M.Sc. Final Year

Mathematics, MM-07

PARTIAL DIFFERENTIAL EQUATIONS AND MECHANICS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

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SYLLABI-BOOK MAPPING TABLE

Partial Differential Equations and Mechanics

Syllabi	Mapping in Book
UNIT - 1: Examples of PDE. Classification, Transport Equation - Initial Value Problem, Non-Homogeneous Equation. Laplace's Equation - Fundamental Solution, Mean Value Formula, Properties of Solutions, Energy Methods. Wave Equation - Solution by Spherical Means, Non-Homogeneous Equations, Energy Methods.	**Unit-1:** Partial Differential Equations - I **(Pages 3-63)**
UNIT - 2: Non-Linear First Order PDE - Complete Integrals, Envelopes, Characteristics, Hamilton - Jacobi Equations (Calculus of Variations, Hamilton's ODE, Legendre Transform, Hopf-Lax Formula, Weak Solutions, and Uniqueness). Representation of Solutions - Separation of Variables, Similarity Solution (Plane and Travelling Waves, Solitons, Similarity Under Scaling), Fourier and Laplace Transform, Hopf - Cole Transform, Hodograph and Legendre Transform, Potential Functions.	**Unit-2:** Partial Differential Equations - II **(Pages 65-126)**
UNIT - 3: Generalized Coordinates, Holonomic and Non-Holonomic Systems. Scleronomic and Rheonomic Systems, Generalized Potential. Lagrange's Equations of First Kind. and Second Kind. Uniqueness of Solution. Energy Equation for Conservative Fields, Hamilton's Variables, Donkin's Theorem, Hamilton Canonical Equations. Cyclic Coordinates Routh's Equations. Poisson's Bracket. Poisson's Identity. Jacobi-Poisson Theorem.	**Unit-3:** Analytical Dynamics: Generalized Co-ordinates **(Pages 127-172)**
UNIT - 4: Hamilton's Principle. Principle of Least Action. Poincare Carton Integral Invariant Whittaker's Wquations. Jacobi's Equations. Statement of Lee Hwa Chung's Theorem. Hamilton - Jacobi Equation. Jacobi Theorem, Method of Separation of Variables. Largrange Brackets Condition of Canonical Character of a Transformation Intern of Lagrange Brackets and Poisson Brackets. Invariance of Lagrange Brackets and Poisson Brackets under Canonical Transformations.	**Unit-4:** Analytical Dynamics: Hamilton's Principle **(Pages 173-197)**
UNIT - 5: Gravitation Attraction and Potential of Rod, Disc, Spherical Shells and Sphere. Surface Integral of Normal Attraction (Application & Gauss Theorem), Laplace and Poisson Equations. Workdone by Self Attracting Systems. Distributions for a given Potential, Equipotential Surfaces. Surface and Solid Harmonics Surface Density in Term of Surface Harmonics.	**Unit-5:** Gravitation **(Pages 199-245)**

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UNIT 5 GRAVITATION **199-245**

INTRODUCTION

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A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering and physics specifically whenever a deterministic relation involving some continuously varying quantities modelled by functions and their rates of change in space and/or time expressed as derivatives is known or postulated. This is evaluated using mechanics where the motion of a body is described by its position and velocity as the time value varies. Mechanics is the branch of physics concerned with the behaviour of physical bodies when subjected to forces or displacements and the subsequent effects of the bodies on their environment. It is a branch of classical physics that deals with the particles that are moving either with less velocity or that are at rest. Newton's laws allow for the given position, velocity, acceleration and various forces acting on the body to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation is also termed as an equation of motion and can be solved explicitly. Differential equations are mathematically examined from several different perspectives typically concerned with their solutions and the set of functions that satisfy the equation.

In mathematics, partial differential equations or PDE are differential equations that contain unknown multivariable functions and their partial derivatives. Partial differential equations are used to formulate problems involving functions of several variables. Partial differential equations can be used to describe a wide variety of phenomenon such as sound, heat, electrostatics, electrodynamics, fluid flow or elasticity. These apparently distinct physical phenomena can be formalized identically in terms of partial differential equations which show that they are governed by the same underlying dynamic. Partial differential equations find their generalization in stochastic partial differential equations. General solutions of the heat equation can be found by the method of separation of variables. Solutions of Laplace's equation are called harmonic functions. The real and imaginary parts of any analytic function are conjugate harmonic functions and they both satisfy the Laplace equation. Laplace's equation is specifically used to find a solution that satisfies arbitrary values on the boundary of a domain. The partial differential equations are used to solve the wave equation and more general hyperbolic partial differential equations which typically have no more derivatives than the data. Some linear, second order partial differential equations can be classified as parabolic, hyperbolic or elliptic.

This book, *Partial Differential Equations and Mechanics*, follows the SIM format wherein each Unit begins with an Introduction to the topic followed by an outline of the 'Objectives'. The detailed content is then presented in a simple and an organized manner, interspersed with 'Check Your Progress' questions to test the understanding of the students. A 'Summary' along with a list of 'Key Terms' and a set of 'Self-Assessment Questions and Exercises' is also provided at the end of each unit for effective recapitulation.

UNIT 1 PARTIAL DIFFERENTIAL EQUATIONS - I

NOTES

Structure

1.0 INTRODUCTION

A Partial Differential Equation (PDE) is a mathematical equation that including two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables. Each type of PDE has certain functionalities that help to identify whether a particular finite element approach is appropriate to the problem being described by the PDE. The solution depends on the equation and several variables contain partial derivatives with respect to the variables. There are three-types of second-order PDEs in mechanics. They are elliptic, Parabolic and hyperbolic PDE.

The transport equation is a partial differential equation of the form $u_t + cu_x = 0$. This equation can be used to model air pollution, dye dispersion, or even traffic flow with u representing the density of the pollutant (or dye or traffic) at position x and time t.

An Initial Value Problem (IVP) is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a

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given point in the domain. Modeling a system in physics or other sciences frequently amounts to solving an initial value problem.

If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise.

In mathematics and physics, 'Laplace's Equation' is a second-order partial differential equation named after Pierre-Simon Laplace, who first studied its properties. The general theory of solutions to Laplace's equation is known as 'Potential Theory'. The solutions of Laplace's equation are the harmonic functions, which are important in multiple branches of physics, notably electrostatics, gravitation, and fluid dynamics.

Based on the ubiquitous nature of the mean value theorem in problems involving the Laplacian, it is clear that an analogous formula for a general divergence form elliptic operator would necessarily be very useful.

Energy methods originate some category of system 'Energy' from a partial differential equation. That energy may then be used to derive such things as existence or uniqueness of the solution, and whether it depends continuously on the data.

The wave equation is a second-order linear partial differential equation for the description of waves—as they occur in classical physics—such as mechanical waves (e.g., water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics. Due to the fact that the second order wave equation describes the superposition of an incoming wave and an outgoing wave (i.e., rather a standing wave field) it is also called 'Two-Way Wave Equation' (in contrast, the 1st order One-way wave equation describes a single wave with predefined wave propagation direction and is much easier to solve due to the 1st order derivatives).

In this unit, you will learn about the partial differential equations, examples and classification of PDE, transport equation, initial value problem, non-homogeneous equation, Laplace's equation, fundamental solution of Laplace equation, mean value formula, properties and energy methods of mean value, wave equation, solution of wave equation by spherical means, non-homogeneous and energy method of wave equation.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Analyse partial differential equations
- Solve various types of partial differential equations
- Know about the classification of PDE
- Understand the transport equation and initial value problems

- Elaborate of the non-homogeneous equations
- Understand the significance of Laplace's equation
- Calculate the mean value formula and thier properties
- Elaborate on the energy methods of mean value formula
- Define wave equation
- Discuss the solution of wave equation by spherical method
- Explain about the non-homogeneous and energy method of wave equation

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1.2 PARTIAL DIFFERENTIAL EQUATIONS: AN INTRODUCTION

In mathematics, a Partial Differential Equation (PDE) is an equation which carry out the relations between the various 'Partial Derivatives' of a multivariable function. Whereas the function is often thought of as an 'Unknown' to be solved for, similarly to how x is thought of as an unknown number to be solved for in an **algebraic equation** similar as like $x^2 - 3x + 2 = 0$.

On the other hand, it is usually impossible to write down explicit formulas for solutions of partial differential equations. There is, similarly, a vast amount of modern mathematical and scientific research on methods to numerically approximate solutions of certain partial differential equations using computers. Partial differential equations also occupy a large sector of pure mathematical research, in which the usual questions are, broadly speaking, on the identification of general qualitative features of solutions of various partial differential equations. Partial differential equations are ubiquitous in mathematically-oriented scientific fields, such as physics and engineering. For example, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrödinger equation, Pauli equation, etc.). They also arise from many purely mathematical considerations, such as differential geometry and the calculus of variations; among other notable applications.

Let $z = f(x, y)$ be a function of two independent variables x and y. Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives; $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.

Any equation which contains one or more partial derivatives is called a **partial differential equation.** $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$; $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for partial differential equation (PDE) of first order and second order respectively.

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We use the following notations for partial derivatives,

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Partial differential equation may be formed by eliminating *(i)* arbitrary constants *(ii)* arbitrary functions.

Example 1.1: Form the parital differential equation by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution: Given, $z = ax + by + a^2 + b^2$ (1)

Here we have two arbitrary constants a and b. Therefore, we need two more equations to eliminate a and b. Differentiating equation (1) partially with respect to x and y respectively we get,

$$\frac{\partial z}{\partial x} = p = a \quad (2)$$

$$\frac{\partial z}{\partial y} = q = b \quad (3)$$

From equations (2) and (3), we get,

$$a = p, b = q$$

Substituting values of a and b in (1) we get,

$$z = px + qy + p^2 + q^2$$

This is the required partial differential equation.

Example 1.2: Eliminate a and b from $z = (x + a)(y + b)$.

Solution: Differentiating partially with respect to x and y,

$$p = y + b, q = x + a$$

Eliminating a and b, we get $z = pq$.

Example 1.3: Form the partial differential equation by eliminating the arbitrary constants in $z = (x - a)^2 + (y - b)^2$.

Solution: Given, $z = (x - a)^2 + (y - b)^2$ (1)

Here we have two arbitrary cosntants a and b. To eliminate these two arbitrary constants we need two more equations connecting a and b. Therefore, differentiating equation (1) partially with respect to x and y, we get,

$$\frac{\partial z}{\partial x} = p = 2(x - a) \quad (2)$$

$$\frac{\partial z}{\partial y} = q = 2(y - b) \tag{3}$$

From equation (2), we get,

$$x - a = \frac{p}{2} \tag{4}$$

From equation (3), we get,

$$y - b = \frac{q}{2} \tag{5}$$

Substituting equations (4) and (5) in Equation (1) we get,

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

Simplifying we get, $4z = p^2 + q^2$

This gives the partial differential equation after elimination of a and b.

Example 1.4: Form the partial differential equation by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Solution: Given, $z = (x^2 + a)(y^2 + b)$ (1)

Here, we have two arbitrary constants a and b.

Differentiating equation (1) partially with respect to x and y we get,

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \tag{2}$$

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \tag{3}$$

From equation (2) we get, $\frac{p}{2x} = y^2 + b$ (4)

From equation (3) we get, $\frac{q}{2y} = x^2 + a$ (5)

Substituting equations (4) and (5) in Equation (1), we get,

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$pq = 4xyz$$

This gives the required partial differential equation.

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Example 1.5: Form the partial differential equation by eliminating *a, b, c* from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

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Solution: Given, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (1)

Differential partially with respect to *x* and *y* we get,

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot p=0 \quad (2)$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}\cdot q=0 \quad (3)$$

Differentiating equation (2) partially with respect to *y*,

$$0+\frac{2}{c^2}(zs+qp)=0$$

$$zs+qp=0$$

Note: More than one partial differential equation is possible in this problem. These partial differential equations are,

$xzr+xp^2-zp=0,\ yzt+yq^2-zq=0$

Formation of Partial Differential Equation by Eliminating Arbitrary Functions

The partial differential equations can be formed by eliminating arbitrary junctions. The following examples will make the concept clear.

Example 1.6: Eliminate arbitrary function from,

$$z=f(x^2+y^2) \quad (1)$$

Solution: Differentiating partially with respect to *x* and *y*, we get,

$$p=f'(x^2+y^2).2x \quad (2)$$

$$q=f'(x^2+y^2).2y \quad (3)$$

Eliminating $f'(x^2+y^2)$ from equation (2) and (3), we get, $py=qx$

Example 1.7: Form the partial differential equation by eliminating the arbitrary function ϕ from $xyz=\phi(x^2+y^2-z^2)$.

Solution: Given, $xyz=\phi(x^2+y^2-z^2)$ (1)

This equation contains only one arbitrary function ϕ and we have to eliminate it.

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Differentiating equation (1) partially with respect to x and y we get,

$$yz + xyp = \phi'(x^2 + y^2 - z^2)(2x - 2zp) \quad (2)$$

$$xz + xyq = \phi'(x^2 + y^2 - z^2)(2y - 2zq) \quad (3)$$

From equation (2), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{yz + xyp}{2x - 2zp} \quad (4)$$

From equation (3), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{xz + xyq}{2y - 2zq} \quad (5)$$

Since, LHS of equations (4) and (5) are equal, we have,

$$\frac{yz + xyp}{2x - 2zp} = \frac{xz + xyq}{2y - 2zq}$$

$$(yz + xyp)(y - zq) = (xz + xyq)(x - zp)$$

i.e., $$y(z + xp)(y - zq) = x(z + yq)(x - zp) \quad (6)$$

On simplifying equation (6) we get,

$$px(y^2 + z^2) - qy(z^2 + x^2) = z(x^2 - y^2)$$

Which gives the required partial differential equation.

Example 1.8: Eliminate the arbitrary function from $z = (x + y) f(x^2 - y^2)$

Solution: Given, $z = (x + y) f(x^2 - y^2)$ (1)

Differentiating partially with respect to x and y we get,

$$p = (x + y)f'(x^2 - y^2)2x + f(x^2 - y^2) \bullet 1 \quad (2)$$

$$q = (x + y)f'(x^2 - y^2)(-2y) + f(x^2 - y^2) \bullet 1 \quad (3)$$

Eliminating $f'(x^2 - y^2)$ from equations (2) and (3) we get,

$$\frac{2x(x + y)}{-2y(x + y)} = \frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)}$$

$$2x[q - f(x^2 - y^2)] = -2y[p - f(x^2 - y^2)]$$

$$xq - xf(x^2 - y^2) = -yp + yf(x^2 - y^2)$$

$$xq + yp = (x + y) f(x^2 - y^2)$$

$$= (x + y)\frac{z}{(x + y)}$$

$$\therefore \quad z = xq + yp$$

This is a required equation.

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Example 1.9: Eliminate the arbitrary function from $z = xy + f(x^2 + y^2)$

Solution: Given, $z = xy + f(x^2 + y^2)$ (1)

Differentiating partially equation (1) with respect to x and y we get,

$$p = y + f'(x^2 + y^2) \bullet 2x \quad (2)$$

$$q = x + f'(x^2 + y^2) \bullet 2y \quad (3)$$

Eliminating $f'(x^2 + y^2)$ from equations (2) and (3) we get,

$$(p - y)y = (q - x)x$$

$$py - y^2 = qx - x^2$$

$$py - qx = y^2 - x^2$$

Which is a required equation.

Example 1.10: Eliminate the arbitrary functions f and ϕ from the relation $z = f(x + ay) + \phi(x - ay)$

Solution: Differentiating partially with respect to x and y we get,

$$p = f'(x + ay) + \phi'(x - ay) \quad (1)$$

$$q = af'(x + ay) - a\phi'(x - ay) \quad (2)$$

Differentiating these again, with respect to x and y we get,

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x + ay) + \phi''(x - ay) \quad (3)$$

$$\frac{\partial^2 z}{\partial y^2} = t = a^2 f''(x + ay) + a^2\phi''(x - ay) \quad (4)$$

From equations (3) and (4) we get,

$$t = a^2 r$$

Equations Solvable by Direct Integration

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $\varphi(x, y, z, a, b) = 0$, is the complete integral of a partial differential equation, then the eliminant of a and b from the equations $\frac{\partial \phi}{\partial a} = 0$, $\frac{\partial \phi}{\partial b} = 0$, is called singular integral.

Let us consider four standard types of non-linear partial differential equations and the procedure for obtaining their complete solution.

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Type I Equations of the form $F(p, q) = 0$. In this type of equations we have only p and q and there is no x, y and z. To solve this type of problems, let us assume that $z = ax + by + c$ be the solution and then proceed as in the following examples.

Example 1.11: .Solve $p^2 + q^2 = 4$

Solution: Given, $p^2 + q^2 = 4$ (1)

Let us assume that $z = ax + by + c$ be a solution of equation (1). (2)

Partially differentiating equation (1) with respect to x and y, we get,

$$\frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b \quad (3)$$

Substituting equation (3) in (1) we get,

$$a^2 + b^2 = 4 \quad (4)$$

To get the complete integral we have to eliminate any one of the arbitrary constants from equation (2).

From equation (4) we get,

$$b = \pm\sqrt{4 - a^2} \quad (5)$$

Substituting equation (5) in (2) we get,

$$z = ax \pm y\sqrt{4 - a^2} + C \quad (6)$$

Which contains only two constants (equal to number of independent variables). Therefore, it gives the complete integral.

To check for Singular Integral:

Differentiating equation (6) partially with respect to a and c and equating to zero, we get,

$$\frac{\partial z}{\partial a} = x \pm \frac{1}{2\sqrt{4 - a^2}}(-2a) = 0 \quad (7)$$

and, $$\frac{\partial z}{\partial c} = 1 = 0$$

Here, $1 = 0$ is not possible.

Hence, there is no singular integral.

Example 1.12: Solve $p^2 + q^2 = npq$

Solution. The solution is, $z = ax + by + c$, where $a^2 + b^2 = nab$

Solving, $b = \dfrac{a\left(n \pm \sqrt{n^2 - 4}\right)}{2}$

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The complete integral is,

$$z = ax + \frac{ay}{2}\left(n \pm \sqrt{n^2 - 4}\right) + c$$

Differentiating partially with respect to c, we see that there is no singular integral, as we get an absurd result.

Example 1.13: Solve $p + q = pq$

Solution: This equation is of the type, $F(p, q) = 0$.

$\therefore$ The complete solution is of the form, $z = ax + by + c$ (1)

Differentiating equation (1) partially with respect to x and y we get,

$$p = a,\ q = b$$

Therefore, the given equation becomes,

$$a + b = ab$$

$$a = b(a - 1); \qquad b = \frac{a}{a-1}$$

Therefore, the complete solution is,

$$z = ax + \left(\frac{a}{a-1}\right)y + c$$

This type of equation has no singular solution.

Let, $c = \varphi(a)$

$$z = ax + \left(\frac{a}{a-1}\right)y + \phi(a) \tag{2}$$

Differentiating partially with respect to a,

$$0 = x + \left[\frac{(a-1)1 - a}{(a-1)^2}\right]y + \phi'(a)$$

$$0 = -\frac{1}{(a-1)^2}y + \phi'(a) \tag{3}$$

The elimination of a between equations (2) and (3) gives the general solution.

Type II Equation of the form $z = px + qy + F(p, q)$ (Clairaut's form). In this type of problems assume that, $z = ax + by + F(a, b)$ be the solution.

Example 1.14: Solve $z = px + qy + ab$

Solution: This equation is of Clairaut's type. Therefore, the complete solution is obtained by replacing p by a and q by b, where a and b are arbitrary constants.

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i.e., the complete solution is, $z = ax + by + ab$ (1)

Differentiating equation (1) partially with respect to a and b, and equating these to zero we get,

$$0 = x + b \quad (2)$$

$$0 = y + a \quad (3)$$

Eliminating a and b from equations (1), (2) and (3) we get,

$$z = -xy - xy + xy$$

i.e., $z + xy = 0$

This gives the singular solution of the given partial differential equation and to get the general solution.

Put, $b = \phi(a)$ in equation (1)

$\therefore$ $z = ax + \phi(a)y + a\phi(a)$ (4)

Differentiating partially with respect to a we get,

$$0 = x + \phi'(a)y + a\phi'(a) + \phi(a) \quad (5)$$

Eliminating a from equations (4) and (5) we get the general solution.

Example 1.15: Obtain the complete solution and singular solution of,

$$z = px + qy + p^2 + pq + q^2$$

Solution: This equation is of Clairaut's form. Therefore, the complete solution is,

$$z = ax + by + a^2 + ab + b^2 \quad (1)$$

Where, a and b are arbitrary constants.

Differentiating equation (1) partially with respect to a and b we get,

$$0 = x + 2a + b \quad (2)$$

$$0 = y + 2b + a \quad (3)$$

$2x - y = 3a$, and $2y - x = 3b$

$$a = \frac{2x - y}{3}, \; b = \frac{2y - x}{3}$$

Substituting this in equation (1) we get,

$$z = \left(\frac{2x - y}{3}\right)x + \left(\frac{2y - x}{3}\right)y + \left(\frac{2x - y}{3}\right)^2$$

$$+\frac{(2x-y)(2y-x)}{9}+\left(\frac{2y-x}{3}\right)^2$$

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Simplifying we get, $3z = xy - x^2 - y^2$. This is the singular solution.

To find singular integral:

Differentiating equation (2) partially with respect to a and b, and then equating to zero, we get,

$$\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}} = 0 \tag{3}$$

$$\frac{\partial z}{\partial b} = y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \tag{4}$$

From equation (3), we get,

$$x^2 = \frac{a^2}{1+a^2+b^2} \tag{5}$$

From equation (4), we get,

$$y^2 = \frac{b^2}{1+a^2+b^2} \tag{6}$$

From equations (5) and (6) we get,

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$\therefore \quad 1-(x^2+y^2) = 1-\frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1}{1+a^2+b^2}$$

i.e., $$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$$

$$\therefore \quad \sqrt{1+a^2+b^2} = \frac{1}{1-x^2-y^2} \tag{7}$$

Substituting equation (7) in (3) and (4) we get,

$$a = \frac{-x}{\sqrt{1-x^2-y^2}},\ b = \frac{-y}{\sqrt{1-x^2-y^2}} \tag{8}$$

Substituting equations (7) and (8) in (2) we get,

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$\therefore \quad z = \sqrt{1-x^2-y^2}$ or, $z^2 = 1 - x^2 - y^2$

$\therefore \quad x^2 + y^2 + z^2 = 1$

This is the singular integral.

Type III Equation of the form, $F(z, p, q) = 0$

Example 1.16: Solve $z = p^2 + q^2$

Solution: Given, $z = p^2 + q^2$ (1)

Assume that, $z = f(x + ay)$ is a solution of equation (1). (2)

Put, $x + ay = u$ in equation (2)

Then, $z = f(u)$ (3)

Partially differentiating equation (3) with respect to x and y we get,

$$p = \frac{dz}{du}, \; q = a\frac{dz}{du} \tag{4}$$

$$\left(\because \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} \right)$$

Substituting equation (4) in (1) we get,

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

i.e., $$\left(\frac{dz}{du}\right)^2 (1 + a^2) = z$$

i.e., $$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

i.e., $$\frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}} \tag{5}$$

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Integrating equation (5) we get,

$$\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} \int du$$

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

i.e., $$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + b$$

This gives the complete integral.

Example 1.17: Solve $ap + bq + cz = 0$

Solution: Given, $ap + bq + cz = 0$ (1)

Let us assume that, $z = f(x + ky)$ (2)

By the solution of equation (2).

Put $x + ky = u$ in equation (2)

$\therefore$ $$z = f(u) \quad (3)$$

$$p = \frac{dz}{du};\ q = k\frac{dz}{du} \quad (4)$$

Substituting equation (4) in (5) we get,

$$a \cdot \frac{dz}{du} + b \cdot k\frac{dz}{du} + c \cdot z = 0$$

i.e., $$\frac{dz}{du}(a + bk) = -cz$$

$\therefore$ $$\frac{dz}{du} = -\frac{cz}{a+bk}$$

i.e., $$\frac{dz}{z} = -\frac{c}{a+bk}du \quad (5)$$

Integrating equation (5) we get,

$$\int \frac{dz}{z} = -\frac{c}{a+bk}\int du$$

$$\log z = -\frac{c}{a+bk}(u) + \log b$$

i.e., $$\log z = A[x + ky] + \log b, \quad \text{where } A = -\frac{c}{a+bk}$$

i.e., $\log z - \log b = A(x + ky)$

$$\log\left(\frac{z}{b}\right) = A(x + ky)$$

$$\frac{z}{b} = e^{A(x+ky)}$$

$\therefore$ $$z = be^{A(x+ky)}$$

This gives the complete integral.

Type IV Equation of the form, $F_1(x, p) = F_2(y, q)$

Example 1.18: Solve the equation, $p + q = x + y$

Solution: We can write the equation in the form, $p - x = y - q$

Let, $p - x = a$, then $y - q = a$

Hence, $p = x + a$, $q = y - a$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

$$= (x + a)dx + (y - a)dy$$

On Integrating,

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b$$

There is no singular integral and the general integral is found as usual.

Example 1.19: Solve $p^2 + q^2 = x + y$

Solution: Given, $p^2 + q^2 = x + y$

$$p^2 - x = y - q^2 = k$$

$\therefore$ $$p^2 - x = k;\ y - q^2 = k$$

$$p = \pm\sqrt{x + k}\ ,\ q = \pm\sqrt{y - k}$$

$$dz = pdx + qdy$$

$$= \pm(\sqrt{x + k})dx \pm(\sqrt{y - k})dy$$

Integrating we get the complete solution.

$$z = \pm\frac{2}{3}(x + k)^{3/2} \pm \frac{2}{3}(y - k)^{3/2} + C$$

$$= \pm\frac{2}{3}\left[(x + k)^{3/2} + (y - k)^{3/2}\right] + C$$

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Example 1.20: Solve $p + q = \sin x + \sin y$

Solution:

$$p - \sin x = \sin y - q = k$$

$$\therefore \quad p = k + \sin x;\ q = \sin y - k$$

$$dz = pdx + qdy$$

$$= (k + \sin x)dx + (\sin y - k)dy$$

On integrating, we get,

$$z = (kx - \cos x) - (ky + \cos y) + C$$

$$z = k(x - y) - (\cos x + \cos y) + C$$

This is the complete solution.

1.2.1 Classification of Partial Differential Equation

PDE has definite functionalities, which is help to determine whether a particular finite element approach is appropriate to the problem being described by the PDE. The solution depends on the equation and several variables contain partial derivatives with respect to the variables. There are three-types are following:

- Elliptic PDE
- Parabolic PDE
- Hyperbolic PDE

We usually come across three-types of second-order PDEs in mechanics. These are classified as elliptic, hyperbolic, and parabolic. The equations of elasticity (without inertial terms) are elliptic PDEs. Hyperbolic PDEs describe wave propagation phenomena. The heat conduction equation is an example of a parabolic PDE.

Each type of PDE has certain characteristics that help determine if a particular finite element approach is appropriate to the problem being described by the PDE. Interestingly, just knowing the type of PDE can give us insight into how smooth the solution is, how fast information propagates, and the effect of initial and boundary conditions.

- In *hyperbolic* PDEs, the smoothness of the solution depends on the smoothness of the initial and boundary conditions. For example, if there is a jump in the data at the start or at the boundaries, then the jump will propagate as a shock in the solution. If, in addition, the PDE is non-linear, then shocks may develop even though the initial conditions and the boundary conditions are smooth. In a system modeled with a hyperbolic PDE, information travels at a finite speed referred to as the wave speed. Information is not transmitted until the wave arrives.

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- In contrast, the solutions of *elliptic* PDEs are always smooth, even if the initial and boundary conditions are rough (though there may be singularities at sharp corners). In addition, boundary data at any point affect the solution at all points in the domain.
- *Parabolic* PDEs are usually time dependent and represent diffusion-like processes. Solutions are smooth in space but may possess singularities. Conversely, information travels at infinite speed in a parabolic system.

Suppose we have a second-order PDE of the form

$$a(x_1,x_2)\frac{\partial^2 u}{\partial x_1^2}+b(x_1,x_2)\frac{\partial^2 u}{\partial x_1 \partial x_2}+c(x_1,x_2)\frac{\partial^2 u}{\partial x_2^2}+d(x_1,x_2)\frac{\partial u}{\partial x_1}+e(x_1,x_2)\frac{\partial u}{\partial x_2}+f(x_1,x_2)u=g(x_1,x_2)$$

Then, the PDE is called **elliptic** if,

$$b^2-4ac<0$$

For example,

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_1 \partial x_2}+\frac{\partial^2 u}{\partial x_2^2}=x_1\frac{\partial u}{\partial x_1}$$

The PDE is called **hyperbolic** if,

$$b^2-4ac>0$$

For example,

$$\frac{\partial^2 u}{\partial x_1^2}+3\frac{\partial^2 u}{\partial x_1 \partial x_2}+\frac{\partial^2 u}{\partial x_2^2}=x_1\frac{\partial u}{\partial x_1}$$

The PDE is called **parabolic** if,

$$b^2-4ac=0$$

For example,

$$\frac{\partial^2 u}{\partial x_1^2}+2\frac{\partial^2 u}{\partial x_1 \partial x_2}+\frac{\partial^2 u}{\partial x_2^2}=x_1\frac{\partial u}{\partial x_1}$$

If this is true at all points in a domain Ω, then the equation is said to be elliptic, parabolic, or hyperbolic in that domain.

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Second Order PDEs

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g.$$

Classification	Type	Canonical form	Characteristics
$b^2 - ac > 0$	Hyperbolic	$\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots = 0$	$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$
$b^2 - ac = 0$	Parabolic	$\frac{\partial^2 u}{\partial \eta^2} + \ldots = 0$	$\frac{dy}{dx} = \frac{b}{a}, \eta = x$ (say)
$b^2 - ac < 0$	Elliptic	$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + \ldots = 0$	$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}, \begin{cases} \alpha = \xi + \eta \\ \beta = i(\xi - \eta) \end{cases}$

Classification of Second Order Linear PDEs

1. **Elliptic:** The eigenvalues are all positive or all negative.
2. **Parabolic:** The eigenvalues are all positive or all negative, negative, save one, which is zero.
3. **Hyperbolic:** There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

Canonical Forms

Transformation of independent variables x and y of Equation (1.1) to new variables ξ, η,

Where $\xi = \xi(x, y)$, $\eta = \eta(x, y)$

- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)$.
- **Parabolic:** $u_{\xi\xi} = \varphi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\eta\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)$
- **Hyperbolic:** $u_{\xi\xi} - u_{\eta\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)$

Examples

(i) $u_t = k(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional heat equation]

(ii) $u_{xx} + u_{yy} + u_{zz} = 0$ [Laplace equation in three dimensions]

(iii) $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional wave equation]

(iv) $u_t + uu_x = \mu u_{xx}$ [non-linear one-dimensional Burger equation]

The order of a partial differential equation is the order of the highest derivative occurring in the equation. All the above examples are second order partial differential equations. $u_t = uu_{xx} + \sin x$ is an example for **third order partial differential equation**.

Consider the example, $au_{xx} + bu_{yy} + cu_{yy} = 0$, $u = u(x,y)$. For a given point (x,y), the equation is said to be 'Elliptic' if $b^2 - ac < 0$ which are used to describe the equations of elasticity without inertial terms. 'Hyperbolic' PDEs describe the

phenomena of wave propagation if it satisfies the condition b^2- ac >0. For 'Parabolic' PDEs, it should satisfy the condition b^2 - ac=0. The heat conduction equation is an example of a parabolic PDE.

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1.3 TRANSPORT EQUATION

The 'Transport Equation' is a partial differential equation of the form is,

$$u_t + cu_x = 0 \quad (1.1)$$

Where, u is a function of two variables (x,t) and the subscripts denote partial derivatives. We will assume that c is a fixed constant. Given an initial condition

$$u(x,0) = f(x) \quad (1.2)$$

We would like to find a function of two variables that satisfies both the transport equation (1.2) and the initial condition for an equation (1.2).

In this section we will see that we can quite easily generalise the solution method and the uniqueness proof we used there to multiple dimensions. Let $d \in \mathbb{N}$. The non- homogenous d-dimensional transport equation is,

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^d : \partial_t u(t,x) - \mathbf{v} \cdot \nabla_x u(t,x) = f(t,x)$$

Whereas, f: $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a function and $v \in \mathbb{R}$ is a vector.

Definition: Let f: $\mathbb{R}^d \to \mathbb{R}$ be a function and $n \in \mathbb{N}$. We say that f is n times continuously differentiable iff all the partial derivatives,

$$\partial_\alpha f, \alpha \in \mathbb{N}_0^d \text{ and } |\alpha| \leq n$$

exist and continuous. We can say that $f \in C^n(\mathbb{R}^d)$.

Theorem 1.1 (Leibniz' Integral Rule) : Let $O \subseteq \mathbb{R}$ be open and $B \subseteq \mathbb{R}^d$ where $d \in \mathbb{N}$ is arbitrary, and let $f \in C^1(O \times B)$ If the conditions

- For all $x \in O$, $\int_B |f(x,y)|dy < \infty$
- For all $x \in O$ and $y \in B$, $\frac{d}{dx} f(x,y)$ exists
- There is a function g: $B \to \mathbb{R}$ such that

$$\forall (x,y) \in O \times B : |\partial_x f(x,y)| \leq |g(y)| \text{ and } \int_B |g(y)|dy < \infty$$

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Then,

$$\frac{d}{dx}\int_B f(x,y)dy = \int_B \frac{d}{dx} f(x,y)$$

Theorem 1.2: If $f \in C^1(\mathbb{R} \times \mathbb{R}^d)$, $g \in C^1(\mathbb{R}^d)$ and $\mathbf{v} \in \mathbb{R}^d$, then the function

$$u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, u(t,x) := g(x + \mathbf{v}t) + \int_0^t f(s, x + \mathbf{v}(t-s))ds$$

Solves the non- homogenous *d*-dimensional transport equation

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^d : \partial_t u(t,x) - \mathbf{v} \cdot \nabla_x u(t,x) = f(t,x)$$

Proof: We represent that *u* is appropriately often differentiable. From the 'Chain Rule' follows that $g(x + vt)$ is continuously differentiable in all the directions $t, x_1 \ldots\ldots x_d$ The existence of

$$\partial_{x_n} \int_0^t f(s, x + \mathbf{v}(t-s))ds, n \in \{1, \ldots, d\}$$

$$\partial_t \int_0^t f(s, x + \mathbf{v}(t-s))ds$$ (By Leibniz Integral Rule)

We will well ahead in this proof show to be equal to

$$f(t,x) + \mathbf{v} \cdot \nabla_x \int_0^t f(s, x + \mathbf{v}(t-s))ds.$$

Which is occurs because of

$$\nabla_x \int_0^t f(s, x + \mathbf{v}(t-s))ds$$

Just consists of the derivatives

$$\partial_{x_n} \int_0^t f(s, x + \mathbf{v}(t-s))ds, n \in \{1, \ldots, d\}$$

1.3.1 Initial Value Problem

In the field of differential equations, an initial value problem is an ordinary differential equation together with specified value, called the initial condition, of the unknown function at a given point in the domain of the solution.

An initial value problem is a differential equation,

$$y'(t) = f(t, y(t)) \text{ with } f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

where Ω is an open set, together with a point in the domain of f $(t_0, y_0) \in \Omega$,

called the initial condition.

A solution to an initial value problem is a function y that is a solution to the differential equation and satisfies,

$$(t_0, y_0) \in \Omega,$$

This statement subsumes problems of higher order, by interpreting *y* as a vector. For derivatives of second or higher order, new variables (elements of the vector *y*) are introduced.

More generally, the unknown function *y* can take values on infinite dimensional spaces, such as Banach spaces or spaces of distributions.

Initial Value Problem of Transport Equation

Let us consider the initial-value problem

$$\left\{\begin{array}{l} u_t + b.Du = 0 \ \ in \ \ R^n \times (0,\infty) \\ \qquad u = g \ \ on \ \ R^n \times \{t = 0\} \end{array}\right\} \tag{1.3}$$

Consequently $b \in R^n$ and g: $R^n \to R$ are known, and the problem is how to calculate *u.*

Given (x,t) as above, the line through (x,t) with direction $(b,1)$ is represented parametrically by, $(x + sb, t+s)$ $(s \in R)$. This line hits the plane $\Gamma := R \times \{t = 0\} n$ when

$s = -t$, at the point $(x - tb,0)$. Since *u* is constant on the line and $u(x\text{-}tb,0) = g(x\text{-}tb)$.we deduce

$$u(x,t) = g(x\text{-}tb) \ (x \in R \ , t \ \geq 0). \tag{1.4}$$

So, that if Equation (1.3) has a appropriately regular solution *u*, it must certainly be given by Equation (1.4). And conversely, it is easy to check directly that if *g* is C^1 then *u* defined by Equation (1.4) is indeed a solution of Equation (1.3).

Theorem 1.3: If $\boldsymbol{f \in C^1}$ $(\mathbb{R} \times \mathbb{R}^d)$, $\boldsymbol{g \in C^1}$ $(\mathbb{R}^d)$, then the function

$$u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, u(t,x) := g(x + \mathbf{v}t) + \int_0^t f(s, x + \mathbf{v}(t-s))ds$$

is the unique solution of **the initial value problem** of the **transport equation**

$$\left\{\begin{array}{ll} \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d : & \partial_t u(t,x) - \mathbf{v} \cdot \nabla_x u(t,x) = f(t,x) \\ \forall x \in \mathbb{R}^d : & u(0,x) = g(x) \end{array}\right.$$

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Proof: We easier to show that,

$$u(0,x) = g(x + \mathbf{v} \cdot 0) + \int_0^0 f(s, x + \mathbf{v}(t-s))ds = g(x)$$

Therefore Theorem1.2, *u* is a solution to the initial value problem of the transport equation. So we proceed to show uniqueness.

Suppose that the *v* is an arbitrary other solution. We show that *v* = *u*, thereby excluding the possibility of a different solution.

We define as *w*: = *u*-*v*. Then

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^d: \quad \partial_t w(t,x) - \mathbf{v} \cdot \nabla_x w(t,x) \quad = (\partial_t u(t,x) - \mathbf{v} \cdot \nabla_x u(t,x)) - (\partial_t v(t,x) - \mathbf{v} \cdot \nabla_x v(t,x))$$

$$= f(t,x) - f(t,x) = 0 \qquad (*)$$

$$\forall x \in \mathbb{R}^d: \qquad w(0,x) = u(0,x) - v(0,x) \quad = g(x) - g(x) = 0$$

Analogous to the proof of uniqueness of solutions for the one-dimensional homogenous initial value problem of the transport equation. Now we define for arbitrary

$$(t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

$$\mu_{(t,x)}(\xi) := w(t - \xi, x + \mathbf{v}\xi)$$

Using the multi-dimensional chain rule, we calculate,

$$\begin{aligned} u'_{(t,x)}(\xi) &:= \frac{d}{d\xi} w(t-\xi, x+\mathbf{v}\xi) \\ &= (\partial_t w(t-\xi, x+\mathbf{v}\xi) \quad \partial_{x_1} w(t-\xi, x+\mathbf{v}\xi) \quad \cdots \quad \partial_{x_d} w(t-\xi, x+\mathbf{v}\xi)) \begin{pmatrix} -1 \\ \mathbf{v} \end{pmatrix} \\ &= -\partial_t w(t-\xi, x+\mathbf{v}\xi) + \mathbf{v} \cdot \nabla_x w(t-\xi, x+\mathbf{v}\xi) \\ &= 0 \end{aligned}$$

Therefore, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ $\mu_{(t,x)} \in (\xi)$ is constant, and thus

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^d: w(t,x) = \mu_{(t,x)}(0) = \mu_{(t,x)}(t) = w(0, x + \mathbf{v}t) \overset{(**)}{=} 0$$

Which shows that *w* = *u*-*v* = 0 and thus *u* = *v*.

Existence and Uniqueness of Solutions

The **Picard-Lindelöf theorem** guarantees a unique solution on some interval containing t_0 if *f* is continuous on a region containing t_0 and y_0 and satisfies the Lipschitz condition on the variable y. The proof of this theorem proceeds by reformulating the problem as an equivalent integral equation. The integral can be considered an operator which maps one function into another, such that the solution

is a fixed point of the operator. The Banach fixed point theorem is then invoked to show that there exists a unique fixed point, which is the solution of the initial value problem.

An older proof of the Picard–Lindelöf theorem constructs a sequence of functions which converge to the solution of the integral equation, and thus, the solution of the initial value problem. Such a construction is sometimes called 'Picard's method' or 'the method of successive approximations'. This version is essentially a special case of the Banach fixed point theorem. The Peano existence theorem however proves that even for f merely continuous, solutions are guaranteed to exist locally in time; the problem is that there is no guarantee of uniqueness.

Picard's Theorem on Existence and Uniqueness

Let $S = \{(x,y) : |x - x_0| \leq a,\ |y - y_0| \leq b\}$, and $a, b > 0$. Let $f : S \longrightarrow \mathbb{R}$ be such that f as well as $\frac{\partial f}{\partial y}$ are continuous on S. Also, let $M, K \in \mathbb{R}$ be constants such that

$$|f| \leq M,\ |\frac{\partial f}{\partial y}| \leq K \ \text{ on } \ S.$$

Let $h = \min\{a, b/M\}$. Then the sequence of successive approximations $\{y_n\}$ for the Initial Value Problem (IVP) uniformly converges on $|x - x_0| \leq h$ to a solution of IVP. Moreover the solution to IVP is unique.

The theorem asserts the existence of a unique solution on a subinterval $|x - x_0| \leq h$ of the given interval $|x - x_0| \leq a$. In a way it is in a neighbourhood of x_0 and so this result is also called the local existence of a unique solution. A natural question is whether the solution exists on the whole of the interval $|x - x_0| \leq a$.

1.3.2 Non-Homogeneous Equation

If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called **non-homogeneous partial differential equation** or **homogeneous** otherwise. The following equation is consider as non-homogeneous,

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2 \tag{1.5}$$

Non Homogeneous for transport equation is defined by:

$$\begin{Bmatrix} u_t + b.Du = 0 \ in \ \ R^n \times (0,\infty) \\ u = g \ \ on \ \ R^n \times \{t = 0\} \end{Bmatrix} \tag{1.6}$$

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Solution of non-homogeneous Equation is,

Let $(t,x) \in R^{n+1}$ and set $z(s)$: $u(x+sb, t+s)$ for $s \in R$. Then

$$z(s) = Du(x+sb,t+s).b + u_t(x+sb,t+s) = f(x+sb,t+s)$$

Consequently,

$$u(x,t) - g(x-tb) = z(0) - z(-t)$$

$$= \int_{-t}^{0} z(s)\,ds$$

$$= \int_{-t}^{0} f(x+sb,t+s)\,ds$$

$$= \int_{0}^{t} f(x+(s-t)b,s)\,ds$$

$$\text{So } u(x,t) = g(x-tb) + \int_{0}^{t} f(x+(s-t)b,s)\,ds \quad (x \in R^n,\ t \geq 0)$$

(1.7)

The general form of the equation of this type is,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad ...(1.8)$$

Where atleast one c_1 and c_2 is non-zero.

