta}{1-X-iZ} \pm \frac{1}{1-X-iZ}\left(\frac{1}{4}Y^4 \sin^4\theta + Y^2\cos^2\theta\,(1-X-iZ)^2\right)^{1/2}}$$

- The presence or existence of the $\pm$ sign in the Appleton–Hartree equation provides two separate solutions for the refractive index. For propagation perpendicular to the magnetic field, i.e., the '+' sign represents the 'Ordinary Mode', and the '–' sign represents the 'Extraordinary Mode'.
- Whistler is Very Low Frequency or VLF electromagnetic (radio) wave which is generated during lightning discharges or thunderstorms and lightning flash.
- This wave propagates through the ionosphere, the portion of the atmosphere where the number of ions is large enough; it begins at a height of about 50 km above the Earth's surface, which is guided by ducts or region along the earth magnetic field.

NOTES

- Frequencies of whistlers are usually much smaller than the electron cyclotron frequency ($\omega << \omega_{ce}$) in the earth ionosphere and is 100 Hz to 10 kHz, with a maximum amplitude usually at 3 kHz to 5 kHz. These waves are electromagnetic waves but they comprise audio frequencies hence can be detected by a sensitive audio amplifier or loudspeaker.
- Because these waves produce sound thus also called as whistling atmospheric radio wave. This wave generates gliding sound or descending pitch whistle from high-to-low-frequency. This is due to that these waves get dispersed in course of time in such a way that the higher frequencies wave moves faster than the lower ones.
- At point of detection, higher frequency wave arrives sooner than the lower ones. When the whistlers are detected at magnetic conjugate points, it is called as short whistlers.
- However, electromagnetic signal may be reflected at the earth surface and get back along the earth magnetic field to a point close to where it is originated. If whistler is detected at this point, it is called as long whistler.
- Initially, whistlers last about half a second, and they may be repeated at regular intervals of several seconds, growing progressively longer and faints with time.
- The polarization rotator based on the Faraday effect, also known as 'Faraday rotator is a magneto-optic effect involving transmission of light through a material wherein a longitudinal static magnetic field is present.
- The state of polarization such as, the axis of linear polarization or the orientation of elliptical polarization is rotated as the wave traverses the device, which is explained by a slight difference in the phase velocity between the left and right circular polarizations.
- Circular birefringence, involving a difference in propagation between opposite circular polarizations, is distinct from linear birefringence which also transforms a wave's polarization but not through a simple rotation.
- Faraday rotation is a rare example of non-reciprocal optical propagation.

4.13 KEY TERMS

- **Plasma physics:** Plasma physics is the study of a state of matter comprising charged particles.
- **Plasma:** Plasma is an electrically conducting medium in which there are roughly equal numbers of positively and negatively charged particles which are produced when the atoms in a gas gets ionized.
- **Plasma oscillation:** Plasma oscillations, also known as Langmuir waves (after Irving Langmuir), are rapid oscillations of the electron density in conducting media such as plasmas or metals in the ultraviolet region.
- **Debye shielding:** A slightly different approach to discussing quasi--neutrality leads to the important quantity called the Debye length.

NOTES

- **Plasma parameters:** Plasma parameters define various characteristics of a plasma, an electrically conductive collection of charged particles that responds collectively to electromagnetic forces.
- **Plasma confinement:** Plasma confinement refers to the act of maintaining a plasma in a discrete volume.
- **Magneto-sonic wave:** A magnetosonic wave, also called a magneto-acoustic wave, is a linear Magneto-HydroDynamic (MHD) wave that is driven by thermal pressure, magnetic pressure, and magnetic tension.
- **Inhomogeneous plasma:** Plasma where at least one of the background quantities is not constant.
- **Alfvén wavr:** An Alfvén wave in a plasma is a low-frequency (compared to the ion cyclotron frequency) travelling oscillation of the ions and the magnetic field.
- **Whistles:** Whistler is Very Low Frequency or VLF electromagnetic (radio) wave which is generated during lightning discharges or thunderstorms and lightning flash.
- **Faraday rotation:** A Faraday rotator is a polarization rotator based on the Faraday effect, a magnetooptic effect involving transmission of light through a material when a longitudinal static magnetic field is present.

4.14 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the terms plasma and plasma oscillation.
2. What do you understand by plasma approximation?
3. State the motion of charged particles in a uniform electric field.
4. Define the term Debye shielding.
5. What are the fundamental plasma parameters?
6. What do you understand by magnetoplasma?
7. What is the importance of plasma confinement?
8. Give the fundamental equations of hydromagnetic waves.
9. State the definitions of magnetosonic wave and Alfvén wave.
10. What is phase and group velocity cut offs?
11. Define the hydrodynamical desorption of plasma.
12. Give the Appleton–Hartree formula.
13. How does the propagation take place through the ionosphere?
14. Write notes on the magnetosphere helicon, whistles, and Faraday rotation.

Long-Answer Questions

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1. What do you understand by plasma? Explain the conditions for plasma existence giving appropriate examples.
2. Discuss the concept of plasma oscillation and plasma parameters giving relevant examples.
3. Explain in detail the Debye shielding of plasma.
4. Describe the wave phenomenon of magnetoplasma giving relevant examples.
5. Discuss in detail about the plasma confinement and hydrodynamical desorption of plasma.
6. Briefly explain the parameters and derivations of magnetosonic wave and Alfvén wave.
7. What is the significance of phase and group velocity cut offs? Explain giving examples.
8. Explain the resonance for electromagnetic wave propagating parallel and perpendicular to the magnetic field in plasma.
9. Elaborate on the Appleton–Hartree formula.
10. Discuss the concept of propagation through ionosphere and magnetosphere helicon giving examples.
11. Describe whistles and Faraday rotation with reference to plasma physics.

4.15 FURTHER READING

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UNIT 5 RELATIVISTIC FORMULATIONS

NOTES

Structure

5.0 INTRODUCTION

In the classical electromagnetism, the covariant formulation is well defined by laws of classical electromagnetism, such as Maxwell's equations and the Lorentz force invariant under the Lorentz transformations typically applying the coordinate systems for rectilinear inertial. These formulations, interpretations and transformations/ alterations prove that the classical electromagnetism laws take the similar form or identical structure in any inertial coordinate system and facilitates in explaining the fields and forces from one frame to another.

The continuity equation elaborates and specifies that the rate of mass accumulation in the volume element equals the rate of mass in minus (–) the rate of mass out. It can be stated as mass balance of fluid flowing through a stationary volume element. Characteristically, this statement refers to the principle of mass conservation for a steady or stable, one-dimensional flow together with one inlet and one outlet, and consequently this equation is termed as the continuity equation for the steady or stable one-dimensional flow.

The continuity equation, as per the electromagnetic theory is considered as the specific empirical law which expresses the 'Local' charge conservation however mathematically it is considered as the specific automatic consequence of Maxwell's equations, even though the charge conservation is characteristically more fundamental in comparison to the Maxwell's equations. It precisely states that, 'The divergence of the current density **J** (in Amperes per square metre) is equal to the negative rate of change of the charge density ρ (in Coulombs per cubic metre)'.

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Characteristically, both the form of 'Motion and Continuity' equations are considered as the fundamental and essential equations from which the well-known unique theory of filtration is derived. The accurately simplified and well abbreviated explanations are given by R.B. Bird, W.E. Stewart, E.N. Lightfoot, in which they have defined the two equations, namely the 'Equation of Continuity' and the 'Equation of Motion' for the isothermal systems involving the equation of mechanical energy.

Electromagnetic field tensor is referred as the mathematical object which describes the electromagnetic field in space-time and is also known as field strength tensor/Faraday tensor/Maxwell bisector's tensor. Additionally, it helps in precisely writing the related physical laws which are extremely succinct or brief.