The cases of this type are considered below:

Case (1): When $b_1 = -a_2$

Then the equation becomes,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{-b_1x + b_2y + c_2}$$

Cross-multiplying we get,

$$(-b_1x + b_2y + c_2)dy = (a_1x + b_1y + c_1)dx$$

$$b_2ydy + c_2dy - a_1xdx - c_1dx = b_1(ydx + xdy)$$

i.e., $$(b_2y + c_2)dy - (a_1x + c_1)dx = b_1d(xy)$$

Integrating we get,

$$b_2\frac{y^2}{2} + c_2y - a_1\frac{x^2}{2} - c_1x = b_1xy + K$$

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Example 1.21: Solve $\frac{dy}{dx} = \frac{2x - y + 3}{x - 3y - 1}$

Solution: Cross-multiplying and rearranging the terms, we get,

$$(xdy + ydx) = (2x + 3)dx + (3y + 1)dy$$

Integrating we get,

$$\int d(xy) = \int (2x+3)dx + \int (3y+1)dy$$

$$xy = x^2 + 3x + \frac{3}{2}y^2 + y + K \text{ is the solution.}$$

Case (2): When $a_1 = b_2$ and $b_1 = a_2$

Then the equation becomes,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{b_1x + a_1y + c_2}$$

$$\frac{dy}{a_1x + b_1y + c_1} = \frac{dx}{b_1x + a_1y + c_2}$$

$$= \frac{d(x+y)}{(a_1 + b_1)(x+y) + c_1 + c_2}$$

$$= \frac{d(x-y)}{(b_1 - a_1)(x-y) + c_2 - c_1}$$

$$\int \frac{d(x+y)}{(a_1 + b_1)(x+y) + c_1 + c_2} = \int \frac{d(x-y)}{(b_1 - a_1)(x-y) + c_2 - c_1}$$

$$\frac{1}{(a_1 + b_1)} \log[(a_1 + b_1)(x+y) + (c_1 + c_2)]$$

$$= \frac{1}{(b_1 - a_1)} \log[(b_1 - a_1)(x-y) + (c_2 - c_1)] + K$$

Example 1.22: Solve $\frac{dy}{dx} = \frac{3x + 4y - 1}{4x + 3y + 2}$

Solution:

$$\frac{dy}{3x+4y-1} = \frac{dx}{4x+3y+2} = \frac{d(x+y)}{7(x+y)+1} = \frac{d(x-y)}{(x-y)+3}$$

$$\int \frac{d(x+y)}{7(x+y)+1} = \int \frac{d(x-y)}{(x-y)+3}$$

$$\frac{1}{7}\log[7(x+y)+1] = \log[(x-y)+3] + \log c$$

$$\frac{1}{7}\log[7(x+y)+1] = \log c[(x-y)+3]$$

$$\log[7(x+y)+1] = \log c[(x-y)+3]^7$$

$$7x + 7y + 1 = K(x - y + 3)^7$$

NOTES

Non-homogeneous Equations with Constant Coefficients

Consider the equation of the form,

$$f(D, D')z = F(x, y) \quad ...(1.9)$$

If $f(D, D')$ is not homogeneous then Equation (1.9) is called a non-homogeneous linear equation.

Compolementry Function (PI) can be found as in homogeneous linear equations.

To find Complementry Function (CF), consider $f(D, D')z = 0$ ...(1.10)

Assume, $z = Ce^{hx+ky}$ as a trial solution.

Substituting in Equation (3) we get, $f(h, k) = 0$

Find k in terms of h [or h in terms of k]

Let the r values of k be,

$$f_1(h), f_2(h),..., f_r(h)$$

Then, $z = C_n e^{hx+f_n(h)y}$, $n = 1, 2, ... , r$

will be the seperate solution of Equation (3).

The general solution of Equation (3) is of the form,

$$z = \sum C_1 e^{hx+f_1(h)y} + \sum C_2 e^{hx+f_2(h)y} + ... + \sum C_r e^{hx+f_r(h)y}$$

Example 1.23: Solve $(D^2 - DD' + D' - 1)z = e^x$

Solution:

To find CF, consider,

$(D^2 - DD' + D' - 1)z = 0$

Let, $z = Ce^{hx+ky}$ as a trial solution.

$$f(h, k) = 0$$

$$h^2 - hk + k - 1 = 0$$

$$h = \frac{k \pm \sqrt{k^2 - 4(k-1)}}{2}$$

$$= \frac{k \pm \sqrt{(k-2)^2}}{2}$$

$$= \frac{k \pm (k-2)}{2} = k - 1, 1$$

$\therefore$ $$CF = \sum C_1 e^{(k-1)x+ky} + \sum C_2 e^{x+ky}$$

$$= e^{-x} \sum C_1 e^{k(x+y)} + e^x \sum C_2 e^{ky}$$

$$= e^{-x} \phi_1(y + x) + e^x \phi_2(y)$$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1} e^x$$

Put, $D = 1, D' = 0, Dr = 0$

$$= \frac{1}{(D - D' + 1)(D - 1)} e^x$$

$$= \frac{1}{D - 1} e^x$$

$$= e^x \frac{1}{D + 1 - 1}(1)$$

$$= xe^x$$

$$\therefore \quad z = e^{-x}\phi_1(y + x) + e^x\phi_2(y) + xe^x$$

Example 1.24: Solve $(D^2 + DD' + D' - 1)z = \cos(x - y)$

Solution:

To find CF, consider

$$(D^2 + DD' + D' - 1)z = 0$$

Assume, $z = Ce^{hx + ky}$ as a trial solution.

$f(h, k) = 0$ becomes, $h^2 + hk + k - 1 = 0$

$$h = \frac{-k \pm \sqrt{k^2 - 4(k - 1)}}{2}$$

$$= \frac{-k \pm \sqrt{(k - 2)^2}}{2}$$

$$= \frac{-k \pm (k - 2)}{2}$$

$$= -1, -k + 1$$

$$\text{CF} = \sum C_1 e^{-x + ky} + \sum C_2 e^{(-k+1)x + ky}$$

$$= e^{-x} \sum C_1 e^{ky} + e^x \sum C_2 e^{k(y - x)}$$

$$= e^{-x}\phi_1(y) + e^x\phi_2(y - x)$$

$$\text{PI} = \frac{1}{D^2 + DD' + D' - 1} \cos(x - y)$$

Put, $D^2 = -a^2 = -1$

$DD' = -ab = 1$

$D'^2 = -b^2 = -1$

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NOTES

$$= \frac{1}{-1+1+D'-1}\cos(x-y)$$

$$= \frac{1}{D'-1}\cos(x-y)$$

$$= \frac{(D'+1)}{D'^2-1}\cos(x-y)$$

$$= -\frac{1}{2}(\sin(x-y)+\cos(x-y))$$

$$\therefore\ z = e^{-x}f_1(y) + e^{x}f_2(y-x) - \frac{1}{2}[\sin(x-y)+\cos(x-y)]$$

Example 1.25: Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{x-y} + x^2y$

Solution: To find CF, consider

$(D^2 + 2DD' + D'^2 - 2D - 2D')z = 0$

$f(h, k) = 0$ becomes,

$h^2 + 2hk + k^2 - 2h - 2k = 0$

$h^2 + 2h(k-1) + k^2 - 2k = 0$

$$h = \frac{-2(k-1) \pm \sqrt{4(k-1)^2 - 4(k^2-2k)}}{2}$$

$$= \frac{-2(k-1) \pm \sqrt{4}}{2}$$

$$= 1 - k \pm 1$$

$$= 2 - k, -k$$

$$\text{CF} = \sum C_1 e^{(2-k)x+ky} + \sum C_2 e^{-kx+ky}$$

$$= e^{2x}\sum C_1 e^{k(y-2x)} + \sum C_2 e^{k(y-x)}$$

$$= e^{2x}\ \varphi_1(y-2x) + \varphi_2(y-x)$$

$$\text{PI}_1 = \frac{1}{(D+D'-2)(D+D')}e^{x-y}$$

$$= \frac{1}{(1-1-2)(D+D')}e^{x-y}$$

$$= -\frac{1}{2}\int e^{x-(y+x)}dx$$

NOTES

$$= -\frac{1}{2}e^{-y}x$$

$$= -\frac{1}{2}xe^{x-y}$$

$$\text{PI}_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}x^2y$$

$$= \frac{1}{-2D\left[1 + \frac{D'}{D} - \frac{D'^2}{2D} - D' - \frac{D}{2}\right]}x^2y$$

$$= \frac{1}{-2D\left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D}\right)\right]}x^2y \qquad \left(\because D'^2(x^2y) = 0\right)$$

$$= \frac{1}{-2D}\left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D}\right)\right]^{-1}x^2y$$

$$= -\frac{1}{2D}\left[1 + \frac{D}{2} + D' - \frac{D'}{D} + \frac{D^2}{4} + DD' - D' - \frac{3}{4}DD' + \frac{3}{4}D^2D'\right]x^2y$$

$$= -\frac{1}{2D}\left[1 + \frac{D}{2} - \frac{D'}{D} + \frac{1}{4}DD' + \frac{D^2}{4} + \frac{3}{4}D^2D'\right]x^2y$$

$$= -\frac{1}{2}\left[\frac{1}{D} + \frac{1}{2} - \frac{D'}{D^2} + \frac{D'}{4} + \frac{D}{4} + \frac{3}{4}DD'\right]x^2y$$

$$= -\frac{1}{2}\left[\frac{x^3}{3}y + \frac{1}{2}x^2y - \frac{x^4}{12} + \frac{x^2}{4} + \frac{xy}{2} + \frac{3x}{2}\right]$$

$$\therefore\ z = e^{2x}\,\varphi_1(y - 2x) + \varphi_2(y - x) - \frac{x}{2}e^{x-y} - \frac{1}{2}\left[\frac{x^3y}{3} + \frac{x^2y}{2} - \frac{x^4}{12} + \frac{x^2}{4} + \frac{xy}{2} + \frac{3x}{2}\right]$$

Check Your Progress

1. What do you understand by PDE?
2. What are the types of PDE?
3. State the transport equation.
4. Define the term initial value problem.
5. How will define by non-homogeneous for transport equation?

NOTES

1.4 LAPLACE'S EQUATION

In mathematics, Laplace's equation is termed as a second order partial differential equation and is named after Pierre-Simon Laplace who initially examined its properties. The equation is written as follows:

$$\Delta\varphi = 0 \quad \text{or} \quad \nabla^2\varphi = 0$$

Here $\Delta = \nabla^2$ is termed as the Laplace operator and φ is considered as a scalar function. Generally, $\Delta = \nabla^2$ is referred as the Laplace-Beltrami or Laplace–de Rham operator. The Laplace's equation and Poisson's equation are the termed as the simple cases of elliptic partial differential equations. Solutions of Laplace's equation are known as harmonic functions whereas the general theory of solutions to Laplace's equation is known as potential theory. In the case of heat conduction, the Laplace equation is termed as the steady-state heat equation.

In three dimensions, the difficulty is to find twice differentiable real valued functions ***f*** of real variables x, y and z. We find the solution using the following coordinate systems:

Cartesian coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Cylindrical coordinates:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Spherical coordinates:

$$\Delta f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} = \theta.$$

Generally, it is written as,

$$\Delta^2\varphi = 0$$

For particular cases in more general contexts we can use the form,

$$\Delta\varphi = 0,$$

Where $\Delta = \nabla^2$ is the Laplace operator. Further,

$$\Delta\varphi = \nabla^2\varphi = \nabla \cdot \nabla\varphi = \text{div grad}\,\varphi,$$

Where $\nabla\cdot$ = div is the divergence and ∇ = grad is the gradient.

If the right hand side is specified as a given function $f(x, y, z)$ then equation is written as follows and is termed as Poisson's equation:

$$\Delta\varphi = f$$

Boundary Conditions

The Dirichlet condition for Laplace's equation consists of finding a solution φ on some domain D such that φ on the boundary of D is equal to some given function. Because the Laplace operator is also used in the heat equation hence we can interpret it by fixing the temperature on the boundary of the domain according to the given specification of the boundary condition. Permit the heat to flow till a stationary state reaches in which the temperature at each point on the domain does not change to any further extent. The temperature distribution in the interior can then be given by the solution to the corresponding Dirichlet problem. Solutions of Laplace's equation are harmonic functions as they are all analytic within the domain where the equation is satisfied.

Laplace Equation in Two Dimensions

The Laplace equation in two independent variables has the following form:

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} \equiv \psi_{xx} + \psi_{yy} = 0.$$

Analytic Functions: The real and imaginary parts of a complex analytic function both satisfy the Laplace equation, i.e., if $z = x + iy$ and if

$$f(z) = u(x, y) + iv(x, y),$$

Then the necessary condition that $f(z)$ be analytic is that the following Cauchy-Riemann equations must be satisfied:

$$u_x = v_y, \quad u_x = -u_y.$$

Here u_x is the first partial derivative of u with respect to x. It follows that,

$$u_{yy} = (-v_x)_y = -(v_y)_x = -(u_x)_x.$$

Consequently u satisfies the Laplace equation. Similarly we can show that v will also satisfy the Laplace equation. Conversely, given a harmonic function, it is the real part of an analytic function, $f(z)$.

1.4.1 Fundamental Solution of Laplace Equation

Let consider the solution of Laplace equation $\Delta u = 0$ in $U = R^n$, having the form $u(x) = v(r)$,

Where $r = |x| = \left(x_1^{\,2} + ...x_n^{\,2}\right)^{1/2}$ and v is to be chosen so that $\Delta u = 0$ holds.

For $i = 1,..... n$

NOTES

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^{\,2} + \ldots\ldots\ldots x_n^{\,2})^{-1/2} 2x_i = \frac{x_i}{r} \qquad (x \neq 0)$$

$$u_{x_i} = v'(r)\frac{x_i}{r}, \qquad u_{x_i x_i} = v''(r)\frac{x_i^{\,2}}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^{\,2}}{r^3}\right)$$

$$For \quad i = 1,\ldots\ldots\ldots\ldots n$$

$$\Delta u = v''(r) + \frac{n-1}{n} v'(r).$$

So that $\Delta u = 0$ iff only iff,

$$v''(r) + \frac{n-1}{r} v'(r) = 0.$$

If $v' \neq 0$, we deduce $\log(v')' = \frac{v''}{v'} = \frac{1-n}{r}$,

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if r >0,we have

$$v(r) = \begin{cases} b\log r + c & (n = 2) \\ \dfrac{b}{r^{n-2}} + c & (n \geq 3) \end{cases}$$

This is also known as 'Fundamental Solution' of 'Laplace Equation' where b and c are constants.

A 'Fundamental Solution' of Laplace's equation satisfies as following equation,

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = -\delta(x - x', y - y', z - z'),$$

Here the 'Dirac Delta Function' δ denotes a unit source concentrated at the points on x', y', z'. There is no such function has this property but it can be consider as limit function whose integrals over space are unity. Generally a different sign convention for this equation is taken. This choice of sign is often convenient to work with because $-\Delta$ is a positive operator. The definition of the fundamental solution thus implies that, if the Laplacian of u is integrated over any volume that encloses the source point, then encloses the source point then we have the following equation,

$$\iiint_V \nabla \cdot \nabla u \, dV = -1.$$

The Laplace equation remains unchanged under a rotation of coordinates and hence we can obtain a fundamental solution that only depends upon the distance r from the source point.

NOTES

$$-1 = \iiint_V \nabla \cdot \nabla u \, dV = \iint_S \frac{du}{dr} \, dS = 4\pi a^2 \frac{du}{dr}\bigg|_{r=a}.$$

It follows that,

$$\frac{du}{dr} = -\frac{1}{4\pi r^2},$$

On a sphere of radius r that is cantered on the source point, and hence

$$u = \frac{1}{4\pi r}.$$

Remember that, with the opposite sign convention (used in physics), this is the potential generated by a point particle, for an inverse-square law force, arising in the solution of Poisson equation. A similar argument shows that in two dimensions,

$$u = -\frac{\log(r)}{2\pi}.$$

Where log(r) denotes the natural logarithm. Note that, with the opposite sign convention, this is the potential generated by a point like sink, which is the solution of the **Euler equations** in two-dimensional incompressible flow.

1.4.2 Mean Value Formula of Laplace Equation

Assume that an open set $U \subset R^n$ and suppose u is harmonic function within U.

Now we derive the important mean-value formulas, which state that $u(x)$ equals both the average of u over the sphere $\partial B(x, r) \subset U$. These contained formulae involving u generate a remarkable number of significance.

Theorem 1.4: If $u \in C^2$ is a harmonic, then

$u(x) = \int_{\partial B(x,r)} u dS = \int_{B(x,r)} u dy$ for each ball $B(x, r) \subset U$.

Proof: Set $\phi(r) : \int_{\partial B(x,r)} u(y) dS(y) = \int_{\partial B(0,1)} u(x + rz) dS(z).$

Then, $\phi'(r) = \int_{\partial B(0,1)} Du(x + rz).z dS(z),$

Consequently, using Green's formulae, we calculate,

NOTES

$$\phi'(r) = \int_{\partial B(x,r)} Du(y).\frac{y-x}{r} dS(y)$$

$$= \int_{\partial B(x,r)} \frac{\partial u}{\partial v} dS(y)$$

$$= \frac{r}{n} \int_{B(x,r)} \Delta u(y) dy = 0$$

Here is ϕ is constant and so that

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \int_{\partial B(x,t)} u(y) dS(y) = u(x).$$

Theorem 1.5: (Opposite to Mean-Value Property)

If $u \in C^2$ which is, satisfies to the $u(x) = \int_{\partial B(x,r)} u dS$ for each ball $B(x, r) \subset U$.

Then u is harmonic.

Proof: If $\Delta u \neq 0$, there exists some ball $B(x, r) \subset U$. such that, say, $\Delta u < 0$ within $B(x, r)$. But then for ϕ

as above, $0 = \phi'(r) = \frac{r}{n} \int_{B(x,r)} \Delta u(y) dy > 0$, which is contradiction.

Theorem 1.6: Mean value formula for two dimensional equation

Let $u(P)$ be a harmonic function on Laplace equation.

$$B = B_a(P_0) = \{P \in R^3 : |P - P_0| \leq a\},$$

$$S = S_a(P_0) = \{P \in R^3 : |P - P_0| = a\}.$$

By,

$$\iiint_{\Omega} v \Delta u dV = \iiint_{\Omega} \nabla v . \nabla u \, dV = \iint_{\partial\Omega} v \frac{\partial u}{\partial n} dS_P$$

It is follow

$$0 = \iiint_{R} \Delta v \, dV = \iint \frac{\partial u}{\partial n} \, dS_P. \qquad (1.11)$$

For the sphere S the normal vector at $P \in S$ is

$$\vec{n} = \frac{P \quad P_0}{a} = \left(\frac{x \quad x_0}{a}, \frac{y \quad y_0}{a} \frac{z \quad z_0}{a} \right).$$

NOTES

When make the change of variables so that,

$$\begin{Bmatrix} x = x_0 + \rho\cos\theta\sin\varphi, \\ y = y_0 + \rho\sin\theta\sin\varphi, \\ z = z_0 + \rho\cos\varphi. \end{Bmatrix}$$

Then for $u(\rho,\theta,\varphi) = u(x_0 + \rho\cos\theta\sin\varphi, y_0 + \rho\sin\theta\sin\varphi, z_0 + \rho\cos\varphi)$

Then we have,

$$\frac{\partial u}{\partial n}|_S = \frac{x-x_0}{a}u_x + \frac{y-y_0}{a}u_y + \frac{z-z_0}{a}u_z$$

$$= \cos\theta\sin\varphi u_x + \sin\theta\sin\varphi u_y + \cos\varphi u_z$$

$$= \frac{\partial}{\partial\rho}u(\rho,\theta,\varphi)|_{\rho=a}$$

Equation (1.11) converts as following

$$0 = \iint_S \frac{\partial}{\partial\rho}u(\rho,\theta,\varphi)|_{\rho=a}dS$$

$$= \int_0^{2\pi}\int_0^{\pi} u_\rho(\rho,\theta,\varphi)|_{\rho=a}a^2\sin\varphi\ d\varphi\ d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} u_\rho(\rho,\theta,\varphi)|_{\rho=a}\ \sin\varphi\ \ d\varphi\ \ d\theta. \qquad (1.12)$$

Equation (1.12) is valid for every $a > 0$ so that we can consider a as a variable r and we have

$$\frac{d}{dr}\left(\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi)\sin\varphi\ \ d\varphi\ \ d\theta\right) = 0$$

Then

$$I(r) = \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi)\sin\varphi\ \ d\varphi\ \ d\theta$$

Letting $r\to 0$, we get

$$\lim_{r\to 0} I(r) = \int_0^{2\pi}\int_0^{\pi} u(0,\theta,\varphi)\sin\varphi\ \ d\varphi\ \ d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} u(P_0)\sin\varphi\ \ d\varphi\ \ d\theta$$

$$= 4\pi u(P_0)$$

NOTES

Further it is follows,

$$4\pi u(P_0) = \int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)\sin\varphi \; d\varphi \; d\theta$$

$$4\pi a^2 u(P_0) = \int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)a^2\sin\varphi \; d\varphi \; d\theta$$

$$= \iint_S u(P)dS_P$$

$$u(P_0) = \frac{1}{4\pi a^2}\iint_S u(P)dS_P.$$

Note: 'Mean Value Property' is also useable in the two dimensional case. Specifically, if $u(x, y)$ is a harmonic function in following equation

$$R^2, P_0(x_0, y_0) \in R^2 \text{ and } K_a = \{P \in R^2 : |P - P_0| \le a\}$$

In a disk, $C_a = \partial K_a$ then $u(P_0) = \frac{1}{2\pi a}\oint_{C_a} u(P)dS_P$ which is known as mean value formula for the two dimensional equation.

Let us consider a measure space (Ω, S, μ). Then, for $A \in S$ with $0 < \mu(A) < \infty$ and $h \in L^1(S, m; \mathbb{R}^n)$ where as $\fint$ symbol denoted as finite part integral.,

$$\fint_S h d\mu = \frac{1}{\mu(S)} \fint_S h\mu \in \mathbb{R}^n$$

is falled the mean value or the integral mean of h on S.

Theorem 1.7: In mean value property theorem consider an open set $\Omega \subset \mathbb{R}^n$, a harmonic function $u \in C^2(\Omega)$, and an arbitrary ball $\overline{B}_r(a) \subset \Omega$. Then, for the mean values on the ball $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$ and the sphere $S_r(a) := \partial B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$, we have

$$u(a) = f_{B_r(a)} u \, dx = \int_{S_r(a)} u \, d\mathcal{H}^{n-1}$$

Proof: For arbitrary $\rho \in (0, r)$, by the change of variables $x = a + \rho\omega$ and the corresponding integral transformation (which in turn follows from the invariance and scaling properties of the Hausdorff measure) we have

$$\fint_{S_\varrho(a)} h(x) \, d\mathcal{H}^{n-1}(x) = \fint_{S_1} h(a+\varrho\omega) \, d\mathcal{H}^{n-1}(\omega)$$

Differentiating of above equation, exchanging derivative and integral (we can do this since ∇u is locally bounded on Ω), and using the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\fint_{\mathrm{S}_\varrho(a)} h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \fint_{\mathrm{S}_1} \frac{\mathrm{d}}{\mathrm{d}\varrho} h(a+\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = \fint_{\mathrm{S}_1} \omega\cdot\nabla h(a+\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega).$$

NOTES

Also using the reverse change of variables, the divergence theorem, and the harmonicity of u we can write

$$\fint_{\mathrm{S}_1} \omega\cdot\nabla h(a+\varrho\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = \fint_{\mathrm{S}_\varrho(a)} \frac{x-a}{\varrho}\cdot\nabla h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \frac{1}{\mathcal{H}^{n-1}(\mathrm{S}_\varrho(a))}\int_{\mathrm{B}_\varrho(a)} \Delta h(x)\;\mathrm{d}x = 0.$$

Observing the above equation combining the least two chanin of equations, we can infer that the continuous mapping $\varrho \mapsto \fint_{\mathrm{S}_\varrho(a)} h\,\mathrm{d}\mathcal{H}^{n-1}$ has zero derivative on $(0,r)$ and hence is constant on $(0,r]$. Also, continuity of u at a implies $\left|\fint_{\mathrm{S}_\varrho(a)} h\,\mathrm{d}\mathcal{H}^{n-1} - h(a)\right| \le \sup_{\mathrm{S}_\varrho(a)} |h-h(a)| \underset{\varrho\searrow 0}{\longrightarrow} 0$ and hence

$$\lim_{\varrho\searrow 0}\fint_{\mathrm{S}_\varrho(a)} h\,\mathrm{d}\mathcal{H}^{n-1} = h(a).$$

Therefore, the constant value of $\varrho \mapsto \fint_{\mathrm{S}_\varrho(a)} h\,\mathrm{d}\mathcal{H}^{n-1}$ is actually equal to $u(a)$, and the claim is verified for spherical means.

With the help of spherical coordinates the mean value property on balls can now be deduced as follows:

$$\begin{aligned}\fint_{\mathrm{B}_r(a)} h\,\mathrm{d}x &= \frac{1}{\omega_n r^n}\int_0^r \int_{\mathrm{S}_\varrho(a)} h\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\varrho \\ &= \frac{1}{\omega_n r^n}\int_0^r n\omega_n \varrho^{n-1} h(a)\,\mathrm{d}\varrho = \frac{n}{r^n}\int_0^r \varrho^{n-1}\,\mathrm{d}\varrho\, h(a) = h(a).\end{aligned}$$

This completes the proof.

Remarks 1: Consider an open set Ω in $\mathbb{R}^n$.

- A function $u \in \mathrm{C}^2(\Omega)$ is said to be sub harmonic [1] on Ω if $\Delta u \ge 0$ holds on Ω. For subharmonic u on Ω and $\overline{\mathrm{B}}_r(a) \subset \Omega$, we can see from the above proof that the mean values on both $\mathrm{B}_\rho(a)$ and $\mathrm{S}_\rho(a)$ are non-decreasing functions of $\rho \in (0,r]$ and that the mean value in equality

  $$u(a) \le \fint_{\mathrm{B}_r(a)} u\,\mathrm{d}x \le \fint_{\mathrm{S}_r(a)} u\,\mathrm{d}\mathcal{H}^{n-1}$$

  is valid. On the other hand we can consider if $\Delta u(a) > 0$ holds true, the above mentioned mean values are even strictly increasing functions, and also the mean value inequality holds in the strict form

NOTES

$$u(a) < \fint_{B_r(a)} u \, dx < \fint_{S_r(a)} u \, d\mathcal{H}^{n-1}.$$

Similarly, superharmonic functions u are defined by the inequality $\Delta u \leq 0$ and satisfy the reverse mean value inequality.

- The respective form of mean value inequality even characterizes sub- and super harmonic functions, respectively, i.e., the converse to the assertions in above equation now spelled out for the subharmonic case - also holds: If we have $u \in C^2(\Omega)$ and either $u(a) \leq \fint_{B_r(a)} u \, dx$ or $u(a) \leq f_{S_r(a)} u \, d\mathcal{H}^{n-1} = u(a)$ holds for every ball $\overline{B}_r(a) \subset \Omega$, then u is subharmonic. Clearly, the combination of the assertions on sub and super harmonicity implies that the mean value property characterizes harmonic functions.

Proof of the Statement for the Subharmonic Case: Let us try to prove the case by contradiction. So, assume that the statement is false, i.e.,

$$u(a) \leq \fint_{B_r(a)} u \, dx$$

$$u(a) \leq \fint_{S_r(a)} u \, d\mathcal{H}^{n-1}$$

for every ball $\overline{B}_r(a) \subset \Omega$,

but still $\Delta u(x_0) < 0$ for some $x_0 \in \Omega$.

Then, by continuity of Δu, we have $\Delta u \leq 0$ on $B_{2\delta}(x_0) \subset \Omega$ for some sufficiently small $\delta > 0$.

Hence, u is superharmonic on $B_{2\delta}(x_0)$ with $\Delta u(x_0) < 0$, and first assertion yields $u(a) > \fint_{B_s(x_0)} u \, dx > \fint_{S_s(x_0)} u \, d\mathcal{H}^{n-1}$

This contradicts the initial assumption on the mean values and thus completes the proof of the claim.

1.4.3 Properties of Solutions

- **Strong maximum principle; Uniqueness.**

  Assume that $u \in C^2(U) \cap C(\overline{U})$ is harmonic within U.

  (a) Furthermore $\max_{\overline{U}} u = \max_{\partial U} u$

(b) However, if U is connected and there exists a point $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u$ then u is constant within U.

Assertion (a) is the maximum principle for Laplace's equation and (b) is the strong maximum principle.

Proof: Supposing that there is a exists point

$$x_0 \in U \text{ with } u(x_0) = M := \max_{\overline{U}} u .$$

Then for,

$0 < r < dist(x_0, \partial U)$, the mean-value property asserts

$$M = u(x_0) = \int_{B(x_0,r)} u dy \leq M.$$

As equivalence holds only if $u \equiv M$ within $B(x_0, r)$ we see $u(y) = M \square y \in B(x_0, r)$. Hence the set $\{x \in U \mid u(x) = M\}\}$ is both open and relatively closed in U, and thus equals U if U is connected. This proves assertion (b), from which (a) follows.

- **Smoothness**- If $u \in C(U)$ satisfies the mean-value property

$$u(x) = \int_{\partial B(x,r)} u dS = \int_{B(x,r)} u dy \text{ for each ball } B(x,r) \subset U, \text{ then}$$

$$u \in C^{\infty}(U)$$

Proof: Let us to be η a function.

Then,

$$\text{Set } u^{\varepsilon} := \eta_{\varepsilon} * u \text{ in } U\varepsilon - \{x \in U \mid dist(x, \partial U) > \varepsilon\}.$$

Theorem 1.8 (Weak Maximum Principle): Consider a bounded open set Ω in $\mathbb{R}^n$ and a subharmonic function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

Then we have the bound

$$u \leq \max_{\partial\Omega} u \quad \text{on} \Omega$$

(or, clearly equivalent, $\sup_{\Omega} u \leq \max_{\partial\Omega} u$).

Theorem 1.9: Strong Maximum Principle Consider a domain Ω in R^n. If a subharmonic function $u \in C^2(\Omega)$ attains its global maximum in Ω, then u is constant on Ω.

Remarks on the Maximum Principles

- Roughly speaking, the weak maximum principle asserts that the maximum is attained at the boundary, and the strong maximum principle asserts that it is attained only at the boundary (apart from the case of constants).
- Boundedness of Ω is essential for the above form of the weak maximum principle and connectedness of Ω is essential for the strong maximum principle otherwise u would merely need to be constant on the connected components of Ω where the maximum is attained.
- Clearly, for superharmonic functions u, the weak minimum principle

  $u \geq \min\delta\Omega u$ on Ω and the analogous strong minimum principle hold true. Precisely, for harmonic functions h, both maximum and minimum principles hold, and particularly this implies the maximum modulus estimate $|h| \leq \max_{\partial\Omega} |h|$ for harmonic h on bounded Ω.

1st **Proof of the Weak Maximum Principle:** The boundedness of Ω implies that $\partial\Omega$ is compact and $\max_{\partial\Omega} u \in \mathbb{R}$ exists. We fix an arbitrary $M \in \mathbb{R}$ with $M > \max_{\partial\Omega} u$ and introduce the auxiliary function

$$v := (u - M)_+^2$$

(with the usual abbreviation $f_+ := \max\{f, 0\}$).

Since v is the composition of $u - M$ and the C^1 function $x \mapsto x_+^2$ on the real line, the chain rule gives

$$v \in C^1(\Omega) \cap C^0(\bar{\Omega})$$

with

$$\nabla v = 2(u - M)\nabla u \text{ on } \{u \geq M\}$$

and

$$\nabla v \equiv 0 \text{ on } \{u \leq M\}$$

Moreover, the definition of v and the choice of M imply spt $v \subset \{x \in \bar{\Omega} : u(x) \geq M\} \subset \Omega$

Using boundedness of Ω once more, we deduce that spt v and $\{u \geq M\}$ are compact subsets of Ω. All in all, using v 'as a test function' for the subharmonicity of u and integrating by parts, we get

$$0 \leq \int_\Omega v\Delta u \, dx = -\int_\Omega \nabla v \cdot \nabla u \, dx = -2\int_{\{u>M\}} (u - M) |\nabla u|^2 \, dx$$

From the resulting inequality we can conclude $\nabla u \equiv 0$ on $\{u > M\}$, and hence u is equal to some constant $> M$ on every connected component of the open set $\{u > M\}$. However, each such component, as it is also open and contained in a compact subset of Ω, posses boundary points in which the value of u is $\leq M$. Hence, the existence of any connected component would lead to discontinuity of u at its boundary and would thus result in a contradiction. This leaves $\{u > M\} = \phi$ as the only possibility and yields $u \leq M$ on Ω. Finally, sending $M \searrow \max_{\partial\Omega} u$, we arrive at the claim.

2^{nd} **Proof of the Weak Maximum Principle:** Let us first assume that $\Delta u > 0$ holds on Ω and prove that there is no maximum point for u in Ω (that is, in the case $\Delta u > 0$ we prove the strong maximum principle). Indeed, if $x_0 \in \Omega$ is such a maximum point, the well-known second order necessary criterion for extremal points asserts that the Hessian $\nabla^2 u(x_0)$ is semi-negative, i.e. has only eigenvalues ≤ 0, and in conclusion we get

$$\Delta u(x_0) = trace\left(\nabla^2 u(x_0)\right) \leq 0$$

This contradicts the initial assumption and proves the absence of maximum points. Under the assumption that Ω is bounded, u posseses, however, a maximum on the compactum Ω, and thus we have shown $u < \max_{\partial\Omega} u$ on Ω.

Now we merely assume that u is subharmonic.

For arbitrary positive ε, we introduce an auxiliary function

$$u_\varepsilon \quad by \quad u_\varepsilon(x) := u(x) + \varepsilon |x|^2$$

for $x \in \Omega$ and record

$$\Delta u_\varepsilon = \Delta u + 2n\varepsilon \geq 2n\varepsilon > 0$$

Thus, the first part of the reasoning applies to u_ε and yields $u_\varepsilon < \max_{\partial\Omega} u_\varepsilon$ on Ω. Using $u < u_\varepsilon$ on the left-hand side of this estimate and writing out the definition of u_ε on its right-hand side, we arrive at

$$u < \max_{x \in \partial\Omega} \left[u(x) + \varepsilon |x|^2 \right] \leq \max_{\partial\Omega} u + \varepsilon \max_{x \in \partial\Omega} |x|^2 \quad on \Omega.$$

Considering the boundedness of Ω, we have $\max_{x \in \partial\Omega} |x|^2$, and sending $\varepsilon \searrow 0$ we can conclude $u \leq \max_{\partial\Omega} u$ on Ω.

NOTES

Proof of the Strong Maximum Principle

$Let M := \sup_{\Omega} u.$

By assumption, we have $M \in \mathbb{R}$ and $\{u = M\} \neq \phi$.

Moreover, $\{u = M\}$ is closed in Ω. Next we demonstrate that it is also open. Indeed, for $a \in \{u = M\}$, we fix a positive radius r with $\overline{\mathrm{B}}_r(a) \subset \Omega$. Then, by the choice of a and the mean value inequality for the subharmonic function u, we get

$$M = u(a) \leq \fint_{\mathrm{B}_r(a)} u \, \mathrm{d}x$$

but by the choice of M we also know $u \leq M$ on $\mathrm{B}_r(a)$.

This is only possible if $u \equiv M$ holds on the whole ball $\mathrm{B}_r(a)$ and we thus have $\mathrm{B}_r(a) \subset \{u = M\}$. All in all, the set $\{u = M\}$ is non-empty, open, and closed in Ω. Since Ω is a domain and thus connected this leaves $\{u = M\} = \Omega$ as the only possibility. We have thus shown that u is constant with value M on Ω

Corollary 1: Continuous Dependence for the Dirichlet Problem

Consider a bounded open set Ω in $\mathbb{R}^n$, and define ℓ as the maximum width of Ω in the sense of the smallest number $\ell \in (0, \infty)$ such that

$$\Omega \subset \left\{ x \in \mathbb{R}^n : | v \cdot (x - a) | \leq \frac{1}{2} \ell \right\}$$

holds for some point $a \in \mathbb{R}^n$ and some unit vector $v \in \mathbb{R}^n$. If $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ solves the Dirichlet problem

$\Delta u = f$ on Ω, $\quad u = \varphi$ on $\partial\Omega$

and $\tilde{u} \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ solves the Dirichlet problem.

$\Delta \tilde{u} = \tilde{f}$ on Ω, $\quad \tilde{u} = \tilde{\varphi}$ on $\partial\Omega$,

then we have the estimate

$$\max_{\overline{\Omega}} | \tilde{u} - u | \leq \max_{\partial\Omega} | \tilde{\varphi} - \varphi | + \frac{1}{8} \ell^2 \sup_{\Omega} | \tilde{f} - f |$$

Proof: Taking into account linearity of the Laplace operator Δ, we can assume

$$\tilde{u} \equiv 0, \quad \tilde{f} \equiv 0, \quad \tilde{\varphi} \equiv 0$$

NOTES

Moreover, it can be checked that Δ is invariant under translations and rotations, and thus we can also assume $a = 0, v = e_1$, that is

$$\Omega \subset \left(-\frac{1}{2}\ell, \frac{1}{2}\ell\right) \times \mathbb{R}^{n-1}$$

We now abbreviate $M := \sup_\Omega |f|$ and set $w(x) := u(x) + \frac{1}{2}Mx_1^2$.

Then, in view of $\Delta w = \Delta u + M = f + M \geq 0$ on Ω, we have that w is subharmonic on Ω. By the weak maximum principle, together with the choices of w and ℓ, we get

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} w \leq \max_{\partial\Omega} w \leq \max_{\partial\Omega} u + \frac{1}{2}M \max_{|x_1| \leq \frac{1}{2}\ell} x_1^2 = \max_{\partial\Omega} \varphi + \frac{1}{8}\ell^2 M.$$

Applying the same reasoning to $-u$ (and relying on $-f + M \geq 0$), we also get

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} \varphi - \frac{1}{8}\ell^2 M.$$

In conclusion we arrive at

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |\varphi| + \frac{1}{8}\ell^2 M.$$

This is the claim.

Corollary 2: Comparison Principle

Consider a bounded open set Ω in $\mathbb{R}^n$ and $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then, the inequalities

$\Delta u \geq \Delta v$ on Ω, $\quad v$ on $\partial\Omega$

imply the inequality

$u \leq$ even on $\bar{\Omega}$.

Proof: From $\Delta(u - v) = \Delta u - \Delta v \geq 0$ on Ω we see that $u - v$ is subharmonic on Ω. By the weak maximum principle we infer $u - v \leq \max_{\partial\Omega}(u - v) \leq 0$ and thus $u \leq v$ on Ω.

Remarks (on the Comparison Principle)

(1) Clearly, the assumption $\Delta u \geq \Delta v$ on Ω is satisfied if u is subharmonic and v superharmonic on Ω. This is the case in typical applications of the comparison principle. Often one of the two functions is even harmonic.

NOTES

(2) For a subharmonic function u on Ω the comparison principle guarantees $u \le h$ on Ω for every harmonic function h which coincides with u on $\partial\Omega$. In view of this property the introduction of the term 'subharmonic' indeed makes sense.

1.4.4 Energy Method of Laplace Equation

Energy methods calculate the some kind of system 'Energy' from a 'Partial Differential Equation'. Which energy may then be used to derive such things as existence or uniqueness of the solution, and whether it depends continuously on the data.

For the Ordinary Differential Equation (ODE) is an equation, which is include the scalar-valued function from the real line to the real line, and some of its derivatives. A Partial Differential Equation (PDE) is a simplification of the ordinary differential equation to the event where the argument is in multiple dimensions. In this case we have partial derivatives in each direction. For example, if the input is an ordered pair, then the input belongs to R^2, so we may have partial derivatives in two different directions.

For any given ODE or PDE, there might be one solution; there might be multiple solutions; or there might not be any at all. By the help of an energy method we can find answer of this question.

Suppose that $\Omega \subset \mathbb{R}^n$ is an open, with bounded set C^∞ smooth boundary $\delta\Omega$. Let T>0, $\Omega_T = \Omega \times (0, T]$. Assume that $a \in C^1(\overline{\Omega})$, $a>0$ on $\overline{\Omega}$, $\phi\Psi \in C^1(\overline{\Omega})$. Suppose that $u \in C^2(\overline{\Omega_T})$ is solution of,

$$u_{tt} - a(x)\Delta u = u^3 \quad \text{on} \quad \Omega_T,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times [0, T],$$

$$u = \phi, \quad u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}.$$

Prove it u is unique.

The PDE is $u_{tt} = C^2\Delta u - q(x)$ arise the study of wave prorogation in a non-homogeneous elastic medium: $q(x)$ is non-negative and proportional to the coefficient of elasticity at x. Let we will discuss about the

Outline on the Suitable Notion of Energy for Solutions

Solution: The energy integral is $\mathcal{E}(t) = \frac{1}{2}\int(|u_t|^2 + c^2|\nabla u|^2 + q(x)u^2)dx.$

Let we differentiate this function of t then obtain following equation

$$\frac{d\mathcal{E}}{dt} = \int_{R^n} \left(u_t u_{tt} + c^2 \sum_{i=1}^{n} u_{x_i} u_{x_i t} + q(x) u u_t \right) dx$$

Integration by parts then yields

$$\frac{d\mathcal{E}}{dt} = \int_{R^n} u_t (u_{tt} - c^2 \Delta u + q(x)) dx = 0$$

However ε(*t*) must constant.