Lorentz force is defined as the force exerted on a charged particle q moving with velocity '*v*' through an electric field **E** and magnetic field **B**. The entire electromagnetic force *F* on the charged particle is termed as the Lorentz force, named after the Dutch physicist Hendrik A. Lorentz, and is given by the equation, $F = qE + qv \times B$.

The electric field of a point charge is considered as the highly essential and basic concept in the field of electromagnetism. Traditionally, the force between two charged points or objects was discovered for scaling together with the product of the charges on points or objects and their precise inverse squared distance. This unique force is now recognized as the Coulomb's law and was precisely discovered around the second half of the 18th century.

Lagrangian mechanics can be precisely defined as a reformulation of classical mechanics. The key difference between the Lagrangian and Hamiltonian mechanics is that Lagrangian mechanics uniquely explain the difference between the Kinetic Energy (KE) and the Potential Energy (PE), whereas the Hamiltonian mechanics evidently describe the sum of the Kinetic Energy (KE) and the Potential Energy (PE).

In this unit, we will study about the covariant formulation of electrodynamics, continuity equation, Lorentz Force, potentials, operators, electromagnetic field tensor, transformation of fields, transformation of field due to a point charge in uniform motion, Lagrangian and Hamiltonian formulations of the motion of a charged particle in an electromagnetic field, radiation from relativistically moving particles.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the covariant formulation of electrodynamics
- State the continuity equation as per electromagnetic theory
- Explain the concept of Lorentz force

- Describe the meaning of electromagnetic field tensor
- Analyse the transformation of fields due to a point charge in uniform motion
- Understand the Lagrangian and Hamiltonian formulations of the motion of a charged particle in an electromagnetic field
- Specify the synchrotron radiation, i.e., radiation from relativistically moving particles

5.2 COVARIANT FORMULATION OF ELECTRODYNAMICS

In the classical electromagnetism, the covariant formulation is well defined by laws of classical electromagnetism, specifically the Maxwell's equations and the Lorentz force invariant under the Lorentz transformations typically applying the coordinate systems for rectilinear inertial.

These notions, derivations, explanations, specified formulations, interpretations and transformations/alterations helps to prove that the classical electromagnetism laws take the similar form or identical structure in any inertial coordinate system and facilitates in explaining the fields and forces from one frame to another. However, this cannot be considered as general or commonly used equation as Maxwell's equations in curved spacetime or non-rectilinear coordinate systems.

In this section, we will use the classical analysis of tensor and the Einstein summation convention to analyse the equations and the Makowski metric form diag $(+1, -1, -1, -1)$. Additionally, this also signifies the classical behaviour of tensor and the Einstein summation convention in which the unique equations are typically restricted as being holding in vacuum, may possibly instead be regarded as the formulations of Maxwell's equations with reference to 'Total Charge' and 'Current'.

For an additional general overall outline of the relationships between classical electromagnetism and special relativity, including various conceptual implications the classical electromagnetism and special relativity theory is used.

Principally, the theory of special relativity has significant role in today's classical electromagnetism theory because it provides formulations and notations about how electromagnetic objects, specifically the electric field and magnetic field get transformed under a Lorentz transformation from one specific inertial frame of reference to another specific inertial frame of reference.

Lorentz Covariant Objects

In physics, the following types of Lorentz tensors can be used for describing specific bodies or particles.

Four Displacements

$$x^\alpha = (ct, \mathbf{x}) = (ct, x, y, z) \, .$$

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Four Velocities

$$u^\alpha = \gamma(c, \mathbf{u}).$$

Here $\gamma(\mathbf{u})$ is termed as the Lorentz factor at the 3-velocity $\mathbf{u}$.

Four Momentum

$$p^\alpha = (E/c, \mathbf{p}) = m_0 u^\alpha$$

Here $\boldsymbol{p}$ is the 3-momentum, E is the total energy, and m_0 is the rest mass.

Four Gradients

$$\partial^\nu = \frac{\partial}{\partial x_\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

The d'Alembertian operator is symbolized by,

$$\partial^2, \partial^2 = \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t} - \nabla^2$$

In the following analysis of tensors, the signs typically depend on the convention that was specifically used in the analysis of metric tensor. Here the convention used can be denoted as (+−−−) in diagonal representation, which is precisely equivalent or analogous to the Minkowski metric tensor:

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

5.3 CONTINUITY EQUATION

The continuity equation elaborates and specifies that the rate of mass accumulation in the volume element equals the rate of mass in minus (–) the rate of mass out. It can be stated as mass balance of fluid flowing through a stationary volume element. Characteristically, this statement refers to the principle of mass conservation for a steady or stable, one-dimensional flow together with one inlet and one outlet, and consequently this equation is termed as the continuity equation for the steady or stable one-dimensional flow.

The continuity equation, as per the electromagnetic theory is considered as the specific empirical law which expresses the 'Local' charge conservation however mathematically it is considered as the specific automatic consequence of Maxwell's equations, even though the charge conservation is characteristically more fundamental in comparison to the Maxwell's equations. It precisely states that, 'The divergence of the current density **J** (in Amperes per square metre) is equal to the negative rate of change of the charge density ρ (in Coulombs per cubic metre)'.

NOTES

While it is generally recognized and acknowledged that the 'Current' is the 'Movement of Charge', essentially and fundamentally the continuity equation states that, 'If the charge moves out of a differential volume typically where the divergence of the current density is positive then the amount of charge within that volume will decrease, which evidences and confirms that the rate of change of charge density will be negative. Therefore, the continuity equation amounts to a conservation of charge'.

If the magnetic monopoles exist, then precisely there would be a continuity equation for monopole currents as well.

Equations of Motion and Continuity

Characteristically, both the form of 'Motion and Continuity' equations are considered as the fundamental and essential equations from which the well-known unique theory of filtration is derived. The accurately simplified and well abbreviated explanations are given by R.B. Bird, W.E. Stewart, E.N. Lightfoot, in which they have defined the two equations, namely the 'Equation of Continuity' and the 'Equation of Motion' for the isothermal systems involving the equation of mechanical energy.

The equation of continuity is simply a mass balance of a fluid flowing through a stationary volume element wherein it states that the rate of mass accumulation in this volume element equals the rate of 'Mass In' minus the rate of 'Mass Out'. In vector form, the balance is as follows:

$$\frac{\partial \rho}{\partial t} = -\left(\blacktriangledown \cdot \rho v\right)$$

Here,

ρ = Density, kg/m^3

t = Time Variable, s

ν = Velocity Vector, m/s

(∇ $\rho\nu$) = Vector Operator signifying the Divergence of the Mass Flux $\rho\nu$

Note: The ∇ uses the units of reciprocal length, m^{-1}.

Somewhat similar to the equation of continuity, the equation of motion is considered as a momentum which is balanced around a unit volume of fluid. Wherein it defines that the rate of momentum which is accumulated equals the rate of 'Momentum In' minus the rate of 'Momentum Out' plus the sum of all the other forces that act on the system. Its vector form is given as:

$$\rho \frac{Dv}{Dt} = -\nabla p - \left[\nabla \cdot \tau\right] + \rho g \qquad \ldots(5.1)$$

Here,

ρ = Density, kg/m^3

t = Time Variable, s

NOTES

$\rho \frac{Dv}{Dt}$ = Rate of Momentum, which is Accumulated Per Unit Volume, kg/m^2/s^2

∇_p = Pressure Force on the Element Per Unit Volume, Pa (Pa = kg/m/s^{12})

$[\nabla\ \tau]$ = Viscous Force on the Element Per Unit Volume, kg/m^2/s^2

τ = Shear Stress Tensor, kg/m/s^2

ρg = Gravitational Force on the Element Per Unit Volume, kg/m^2/s^2

g = Acceleration of Gravity (9.807 m/s^2)

$[\nabla\ \tau]$ can be revised in terms of the fluid viscosity μ, assuming constant μ and constant ρ.

$$[\nabla \cdot \tau] = \mu \nabla^2 v \qquad \ldots(5.2)$$

For the case of constant μ and constant ρ, Equation (5.1) becomes:

$$\rho \frac{Dv}{Dt} = -\nabla p - [\mu \nabla^2 \cdot v] + \rho g \qquad \ldots (5.3)$$

Equation (5.3) is known as the **Navier–Stokes** equation.