Prove the Corresponding Energy Inequality

For any time

$$\tau \in [0, t_0], \text{ let } \bar{B}_\tau = \{x \in R^n : |x - x_0| \leq c(t_0 - \tau)\}.$$

Consider the energy function is,

$$\mathcal{E}_{x_0,t_0}(\tau) = \frac{1}{2} \int_{B_\tau} (u_t^2 + c^2 |\nabla u|^2 + q(x) u^2)|_{t=\tau} dx \qquad for\ 0 \leq \tau \leq t_0.$$

(1.13)

We claim that Equation (1.13) is a non-increasing function of τ, i.e., the following energy inequality holds:

$$\mathcal{E}_{x_0,t_0}(\tau) \leq \mathcal{E}_{x_0,t_0}(0) \qquad for\ 0 \leq \tau \leq t_0. \qquad (1.14)$$

To prove Equation (1.13), then we introduce the following notations

$$\Omega_\tau = \{(x,t) : |x - x_0| < c(t_0 - t), 0 < t < \tau\}$$
$$C_\tau = \{(x,t) : |x - x_0| = c(t_0 - t), 0 < t < \tau\}.$$

Note that $\partial\Omega_\tau = C_\tau \cup \bar{B}_0 \cup (\bar{B}_\tau \times \{\tau\})$ where the unions are disjoint. Consequently, the exterior unit is normal ν on δΩ is given on $\bar{B}_\tau \times \{\tau\}$ by $\nu - \langle 0, \ldots, 0, 1 \rangle$, and on $\bar{B}_0$ by $\nu = \langle 0, \ldots, 0, -1 \rangle$. On C_τ, $\nu = \langle \nu_1, \ldots, \nu_n, \nu_{n+1} \rangle$ satisfies $c^2(\nu_1^2 + \cdots + \nu_n^2) = \nu_{n+1}^2$ together with the unit length condition $\nu_1^2 + \cdots + \nu_n^2 + \nu_{n+1}^2 = 1$, that implies,

$$\nu_1^2 + \cdots + \nu_n^2 = \frac{\nu_{n+1}^2}{c^2} = \frac{1}{1 + c^2}$$

NOTES

Given a solution u, we define the vector field

$$V = \langle 2c^2 u_t u_{x_1}, \cdots, 2c^2 u_t u_{x_n}, -(c^2|\nabla u|^2 + u_t^2 + q(x)u^2) \rangle$$

If we use the divergence, we get,

$$div \cdot V = 2c^2\left(u_{tx_1}u_{x_1} + u_t u_{x_1x_1} + \cdots + u_{tx_n}u_{x_n} + u_t u_{x_nx_n}\right) - 2c^2\left(u_{tx_1}u_{x_1} + \cdots + u_{tx_n}u_{x_n}\right) - 2u_t u_{tt} - 2q(x)uu_t = 0$$

The divergence theorem therefore implies

$$\int_{\partial\Omega_\tau} V \cdot \nu dS = 0 .$$

So that the C_τ is hold the following inequality,

$$2u_t\left(u_{x_1}\nu_1 + \cdots + u_{x_n}\nu_n\right) \le \frac{c}{\sqrt{1+c^2}}|\nabla u|^2 + \frac{1}{c\sqrt{1+c^2}}(u_t^2 + q(x)u^2)$$

We may calculate on C_τ,

$$V \cdot \nu = 2c^2 u_t\left(u_{x_1}\nu_1 + \cdots + u_{x_n}\nu_n\right) - (c^2|\nabla u|^2 + u_t^2 + q(x)u^2)\nu_{n+1} \le 0 ,$$

So in actual condition is,

$$\int_{C_\tau} V \cdot \nu dS \le 0 .$$

Furthermore we have that,

$$0 \le \int_{B_0} V \cdot \nu dS + \int_{\partial B_\tau \times (\tau)} V \cdot \nu dS = \int_{B_0} (u_t^2 + c^2|\nabla u|^2 + q(x)u^2)|_{t=0} dx - \int_{B_\tau} (u_t^2 + c^2|\nabla u|^2 + q(x)u^2)|_{t=\tau} dx$$

Which is, prove suitable notion of energy for solutions.

Use the Energy Method to Prove That Solutions are Uniquely Determined by Their Cauchy Data

Let both u_1 and u_2 to be solution to $u_{tt} = c^2\Delta u - q(x)$ on $C^2(R^n \times (0,\infty))$ with initial conditions

$u(x,0) = g(x),\ u_t(x,0) = h(x)$ for $x \in R^n$. Let $w \equiv u_1 - u_2$, then we get

$$\begin{cases} w_{tt} = c^2\Delta w - q(x)w, & for\ x \in R^n, t > 0 \\ w(x,0) = w_t(x,0) = 0, & for\ x \in R^n \end{cases}$$

Thus, we know that $\varepsilon(t)= \varepsilon(0)=0$ and it follows that

$$0 = \frac{1}{2}\int_{R^n} (w_t^2 + c^2|\nabla w|^2 + q(x)w^2)dx$$

Since $q(x)$ is non-negative, the third term implies that $w(x,t) \equiv 0$ and thus $u_1(x,t) \equiv u_2(x,t)$.

NOTES

1.5 WAVE EQUATION

The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves. The wave equation is considered as a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a scalar function $u=(x_1, x_2,...,x_n,t)$ that satisfies,

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

Here ∇^2 is the spatial Laplacian and c is a fixed constant equal to the propagation speed of the wave and is also known as the non-dispersive wave equation. For a sound wave in air at 20°C this constant is about 343 m/s (speed of sound). For a spiral spring, it can be as slow as a meter per second. The differential equations for waves are based on the speed of wave propagation that varies with the frequency of the wave. This specific phenomenon is known as dispersion. In such a case, c must be replaced by the phase velocity as shown below:

$$v_{\mathrm{p}} = \frac{\omega}{k}$$

The speed can also depend on the amplitude of the wave which will lead to a non-linear wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = c(u)^2\nabla^2 u$$

A wave can be superimposed onto another movement. In that case the scalar u will contain a Mach factor which is positive for the wave moving along the flow and negative for the reflected wave.

The elastic wave equation in three dimensions describes the propagation of waves in an isotropic homogeneous elastic medium. Most of the solid materials are elastic, hence this equation is used to analyse the phenomena such as seismic waves in the Earth and ultrasonic waves which detect flaws in materials. In its linear form, this equation has a more complex form compared to the equations discussed above because it accounts for both longitudinal and transverse motion using the notation:

NOTES

$$\rho\ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u})$$

Where:

- λ and μ are termed as Lamé parameters which describe the elastic properties of the medium.
- ρ is the density.
- **f** is the source function or driving force.
- **ü** is the displacement vector.

In this equation, both the force and the displacement are vector quantities. Hence, this equation is also termed as the vector wave equation.

General Solution of One Dimensional Wave Equation

The one dimensional wave equation for a partial differential equation has a general solution of the form that defines new variables as,

$$\xi = x - ct \quad ; \quad \eta = x + ct$$

It changes the wave equation into,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

This leads to the general solution of the form,

$$u(\xi, \eta) = F(\xi) + G(\eta) \quad \Rightarrow \quad u(x,t) = F(x - ct) + G(x + ct)$$

Basically, solutions of the one dimensional wave equation are sums of a right traveling function F and a left traveling function G. Here the term 'Traveling' refers the shape of the individual arbitrary functions with respect to x which stays constant, though the functions are transformed left and right with time at the speed c.

As per the Helmholtz equation, named for Hermann von Helmholtz, is the elliptic partial differential equation of the form $\nabla^2 A + k^2 A = 0$, where ∇^2 is the Laplace operator, k is the wavenumber and A is the amplitude.

On the other hand wave equation is given by,

$$u_{tt} - \Delta u = 0 \tag{1.15}$$

And the non-homogeneous wave equation is represented by,

$$u_{tt} - \Delta = f \tag{1.16}$$

Focused on suitable initial and boundary conditions. Here is $t>0$ and $x \in U$, where $U \subset R^n$ is open.

The unknown is $u : \overline{U} \times [0,\infty) \to R,,t)\ u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1 \ldots . x_n)$.

NOTES

In Equation (1.16) the function $f : U \times [0,\infty) \to R$ is given. A common abbreviation is to write

$$u = u_{tt} - \Delta u$$

Physical Interpretation- The wave equation is a basic model for a vibrating string (n=1), membrane (n=2).or elastic solid (n=3). In these physical interpretation $u(x, t)$ represents the displacement in some direction of the point x at time $t \geq 0$.

Let V represent any smooth sub region of U. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u \, dx = \int_V u_{tt} \, dx|$$

So that net contact force is, $-\int_{\partial V} F.v\, dS,$

Whereas F denotes the force acting on V through ∂V and the mass density is taken to be unity. Newton's law states that "the mass times the acceleration equals the net force":

$$\int_V u_{tt} dx = -\int_V F.v\, dS,$$

This identity obtains for each sub-region V and so u_{tt} = -div$F$.

For elastic bodies, F is a function of the displacement gradient Du; whence u_{tt} + div$F$ (Du) = 0. For small Du, the linearization $F(Du) \approx -aDu$ is often appropriate; and so $u_{tt} - a\Delta u = 0$.

This is the wave equation if a =1.

Note: u=displacement, u_t=velocity at time t=0.

1.5 .1 Solution by Spherical Means

For the wave equation firstly solving u_{tt} -Δu = 0

For n=1 directly and then for $n \geq 2$ by the method of spherical.

- **Solution for n=1, D'Alembert's formula:**

Initial-value problem for the one-dimensional wave equation in all of R:

$$\left\{\begin{array}{l} u_{tt} - u_{xx} = 0 \quad in \quad R \times (0,\infty) \\ b = g, \;\; u_t = h \quad on \;\; R \times \{t = 0\} \end{array}\right\} \qquad (1.17)$$

NOTES

Where g, h are given,

PDE in Equation (1.17) can be 'Factored' as following equation is,

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u = u_{tt} - u_{xx} = 0 \tag{1.18}$$

Assume that

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u = v \tag{1.19}$$

From Equation (1.18) $v_t + v_x = 0 \qquad (x \in R,\ t > 0).$

Which is, transport equation with constant coefficients.

at n=1, b=1, we find

$v = a(x\text{-}t)$ (1.20)

For $a(x)$: = $v(x,0)$, Combining now Equation (1.18) to (1.20), we obtain $u_t - u_x = a(x-t)$ in $R\times(0,\infty)$.

Which also known as a **non-homogeneous transport equation** ;(with n=1, b= -1, $f(x, t) = a(x\text{-}t)$) implies

$$\begin{aligned} u(x,t) &= \int_0^t a(x+(t-s)-s)ds + b(x+t) \\ &= \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t), \end{aligned} \tag{1.21}$$

Where $b(x)$: = $u(x,0)$.

First initial condition in Equation (1.17) gives $b(x) = g(x)$ $(x \in R)$;

Whereas the second initial condition and Equation (1.19) implies

$$\begin{aligned} a(x) = v(x,0) &= u_t(x,0) - u_x(x,0) \\ &= h(x) - g'(x) \qquad (x \in R). \end{aligned}$$

Put in Equation (1.21) which gives that,

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)dy + g(x+t)$$

Consequently,

$$u(x,t)=\frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy \qquad (x\in R, t\geq 0)$$

• Spherical Means

NOTES

Now assume that $n\geq 2$, $m\geq 2$ and $u\in C^m\ (R^n\times[0,\infty))$ solves the initial value problem:

$$\text{Value problem} = \left\{\begin{matrix} u_{tt}-\Delta u=0 \quad in \qquad R^n\times(0,\infty) \\ b=g,\ u_t=h \quad on \quad R^n\times\{t=0\}. \end{matrix}\right\}$$

We propose that to derive an explicit formula for u in terms of g, h. The study to first the average of u over certain spheres. These averages, taken as functions of the time t and the radius r, turn out to solve the *Euler- Poisson-Darboux equation*, a PDE which we can for odd convert into the ordinary one-dimensional wave equation. Applying D'Alembert's formula, or more precisely its variant in Equation (1.17), eventually leads us to a formula for the solution.

1.5.2 Non-Homogeneous Equation

Initial problem for the non-homogeneous wave equation is defined by,

$$\left\{\begin{matrix} u_{tt}-\Delta u=f \quad in \quad R^n\times(0,\infty) \\ u=0, u_t=0 \quad on \quad R^n\times\{t=0\}. \end{matrix}\right\}$$

Modified by Duhamel's principle we define $u=u(x,t;s)$ to be the solution of

$$\left\{\begin{matrix} u_{tt}(.;s)-\Delta u(.;s)=0 \quad in \quad R^n\times(s,\infty) \\ u(.;s)=0, u_t(.;s)=f=(.;s0 \quad on \quad R^n\times\{t=s\} \end{matrix}\right\}$$

Hence,

$$u(x,t)=\int_0^t u(x,t;s)ds \qquad (x\in R^n,\ t\geq 0)$$

Duhamel's principle emphasises, i.e., a solution of

$$\left\{\begin{matrix} u_{tt}-\Delta u=f \quad in \quad R^n\times(0,\infty) \\ u=0,\ u_t=0 \quad on \quad R^n\times\{t=0\} \end{matrix}\right\}$$

Theorem 1.10: Solution of non-homogeneous wave equation-

Supposing $n\geq 2$

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And,

$$f \in C^{[n/2]+1}(R^n \times [0,\infty))$$

Define u by,

$$u(x,t) := \int_0^t u(x,t;s)ds \qquad (x \in R^n,\ t \geq 0)$$

- $u \in C^2(R^n \times [0,\infty))$,
- $u_{tt} - \Delta u = f \quad in \quad R^n \times (0,\infty)$,
- $\lim_{\substack{(x,t)\to(x^0,0)\\ x\in R^n, t>0}} u(x,t) = 0, \qquad \lim_{\substack{(x,t)\to(x^0,0)\\ x\in R^n, t>0}} u_t(x,t) = 0 \qquad for \quad each \quad point \quad x^0 \in R^n$

Proof: If n is odd, then $\left[\frac{n}{2}\right]+1 = \frac{n+1}{2}$,

By Solution of wave in odd dimension $u(.,.;s) \in C^2(R^n \times [0,\infty))$.

For individually $s \geq 0$ and so that,

$u \in C^2(R^n \times [0,\infty))$. If n is even, $\left[\frac{n}{2}\right]+1 = \frac{n+2}{2}$

Consequently,

$$u \in C^2(R^n \times [0,\infty))$$

We then calculate:

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)ds = \int_0^t u_t(x,t;s)ds,$$

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)ds$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)ds$$

Moreover,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)ds = \int_0^t u_{tt}(x,t;s)ds$$

Thus,

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t) \qquad (x \in R^n, t > 0),$$

Subsequently,

$$u(x,0) = u_t(x,0) = 0 \quad for \quad x \in R^n$$

1.5.3 Energy Methods

(i) Uniqueness

Let $U \subset R^n$ be bounded, open set with a smooth boundary ∂U, and as usual set $U_T = U \times (0,T]$, $\Gamma T = \overline{U}_T - U_T$, Where $T>0$.

Theorem 1.11: (Uniqueness for Wave Equation) - There exists at most one function $u \in C^2(\overline{U}_T)$ solving,

$$\begin{Bmatrix} u_{tt} - \Delta u = f \ \ in \ \ U_T \\ u = g \ \ on \ \ \Gamma T \\ u_t = h \ \ on \ U \times \{t = 0\} \end{Bmatrix}$$

Proof: If $\overline{u}$ is another such solution, then $w := u - \overline{u}$ solves

$$\begin{Bmatrix} w_{tt} - \Delta w = f \ \ in \ \ U_T \\ w = g \ \ on \ \ \Gamma T \\ w_t = 0 \ \ on \ U \times \{t = 0\} \end{Bmatrix}$$

Define the 'Energy' $e(t) := \frac{1}{2}\int_U w_t^2(x,t) + |Dw(x,t)|^2 dx \quad (0 \le t \le T)$

So we get,

$$\dot{e}(t) = \int_U w_t w_{tt} + Dw.Dw_t dx$$
$$= \int_U w_t (w_{tt} - \Delta w) dx = 0$$

There is no boundary term since $w = 0$, and hence $w_t = 0$ on $\partial U \times [0,T]$. Thus for all $0 \le t \le T$, $e(t) = e(0) = 0$, and so w_t, Dw = 0 within U_T.

Since $w = 0$ on $U \times \{t = 0\}$ So we conclude that, $w = u - \overline{u} \equiv 0 \ \ in \ \ U_T$.

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(ii) Domain of Dependence

Let us examine again the domain of dependence of solutions to the wave equation in all of space.

For this, suppose $u \in C^2$ solves

Fix $x_0 \in R^n$, $t_0 > 0$ and consider the cone as per following equation $C = \{(x,t) | 0 \le t \le t_0, |x - x_0| \le t_0 - t$

Theorem 1.12: (Finite Propagation Speed)- If $u \equiv u_t \equiv 0$ on B (x_0, t_0) then $u \equiv 0$ within the cone C.

Proof:

Define $e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^{\,2}(x,t) + |Du(x,t)|^2 dx \quad (0 \le t \le t_0)$

Then,

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} u_t u_{tt} + Du.Du_t dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^{\,2} + |Du|^2 dS$$

$$= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u)dx + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial v} u_t dS - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^{\,2} + |Du|^2 dS$$

$$- \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial v} u_t - \frac{1}{2} u_t^{\,2} - \frac{1}{2}|Du|^2 dS. \qquad (1.22)$$

Further,

$$\frac{\partial u}{\partial v} u_t \le |u_t||Du| \le \frac{1}{2} u_t^{\,2} + \frac{1}{2}|Du|^2, \qquad (1.23)$$

By the Cauchy- Schwarz and Cauchy inequalities, from Equations (1.22) and (1.23) we, find

$\dot{e}(t) \le 0$ and so $e(t) \le e(0) = 0$ for all $0 \le t \le t_0$.

Thus u_t, Du=0 t and consequently u≡0 within the cone C.

Check Your Progress

6. State the Laplace equation.
7. What do you understand by energy method in PDE?
8. What is wave equation in PDE?
9. Give the physical interpretation of wave equation.
10. How will you define the initial problem for the non-homogeneous wave equation?

1.6 ANSWERS TO 'CHECK YOUR PROGRESS'

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1. In mathematics, a Partial Differential Equation (PDE) is an equation which carry out the relations between the various 'Partial Derivatives' of a multivariable function. Whereas the function is often thought of as an 'Unknown' to be solved for, similarly to how x is thought of as an unknown number to be solved for in an algebraic equation similar as like $x^2 - 3x + 2 = 0$.
2. There are three-types are following:
   - Elliptic PDE
   - Parabolic PDE
   - Hyperbolic PDE
3. The 'Transport Equation' is a partial differential equation of the form is,

   $u_t + cu_x = 0$

   Where, u is a function of two variables (x,t) and the subscripts denote partial derivatives. We will assume that c is a fixed constant.
4. In the field of differential equations, an initial value problem is an ordinary differential equation together with specified value, called the initial condition, of the unknown function at a given point in the domain of the solution.
5. If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise. The following equation is consider as non-homogeneous,

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

   Non-homogeneous for transport equation is defined by:

$$\begin{Bmatrix} u_t + b.Du = 0 \text{ in } R^n \times (0,\infty) \\ u = g \quad on \quad R^n \times \{t = 0\} \end{Bmatrix}$$

6. In mathematics, Laplace's equation is termed as a second order partial differential equation and is named after Pierre-Simon Laplace who initially examined its properties. The equation is written as follows:

   $\Delta\varphi = 0$ or $\nabla^2\varphi = 0$

   Here $\Delta = \nabla^2$ is termed as the Laplace operator and φ is considered as a scalar function.

7. Energy methods calculate the some kind of system 'Energy' from a 'Partial Differential Equation'. Which energy may then be used to derive such things as existence or uniqueness of the solution, and whether it depends continuously on the data.
8. The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves. The wave equation is considered as a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a scalar function $u=(x_1, x_2,...,x_n,t)$ that satisfies,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Here ∇^2 is the spatial Laplacian and c is a fixed constant equal to the propagation speed of the wave and is also known as the non-dispersive wave equation.

9. The wave equation is a basic model for a vibrating string (n=1), membrane (n=2).or elastic solid (n=3). In these physical interpretation $u(x, t)$ represents the displacement in some direction of the point x at time $t \geq 0$.
10. Initial problem for the non-homogeneous wave equation is defined by,

$$\left\{\begin{matrix} u_{tt} - \Delta u = f \quad in \quad R^n \times (0,\infty) \\ u = 0, u_t = 0 \quad on \quad R^n \times \{t = 0\}. \end{matrix}\right\}$$

1.7 SUMMARY

- In mathematics, a Partial Differential Equation (PDE) is an equation which carry out the relations between the various 'Partial Derivatives' of a multivariable function. Whereas the function is often thought of as an 'Unknown' to be solved for, similarly to how x is thought of as an unknown number to be solved for in an algebraic equation similar as like x^2 " $3x + 2 = 0$.
- On the other hand, it is usually impossible to write down explicit formulas for solutions of partial differential equations. There is, similarly, a vast amount of modern mathematical and scientific research on methods to numerically approximate solutions of certain partial differential equations using computers. Partial differential equations also occupy a large sector of pure mathematical research, in which the usual questions are, broadly speaking, on the identification of general qualitative features of solutions of various partial differential equations.

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- Partial differential equations are ubiquitous in mathematically-oriented scientific fields, such as physics and engineering.
- PDE is also arise from many purely mathematical considerations, such as differential geometry and the calculus of variations; among other notable applications.
- A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.
- PDE has definite functionalities, which is help to determine whether a particular finite element approach is appropriate to the problem being described by the PDE. The solution depends on the equation and several variables contain partial derivatives with respect to the variables.
- We usually come across three-types of second-order PDEs in mechanics. These are classified as elliptic, hyperbolic, and parabolic. The equations of elasticity (without inertial terms) are elliptic PDEs. Hyperbolic PDEs describe wave propagation phenomena. The heat conduction equation is an example of a parabolic PDE.
- In contrast, the solutions of elliptic PDEs are always smooth, even if the initial and boundary conditions are rough (though there may be singularities at sharp corners). In addition, boundary data at any point affect the solution at all points in the domain.
- Parabolic PDEs are usually time dependent and represent diffusion-like processes. Solutions are smooth in space but may possess singularities. Conversely, information travels at infinite speed in a parabolic system.
- In the field of differential equations, an initial value problem is an ordinary differential equation together with specified value, called the initial condition, of the unknown function at a given point in the domain of the solution.
- The Picard-Lindelöf theorem guarantees a unique solution on some interval containing t_0 if f is continuous on a region containing t_0 and y_0 and satisfies the Lipschitz condition on the variable y. The proof of this theorem proceeds by reformulating the problem as an equivalent integral equation. The integral can be considered an operator which maps one function into another, such that the solution is a fixed point of the operator. The Banach fixed point theorem is then invoked to show that there exists a unique fixed point, which is the solution of the initial value problem.
- If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise.
- In mathematics, Laplace equation is a second order partial differential equation.

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- Laplace equation and Poisson equation are examples of elliptic partial differential equations. The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great important in many fields of science.
- The Dirichlet condition for Laplace's equation consists of finding a solution j on some domain *D* such that j on the boundary of *D* is equal to some given function. Because the Laplace operator is also used in the heat equation hence we can interpret it by fixing the temperature on the boundary of the domain according to the given specification of the boundary condition.
- The definition of the fundamental solution thus implies that, if the Laplacian of *u* is integrated over any volume that encloses the source point, then encloses the source point then we have the following equation,

$$\iiint_V \nabla \cdot \nabla u \, dV = -1.$$

- The Laplace equation remains unchanged under a rotation of coordinates and hence we can obtain a fundamental solution that only depends upon the distance *r* from the source point.
- Energy methods calculate the some kind of system 'Energy' from a 'Partial Differential Equation'. Which energy may then be used to derive such things as existence or uniqueness of the solution, and whether it depends continuously on the data.
- For the Ordinary Differential Equation (ODE) is an equation, which is include the scalar-valued function from the real line to the real line, and some of its derivatives. A Partial Differential Equation (PDE) is a simplification of the ordinary differential equation to the event where the argument is in multiple dimensions. In this case we have partial derivatives in each direction.
- The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves. The wave equation is considered as a hyperbolic partial differential equation.
- For a sound wave in air at 20°C this constant is about 343 m/s (speed of sound). For a spiral spring, it can be as slow as a meter per second. The differential equations for waves are based on the speed of wave propagation that varies with the frequency of the wave. This specific phenomenon is known as dispersion.
- The elastic wave equation in three dimensions describes the propagation of waves in an isotropic homogeneous elastic medium. Most of the solid materials are elastic, hence this equation is used to analyse the phenomena such as seismic waves in the Earth and ultrasonic waves which detect flaws in materials.

- Basically, solutions of the one dimensional wave equation are sums of a right traveling function F and a left traveling function G. Here the term 'Traveling' refers the shape of the individual arbitrary functions with respect to x which stays constant, though the functions are transformed left and right with time at the speed c.
- If $u \equiv u_t \equiv 0$ on B (x_0, t_0) then $u \equiv 0$ within the cone C.

1.8 KEY TERMS

- **Partial Differential Equation (PDE):** In mathematics, a Partial Differential Equation (PDE) is an equation which carry out the relations between the various 'Partial Derivatives' of a multivariable function.
- **Hyperbolic PDEs:** In hyperbolic PDEs, the smoothness of the solution depends on the smoothness of the initial and boundary conditions.
- **Parabolic PDEs:** Parabolic PDEs are usually time dependent and represent diffusion-like processes. Solutions are smooth in space but may possess singularities. Conversely, information travels at infinite speed in a parabolic system.
- **Initial value problem:** In the field of differential equations, an initial value problem is an ordinary differential equation together with specified value, called the initial condition, of the unknown function at a given point in the domain of the solution.
- **Non-homogeneous PDE:** If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise.
- **Wave equation:** The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves. The wave equation is considered as a hyperbolic partial differential equation.

1.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define partial differential equation.
2. Give the classification of PDE.
3. State the transport equation.
4. What is initial value problem?
5. How are non-homogeneous linear equations with constant coefficients formed?

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6. Define the significance of Laplace's equation.
7. How will you calculate the mean value formula?
8. Give the properties of solutions in Laplace equation.
9. What do you understand by wave equation?
10. Determine the energy method in wave equation.

Long-Answer Questions

1. Obtain a partial differential equation by eliminating the arbitrary constants of the following:

   (i) $z = ax + by + \sqrt{a^2 + b^2}$ *(ii)* $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$

   (iii) $z = xy + y\sqrt{x^2 - a^2} + b$ *(iv)* $z = ax^3 + by^3$

   (v) $(x-a)^2 + (y-b)^2 + z^2 = a^2 + b^2$ *(vi)* $2z = (ax + y)^2 + b$

2. Eliminate the arbitrary function from the following:

   (i) $z = e^y f(x + y)$ *(ii)* $z = f(my - lx)$

   (iii) $z = f(x^2 + y^2 + z^2)$ *(iv)* $z = x + y + f(xy)$

   (v) $z = f(x) + e^y g(x)$ *(vi)* $z = f(x + 4y) + g(x - 4y)$

   (vii) $z = f(2x + 3y) + y\, g(2x + 3y)$ *(viii)* $z = f(x + y) \bullet \phi(x - y)$

3. Solve the following differential equations:

   (i) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ *(ii)* $y^2zp + x^2zq = y^2x$

   (iii) $x^2p - y^2q = (x - y)z$ *(iv)* $xp + yq = 2z$

   (v) $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

4. Eliminate the arbitrary function(s) from the following and form the partial differential equations:

   (i) $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ *(ii)* $z = f(x^2 + y^2 + z^2)$

   (iii) $u = e^y f(x - y)$ *(iv)* $z = f(\sin x + \cos y)$

   (v) $\phi(x + y + z, x^2 + y^2 - z^2) = 0$ *(vi)* $z = f(2x + 3y) + \phi(y + 2x)$

   (vii) $u = f(x^2 + y) + g(x^2 - y)$ *(viii)* $u = x\, f(ax + by) + g(ax + by)$

5. Find the complete solution of the following partial differential equations:

   (i) $pq + p + q = 0$ *(ii)* $p^3 = q^3$

   (iii) $p = e^q$ *(iv)* $z = px + qy + p^2 + pq + q^2$

   (v) $z = px + qy + \log pq$ *(vi)* $z = px + qy + p^2 - q^2$

   (vii) $z^2 = 1 + p^2 + q^2$

6. Explain in detail about the transport equation in the terms of initial value problem and non-homogeneous equation with appropriate example.
7. Why is initial value problems solved by Picard method? Give reasons.
8. Discuss about the Laplace equation giving fundamental solution of Laplace equation with examples.
9. Elaborate on the wave equation in the terms of solution by spherical means, non-homogeneous equation and Energy mehods.
10. Solve the one dimensional wave equation $\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$ for a string of length l fixed at both ends. The boundary conditions are

$$\{f(x,t)\}_{t=0} = \frac{2hx}{l}, 0 \le x < \frac{l}{2}$$

$$= \frac{2h}{l}(l-x), \frac{l}{2} \le x < l.$$

Discuss the nature of the solution.

1.10 FURTHER READING

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UNIT 2 PARTIAL DIFFERENTIAL EQUTIONS-II

Structure

2.0 INTRODUCTION

If the ODE (Ordinary Differential Equations) has a product of the unknown function times any of its derivatives, the ODE is non-linear.

A solution of an *n* the order ordinary differential equation which depand on *n* arbitrary constant as well as the independent variable, which is also know as complete primitive. Where as a solution of a PDE of the first order that contains as many arbitrary constants as there are independent variables also known as complete integrals.

In geometry, an envelope of a planar family of curves is a curve, i.e., tangent to every member of the family at some point, and these points of tangency together form the total envelope. Typically, a point on the envelope can be thought of as the intersection of two 'Infinitesimally Adjacent' curves, meaning the limit of intersections of nearby curves. This idea can be generalized to an envelope of surfaces in space, and so on to higher dimensions. Envelopes can be used to construct more

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complicated solutions of first order Partial Differential Equations (PDEs) from simpler ones.

In mathematics, the Hamilton-Jacobi equation is an essential condition which is, describe extremal geometry in simplifications of problems from the calculus of variations. It can be understood as a special case of the Hamilton-Jacobi-Bellman equation from dynamic programming.

The Hamiltonian often has a physical meaning for the system of ODEs, i.e., Modeling a particular real-world situation, since it represent a quantity, i.e., being conserved over time. Hamilton's ODE next we introduce the Hamiltonian and intimately connect it with the Lagrangian. This will give us a tool to convert the Euler-Lagrange equations into Hamilton's equations; a system of n second-order ODE to a system of $2n$ system of first-order ODE. In mathematics the 'Legendre Transform' named after Adrien-Marie Legendre, is an involute transformation on the real valued convex function of one real variable. On the other hand Hop F Lax formula is state that "With optimal control problem whereas controlled dynamics is given by a time change 'stochastic process' describing the trajectory of particle subject to random trapping effects.

The major restriction involved in the Hopf-Lax formula is the fact that F is not allowed to depend on t, x or u. The reason for that the formula is derived from a variational characterization of the function u in which straight lines are proved to be the optimal trajectories. When time or space dependence is allowed it is very unlikely that this will be the case.

In mathematics, separation of variables (also known as the Fourier method) is any of several methods for solving ordinary and partial differential equations, in which algebra allows one to modified an equation so that each of two variables occurs on a different side of the equation. Similarity solutions to PDEs are solutions which depend on certain groupings of the independent variables, rather than on each variable separately. Additionally the concept of soliton is related with solutions for non-linear partial differential equations. The soliton solution of a non-linear equation usually is used a single wave.

In the case of partial differential equations, the Fourier transform is the technique which are related to reduce by one the number of variables with respect to which differentiation occurs. On the other hand the Laplace transform can be helpful in solving ordinary and partial differential equations because it can replace an ODE with an algebraic equation or replace a PDE with an ODE. Another reason that the Laplace transform is useful is that it can help deal with the boundary conditions of a PDE on an infinite domain.

The Hopf-Cole transformation turning the strongly non-linear Burgers equation into the linear heat equation plays an important role in the development of mathematical sciences. Hodograph transformation is a technique used to transform non-linear partial differential equations into linear form. It consists of interchanging the dependent and independent variables in the equation to achieve linearity. The

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Hamiltonian equations it is also possible to compute a Legendre transformation of the position $\vec{q}$ variable $\vec{p}$ to a variable. The independent variables in a differential expression can be changed by a Legendre transformation. The term potential function may refer to a mathematical function whose values are a physical potential and the class of functions known as harmonic functions, which are the topic of study in potential theory.

In this unit, you will learn about the non-linear first order PDE, complete integrals, envelopes, Hamilton - Jacobi equations, calculus of variations, Hamilton's ODE, Legendre transform, Hopf-Lax formula, weak solution, uniqueness, separation of variables, similarity solution of travelling wave, soliton, similarity under scaling, Fourier and Laplace transform, Hopf - Cole transform, hodograph and Legendre transform and potential functions.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the non-linear first order PDE
- Explain about the complete integral
- Define envelops
- Analyse the Hamilton-Jacobi equation
- Discuss about the calculus of variations
- Elaborate on the Hamilton's ODE
- Know about the Legendre transform for Hamiltion's-Jacobi equation
- Learn about the Hopf-Lax formula
- Determine the weak solutions and uniqueness
- Exaplin about the separation of variables
- Identify the similarity soultions of travelling wave, soliton and under scaling
- Interpret the Fourier and Laplace transform
- Describe the Hopf - Cole transform
- Discuss about the hodograph and Legendre transform
- Define potential functions

2.2 BASIC CONCEPT OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Qualitative theory of differential equations studies the properties of solutions of ordinary differential equations without finding the solutions themselves.

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The foundations of the qualitative theory of differential equations were laid at the end of the 19th century by H. Poincare and A.M. Lyapunov. Poincare made extensive use of geometric methods, regarding the solutions of systems of differential equations as curves in an appropriate space. On this basis he created a general theory of the behaviour of solutions of second-order differential equations and solved a number of fundamental problems on the dependence of solutions on parameters. Lyapunov studied the behaviour of solutions in a neighbourhood of an equilibrium position and founded the modern theory of stability of motion. The geometric approach of Poincare was developed in the 1920s by George Birkhoff, who discovered many important facts in the qualitative theory of higher-dimensional systems of differential equations.

Non-Linear Systems

General systems of non-linear differential equations are considered in the normal form:

$$\frac{dy}{dx} = Y(y, x), \qquad y \in \mathbf{R}^n . \qquad \ldots(2.1)$$

Autonomous systems are given by the equation,

$$\frac{dy}{dx} = Y(y) . \qquad \ldots(2.2)$$

The space of vectors y for the system Equation (2.2) is called phase space. The system Equation (2.1) can be reduced to the autonomous form Equation (2.2) by increasing the order by one. An autonomous system of the form Equation (2.2) defines a dynamical system if all its solutions can be extended to the whole axis $-\infty < x < +\infty$.

Let $y = y(x, y_0)$ be the solution to Equation (2.2) with initial data $x = 0$, $y = y_0$. The curve $y = y(x, y_0)$, $-\infty < x < +\infty$, in the phase space is called a trajectory, while the parts corresponding to $x \geq 0$, $x \geq 0$ are called semi trajectories. A special role is played by trajectories which degenerate to a point $y(x, y_0) \equiv y_0$ when $Y(y_0) = 0$. Such points are called equilibrium positions. Another important type of trajectory is that of a periodic solution, representing a closed curve in the phase space. A closed trajectory is called a limit cycle if at least one other trajectory converges to it.

An important problem in the qualitative theory of non-linear systems is the study of the asymptotic behaviour of all solutions as $x \to \pm\infty$. For autonomous systems of the form Equation (2.2), this problem reduces to the study of the structure of the limit sets of all the semi trajectories and the ways the trajectories approach these sets. The limit set of each semi trajectory is closed and invariant. A subset of the phase space is called invariant if it consists of complete trajectories. If a semi trajectory is bounded, then its limit set is connected.

If $n = 2$, i.e., when the phase space is a plane, Poincare and I. Bendixson have given an exhaustive description of the possible arrangements of the trajectories. Under the hypothesis that the equation $Y(y) = 0$ has only a finite number of solutions

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in any bounded part of the plane, they proved that the limit set of any bounded semi trajectory can only be one of the following three types:

(i) A single equilibrium state;

(ii) A single closed trajectory; or

(iii) A finite number of equilibrium states and trajectories converging to these equilibrium states as $x \to \pm \infty$.

Poincare and A. Denjoy considered the case of a first-order equation of the type Equation (2.1) whose right-hand side is periodic in both arguments y and x. The structure of the solutions in this case depends essentially on the rotation number, defined by the formula

$$\mu = \lim_{x \to \infty} \frac{y(x, y_0)}{x}.$$

If μ is rational, then there exists a periodic solution and if μ is irrational, then all solutions are quasi-periodic functions with two frequencies.

For $n > 2$ it is not possible to give such a clear description of the behaviour of the trajectories. There is, however, a lot of information about the limiting behaviour of higher-dimensional autonomous systems. Let a closed bounded invariant set of the phase space be called minimal if it contains no proper subset with the same properties. Then each minimal set is the closure of a recurrent trajectory. Thus, the limit set of each bounded semi-trajectory contains a recurrent trajectory.

In the important particular case when the system has an invariant measure, the study of general regularity of the behaviour of the solutions has been carried out in great detail.

Of special interest for applications are structurally-stable systems, i.e., systems which are stable under a perturbation of the right-hand sides which is small in the sense of C^1. For $n = 2$, in any bounded part of the plane there are only a finite number of periodic solutions. For $n > 2$ the behaviour of a structurally-stable system is considerably more complicated. S. Smale has given an example of a structurally-stable system having an infinite number of periodic solutions in a bounded part of the phase space.

Numerous investigations have been devoted to the study of global properties of concrete systems of differential equations. In connection with investigations in the theory of automatic control, a new branch of the qualitative theory of differential equations evolved in the 1950s, namely the theory of stability of motion in the large. An important role in the theory of oscillations is played by dissipative systems of the form (2.1) for which all solutions fall into some bounded domain as time increases. The properties of dissipative systems have been studied in great detail. Relatively reliable methods have been constructed enabling one to establish the dissipativeness of concrete systems.

One of the problems in the qualitative theory of differential equations is that of the existence of periodic solutions. For the proof of the existence of such solutions

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use is often made of topological devices, in particular the various criteria for the existence of a fixed point.

A complete qualitative study of non-linear systems of differential equations has only been achieved in very special cases. For example, it has been proved that the Lienard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ has, under very natural hypotheses, a unique periodic solution, while all its other solutions converge to this periodic one.

With regard to the Van der Pol equation with perturbation,

$$\ddot{x} + k(x^2 - 1)\dot{x} + x = kb\lambda \sin \lambda t,$$

the following interesting facts have been established for large values of the parameter *k*. For a special choice of the parameter *b* the equation has two asymptotically-stable solutions with periods $(2n+1)2\pi / \lambda$ and $(2n-1)2\pi / \lambda$, where *n* is a sufficiently large integer, and the majority of remaining solutions converge to these two. In addition, there is a countable set of unstable periodic solutions and a continuum of recurrent non-periodic ones.

Lyapunov's Method to Determine Stability for Non-linear Systems

An equilibrium point is Lyapunov stable if all solutions of the dynamical system that start out near an equilibrium point x_e stay near x_e forever. More strongly, if x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable.

The general study of the stability of solutions of differential equations is known as stability theory. Lyapunov stability theorems give only sufficient condition.

Lyapunov, in his original 1892 work proposed two methods for demonstrating stability. The first method developed the solution in a series which was then proved convergent within limits. The second method, which is almost universally used nowadays, makes use of a *Lyapunov function V(x)* which has an analogy to the potential function of classical dynamics. It is introduced as follows. Consider a function $V(x) : R^n \to R$ such that

- $V(x) \geq 0$ with equality if and only if $x = 0$ (positive definite.)
- $\dot{V}(x) = \frac{d}{dt}V(x) \leq 0$ with equality if and only if (negative definite).

Then *V(x)* is called a Lyapunov function candidate and the system is asymptotically stable in the sense of Lyapunov. $V(0) = 0$ is required otherwise $V(x) = 1/(1 + |x|)$ would prove that $\dot{x}(t) = x$ is locally stable. An additional condition called properness or radial unboundedness is required in order to conclude global asymptotic stability.

It is easier to visualize this method of analysis by thinking of a physical system (for example, vibrating spring and mass) and considering the energy of

such a system. If the system loses energy over time and the energy is never restored then eventually the system must grind to a stop and reach some final resting state. This final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.

Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, providing a Lyapunov function can be found to satisfy the constraints.

Normally non-linear first-order PDE of the form $F(Du,u,x) = 0$, where $x \in U$ and U is an open subset of R^n.

Where is $F : R^n \times R \times \overline{U} \to R$ is given and $u : \overline{U} \to R$ is the unknown, $u = u(x)$.

We can consider,

$$F = F(p,z,x) = F(p_1 .....p_n, z, x_1, ...x_n)$$

For $p \in R^n$, $z \in R, x \in U$. So that, 'p' is the name of the variable for which we substitute the gradient Du(x), and 'z' is the variable for which we substitute $u(x)$. We also assume that F is smooth and set

$$D_p F = (F_{p_1}, ...F_{p_n})$$
$$D_z F = F_z$$
$$D_x F = (F_{x_1}, ...F_{x_n}).$$

We are concerning with discovering solution u of the PDE $F(Du,u,x) = 0$ in U, usually subject to the boundary condition $u = g$ on Γ where Γ is some given subset of ∂U and g : $\Gamma \to$R is prescribed.

2.2.1 Complete Integrals

Analysis of PDE

$$F(Du, u, x) = 0 \qquad (2.3)$$

Let us consider an open set $A \subset R^n$. Assume that for each parameter $a = (a_1, \ldots, a_n) \in A$ we have a C^2 solution $u = u(x;a)$ of the partial differential Equation 2.3.

Definition of Complete Integral: A C^2 function $u = u(x;a)$ is called a complete integral in $U \times$ A provided,

1. $u(x;a)$ solves the PDE Equation (2.3) for each $a \in A$ and
2. $rank\left(D_a u, D^2_{xa} u\right) = n$, where $x \in U, a \in A$.

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Remark 1.

We use the following shorthand notation.

$$\left(D_a u, D_{xa}^2 u\right) := \begin{pmatrix} u_{a_1} & u_{x_{12}a_1} & u_{x_n a_1} \\ \vdots & \vdots & \vdots \\ u_{a_n} & u_{x_{12}a_{n1}} & u_{x_n a_n} \end{pmatrix}.$$

Thus a complete integral is a solution of a partial differential equation of the first order that contains as many arbitrary constants as there are independent variables. The second condition guarantees that the solution $u(x;a)$ depends on all of the n independent parameters a (this can be seen by assuming otherwise and then checking that the determinant of each submatrix constructed from the assumed new solution equals zero; consequently the rank is strictly less than n).

For example,

The very important Hamilton-Jacobi equation that forms an alternative formulation of classical mechanics in its simplest form is the partial differential equation

$$u_t + H(Du) = 0,$$

where $H : R^n \to R$ is a given Hamiltonian and $u = u(x)$. By setting $x = \left(x_1,..,x_n\right) \in R^n, t \in R, t = x_{n+1}$ and $Du = D_x u = \left(u_{x_1},\ldots,u_{x_n}\right)$; a complete integral of the PDE is

$$u(x,t;a,b) = a \cdot x - tH(a) + b,$$

where $a \in R^n, b \in R$ and $t \geq 0$. This can be verified by simple calculations:

$u_t = -H(a), Du = a, D_a u = x, D_{xa}^2 = diag\left(1_1,\ldots,1_n\right)$,

and s_0 $rank\left(D_a u, D_{xa}^2 u\right) = n$.

Remark 2.

The above formulation can also be acquired by assuming that the variables of u can be separated additively. Let us look for a solution of the form:

$$u(x,t) = w(x) + v(t) \quad x \in R^n, t \geq 0.$$

Now

$$0 = u_t(x,t) + H(Du(x,t)) = v'(t) + H(Dw(x))$$

if and only if

$$\Pi(Dw(x)) = \alpha = -v'(t) \quad t > 0$$

for some constant α. Consequently if $\Pi(Dw) = \alpha$ for some $\alpha \in R$, then

$$u(x,t) = w(x) - \alpha t + b$$

will solve

$$u_t + H(Du) = 0$$

for any constant b. Setting $w(x) = a \cdot x$ for some $a \in R^n$ and $\alpha = H(a)$, we get the same solution as above.

Very often the convention $t = x_{n+1}$ is used in literature to give a special meaning to the last position. This is because the parameter is usually reserved for time, which is one of the hugely important single variables used in most PDE:s. This text will follow this convention.

We shall next construct new solutions from complete integrals. These solutions will turn out to be somewhat more complicated due to the fact that they will depend on an arbitrary function of $n-1$ variables instead of just n parameters as in the definition of the complete integral. In general, the new solutions will come out as envelopes of complete integrals of other m-parameter families of solutions.

Definition

Let $u = u(x;a)$ be a C^1 function of $x \in U, a \in A$, where $U \subset R^n$ and $A \subset R^m$ are open sets. Consider the equation

$$D_a u(x;a) = 0 \tag{2.4}$$

Suppose that we can solve Equation (2.4) for the parameter a as a C^1 function of x, $a = \phi(x)$ and so:

$$D_a u(x;\phi(x)) = 0.$$

We then call

$$v(x) := u(x;\phi(x))$$

the envelope of the functions $\{u(\cdot;a)\}_{a\in A}$. It's worth noting that the function u is now assumed to be C^1 instead of C^2, which is much less restrictive. The assumption that the parameters a are a function of x also reduces the amount of independent variables from $n+m$ to just $n+1$ (since the function ϕ itself varies).

With the help of the newly defined envelopes we can now construct new solutions of the nonlinear first-order PDE.

2.2.2 Envelopes

Envelope, in mathematics, a curve, i.e., tangential to each one of a family of curves in a plane or, in three dimensions, a surface, i.e., tangent to each one of a family of

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surfaces. For example, two parallel lines are the envelope of the family of circles of the same radius having centres on a straight line.

Let $u = u(x;a)$ be a C^1 function of $x \in U$, $a \in A$, where $U \subset R^n$ and $A \subset R^m$ are open sets.

Consider the vector equation

$$D_a u\,(x;a) = 0 \ (x \in U, a \in A). \tag{2.5}$$

Assume that we can solve Equation (2.5) for the parameter a as a C^1 function of x,

$$a = \phi(x); \tag{2.6}$$

Subsequently,

$$D_a u(x;\phi(x)) = 0 \qquad (x \in U) \tag{2.7}$$

Then, we said

$$v(x) := u(x;\phi(x)) \qquad (x \in U) \tag{2.8}$$

The envelope of the functions $\{u(.;a)\}_{a \in A}$.

Example 2.1: Consider the partial differential equation is $u^2(1+|Du|^2) = 1$.

Solution: A complete integral is $u(x,a) = \pm(1-|x-a|^2)^{1/2} \qquad (|x-a| < 1)$

Then we calculate the $D_a u = \dfrac{\mp(x-a)}{(1-|x-a|^2)^{1/2}} = 0$

Provided $a = \phi(x) = x$.

Therefore $v \pm 1$ are singular integrals of Equation (2.5).

The equation f(x, y, a) = 0 represents, in general, a curve in the xy plane for any given value of a. For different values of a, the relation f(x, y, a) = 0 represents a system of curves, called a one-parameter family of curves. The curve which touches every member of the family is called the *envelope of the family* f(x, y, a) = 0 where a is a parameter. For example, consider the family of circles of a fixed radius a whose centres lie on the x-axis. when the lines y = ± a touch each member of the family

$$(x - \alpha)^2 + y^2 = a^2, \tag{2.9}$$

where α is the parameter of the family. Similarly, consider the family of straight lines

$$x \cos\alpha + y \sin\alpha = 1 \tag{2.10}$$

where α is a parameter. A unit circle touches every member of the family of straight lines.

Although the above two examples are geometrically clear, it is often not possible to visualize the envelope. We will develop a mathematical way to obtain the equation of the envelope of the family f(x, y, a) = 0.

Consider two members of the family which are infinitesimally close to each other,

$$f(x, y, a) = 0 \text{ and } f(x, y, a + \Delta a) = 0$$

These two curves intersect at a point which is also a point of intersection of

f(x, y, a)= 0 and $\frac{1}{\Delta a}[f(x, y, a + \Delta a) - f(x, y, a)] = 0$

As $\Delta a \to 0$, we see that this point of intersection tends to a limiting position given by the equation

$$f(x, y, a) = 0, \frac{\partial f}{\partial a}(x, y, a) = 0 \tag{2.11}$$

Geometrically, it is the point on the curve f(x, y, a) = 0 approaching the intersecting point of f(x, y, a) = 0 and f(x, y, a + Δa) = 0 as $\Delta a \to 0$. To understand clearly what the limiting position of the point of intersection of f(x, y, a) = 0 and f(x, y, a + Δa) = 0 mean,let us consider a simple example.

$y = mx\frac{a}{m}$ is a family of straight lines with m as the parameter (a is a constant)

$$y = (m + \Delta m)x + \frac{a}{m + \Delta m}.$$

Solving these two equations (non-parallel lines) for their point of intersection,

$$x = \frac{a}{m(m + \Delta m)} \text{ and } y = \frac{a(2m + \Delta m)}{m(m + \Delta m)}.$$

Limiting pointing of this is $x = \frac{a}{m^2}, y = \frac{2a}{m}$.

The point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ lies on $y = mx + \frac{a}{m}$.

As the parameter a varies, the point of intersection of the pair in (2.11) will trace out a curve, whose equation g(x, y) = 0 is obtained by eliminating a between the equations represented by (2.11). This curve is the envelope of the one-parameter family of curves f(x, y, a) = 0. For example, in the case of the family (2.9),

$(x - a)^2 + y^2 = a^2$, (α, a are the parameters)

$2(x - a)(-1) = 0$ (differentiating w.r.t. α)

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Eliminating α from these two relations, we get $y^2 = a^2$, which represents the pair of lines $y = \pm a$, as the envelope of the given family of circles (2.1). Similarly, for the family of straight lines,

$$x\cos\alpha + y\sin\alpha = 1 .$$

Differentiating w.r.t α,

$$-x\sin\alpha + y\cos\alpha = 0 \Rightarrow \frac{\sin\alpha}{y} = \frac{\cos\alpha}{x} = \frac{1}{\sqrt{x^2+y^2}} .$$

Thus, $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$.

Substituting these in the equation of the family, $\frac{x^2+y^2}{\sqrt{x^2+y^2}} = 1 \Rightarrow x^2 + y^2 = 1$.

Envelope of the family $a(x,y)m^2 + b(x,y)m + c(x,y) = 0$ **;** m is a parameter

Differentiating the equation of the family with respect to the parameter m,

$$2am + b = 0 \Rightarrow m = -\frac{b}{2a}$$

Substituting this in the equation of the family, $a\frac{b^2}{4a^2} - b\frac{b}{2a} + c = 0$.

That is, $b^2 - 4ac = 0$ (discriminant of the quadratic)

Then the envelope of the family is $b^2(x,y) - 4a(x,y)c(x,y) = 0$.

Example 2.2: Find the envelope of $x - y\sin\theta = a\cos\theta$, where θ is the parameter.

Solution: $f(x,y,\theta) = x - y\sin\theta - a\cos\theta = 0$

$$\frac{\partial f}{\partial\theta} = 0 \Rightarrow -y\cos\theta + a\sin\theta = 0 .$$

From which we get,

$$\frac{\sin\theta}{y} = \frac{\cos\theta}{a} = \frac{1}{\sqrt{a^2+y^2}} .$$

$$x - y\frac{y}{\sqrt{a^2 + y^2}} - a\frac{a}{\sqrt{a^2 + y^2}} = 0$$, that is,

$$x - \sqrt{a^2 + y^2} = 0.$$

That is, $x^2 - y^2 = a^2$; a rectangular hyperbola.

Example 2.3: Show that the envelope of the circles whose centres lie on the parabola $y^2 = 4ax$ and which pass through the vertex is $y^2(x + 2a) + x^3 = 0$.

Solution: $(at^2, 2at)$ is any point on the parabola which is the centre of a member of the family under consideration. Since each circle passes through (0, 0), the vertex of the parabola, the square of the radius of the circles is, $a^2t^4 + 4a^2t^2$ and the equation of a member of the family is

$$(x - at^2)^2 + (y - 2at)^2 = a^2t^2(t^2 + 4)$$ with t as parameter.

This simplifies to $x^2 + y^2 - 2axt^2 - 4ayt = 0$

To find the envelope of this family, we notice that it is a quadratic in t and thus the envelope is given by making the discriminant of the quadratic equal to zero.

Thus, the envelope is $16a^2y^2 + 4(x^2 + y^2)2ax = 0$.

That is, $y^2(x + 2a) + x^3 = 0$.

2.2.3 Characteristics

Non-linear first-order PDE $F(Du, u, x) = 0$ in U (2.12)

Focused on the boundary condition $u = g$ on Γ (2.13)

Whereas $\Gamma \subseteq \partial U$ and g: $\Gamma \to R$ are given. After this assume that F, g are '**Smooth Function'**.

Let us suppose that it is described parametrically by the function

$x(s) = (x^1(s), \ldots\ldots\ldots x^n(s))$,

The parameter's lying in some subinterval of R. Assuming u is a C^2 solution of Equation (2.12), we define also

$z(s) := u(x(s))$ (2.14)

In addition, set $p(s) := Du(x(s))$ (2.15)

i.e., $p(x) = (p^1(s), \ldots.p^n(s))$

Where $p^i(s) = u_x{}^i(x(s))$ $(i = 1, \ldots\ldots, n)$ (2.16)

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So $z(.)$ gives the values of u along the curve and $p(.)$ records the values of the gradient Du. We must choose function $x(.)$ in such a way that we can calculate $z(.)$ and $p(.)$.

For this, first differentiate for an Equation (2.16):

$$p^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s))x^j(s) \tag{2.17}$$

This expression is not too capable, since it contains the second derivatives of u.

Now differentiate PDE in Equation (2.12) with respect to x_i:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du,u,x)u_{x_j x_i} + \frac{\partial F}{\partial z}(Du,u,x)u_{x_i} + \frac{\partial F}{\partial x_i}(Du,u,x) = 0 \tag{2.18}$$

We are able to employ this identity to get rid of the 'Dangerous' second derivative terms in Equation (2.17), provided we first set.

$$x^j(s) = \frac{\partial F}{\partial p_j}(p(s),z(s),x(s)) \qquad (j = 1,.....n) \tag{2.19}$$

Supposing now in Equation (2.19) holds, we estimate Equation (2.18) at $x = x(s)$, obtaining thereby from Equations (2.14) and (2.15) the identity is,

$$\therefore \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(p(s),z(s),x(s))u_{x_i x_j}(x(s)) + \frac{\partial F}{\partial z}(p(s),z(s),x(s))p^i(s) + \frac{\partial F}{\partial x_i}(p(s),z(s),x(s)) = 0$$

Substitute the expression and Equation (2.17) into (2.15): we get

$$p^i(s) = -\frac{\partial F}{\partial x_j}(p(s),z(s),x(s)) - \frac{\partial F}{\partial z}(p(s),z(s),x(s))p^i(s) \qquad (i = 1,.....,n) \tag{2.20}$$

Finally we differentiate Equation (2.14), we get

$$\begin{aligned} \dot{z}(s) &= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}((s))\dot{x}^j(s) \\ &= \sum_{j=1}^{n} p^i(s)\frac{\partial F}{\partial p_j}(p(s),z(s),x(s)) \end{aligned} \tag{2.21}$$

The second equality holds by Equations (2.16) and (2.19).

We summarize by modifying Equation (2.19) to (2.21) in vector representation:

$$\left\{\begin{array}{l}(a)\ \dot{p}(s) = -D_x F(p(s), z(s), (s)) - D_z F(p(s), z(s), x(s))p(s) \\ (b)\ \dot{z}(s) = D_p F(p(s), z(s), (s)).p(s) \\ (c)\ \dot{x}(s) = D_p F(p(s), z(s), x(s))\end{array}\right\} \quad (2.22)$$

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This essential system of $2n+1$ first-order ODE encompasses the characteristic equations of the non-linear first-order PDE Equation (2.12). The function $p(.)$ $(p^1(.),\ldots\ldots p^n(.))$, $z(.)$, $x(.) = (x^1(.),\ldots\ldots x^n(.))$ is called the *characteristic*. We will sometimes refer to $x(.)$ as the projected characteristic: it is the projection of the full *characteristics* $(p(.), z(.), x(.)) \subset R^{2n+1}$ onto the physical region $u \subset R^n$.

Check Your Progress

1. Give the normal form of non-linear differential equation.
2. When a subset of the phase space is called invariant?
3. State the Lyapunov's method.
4. Define the complete integral.
5. What do you understand by envelopes?

2.3 HAMILTON - JACOBI EQUATIONS

The initial value problem for the Hamilton-Jacobi equation:

$$\left\{\begin{array}{l}u_t + H(Du) = 0 \quad in \quad R^n \times (0,\infty) \\ u = g \quad on\ R^n \times \{t = 0\}.\end{array}\right\}$$

Where as $u : R^n \times [0,\infty) \to R$ is the unknown, $u=u(x,t)$ and $Du = D_x u = (u_{x1},\ldots\ldots,u_{xn})$.

We are assumed the Hamiltonian $H : R^n \to R$ and the initial function is $g : R^n \to R$.

2.3.1 Calculus of Variations

Calculus of variations is a field of mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions. Basically, a functional is a mapping from a set of functions to the real numbers. Functionals are often formed as definite integrals involving unknown functions and their derivatives. The interest is in *extremal* functions that make the functional attain a maximum or minimum value or *stationary* functions where the rate of change of the functional is precisely zero. The simplest example of such a problem is to find the curve of

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shortest length or geodesic, connecting two points. If there are no constraints, the solution is obviously a straight line between the points. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious and possibly many solutions may exist. Such solutions are known as geodesics. The minimal curve problem asks us to find the function $y = u(x)$ that minimizes the arc length functional among all reasonable functions satisfying the prescribed boundary conditions.

The calculus of variations and its extensions are used to find the optimum function that gives the best value of the model and satisfies the constraints of a system. The first calculus of variations problem, the Brachistochrone problem, was posed and solved by Johannes Bernoulli in 1696. In this problem the optimum curve was determined to minimize the time traveled by a particle sliding without friction between two points. The shape of the curve between two points is to be determined to minimize the time of a particle sliding along a wire without frictional resistance. The particle is acted upon only by gravitational forces as it travels between the two points.

In mathematics, the maximum or minimum of a function was determined to be an optimal point or set of points. In the calculus of variations the maximum or minimum value of a functional is determined to be an optimal function. A functional is a function of a function and depends on the entire path of one or more functions rather than a number of discrete variables. For the calculus of variations the functional is an integral, and the function that appears in the integrand of the integral is to be selected to maximize or minimize the value of the integral. The minimum of this functional is a function $y(x)$ that gives the shortest distance between two points $[x_0, y(x_0)]$ and $[x_1, y(x_1)]$. Minimization problems that can be analysed by the calculus of variations to characterize the equilibrium configurations of almost all continuous physical systems, ranging from elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, etc.

Calculus of Variation

Suppose that $L : R^n \times R^n \to R$ is a given smooth function, then is called the **Lagrangian**.

Hence,

$$L = L(q, x) = L(q_1, ..., q_n, x_1, ..., x_n) \qquad (q, x \in R^n)$$

And,

$$\begin{Bmatrix} D_q L = (L_{q1}, ...L_{qn}) \\ D_x L = (L_{x1}, ..., L_{xn}) \end{Bmatrix}$$

Whereas 'q' is the name of the variable for which we substitute $w(s)$, and 'x' is the variable for which we substitute $w(s)$.

Now fix two points $x, y \in R^n$ and a time $t>0$, we introduce then the action functional.

$$I[w(.)] = \int_0^t L(w(s), w(s))ds$$

2.3.2 Hamilton's Ode

The **Hamilton-Jacobi equation** is a first-order non-linear partial differential equation of the form $H(x,u_x(x,\alpha,t),t)+ut(x,\alpha,t)=K(\alpha,t)$ with independent variables $(x,t)\in\mathbb{R}^n\times\mathbb{R}|$ and parameters $\alpha\in\mathbb{R}^n$ | . It has wide applications in optics, mechanics, and semi-classical quantum theory. Its solutions determine infinite families of solutions of Hamilton's ordinary differential equations, which are the equations of motion of a mechanical system or an optical system in the ray approximation.

Remember that the general form of a planar differential equation (i.e., a system of two first-order ODEs) is,

$$\begin{aligned} \dot{p} &= F(p,q,t), \\ \dot{q} &= G(p,q,t), \end{aligned} \tag{2.23}$$

Where, in keeping with a tradition in the theory of ODEs, $\dot{f}$ denotes the derivative of a quantity f with respect to the time variable t. The system is called a **Hamiltonian system** if there is a function

$$H = H(p,q,t)$$

(called the Hamiltonian associated with the system) such that the functions F and G are satisfied following equation,

$$F(p,q,t) = -\frac{\partial H}{\partial q}, \quad G(p,q,t) = \frac{\partial H}{\partial p}$$

In this case the system has the form,

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q}, \\ \dot{q} &= \frac{\partial H}{\partial p}. \end{aligned} \tag{2.24}$$

The variable p is sometimes called a generalized coordinate, and the variable q is called the generalized momentum associated to p.

For Equation (2.23) is should be Hamiltonian so we consider that F and G are continuously differentiable, it is not difficult to see that a necessary condition is that

$$\frac{\partial F}{\partial p} = -\frac{\partial G}{\partial q} \tag{2.25}$$

NOTES

Or,

$$\frac{\partial F}{\partial p} + \frac{\partial G}{\partial q} = 0$$

Since both sides of the equation are equal to $-\frac{\partial^2 H}{\partial p \partial q}$ equivalently, this condition can be written as

$$\operatorname{div} \mathbf{V} = 0,$$

Where **V** represents the planar vector field **V** = (*F, G*) (we understand the first coordinate of **V** as the '*p*-coordinate' and the second coordinate as the '*q*-coordinate'), and div **V** denotes the divergence of **V**. In physics, a vector field with this property is called divergence-free or solenoidal. Yet another way to write Equation (2.25) is,

$$\operatorname{curl} \mathbf{W} = 0,$$

Where **W** is the vector field **W** = (*G*, –*F*), and curl is the (2-dimensional version of the) curl operator, defined by curl(*A, B*) = $\frac{\partial A}{\partial q} - \frac{\partial B}{\partial p}$. A vector field with this property is called curl-free or irrotational.

Lemma 1: If the Equation (2.23) is defined on a simply connected domain, the Equation (2.25) is both necessary and sufficient for the system to be Hamiltonian.

Proof: This is a slight reformulation of a familiar fact from vector calculus that says that in a simply connected domain, a vector field **W** = (*A, B*) is curl-free iff and only iff it is conservative. A conservative vector field is one for which the line integral of the field between two points is independent of the contour connecting them, or equivalently, such that the line integral on any closed contour vanishes. Such a vector field can always be signified as **W** = ∇*H* (the gradient of *H*) for some scalar function *H*; one simply defines *H*(*p, q*) as the line integral (which for a conservative field is independent of the path of integration).

$$H(p,q) = \int_{(p_0,q_0)}^{(p,q)} \mathbf{W} \cdot \mathbf{ds} = \int_{(p_0,q_0)}^{(p,q)} A\,dp + B\,dq$$

Between some fixed but arbitrary initial point (p_0, q_0) and the point (*p, q*). The fact that **W** = ∇*H* is immediate from the fundamental theorem of calculus. In this fact, **W** = (*G*, –*F*) so the equation **W** = ∇*H* gives exactly the pair of equations F = $-\frac{\partial H}{\partial q}$, $G = \frac{\partial H}{\partial p}$ with *H* serving as the desired Hamiltonian.

Euler-Lagrange Equation

Given a function of three variables $L = L(\dot{q}, q, t)$, the differential equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{2.26}$$

is called the Euler-Lagrange equation.

Note: The notation here may be slightly confusing: for the purpose of computing L ($\dot{q}$, q, t) and finding $\frac{\partial L}{\partial \dot{q}}$ one must think of $\dot{q}$ as an independent variable that has no connection to q. But once $\frac{\partial L}{\partial \dot{q}}$ is evaluated, to apply the time-derivative d/dt one should think of $\dot{q}$ as the time-derivative of q. This leads to a second-order ordinary differential equation for the quantity q. The function L is called the 'Lagrangian'.

Equivalence of the Lagrange and Hamilton Formalisms

We now wish to show that the Euler-Lagrange equation is equivalent to the idea of a Hamiltonian system. Start with the Equation (2.26).

Denote $p = \frac{\partial L}{\partial \dot{q}}$. The Hamiltonian will be defined by,

$$H(p, q, t) = p\dot{q} - L(\dot{q}, q, t), \tag{2.27}$$

Where $\dot{q}$ is again interpreted as a symbol representing an independent variable, which is extracted from p, q, t by inverting the relation $p = \frac{\partial L}{\partial \dot{q}}$ (i.e., this relation defines a transformation from the system of variables $\dot{q}$, q, t to the system p, q, t). Then, using the chain rule we can calculate

$$\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - p\frac{\partial \dot{q}}{\partial p} = \dot{q},$$
$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\frac{dp}{dt} = -\dot{p},$$

Which shows that we indeed get the Hamiltonian system in Equation (2.24).

Going in the other direction, if we start with a Hamiltonian system, we can construct a Lagrangian by setting

$$L(\dot{q}, q, t) = p\dot{q} - H(p, q, t) \tag{2.28}$$

Where in this definition $p = p(q, \dot{q}, t)$ is interpreted as a function of the independent variables q, $\dot{q}$, t, defined by the implicit equation $\dot{q} = \frac{\partial H}{\partial p}$.

Again computing using the chain rule and the Hamiltonian Equations (2.24), we now have that

$$\frac{\partial L}{\partial \dot{q}} = p + \dot{q}\frac{\partial p}{\partial \dot{q}} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial \dot{q}} = p,$$
$$\frac{\partial L}{\partial q} = \dot{q}\frac{\partial p}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q} - \frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q} = \dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

so we have recovered the Euler-Lagrange equation (2.26).

Assume that the C^2 function $x(\cdot)$ is a critical point of the action functional, and thus solves the Euler-Lagrange equations. We define a generalized momentum

NOTES

$p(\cdot)$ corresponding to the position $x(\cdot)$ and velocity $\dot{x}(\cdot)$ by:

$$p(s) := D_q L(\dot{x}(s), x(s)), \quad 0 \le s \le t.$$

Regarding this we need to still make one important assumption:

$$\begin{cases} \text{Suppose for all } x, p \in \mathbb{R}^n \text{ that the equation} \\ p = D_v L(q, x) \text{can be } q \text{ solved for } q \\ \text{as a smooth function of } p \text{ and } x : q = q(p, x). \end{cases}$$

Definition

The Hamiltonian H associated with the Lagrangian L is

$$H(p, x) := p \cdot q(p, x) - L(q(p, x), x), \quad p, x \in R^n,$$

Where the function $q(\cdot, \cdot)$ is defined implicitly by the hypothesis.

Assume for all $x, p \in R^n$ that the equation $P = D_q L(q, x)$ can be uniquely solved for q as a smooth function of P and x, $q = q(p, x)$.

Derivation of Hamilton's ODE: The functions $x(\cdot)$ and $p(\cdot)$ satisfy the coupled system of $2n$ first-order Hamilton's equations:

$$\begin{cases} \dot{p}(s) = -D_x H(p(s), x(s)) \\ \dot{x}(s) = D_p H(p(s), x(s)) \end{cases}$$

For $0 \le s \le t$. Furthermore, the mapping $s \to H(p(s), x(s))$ is constant.

Proof: Set $x(s) = q(p(x), x(s))$

We consider as $q(.) = (q^1(.), \ldots\ldots q^n(.))$ for $i = 1, \ldots\ldots, n$:

$$\frac{\partial H}{\partial x_i}(|p, x) = \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial x_i}(q, x)$$

$$= -\frac{\partial L}{\partial x_i}(q, x)$$

Subsequently,

$$\frac{\partial H}{\partial p_i}(p, x) = q^i(p, x) + \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial p_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial p_i}(p, x)$$

$$= q^i(p, x)$$

Then,

$$\frac{\partial H}{\partial p_i}(p(s), x(s)) = q^i(p(s), x(s)) = \dot{x}^i(s);$$

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$$\frac{\partial H}{\partial p_i}(p(s),x(s)) = -\frac{\partial L}{\partial x_i}(q(p(s),x(s)),x(s)) = -\frac{\partial L}{\partial x_i}(\dot{x}(s),x(s))$$

$$= -\frac{d}{ds}\left(\frac{\partial L}{\partial q_i}(\dot{x}(s),x(s))\right)$$

$$= -\dot{p}^i(s).$$

To conclude we get that,

$$\frac{d}{ds}H(p(s),x(s)) = \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\dot{p}^i + \frac{\partial H}{\partial x_i}\dot{x}^i$$

$$= \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial x_i}\right) + \frac{\partial H}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\right) = 0$$

Theorem 2.1: The constancy of H with respect to the parameter s means that the trajectories lie on the contour lines $H(p,x) = C$. This can be viewed as the conservation of energy of the system it describes (time translation invariance). Thus the equations are particularly useful in identifying conserved quantities for mechanical systems; this being true even when the problem itself cannot be solved completely.

Proof: From $p(s) = D_v L(\dot{x}(s),x(s)), p = D_v L(v,x)$ and $v = v(p,x)$, we get that $\dot{x}(s) = v(p(s),x(s))$. Denote $v(\cdot) = \left(v^1(\cdot),\ldots,v^n(\cdot)\right)$ and compute the partial derivatives of H for $i = 1,\ldots,n$:

$$H_{x_i}(p,x) = \sum_{k=1}^{n} p_k v^k_{x_i}(p,x) - L_{v_k}(v,x)v^k_{x_i}(p,x) - L_{x_i}(v,x) = -L_{x_i}(v,x)$$

and

$$H_{p_i}(p,x) = v^i(p,x) + \sum_{k=1}^{n} p_k v^k_{p_i}(p,x) - L_{v_k}(v,x)v^k_{p_i}(p,x) = v^i(p,x),$$

Thus,

$$H_{p_i}(p(s),x(s)) = v^i(p(s),x(s)) = \dot{x}^i(s)$$

and by the Euler-Lagrange equations

$$H_{x_i}(p(s),x(s)) = -L_{x_i}(v(p(s),x(s)),x(s))$$

$$= -L_{x_i}(\dot{x}(s),x(s)) = -\frac{d}{ds}\left(L_{v_i}(\dot{x}(s),x(s))\right)$$

$$= -\dot{p}^i(s).$$

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For the last assertion simply take the derivative with respect to s and use the identities acquired above;

$$\frac{d}{ds}H(p(s),x(s)) = \sum_{i=1}^{n} H_{p_i}\dot{p}^i + H_{x_i}\dot{x}^i = \sum_{i=1}^{n} H_{p_i}\left(-H_{x_i}\right) + H_{x_i}\left(H_{p_i}\right) = 0.$$

2.3.3 Legendre Transform

Assume that the **Lagrangian** $L : R^n \to R$ satisfies these conditions:

The mapping $q \to L(q)$ is convex and $\lim_{|q|\to\infty} \frac{L(q)}{|q|} = +\infty$.

The convexity implies L is continuous.

Therefore the Legendre transform of L is,

$$L^*(p) := \sup_{q\in R^n}\{p \cdot q - L(q)\} \qquad (p \in R^n)$$

2.3.4 Hopf-Lax formula

Let us update the characteristic equations of the Hamilton-Jacobi is, for more compatly

$$\begin{cases} \dot{p}(s) = D_x H\left(p(s),x(s)\right) \\ \dot{z}(s) = D_p H\left(p(s),x(s)\right)\cdot p(x) \\ \dot{x}(s) = D_p H\left(p(s),x(s)\right) \end{cases} \tag{2.29}$$

In equation (2.29) since $H(p,x) = H(p)$, we have that

$$\begin{cases} \dot{p}(s) = 0 \\ \dot{z}(s) = D_p H(p)\cdot p(s) - H(p) \\ \dot{x}(s) = D_p H(p). \end{cases}$$

Inserting $\dot{x}$ to the expression for $\dot{z}$ and using the Legendre transform further gives us

$$\dot{z} = \dot{x}\cdot p(s) - H(p) = L(\dot{x}).$$

The characteristics provide a smooth solution u for at least short times $t > 0$ so that $z(t) = u(x(t),t)$ and therefore by integrating $\dot{z}$ we get:

$$u(x,t) = \int_0^t L(\dot{x})ds + g(x(0)).$$

We have thus turned the characteristic equations of the Hamilton-Jacobi PDE to a variational problem of the Lagrangian. This however should not come as a surprise since the calculus of variations problem led to Hamilton's ODE for the associated Hamiltonian (those being part of the characteristic equations of the Hamilton-Jacobi PDE).

Let us try to to somehow extend the solutions to times further than where the smoothness ends. Given $x \in \mathbb{R}^n$ and $t > 0$, our goal is to minimize the modified action

$$\int_0^t L(\dot{w}(s))ds + g(w(0))$$

among curves $w(\cdot)$ satisfying $w(t) = x$. We define

$$u(x,t) := \inf\left\{\int_0^t L(\dot{w}(s))ds + g(w(0)) \mid w(t) = x\right\}, \tag{2.30}$$

where the infimum is taken over all C^1 functions $w(\cdot)$. So in what sense does u defined by Equation (2.30) actually solve the Hamilt on-Jacobi PDE. Recall that we are assuming that H is smooth, convex and superlinear. Furthermore we assume that the initial condition satisfies:

$g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous,

with

$$Lip(g) := \sup_{x,y\in\mathbb{R}^n, x\neq y}\left\{\frac{|g(x)-g(y)|}{|x-y|}\right\} < \infty.$$

The vector p is constant along the characteristic curve, and so $\dot{x} = D_p H$ is a constant vector, and therefore the projected characteristics $x(t)$ are straight lines. Thus if $x(0) = y$ and $x(t) = x$, we must have $\dot{x} = \frac{x-y}{t}$ and as a consequence:

$$\dot{z} = L(\dot{x}) = L\left(\frac{x-y}{t}\right) \Rightarrow z(t) = z(0) + tL\left(\frac{x-y}{t}\right) = g(y) + tL\left(\frac{x-y}{t}\right).$$

The only unknown term in this expression is y, so the problem turns into minimizing $z(t)$. The intuition behind this is the following: imagine that instead of starting from time $t = 0$, we start at say $t = -1$. All of our trajectories now start from $t = -1$, pass y at $t = 0$, and end at x at some defined $t > 0$. The initial function g can be thought of tracking the work done from point $t = -1$ to point $t = 0$; so now the correct trajectory is the one minimizing this work.

Theorem 2.2: (Hopf-Lax formula) If $x \in \mathbb{R}^n$ and $t > 0$, then the solution $u = u(x,t)$ of the minimization problem in Equation (2.30) is

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}. \tag{2.31}$$

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Proof: Fix any $y \in \mathbb{R}^n$ and define $w(s) := y + \frac{s}{t}(x-y)$, where $0 \le s \le t$. Then Equation (2.30) implies

$$u(x,t) \le \int_0^t L(\dot{w})ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y),$$

which implies that

$$u(x,t) \le \inf_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}.$$

For the other direction let $w(\cdot)$ be any C^1 function satisfying $w(t) = x$. Since L is convex, by Jensen's inequality we have that

$$L\left(\frac{1}{t}\int_0^t \dot{w}(s)ds\right) \le \frac{1}{t}\int_0^t L(\dot{w}(s))ds.$$

Defining $y = w(0)$ and adding $g(y)$ to both sides, we get

$$tL\left(\frac{x-y}{t}\right) + g(y) \le \int_0^t L(\dot{w}(s))ds + g(y).$$

Taking infimum on both sides now gives us

$$\inf_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\} \le u(x,t).$$

Fix *t* and *x* and define function *h* by,

$$h(y) = tL\left(\frac{x-y}{t}\right) + g(y),\ y \in \mathbb{R}^n.$$

The final claim is then that in the expression for u, the infimum can be replaced by an actual minimum. That is:

$$u(x,t) = \inf_{y\in\mathbb{R}}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\} = \min_{y\in\mathbb{R}}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}.$$

Fix *t* and *x* is define the function of *h* by,

$$h(y) = tL\left(\frac{x-y}{t}\right) + g(y), y \in \mathbb{R}^n$$

The function h is continuous as a sum of continuous functions, and since g is Lipschitz, we have the estimate:

$$\begin{aligned} h(y) &\ge tL\left(\frac{x-y}{t}\right) - \| g \|_{Lip} | x-y | - | g(x) | \\ &= | x-y | \left(\frac{L\left(\frac{x-y}{t}\right)}{\frac{|x-y|}{t}} - \| g \|_{Lip} - \frac{|g(x)|}{|x-y|} \right) \end{aligned}$$

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Let $|y| \to \infty$ and notice that $\frac{L\left(\frac{x-y}{t}\right)}{\frac{|x-1|}{t}|} = \infty$ by superlinearity, with $\frac{|g(x)|}{|x-y|} = 0$ clearly. This implies that $h(y) \to \infty$, and that there exists an $M > 0$ such that if $|y| > M$, then $h(y) > u(x,t) + \lambda$, where $\lambda > 0$ is fixed. Now by the extreme value theorem h attains its minimum on $\bar{B}(0,M)$ at some point y_0, and so $\inf_{|y| \le M} h(y) \ge h(y_0)$. By the definition of the infimum there exists a $\bar{y}' \in \bar{B}(0,M)$ such that $h(y') < u(x,t) + \lambda/2$. Now:

$$\inf_{|y|>M} h(y) \ge u(x,t) + \lambda > u(x,t) + \lambda/2 > h(y') \ge h(y_0).$$

Combining both estimates we have that $u(x,t) \ge h(y_0)$, and hence the minimum is attained for every fixed x and t.

Remark: In fact the Hopf-Lax Equation (2.31) is describe

$$u(x,t) = \min_{y \in B(x,Rt)} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\},$$

for $R = \sup_{\mathbb{R}^n} |DH(Dg)|$ and $L = H^*$. The minimizer y is thus always bounded with respect to the time t, and therefore shows the finite propagation speed for the Hamilton-Jacobi equations with convex Hamiltonian and Lipschitz continuous initial function g. This can be proven via the subdifferential of H.

The explicit formula of the Hopf-Lax was possible because of the special structure in the Hamilton-Jacobi characteristic equations, namely the Hamilton's equations, that allowed us to turn the ODE into a variational problem. Because of this explicitness, the formula has some useful properties. Our ultimate goal is to show that the formula produces a reasonably defined weak solution of the initial-value problem for the Hamilton-Jacobi equation, since it does not in general have a smooth solution u lasting for all times $t > 0$.

Lemma 1: (Functional Identity). For each $x \in \mathbb{R}^n$ and $0 \le s \le t$, we have

$$u(x,t) = \min_{y \in R^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}.$$

Proof omitted.

This identity tells us that to compute $u(\cdot,t)$), we can just calculate u at time s and then use $u(\cdot,s))$ as the initial condition on the remaining time interval $[s,t]$.

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Lemma 4: The function u is Lipschitz continuous in $\mathbb{R}^n \times [0,\infty)$, and

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

Proof omitted:

Now Rademacher's theorem says that a Lipschitz function is differentiable almost everywhere, so u defined by the Hopf-Lax formula in Equation (2.31) is differentiable almost everywhere (a.e.), $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Now we are ready to show that the Hopf-Lax formula actually provides us a solution to the Hamilton-Jacobi PDE wherever u is differentiable.

Theorem 2.3: Suppose $x \in \mathbb{R}^n, t > 0$, and u defined by the Hopf-Lax formula Equation (2.31) is differentiable at a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Then

$$u_t(x,t) + H(Du(x,t)) = 0.$$

Proof: Fix $v \in \mathbb{R}^n, h > 0$. By Lemma 1 we have

$$u(x+hv, t+h) = \min_{y \in \mathbb{R}^n} \left\{ hL\left(\frac{x+hv-y}{h}\right) + u(y,t) \right\} \le hL(v) + u(x,t),$$

where we chose $x = y$ (not necessarily the minimizer). This gives us the difference quotient

$$\frac{u(x+hv,t+h) - u(x,t)}{h} \le L(v).$$

Let $h \to 0^+$ so that the directional derivative becomes

$$v \cdot Du(x,t) + u_t(x,t) \le L(v).$$

This inequality is valid for all $v \in \mathbb{R}^n$, so by $H = L^*$ we have that:

$$u_t(x,t) + H(Du(x,t)) = u_t(x,t) + \max_{v \in \mathbb{R}^n} \{v \cdot Du(x,t) - L(v)\} \le 0.$$

For the other direction we choose a $z \in \mathbb{R}^n$ such that it is the minimizer of the Hopf-Lax formula: $u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$. Fix $h > 0$ and set $s = t - h$ and $y = \frac{s}{t}x + \left(1 - \frac{s}{t}\right)z$, so that $\frac{x-z}{t} = \frac{y-z}{s}$, and thus:

$$\begin{aligned} u(x,t) - u(y,s) &\ge tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z) \right] \\ &= (t-s)L\left(\frac{x-z}{t}\right). \end{aligned}$$

This again gives us a difference quotient by reorganization:

$$\frac{u(x,t)-u\left(\left(1-\frac{h}{t}\right)x+\frac{h}{t}z,t-h\right)}{h}\geq L\left(\frac{x-z}{t}\right).$$

Since $\left(1-\frac{h}{t}\right)x+\frac{h}{t}z=x-h\left(\frac{x-z}{t}\right)$, we can let $h\to 0^{+}$ to achieve

$$\frac{x-z}{t}\cdot Du(x,t)+u_t(x,t)\geq L\left(\frac{x-z}{t}\right),$$

and so

$$\begin{aligned} u_t(x,t)+H(Du(x,t)) &= u_t(x,t)+\max_{v\in R^n}\{v\cdot Du(x,t)-L(v)\} \\ &\geq u_t(x,t)+\frac{x-z}{t}\cdot Du(x,t)-L\left(\frac{x-z}{t}\right) \\ &\geq 0. \end{aligned}$$

This inequality completes the proof.

From this follows a useful lemma for comparing solutions based on their initial functions. The solution u depends monotonically on g : if the initialdata g is increased pointwise, then so is the solution u.

Lemma 2: L^{∞} (Contraction Inequality): Let u^1 and u^2 be two solutions of the initial value problems

$$\begin{cases} u_t^i+H\left(Du^i\right)=0 & \text{a.e.in } \mathbb{R}^n\times(0,\infty) \\ u^i=g^i & \text{on } \mathbb{R}^n\times\{t=0\}(i=1,2), \end{cases}$$

given by the Hopf-Lax formula. Then we have the following inequality

$$\sup_{\mathbb{R}^n}\left|u^1(\cdot,t)-u^2(\cdot,t)\right|\leq\sup_{\mathbb{R}^n}\left|g^1-g^2\right|,\quad \text{for } t>0.$$

Furthermore if $g^2\leq g^1$, then $u^2\leq u^1$.

Proof: By the Hopf-Lax formula we have:

$$u^1(x,t)\leq\left\{tL\left(\frac{x-y}{t}\right)+g^1(y)\right\}\text{and } u^2(x,t)\leq\left\{tL\left(\frac{x-y}{t}\right)+g^2(y)\right\},$$

where equalities hold with y_1 and y_2 respectively. Especially we have $u^1(x,t)\leq\left\{tL\left(\frac{x-y_2}{t}\right)+g^1\left(y_2\right)\right\}$ and $u^2(x,t)\leq\left\{tL\left(\frac{x-y_1}{t}\right)+g^2\left(y_1\right)\right\}$. From these we have the estimates $u^1(x,t)-u^2(x,t)\leq g^1\left(y_2\right)-g^2\left(y_2\right)$ and

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$u^1(x,t) - u^2(x,t) \geq g^1(y_1) - g^2(y_1)$. Taking supremum over both inequalities gives us the first claim.

The second claim is immediate from our second inequality above combined with the assumption $g^2 \leq g^1$.

Let us summarize the section with the Hopf-Lax formula theorem.

Theorem 2.4: The function u defined by the Hopf-Lax formula in Equation (2.31) is Lipschitz continuous, differentiable a.e. in $\mathbb{R}^n \times (0,\infty)$, and solves the initial-value problem

$$\begin{cases} u_t + H(Du) = 0 & \text{a.e.in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Remark: By using the Legendre transform on L, we can write the Hopf-Lax formula for Equation (2.31) as,

$$u(x,t) = \min_{y \in \mathbb{R}^n} \max_{z \in \mathbb{R}^n} \{z \cdot (x-y) - tH(z) + g(y)\}.$$

For each fixed y, z this function solves the PDE . Thus the Hopf Lax formula builds a solution of by taking appropriate two parameter envelopes of these functions using minima and maxima. This is evident since for a fixed pair y, z (with the choices $a = z$ and $b = g(y) - z \cdot y$).

2.3.5 Weak Solution

In this section we show that semi-concavity conditions of the kinds discovered for the Hopf-Lax solution.

Definition 1: A Lipschitz continuous function $u : R^n \times [0,\infty) \to R$ is a weak solution of the initial value problem:

$$\left\{ \begin{array}{rl} u_t + H(Du) = 0 & in \quad R^n \times (0,\infty) \\ u = g & on \quad R^n \times \{t = 0\} \end{array} \right\} \tag{2.32}$$

Provided

- $u(x,0) = g(x) \qquad (x \in R^n),$
- $u_t(x,t) + H(Du(x,t)) = 0 \qquad for \;\; a.e.\; (x,t) \in R^n \times (0,\infty),$
- $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C(1 + \frac{1}{t})|z|^2$

For some constant $C \geq 0$ and all $x, z \in R^n, t > 0$.

Considering at the theorem 2.4, it is tempting to define a suitable weak solution *u* in a similar manner. This definition would however produce problems with uniqueness of the solutions as the Example 2.4 shows.

Example 2.4: Consider the initial value problem is,

$$\begin{cases} u_t + |u_x|^2 = 0 \ \ in \ \mathbb{R} \times (0, \infty) \\ u = 0 \ \ on \ \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Besides the trivial solution $u_0(x; t) \equiv 0$; we have

$$u_1(x,t) = \begin{cases} 0 & if \ |x| \geq t \\ x - t & if \ 0 \leq x \leq t \\ -x - t & if \ -t \leq x \leq 0 \end{cases}$$

which is, Lipschitz continuous and solves the PDE everywhere except on the lines $x = 0; \pm t$. There are actually infinitely many Lipschitz functions satisfying Equation (1) as can be seen from this family of solutions:

$$u_a(x,t) = \begin{cases} 0 & if \ |x| \geq t \\ ax - a^2 t & if \ 0 \leq x \leq t \\ -ax - a^2 t & if \ -t \leq x \leq 0, \end{cases}$$

Where $a \in \mathbb{R}$.