For the case of $[\nabla\ \tau] = 0$, Equation (5.3) reduces to:

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho g \qquad \ldots (5.4)$$

Equation (5.4) is known as the **Euler equation**.

The continuity equation is referred as a mathematical expression which defines that the 'Total Mass' of 'Gas' in any 'Deformable Box' should remain always constant. This law of conservation of mass, enables to give an exceptional and unique relation between the time rate of change of the velocities that are incrementing at the surfaces of the box.

5.4 LORENTZ FORCE

Lorentz force is defined as the force exerted on a charged particle q moving with velocity 'v' through an electric field **E** and magnetic field **B**. The entire electromagnetic force F on the charged particle is termed as the Lorentz force, named after the Dutch physicist Hendrik A. Lorentz, and is given by the equation,

$$F = q\mathbf{E} + qv \times \mathbf{B}.$$

The **Lorentz force,** also sometimes termed as the **electromagnetic force,** is the combined electric force and the magnetic force on a point charge because of the electromagnetic fields. When a particle having charge q moves with a velocity '***v***' in an electric field **E** and a magnetic field **B**, then it experiences a force of the form and is expressed as, $F = q\mathbf{E} + qv \times \mathbf{B}$.

As per SI units, the electromagnetic force on a charge 'q' is referred as a combined force in the direction of the electric field **E** which is proportional to the magnitude of the field and the quantity of charge, whereas a force which is at right angles to the magnetic field **B** and the velocity ***v*** of the charge, proportional to the magnitude of the field, the charge, and the velocity. As such the variations or differences can be defined on this basic and the formula explains the 'Magnetic Force' on a current carrying wire which is also occasionally termed as the 'Laplace Force'. The ElectroMotive Force (EMF) in a wired loop that moves through a magnetic field and the force that exist on a moving charged particle, is characteristically termed as an expression of 'Faraday's Law of Induction'.

Historians recommend that the law is implicitly given by James Clerk Maxwell in the scientific paper, published in the year 1865. Hendrik Lorentz arrived at a complete derivation in 1895, identifying the contribution of the electric force a few years after Oliver Heaviside correctly identified the contribution of the magnetic force.

5.4.1 Lorentz Force Law as the Definition of E and B

The Lorentz force law is specifically used as the definition and explanation of the electric field and magnetic field, **E** and **B**, respectively. Specifically, the Lorentz force is recognized with the help of the following empirical statement:

The electromagnetic force ***F*** *on a test charge at a given point and time is a certain function of its charge q and velocity* **v***, which can be parameterized by exactly two vectors* **E** *and* **B***, in the functional form* $F = q(\mathbf{E} + v \times \mathbf{B})$.

This is logically valid for those particles which are approaching or advancing to the speed of light, i.e., with the magnitude of **v**, $|\mathbf{v}| \approx c$. Consequently, the two vector fields **E** and **B** can be by defined through space and time, and hence are termed as the 'Electric Field' and 'Magnetic Field'. These specific fields are typically defined universally in space and time with respect to the condition that what type of test charge a force would receive irrespective of the condition that whether a charge is present to experience the force.

As per definition of electric field **E** and magnetic field **B**, principally the Lorentz force is only considered as a definition or explanation because a real physical particle (as contrasting to the hypothetical 'Test Charge' of infinitesimally small mass and charge) may generate or produce its own finite **E** field and **B** field, which can amend or modify the electromagnetic force experienced by it.

Additionally, if the acceleration is experienced by the charge when forced into a curved trajectory, then the radiation emitted by it be the reason for losing the Kinetic Energy (KE), such as the Bremsstrahlung and synchrotron light. These unique effects happen through a direct effect termed as the radiation reaction force and indirect effect which affects the motion of adjacent charges and currents. Figure 5.1 illustrates the trajectory of a particle with either a positive charge or a negative charge 'q' under the effect of a magnetic field **B**.

NOTES

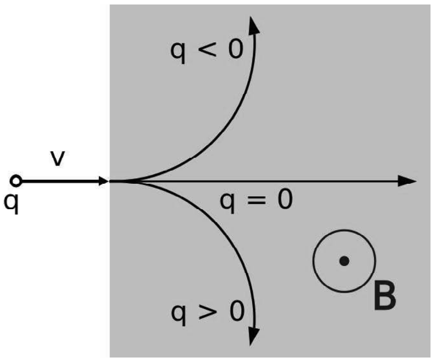

Fig. 5.1 Charged Particles Experiencing the Lorentz Force

5.4.2 Significance of the Lorentz Force

Although the modern Maxwell's equations explain that how the electrically charged particles and currents or moving charged particles produce the electric field and magnetic field, the law of Lorentz force concludes the definition by describing the force that acts on a moving point charge **q** when the electromagnetic fields are present. Furthermore, the Lorentz force law illustrates the effect of electric field **E** and magnetic field **B** upon a point charge, however such electromagnetic forces cannot give the complete explanation. Possibly, the 'Charged Particles' are feasibly coupled to other forces, particularly gravity force and nuclear forces. Consequently, the Maxwell's equations do not remain distinct from other physical laws, although they are coupled to them through the charge density and current density. The answer of a point charge to the Lorentz law is defined as the generation or creation of **E** and **B** by means of currents and charges.

Additionally, the Lorentz force is considered as inadequate or insufficient to explain the collective actions of charged particles for real materials, both in principle and as a matter of computation. In a medium, the charged particles not only respond to the **E** and **B** fields, but the charged particles also generate these fields. Complex transport equations are solved for determining the time and spatial response of charges, such as the Boltzmann equation, the Fokker–Planck equation or the Navier–Stokes equations.

Force on a Current Carrying Wire

When a wire carrying an electric current is placed in a magnetic field, then each of the moving charges, which comprise the current, experiences the Lorentz force, and together they can create a macroscopic force on the wire (sometimes called the **Laplace force**). By combining the Lorentz force law above with the definition of electric current, the following equation results, in the case of a straight, stationary wire.

$$\mathrm{F} = I - \ell \times \mathbf{B}$$

Where $-\ell$ is a vector, whose magnitude is the length of wire, and whose direction is along the wire, aligned with the direction of conventional current charge flow I.

NOTES

If the wire is curved and not straight, then the force on it can be calculated by applying this formula to each infinitesimal segment of wire $d-\ell$, and then adding up all these forces by means of integration. Formally, the net force on a stationary, rigid wire carrying a steady current I is,

$$\mathbf{F} = I \int \mathrm{d}\boldsymbol{\ell} \times \mathbf{B}$$

This is the net force. Additionally, there will typically be torque along with other effects if the wire is not perfectly rigid.

One significant application of this is defined as the Ampère's force law, which explains how two current-carrying wires attract or repel each other, because each experiences a Lorentz force from the other's magnetic field.