Hence for uniqueness we must require more than just solvability of the PDE a.e and Lipschitz continuity of *g*. The next lemma shows that *u* will inherit a form of one-sided second-derivative estimate from the initial function *g*, granted that *g* be semiconcave. Semiconcavity will turn out to be a sufficient condition for the uniqueness to hold.

Lemma 3 (**Semiconcavity**): Suppose there exist a constant *C* such that

$$g(x + y) - 2g(x) + g(x - z) \leq C|z|^2 \tag{2.33}$$

For all $x, z \in \mathbb{R}^n$. Let *u* be defined by the Hopf-Lax Equation (2.31).

Then,

$$u(x + z, t) - 2u(x, t) + u(x - z, t) \leq C|z|^2$$

For all $x, z \in \mathbb{R}^n, \ t > 0$.

Proof omitted.

A function *g* is called semiconcave provided that in Equation (2.33) holds. If we were to assume that *g* was twice continuously differentiable with $\sup_{\mathbb{R}^n} |D^2 g| < \infty$, then the semiconcavity condition would automatically hold another characterisation would be that *g* is semiconcave if and only if the mapping $x \to g(x) - \frac{C}{2}|x|^2$ is concave for some constant C$\in \mathbb{R}$.

NOTES

Definition 2: A convex C^2 function $H : \mathbb{R}^n \to \mathbb{R}$ is called uniformly convex with a constant $\theta > 0$ if

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \quad for\ all\ p, \xi \in \mathbb{R}^n \tag{2.34}$$

Without g being semiconcave, the uniform convexity of H will force u to become semiconcave for $t > 0$ which is exactly what happened before with g. This regularizes the Hopf-Lax and provides uniqueness of the solution.

Lemma 4 (**Semiconcavity**): Suppose that H is uniformly convex with a constant θ and u is defined by the Hopf-Lax formula.

Then,

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leq \frac{1}{\theta t}|z|^2$$

For all $x, z \in \mathbb{R}^n,\ t > 0$.

Proof omitted. These semiconcavity conditions will ensure unique solutions from the Hopf-Lax formula.

Definition 3: We say that a Lipschitz continuous function u: $\mathbb{R}^n \times [0, \infty)$ $\mathbb{R}$ is a weak solution of the initial-value problem:

$$\begin{cases} u_t + H(Du) = 0 & in\ \mathbb{R}^n \times (0, \infty) \\ u = g & on\ \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{2.35}$$

Provided

- $u(x, 0) = g(x) \quad for\ x \in \mathbb{R}^n,$
- $u_t(x,t) + H(Du(x,t)) = 0 \quad for\ a.e.\ (x,t) \in \mathbb{R}^n \times (0,\infty)\ and$
- $u(x+z,t) - 2u(x,t) + u(x-z,t) \leq C(1 + \frac{1}{t})|z|^2$

for some constant $C \geq 0\ and\ all\ x, z \in \mathbb{R}^n,\ t > 0$.

Uniqueness of Weak Solution

Assume H is C^2 and H is convex and $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$ and satisfies g: $R^n \to R$ is Lipschitz continuous. Then there exists at most one weak solution of the initial-value problem in Equation (2.32).

Proof 1: Suppose that u and $\bar{u}$ are two weak solutions of an Equation (2.32) and write w:= u- $\bar{u}$.

NOTES

Observe now at any point (*y, s*) where both *u* and are differentiable and solve our PDE, we have

$$\begin{aligned} w_t(y,s) &= u_t(y,s) - \overline{u_t}(y,s) \\ &= -H(Du(y,s)) + H(D\bar{u}(y,s)) \\ &= -\int_0^t \frac{d}{dr} H(rDu(y,s) + (1-r)D\bar{u}(y,s))dr \\ &= -\int_0^t D\,H(rDu(y,s) + (1-r)D\bar{u}(y,s)dr \,.\, (Du(y,s) - D\bar{u}(y,s)) \\ &: -b(y,s).Dw(y,s). \end{aligned}$$

Consequently,

$w_t + b.\ Dw = 0$ a.e. (2.36)

2. Write $v := \phi(w) \geq 0$, Where $\phi : R \to [0,\infty)$ is smooth function to be selected later, we multiply in Equation (3.36) by to discover

$$v_t + b.\ Dv = 0 \tag{2.37}$$

3. Now choose ε>0 and define $u^\varepsilon := \eta_\varepsilon * u,\ \bar{u}^\varepsilon := \eta_\varepsilon * \bar{u}$ where η_ε is the standard mollifier in the *x* and *t* variables.

$$\left|Du^\varepsilon\right| \leq Lip(u), \quad \left|D\bar{u}^\varepsilon\right| \leq Lip(\bar{u}) \tag{2.38}$$

Or,

$$Du^\varepsilon \to Du, \quad D\bar{u}^\varepsilon \to D\bar{u} \text{ a.e., as } \varepsilon \to 0 \tag{2.39}$$

Furthermore inequality (c) in the definition of weak solution implies

$$D^2u^\varepsilon,\ D^2\bar{u}^\varepsilon \leq C\left(1+\frac{1}{s}\right)I \tag{2.40}$$

For an appropriate constant *C* and all $\varepsilon \geq 0,\ y \in R^n,\ s > 2\varepsilon$.

4. Write $b_\varepsilon(y,s) : \int_0^1 DH(rDu^\varepsilon(g,s) + (1-r)D\bar{u}^\varepsilon(ys)dr$

Consequently,

In Equation (3.37) becomes as follow

$v_t + b_\varepsilon.\ Dv = (b_\varepsilon - b).Dv$

Then,

$v_t + div(vb_\varepsilon) = (divb_\varepsilon)v + (b_\varepsilon - b).Dv$

NOTES

5. Let

$$divb_\varepsilon = \int_0^1 \sum_{k,l=1}^{n} H_{p_k p_l}(rDu^\varepsilon + (1-r)D\tilde{u}^\varepsilon)\,(ru^\varepsilon{}_{x_l x_k} + (1-r)\tilde{u}^\varepsilon{}_{x_l x_k})dr$$

$$\leq C\left(1+\frac{1}{s}\right)$$

For some constant C, in view of (2.38), (2.40). Here we note that H convex implies $D^2 H \geq 0$.

6. Fix $x_0 \in R^n$, $t_0 > 0$ and set $R := \max\{|DH(p)| \, | \, p| \leq \max(Lip(u), Lip(\tilde{u}))\}$

Define also the cone is,

$$C : \{(x,t) : 0 \leq t \leq t_0, \, |x - x_0| \leq R(t_0 - t)\}$$

Then,

$$e(t) = \int_{B(x_0, R(t_0-t))} v(x,t)\,dx$$

So that calculate for a.e. $t > 0$:

$$\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,R(t_0-t))} v_t\,dx - R\int_{\partial B(x_0,R(t_0-t))} v\,dS \\
&= \int_{B(x_0,R(t_0-t))} -div(vb_\varepsilon) + (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx - R\int_{\partial B(x_0,R(t_0-t))} v\,dS \\
&= -\int_{\partial B(x_0,R(t_0-t))} v(b_\varepsilon.\nu + R)dS + \int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx \\
&\leq \int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx \\
&\leq C(1+\frac{1}{t})e(t) + \int_{B(x_0,R(t_0-t))} (b_\varepsilon - b).Dv\,dx
\end{aligned}$$

The last term on the right hand side goes to zero as $\varepsilon \to 0$, for a.e. $t > 0$, according to Equations (2.38) and (2.39) we get the dominated convergence theorem,

Thus,

$$\dot{e}(t) \leq C(1+\frac{1}{t})e(t) \qquad \text{for a.e. } 0 < t < t_0 \tag{2.41}$$

7. Fix $0 < \varepsilon < r < t_0$ and choose the function $\phi(z)$ to equal zero if $|z| \leq \varepsilon[Lip(u) + Lip(\tilde{u})]$ and to be positive otherwise.

Since $u \equiv \tilde{u}$ on $R^n \times \{t = 0\}$

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \quad at \quad \{t = \varepsilon\}.$$

Thus $e(\varepsilon) = 0$

Subsequently,

Gronwall's inequality and in Equation (2.42) imply $e(r) \le e(\varepsilon)e^{\int_\varepsilon^r C(1+\frac{1}{s})ds} = 0$

Hence,

$$|u - \tilde{u}| \le \varepsilon[Lip(u) + Lip(\tilde{u})] \qquad on \ B(x_0, R(t_0 - r))$$

This inequality is valid for all ε>0 and $u \equiv \tilde{u}$ in $B(x_0, R(t_0 - r))$ so

Consequently, in particular, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$.

Check Your Progress

6. Give the initial value problem for the Hamilton-Jacobi equation.
7. What do you understand by calculus of variation?
8. What is geodesics?
9. Define the Hamilton-Jacobi equation in first-order non-linear PDE.
10. State the weak solution of the initial value problem.

2.4 REPRESENTATION OF SOLUTION

An Ordinary Differential Equation (ODE) is an equation that includes some ordinary derivatives (as opposed to partial derivatives) of a function. Often, our goal is to solve an ODE, i.e., determine what function or functions satisfy the equation.

We know that what is derivative of a function and how we can determine the function itself. We need to determine the antiderivative. For example, if you are given

$$\frac{dx}{dt}(t) = \cos t$$

then what is the function $x(t)$ Since the antiderivative of cos t is sin t, then $x(t)$ must be sin t. But there is always an arbitrary constant that we cannot determine if we only know the derivative. Therefore, all we can detetermine from the above equation is that

$$x(t) = \sin t + C$$

NOTES

for some arbitrary constant C. We can verify that indeed $x(t)$ satisfies the equation dx/dt=cos t.

In general, solving an ODE is more complicated than simple integration. Even so, the basic principle is always integration, as we need to go from derivative to function. Usually, the difficult part is determining what integration we need to do.

2.4.1 Separation of Variables

The method of separation of variables tries to construct a solution u to a given partial differential equation as some sort of combination of functions of fewer variables.

Example 2.5: Let us consider $U \subset R^n$ be a bounded, open set with smooth boundary. Then find the solution of initial-value problem for the heat equation by the separation of variables.

$$\left\{\begin{array}{rl} u_t - \Delta u = 0 & in \;\; U \times (0,\infty) \\ u = 0 & on \;\; \partial U \times [0,\infty) \\ u = g & on \;\; U \times \{t = 0\} \end{array}\right\} \tag{1}$$

Here $g : U \to R$ is given.

Solution: Assume there exists a solution having the multiplicative form

$$u(x,t) = v(t)w(x) \qquad (x \in U, \;\; t \geq 0); \tag{2}$$

So that,

$$u_t(x,t) = v'(t)w(x) \quad and \quad \Delta u(x,t) = v(t)\Delta w(x)$$

Consequently,

$$0 = u_t(x,t) - \Delta u(x,t) = v'(t)w(x) - v(t)\Delta w(x)$$

iff,

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} \tag{3}$$

for all $x \in u$ and $t > 0$ such that $w(x), v(t) \neq 0$.

Let us suppose that

$$\frac{v'(t)}{v(t)} = \mu = \frac{\Delta w(x)}{w(x)} \qquad (t \geq 0, \;\; x \in U)$$

NOTES

So that,

$$v' = \mu v \tag{4}$$

$$\Delta w = \mu w \tag{5}$$

If μ is known, the solution of Equation (4) is $v = de^{\mu t}$ for an arbitrary constant d. Consequently we need only investigate equation (5).

Let λ is an eigen value of the operator -Δ on U provided there exists a function w, not identically equal to zero, solving

$$\left\{\begin{array}{l} -\Delta w = \lambda w \quad in \quad U \\ \quad w = 0 \quad on \quad \partial U \end{array}\right\}$$

Where w is corresponding Eigen function. Hence if λ an Eigen value and w is corresponding Eigen function then we set $u = -\lambda$ above, to find

$$u = de^{-\lambda t} w \tag{6}$$

Solves

$$\left\{\begin{array}{l} u_t - \Delta u = 0 \quad in \quad U \times (0,\infty) \\ \quad u = 0 \quad on \quad \partial U \times [0,\infty) \end{array}\right\} \tag{7}$$

Thus the function u defined by Equation (6) solves problem Equation (1) ,provided $g = dw$.

More generally, if $\lambda_1, \ldots\ldots \lambda_m$ are Eigen values, $w_1, \ldots\ldots w_m$ corresponding Eigen functions and $d_1, \ldots\ldots, d_m$ are constants, then

$$u = \sum_{k=1}^{m} d_k e^{-\lambda_k t} w_k$$

Solves (7) with the initial condition,

$$u(.,0) = \sum_{k=i}^{m} d_k w_k$$

If we can find m, ... w_1 etc. such that

$$\sum_{k=i}^{m} d_k w_k = g$$

We can hope generalize further by trying to find a countable sequence, $\lambda_1 \ldots\ldots \lambda_n$ are Eigen values, $w_1, \ldots\ldots w_n$ corresponding Eigen functions and $d_1, \ldots\ldots$ d_1 are constants,

NOTES

So that,

$$\sum_{k=i}^{\infty} d_k w_k = g \text{ in U}$$

Hence,

$$u = \sum_{k=1}^{\infty} d_k e^{-\lambda_k t} w_k$$

Will be the solution of the initial-value problem (1). This is an attractive *representation formula for the solution*.

2.4.2 Similarity Solution

For linear partial differential equations there are numerous techniques for reducing the PDE to an ODE (or at least a PDE in a smaller number of independent variables). These include various integral transforms and Eigen function expansions. Such techniques are much less predominant in dealing with non-linear PDE's. However, there is an approach which identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately.

Plane and Traveling Waves

Let us the first a partial differential equation involving the two variables $x \in R,\ t \in R$. A solution u of the form is,

$$u(x,t) = v(x - \sigma t) \qquad (x \in R,\ \ t \in R) \tag{2.42}$$

Equation (2.42) is called a 'Traveling Wave'. More generally, a solution u of a PDE in the n+1 variables

$x = (x = (x_1, \ldots\ldots x_n) \in R^n, t \in R$ having the form

$$u(x,t) = v(y.x - \sigma t) \qquad (x \in R^n,\ \ t \in R) \tag{2.43}$$

Equation (2.43) is called a 'Plane Wave'.

Soliton

Exponential Solution: A complex-valued plane wave solutions of the form $u(x,t) = e^{i(y.x+\omega t)}$ (2.43(a))

Where $\omega \in C$ and $y = (y_1 \ldots\ldots y_n) \in R^n$, ω being the frequency and $\{y_i\}_{i=1}^n$ the waves numbers.

Consider the Korteweg-de Vries (KDV) equation in the form is,

$$u_t + 6uu_x + u_{xxx} = 0 \quad in \quad R \times (0, \infty) \tag{2.44}$$

NOTES

This non-linear dispersive equation being a model for surface waves in water. A traveling wave solution having the structure

$$u(x,t) = v(x-\sigma t) \quad (x \in R, t > 0). \tag{2.45}$$

Then u solves the KDV Equation (2.43) ,provided *v* satisfies the ODE

$$-\sigma v' + 6vv' + v''' = 0 \qquad (' = \frac{d}{ds}) \tag{2.46}$$

Integrate in Equation (2.46) by first stating

$$-\sigma v + 3v^2 + v'' = a \tag{2.47}$$

a denotes some constant. Multiply this equality by *v*' to obtain

$$-\sigma v v' + 3v^2 v' + v'' v' = av'$$

And so deduce

$$\frac{(v')^2}{2} = -v^3 + \frac{\sigma}{2}v^2 + av + b \tag{2.48}$$

Where *b* is another arbitrary constant. We examine in Equation (2.48) by observing now only for solutions *v* which satisfy is,

$$v, v', v'' \to 0 \ \ as \pm\infty$$

Then by Equations (2.47) and (2.48) imply $a = b = 0$. Equation (2.48) there upon simplifies to read

$$\frac{(v')^2}{2} = v^2(-v + \frac{\sigma}{2}).$$

.

Hence $v' = \pm v(\sigma - 2v)^{1/2}$

We take the minus sign above for computational convenience, and obtain then this implicit formula

$$s = -\int_1^{v(s)} \frac{dz}{z(\sigma - 2z)^{1/2}} + c \quad \text{for v} \tag{2.49}$$

for some constant c.

Now substitute $z = \frac{\sigma}{2}\sec h^2\theta$. It follows that $\frac{dz}{d\theta} = -\sigma \sec h^2\theta \tanh\theta$ and

$$z(\sigma - 2z)^{1/2} = \frac{\sigma^{\frac{3}{2}}}{2}\sec h^2\theta \tanh\theta.$$

NOTES

So that Equation (2.49) is become

$$s = \frac{2}{\sqrt{\sigma}}\theta + c, \tag{2.50}$$

Where θ is implicitly given by the relation

$$\frac{\sigma}{2}\sec h^2\theta = v(s) \tag{2.51}$$

We lastly combine (2.50) and (2.51), to calculate

$$v(s) = \frac{\sigma}{2}\sec h^2\left(\frac{\sqrt{\sigma}}{2}(s-c)\right) \qquad (s \in R). \tag{2.52}$$

On the other hand, it is routine to check *v* so defined actually solves the ODE in Equation (2.46). The outcome is that

$$u(x,t) = \frac{\sigma}{2}\sec h^2\left(\frac{\sqrt{\sigma}}{2}(x-\sigma t-c)\right) \qquad (x \in R, t \geq 0)$$

is a solution of the KDV equation for each $c \in R$, $\sigma > 0$. A solution of this form is called a **'Soliton'**.

Traveling Waves: The scalar reaction-diffusion equation is,

$$u_{tt} - u_{xx} = f(u) \qquad in \quad R \times (0,\infty) \tag{2.53}$$

Where $f : R \to R$ has a *cubic-like shape*.

Similarity under Scaling

Consider porous medium equation is,

$$u_t - \Delta(u^\gamma) = 0 \qquad in \quad R^n \times (0,\infty) \tag{2.54}$$

Where $u \geq 0$ and $\gamma > 1$ are constant.

From the fundamental solution of the heat equation, a solution *u* having the form,

$$u(x,t) = \frac{1}{t^\alpha}v(\frac{x}{t^\beta}) \qquad (x \in R^n, t > 0) \tag{2.55}$$

Where the constants α, β and the function $v : R^n \to R$ must be determined.

Solution *u* of Equation (2.54) invariant under the dilation scaling

$$u(x,t) \to \lambda^a u(\lambda^\beta x, \lambda t)$$

So that,

$$u(x,t) = \lambda^a u(\lambda^\beta x, \lambda t) \text{ for all } \lambda > 0,\ x \in R^n,\ t > 0$$

We obtain Equation (2.54) into (2.55),

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y.Dv(y) + t^{-(\alpha\gamma+2\beta)} \Delta(v^{\gamma})(y) = 0 \quad (2.56)$$

For $y = t^{-\beta} x$.

In order to convert Equation (2.56) into an expression involving the variable *y* alone.

NOTES

Let us assume that,

$$\alpha + 1 = \alpha\gamma + 2\beta \quad (2.57)$$

Then Equation (2.56) reduces to

$$\alpha v + \beta y.Dv + \Delta(v^{\gamma}) = 0 \quad (2.58)$$

At this point we have effected a reduction from *n*+1 to *n* variables. We simplify further by supposing *v* is radial, i.e. $v(y) = w(|y|)$ for some $w : R \to R$

Then Equation (2.58) becomes is,

$$\alpha w + \beta r w' + (w^{\gamma})'' + \frac{n-1}{r}(w^{\gamma})' = 0 \quad (2.59)$$

Whereas,

$$r = |y|, \quad and \quad ' = \frac{d}{dr}$$

So that,

$$\alpha = n\beta \quad (2.60)$$

Equation (2.59) thereupon simplifies to read

$$(r^{n-1}(w^{\gamma})')' + \beta(r^{n} w)' = 0$$

Thus,

$$r^{n-1}(w^{\gamma})' + \beta(r^{n} w) = a$$

for some constant *a*.

Supposing $\lim_{r\to\infty} w, w' = 0$, we conclude **a**=0,

Whereas,

$$(w^{\gamma})' = -\beta r w$$

But then,

$$(w^{\gamma-1})' = -\frac{\gamma-1}{\gamma}\beta r$$

NOTES

Subsequently,

$$w^{\gamma-1} = b - \frac{\gamma-1}{2\gamma}\beta r^2$$

Where b is constant and so that

$$w = \left(b - \frac{\gamma-1}{2\gamma}\beta r^2 \right)^{+\frac{1}{\gamma-1}} \tag{2.61}$$

Where we took the positive part of the right hand side of Equation (2.61) to ensure $w \geq 0$. Remembering $v(y) = w(r)$ and Equation (2.55), we obtain

$$u(x,t) = \frac{1}{t^{\alpha}}\left(b - \frac{\gamma-1}{2\gamma}\beta\frac{|x|^2}{t^{2\beta}} \right)^{+\frac{1}{\gamma-1}} \qquad (x \in R^n,\ t > 0) \tag{2.62}$$

Where from Equations (2.57) and (2.60),

$$\alpha = \frac{n}{n(\gamma-1)+2}, \qquad \beta = \frac{1}{n(\gamma-1)+2} \tag{2.63}$$

Equations (2.62) and (2.63) are Barenblatt s solution to the porous medium equation.

2.4.3 Fourier and Laplace Transformation

Laplace Transform

In mathematics, the Laplace transform, named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s (complex frequency). The transform has many applications in science and engineering because it is a tool for solving differential equations. In particular, it transforms linear differential equations into algebraic equations and convolution into multiplication.

For suitable functions f, the Laplace transform is the integral

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Definition of Fourier transform on L^1 - If $u \in L^1(R^n)$, we express its Fourier transform from Equation (2.64)

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-ixy}u(x)dx \qquad (y \in R^n) \tag{2.64}$$

And it's inverse Fourier Transform,

$$\bar{u}(y) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{ixy}u(x)dx \qquad (y \in R^n) \tag{2.65}$$

NOTES

So that,

$$\left|e^{\pm ix.y}\right| = 1 \quad and \quad u \in L^1(R^n)$$

Express the Fourier transform on L^2 - Choose a sequence is,

$$\{u_k\}_{k=1}^{\infty} \subset L^1(R^n) \cap L^2(R^n)$$

By,

$$u_k \to u \quad in \quad L^2(R^n)$$

This sequence therefore converges to a limit, which we define to be $\hat{u}$:

$$\hat{u}_k \to \hat{u} \quad in \quad L^2(R^n)$$

The definition of $\overline{u}$ does not depend upon the choice of approximating sequence $\{u_k\}_{k=1}^{\infty}$.

Laplace Transform

Let us we define Laplace transform to be

$$u^*(s) = \int_0^{\infty} e^{-st} u(t)\, dt \qquad (s \geq 0)$$

Where, the Fourier transform is most appropriate for functions defined on all of R (or R^n), the Laplace Transform is useful for functions defined only on R_+. In Practice this means that for a Partial differential equation involving time, it may be useful to perform a Laplace transform in t, holding the space variables x fixed.

2.4.4 Hopf- Cole Transform

The Cole-Hopf transform provides an interesting method to solve the viscous Burgers' equation. The viscous Burgers' equation was presented in 1940 and in 1950 Hopf and in 1951 Cole independently gave the method which is known as the Cole-Hopf transformation to solve the viscous Burgers' equation.

Hopf introduced the transformation by first rewriting Burgers' equation as below:

$$u_t = \left(\mu u_x - \frac{u^2}{2}\right)_x \tag{2.66}$$

He then introduced the dependent variable $\varphi = \varphi(x, t)$ where

$$\varphi(x,t) = \exp\left\{-\frac{1}{2\mu}\int u \partial x\right\} \tag{2.67}$$

which is inverted to be

$$u(x,t) = -2\mu\frac{\varphi_x}{\varphi} \tag{2.68}$$

Substituting Equations (2.68) into (2.66) and simplifying, we get

NOTES

$$-2\mu\left(\frac{\varphi_t}{\varphi}\right)_x = -\left(2\mu^2\frac{\varphi_{xx}}{\varphi}\right)_x \tag{2.69}$$

Now, integrating with respect to x and then redefining φ as $\varphi \cdot \exp\left\{-\int C dt\right\}$ where $C(t)$ is the integration constant with respect to x, then (2.69) can be rewritten as

$$\varphi_t = \mu\varphi_{xx} \tag{2.70}$$

which is the heat equation.

Hopf Statement

If u solves Equation 2.66 in an open rectangle R of the $x, t-p$ lane and if u, u_x, u_{xx} are continuous in R then there exists a positive function φ as can be seen from Equation 2.67 that solves the heat equation in R and for which $\varphi, \varphi_x, \varphi_{xx}, \varphi_{xxx}$ are continuous in R. One can easily show the converse, i.e., every positive solution φ of Equation 2.70 with the mentioned properties goes over into a solution of Equation 2.66 of the described general kind. Let u be a regular solution of Equation 2.66) in an x, t-domain D if u, u_x, u_{xx} (and consequently u_t) are continuous in D.

We can solve the heat equation (2.70) using boundary conditions and initial conditions either from the original problem with the transformation applied to the conditions or by solving the equations over an infinite domain assuming that a solution is known for $u(x, t=0) = g(x)$ to then solve for all of $t > 0$.

Hopf proved that $\varphi \in C^{\infty}(R), \forall t > 0$ and the same for u. One can show uniqueness by assuming that $u(x,t)$ is a regular solution of (2.66) in $0 < t < T$ and satisfies

$$\int_0^x u(\varepsilon,t)d\varepsilon \to \int_0^a u_0(\varepsilon)d\varepsilon \text{ as } x \to a, t \to 0$$

where

$$\int_0^x u_0(\varepsilon)d\varepsilon = o\left(x^2\right) \text{for } |x| \text{ large}$$

and u_0 is integrable in every finite x-interval, then $u = -2v\left(\varphi_x / \varphi\right)$. By discretizing the space into strips and then looking at the sign of the limit as t approaches 0 by making use of Widder's theorem on non-negative solutions to the heat equation, Hopf shows that "$\varphi(x,t)$ is uniquely determined by the initial values $u(x, t)$ is, therefore, completely unique".

NOTES

By choosing $f(x) = -2v\dfrac{\partial_x \varphi(x,t)}{\varphi(x,t)}$ we can then solve the ODE to get

$$\varphi(x,0) = h(x) = Ce^{-F(x)}$$

where

$$F(x) = \frac{1}{2v}\int_0^x f(s)ds$$

We can then solve for $\varphi(x,t)$ to get

$$\varphi(x,t) = \frac{C}{\sqrt{4\pi vt}}\int_R h(s)e^{-\frac{(x-s)^2}{4vt}ds}$$

The solution to $u(x,t)$ can then be determined from Eqaution (2.68).

A parabolic PDE with quadratic non-linearity,

Initial value problem for a quasilinear parabolic equation is,

$$\left\{\begin{array}{l} u_t - a\Delta u + b|Du|^2 = 0 \ \ in \ \ R^n \times (0,\infty) \\ u = g \ \ on \ \ R^n \times \{t = 0\} \end{array}\right\} \tag{2.71}$$

Where a>0. This kind of non-linear PDE arises in stochastic optimal control theory.

$w := \phi(u)$ Supposing for the moment u is a smooth solution of Equation (2.71), we set

Where as $\phi : R \to |R$ a smooth function, as yet un-specified. We will try to choose ϕ so that w solves a linear equation. We have $w_t = \phi'(u)u_t$, $\Delta w = \phi'(u)\Delta u + \phi''(u)|Du|^2$ and consequently Equation (2.71) implies

$$\begin{aligned} w_t = \phi'(u)u_t &= \phi'(u)[a\Delta u - b|Du|^2] \\ &= a\Delta w - [a\varphi''(u) + b\phi'(u)]|Du|^2 \\ &= a\Delta w \end{aligned}$$

It only if we choose ϕ is satisfy $a\phi'' + b\phi' = 0$. We solve this differential equation by setting $\phi = e^{\frac{-bz}{a}}$. Thus we see that if u solves Equation (2.71), then

$$w = e^{\frac{-bu}{a}} \tag{2.72}$$

NOTES

Solve the initial- value problem for the heat equation (with conductivity a):

$$\left\{\begin{array}{l} w_t - a\Delta w = 0 \quad in \quad R^n \times (0,\infty) \\ \qquad w = e^{\frac{-bu}{a}} \quad on \quad R^n \times \{t=0\} \end{array}\right\} \tag{2.73}$$

Equation (2.72) is also known as the **Cole-Hopf transformation**.

Now the unique bounded solution of Equation (2.73) is,

$$w(x,t) = \frac{1}{(4\pi at)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-b}{a}g(y)} dy \qquad (x \in R^n, \ t > 0)$$

And since Equation (2.72) implies,

$$u = -\frac{a}{b}\log w$$

We get thereby the explicit formula is,

$$u(x,t) = -\frac{a}{b}\log\left(\frac{1}{(4\pi at)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-b}{a}g(y)} dy\right) \qquad (x \in R^n, \ t > 0)$$

For a solution of quasi-linear initial-value problem Equation (2.71).

2.4.5 Hodograph and Legendre Transform

Equations of Motion

Consider unsteady, plane, incompressible and viscous electrically conducting fluid flow through porous medium in the presence of transverse magnetic field. Darcy-Brinkman-Lapwood equations are

$$\nabla \cdot \mathbf{V} = 0 \tag{2.74}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right] = -\nabla P' + \mu\nabla^2\mathbf{V} + \bar{\mu}(\nabla\times\mathbf{H})\times\mathbf{H} - \frac{\mu}{k}\mathbf{V} \tag{2.75}$$

$$\nabla\times(\mathbf{V}\times\mathbf{H}) + \frac{1}{\mu_e\sigma}\nabla^2\mathbf{H} = \frac{\partial\mathbf{H}}{\partial t} \tag{2.76}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{2.77}$$

where

$\mathbf{V}$ = fluid velocity,

ρ is constant fluid density,

P' = fluid pressure,

K = permeability of porous medium,

$\mathbf{H}$ = magnetic field,

μ = coefficient of viscosity and

μ_e = magnetic permeability.

Here we consider unsteady flow in (X,Y) plane and applied transverse magnetic field and take

NOTES

$$\mathbf{V} = (U(X,Y,t), V(X,Y,t), 0) \tag{2.78}$$

and

$$\mathbf{H} = (0,0,H) \tag{2.79}$$

Since

$$P = \left(P' + \mu_e \frac{H^2}{2} \right) \tag{2.80}$$

We get

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right] = -\nabla P + \mu \nabla^2 \mathbf{V} - \frac{\mu}{K} \mathbf{V} \tag{2.81}$$

The above equation can be rewritten in (X, Y) as under.

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{2.82}$$

$$\rho \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} \right] U = -\frac{\partial P}{\partial X} + \mu \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) U - \frac{\mu}{K} U \tag{2.83}$$

$$\rho \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} \right] V = -\frac{\partial P}{\partial X} + \mu \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) V - \frac{\mu}{K} V \tag{2.84}$$

Doing the following transformations:

$$x = (X - Ct), \quad y = Y, u = U - C, \quad v = V \tag{2.85}$$

Then we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.86}$$

$$\rho \left[u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] u = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u - \frac{\mu}{K} u \tag{2.87}$$

$$\rho \left[u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] v = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v - \frac{\mu}{K} v \tag{2.88}$$

and

$$u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} - \nu_H \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) = 0 \tag{2.89}$$

NOTES

where $\nu_H = \frac{1}{\mu_e \sigma}$. Now Introducing the velocity function

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.90}$$

and eliminating p we get

$$\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} \right) = \left(\nu \nabla^2 \omega - \frac{\nu}{K}\omega \right) \tag{2.91}$$

where $\nu = \frac{\mu}{\rho}$ is kinematical coefficient of viscosity.

Equation in Hodograph plane

Let us consider $u = u(x, y)$ and $v = v(x, y)$ such that

$$J = \frac{\partial(u,v)}{\partial(x,y)} \neq 0. \tag{2.92}$$

In this case we interchange the roles of dependent and independent variables. Hence we have the following relations:

$$\frac{\partial u}{\partial x} = J\frac{\partial y}{\partial v}, \frac{\partial u}{\partial y} = -J\frac{\partial x}{\partial v}, \ \frac{\partial v}{\partial x} = -J\frac{\partial y}{\partial u}, \ \frac{\partial v}{\partial y} = J\frac{\partial x}{\partial u} \tag{2.93}$$

and

$$\frac{\partial g}{\partial x} = \frac{\partial(g,y)}{\partial(x,y)} = J\frac{\partial(g,y)}{\partial(u,v)}, \ \frac{\partial g}{\partial y} = -\frac{\partial(g,x)}{\partial(x,y)} = J\frac{\partial(x,g)}{\partial(u,v)} \tag{2.94}$$

where $x = x(u,v)$, $y = y(u,v)$ and $g = g(x,y) = g(x(u,v)$, $y(u,v)) = \overline{g}(u,v)$ is any continuously differentiable function. Thus,

$$J = J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \left[\frac{\partial(x,y)}{\partial(u,v)} \right]^{-1} = j(u,v) \tag{2.95}$$

Now we can write $\omega(x,y) = \overline{\omega}(u,v), \quad H(x,y) = \overline{H}(u,v)$. Then Equations (2.85) and (2.86) become.

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0 \tag{2.96}$$

$$j\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} \right) = \overline{\omega} \tag{2.97}$$

$$\nu\left[\frac{\partial(jQ_2,y)}{\partial(u,v)}+\frac{\partial(x,jQ_1)}{\partial(u,v)}\right]-\frac{\nu}{K}\frac{\bar{\omega}}{j}=uQ_2+\nu Q_1 \tag{2.98}$$

$$uG_1+\nu G_2-\nu_H\left[\frac{\partial(jG_1,y)}{\partial(u,v)}+\frac{\partial(x,jG_2)}{\partial(u,v)}\right]=0 \tag{2.99}$$

NOTES

where

$$Q_1=Q_1(u,v)=\frac{\partial(x,\bar{\omega})}{\partial(u,v)}\quad Q_2=Q_2(u,v)=\frac{\partial(\bar{\omega},y)}{\partial(u,v)} \tag{2.100}$$

and

$$G_1=G_1(u,v)=\frac{\partial(\bar{H},y)}{\partial(u,v)}\quad G_2=G_2(u,v)=\frac{\partial(x,\bar{H})}{\partial(u,v)} \tag{2.101}$$

The hodograph transform is a technique for converting certain quasi-linear systems of PDE into linear systems, by reversing the roles of the dependent and independent variables. Equation of steady, two-dimensional fluid flow as fallow Equation (2.102)

$$\left\{\begin{array}{l}(a)\ \ \left(\sigma^2(u)-(u^1)^2\right)u^1_{x1}-u^1u^2\left(u^1_{x2}+u^2_{x1}\right)+\left(\sigma^2(u)-(u^2)^2\right)u^2_{x2}=0\\(b)\ \ u^1_{x2}-u^2_{x1}=0\end{array}\right\} \tag{2.102}$$

In R^2. The unknown is the velocity field $u = (u^1, u^2)$ and the function $\sigma(.): R^2 \to R$ the local sound speed, is given.

The Equation (2.102) is known as quasi-linear.

Let us now, however, no longer regard u^1 and u^2 as functions of x_1 and x_2

$$u^1=u^1(x_1,x_2),\ \ u^2=u^2(x_1,x_2) \tag{2.103}$$

But relatively regard x^1 and x^2 as functions of u_1 and u_2;

$$x^1=x^1(u_1,u_2),\ \ x^2=x^2(u_1,u_2) \tag{2.104}$$

According to the Inverse Function Theorem, we can locally at least, invert Equations (2.103) to yield Equation (2.104), provided

$$J=\frac{\partial(u^1,u^2)}{\partial(x_1,x_2)}=u^1_{x1}-u^2_{x2}-u^1_{x2}-u^2_{x1}\neq 0 \tag{2.105}$$

In some region of R^2.

Assuming now Equation (2.105) holds, we calculate

$$\left\{\begin{array}{l}u^2_{x2}=Jx^1_{u1},\ \ u^1_{x1}=-Jx^2_{u1}\\u^1_{x2}=-Jx^1_{u2},\ \ u^1_{x1}=-Jx^2_{u2}\end{array}\right\} \tag{2.106}$$

NOTES

We pull-out of Equation (2.106) into (2.102) , to determine

$$\left\{\begin{array}{ll}(a) & \left(\sigma^2(u)-(u^1)^2\right)x^2_{u2}+u_1u_2\left(x^1_{u2}+x^2_{u1}\right)+\left(\sigma^2(u)-(u_2)^2\right)x^1_{u1}=0\\ (b) & x^1_{u2}-x^2_{u1}=0\end{array}\right\}$$

This is a linear system for $x=(x^1,x^2)$ as a function of $u=(u_1,u_2)$.

Equation in Legendre transform function

Equation (2.77) implies the existence of stream function Ψ satisfying

$$U=\frac{\partial\Psi}{\partial Y},\quad V=-\frac{\partial\Psi}{\partial X} \tag{2.107}$$

and similarly Equation (2.81) implies existence of ψ such that

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{2.108}$$

and

$$\Psi=\Psi(X,Y,t)=\psi(X-Ct)-CY+Constant \tag{2.109}$$

In a similar manner the Equation (2.91) implies

$$\frac{\partial L}{\partial u}=-y,\quad \frac{\partial L}{\partial v}=x, \tag{2.110}$$

where $\psi=\psi(x,y)$ and $L=L(u,v)$ are related by

$$L(u,v)=vx-uy+\psi(x,y) \tag{2.111}$$

Introducing $L(u,v)$ in (2.91) to (2.94) we see that the Equation (2.91) is identically satisfied and the remaining equations are as below:

$$j\left(\frac{\partial^2L}{\partial u^2}+\frac{\partial^2L}{\partial v^2}\right)=\bar{\omega} \tag{2.112}$$

and

$$\nu\left[\frac{\partial\left(\frac{\partial L}{\partial u},jQ_2\right)}{\partial(u,v)}+\frac{\partial\left(\frac{\partial L}{\partial v},jQ_1\right)}{\partial(u,v)}\right]-\frac{\nu}{K}\frac{\omega}{j}=uQ_2+vQ_1 \tag{2.113}$$

Again

$$uG_1+vG_2-\nu_H\left[\frac{\partial\left(\frac{\partial L}{\partial u},jG_1\right)}{\partial(u,v)}-\frac{\partial\left(\frac{\partial L}{\partial v},jG_2\right)}{\partial(u,v)}\right]=0 \tag{2.114}$$

NOTES

where

$$Q_1 = Q_1(u,v) = \frac{\partial\left(\frac{\partial L}{\partial u}, \bar{\omega}\right)}{\partial(u,v)}, \quad Q_2 = Q_2(u,v) = \frac{\partial\left(\frac{\partial L}{\partial v}, \bar{\omega}\right)}{\partial(u,v)} \tag{2.115}$$

$$G_1 = G_1(u,v) = \frac{\partial\left(\frac{\partial L}{\partial u}, \bar{H}\right)}{\partial(u,v)}, \quad G_2 = G_2(u,v) = \frac{\partial\left(\frac{\partial L}{\partial v}, \bar{H}\right)}{\partial(u,v)} \tag{2.116}$$

and

$$j = \left[\frac{\partial^2 L}{\partial u^2}\frac{\partial^2 L}{\partial v^2} - \left(\frac{\partial^2 L}{\partial u \partial u}\right)^2\right]^{-1} \tag{2.117}$$

Summing up above we have the following theorem.

2.4.6 Potential Functions

A mathematical function whose values represent physical potentials is referred to as a potential function. It also be defined as a function lying in the theory in the category of function harmonic in nature and is usually studied as a part of the potential theory.

Another technique is to utilize a potential function to convert a non-linear system of PDE into a single linear PDE. We consider as an example Euler s equations for in viscid, incompressible fluid flow in Equation (2.118)

$$\left\{\begin{array}{lll} (a) & u_t + u.Du = -Dp + f & in \quad R^3 \times (0,\infty) \\ (b) & div\ u = 0 & in \quad R^3 \times (0,\infty) \\ (c) & u = g & on \quad R^3 \times \{t = 0\} \end{array}\right\} \tag{2.118}$$

Now the unknown are the velocity field $u = (u^1, u^2, u^3)$ and the scalar pressure p; the external force $f = (f^1, f^2, f^3)$ and the initial velocity $g = (g^1, g^2, g^3)$ are given. Here D as usual denotes the gradient in the spatial variables $x = (x_1, x_2, x_3)$.

The vector Equation (2.107) means

$$u_t^i + \sum_{j=1}^{3} u^j u_{xj}^i = -p_{xi} + f^i \qquad (i = 1,2,3) \tag{2.119}$$

We will assume $div\, g = 0$

NOTES

If furthermore there exists a scalar function $h : R^3 \times (0,\infty) \to R$ such that

$$F = Dh \tag{2.120}$$

Now we have to find the solution (u,p) of Equation (2.107) for which the velocity field u is also derived from a potential, say $u=Dv$

As curl u=0 flow will be irrotational.

And so v must be harmonic as a function of x, for each time t>0.

If $u.Du = \frac{1}{2}D\left(|Dv|^2\right)$ consequently (1) (a) reads $D\left(v_t + \frac{1}{2}|Dv|^2\right) = D(-p+h)$

Consequently we take,

$$v_t + \frac{1}{2}|Dv|^2 + p = h \tag{2.121}$$

This is Bernoulli's law.

Theorem 2.5: If $L(u,v)$ is the Legendre transform of a stream function of the equation of motion to governing the plane unsteady flow of a finitely conducting viscous incompressible fluid. Then $L(u,v)$ must satisfy the existence of ψ and $\bar{H}$ satisfy the similar manner.

Proof: Then we have

$$q = \sqrt{u^2 + v^2}, \quad \theta = \tan^{-1}\left(\frac{v}{u}\right) \tag{2.122}$$

and hence

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta} \tag{2.123}$$

Again we define $L^*(q,\theta), \omega^*(q,\theta)$ and $j^*(q,\theta)$ to be the Legendre transform function, vorticity function and Jacobean function in (q,θ) plane. Now using the relations

$$\frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial\left(F^*,G^*\right)}{\partial(q,\theta)}\frac{\partial(q,\theta)}{\partial(u,v)} = \frac{1}{q}\frac{\partial\left(F^*,G^*\right)}{\partial(q,\theta)} \tag{2.124}$$

where $F(u,v) = F^*(q,\theta), G(u,v) = G^*(q,\theta)$ are continuously differentiable functions.