5.4.3 Lorentz Force in Space-Time Algebra (STA)

Because, the electric field **E** and magnetic field **B** depend on the velocity of an viewer, therefore the relativistic form of the Lorentz force law is appropriately exhibited beginning from a coordinate-independent expression for the electromagnetic and magnetic fields $\mathcal{F}$, and an arbitrary time-direction, γ_0. This can be established through Space-Time Algebra (or the geometric algebra of space-time), a kind of Clifford algebra typically defined on a pseudo-Euclidean space, as

$$\mathbf{E} = (\mathcal{F} \cdot \gamma_0)\, \gamma_0$$

And

$$i\mathbf{B} = (\mathcal{F} \wedge \gamma_0)\, \gamma_0$$

$\mathcal{F}$, is a space-time bivector, an oriented plane segment, similar to a vector is an oriented line segment, which holds six degrees of freedom corresponding to boosts, and rotations in space-time planes. As per the space algebra, the dot product with the vector γ_0 pulls a vector from the translational portion, while the wedge-product creates a trivector which is considered dual to a vector specifically the usual magnetic field vector. The relativistic velocity is given by the time-like changes in a time-position vector $v = \dot{x}$

5.4.4 Lorentz Force in Terms of Potentials

The electric field **E** and the magnetic field **B** can be replaced by the magnetic vector potential **A** and the scalar electrostatic potential ϕ by equation of the form,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Where,

∇ = Gradient

∇ = Divergence

$\nabla \times$ = Curl

NOTES

The force now given as,

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A})\right]$$

Using an identity for the triple product this can be rewritten as,

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\,\mathbf{A}\right]$$

Using the chain rule, the total derivative of **A** is:

$$\mathbf{F} = q\left[-\nabla_{\mathbf{x}}(\phi - \dot{\mathbf{x}}\cdot\mathbf{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{\mathbf{x}}}(\phi - \dot{\mathbf{x}}\cdot\mathbf{A})\right]$$

Where,

$$\nabla_{\mathbf{x}} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

And,

$$\nabla_{\dot{\mathbf{x}}} = \hat{x}\frac{\partial}{\partial \dot{x}} + \hat{y}\frac{\partial}{\partial \dot{y}} + \hat{z}\frac{\partial}{\partial \dot{z}}.$$

5.5 ELECTROMAGNETIC FIELD TENSOR

Electromagnetic field tensor is the mathematical object which describes the electromagnetic field in space-time, also known as field strength tensor/Faraday tensor/Maxwell bisector's tensor, and additionally it helps to precisely write the related physical laws which are extremely succinct or brief.

The electromagnetic tensor or electromagnetic field tensor (sometimes termed as the field strength tensor, Faraday tensor or Maxwell bivector) as per electromagnetism is a mathematical object which describes the electromagnetic field in space-time. The field tensor was first used after the four-dimensional tensor formulation of special relativity was introduced by Hermann Minkowski. The tensor allows related physical laws to be written very concisely.

Lorentz boost established the transformation of electric and magnetic fields even before Einstein developed the theory of relativity. It is known that E-fields (Electric fields) can transform into B-fields (Magnetic fields) and vice versa. For example, a point charge being at rest gives an 'Electric Field or E-Field'. If we boost to a frame in which the charge is moving, then there is an 'Electric Field' and a 'Magnetic Field'. This means that the E-field cannot be a Lorentz vector. Both the electric and magnetic fields are required to be put together into one (tensor) object to properly handle Lorentz transformations and to write our equations in a covariant approach.

NOTES

Fundamentally, the electromagnetic tensor, conventionally or traditionally labelled or denoted as ***F***, is typically defined as the exterior derivative of the electromagnetic four-potential '***A***' which is referred as a differential 1-form denoted as,

$$F \stackrel{\text{def}}{=} \mathrm{d}A$$

Consequently, '*F*' is referred as a differential 2-form, i.e., precisely an antisymmetric rank-2 tensor field on the Minkowski space. In the component form, it is expressed as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Where '∂' is referred as the four-gradient and '***A***' is defined as the four-potential.

The SI units for Maxwell's equations and the physicist's particle sign convention for the representing the signature of Minkowski space (+ – – –) is used for explanation and evaluation.

The simplest approach and the correct method to do this is to make the electric and magnetic fields components of a rank 2 (antisymmetric) tensor in matrix form is shown below.

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

The electric and magnetic fields can evidently be written in terms of the vector potential $A_\mu = (\vec{A}, i\phi)$, which is precisely a Lorentz vector, and is denoted as,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

Remember that this formulation is inevitably considered as antisymmetric with reference to the interchange of the indices properties. Consequently, the first two Maxwell equations can be certainly and inevitably satisfied for the fields that are precisely derived from a vector potential. Subsequently, the other two Maxwell equations can be expressed or written in terms of the 4-vector $j_\mu = (\vec{j}, ic\rho)$ specifically denoted as,

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{j_\mu}{c}$$

Therefore, as per the theories of electricity and magnetism, it can be stated that,

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{j_\mu}{c}$$

NOTES

Now we will verify some of the terms that are specified in the given equations. Obviously, in the field tensor all the diagonal terms must be zero by the antisymmetry property. Consider the following examples on the off-diagonal conditions for the field tensor with reference to the potential.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = (\vec{\nabla} \times \vec{A})_z = B_z$$
$$F_{13} = \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} = -(\vec{\nabla} \times \vec{A})_y = -B_y$$

$$F_{4i} = \frac{\partial A_i}{\partial x_4} - \frac{\partial A_4}{\partial x_i} = \frac{1}{ic}\frac{\partial A_i}{\partial t} - \frac{\partial (i\phi)}{\partial x_i}$$

$$= -i\left(\frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i}\right)$$

$$= -i\left(\frac{\partial \phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right) = iE_i$$

Consequently, the Maxwell equation states for the last row in the given tensor where,

$$\frac{\partial F_{4\nu}}{\partial x_\nu} = \frac{j_4}{c}$$
$$\frac{\partial F_{4i}}{\partial x_i} = \frac{ic\rho}{c}$$
$$\frac{\partial (iE_i)}{\partial x_i} = i\rho$$
$$\frac{\partial E_i}{\partial x_i} = \rho$$
$$\vec{\nabla} \cdot \vec{E} = \rho$$

5.6 TRANSFORMATION OF FIELDS DUE TO A POINT CHARGE

The electric field of a point charge is considered as the highly essential and basic concept in the field of electromagnetism. Traditionally, the force between two charged points or objects was discovered for scaling together with the product of the charges on points or objects and their precise inverse squared distance. This unique force is now recognized as the Coulomb's law and was precisely discovered around the second half of the 18th century.

NOTES

However, the concept of field is slightly different. If the charge is given, then the force acting on an object under study with unit charge can be calculated. In the standard terms this unit charge is termed as the electric field.

Since the components of the electric and magnetic fields are precisely associated with the elements of a rank-2 tensor, hence the transformation law for these fields precisely follows according to the general tensor transformation law for the rank-2 tensors. Initially, we state the general rule in the standard notion and then consider some specific examples. Under an increase by means of a three-velocity $\vec{v}$, both the electric and magnetic fields $\vec{E}$ and $\vec{B}$ precisely transform to the forms $\vec{E}$ and $\vec{B}$.

Example 5.1 Illustrate a line of charge.