Corollary: If $L^*(q,\theta)$ is the Legendre transform function of a stream function of the equation of motion (2.81) to (2.83) then $L^*(q,\theta)$ must satisfy

$$\nu\left[\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta},j^*Q_1^*\right)}{\partial(q,\theta)}+\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q}-\frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta},j^*Q_2^*\right)}{\partial(q,\theta)}\right]$$

$$-\frac{\nu}{K}\frac{\omega^* q}{j^*}=q^2\left(\sin\theta Q_1^*+\cos\theta Q_2^*\right) \tag{2.125}$$

And if $\bar{H}^*(q,\theta)$ is Legendre transform of magnetic field, then equation (2.86) becomes

$$q\left(\cos\theta G_1^*+\sin\theta G_2^*\right)$$

$$-\frac{\nu_H}{q}\left[\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q}-\frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta},j^*G_1^*\right)}{\partial(q,\theta)}+\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta},j^*G_2^*\right)}{\partial(q,\theta)}\right]=0 \tag{2.126}$$

$$G_1^*(q,\theta)=\frac{1}{q}\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q}-\frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta},H^*\right)}{\partial(q,\theta)}\quad G_2^*(q,\theta)=\frac{1}{q}\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta},H^*\right)}{\partial(q,\theta)} \tag{2.127}$$

Corollary: $L^*(q,\theta)$ and $H^*(q,\theta)$ must satisfy Equation (2.116) and

$$\cos\theta G_1^*+\sin\theta G_2^*=0 \tag{2.128}$$

After figuring out the solution of $L^*(q,\theta)$ of the system of equation (2.116), (2.117) we employ

$$x=\sin\theta\frac{\partial L^*}{\partial q}+\frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta},\quad y=\frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}-\cos\theta\frac{\partial L^*}{\partial q} \tag{2.129}$$

and Equation (2.113) to get $u(x,y)$ and $v(x,y)$ in the physical plane.

Example 2.6: Let $L(u,v)=F(u)+G(v)$

(2.130)

such that first and second derivative of F(u) and G(v) are non-zero. Now putting this value of $L(u,v)$ in (2.107), (2.110), (2.111) we get

$$\bar{\omega}=\frac{1}{F''(u)},j=\frac{1}{F''(u)G''(v)}$$

$$Q_1=\frac{F'''(u)G''(v)}{F''^2(u)},\quad Q_2=\frac{G'''(v)F''(u)}{G''^2(v)} \tag{2.131}$$

$$G_1=F''(u)\frac{\partial\bar{H}}{\partial v},\quad G_2=F''(v)\frac{\partial\bar{H}}{\partial u}$$

NOTES

Now using equations (2.124), (2.125) and (2.92) we get

$$\nu\left[\frac{1}{G''}\left(\frac{G'''}{G''^2}\right)'+\frac{1}{F''}\left(\frac{F'''}{F''^2}\right)'\right]+\frac{\nu}{K}\frac{\left(G''+F''\right)}{F''G''}=\nu\frac{F'''}{F''^3}+u\frac{G'''}{G''^3}. \quad (2.132)$$

If Equation (2.125) defines the Legendre transformation function such that $F'''(u)=0$ and $G'''(v)=0$ then (2.127) is satisfied only when $\eta_2=0$ or $G''(v)+F''(u)=0$. Since we have taken the porous media, hence $\eta_2\neq 0$ so we must have $G''(v)+F''(v)=0$. Now $F'''(u)=G'''(v)=0\Leftrightarrow F''(u)=K_1$ and $G''(v)=K_2$, where K_1,K_2 are arbitrary constants. Then, $G''(v)+F''(u)=0\Rightarrow K_1=-K_2$ and

$$L(u,v)=C_1u^2+C_2u+C_3+D_1v^2+D_2v+D_3 \quad (2.133)$$

and $D_1=-C_1$. Now using Equation (2.100), we get

$$u=-\frac{1}{2C_1}\left(y+C_2\right),\quad v=-\frac{1}{2D_1}\left(x-D_2\right) \quad (2.134)$$

Again by using Equation (2.98) we get the stream function as

$$\psi=\frac{\left(x-D_2\right)^2}{4C_1}-\frac{\left(y+C_2\right)^2}{4C_1} \quad (2.135)$$

The time dependent stream function is given by

$$\Psi(X,Y,t)=\frac{\left(X-Ct-D_2\right)^2}{4C_1}-\frac{\left(Y+C_2\right)^2}{4C_1}. \quad (2.136)$$

From Equations (2.126), we have

$$G_1=2C_1\frac{\partial\bar{H}}{\partial v},\quad G_2=2C_1\frac{\partial\bar{H}}{\partial u} \quad (2.137)$$

Further we consider

$$-\frac{u}{C_1}\frac{\partial\bar{H}}{\partial v}+\frac{v}{C_1}\frac{\partial\bar{H}}{\partial u}-\nu_H\left[\frac{1}{2C_1^2}\frac{\partial^2\bar{H}}{\partial v^2}+\frac{1}{2C_1^2}\frac{\partial^2\bar{H}}{\partial u^2}\right]=0 \quad (2.138)$$

Let $\bar{H}(u,v)=F(u)+G(v)$ then. Putting $\bar{H}(u,v)$ in Equation (2.132), we get

$$-\frac{u}{C_1}G'(v)+\frac{v}{C_1}F'(u)-\nu_H\left[\frac{1}{2C_1^2}\left\{G''(v)+F''(u)\right\}\right]=0. \quad (2.139)$$

Differentiating twice with respect to u we get

$$\frac{F'''(u)}{C_1}v+\frac{\nu_H}{2C_1^2}F^{(4)}(u)=0. \tag{2.140}$$

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Now the above equation is true for all v if $\frac{F'''(u)}{C_1}=0$ and $\frac{\nu_H}{2C_1^2}F^{(4)}(u)=0$. Hence $F(u)=\left(\frac{C_2}{2}u^2+C_3u+C_4\right)$

Now equation (2.134) becomes

$$\left\{-\frac{G'}{C_1}-\frac{C_2v}{C_1}\right\}u-\left\{\frac{C_3}{C_1}v+\frac{\nu_H}{2C_1^2}G''+\frac{\nu_H}{2C_1^2}C_2\right\}=0 \tag{2.141}$$

Now this equation is true for all u, if and only if,

$$-\frac{G'}{C_1}-\frac{C_2v}{C_1}=0,\frac{C_3}{C_1}v+\frac{\nu_H}{2C_1^2}G''+\frac{\nu_H}{2C_1^2}C_2=0. \tag{2.142}$$

Now solving above equation, we get $G(v)=\left(-\frac{C_2}{2}v^2+C_5\right),C_3=0$ and hence

$$\bar{H}(u,v)=\frac{C_2}{2}\left(u^2-v^2\right)+C_6, \tag{2.143}$$

where $C_6=C_5+C_4$. Now using equation (2.128), we have

$$H(x,y)=\frac{C_2}{8C_1^2}\left[\left(y+C_2\right)^2-\left(x-D_2\right)^2\right]+C_6. \tag{2.144}$$

Similarly by using transformations Equation (2.144) magnetic field is as under:

$$H(X,Y)=\frac{C_2}{8C_1^2}\left[\left(Y+C_2\right)^2-\left(X-Ct-D_2\right)^2\right]+C_6. \tag{2.145}$$

Using Equations (2.107),(2.90), we have

$\omega=0.$

The total pressure by integrating with respect to x and y respectively is:

$$p=\frac{\mu}{2C_1K}\left[2xy+C_2x-D_2y\right]-\frac{\rho}{2C_1}\left[\left(x-D_2\right)^2+\left(y+C_2\right)^2\right]+D_3 \tag{2.146}$$

where D_2,C_2,D_3 are arbitrary constants.

Here time dependent pressure is given by:

$$P=\frac{\mu}{2C_1K}\left[2(X-Ct)Y+C_2(X-Ct)-D_2Y\right]$$

NOTES

$$-\frac{\rho}{2C_1}\left[\left((X-Ct)-D_2\right)^2+\left(y+C_2\right)^2\right]+D_3. \tag{2.147}$$

The fluid pressure can be calculated from.

$$P' = P-\frac{\mu_e H^2}{2}. \tag{2.148}$$

Hence,

$$P' = \frac{\mu}{2C_1K}\left[2(X-Ct)Y+C_2(X-Ct)-D_2Y\right]$$

$$-\frac{\rho}{2C_1}\left[\left((X-Ct)-D_2\right)^2+\left(y+C_2\right)^2\right]+D_3$$

$$-\frac{1}{2}\mu_e\left\{\frac{C_2}{8C_1^2}\left[\left(Y+C_2\right)^2-\left(X-Ct-D_2\right)^2\right]+C_6\right\}^2 \tag{2.149}$$

Example 2.7: Let $L(u,v)=vF(u)+G(u),$ such that $F(u)\neq 0$ .Also, first and second derivative of F(u) and G(v) are non zero.

Now substituting this value of $L(u,v)$ in (2.107), (2.110) we get

$$\bar{\omega} = -\frac{\left(vF''+G''\right)}{F'^2}, j=-\frac{1}{F'^2}$$

$$Q_1 = -\frac{F''}{F''}, \quad Q_2 = \frac{v\left(F'F'''-3F'^2\right)+F'G'''-3F''G''}{F'^2}. \tag{2.150}$$

Substituting these values in equation (2.41), we get

$$v\left[v\left(\frac{F^{iv}}{F'^2}-10\frac{F'''F''}{F'^3}+15\frac{F''^3}{F'^4}\right)+\left(\frac{G^{iv}}{F'^2}-6\frac{G'''F''}{F'^3}-4\frac{G''F'''}{F'^3}+15\frac{G''F''^2}{F'^4}\right)\right] \tag{2.151}$$

$$-\frac{\nu}{K}\left(vF''+G''\right) \tag{2.152}$$

$$=-v\frac{F''}{F'}+\left[uv\left(\frac{F'''}{F'}-3\frac{F'^2}{F'^2}\right)+u\left(\frac{G'''}{F'}-3\frac{G''F''}{F'^2}\right)\right]. \tag{2.153}$$

Equation (2.144) gives rise to following differential equations

$$v\left[F^{iv}F'^2-10F'''F''F'+15F''^3-\frac{1}{K}F''F'^4\right]$$

$$=u\left(F'''F'^3-3F''^2F'^3\right)-F''F'^3 \tag{2.154}$$

and

$$\nu\left[G^{iv}F'^2-6G'''F''F'-4G''F'''F'+15G''F''^2-\frac{1}{K}G''F'^4\right]$$

$$=u\left(G'''F'^3-3G''F''F'^2\right) \tag{2.155}$$

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This system is coupled non-linear partial differential equation in two unknown F and G and quite complex to solve, so we assume $F(u)$ as

$$F(u)=\frac{1}{a}\ln\frac{u-A}{KA}, \tag{2.156}$$

where A and a are constants. Using this value of $F(u)$, (2.146) is satisfied if and only if $A=\nu b$, where $b=a-\frac{1}{Ka}$. So, we must have

$$F(u)=\frac{1}{a}\ln\frac{u-\nu b}{K\nu b}. \tag{2.157}$$

Using equations (2.147) and (2.149) we can write

$$G^{iv}+\frac{(6\nu a-u)}{\nu a(u-\nu b)}G'''+\frac{(c\nu a-3u)}{\nu a(u-\nu b)^2}G''=0, \tag{2.158}$$

where $c=\left(7-\frac{1}{a^2K}\right)$.

Using $G'''=H$, we get

$$H''+\frac{(6\nu a-u)}{\nu a(u-\nu b)}H'+\frac{(c\nu a-3u)}{\nu a(u-\nu b)^2}H=0, \tag{2.159}$$

which is linear differential equation of order two with a regular singular point at $u=\nu b$, so using Frobenious method we have solution as below:

$$H(u)=A\phi_1(u-\nu b)+B\phi_2(u-\nu b), \tag{2.160}$$

where A and B are arbitrary constants.

$$\phi_2(u-\nu b)=(u-\nu b)^{r_2}\left(c_0'+c_1'(u-\nu b)+c_2'(u-\nu b)^2+\ldots\right). \tag{2.161}$$

In the above equations r_1, r_2 are roots of Indicial equation.

$$r^2+\left(5-\frac{b}{a}\right)r+\left(c-\frac{3b}{a}\right)=0 \tag{2.162}$$

and $c_0, c_1, c_2\ldots; c_0', c_1', c_2'\ldots;$ are arbitrary constants.

$$c_m\left[(r+m)^2+\left(5-\frac{b}{a}\right)(r+m)+\left(c-\frac{3b}{a}\right)\right]=\frac{1}{\nu a}c_{m-1}[r+m-2]. \tag{2.163}$$

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Now after getting $H(u)$ as above Equation (2.160), we can get $G(u)$ from the relation $G'' = H$ and hence $L(u,v)$ from Equation (2.111). Further we can get u and v from Equation (2.100) and steady state stream function ψ from Equation (2.101).

CHECK YOUR PROGRESS

11. What is separation of variables?
12. Define the travelling wave.
13. Give the initial value problem for a quasilinear parabolic equation.
14. Define the hodograph transform.

2.5 ANSWERS TO 'CHECK YOUR PROGRESS'

1. General systems of non-linear differential equations are considered in the normal form:

   $\frac{dy}{dx} = Y(y,x), \quad y \in \mathbf{R}^n$.

2. A subset of the phase space is called invariant if it consists of complete trajectories. If a semi trajectory is bounded, then its limit set is connected.
3. An equilibrium point is Lyapunov stable if all solutions of the dynamical system that start out near an equilibrium point x_e stay near x_e forever. More strongly, if x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable.
4. Let us consider an open set $A \subset R^n$. Assume that for each parameter $a = (a_1,\ldots,a_n) \in A$ we have a C^2 solution $u = u(x;a)$ of the partial differential Equation 2.3.

   Definition of Complete Integral: A C^2 function $u = u(x;a)$ is called a complete integral in $U \times$ A provided,

   1. $u(x;a)$ solves the PDE Equation (2.3) for each $a \in A$ and
   2. $rank\left(D_a u, D^2_{xa} u\right) = n$, where $x \in U, a \in A$.

5. Envelope, in mathematics, a curve, i.e., tangential to each one of a family of curves in a plane or, in three dimensions, a surface, i.e., tangent to each one of a family of surfaces. For example, two parallel lines are the envelope of the family of circles of the same radius having centres on a straight line.

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6. The initial value problem for the Hamilton-Jacobi equation:

$$\left\{\begin{matrix} u_t + H(Du) = 0 \quad in \quad R^n \times (0,\infty) \\ u = g \quad on \ R^n \times \{t = 0\}. \end{matrix}\right\}$$

   Where as $u : R^n \times [0,\infty) \to R$ is the unknown, $u=u(x,t)$ and $Du = D_x u = (u_{x1},.....,u_{xn})$.

7. Calculus of variations is a field of mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions. Basically, a functional is a mapping from a set of functions to the real numbers. Functionals are often formed as definite integrals involving unknown functions and their derivatives.

8. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious and possibly many solutions may exist. Such solutions are known as geodesics.

9. The Hamilton-Jacobi Equation is a first-order non-linear partial differential equation of the form $H(x,u_x(x,\alpha,t),t)+ut(x,\alpha,t)=K(\alpha,t)$ with independent variables (x,t)" $\mathbb{R}^n \times \mathbb{R}|$ and parameters α" $\mathbb{R}^n$ | . It has wide applications in optics, mechanics, and semi-classical quantum theory. Its solutions determine infinite families of solutions of Hamilton's ordinary differential equations, which are the equations of motion of a mechanical system or an optical system in the ray approximation.

10. A Lipschitz continuous function $u : R^n \times [0,\infty) \to R$ is a weak solution of the initial value problem:

$$\left\{\begin{matrix} u_t + H(Du) = 0 \qquad in \quad R^n \times (0,\infty) \\ u = g \qquad on \ R^n \times \{t = 0\} \end{matrix}\right\}$$

11. The method of separation of variables tries to construct a solution u to a given partial differential equation as some sort of combination of functions of fewer variables.

12. Let us the first a partial differential equation involving the two variables $x \in R,\ t \in R$. A solution u of the form is,

$$u(x,t) = v(x - \sigma t) \qquad (x \in R, \quad t \in R)$$

    is called a 'Traveling Wave'.

13. Initial value problem for a quasilinear parabolic equation is,

$$\left\{\begin{matrix} u_t - a\Delta u + b|Du|^2 = 0 \ \ in \quad R^n \times (0,\infty) \\ u = g \quad on \ R^n \times \{t = 0\} \end{matrix}\right\}$$

14. The hodograph transform is a technique for converting certain quasi-linear systems of PDE into linear systems, by reversing the roles of the dependent and independent variables.

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2.6 SUMMARY

- Qualitative theory of differential equations studies the properties of solutions of ordinary differential equations without finding the solutions themselves.
- Lyapunov studied the behaviour of solutions in a neighbourhood of an equilibrium position and founded the modern theory of stability of motion. The geometric approach of Poincare was developed in the 1920s by George Birkhoff, who discovered many important facts in the qualitative theory of higher-dimensional systems of differential equations.
- A subset of the phase space is called invariant if it consists of complete trajectories. If a semi trajectory is bounded, then its limit set is connected.
- A finite number of equilibrium states and trajectories converging to these equilibrium states as $x \to \pm \infty$.
- In the important particular case when the system has an invariant measure, the study of general regularity of the behaviour of the solutions has been carried out in great detail.
- Of special interest for applications are structurally-stable systems, i.e., systems which are stable under a perturbation of the right-hand sides which is small in the sense of C^1. For $n = 2$, in any bounded part of the plane there are only a finite number of periodic solutions. For $n > 2$ the behaviour of a structurally-stable system is considerably more complicated. S. Smale has given an example of a structurally-stable system having an infinite number of periodic solutions in a bounded part of the phase space.
- One of the problems in the qualitative theory of differential equations is that of the existence of periodic solutions. For the proof of the existence of such solutions use is often made of topological devices, in particular the various criteria for the existence of a fixed point.
- A complete qualitative study of non-linear systems of differential equations has only been achieved in very special cases. For example, it has been proved that the Lienard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ has, under very natural hypotheses, a unique periodic solution, while all its other solutions converge to this periodic one.
- The general study of the stability of solutions of differential equations is known as stability theory. Lyapunov stability theorems give only sufficient condition.
- Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, providing a Lyapunov function can be found to satisfy the constraints.

- Normally non-linear first-order PDE of the form $F(Du,u,x) = 0$, where $x \in U$ and U is an open subset of R^n.
- We are concerning with discovering solution u of the PDE $F(Du,u,x) = 0$ in U,usually subject to the boundary condition $u = g$ on Γ where Γ is some given subset of ∂U and g : ΓR is prescribed.
- Envelope, in mathematics, a curve, i.e., tangential to each one of a family of curves in a plane or, in three dimensions, a surface, i.e., tangent to each one of a family of surfaces. For example, two parallel lines are the envelope of the family of circles of the same radius having centres on a straight line.
- Let $u = u(x;a)$ be a C^1 function of $x \in U$, $a \in A$, where $U \subset R^n$ and $A \subset R^m$ are open sets.
- The equation $f(x, y, a) = 0$ represents, in general, a curve in the xy plane for any given value of a. For different values of a, the relation $f(x, y, a) = 0$ represents a system of curves, called a one-parameter family of curves. The curve which touches every member of the family is called the envelope of the family $f(x, y, a) = 0$ where a is a parameter.
- Although the above two examples are geometrically clear, it is often not possible to visualize the envelope. We will develop a mathematical way to obtain the equation of the envelope of the family $f(x, y, a) = 0$.
- Geometrically, it is the point on the curve $f(x, y, a) = 0$ approaching the intersecting point of $f(x, y, a) = 0$ and $f(x, y, a + Da) = 0$ as $Da \to 0$. To understand clearly what the limiting position of the point of intersection of $f(x, y, a) = 0$ and $f(x, y, a + Da) = 0$ mean.
- The interest is in *extremal* functions that make the functional attain a maximum or minimum value or *stationary* functions where the rate of change of the functional is precisely zero.
- The minimal curve problem asks us to find the function $y = u(x)$ that minimizes the arc length functional among all reasonable functions satisfying the prescribed boundary conditions.
- The calculus of variations and its extensions are used to find the optimum function that gives the best value of the model and satisfies the constraints of a system. The first calculus of variations problem, the Brachistochrone problem, was posed and solved by Johannes Bernoulli in 1696. In this problem the optimum curve was determined to minimize the time traveled by a particle sliding without friction between two points.
- In mathematics, the maximum or minimum of a function was determined to be an optimal point or set of points. In the calculus of variations the maximum or minimum value of a functional is determined to be an optimal function. A functional is a function of a function and depends on the entire path of one or more functions rather than a number of discrete variables.

- Hamilton-Jacobi equation has wide applications in optics, mechanics, and semi-classical quantum theory. Its solutions determine infinite families of solutions of Hamilton's ordinary differential equations, which are the equations of motion of a mechanical system or an optical system in the ray approximation.
- In physics, a vector field with this property is called divergence-free or solenoidal.
- Hence for uniqueness we must require more than just solvability of the PDE a.e and Lipschitz continuity of g. The next lemma shows that u will inherit a form of one-sided second-derivative estimate from the initial function g, granted that g be semiconcave. Semiconcavity will turn out to be a sufficient condition for the uniqueness to hold.
- (Semiconcavity): Suppose that H is uniformly convex with a constant θ and u is defined by the Hopf-Lax formula.
- An Ordinary Differential Equation (ODE) is an equation that includes some ordinary derivatives (as opposed to partial derivatives) of a function. Often, our goal is to solve an ODE, i.e., determine what function or functions satisfy the equation.
- In general, solving an ODE is more complicated than simple integration. Even so, the basic principle is always integration, as we need to go from derivative to function. Usually, the difficult part is determining what integration we need to do.
- For linear partial differential equations there are numerous techniques for reducing the PDE to an ODE (or at least a PDE in a smaller number of independent variables). These include various integral transforms and Eigen function expansions. Such techniques are much less predominant in dealing with non-linear PDE's. However, there is an approach which identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately.
- Where, the Fourier transform is most appropriate for functions defined on all of R (or R^n), the Laplace Transform is useful for functions defined only on R_+. In Practice this means that for a Partial differential equation involving time, it may be useful to perform a Laplace transform in t, holding the space variables x fixed.
- Another technique is to utilize a potential function to convert a nonlinear system of PDE into a single linear PDE.

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2.7 KEY TERMS

- **Envelope:** Envelope, in mathematics, a curve, i.e., tangential to each one of a family of curves in a plane or, in three dimensions, a surface, i.e., tangent to each one of a family of surfaces. For example, two parallel lines are the envelope of the family of circles of the same radius having centres on a straight line.
- **Calculus of variations:** Calculus of variations is a field of mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions. Basically, a functional is a mapping from a set of functions to the real numbers. Functionals are often formed as definite integrals involving unknown functions and their derivatives.
- **Hamilton-Jacobi equation:** It has wide applications in optics, mechanics, and semi-classical quantum theory. Its solutions determine infinite families of solutions of Hamilton's ordinary differential equations, which are the equations of motion of a mechanical system or an optical system in the ray approximation.
- **Separation of variables:** The method of separation of variables tries to construct a solution u to a given partial differential equation as some sort of combination of functions of fewer variables.
- **Potential function:** A mathematical function whose value represent physical potentials is referred to as a potential function.

2.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is non-linear PDE of the first order?
2. Give the types of bounded semi trajectory.
3. How will you determine the stability for non-linear system?
4. What is complete integral?
5. Define envelopes.
6. Give the derivation of characteristics ODE.
7. What is calculus of variation?
8. Define Hamilton's ODE.
9. Give the uniqueness of weak solution.
10. State the similarity solution of plane and travelling wave.

11. What do you understand by Hopf-Cole transform?
12. Define the hodograph.

Long-Answer Questions

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1. Discuss about the non-linear PDE of the first order with the help of examples.
2. Elaborate on the complete integral and envelop giving examples.
3. Describe the Hamilton - Jacobi equations in the terms of calculus variable, Legendre transform and weak solution.
4. Explain in detail about the representation of solution with appropriate examples.

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UNIT 3 ANALYTICAL DYNAMICS: GENERALIZED CO-ORDINATES

Structure

3.0 INTRODUCTION

In classical mechanics, analytical dynamics is also known as dynamics or simply dinamics is, concerned with the relationship between motion of bodies and its causes, mainly the force ..... on the bodies and the properties of the bodies particular mass and momentum of inertia. The foundation of modern day dynamic is Newtonian mechanics and its reformulation as Lagrangian mechanics and Hamiltonian mechanics.

The term generalized coordinates refers to the parameters that describe the configuration of the system relative to some reference configuration. These parameters must uniquely define the configuration of the system relative to the reference configuration. This is done assuming that can be done with a single chart. The generalized velocities are the time derivatives of the generalized coordinates of the system. A holonomic, non-holonomic, scleronomic, and rheonomic systems depends only on the coordinates and time. It does not depend

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on the velocities. Holonomic system are systems which all constraints are integrable into positional constraints. Non-holonomic system are system which have constraints that are non-integrable into positional constraints.

The generalised potential is also known as from Newtonian physics that apparent forces appear when the motion of masses is described by using a non-inertial frame of reference. The generalized potential of such forces is rigorously analysed focusing on their mathematical aspects.

Ordinary second-order differential equations which describe the motions of mechanical systems under the action of forces applied to them. The equations were established by J.L. Lagrange in two forms: Lagrange's equations of the first kind, or equations in Cartesian coordinates with undetermined Lagrange multipliers, and of the second kind, or equations in generalized Lagrange coordinates.

Hamiltonian formalism uses $\dot{q}_i$ and p_i as dynamical variables, here p_i is generalized momenta and is additional dynamical variables. Hamilton's canonical equations of motion are the differential equations of motion of a mechanical system in which the variables are the generalized momenta, and other is generalized. The Hamiltonian formulation is obtain from the Lagrangian function L by replacing the generalised velocity by the conjugate momentum this is done by Dokin's theorem. A cyclic coordinate is one that does not obviously appear in the Lagrangian. The term cyclic is a natural name when one has cylindrical or spherical symmetry. In Hamiltonian mechanics a cyclic coordinate often is called an ignorable coordinate. In classical mechanics, Routh's method or Routhian mechanics is a hybrid formulation of Lagrangian mechanics and Hamiltonian mechanics developed by Edward John Routh. Respectively, the Routhian is the function which replaces both the Lagrangian and Hamiltonian functions.

In mathematics and classical mechanics, the 'Poisson Bracket' is an important binary operation in Hamiltonian mechanics, playing a central role in Hamilton's equations of motion, which govern the time evolution of a Hamiltonian dynamical system. Jacobi referred to Poisson's theorem as "one of the most remarkable theorems of the whole of integral calculus". In the particular case when $H = T–U$, it is the fundamental theorem of analytical mechanics. Of course, the Jacobi–Poisson theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the first integral obtained is simply a function of the original integrals.

In this unit, you will learn about the basics of generalized coordinates, holonomic, non-holonomic, scleronomic, and rheonomic systems, generalized potential, Lagrange's equations of first and second kind, uniqueness of solution, energy equation for conservative, Hamilton's variables, Donkin's theorem, Hamilton canonical equation Poission's bracket, identity and Jacobi-Poisson theorem.

3.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand the generalized co-ordinates
- Explain the holonomic and non-holonomic systems
- Know about the scleronomic and rheonomic system
- Analyse the generalized potential and Lagrange's equations of first and second kind
- Calculate the uniqueness of solution
- Describe the energy equation for conservation fields
- Learn about the Hamilton's variables
- State the Donkin's theorem
- Elaborate on the Poisson's bracket and their identity
- Know about the Jacobi-Poisson theorem

3.2 BASICS OF GENERALIZED COORDINATES

When we try to describe the configuration of a system of particles, we need some variables. We should choose the least number of possible variables to describe the configuration satisfactorily. These least number of variables are called 'Generalized co-ordinates'. These are nothing but a set of minimum co-ordinates that describe the configuration of a system. Uses of generalized co-ordinates follow some rules. These are as follows:

(*i*) The values of generalized co-ordinates (variables) define and determine the configuration of the system.

(*ii*) The generalize co-ordinates may be varied (as they are variables): arbitrarily and independent of each other, irrespective of constraints.

(*iii*) We can choose generalized co-ordinates at random to simplify the mathematical methods to describe the configuration of the system.

Notation of Generalized Co-ordinates

A set of generalized co-ordinates of n dimension are denoted by q_1, q_2, q_3, ..., q_n; i.e., by q_i, where $i = 1, 2, 3, ...$ up to n.

If we consider that a particle moves in a plane, we may describe its co-ordinates by Cartesian or polar co-ordinates as follows:

Let, in a Cartesian co-ordinate system, the position of a particle at an instant be denoted by $P(x, y)$, whereas in polar co-ordinates system, it is denoted by $P(r, \theta)$. When we describe the position of the particle at an instant by generalized co-ordinates, we can write:

$$\left.\begin{aligned} q_1 &= x \\ q_2 &= y \end{aligned}\right\} \text{ in Cartesian co-ordinates}$$

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and

$$q_1 = r\,[= \sqrt{(x^2 + y^2)}]$$

$$q_2 = \theta\,[= \tan^{-1} y/x]$$

Now, we consider a system of particles in $3N$ dimensions (i.e., 3D of N particles). The generalized co-ordinates of a system of particles can be described as:

$$\left.\begin{aligned} q_1 &= q_1(x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; .....x_N, y_N, z_N; t) \\ q_2 &= q_2(x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; .....x_N, y_N, z_N; t) \\ &..... \\ &..... \\ q_{3N} &= q_{3N}(x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; .....x_N, y_N, z_N; t) \end{aligned}\right\} \quad ...(3.1)$$

The above Equation (3.1) is the transformation equation from Cartesian co-ordinates to generalized co-ordinates in $3N$ dimensions.

The transformation equation of Cartesian co-ordinates from generalized co-ordinates in $3N$ dimensions is as follows:

$$\left.\begin{aligned} x_1 &= x_1(q_1, q_2, q_3; ....., q_{3N}; t) \\ y_1 &= y_1(q_1, q_2, q_3, ..., q_{3N}; t) \\ &..... \\ &..... \\ z_1 &= z_1(q_1, q_2, q_3, ..., q_{3N}; t) \end{aligned}\right\} \quad ...(3.2)$$

In general, $\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, ..., q_{3N}; t) = \vec{r}_i(q_j; t)$ where $i = 1, 2, 3, ...N$ and $j = 1, 2, 3, ... 3N$ (degrees of freedom).

Generalized Displacement

Let us consider a system of N particles, which moves an infinitesimally small distance δr_i from its original position in the Cartesian co-ordinate system.

The position vector of the system can be defined as:

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, ..., q_{3N}; t) = \vec{r}_i(q_j; t)$$

If we consider that the time is fixed, then

$$\delta\vec{r}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \quad (\text{as } t \text{ is fixed, } \delta t = 0) \quad ...(3.3)$$

The term δq_j in the above equation is called *generalized displacement*.

The derivation has been done according to the Euler's theorem of partial derivative. According to the theory, if $f = f(x, y, z)$, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

or

$$f = f(x_1, x_2, x_3, ..., x_j)$$

$$\delta f = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \frac{\partial f}{\partial x_3}\delta x_3 + ... + \frac{\partial f}{\partial x_j}\delta x_j$$

$$= \sum_j \frac{\partial f}{\partial x_j}\delta x_j \Bigg].$$

NOTES

Generalized Velocity

Generalized velocity may be described in terms of time derivative of the generalized displacement δq_j.

We have $\vec{r}_i = \vec{r}_i(q_j; t)$

So, $$\frac{\delta \vec{r}_i}{\delta t} = \dot{\vec{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\delta q_j}{\delta t} + \frac{\partial \vec{r}_i}{\partial t}$$

$$\Rightarrow \quad \dot{\vec{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \qquad ...(3.4)$$

The term $\dot{q}_j \left(= \frac{\delta q_j}{\delta t} \right)$ is called *generalized velocity*.

If the system has K number of constraints, then the number of degrees of freedom is $F = 3N - K$. So, the Equation (3.4) has the form:

$$\dot{\vec{r}}_i = \sum_{j=1}^{F=3N-K} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \qquad ...(3.5)$$

Generalized Acceleration

Again, if we explicitly differentiate the equation of generalized velocity with respect to time, we will get the equation of generalized acceleration.

Differentiating Equation (3.4), we get

$$\frac{\delta^2 \vec{r}_i}{\delta t^2} = \ddot{\vec{r}}_i = \frac{d}{dt}\left(\sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$= \sum_{j=1}^{3N} \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right)\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial r_i}{\partial q_j}\frac{d}{dt}(\dot{q}_j) + \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial t} \right)$$

$$= \sum_{j=1}^{3N} \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial \dot{\vec{r}}_i}{\partial t} \qquad ...(3.6)$$

The term $\ddot{q}_j$ in the above equation is called *generalized acceleration* (when no constraints are present).

Generalized Force

From the equation of generalized displacement Equation (3.3), we can derive the component of generalized force.

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The generalize displacement equation is

$$\delta\vec{r}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

Now, if we consider that a force $\vec{F}_i$ is applied on the *i*-th particle of the system, then in equilibrium condition the virtual work due to the force $\vec{F}_i$ is defined by,

$$W_i = \sum_{i=1}^{N} \vec{F}_i . \delta\vec{r}_i = \sum_{\substack{i=1 \\ j=1}}^{\substack{3N \\ N}} \vec{F}_i . \left(\frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{j=1}^{3N} \vec{Q}_j . \delta q_j \qquad ...(3.7)$$

where $$\vec{Q}_j = \sum_{\substack{i=1 \\ j=1}}^{\substack{3N \\ N}} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j} \qquad ...(3.8)$$

is called the component of *generalized force*.

Generalised Potential

For conservative forces, potential function

$$V = V(q_1, q_2 \ldots q_n)$$

Therefore, $\delta W = -\delta V$

$$= -\Sigma \left(\frac{\partial V}{\partial x_i} \delta x_i \right)$$

$$= -\Sigma \left(\frac{\partial V}{\partial q_j} \right) \delta q_i$$

Also $\delta W = \Sigma Q_j \delta q_j$ where Q_i are generalized forces.

$$\Rightarrow \Sigma Q_j \delta q_j = -\Sigma \left(\frac{\partial V}{\partial q_j} \right) \delta q_j$$

$$\Rightarrow \quad Q_j = -\frac{\partial V}{\partial q_j}$$

3.2.1 Holonomic and Non-Holonomic Systems

Holonomic system are systems which all constraints are integrable into positional constraints. Non-holonomic systems are systems which have constraints that are non-integrable into positional constraints.

A system of material points, i.e., either not constrained by any constraint or constrained only by geometric constraints. The latter impose restrictions on the positions of the points of the system and may be represented by relations of the type as follow

$$f_s(x_1 \dots x_{3N}, t) = 0, \;\; s = 1 \dots k; \;\; f_s(x,t) \in C^2 \qquad \text{...(3.9)}$$

Here t is the time, x_i are the Cartesian coordinates of the point and N is the number of points in the system. If $\partial f_s/\partial t$ a $\equiv 0$, the constraints are said to be stationary; otherwise they are called **non-stationary**. Any position of the system for which the coordinates of the points obey Equations (3.9) is called possible for the given moment t. The constraints in Equation (3.9) impose restrictions not only on the positions x_v, but also on the velocities v_v and on the accelerations w_v of the points is,

$$\left.\begin{aligned} \frac{df_s}{dt} &= \sum_{\nu=1}^{N} \operatorname{grad} f_s \cdot v_\nu + \frac{\partial f}{\partial t} = 0, \\ \frac{d^2 f_s}{dt^2} &= \sum_{\nu=1}^{N} \operatorname{grad} f_s \cdot w_\nu + \dots = 0. \end{aligned}\right\} \qquad \text{...(3.10)}$$

The velocities and accelerations satisfying Equations (3.10) are said to be kinematically possible in a given position x_v of the system for a given moment of time t. Infinitesimal displacements δr_v which satisfy conditions of the type

$$\sum_{\nu=1}^{N} \operatorname{grad} f_s \cdot \delta r_\nu = 0, \;\; s = 1 \dots k, \qquad \text{...(3.11)}$$

For *stationary constraints* real displacements are found amongst the possible displacements, while for non-stationary constraints they are not found among the possible displacements, in general. Possible displacements are useful for converting a *holonomic system* from one position of the system, which is possible for a given t to another infinitely-close position which is possible at the same moment t.

The number of independent variations of the points of the system is said to be the number of its degrees of freedom; for a holonomic system this coincides with the number n=3N–k of independent arbitrary parameters q_i with the help of Equations (3.9) may be represented in the form of relations of the type of,

$$x_\nu = x_\nu(q_1 \dots q_n, t), \;\; \nu = 1 \dots 3N, \;\; x_\nu(q,t) \in C^2 \qquad \text{...(3.12)}$$

The parameters q_i are called generalized, or Lagrangian, coordinates of the system; they are called holonomic coordinates, in distinction from non-holonomic, or quasi-, coordinates π_s, introduced by non-integrable relations of the type

$$d\pi_s = \sum_{i=1}^{n} a_{si} dq_i, \;\; a_{si}(q_i, t) \in C^1 \qquad \text{..(3.13)}$$

NOTES

Constraints that can be analytically expressed by Equation (3.9) are said to be retaining, or two-sided, in distinction from non-retaining, or one-sided, constraints, which can be expressed by inequalities of the type

$$f(x, t) \geq 0,$$

and impose the following conditions on the possible displacements is,

$$\sum_{\nu=1}^{N} \text{grad } f_s \cdot \delta r_\nu \geq 0.$$

The possible displacements of a system with two-sided constraints are reversible; among the possible displacements of a system with one-sided constraints there are irreversible ones. The motions of holonomic systems are described by the *Lagrange equations* (in mechanics) (of the first and second kinds), by the Hamilton equations in Lagrangian coordinates has been already discussed.

Non-holonomic System

A *non-holonomic system* in mathematics is a physical system whose state depends on the path taken in order to achieve it. Such a system is labelled by a set of parameters subject to differential constraints, such that when the system evolves along a path in its parameter space (the parameters varying continuously in values) but finally returns to the original set of parameter values at the start of the path, the system itself may not have returned to its original state.

It is probable to model the wheel mathematically with a system of constraint equations, and then prove that that system is non-holonomic.

First, we define the configuration space. The wheel can change its state in three ways: having a different rotation about its axle, having a different steering angle, and being at a different location. We may say that ϕ is the rotation about the axle, θ is the steering angle relative to the x-axis, and x and y define the spatial position. Thus, the configuration space is,

$$\vec{u} = [x \; y \; \theta \; \phi]^T$$

We must now relate these variables to each other. We notice that as the wheel changes its rotation into their position. The variation in rotation and position implying velocities must be present, we try to relate angular velocity and steering angle to linear velocities by taking simple time-derivatives of the appropriate terms is,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\dot{\phi}\cos\theta \\ r\dot{\phi}\sin\theta \end{pmatrix}$$

The velocity in the x direction is equal to the angular velocity times the radius times the cosine of the steering angle, and the y velocity is similar. Now we do some algebraic manipulation to transform the equation to Pfaffian form so it is possible to test whether it is holonomic, starting with as follows

$$\begin{pmatrix} \dot{x} - r\dot{\phi}\cos\theta \\ \dot{y} - r\dot{\phi}\sin\theta \end{pmatrix} = \vec{0}$$

Then, let it is separate the variables from their coefficients (left side of equation, derived from above equation). We also understand that we can multiply all terms by d*t* so we end up with only the differentials (right side of equation) is,

$$\begin{pmatrix} 1 & 0 & 0 & -r\cos\theta \\ 0 & 1 & 0 & -r\sin\theta \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \vec{0} = \begin{pmatrix} 1 & 0 & 0 & -r\cos\theta \\ 0 & 1 & 0 & -r\sin\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}\theta \\ \mathrm{d}\phi \end{pmatrix}$$

The right side of the equation is now in Pfaffian form is,

$$\sum_{s=1}^{n} A_{rs}\, du_s = 0;\ r = 1, 2$$

We now use the universal test for holonomic constraints. If this system were holonomic, we might have to do up to eight tests. Though, we can use mathematical intuition to try our best to prove that the system is non-holonomic on the first test. Considering the test equation is:

$$A_\gamma\left(\frac{\partial A_\beta}{\partial u_\alpha} - \frac{\partial A_\alpha}{\partial u_\beta}\right) + A_\beta\left(\frac{\partial A_\alpha}{\partial u_\gamma} - \frac{\partial A_\gamma}{\partial u_\alpha}\right) + A_\alpha\left(\frac{\partial A_\gamma}{\partial u_\beta} - \frac{\partial A_\beta}{\partial u_\gamma}\right) = 0$$

we can see that if any of the terms A_α, A_β and $A\gamma$ were zero, that the part of the test equation would be trivial to solve and would be equal to zero. Consequently, it is often best practice to have the first test equation have as many non-zero terms as possible to maximize the chance of the sum of them not equalling zero. Therefore, we choose

$A\alpha = 1$

$A\beta = 0$

$A\gamma = -r\cos\theta$

$u\alpha = dx$

$u\beta = d\theta$

$u\gamma = d\phi$

We substitute into our test equation is,

$$(-r\cos\theta)\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial\theta}(1)\right) + (0)\left(\frac{\partial}{\partial\phi}(1) - \frac{\partial}{\partial x}(-r\cos\theta)\right) + (1)\left(\frac{\partial}{\partial\theta}(-r\cos\theta) - \frac{\partial}{\partial\phi}(0)\right) = 0$$

and simplify,

$r\sin\theta = 0$

We can easily realise that this system, as termed, is non-holonomic, because $\sin\theta$ is not always equal to zero.

A mechanical system is scleronomous if the equations of constraints do not contain the time as an explicit variable and the equation of constraints can be described by generalized coordinates. Such constraints are called scleronomic constraints. The opposite of scleronomous is rheonomous.

Constraints as already discussed, we can solve all the problems of mechanics by the proper exploitation of the equation below:

$$m_i\ddot{\vec{r}}_i = \sum_{\substack{i,j \\ i\neq j}} \vec{F}_{ij} + \sum_i \vec{F}_i^{(e)} \qquad ...(3.14)$$

To prove the various theories (i.e., conservation of linear momentum, conservation of angular momentum, conservation of energy, etc.), we consider that the internal force between *i*-th and *j*-th particles is zero; and also the distance between the *i*-th and *j*-th particles is constant about origin *O*. This is also considered for the motion of a rigid body; i.e., $|\vec{r}_{ij}| = |\vec{r}_i - \vec{r}_j| =$ Constant. So, this is the restriction imposed to the rigid body motion. Another example is when a particle placed on the surface of a solid sphere is restricted to move only on the surface of the sphere.

So, the restrictions employed on the motion of a particle or system of particles are *constraints*. We can describe constraints as follows: 'The limitations or geometrical restrictions on the motion of a particle or system of particles generally known as Constraints.' Constraints simplify our mathematical solvation of various problems in mechanics.