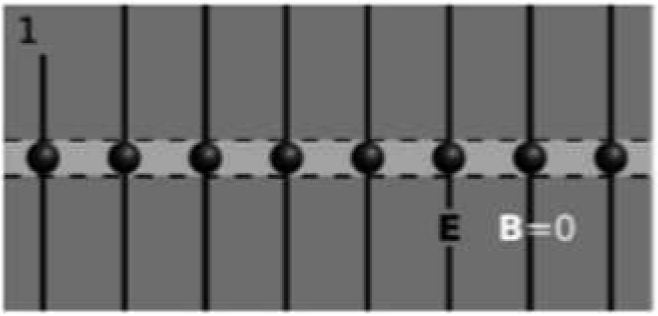

Fig. 5.2 (a): *A Line of Charge*

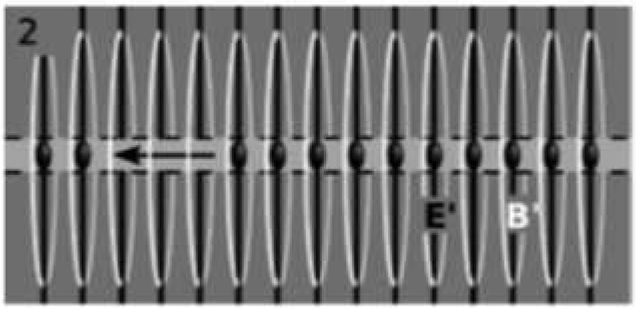

Fig. 5.2 (b): *A Line of Charge*

Figure 5.2 shows a line of charges. At a given nearby point, it creates an electric field E that points outward, as measured by an observer 'O' who is at rest relative to the charges. This field is represented in the Figure 5.2 (b) by means of its design of field lines, which begin on the charges and radiate outward analogous to the bristles of a bottle brush. Because the charges are at rest, therefore the magnetic field is zero. To find the magnitude of the field at a specific distance is referred as a simple and direct application of Gauss's law.

Now consider an observer 'O′', Figure 5.2 (b), moving at velocity '***v***' to the right relative to 'O'. Essentially, it is not required to specify that how the field was created, but we can simply transform the fields, at the point in space into the new frame.

The result is $\mathbf{E}' = \gamma\mathbf{E}$ and $\mathbf{B}'_\perp = \gamma\nu \times \mathbf{E}$

NOTES

In this frame, the electric field is more intense, and there is also a magnetic field, whose design of white field lines forms circles lying in planes perpendicular to the line. If it is obvious that the field was created by means of the line of charge, which is moving according to 'O′', then we can easily explain these conclusions which arise from two effects. First, the line of charge has been length-contracted. This causes the density of charge per unit length to increase by a factor of γ, with a proportional increase in the electric field. In the field-line description, we simply have more charges in the above given Figure (5.2), so there are more field lines coming out of them. Second, the line of charge is moving to the left in this frame, so it forms an electric current, and this current is the cause of the magnetic field B′.

Example 5.2 Illustrate a moving charge.

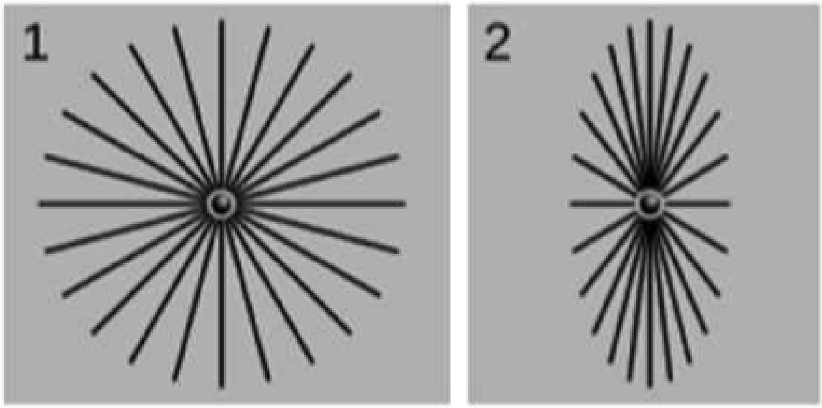

Fig. 5.3 *A Moving Charge*

The Figure 5.3 (1) shows the electric field lines of a charge, in the charge's rest frame K. In Figure 5.3 (2) we see the same electric field, in a frame K′ in which the charge is moving along the X-axis, which points to the right, at 90% of ***c***. In this frame there is also a magnetic field, which is not shown. This electric field, which is time varying is shown as a hyperplane of simultaneity ***t′*=0 of K′**. Remarkably, these field lines all point toward the charge's present position in K′.

Disturbances in the electromagnetic field propagate at ***c***, not instantaneously, so one might have expected the field at a certain location ***P*** in the Figure (5.3) to point toward a location at a distance ***r*** that the charge had occupied at an earlier time ***t′*=–*r*/*c***. This would have produced a set of curved field lines. To see that this is not possible, consider the point (0, 0, *h*, 0) in the Minkowski coordinates of K, i.e., a point on the Y-axis. After a Lorentz transformation along X, the coordinates of this point in K′ are still (0, 0, *h*, 0), so in K′ as well it lies on a line that passes transversely through the present position of the charge. Since this point has $E_x = 0$ and $\vec{B} = 0$ in K, application of the transformation laws shows that $E'_x = 0$ as well, so that the field points toward the charge's present position, not its past position.

An analogous but more complicated calculation illustrates that the field at intermediate angles is also in the instantaneously radial direction. More accurately, the Poynting vector $\vec{E} \times \vec{B}$ then has no radial component, which is as expected because energy should be transported forward but not radiated outward.

NOTES

One might worry that this would indicate that the information about the charge's position was propagating instantaneously, contradicting relativity. But this is a charge that has always been in its current state of motion and always will be. If the charge's motion had been disturbed by some external force at a time later than $t' = -r/c$, the field lines in K would still be pointing toward the location that the charge had previously occupied while at rest, and the field in K′ would be pointing toward its linearly extrapolated position.

5.7 LAGRANGIAG AND HAMILTONIAN FORMULATION OF THE MOTION OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD

Lagrangian mechanics can be precisely defined as a reformulation of classical mechanics. The key difference between the Lagrangian and Hamiltonian mechanics is that Lagrangian mechanics uniquely explain the difference between the Kinetic Energy (KE) and the Potential Energy (PE), whereas the Hamiltonian mechanics evidently describe the sum of the Kinetic Energy (KE) and the Potential Energy (PE).

Motion of a charged particle simultaneously in the existence of both the electric field and the magnetic field has several manifestations and expressions that specifically range from straight line motion to the cycloid and other complex motion. Both the electric field and the magnetic field typically impart acceleration to the charged particles. But the magnetic field has a prerequisite as acceleration due to magnetic field can be related only to the change of direction of motion. Magnetic force is considered being normal always to the velocity of the particle which tends to move the particle about a circular trajectory. Alternatively, the electric force is precisely defined along the electric field and is competent to bring about change in both direction and magnitude depending upon the initial direction of velocity of the charged particle with respect to electric field. If velocity and electric vectors are at an angle, then the particle follows a parabolic path.

One of the significant orientations and angles of the electric field and the magnetic field is referred as 'Crossed Fields'. The term 'Crossed Fields' is precisely used to specify the simultaneous presence of both the electric field and the magnetic field at the right angle. The conduct and action of charged particles, such as electrons under crossed fields has significant importance and consequences in the study of electromagnetic measurement and applications, for example the determination of specific charge of electron, cyclotron, etc.

Evidently, the elementary charged particles have mass of the order of approximately 10^{-28} kg or less. Therefore, even the small electric force or the magnetic force is capable of generating exceptionally high acceleration of the order of 10^{12} m/s^2 or even more. Using the appropriate set up of unit, these particles achieve velocity comparable to speed of light.

Some of the important applications or phenomena associated with simultaneous presence of two fields include:

- Motion of a charged particle in the electric field and the magnetic field.
- Measurement of specific charge of an electron (J. J. Thomson Experiment).

NOTES

- Acceleration of charged particles (cyclotron).

Characteristically, the classical mechanics fundamentally includes two significant formulations namely the **Lagrangian formulation** and the **Hamiltonian formulation**.

Additionally, with respect to the classical mechanics, both the Lagrangian formulation and the Hamiltonian formulation are considered equivalent to the form of Newtonian formulations.

The Lagrangian and Hamiltonian Equations of Motion

In the **Lagrangian formulations**, the equations of motion are precisely obtained by means of the Euler-Lagrange equation, which defines that how a particle or object describes the trajectory, i.e., path in space, which the particle or object will take during the motion.