Generally, constraints are classified into the following four groups:

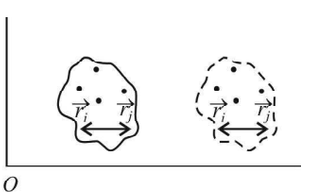
(*i*) Holonomic Constraints: If the conditions of any constraint can be expressed by an equation with the co-ordinates of the particles as well as time explicitly having the form:

$$f(r_1, r_2, r_3, ...; t) = 0 \qquad ...(3.15)$$

Then the constraints are called *holonomic*.

For examples:

1. *Motion of a rigid body*: In this case the distance between any two particles of a rigid body remains always constant, i.e.,



$|\vec{r}_i - \vec{r}_j| = r_{ij}$ (constant)

where $i, j = 1, 2, 3, ..., n$; $i \neq j$.

2. *Particle moving on the surface of sphere*: In this case the particle is restricted to move on the surface of the sphere, i.e.,

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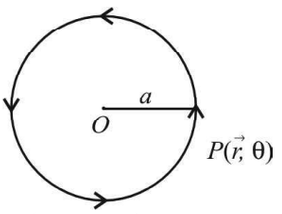
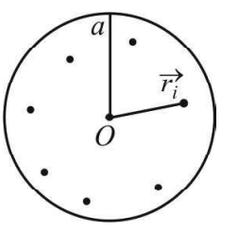


$|\vec{r}| = a$ (constant)

(*ii*) Non-holonomic Constraints: If the conditions of the constraints cannot be expressed by an equation, then they are termed as *non-holonomic* constraints.

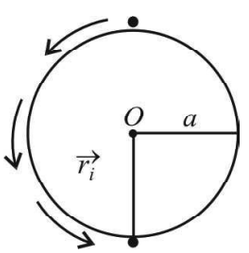


For examples:

1. *Particles confined to move within a circle with rigid wall:* In this case the particles can move only in the circle with rigid wall, i.e.,

   $$|\vec{r}_i| \leq a$$

   where $\vec{r}_i$ is the radius vector of *i*-th particle and *a* is the radius of the circle with rigid wall.
2. *Particle falling under gravity from the top of a sphere*: In this case the condition for radius vector is given by,



$$|\vec{r}_i| \geq a$$

where *a* is the radius of the sphere.

3.2.2 Scleronomic and Rheonomic Systems

Scleronomic System

The scleronomousis is a mechanical system which equations of constraints do not contain the time as an explicit variable and the equation of constraints can be explain by 'Generalized Coordinates'. Such constraints are called **scleronomic constraints**. The opposite of scleronomous is **rheonomous**. Additionally a mechanical system is *rheonomous* if its equations of constraints contain the time as an explicit variable. Such constraints are called rheonomic constraints. The opposite of rheonomous is scleronomous.

NOTES

Generalized Coordinates of Scleronomic System

In three dimensional space, a particle with mass m, velocity v, has kinetic energy T,

$$T = \frac{1}{2}mv^2 .$$

Velocity is the derivative of position r with respect to time t. Use chain rule for several variables is,

$$v = \frac{dr}{dt} = \sum_i \frac{\partial r}{\partial q_i}\dot{q}_1 + \frac{\partial r}{\partial t}$$

Where $\dot{q}_i$ are *generalized coordinates*.

Consequently,

$$T = \frac{1}{2}m\left(\sum_i \frac{\partial r}{\partial q_i}\dot{q}_i \frac{\partial r}{\partial t}\right)^2 .$$

Rearranging the terms carefully,

$$T = T_0 + T_1 + T_2 :$$

$$T_0 = \frac{1}{2}m\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2,$$

$$T_1 = \sum_i m\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial q_i}\dot{q}_i,$$

$$T_2 = \sum_{i,j}\frac{1}{2}m\frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_j}\dot{q}_i\dot{q}_j,$$

Where T_0, T_1, T_2 are respectively homogeneous functions of degree 0, 1, and 2 in *generalized velocities*. If this system is scleronomous, then the position does not depend explicitly with time so that,

$$\frac{\partial r}{\partial t} = 0.$$

Hence, only term of T_2 does not vanish:

$$T = T_2$$

Kinetic energy is a homogeneous function of degree 2 in generalized velocities.

For example,

The 'Simple Pendulum' exhibits Simple Harmonic Motion (SHM) as the acceleration of the pendulum bob is directly proportional to the displacement from the mean position and is always directed towards it. The time period (T) of a simple pendulum for oscillations of small amplitude, is the general express by Equation (3.16)

$$T = 2\pi\sqrt{L/G} \qquad ...(3.16)$$

Whereas L is the length of the pendulum and g is the acceleration of gravity.

As shown in Figure (1), a simple pendulum is a system composed of a weight and a string. The string is attached at the top end to a pivot and at the bottom end to a weight. Being inextensible, the string's length is a constant. Therefore, this system is scleronomous; it obeys scleronomic constraint so that,

$$\sqrt{x^2 + y^2} - L = 0,$$

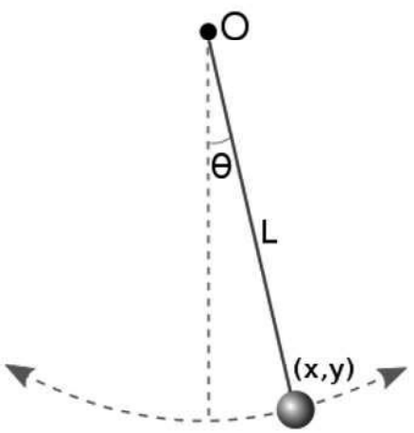


Fig. 1 *A Simple Pendulum*

Where (x, y) is the position of the weight and L is length of the string.

Let us assume the top end of the string is attached to a *pivot point* undergoing a simple harmonic motion shown in Figure (2),

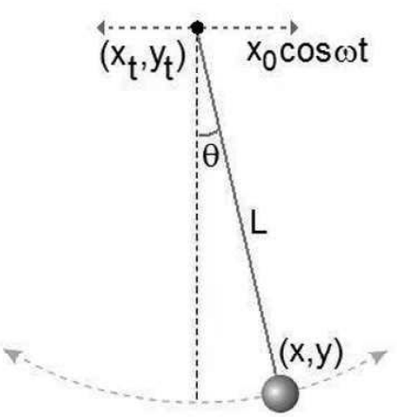


Fig. 2 *A Simple Pendulum with Oscillating Pivot Point*

So that,

$x_t = x_0 \cos \omega t$,

Here x_0 is amplitude, ω is angular frequency, and t is time.

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Though the top end of the string is not fixed, the length of this inextensible string is still a constant. The distance between the top end and the weight must stay the same. Therefore, this system is *rheonomous* as it obeys rheonomic constraint explicitly dependent on time is,

$$\sqrt{(x - x_0 \cos \omega t)^2 + y^2} - L = 0 .$$

Constraints

(*iii*) Scleronomic Constraints: If the constraints do not explicitly depend on time, then they are called *scleronomic*. Holonomic and non-holonomic are also the examples of scleronomic constraints. So, scleronomic constraints are the special type of holonomic and non-holonomic constraints.

(*iv*) Rheonomic Constraints: If the constraints explicitly depend on time, then they are said to be rheonomic type.

Difficulties of Constraints

In the solution of mechanical problems, the constraints show two types of difficulties:

1. The co-ordinates r_i are connected by the equation (3.14) of constraints conditions; therefore, they are not independent.
2. Generally, the forces associated in a mechanical problem are required to maintain the constraints in the system. These forces are not known to us primarily, i.e., the forces associated in a given mechanical problem are not specified directly.

   The first difficulty can be solved by the use of generalized co-ordinates for holomonic type. The second difficulty can be overcome by eliminating the forces from the equation of motion of a mechanical problem at an early stage by the proper exploitation of mechanics.

3.2.3 Generalized Potential

Kinetic Energy

For the kinetic energy T_i of a rigid body K_i with the mass m_i the inertial tensor $\mathbf{I}_i$ the absolute center of gravity velocity $\mathbf{v}_{Mi}$ and the angular velocity w_i, it yields

$$T = \underbrace{\frac{1}{2} m_i \mathbf{v}_{Mi}^2}_{\text{translational}} + \underbrace{\frac{1}{2} \mathrm{w}_i^T \mathbf{I}_i \mathrm{w}_i}_{\text{rotational}} . \tag{3.17}$$

Kinetic energy consists of the translational and rotational parts. Because the kinetic energy is independent from the used coordinate system, it is not important

in which coordinate system the particular energy parts are calculated. On the other hand it is obvious that the angular velocities have to be specified in the same system, as those of the inertial tensor.

The kinetic energy of a multi-body system is obtained by the sum of kinetic energies of the single bodies. Taking the mass midpoint of the single bodies as the reference point, we get

$$T = \frac{1}{2}\sum_{i=1}^{p}\left(m_i \mathbf{v}_{Mi}^2 + \mathrm{w}_i^T \mathbf{I}_i \mathrm{w}_i\right) \tag{3.18}$$

Potential Energy

If the work obtained by the applied forces is independent from the distance covered, the forces have a potential and can thus be determined by differentiation. It gives:

$$\mathbf{f}^e = -\nabla U = \begin{bmatrix} -\dfrac{\partial U}{\partial x} \\ -\dfrac{\partial U}{\partial y} \\ -\dfrac{\partial U}{\partial z} \end{bmatrix} \tag{3.19}$$

with the potential energy $U = U(x, y, z)$ which is a scalar local function.

The potential energy of a multi-body system can be obtained from the sum of the potential energies of the single bodies

$$U = \sum_{i=1}^{p} U_i \tag{3.20}$$

Forces which in accordance with equation (3.18) can be obtained by differentiating a potential possess energy and are therefore referred to as conservative. Non-conservative forces modify their mechanical total energy. If we specially deal with forces which eliminate energy, we refer to them as dissipative forces.

Examples of conservative forces are weights

$f_G = -mg$

and spring forces

$f_F = -cs.$

The appropriate potentials are

$U_G = mgz$

NOTES

and

$$U_F = \frac{1}{2} cs^2.$$

Potentials can be determined at any arbitrary point except for one additive constant. If a multi-body system has only conservative forces, the whole system is referred to as conservative. Therefore, it yields the theorem of conservation of mechanical energy:

$$T + U = T_0 + U_0 = \text{constant} \tag{3.21}$$

The given energy expressions are used for the derivation of the equations of motion. Here, in contrast to the synthetic method the single bodies are not free cut, but the system will be considered as a whole.

The kinetic energy will be represented subject to the *generalized coordinates* and, if necessary, subject to time.

$$T(\mathbf{y}, \dot{\mathbf{y}}, t) = \frac{1}{2} \sum_{i=1}^{p} \left(m_i \mathbf{v}_{Mi}^2(\mathbf{y}, t) + \mathrm{w}_i^T(\mathbf{y}, t) \mathbf{I}_i(\mathbf{y}) \mathrm{w}_i(\mathbf{y}, t) \right) \tag{3.22}$$

The generalized forces result from imposed forces and moments.

$$Q_k = \sum_{i=1}^{p} \left(\left[\frac{\partial \mathbf{r}_i}{\partial y_k} \right]^T \mathbf{f}_i^e + \left[\frac{\partial \mathbf{s}_i}{\partial y_k} \right]^T \mathbf{I}_i^e \right) = \sum_{i=1}^{p} \left(\left[\mathbf{J}_{Ti} \right]_k^T \mathbf{f}_i^e + \left[\mathbf{J}_{Ri} \right]_k^T \mathbf{I}_i^e \right) \tag{3.23}$$

By means of these quantities we obtain Lagrange's equation of motion of second kind

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{y}}_k} \right) - \frac{\partial T}{\partial \mathbf{y}_k} = Q_k, \quad k = 1, \ldots, f \tag{3.24}$$

Annotation

- The number of equations of motion is same as to the number of degrees of freedom of the system. ALso, it is not necessary to introduce the reaction forces as they cannot be calculated.
- In order to write the motion equations, we have to calculate the partial and total differentiations of the function $T(\mathbf{y}, \dot{\mathbf{y}}, t)$. Chain rule is required to perform the total differentiation of T with respect to time *t*.

In conservative systems the calculation of generalized forced with respect to eqution (3.23) can be avoided because these can also be analogous to Equation (3.19) calculated by finding derivative of the potential energy U in accordance with the generalized coordinate. We get,

$$Q_k = -\frac{\partial U}{\partial y_k}. \tag{3.25}$$

From Lagrange's function $L = T - U$, we obtain Lagrange's motions equation in classical form

$$\frac{d}{dt}\quad\frac{(\partial L)}{(\partial \dot{y}_k)} - \frac{\partial L}{\partial y_k} = 0, \quad k = 1,\ldots, f \tag{3.26}$$

If, in addition to conservative forces, non-conservative forces appear, these (and only these) are considered by the expression of Equation (3.23) on the right hand side of Equation (3.29).

For example,

The kinetic energy of a total system

$$T = \frac{1}{2}m\mathbf{v}_1^2 + \frac{1}{2}m\mathbf{v}_2^2 + \frac{1}{2}I_z\omega_1^2 + \frac{1}{2}I_z\omega_1^2$$

$$= \frac{1}{2}ml^2\dot{\alpha}^2 + \frac{1}{2}ml^2\left(4\dot{\alpha}^2 + \dot{\beta}^2 + 4\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)\right) + \frac{1}{2}I_z\dot{\alpha}^2 + \frac{1}{2}I_z\dot{\beta}^2 \tag{1}$$

and the potential energy results into

$U = -mgl(3\cos\alpha + \cos\beta)$

We obtain the following expression for the partial derivation

$$\frac{\partial L}{\partial \dot{\alpha}} = \left(I_z + 5ml^2\right)\alpha^2 - 2ml^2\dot{\beta}\cos(\alpha-\beta) + I_z\dot{\alpha}$$

$$\frac{\partial L}{\partial \dot{\beta}} = 2ml^2\dot{\alpha}\cos(\alpha-\beta) + ml^2\dot{\beta} + I_z\dot{\beta}$$

$$\frac{\partial L}{\partial \alpha} = -2ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - 3mgl\sin\alpha$$

$$\frac{\partial L}{\partial \beta} = 2ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - mgl\sin\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \left(I_z + 5ml^2\right)\ddot{\alpha} + 2ml^2\ddot{\beta}\cos(\alpha-\beta) + 2ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - 2ml^2\dot{\beta}^2\sin(\alpha-\beta) \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = 2ml^2\ddot{\alpha}\cos(\alpha-\beta) + ml^2\ddot{\beta} + I_z\ddot{\beta} + 2ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) + 2ml^2\dot{\alpha}^2\sin(\alpha-\beta).$$

(3)

NOTES

If we insert these statements into Lagrange's Equations (3.26), we obtain the following required equation of motion

$$\left(I_z + 5ml^2\right)\ddot{\alpha} + 2ml^2\ddot{\beta}\cos(\alpha-\beta) - 2ml^2\ddot{\beta}^2\sin(\alpha-\beta) = -3ml\sin\alpha$$

$$2ml^2\ddot{\alpha}\cos(\alpha-\beta) + \left(I_z + ml^2\right)\ddot{\beta} - 2ml^2\dot{\alpha}^2\sin(\alpha-\beta) = -mgl\sin\beta$$

(4)

In this case we also obtain the motion equations:

$$\mathbf{M}(\mathbf{y},t) = \begin{bmatrix} I_z + 5mI^2 & 2ml^2\cos(\alpha-\beta) \\ 2ml^2\cos(\alpha-\beta) & ml^2 \end{bmatrix},$$

$$\mathbf{k}(\mathbf{y},\dot{\mathbf{y}},t) = \begin{bmatrix} -2ml^2\dot{\beta}^2\sin(\alpha-\beta) \\ -2ml^2\dot{\beta}^2\sin(\alpha-\beta) \end{bmatrix},$$

$$\mathbf{q}(\mathbf{y},\dot{\mathbf{y}},t) = \begin{bmatrix} -3mgl\sin\alpha \\ -mgl\sin\beta \end{bmatrix}.$$

Annotation

- The motion equations, which result from Lagrange's equations, are completely identical with the equations (with identical generalized coordinates) calculated in Section, which we obtained by means of d'Alembert's principle from Newton-Euler Equations. This is true for the general case also. Both methods exclusively differ in respect to their approaches, not in respect to their results.
- In contrast to Newton-Euler's formalism, Lagrange's equations (in this form) allow for the calculation of reaction forces (position forces). Conversely, the consideration of these forces is not necessary when we set up equations, which accrues considerable advantages in practice.

Check Your Progress

1. What do you understand by generalised co-ordinates?
2. Give the rules of generalised co-ordinates.
3. Define the holonomic and non-holonomic system.
4. Differentiate between scleronomic and rheonomic system.

3.3 LAGRANGE EQUATIONS OF FIRST KIND AND SECOND KIND

NOTES

Lagrange Equaton of First Kind

Consider a holonomic system of n material points $A_1(x_1, y_1, z_1),...,$ $A_n(x_n, y_n, z_n)$. Denote the forces acting on the points of the system by $P_1,...,P_n$ and let $m_1,...m_n$, be the masses of these points. Assuming that the constraints are bilateral, defined by Equation (3.27)

$$F_j(x_1, y_1, z_1,...x_n, y_n, z_n, t) = 0 \qquad (j = 1,2,...m) \tag{3.27}$$

The virtual displacement of the system satisfy the equations:

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\delta x_i + \frac{\partial F_j}{\partial y_i}\delta y_i + \frac{\partial F_j}{\partial z_i}\delta z_i\right) = 0 \qquad (j = 1,2,...,m) \tag{3.28}$$

From D'Alembert's principle we have,

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\delta_{z_i}] = 0 \tag{3.29}$$

Equation (3.29) holds for every set of numbers $\delta x_i, \delta y_i, \delta z_i$ satisfying the system of Equations (3.28). There exist Lagrange's multipliers $\lambda_1,...,\lambda_m$ such that Equation (2.27) are satisfied each moment t where it is necessary to substitute $P_{i_x} - m_i\ x_i^{\cdot\cdot}$ for P_{i_x},

$$\begin{aligned}
&P_{i_x} - m_i x_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial x_i} = 0,\\
&P_{i_y} - m_i y_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial y_i} = 0,\\
&P_{i_z} - m_i z_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial z_i} = 0, \qquad (i = 1,2,...,n).
\end{aligned} \tag{3.30}$$

The numbers $\lambda_1,...,\lambda_m$, depend on t and hence are functions of time. Consequently, $\lambda_1 = \lambda_1(t),...,\lambda_m = \lambda_m(t)$. From Equation (3.30) we get,

$$m_i x_i^{\cdot\cdot} = P_{i_x} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial x_i},$$

NOTES

$$m_i y_i^{\cdot\cdot} = P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i},$$

$$m_i z_i^{\cdot\cdot} = P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \qquad (i = 1, 2, ..., n).$$

These equations are known as Lagrange's equations of the first kind.

Lagrange Equation of Second Kind

Let us the place of a mechanical system is defined by n 'Generalized Coordinates' $q_j (j=1,\ldots, n)$, while its motion is subject to m *non-holonomic* constraints is,

$$q_{s+\nu}^{(k)} = \varphi_\nu (t, q_j, \dot{q}_j, ..., q_j^{(k-1)}, q_i^{(k)}), \tag{3.31}$$

Here is, $\nu = 1,..., m$ and s is the number of degrees of freedom $s = n - m$; $i = 1,..., s$;

$(L)^k = \partial^k (L)/\partial t^k$, where t is time; $k \geq 1$.

The constraints in Equation (3.31) are of *Chaplygin's type*, since q_i are independent coordinates and $q_{s+\nu}$ are dependent ones.

So that,

$$(k+1)\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_i^{(k)}} = -\Lambda_\nu \frac{\partial \varphi_\nu}{\partial q_i^{(k)}}, \quad (k+1)\frac{\partial L^{(k+1)}}{\partial q_{s+\nu}^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_{s+\nu}^{(k)}} = \Lambda_\nu, \tag{3.32}$$

Where L is a Lagrangian of the system and λ_ν are unknown *Lagrange's multipliers.* Extending them to the system of N point with variable mass, one obtains is,

$$\begin{aligned} (k+1)\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_i^{(k)}} &= -\Lambda_\nu \frac{\partial \varphi_\nu}{\partial q_i^{(k)}} + R_i, \\ (k+1)\frac{\partial L^{(k+1)}}{\partial q_{s+\nu}^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_{s+\nu}^{(k)}} &= \Lambda_\nu + R_{s+\nu}. \end{aligned} \tag{3.33}$$

Therefore, generalized reactive forces

$$R_j = \sum_{I=1}^{N} \dot{M}_I \left(\vec{V}_I - \frac{d\vec{r}_I}{dt} \right) \frac{\partial \vec{r}_I}{\partial q_j}$$

it is an appear as a consequence of mass variation $\dot{M}_I$ of I[th] point and relative velocity of their change to be a $(\vec{V}_I - d\,\vec{r}_I / dt)$ where $\vec{V}_I$ is the absolute velocity of an added or separated particle

NOTES

Note: the assumption of the reactive force in this form requires treating mass as a constant during the differentiation of the Lagrange's function.

After eliminating the multipliers the Equation (3.33) transformed into as follow,

$$(k+1)\frac{\partial L_*^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L_*^{(k)}}{\partial q_i^{(k)}} = \left(R_i + R_{s+\nu}\frac{\partial \varphi_\nu}{\partial q_i^{(k)}} \right)_*,$$

Where L_* denotes the terms obtained after excluding $q^{(k)}_{s+v}$ and $q^{(k+1)}_{s+v}$ from Equation (3.33).

3.3.1 Uniqueness of Solution

Let consider the $\mathcal{J} : W^{1,p} \longrightarrow \mathbb{R}$.

So that, it will be a follow hypotheses is,

- $L(t, x, v)$ be a regular.
- $\alpha \odot |v|^p \leq L(t,x,v) \leq \beta \odot (|x|^p \oplus |v|^p)$ with α, β > 0;
- (t, x, v) is convex on for any $\mathbb{R}^2$.

Therefore,

$$\text{Inf}\{\mathcal{J}(x) : x \in W^{1,p}(X)\}$$

has at minimum one solution. In addition, if $L(t, x, v)$ is strictly convex on $\mathbb{R}^2$ for any $t \in X$, this *solution is unique*.

Proof: Assume that there is exist $\tilde{x}_1, \tilde{x}_2 \in W^{1,p}$ such that

$$\text{Inf}\{\mathcal{J}(\tilde{x}) : x \in \mathrm{W}^{1,p}\} = \mathcal{J}(\tilde{x}_1) = \mathcal{J}(\tilde{x}_2) = m \in \mathbb{R},$$

and show that this implies is,

$$\tilde{x}_1 = \tilde{x}_2 .$$

Let $\tilde{y} = (\tilde{x}_1 \oplus \tilde{x}_2) \oslash 2$, so that $\overline{y} \in W^{1,p}$.

Since L is convex, we can say that $\tilde{y}$ is also a minimum of $\mathcal{J}$ because

$$m \leq \mathcal{J}(\tilde{y}) \leq (\mathcal{J}(\tilde{x}_1) \oslash 2) \oplus (\mathcal{J}(\tilde{x}_2) \oslash 2) = m$$

We thus obtain,

$$\int_X^{\oplus} \left[\left(L\left(t, \tilde{x}_1, \frac{d^{\oplus}\tilde{x}_1}{dt}\right) \oslash 2 \right) \oplus \left(L\left(t, \tilde{x}_1, \frac{d^{\oplus}\tilde{x}_2}{dt}\right) \oslash 2 \right) \right.$$
$$\left. \ominus L\left(t, (\tilde{x}_1 \oplus \tilde{x}_2) \oslash 2, \frac{d^{\oplus}(\tilde{x}_1 \oplus \tilde{x}_2)}{dt} \oslash 2\right) \right] \odot dt = \mathbf{0} \cdot$$

The convexity of L confirms that the integrate is non-negative. As the integral is null, so the only possibility is,

NOTES

$$\left(L\left(t,\tilde{x}_1,\frac{d^{\oplus}\tilde{x}_1}{dt}\right)\oslash 2\right)\oplus\left(L\left(t,\tilde{x}_1,\frac{d^{\oplus}\tilde{x}_2}{dt}\right)\oslash 2\right)$$
$$\ominus L\left(t,(\tilde{x}_1\oplus\tilde{x}_2)\oslash 2,\frac{d^{\oplus}(\tilde{x}_1\oplus\tilde{x}_2)}{dt}\oslash 2\right)=\mathbf{0}\text{ on }X\cdot$$

We now use the strict convexity of L to get that $\tilde{x}_1=\tilde{x}_2$ and $\frac{d^{\oplus}\tilde{x}_1}{dt}=\frac{d^{\oplus}\tilde{x}_2}{dt}$ on X as asserted.

The proof is complete.

3.3.2 Energy Equation for Conservative Fields

The D'Alembert's equation may be defined as:

$$\sum_i \vec{P}_i.\delta\vec{r}_i = \sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i$$

$$\Rightarrow \sum_i \vec{P}_i.\delta\vec{r}_i = \sum_i m_i\ddot{\vec{r}}_i.\delta\vec{r}_i$$

$$= \sum_{i,j} m_i\ddot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$$= \sum_{i,j}\left[\left\{\frac{d}{dt}\left(m_i\dot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j}\right)-m_i\dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)\right\}\right]\delta q_j \tag{3.34}$$

From the equation of generalized velocity, we have

$$\left.\begin{aligned}\dot{\vec{r}}_i = \vec{v}_i = \sum_j\frac{\partial\vec{r}_i}{\partial q_j}\dot{\vec{q}}_j+\frac{\partial\vec{r}_i}{\partial t};\\ \frac{\partial\vec{v}_i}{\partial\dot{q}_j}=\frac{\partial\vec{r}_i}{\partial q_j}\end{aligned}\right\} \tag{3.35}$$

So,

Putting these values in Equation (3.34), we have

$$\sum_i \vec{P}_i.\delta\vec{r}_i = \sum_{i,j}\left[\left\{\frac{d}{dt}\left(m_i\vec{v}_i.\frac{\partial\vec{v}_i}{\partial\dot{q}_j}\right)-m_i\vec{v}_i.\frac{\partial\vec{v}_i}{\partial q_j}\right\}\right]\delta q_j$$

$$= \sum_j\left[\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{q}_j}\left(\sum_i\frac{1}{2}m_iv_i^2\right)\right\}-\frac{\partial}{\partial q_j}\left(\sum_i\frac{1}{2}m_iv_i^2\right)\right]\delta q_j \tag{3.36}$$

From the equation of generalized force, we have

$$\vec{Q}_j = \sum_{i,j}\vec{F}_i\frac{\partial\vec{r}_i}{\partial q_j}$$

NOTES

$$\Rightarrow \qquad \vec{Q}_j.\delta q_j = \sum_{i,j} \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_i \vec{F}_i.\delta \vec{r}_i$$

[by the equation of generalized displacement

$$= \sum_i \vec{P}_i.\delta \vec{r}_i$$

[by D'Alembert's equation]

Substituting the value of $\sum_i \vec{P}_i.\delta \vec{r}_i$ in Equation (3.36), we have

$$\vec{Q}_j.\delta q_j = \sum_j \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_i}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j$$

$$= \sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \right] \delta q_j \qquad \left[\text{as } \sum_i \frac{1}{2} m_i v_i^2 = T \text{ (K.E.)} \right]$$

$$\Rightarrow \qquad \sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - \vec{Q}_j \right] \delta q_j = 0$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = \vec{Q}_j \qquad (3.37)$$

The Equation (3.37) is often termed as Lagrange's equation, when the system is conservative, i.e., when the forces are derivable from a scalar function (Potential) V, which is the potential energy of the system:

i.e., $\vec{F}_i = -\vec{\nabla}_i V$

So, from the generalized force equation, we can write

$$\vec{Q}_j = \sum_i \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\sum_i \vec{\nabla}_i V.\frac{\partial \vec{r}_i}{\partial q_i}$$

$$= -\frac{\partial V}{\partial q_j}$$

So, Equation (3.37) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial (T - V)}{\partial q_j} = 0$$

The potential energy V is the function of position only and hence independent of generalized velocities. So, we can include the potential energy V in the partial derivative with respect to $\dot{q}_j$; so, the above equation takes the form:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(T - V) \right] - \frac{\partial}{\partial q_j}(T - V) = 0$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{3.38}$$

[for conservative system]

NOTES

where $T - V = L$ is called Lagrangian and the Equation (3.38) is termed as the *Lagrange's equation of motion* for a conservative system.

For the non-conservative force field, the Lagrange's equation of motion is termed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{3.39}$$

[for non-conservative system]

Thus, the Lagrangian of a system is the difference between kinetic energy (T) and potential energy (V), i.e., $L = T - V$.

From the above discussion, we can define L as the functions of q_j, $\dot{q}_j$ and t [as L depends on T and V]; hence we can write

$$L = L(q_j, \dot{q}_j; t) \tag{3.40}$$

Generalized Momentum from Lagrangian

The Lagrange's equation of motion is defined as $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$ for conservative system, where $L = T - V$.

So,
$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - V) = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

[as the potential energy V depends only on position]

$$= \frac{\partial}{\partial \dot{q}_j}\sum_j\left(\frac{1}{2}m_j\dot{q}_j^2\right) = \sum_j m_j\dot{q}_j = P_j \tag{3.41}$$

P_j in the above equation is called the *generalized momentum*.

Conservation of Generalized Momentum

We have, $L = L(q_j, \dot{q}_j; t)$

If we consider that the generalized co-ordinate q_j is *cyclic* or *ignorable*, then $L \neq L(q_j)$.

So, $$\frac{\partial L}{\partial q_j} = 0$$

We can write the Lagrange's equation of motion as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \text{ for conservative system.}$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_j} = \text{Constant} \tag{3.42}$$

From Equations (3.31) and (3.32), we can write

P_j= Constant; i.e., generalized momentum is conserved.

[If the Lagrangian of a system does not contain generalized co-ordinate q_j, then obviously for such system $\frac{\partial L}{\partial q_j} = 0$. Such a co-ordinate is referred to as an ignorable or cyclic co-ordinate. And for that type of co-ordinate system, the Lagrangian L has the form $L = L(\dot{q}_j, t)$].

Conservation of Generalized Linear Momentum

We have, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

$$\Rightarrow \quad \frac{d}{dt}(P_j) = \frac{\partial}{\partial q_j}(T - V) = -\frac{\partial V}{\partial q_i}$$

[as $\frac{\partial L}{\partial \dot{q}_j} = P_j$ by Equation (3.31) and K.E. (T) is independent of position]

$$\Rightarrow \quad \dot{P}_j = -\frac{\partial V}{\partial q_j} = Q_j \qquad \text{[By Equation (3.38)]}$$

where Q_j defines generalized force and for a conservative system Q_j must vanish.

So, $\dot{P}_j = 0$

$\Rightarrow \quad P_j = \text{Constant},$

i.e., generalized linear momentum is conserved.

Conservation of Generalized Angular Momentum

From the conservation of generalized linear momentum, we have

$$P_j = \text{Constant}$$

But we have $\vec{P}_j = \hat{n}.\vec{L}$, where $\vec{L}$ denotes the total angular momentum.

So, we can write

$$\hat{n}.\vec{L} = \text{Constant}$$

or, $\vec{L}$ is conserved, i.e., generalized angular momentum is conserved.

[as $P_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$ (V depends only on Position, q_j)

NOTES

$$= \sum_i m_i \vec{v}_i . \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i . \hat{n} \times \vec{r}_i$$

$$= \hat{n}_i . \sum_i \vec{r}_i \times \vec{P}_i$$

$$= \hat{n}.\vec{L} \text{ ; because } | d\vec{r}_i | = AB\, dq_j = r_i \sin\theta\, dq_j$$

$$\Rightarrow \quad \frac{| d\vec{r}_i |}{| dq_j |} = r_i \sin\theta = \hat{n} \times \vec{r}_i \,]$$

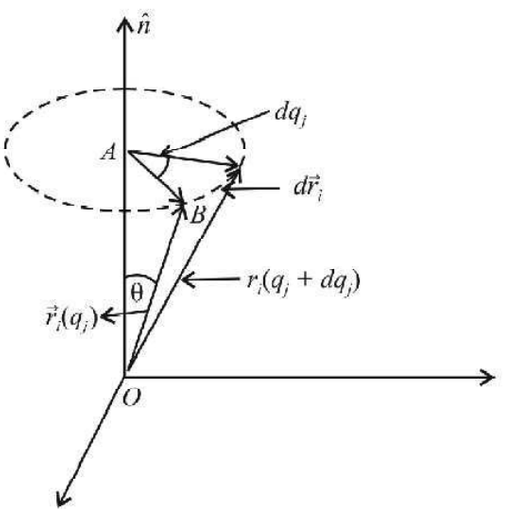

Fig 3.1 *Displacement of Position Vectors for a Rotational Motion*

Conservation of Generalized Energy

We have $$L = L(q_j, \dot{q}_j; t)$$

$$= L(q_1, q_2, ..., q_j; \dot{q}_1, \dot{q}_2, ..., \dot{q}_j; t)$$

Let us assume that the Lagrangian L does not explicitly depend on time.

So, $$L = L(q_1, q_2, ..., q_j; \dot{q}_1, \dot{q}_2, ..., \dot{q}_j)$$

$$\Rightarrow \quad \frac{dL}{dt} = \frac{\partial L}{\partial q_1}\dot{q}_1 + \frac{\partial L}{\partial q_2}\dot{q}_2 + ... + \frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_1}\ddot{q}_1$$

$$+ \frac{\partial L}{\partial \dot{q}_2}\ddot{q}_2 + ... + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \text{ [as per Euler's theorem]}$$

$$= \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \qquad (3.43)$$

From Lagrange's equation of motion, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad \text{[for conservative system]}$$

Putting the value of $\frac{\partial L}{\partial q_j}$ in Equation (3.43), we get

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\Rightarrow \quad L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \text{Constant} \tag{3.44}$$

but $$\frac{\partial L}{\partial \dot{q}_j} = P_j = \sum_j m_j \dot{q}_j \qquad \text{[by Equation (3.41)]}$$

Putting the value of $\frac{\partial L}{\partial \dot{q}_j} = \sum m_j \dot{q}_j$ in Equation (3.44), we get

$$L - \sum_j \dot{q}_j m_j \dot{q}_j = \text{Constant}$$

$$\Rightarrow \quad L - \sum_j m_j \dot{q}_j^2 = \text{Constant}$$

$$\Rightarrow \quad T - V - 2T = \text{Constant} \qquad [\text{as } L = T - V \text{ and } T = \sum_j \frac{1}{2} m_j \dot{q}_j^2]$$

$$\Rightarrow \quad T + V = \text{Constant} \tag{3.45}$$

i.e., total generalized energy is conserved.

So, conservation of energy states that if the Lagrangian function does not contain time explicitly, the total generalized energy of the system is *conserved*.

Check Your Progress

5. State the first kind of Lagrange equation.
6. Define the Lagrange equation of motion for non-conservative force field.

3.4 HAMILTON'S VARIABLES

We have already discussed Lagrangian and Lagrange's equations of motion, which consist of a set of $2n$ dimensional second order differential equations. Here, in this section, we will discuss about a set of $2n$ dimensional generalized co-ordinate system of the first order, with a function H (Hamiltonian), which is termed as 'Hamilton's canonical equations of motion'. The Hamiltonian is said to be the foundation of statistical and quantum mechanics.

From the conservation of generalized energy, we have

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \text{Constant}$$

$$\Rightarrow \frac{d}{dt}\left[L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right] = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left[L - \sum_j \dot{q}_j P_j \right] = 0 \qquad \left[\text{as } P_j = \frac{\partial L}{\partial \dot{q}_j} \right]$$

$$\Rightarrow \quad \frac{d}{dt}\left[\sum_j \dot{q}_j P_j - L \right] = 0 \Rightarrow \sum_j \dot{q}_j . P_j - L = \text{Constant} \qquad (3.46)$$

(Thus, the quantity $(\Sigma \dot{q}_j P_j - L)$ is constant in motion with the condition that the Lagrangian L does not contain time explicitly. This quantity is designated by H (Hamiltonian function)).

So,
$$H = \sum_j \dot{q}_j . P_j - L(q_j, \dot{q}_j)$$

$$= \sum_j \dot{q}_j . m_j \dot{q}_j - L(q_j, \dot{q}_j)$$

$$= \sum_j m_j . \dot{q}_j^2 - L(q_j, \dot{q}_j)$$

$$= 2T - (T - V) \qquad \left[\text{as } T = \sum_j \frac{1}{2} m_j \dot{q}_j^2 \right]$$

$$= T + V = \text{Total energy of the system}$$

Hence, the Hamiltonian H is nothing but the total energy of the system.

From the above discussion, we can define H as the function of q_j, p_j and t.

Therefore, $H = H(q_j, P_j; t)$ (3.47)

3.4.1 Hamilton's Canonical Equations of Motion

As per the discussions in the previous section by Equation 3.47, the Hamiltonian H is the function of the position co-ordinates (q_j), the momentum co-ordinates (P_j) and the time (t), i.e.,

$$H = H(q_j, P_j; t);$$

So,
$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial P_j} dP_j + \frac{\partial H}{\partial t} dt$$

(according to Euler's theorem) (3.48)

Also, we have

$$H = \sum_j \dot{q}_j P_j - L$$

So,
$$dH = \sum_j P_j d\dot{q}_j + \sum_j \dot{q}_j dP_j - dL \quad (3.49)$$

But, we have Lagrangian $L = L(q_j, \dot{q}_j; t)$

So,
$$dL = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt$$

(according to Euler's theorem) (3.50)

Putting the value of dL from Equation (3.50) to Equation (3.49), we get

$$dH = \sum_j P_j d\dot{q}_j + \sum_j \dot{q}_j dP_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$= \sum_j P_j d\dot{q}_j + \sum_j \dot{q}_j . dP_j - \sum_j \dot{P}_j dq_j - \sum P_j d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$\left(\text{as } \frac{\partial \mathrm{L}}{\partial \dot{q}_j} = P_j \text{ and } \frac{\partial L}{\partial q_j} = \dot{P}_j \right)$$

$$= \sum_j \dot{q}_j dP_j - \sum_j \dot{P}_j dq_j - \frac{\partial L}{\partial t} dt \quad (3.51)$$

Comparing the coefficients of Equations (3.48) and (3.49), we get

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial P_j} \\ -\dot{P}_j &= \frac{\partial H}{\partial q_j} \\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t} \end{aligned}\right\} \quad (3.52)$$

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Equation (3.52) is termed as 'Hamilton's canonical equations of motion'. And the equation is a set of $2n$ dimensional first order differential equations with generalized coordinates.

Physical Significance of Hamiltonian (H)

(a) We have, $H = H(q_j, P_j; t)$

$$\Rightarrow \quad \frac{dH}{dt} = \sum_j \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial H}{\partial P_j}\dot{P}_j + \frac{\partial H}{\partial t}$$

$$= -\sum_j \dot{P}_j \dot{q}_j + \sum_j \dot{q}_j . \dot{P}_j - \frac{\partial L}{\partial t}$$

[from Equation (3.52)]

$$= -\frac{\partial L}{\partial t} \qquad (3.53)$$

Now, if we consider that Lagrangian L does not contain time (t) explicitly, then

$$\frac{\partial L}{\partial t} = 0$$

Hence, Equation (3.53) takes the form

$$\frac{dH}{dt} = 0$$

$$\Rightarrow \quad H = \text{Constant}$$

Thus, we can conclude that if Lagrangian L does not contain time (t) explicitly, the Hamiltonian H [i.e., total energy $(T + V)$] is conserved.

(b) Now, if we consider that Lagrangian L does not contain position co-ordinates (q_j), then

$$\frac{\partial L}{\partial q_j} = 0;$$

But, we know $\frac{\partial L}{\partial q_j} = \dot{P}_j$;

So, we can write

$$\dot{P}_j = 0$$

$$\Rightarrow \quad P_j = \text{Constant}$$

We know, $\dot{P}_j = -\frac{\partial H}{\partial q_j}$;

If P_j is constant, then

$$\dot{P}_j = -\frac{\partial H}{\partial q_j} = 0$$

$$\Rightarrow \quad \frac{\partial H}{\partial q_j} = 0$$

Thus, we can conclude that if Lagrangian L does not contain position co-ordinates (q_j) [i.e., if a co-ordinate is cyclic or ignorable in L], then Hamiltonian H also does not contain position co-ordinates (q_j), i.e., the co-ordinates are also cyclic or ignorable in H.

3.4.2 Donkin's Theorem

The Hamiltonian formulation is obtained from the Lagrangian function L by replacing the generalized velocity $\dot{q}_i$ by the conjugate momenta p_i. This can be done by using Donkin's theorem.

Let $X(x_1, x_2, ..., x_n)$ be a function of independent variables $x_1, x_2, ..., x_n$, with Hessian,

$$\det\left(\frac{\partial^2 X}{\partial x_i \partial x_j}\right)_{i,j=1}^{n} \neq 0 \tag{3.54}$$

Let there be a transformation of variables generated by $X(x_1, x_2, ..., x_n)$ given by,

$$y_i = \frac{\partial X}{\partial x_i}, \quad i = 1, 2, ..., n \tag{3.55}$$

Then there exists a transformation which is inverse of the transformation given by Equation (3.55) which likewise generates some function $Y(y_1, y_2, ..., y_n)$. Here $y_1, y_2, ..., y_n$ are also independent, i.e.,

$$x_i = \frac{\partial Y}{\partial y_i}, \quad i = 1, 2, ..., n \tag{3.56}$$

Then, the following relation exists between the inverse transformation Y and its generator function X:

$$Y = \sum_{i=1}^{n} x_i y_i - X \tag{3.57}$$

If the function X contains the parameters $\alpha_1, \alpha_2, ..., \alpha_m$, i.e., if $X = X(x_1, x_2, ..., x_n; \alpha_1, ..., \alpha_m,)$ then Y also contains these parameters, i.e., $Y = Y(y_1, y_2, ..., y_n; \alpha_1, ..., \alpha_m,)$ and

$$\frac{\partial Y}{\partial \alpha_j} = -\frac{\partial X}{\partial \alpha_j}, \quad j = 1, 2, ..., m \tag{3.58}$$

Proof: It can be clearly seen that the Hessian of X shown in Equation (3.54) coincides with the Jacobian of the right hand side of Equation (3.55). Therefore, the condition (3.54) shows that using Equation (3.55), it is possible to express the variables $x_1, x_2, ...$ in terms of $y_1, y_2, ...$, or

$$x_1 = f_i\,(y_1, y_2, \ldots, y_n) \qquad i = 1, \ldots, n \qquad (3.59)$$

If the function Y is defined by Equation (3.57) in which the variables x_i are replaced by the expression given in Equation (3.59), then

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$$\frac{\partial Y}{\partial y_i} = \frac{\partial}{\partial y_i}\left(\sum_{l=1}^{n} x_l y_l - X\right)$$

$$\sum_{l=1}^{n} \frac{\partial x_l}{\partial y_i} y_l + x_i - \sum_{l=1}^{n} \frac{\partial X}{\partial x_l}\frac{\partial x_l}{\partial y_i}\left(\frac{\partial y_l}{\partial y_i} = 0 \text{ and } x_i y_i \text{ yields } x_i;\ y_l = \frac{\partial X}{\partial x_i}\right)$$

$$= x_i$$

Hence Equation (3.56) holds true. Now if contains the parameters α's in addition to x's then the parameters occur in the direct transformation given in Equation (3.55) and in the same way in the inverse also:

$$x_i = f_i(y_1, y_2, \ldots, y_n, \alpha_1, \alpha_2, \ldots, \alpha_m), \quad i = 1, \ldots, n$$

The function Y is determined by Equation (3.57) in which now the x_i's are replaced by $f_i(y_1, y_2, \ldots y_n; \alpha_1, \alpha_2, \ldots, \alpha_m)$ and so,

$$\frac{\partial Y}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j}\left(\sum_{i=1}^{n} x_i y_i - X\right)$$

$$= \sum_{i=1}^{n} \frac{\partial x_i}{\partial \alpha_j} y_i - \sum_{i=1}^{n} \frac{\partial X}{\partial x_i}\frac{\partial x_i}{\partial \alpha_j} - \frac{\partial X}{\partial \alpha_j}$$

Now as y's and α's are independent, $\frac{\partial y}{\partial \alpha} = 0$ and since $y_i = \frac{\partial X}{\partial x_i}$, we have $\frac{\partial Y}{\partial \alpha_j} = -\frac{\partial X}{\partial \alpha_j}$

This proves Equation (3.58).