Fundamentally, the Euler-Lagrange equation is essentially defined as a second order differential equation, i.e., it involves the second derivatives and is represented as,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

In this equation, 'q' and 'q' are basically only the position of particle and its time derivative, i.e., velocity, which is specifically denoted by simply putting a dot above 'q'. More precisely, these are referred as the generalized or simplified position and velocity coordinates.

Additionally, the 'L' denotes or specifies the Lagrangian, which is basically a function that essentially describes the motion through the difference of Kinetic Energy (KE) and Potential Energy (PE).

In the **Hamiltonian formulation**, the following two different but identical equations of motion are considered:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Both of the above mentioned Hamiltonian equations are defined as the first order differential equations with regard to time.

The notion 'H' in the above equation is defined as the Hamiltonian, which uniquely represents or symbolizes the total energy and is generally represent in the form,

$$H = \sum_i p_i \dot{q}_i - L$$

NOTES

Characteristically, the Lagrangian mechanical formulations were developed by the Italian mathematician Joseph-Louis Lagrange in the year 1788, while Hamiltonian mechanical formulations were developed by William Rowan Hamilton in the year 1833. Furthermore, the Lagrangian formulations uses the Cartesian coordinates in the evaluations or calculations on equation of motion, whereas the Hamiltonian formulations typically uses the canonical coordinates.

5.8 RADIATION FROM RELATIVISTICALLY MOVING PARTICLES

'Synchrotron Radiation' which is also known as 'Magneto Bremsstrahlung Radiation' is the electromagnetic radiation produced when relativistic charged particles are subject to an acceleration uniquely perpendicular to their velocity, denoted as ($a \perp v$). It is produced artificially in some types of particle accelerators, or naturally by fast electrons moving through magnetic fields. The radiation produced in this process has a characteristic polarization and the frequencies are generated, ranging over a large portion of the electromagnetic spectrum.

Synchrotron radiation or the Bremsstrahlung radiation is typically emitted by means of a charged particle whenever the acceleration is precisely parallel to the direction of motion. 'Gyromagnetic Radiation' is the general term for radiation emitted by particles in a magnetic field, for which synchrotron radiation is the ultra-relativistic special case. Radiation emitted by charged particles moving non-relativistically in a magnetic field is called 'Cyclotron Emission'. For particles in the mildly relativistic range (~85% of the speed of light), the emission is termed 'Gyro-Synchrotron Radiation'. As per 'Astrophysics', 'Synchrotron Emission' occurs, due to ultra-relativistic motion of a charged particle around the black hole. Whereas If the source follows a circular geodesic around the black hole, the 'Synchrotron Radiation' occurs for orbits close to the photosphere where the motion is in the ultra-relativistic regime.

Synchrotron radiation was first observed by technician Floyd Haber, on April 24th, 1947 at the 70 MeV electron synchrotron of the General Electric research laboratory in Schenectady, New York. While this was not the first synchrotron built, it was the first with a transparent vacuum tube, allowing the radiation to be directly observed.

A direct consequence of Maxwell's equations is that the accelerated charged particles will always emit electromagnetic radiation. Synchrotron radiation is the special case of charged particles moving at relativistic speed undergoing acceleration uniquely perpendicular to their direction of motion, typically in a magnetic field. In such a field, the force due to the field is always perpendicular to both the direction of motion and to the direction of field, as defined by the Lorentz force law.

Principally, the power that is typically carried by means of the radiation is obtained precisely by the relativistic Larmor formula and is represented in SI units as,

$$P_\gamma = \frac{1}{6\pi\epsilon_o}\frac{q^2 a^2}{c^3}\gamma^4$$

NOTES

Where,

ε_0 is the Vacuum Permittivity.

q is the Particle Charge.

a is the Magnitude of the Acceleration.

c is the Speed of Light.

Υ is the Lorentz Factor.

The force on the emitting electron is uniquely given by the Abraham–Lorentz–Dirac force.

In the field of physics and precisely of electromagnetism, the Abraham–Lorentz force, also termed as the Lorentz–Abraham force, is defined as the recoil force precisely on an accelerating charged particle uniquely caused by the particle emitting electromagnetic radiation. It is also sometimes termed as the radiation reaction force, radiation damping force or the self-force. The Abraham–Lorentz force is named after the physicists Max Abraham and Hendrik Lorentz.

Fundamentally, when the radiation is precisely emitted by means of a particle moving in a plane, then the radiation is linearly polarized when typically observed in that precise plane, and circularly polarized when typically observed at a small angle.

In the higher studies of particle formulations, the 'Supermassive Black Holes' have been recommended for the production of the synchrotron radiation, by means of ejection of jets typically produced through the gravitationally accelerating ions precisely through the super contorted 'Tubular' polar areas of the magnetic fields. Such specific jets, the nearest or closest being the Messier 87, have been checked and validated by means of the Hubble telescope while apparently being superluminal, travelling at $6 \times c$ (six times the speed of light) from our planetary frame. This phenomenon is typically caused because the jets travel extremely close to the speed of light and at an extremely small angle towards the observer. Consequently, the high-velocity jets typically emit light at every point of their path, subsequently the light emitted by the high-velocity jets does not approach or reach the observer as quickly as the jet itself.

Check Your Progress

1. Define the covariant formulation of classical electromagnetism.
2. State the types of Lorentz tensors used for describing specific particles.
3. What does the equation of continuity state?
4. Give the equation of continuity in vector form.
5. What is Lorentz force?
6. Define the Lorentz force law.
7. What do you mean by electromagnetic field tensor?
8. State about the electric field of a point charge.
9. What is the key difference between Lagrangian and Hamiltonian mechanics?
10. Define the term synchrotron radiation.

5.9 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. In the classical electromagnetism, the covariant formulation is defined by laws of classical electromagnetism, specifically the Maxwell's equations and the Lorentz force invariant under the Lorentz transformations typically applying the coordinate systems for rectilinear inertial.
2. In physics, the following types of Lorentz tensors can be used for describing specific bodies or particles.

   Four Displacements

   $$x^{\alpha} = (ct, \mathbf{x}) = (ct, x, y, z) .$$

   Four Velocities

   $$u^{\alpha} = \gamma(c, \mathbf{u}),$$

   Here γ(**u**) is termed as the Lorentz factor at the 3-velocity **u**.

   Four Momentum

   $$p^{\alpha} = (E/c, \mathbf{p}) = m_0 u^{\alpha}$$

   Here ***p*** is the 3-momentum, *E* is the total energy, and m_0 is the rest mass.

   Four Gradients

   $$\partial^{\nu} = \frac{\partial}{\partial x_{\nu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

   The d'Alembertian operator is symbolized by,

   $$\partial^2, \partial^2 = \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t} - \nabla^2$$

3. The continuity equation, as per the electromagnetic theory is considered as the specific empirical law which expresses the 'Local' charge conservation however mathematically it is considered as the specific automatic consequence of Maxwell's equations, even though the charge conservation is characteristically more fundamental in comparison to the Maxwell's equations. It precisely states that, 'The divergence of the current density **J** (in Amperes per square metre) is equal to the negative rate of change of the charge density ρ (in Coulombs per cubic metre)'.
4. The equation of continuity is simply a mass balance of a fluid flowing through a stationary volume element wherein it states that the rate of mass accumulation in this volume element equals the rate of 'Mass In' minus the rate of 'Mass Out'. In vector form, the balance is as follows:

   $$\frac{\partial \rho}{\partial t} = -\left(\blacktriangledown \cdot \rho v\right)$$

NOTES

Here,

ρ = Density, kg/m^3

t = Time Variable, s

ν = Velocity Vector, m/s

$(\nabla \rho\nu)$ = Vector Operator signifying the Divergence of the Mass Flux $\rho\nu$

The ∇ uses the units of reciprocal length, m^{-1}.

5. The Lorentz force is defined as the force exerted on a charged particle q moving with velocity 'v' through an electric field **E** and magnetic field **B**. The entire electromagnetic force F on the charged particle is termed as the Lorentz force, named after the Dutch physicist Hendrik A. Lorentz, and is given by the equation,

   $F = q\mathbf{E} + qv \times \mathbf{B}$.

   The Lorentz force**,** also sometimes termed as the electromagnetic force**,** is the combined electric force and the magnetic force on a point charge because of the electromagnetic fields.

6. The Lorentz force law is specifically used as the definition and explanation of the electric field and magnetic field, **E** and **B**, respectively. Specifically, the Lorentz force is recognized with the help of the following empirical statement:

   The electromagnetic force ***F*** on a test charge at a given point and time is a certain function of its charge ***q*** and velocity ***v****,* which can be parameterized by exactly two vectors **E** and **B***,* in the functional form $F = q(\mathbf{E} + v \times \mathbf{B})$.

7. The electromagnetic tensor or electromagnetic field tensor (sometimes called as the field strength tensor, Faraday tensor or Maxwell bivector) as per electromagnetism is a mathematical object which describes the electromagnetic field in space-time. The field tensor was first used after the four-dimensional tensor formulation of special relativity was introduced by Hermann Minkowski.

8. The electric field of a point charge is considered as the highly essential and basic concept in the field of electromagnetism. Traditionally, the force between two charged points or objects was discovered for scaling together with the product of the charges on points or objects and their precise inverse squared distance. This unique force is now recognized as the Coulomb's law and was precisely discovered around the second half of the 18th century.

9. Lagrangian mechanics can be precisely defined as a reformulation of classical mechanics. The key difference between the Lagrangian and Hamiltonian mechanics is that Lagrangian mechanics uniquely explain the difference between the Kinetic Energy (KE) and the Potential Energy (PE), whereas the Hamiltonian mechanics evidently describe the sum of the Kinetic Energy (KE) and the Potential Energy (PE).

10. The 'Synchrotron Radiation' which is also known as 'Magneto Bremsstrahlung Radiation' is the electromagnetic radiation produced when relativistic charged particles are subject to an acceleration uniquely perpendicular to their velocity, denoted as $(a \perp v)$. It is produced artificially

in some types of particle accelerators, or naturally by fast electrons moving through magnetic fields. The radiation produced in this process has a characteristic polarization and the frequencies are generated, ranging over a large portion of the electromagnetic spectrum.

NOTES

5.10 SUMMARY

- In the classical electromagnetism, the covariant formulation is well defined by laws of classical electromagnetism, specifically the Maxwell's equations and the Lorentz force invariant under the Lorentz transformations typically applying the coordinate systems for rectilinear inertial.
- These notions, derivations, explanations, specified formulations, interpretations and transformations/alterations helps to prove that the classical electromagnetism laws take the similar form or identical structure in any inertial coordinate system and facilitates in explaining the fields and forces from one frame to another.
- However, this cannot be considered as general or commonly used equation as Maxwell's equations in curved spacetime or non-rectilinear coordinate systems.
- The classical analysis of tensor and the Einstein summation convention is used to analyse the equations and the Makowski metric form diag (+1, –1, –1, –1). Additionally, this also signifies the classical behaviour of tensor and the Einstein summation convention in which the unique equations are typically restricted as being holding in vacuum, may possibly instead be regarded as the formulations of Maxwell's equations with reference to 'Total Charge' and 'Current'.
- Principally, the theory of special relativity has significant role in today's classical electromagnetism theory because it provides formulations and notations about how electromagnetic objects, specifically the electric field and magnetic field get transformed under a Lorentz transformation from one specific inertial frame of reference to another specific inertial frame of reference.
- In the analysis of tensors, the signs typically depend on the convention that was specifically used in the analysis of metric tensor. Here the convention used can be denoted as (+– – –) in diagonal representation, which is precisely equivalent or analogous to the Minkowski metric tensor:

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- The continuity equation elaborates and specifies that the rate of mass accumulation in the volume element equals the rate of mass in minus (–) the rate of mass out. It can be stated as mass balance of fluid flowing through a stationary volume element.

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- Characteristically, this statement refers to the principle of mass conservation for a steady or stable, one-dimensional flow together with one inlet and one outlet, and consequently this equation is termed as the continuity equation for the steady or stable one-dimensional flow.
- The continuity equation, as per the electromagnetic theory is considered as the specific empirical law which expresses the 'Local' charge conservation however mathematically it is considered as the specific automatic consequence of Maxwell's equations, even though the charge conservation is characteristically more fundamental in comparison to the Maxwell's equations.
- It precisely states that, 'The divergence of the current density **J** (in Amperes per square metre) is equal to the negative rate of change of the charge density ρ (in Coulombs per cubic metre)'.
- While it is generally recognized and acknowledged that the 'Current' is the 'Movement of Charge', essentially and fundamentally the continuity equation states that, 'If the charge moves out of a differential volume typically where the divergence of the current density is positive then the amount of charge within that volume will decrease, which evidences and confirms that the rate of change of charge density will be negative. Therefore, the continuity equation amounts to a conservation of charge'.
- Characteristically, both the form of 'Motion and Continuity' equations are considered as the fundamental and essential equations from which the well-known unique theory of filtration is derived.
- The accurately simplified and well abbreviated explanations are given by R.B. Bird, W.E. Stewart, E.N. Lightfoot, in which they have defined the two equations, namely the 'Equation of Continuity' and the 'Equation of Motion' for the isothermal systems involving the equation of mechanical energy.
- The equation of continuity is simply a mass balance of a fluid flowing through a stationary volume element wherein it states that the rate of mass accumulation in this volume element equals the rate of 'Mass In' minus the rate of 'Mass Out'.
- The Lorentz force is defined as the force exerted on a charged particle q moving with velocity 'v' through an electric field **E** and magnetic field **B**. The entire electromagnetic force F on the charged particle is termed as the Lorentz force, named after the Dutch physicist Hendrik A. Lorentz, and is given by the equation,

  $F = q\mathbf{E} + qv \times \mathbf{B}$.
- The Lorentz force, also sometimes termed as the electromagnetic force**,** is the combined electric force and the magnetic force on a point charge because of the electromagnetic fields.

  Fs per SI units, the electromagnetic force on a charge 'q' is referred as a combined force in the direction of the electric field **E** which is proportional to the magnitude of the field and the quantity of charge, whereas a force which is at right angles to the magnetic field **B** and the velocity ***v*** of the charge, proportional to the magnitude of the field, the charge, and the velocity.

- The Lorentz force law is specifically used as the definition and explanation of the electric field and magnetic field, **E** and **B**, respectively. Specifically, the Lorentz force is recognized with the help of the following empirical statement:
- The electromagnetic force ***F*** on a test charge at a given point and time is a certain function of its charge ***q*** and velocity ***v***, which can be parameterized by exactly two vectors **E** and **B**, in the functional form $F = q(\mathbf{E} + v \times \mathbf{B})$.
- As per definition of electric field **E** and magnetic field **B**, principally the Lorentz force is only considered as a definition or explanation because a real physical particle (as contrasting to the hypothetical 'Test Charge' of infinitesimally small mass and charge) may generate or produce its own finite **E** field and **B** field, which can amend or modify the electromagnetic force experienced by it.
- When a wire carrying an electric current is placed in a magnetic field, then each of the moving charges, which comprise the current, experiences the Lorentz force, and together they can create a macroscopic force on the wire (sometimes called the Laplace force).
- By combining the Lorentz force law above with the definition of electric current, the following equation results, in the case of a straight, stationary wire.

  $$F = I - \ell \times \mathbf{B}$$

  Where $-\ \ell$ is a vector, whose magnitude is the length of wire, and whose direction is along the wire, aligned with the direction of conventional current charge flow *I*.
- Electromagnetic field tensor is the mathematical object which describes the electromagnetic field in space-time, also known as field strength tensor/ Faraday tensor/Maxwell bisector's tensor, and additionally it helps to precisely write the related physical laws which are extremely succinct or brief.
- The electromagnetic tensor or electromagnetic field tensor (sometimes called as the field strength tensor, Faraday tensor or Maxwell bivector) as per electromagnetism is a mathematical object which describes the electromagnetic field in space-time.
- The field tensor was first used after the four-dimensional tensor formulation of special relativity was introduced by Hermann Minkowski. The tensor allows related physical laws to be written very concisely.
- Lorentz boost established the transformation of electric and magnetic fields even before Einstein developed the theory of relativity. It is known that E-fields (Electric fields) can transform into B-fields (Magnetic fields) and vice versa. For example, a point charge being at rest gives an 'Electric Field or E-Field'.

NOTES

- If we boost to a frame in which the charge is moving, then there is an 'Electric Field' and a 'Magnetic Field'. This means that the E-field cannot be a Lorentz vector. Both the electric and magnetic fields are required to be put together into one (tensor) object to properly handle Lorentz transformations and to write our equations in a covariant approach.
- The electric field of a point charge is considered as the highly essential and basic concept in the field of electromagnetism.
- Traditionally, the force between two charged points or objects was discovered for scaling together with the product of the charges on points or objects and their precise inverse squared distance. This unique force is now recognized as the Coulomb's law and was precisely discovered around the second half of the 18th century.
- However, the concept of field is slightly different. If the charge is given, then the force acting on an object under study with unit charge can be calculated. In the standard terms this unit charge is termed as the electric field.
- Lagrangian mechanics can be precisely defined as a reformulation of classical mechanics. The key difference between the Lagrangian and Hamiltonian mechanics is that Lagrangian mechanics uniquely explain the difference between the Kinetic Energy (KE) and the Potential Energy (PE), whereas the Hamiltonian mechanics evidently describe the sum of the Kinetic Energy (KE) and the Potential Energy (PE).
- Evidently, the elementary charged particles have mass of the order of approximately $10^{"28}$ kg or less. Therefore, even the small electric force or the magnetic force is capable of generating exceptionally high acceleration of the order of 10^{12} m/s^2 or even more.
- In the Lagrangian formulations, the equations of motion are precisely obtained by means of the Euler-Lagrange equation, which defines that how a particle or object describes the trajectory, i.e., path in space, which the particle or object will take during the motion.
- Characteristically, the Lagrangian mechanical formulations were developed by the Italian mathematician Joseph-Louis Lagrange in the year 1788, while Hamiltonian mechanical formulations were developed by William Rowan Hamilton in the year 1833. Furthermore, the Lagrangian formulations uses the Cartesian coordinates in the evaluations or calculations on equation of motion, whereas the Hamiltonian formulations typically uses the canonical coordinates.
- The 'Synchrotron Radiation' which is also known as 'Magneto Bremsstrahlung Radiation' is the electromagnetic radiation produced when relativistic charged particles are subject to an acceleration uniquely perpendicular to their velocity, denoted as ($a \perp v$). It is produced artificially in some types of particle accelerators, or naturally by fast electrons moving through magnetic fields.
- The radiation produced in this process has a characteristic polarization and the frequencies are generated, ranging over a large portion of the electromagnetic spectrum.

- Synchrotron radiation was first observed by technician Floyd Haber, on April 24th, 1947 at the 70 MeV electron synchrotron of the General Electric research laboratory in Schenectady, New York. While this was not the first synchrotron built, it was the first with a transparent vacuum tube, allowing the radiation to be directly observed.
- A direct consequence of Maxwell's equations is that the accelerated charged particles will always emit electromagnetic radiation.
- Synchrotron radiation is the special case of charged particles moving at relativistic speed undergoing acceleration uniquely perpendicular to their direction of motion, typically in a magnetic field. In such a field, the force due to the field is always perpendicular to both the direction of motion and to the direction of field, as defined by the Lorentz force law.

5.11 KEY WORDS

- **Covariant formulation:** In the classical electromagnetism, the covariant formulation is defined by laws of classical electromagnetism, specifically the Maxwell's equations and the Lorentz force invariant under the Lorentz transformations typically applying the coordinate systems for rectilinear inertial.
- **Continuity equation:** The continuity equation, as per the electromagnetic theory is considered as the specific empirical law which precisely states that, 'The divergence of the current density **J** (in Amperes per square metre) is equal to the negative rate of change of the charge density ρ (in Coulombs per cubic metre)'.
- **Lorentz force:** The Lorentz force is defined as the force exerted on a charged particle q moving with velocity 'v' through an electric field **E** and magnetic field **B**. The entire electromagnetic force F on the charged particle is termed as the Lorentz force, named after the Dutch physicist Hendrik A. Lorentz, and is given by the equation,

  $F = q\mathbf{E} + qv \times \mathbf{B}$.
- **Lorentz force law:** The Lorentz force law is specifically used as the definition and explanation of the electric field and magnetic field, **E** and **B**, respectively. Specifically, the Lorentz force states that, 'The electromagnetic force $\boldsymbol{F}$ on a test charge at a given point and time is a certain function of its charge $\boldsymbol{q}$ and velocity $\boldsymbol{v}$, which can be parameterized by exactly two vectors **E** and **B**, in the functional form $F = q(\mathbf{E} + v \times \mathbf{B})$'.
- **Electromagnetic tensor:** The electromagnetic tensor or electromagnetic field tensor (sometimes called as the field strength tensor, Faraday tensor or Maxwell bivector) as per electromagnetism is a mathematical object which describes the electromagnetic field in space-time.
- **Electromagnetic field tensor:** Electromagnetic field tensor is the mathematical object which describes the electromagnetic field in space-time, also known as field strength tensor/Faraday tensor/Maxwell bisector's

tensor, and additionally it helps to precisely write the related physical laws which are extremely succinct or brief.

- **Synchrotron radiation:** Synchrotron radiation is the special case of charged particles moving at relativistic speed undergoing acceleration uniquely perpendicular to their direction of motion, typically in a magnetic field, denoted as (a ¥" v). It is produced artificially in some types of particle accelerators, or naturally by fast electrons moving through magnetic fields.

5.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What do you mean by covariant formulation?
2. Define the term continuity equation.
3. How is the Lorentz force law used as the definition of electric and magnetic fields E and B**?**
4. State the Lorentz force in space-time algebra.
5. Write a note on Lorentz force in terms of potentials.
6. What is electromagnetic field tensor?
7. Define the transformation of field due to a point charge in uniform motion.
8. Differentiate between the Lagrangian and Hamiltonian formulations of the motion.
9. How does the radiation from relativistically moving particles defined?
10. What creates synchrotron radiation?

Long-Answer Questions

1. Briefly explain the significance of covariant formulation with reference to electrodynamics giving appropriate examples.
2. Discuss in detail about the continuity equation giving relevant examples.
3. Explain the significance of Lorentz force.
4. Elaborate on the Lorentz force law giving the potentials and operators giving examples.
5. Briefly explain the electromagnetic field tensor giving derivations and explanations.
6. Evaluate the transformation of fields due to a point charge.
7. Explain the transformation of field due to a point charge in uniform motion giving examples.
8. Discuss in detail the Lagrangian and Hamiltonian formulations of the motion of a charged particle in an electromagnetic field.
9. Briefly discuss about the radiation taking place from relativistically moving particles.

5.13 FURTHER READINGS

NOTES

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