3.4.3 Cyclic Coordinates and Routh's Equations

Generalized coordinates of a certain physical system that do not occur obviously in the expression of the individual function of this system. When one uses the corresponding equations of motion, one may obtain at once for every cyclic coordinate the integral of motion corresponding to it. For example, if the Lagrange function $L(q_i, \dot{q}_i, t)$, where the q_i are 'Generalized Coordinates', the $\dot{q}_i$ is the generalized velocities, and t the time, does not contain q_j explicitly, then q_j is a cyclic coordinate, and the j-th Lagrange equation has the form (d/dt) $(\partial L / \partial \dot{q}_j) = 0$, which at once gives an integral of motion is,

$$\frac{\partial L}{\partial \dot{q}_j} = \text{constant}$$

The concept of a cyclic coordinate (angle coordinate, angle variable) ties in with action-angle coordinates in the theory of completely-integrable Hamiltonian systems. Each such system (with finite degrees of freedom) can be transformed into one with coordinates (y_k, x_k) such that the Hamiltonian has the form $H(y_1,\ldots,y_n)$, i.e. does not contain $x_1,\ldots,x_n$. Then the y_k are called the action coordinates and the x_k the angle coordinates.

In classical mechanics, *Routh's procedure* or *Routhian mechanics* is a hybrid formulation of Lagrangian mechanics and Hamiltonian mechanics developed by Edward John Routh. Similarly, the Routhian is the function which replaces both the Lagrangian and Hamiltonian functions. Routhian mechanics is equivalent to Lagrangian mechanics and Hamiltonian mechanics. It offers an alternative way to solve mechanical problems.

In the method devised by Routh, the q, $\dot{q}$ basis of Lagrange is changed to q, p basis only for those coordinates which are cyclic and their equations of motion are obtained in Hamiltonian form, while the remaining coordinates which are not cyclic are represented by Lagrange's equations. If $q_1, q_2, ..., q_s$ are cyclic coordinates, then a new function R which is called Routhian is defined by the following relation:

$$R(q_1, q_2, ..., q_n, p_1, p_2, ..., p_a, \dot{q}_{s+1}, ..., \dot{q}_n, t) = \sum_{i=1}^{s} p_i \dot{q}_i - L$$

Routhian formulation is a path to both the Lagrangian and Hamiltonian formulations.

Noether's theorem states that if the coordinate q_j is cyclic, and if the Lagrange multiplier plus generalized force contributions for the jth coordinates are zero, then the canonical momentum of the cyclic variable, p_j, is a constant of motion. Therefore, both (q_j, p_j) are constants of motion for cyclic variables, and these constant (q_j, p_j) coordinates can be factored out of the Hamiltonian $H(p, q, t)$. This reduces the number of degrees of freedom included in the Hamiltonian. For this reason, cyclic variables are called ignorable variables in Hamiltonian mechanics. This advantage does not apply to the $(q_j, \dot{q}_j)$ variables used in Lagrangian mechanics since $\dot{q}$ is not a constant of motion for a cyclic coordinate. The ability to eliminate the cyclic variables as unknowns in the Hamiltonian is a valuable advantage of Hamiltonian mechanics that is exploited widely for solving problems.

It is advantageous to have the ability to exploit both the Lagrangian and Hamiltonian formulations simultaneously when handling systems that involve a mixture of cyclic and non-cyclic coordinates. The equations of motion for each independent generalized coordinate can be derived independently of the remaining generalized coordinates. Thus it is possible to select either the Hamiltonian or the Lagrangian formulations for each generalized coordinate. Routh devised an elegant, and useful, hybrid technique that separates the cyclic and non-cyclic generalized coordinates in order to simultaneously exploit the differing advantages of both the Hamiltonian and Lagrangian formulations of classical mechanics. The Routhian

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reduction approach partitions the $\sum_{i=1}^{n} p_i \dot{q}_i$ kinetic energy term in the Hamiltonian into a cyclic group, plus a non-cyclic group, i.e.

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = \sum_{i=1}^{n} p_i \dot{q}_i - L = \sum_{cyclic}^{s} p_i \dot{q}_i + \sum_{noncyclic}^{n-s} p_i \dot{q}_i - L$$

The new function is called Routhian , that include only one of the two partitions of the kinetic energy terms. This makes the Routhian a Hamiltonian for the coordinates for which the kinetic energy terms are included, while the Routhian acts like a negative Lagrangian for the coordinates where the kinetic energy term is omitted.

$$R_{cyclic}(q_1, ..., q_n; \dot{q}_1, ..., \dot{q}_s; p_{s+1}, ..., p_n; t) \equiv \sum_{cyclic}^{m} p_i \dot{q}_i - L$$

$$R_{noncyclic}(q_1, ..., q_n; p_1, ...p_s; \dot{q}_{s+1}, ..., \dot{q}_n; t) \equiv \sum_{noncyclic}^{s} p_i \dot{q}_i - L$$

The first, Routhian, called R_{cyclic}, includes the kinetic energy terms only for the cyclic variables, and behaves like a Hamiltonian for the cyclic variables, and behaves like a Lagrangian for the non-cyclic variables. The second Routhian, called $R_{non\text{-}cyclic}$, includes the kinetic energy terms for only the non-cyclic variables, and behaves like a Hamiltonian for the non-cyclic variables, and behaves like a negative Lagrangian for the cyclic variables. These two Routhians complement each other in that they make the Routhian either a Hamiltonian for the cyclic variables, or the converse where the Routhian is a Hamiltonian for the non-cyclic variables. The Routhians use $(q_i, \dot{q}_i)$ to denote those coordinates for which the Routhian behaves like a Lagrangian, and $(q_i p_i)$ for those coordinates where the Routhian behaves like a Hamiltonian. For uniformity, it is assumed that the degrees of freedom between $1 \leq i \leq s$ are non-cyclic, while those between $s+1 \leq i \leq n$ are ignorable cyclic coordinates.

Check Your Progress

7. Give the Hamiltonian canonical equation of motion.
8. What happen when Lagrangian L does not contain time t?
9. State the Dokin's theorem.
10. What is cyclic coordinate?

3.5 POISSON BRACKETS

Definition: Let (q, p) be canonical variables and $u = u(q, p)$, $v = v(q, p)$. Then we define,

$$\{u, v\}_{q,p} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

The Poisson brackets map two functions into a new function. It is a bilinear antisymmetric product. In symplectic form this becomes,

$$\{u, v\}_{\eta} = (D_{\eta}u)^T J (D_{\eta}v) \sum_{i,j=1}^{2n} \frac{\partial u}{\partial \eta_i} J_{ij} \frac{\partial v}{\partial \eta_j}$$

Particularly,

$$\{q_i, q_j\}_{q,p} = (p_i, p_j)_{q,p} = 0$$

And

$$\{q_i, p_j\}_{q,p} = \delta_{ij}$$

In symplectic notation we write,

$$\{\eta_i, \eta_j\}_{\eta} = J_{ij}$$

It is more convenient to do it in the symplectic notation, even though it can easily be verified within the (q, p) notation. Thus for a canonical transformation $\xi(\eta)$,

$$\{u, v\}_{\xi} = (\mathrm{D}_{\xi} u)^T J (\mathrm{D}_{\xi} v) = (\mathrm{D}_{\eta} u)^T (\mathrm{D}_{\xi} \eta)^T J (\mathrm{D}_{\xi} \eta)(\mathrm{D}_{\eta} v)$$

Note that $(\mathrm{D}_{\xi} \eta) = (\mathrm{D}_{\eta} \xi)^{-1}$, the center part is found to be equal to J (the symplectic condition), and we obtain

$$\{u, v\}_{\xi} = (u, v)_{\eta}$$

i.e., Poisson brackets are invariant under canonical transformations. Hamilton's equations have the property of remaining invariant under canonical transformations. Similarly, equations expressed in terms of Poisson brackets will be invariant under such transformations. It is indeed possible to reformulate classical mechanics completely in terms of Poisson brackets. The remarkable fact is that this reformulation can be extended with little change into the law of quantum mechanics. The Poisson bracket can be observed as a product operation between functions of phase-space and perhaps time. This product along with the vector field character of functions produces algebra. The algebraic properties of the Poisson bracket are:

1. Antisymmetry
2. Bilinearity
3. $\{uv, w\} = u\{v, w\} + (u, w)v$ from the law of differentiation.
4. Jacobi's identity, $\{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\} = 0$.

Note:

$$\{u, \{v, w\}\} = \frac{\partial u}{\partial \eta_i} J_{ij} \frac{\partial}{\partial \eta_j} \left(\frac{\partial v}{\partial \eta_k} J_{kl} \frac{\partial w}{\partial \eta_l} \right) = \frac{\partial u}{\partial \eta_i} J_{ij} J_{kl} \left(\frac{\partial^2 v}{\partial \eta_j \partial \eta_k} \frac{\partial w}{\partial \eta_l} + \frac{\partial v}{\partial \eta_k} \frac{\partial^2 w}{\partial \eta_j \partial \eta_l} \right)$$

with summation over repeated indexes. To do this we need to add the cyclic permutations,

$$\{v, \{w, u\}\} = \frac{\partial v}{\partial \eta_i} J_{ij} J_{kl} \left(\frac{\partial^2 w}{\partial \eta_j \partial \eta_k} \frac{\partial u}{\partial \eta_l} + \frac{\partial w}{\partial \eta_k} \frac{\partial^2 u}{\partial \eta_j \partial \eta_l} \right)$$

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$$\{w,\{u,v\}\}=\frac{\partial w}{\partial \eta_i}J_{ij}J_{kl}\left(\frac{\partial^2 u}{\partial \eta_j \partial \eta_k}\frac{\partial v}{\partial \eta_l}+\frac{\partial u}{\partial \eta_k}\frac{\partial^2 v}{\partial \eta_j \partial \eta_l}\right)$$

Consider, for example, all the terms where w is differentiable twice:

$$\frac{\partial u}{\partial \eta_i}J_{ij}J_{kl}\frac{\partial v}{\partial \eta_k}\frac{\partial^2 w}{\partial \eta_j \partial \eta_l}+\frac{\partial v}{\partial \eta_i}J_{ij}J_{kl}\frac{\partial^2 w}{\partial \eta_j \partial \eta_k}\frac{\partial u}{\partial \eta_l}$$

Since the indexes are all dummy indexes, we may rename them in the second term, $i\to k,\ l\to i,\ k\to j$ and $j\to l$

$$\frac{\partial u}{\partial \eta_i}J_{ij}J_{kl}\frac{\partial v}{\partial \eta_k}\frac{\partial^2 w}{\partial \eta_j \partial \eta_l}+\frac{\partial v}{\partial \eta_k}J_{kl}J_{ji}\frac{\partial^2 w}{\partial \eta_l \partial \eta_j}\frac{\partial u}{\partial \eta_i}=0$$

where we have used the antisymmetry of J.

Poisson Bracket Formulation of Mechanics

Let $u(q,p,t)$ be any differentiable function. Its time derivative along a Hamiltonian trajectory is,

$$\frac{d}{dt}u(q(t),p(t),t)=\sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{dq_i}{dt}+\frac{\partial u}{\partial p_i}\frac{dp_i}{dt}\right)+\frac{\partial u}{\partial t}=\{u,H\}_{q,p}+\frac{\partial u}{\partial t}$$

This equation, which governs the dynamics of any observable under Hamiltonian dynamics includes the laws of mechanics. Particularly,

$$\frac{d\eta}{dt}=\{\eta,H\}_{\eta}$$

An immediate consequence of this is,

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}$$

which is the conservation of the Hamiltonian when it is not explicitly time-dependent. Additionally, any function $u(q,p,t)$ is a constant of motion iff,

$$\{u,H\}_{q,p}=\frac{\partial u}{\partial t}=0$$

Moreover, it follows that if both u, v are constants of motion so is $\{u,v\}$. Indeed, by Jacobi's identity and the antisymmetry of the Poisson bracket,

$$\{H,(u,v)\}=-\{u,(v,H)\}-\{v,(H,u)\}=\left\{u,\frac{\partial v}{\partial t}\right\}-\left\{v,\frac{\partial u}{\partial t}\right\}=\frac{\partial}{\partial t}\{u,v\}$$

This may give a means to generate many invariants of motion. In most cases, however, this process terminates fast, yielding trivial variables. A class of systems

for which infinitely many conserved quantities can be thus generated are integrable systems.

3.5.1 Poisson's Bracket for Hamilton's Equation Motion

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Let us consider an arbitrary function $f(\mathbf{q},\mathbf{p},t)$. Then its time evolution is given by

$$\frac{df}{dt} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t} \tag{3.60}$$

where the first equality used the definition of total time derivative together with the chain rule, and the second equality used Hamilton's equations of motion.

Let $f(\mathbf{q},\mathbf{p},t)$ and $g(\mathbf{q},\mathbf{p},t)$ be any two functions; we then define their Poisson bracket $\{f,g\}$ to be

$$\{f,g\} \stackrel{\text{def}}{=} \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{3.61}$$

The time-evolution Equation (3.60) can then be rewritten in compact form as

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t} \tag{3.62}$$

3.5.2 Poission's Identity

While discussing Poisson brackets, we shall often just consider functions $f(\boldsymbol{q},\boldsymbol{p})$ and $g(\boldsymbol{q},\boldsymbol{p})$ and not bother to discuss explicit time-dependence.

Here are some fundamental properties of the Poisson bracket:

- **Bilinearity:** We have

  $$\{\alpha_1 f_1 + \alpha_2 f_2, g\} = \alpha_1\{f_1,g\} + \alpha_2\{f_2,g\} \tag{3.63}$$

  and in a similar way for g.

- **Anticommutativity:** We have

  $$\{f,g\} = -\{g,f\}. \tag{3.64}$$

  In particular it follows that $\{f,f\} = 0$.

- **Jacobi Identity:** For any three functions f,g,h we have

  $$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \tag{3.65}$$

  or equivalently (using anticommutativity)

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$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0 \tag{3.66}$$

- **Product Identity or Leibniz Rule:** For any three functions f,g,h we have

$$\{fg,h\}=f\{g,h\}+g\{f,h\} \tag{3.67}$$

The Poisson bracket $\{f,g\}$ involves first derivatives of f and of g.

- **Fundamental Poisson brackets:** The Poisson brackets among the canonical coordinates $\boldsymbol{q}=(q_1,\ldots,q_n)$ and $\boldsymbol{p}=(p_1,\ldots,p_n)$ are

$$\{q_i,q_j\}=0$$

$$\{p_i,p_j\}=0$$

$$\{q_i,p_j\}=\delta_{ij} \tag{3.68}$$

where δ_{ij} is the Kronecker delta, i.e.

$$\delta_{ij}=\begin{cases}1 & \text{if} i=j\\ 0 & \text{if} i\neq j\end{cases}$$

The three properties of bilinearity, anticommutativity and the Jacobi identity play such a fundamental role in many areas of mathematics that they have been given a name: an algebraic structure involving a 'Product' that is bilinear, anticommutative and satisfies the Jacobi identity is called a Lie algebra.

We can now prove an important result in Hamiltonian dynamics:

Total Time Derivative of a Poisson Bracket

For any two functions $f(\boldsymbol{q},\boldsymbol{p},t)$ and $g(\boldsymbol{q},\boldsymbol{p},t)$, we have

$$\frac{d}{dt}\{f,g\}=\left\{\frac{df}{dt},g\right\}+\left\{f,\frac{dg}{dt}\right\}. \tag{3.69}$$

Proof: From the fundamental time-evolution Equation (3.62) applied to $\{f,g\}$, we have

$$\frac{d}{dt}\{f,g\}=\{\{f,g\},H\}+\frac{\partial}{\partial t}\{f,g\} \tag{3.70}$$

The first term on the right-hand side can be transformed using the Jacobi identity and anticommutativity:

$$\{\{f,g\},H\}=-\{\{g,H\},f\}-\{\{H,f\},g\}=\{f,\{g,H\}\}+\{\{f,H\},g\} \tag{3.71}$$

And for the second term on the right-hand side, we use the fact that $\partial/\partial t$

commutes with the partial derivatives $\partial / \partial q_i$ and $\partial / \partial p_j$ occurring in the definition of the Poisson bracket; it therefore follows that

$$\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} \tag{3.72}$$

If we add (3.71) and (3.72) and use the fundamental time evolution equation (3.62) for f and for g, we obtain (3.69).

In particular, if f and g are constants of motion, then so is $\{f,g\}$. So this gives us a method to obtain new constants of motion, given old ones. These new constants of motion are not guaranteed to be nontrivial. (For instance, we might have $\{f,g\} = 0$.)

3.5.3 Jacobi-Poisson Theorem

The following equation is satisfied by the partial derivative of any Poisson bracket:

$$\frac{\partial}{\partial t}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] \tag{3.73}$$

The Jacobi's identity is given by,

$$(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0 \tag{3.74}$$

If u and v are any two constants of motion of any given holonomic dynamical system, then their Poisson bracket $[u, v]$ is also a constant of motion. This is known as the **Jacobi-Poisson's theorem**. It is also called Poisson's second theorem on the Poisson bracket relations.

Proof: Consider

$$\frac{d}{dt}(u,v) = \frac{\partial}{\partial t}(u,v) + ((u,v),H)$$

Using Equations (3.73) and (3.74), we get

$$\frac{d}{dt}(u,v) = \left(\frac{\partial u}{\partial t}, v\right) + \left(u, \frac{\partial v}{\partial t}\right) - ((v,H),u) - ((H,u),v)$$

$$= \left(\frac{\partial u}{\partial t} + (u,H), v\right) + \left(u, \frac{\partial v}{\partial t} + (v,H)\right)$$

$$= \left(\frac{du}{dt}, v\right) + \left(u, \frac{dv}{dt}\right)$$

$$= 0$$

because both du/dt and dv/dt vanish (since u and v are both constants of motion).

This theorem finds intense significance in determining new constants of motion. For example, if we have got any two independent constants of motion, then a third one can be constructed from the Poisson bracket of these two, which may result in either a new constant of motion or trivially either of the first two. If the former is true, then we can make another pair of new Poisson brackets and if we might in this way generate all the hidden constants of the motion. It should be remembered that a dynamical system having n degrees of freedom can have at the most $2n-1$ independent constants of motion, which are functions of p_i and q_i's only, and one constant of motion that must involve time explicitly.

Check Your Progress

11. Define the Poisson bracket.
12. Give the algebraic properties of Poisson bracket.

3.6 ANSWERS TO 'CHECK YOUR PROGRESS'

1. When we try to describe the configuration of a system of particles, we need some variables. We should choose the least number of possible variables to describe the configuration satisfactorily. These least number of variables are called 'Generalized co-ordinates'. These are nothing but a set of minimum co-ordinates that describe the configuration of a system.
2. Uses of generalized co-ordinates follow some rules. These are as follows:
   (*i*) The values of generalized co-ordinates (variables) define and determine the configuration of the system.
   (*ii*) The generalize co-ordinates may be varied (as they are variables): arbitrarily and independent of each other, irrespective of constraints.
   (*iii*) We can choose generalized co-ordinates at random to simplify the mathematical methods to describe the configuration of the system.
3. Holonomic system are systems which all constraints are integrable into positional constraints. Non-holonomic systems are systems which have constraints that are non-integrable into positional constraints.
4. The scleronomousis is a mechanical system which equations of constraints do not contain the time as an explicit variable and the equation of constraints can be explain by 'Generalized Coordinates'. Such constraints are called scleronomic constraints. The opposite of scleronomous is rheonomous. Additionally a mechanical system is rheonomous if its equations of constraints contain the time as an explicit variable. Such constraints are called rheonomic constraints. The opposite of rheonomous is scleronomous.
5. Consider a holonomic system of n material points $A_1(x_1, y_1, z_1),...,$ $A_n(x_n, y_n, z_n)$. Denote the forces acting on the points of the system by $P_1,..., P_n$ and let $m_1,...m_n$, be the masses of these points.

6. For the non-conservative force field, the Lagrange's equation of motion is termed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \quad \text{[for non-conservative system]}$$

Thus, the Lagrangian of a system is the difference between kinetic energy (T) and potential energy (V), i.e., $L = T - V$.

7. Comparing the coefficients of equations of $H = H(q_j, p_j, t)$, we get

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial P_j} \\ -\dot{P}_j &= \frac{\partial H}{\partial q_j} \\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t} \end{aligned}\right\}$$

The above equation is termed as 'Hamilton's canonical equations of motion'. And the equation is a set of $2n$ dimensional first order differential equations with generalized coordinates.

8. Now, if we consider that Lagrangian L does not contain time (t) explicitly, then

$$\frac{\partial L}{\partial t} = 0$$

Hence, equation of $\frac{\partial L}{\partial t}$ takes the form

$$\frac{dH}{dt} = 0$$

$\Rightarrow \quad H = \text{Constant}$

Thus, we can conclude that if Lagrangian L does not contain time (t) explicitly, the Hamiltonian H [i.e., total energy ($T + V$)] is conserved.

9. The Hamiltonian formulation is obtained from the Lagrangian function L by replacing the generalized velocity $\dot{q}_i$ by the conjugate momenta p_i. This can be done by using Donkin's theorem.

10. The concept of a cyclic coordinate (angle coordinate, angle variable) ties in with action-angle coordinates in the theory of completely-integrable Hamiltonian systems. Each such system (with finite degrees of freedom) can be transformed into one with coordinates (y_k, x_k) such that the Hamiltonian has the form $H(y_1,\ldots,y_n)$, i.e. does not contain $x_1,\ldots,x_n$. Then the y_k are called the action coordinates and the x_k the angle coordinates.

11. Definition: Let (q, p) be canonical variables and $u = u(q, p)$, $v = v(q, p)$. Then we define,

$$\{u, v\}_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

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The Poisson brackets map two functions into a new function. It is a bilinear antisymmetric product.

12. The algebraic properties of the Poisson bracket are:
    (i) Antisymmetry
    (ii) Bilinearity
    (iii) $\{uv, w\} = u\{v, w\} + (u, w)v$ from the law of differentiation.
    (iv) Jacobi's identity, $\{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\} = 0$.

3.7 SUMMARY

- The values of generalized co-ordinates (variables) define and determine the configuration of the system.
- The generalize co-ordinates may be varied (as they are variables): arbitrarily and independent of each other, irrespective of constraints.
- Holonomic system are systems which all constraints are integrable into positional constraints. Non-holonomic systems are systems which have constraints that are non-integrable into positional constraints.
- For stationary constraints real displacements are found amongst the possible displacements, while for non-stationary constraints they are not found among the possible displacements, in general. Possible displacements are useful for converting a holonomic system from one position of the system, which is possible for a given *t* to another infinitely-close position which is possible at the same moment *t*.
- The number of independent variations of the points of the system is said to be the number of its degrees of freedom.
- The possible displacements of a system with two-sided constraints are reversible; among the possible displacements of a system with one-sided constraints there are irreversible ones.
- A non-holonomic system in mathematics is a physical system whose state depends on the path taken in order to achieve it. Such a system is labelled by a set of parameters subject to differential constraints, such that when the system evolves along a path in its parameter space (the parameters varying continuously in values) but finally returns to the original set of parameter values at the start of the path, the system itself may not have returned to its original state.
- A mechanical system is scleronomous if the equations of constraints do not contain the time as an explicit variable and the equation of constraints can

be described by generalized coordinates. Such constraints are called scleronomic constraints. The opposite of scleronomous is rheonomous.

- The scleronomousis is a mechanical system which equations of constraints do not contain the time as an explicit variable and the equation of constraints can be explain by 'Generalized Coordinates'. Such constraints are called scleronomic constraints. The opposite of scleronomous is rheonomous.
- The 'Simple Pendulum' exhibits Simple Harmonic Motion (SHM) as the acceleration of the pendulum bob is directly proportional to the displacement from the mean position and is always directed towards it.
- The forces associated in a mechanical problem are required to maintain the constraints in the system. These forces are not known to us primarily, i.e., the forces associated in a given mechanical problem are not specified directly.
- The kinetic energy of a multi-body system is obtained by the sum of kinetic energies of the single bodies.
- If the work obtained by the applied forces is independent from the distance covered, the forces have a potential and can thus be determined by differentiation.
- Non-conservative forces modify their mechanical total energy. If we specially deal with forces which eliminate energy, we refer to them as dissipative forces.
- The number of equations of motion is same as to the number of degrees of freedom of the system. Also, it is not necessary to introduce the reaction forces as they cannot be calculated.
- In order to write the motion equations, we have to calculate the partial and total differentiations of the function $T(\mathbf{y}, \dot{\mathbf{y}}, t)$. Chain rule is required to perform the total differentiation of T with respect to time t.
- In contrast to Newton-Euler's formalism, Lagrange's equations (in this form) allow for the calculation of reaction forces (position forces). Conversely, the consideration of these forces is not necessary when we set up equations, which accrues considerable advantages in practice.
- The Lagrange's equations of the first kind are $m_i x_{\ddot{i}} = P_{i_x} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i}$, $m_i y_{\ddot{i}} = P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i}$, and $m_i z_{\ddot{i}} = P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i}$ for $(i = 1, 2, ..., n)$.
- Lagrange's equation of motion for a conservative system is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}$.
- Where $T - V = L$ is called Lagrangian and the equation for conservative is termed as the Lagrange's equation of motion for a conservative system.
- Thus, the Lagrangian of a system is the difference between kinetic energy (T) and potential energy (V), i.e., $L = T - V$.

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- So, conservation of energy states that if the Lagrangian function does not contain time explicitly, the total generalized energy of the system is conserved.
- Thus, the quantity $(\Sigma \dot{q}_j P_j - L)$ is constant in motion with the condition that the Lagrangian L does not contain time explicitly. This quantity is designated by H (Hamiltonian function).
- We can conclude that if Lagrangian L does not contain time (t) explicitly, the Hamiltonian H [i.e., total energy $(T+V)$] is conserved.
- We can conclude that if Lagrangian L does not contain position co-ordinates (q_j) [i.e., if a co-ordinate is cyclic or ignorable in L], then Hamiltonian H also does not contain position co-ordinates (q_j), i.e., the co-ordinates are also cyclic or ignorable in H.
- The Hamiltonian formulation is obtained from the Lagrangian function L by replacing the generalized velocity $\dot{q}_i$ by the conjugate momenta p_i. This can be done by using Donkin's theorem.
- In the method devised by Routh, the q, $\dot{q}$ basis of Lagrange is changed to q, p basis only for those coordinates which are cyclic and their equations of motion are obtained in Hamiltonian form, while the remaining coordinates which are not cyclic are represented by Lagrange's equations.
- Routhian formulation is a path to both the Lagrangian and Hamiltonian formulations.
- Poisson brackets are invariant under canonical transformations. Hamilton's equations have the property of remaining invariant under canonical transformations. Similarly, equations expressed in terms of Poisson brackets will be invariant under such transformations. It is indeed possible to reformulate classical mechanics completely in terms of Poisson brackets.
- The Poisson bracket can be observed as a product operation between functions of phase-space and perhaps time. This product along with the vector field character of functions produces algebra.
- If u and v are any two constants of motion of any given holonomic dynamical system, then their Poisson bracket $[u, v]$ is also a constant of motion. This is known as the Jacobi-Poisson's theorem. It is also called Poisson's second theorem on the Poisson bracket relations.
- If the former is true, then we can make another pair of new Poisson brackets and if we might in this way generate all the hidden constants of the motion. It should be remembered that a dynamical system having n degrees of freedom can have at the most $2n - 1$ independent constants of motion, which are functions of p_i and q_i's only, and one constant of motion that must involve time explicitly.

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3.8 KEY TERMS

- **Generalized Co-Ordinates:** These are nothing but a set of minimum co-ordinates that describe the configuration of a system.
- **Non-Holonomic Constraints:** Non-holonomic constraints is the conditions of the constraints that cannot be expressed by an equation, then they are termed as non-holonomic constraints.
- **Scleronomic Constraints:** There are the constraints that do not explicitly depend on time, then they are called scleronomic.
- **Rheonomic Constraints:** Rheonomic constraints is the constraints that explicitly depend on time, then they are said to be rheonomic type.
- **Cyclic coordinate:** The concept of a cyclic coordinate (angle coordinate, angle variable) ties in with action-angle coordinates in the theory of completely-integrable Hamiltonian systems.
- **Routhian mechanics:** In classical mechanics, Routh's procedure or Routhian mechanics is a hybrid formulation of Lagrangian mechanics and Hamiltonian mechanics developed by Edward John Routh.

3.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the terms generalized co-ordinates and generalized displacement.
2. Differentiate between holonomic and non-holonomic systems.
3. Give the notation of generalized co-ordinates.
4. What is Scleronomic system?
5. Define kinetic and potential energy.
6. How will you define the Lagrange's equations of first and second kind?
7. What are Lagrange's multipliers?
8. Give the Lagrange's equations of motion for a energy conservative system.
8. What do you understand by uniqueness of solution.
9. How will you define the Hamilton's variables?
10. State the is Donkin's theorem?
11. Define the Poission's bracket.
12. State the Jacobi-Poisson Theorem.

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Long-Answer Questions

1. Explain in detail about the generalized co-ordinates with the help of their notations and examples.
2. Describe the advantages of using generalized co-ordinates with the help of examples.
3. Elaborate on the holonomic and non-honlonomic systems with examples.
4. What are constraints? Define various constraints with examples. How do they affect the motion of a system of particles?
5. Analyse the Lagrange's equations of first and second kind with the help of examples.
6. Discuss the detail about the Lagrange's equation of motion from D'Alembert's principle for conservative and non-conservative systems.
7. What is Hamilton's variables? Discuss the Hamilton's canonical equations of motion with the help of examples.
8. Analyse the Donkin's theorem. Give appropriate examples.
9. What do you understand by the Poission's identity? Give appropriate examples.

3.10 FURTHER READING

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UNIT 4 ANALYTICAL DYNAMICS: HAMILTON'S PRINCIPLE

NOTES

Structure

4.0 INTRODUCTION

The Hamilton's Principle is state that the dynamics of a physical system are determine by avariatinal problem for a functional based on a single function.

Least action refers to the absolute value of the action functional being minimized. The principle can be used to derive Newtonian, Lagrangian and Hamiltonian equations of motion, and even general relativity. The canonical formalism for constrained systems with a finite number of degrees of freedom by making use of the Poincare-Cartan integral invariant method. In mathematics, a Whittaker function is a special solution of Whittaker's equation, a modified form of the confluent hypergeometric equation introduced by Whittaker (1903) to make the formulas involving the solutions more symmetric. The Lee Hwa Chung state that any other universal integral invariant differ from one of the enumerated integral by constant factor.

In mathematics, the Hamilton–Jacobi equation is a necessary condition describing extremal geometry in generalizations of problems from the calculus of variations. It can be understood as a special case of the Hamilton–Jacobi–Bellman equation from dynamic programming. On other hand Jacobi's theorem states the Hamilton partial differential equation for a given Hamiltonian determines in a simple

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way all the trajectories of the Hamiltonian flow. It is usually proved by appealing to the theory of canonical transformations.

Lagrange brackets are certain expressions closely related to Poisson brackets that were introduced by Joseph Louis Lagrange in 1808–1810 for the purposes of mathematical formulation of classical mechanics, but unlike the Poisson brackets, have fallen out of use.

In mathematics and classical mechanics, the Poisson bracket is an important binary operation in Hamiltonian mechanics, playing a central role in Hamilton's equations of motion, which govern the time evolution of a Hamiltonian dynamical system. The Poisson bracket also distinguishes a certain class of coordinate transformations, called canonical transformations, which map canonical coordinate systems into canonical coordinate systems. A 'Canonical coordinate system' consists of canonical position and momentum variables.

In this unit, you will learn about the introduction to Hamilton's principle and principle of least action, Poincare-Cartan integral invariant and Whittaker's equation, Lee Hwa Chung's theorem, Hamilton - Jacobi equation, canonical transformation interm of Lagranege and Poisson bracket, invariance of Lagrange and Poisson baracket under canonical transformation.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Introduce the Hamilton's principle and principle of least action.
- State the Poincare-Cartan integral invariant and Whittaker's equations
- Define the Lee Hwa Chung's theorem.
- Explain about the Hamilton-Jacobi equation
- Elaborate on the Jacobi's theorem and method of separation of variables
- Describe the Lagrange brackets and canonical transformation of Lagrange and Poisson bracket
- Discuss invariance of Lagrange and Poisson brackets under canonical transformations

4.2 INTRODUCTION TO HAMILTON'S PRINCIPLE

Hamilton's principle is an integral principle. This means that it considers the entire motion of a system between time t_1 and t_2 . The instantaneous configuration of the system is described by the values of n generalized coordinates $q_1,..., q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is called the configuration space. With time, the system point moves in this configuration space, tracing out a curve which describes the path of motion of the system. The configuration space can be

very different from the physical 3-dimensional space, where only three coordinates are needed to describe a position at any given time. Hamilton's principle considers the motion of a mechanical system, described by a scalar potential that may be a function of the coordinates, velocities and time. The integral, which is frequently also referred to as the action in an essential 1-dimensional form from t_1 to t_2, is

$$A = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt \tag{4.1}$$

where $\mathcal{L} = T - V$ is the **Lagrangian**, T and V being respectively the kinetic and potential energy. The dot indicates derivative with respect to time. The dependence of x on t is not fixed, i.e., although the integral is from t_1 to t_2, the exact path of integration is not known. The correct path of motion of the system is such that the action has a stationary value which means that the integral along the given path has the same value to within first-order infinitesimals as that along all neighbouring paths. The difference between two paths for a given value of t, δx , is termed as the variation of x and is usually described by introducing a new function $\eta(t)$ to define the arbitrary deformation of the path and a scale factor α to give the magnitude of the variation. The function $\eta(t)$ is arbitrary although it must satisfy the boundary values, $\eta(t_1) = \eta(t_2) = 0$ and it must be twice differentiable. The paths can then be described as

$$x(t, \alpha) = x(t, 0) + \delta x = x(t, 0) + \alpha \eta(t) \tag{4.2}$$

We have a stationary value of the action when the derivative of A with respect to the scale factor α is zero, i.e.,

$$\left(\frac{\partial A}{\partial \alpha} \right)_{\alpha = 0} = 0 \tag{4.3}$$

The α-dependence of the integral is contained in $x(t, \alpha)$ and $\dot{x}(t, \alpha)$, and therefore

$$\frac{\partial A}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \alpha} \right] dt \tag{4.4}$$

By inserting Equation (4.2) and integrating the second term by parts, we obtain

$$\frac{\partial A}{\partial \alpha} = \int_{t_1}^{t_2} \eta(t) \cdot \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right] dt + \left[\eta(t) \frac{\partial \mathcal{L}}{\partial \dot{x}} \right]_{t_1}^{t_2} \tag{4.5}$$

The integrated part vanishes because the end-points are fixed. The condition for stationary values, Equation (4.3), is therefore given by

$$\int_{t_1}^{t_2} \eta(t) . \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right] dt = 0 \tag{4.6}$$

From the fundamental theorem of variational calculus, if the integral in Equation (4.6) vanishes for every $\eta(t)$ continuously differentiable in the interval (t_1, t_2), then

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the content of the brackets in the Equation (4.6) must identically disappear in the same interval. It therefore follows that A can have stationary values only if,

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \tag{4.7}$$

which is the Euler-Lagrange differential equation.

4.2.1 Principle of Least Action

The stationary-action principle is also known as the principle of least action is a variational principle that give state us when applied to the action of a mechanical system, yields the equations of motion for that system. The principle states that the trajectories (i.e. the solutions of the equations of motion) are stationary points of the system's action functional. The term 'Least Action' is a historical misnomer since the principle has no minimality requirement: the value of the action functional need not be minimal (even locally) on the trajectories. Least action refers to the absolute value of the action functional being minimized. The principle can be used to derive Newtonian, Lagrangian and Hamiltonian equations of motion, and even general relativity. According to a relativity, the different action must be minimized or maximized.

Statement

The starting point is the action, denoted $\boldsymbol{S}$, of a physical system. It is defined as the integral of the Lagrangian L between two instants of time t_1 and t_2 – technically a functional of the N generalized coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ which are functions of time and define the configuration of the system,

$$\mathbf{q} : \mathbf{R} \to \mathbf{R}^N$$

$$\mathcal{S}[\mathbf{q}, t_1, t_2] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt$$

Here is dot denotes the time derivative, and t is time.

Mathematically the principle is,

$$\delta \mathcal{S} = 0,$$

Where δ (lowercase Greek delta) means a small change. The path taken by the system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary (no change) to first order.

Stationary action is not always a minimum, despite the historical name of least action. It is a minimum principle for sufficiently short, finite segments in the path. In applications the statement and definition of action are taken together

$$\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt = 0.$$

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The action and Lagrangian both contain the dynamics of the system for all times. The term 'Path' simply refers to a curve traced out by the system in terms of the coordinates in the configuration space, i.e. the curve $\mathbf{q}(t)$, parameterized by time.

Action of a dynamical system over an interval $t_1 < t < t_2$ is

$$A = \int_{t_1}^{t_2} 2T \, dt$$

Where T = K.E.

This principle states that the variation of action along the actual path between given time interval is least, i.e.,

$$\delta \int_{t_1}^{t_2} 2T \, dt = 0$$

Now we know that T + V = E (constant)

V = P.E. and L = T–V

By Hamilton's principle, we have

$$\int_{t_1}^{t_2} \delta L \, dt = 0 \text{ or } \int_{t_1}^{t_2} \delta(T - V) dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta (T - E + T) \, dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} [\delta(2T) - \delta E] \, dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \delta(2T) \, dt = 0 \qquad [\text{using } E = \text{constant} \quad \therefore \ \delta E = 0]$$

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} 2T \, dt = 0$$

4.2.2 Poincaré-Cartan Integral Invariant

Henri Poincaré has shown that for any Hamiltonian system the form $w = \sum_i dp_i dq_i$ possesses the property that $\int_D^W$ over any two-dimensional manifold D in

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$(p_1,...,p_n,q_1,...q_n)$ -space is independent of *t.* A differential form with such a property is called a Poincaré integral invariant. If $F(x_1,...,x_k,t,dk_1,...,dx_k)$ is a Poincaré integral invariant of a system in which derivatives are identified with functions of the variables and *t*, then $F(x_1,...,x_k,t,\ dx - A_1\ dt,...,dx - A_k\ dt)$ is an invariant form of the system expressible in terms of differentials and the functions A_k and is called the related Cartan form. The associated Cartan form of $\sum_i dp_i dq_i$ can therefore be expressed in terms of the differentials of *H*. It is not only an invariant of the system in associated differential form but also actually determines the system, which is the first associated characteristic system of *w'*.

4.2.3 Whittaker's Equations

Consider an arbitrary system for which the function *H* is not explicitly dependent on the time in which case we have generalized integral of energy,

$$H(q_i, p_i) = h \tag{4.8}$$

This integral is equivalent to the integral of conservation of momentum, i.e., $p_1 = c$. We obtain $p_1 = c$ when q_1 is the cyclic coordinate, i.e., when

$$\frac{\partial H}{\partial q_1} = 0$$

Also from the analogy between the variable of time t and a cyclic coordinate, we can reduce the order of the set of differential equation of motion with the help of integral of energy given in Equation (4.8). For this, consider a non-extended 2*n* dimensional phase space in which $q_i, p_i\ (i = 1,...,n)$ are the coordinates of points. Consider here only those points of the phase space whose coordinates satisfy Equation (4.8) with fixed value of constant h_0, i.e.,

$$H = H(q_i^o, p_i^o) = h_o \tag{4.9}$$

The Poincare-Cartan integral invariant is

$$I_1 = \oint \sum_{i=1}^{n} p_i \delta q_i \tag{4.10}$$

From Equations (4.8) and (4.9),

$$\oint H\delta t = h_o \oint \delta t = 0$$

as *t* is not variable on a closed curve and initial as well as terminal points are same. On solving Equation (4.8) for p_1, we get

$$p_1 = -\bar{K}\ (q_1,...q_n, p_n, p_2,..., p_n, h_o)$$

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Putting the above value in Equation (4.10), we get

$$I_1 = \oint \left[\sum_{j=2}^{n} p_j \delta q_j - \bar{K}\, \delta q_1 \right]$$

This integral also has the form of Poincare-Cartan integral, where the basic coordinates and momenta are q_j and p_j (j = 2, …, n) and the variable is q_1. This implies that the motion of the generalized conservative system should satisfy the system of (2n–2) differential equations given by,

$$\frac{\partial q_j}{dq_1} = \frac{\partial \bar{K}}{\partial p_j}, \frac{dp_j}{dq_1} = -\frac{\partial \bar{K}}{\partial q_j}, \qquad (j = 2, ..., n)$$

These equations are termed as Whittaker's equations.

Lagrangian and the Action

For every mechanical system, there exists a function of the generalized coordinates (and velocities) called the **Lagrangian** $L = L(q_i, \dot{q}_i)$ which, when integrated over a period of time, defines the **action**, A, associated with the system,

$$A = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt$$

We can apply the calculus of variations to derive the Euler-Lagrange equations of motion,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \forall i = 1, ..., N$$

to find the unique trajectory defined by the coordinates q_i. The Lagrangian, as defined, is very general and in this form, applies even to quantum mechanics. For all known classical systems, the Lagrangian can be written as the difference between the total kinetic energy, T, of the system and the potential energy, U, i.e., for the classical systems, $L = T - U$.

4.3 LEE HWA CHUNG'S THEOREM

Lee Hwa Chung stated that any other universal integral invariant differs from one of the enumerated integrals by a constant factor. This refers as Lee Hwa Chung theorem. Therefore if,

$$L' = \oint \sum_{i=1}^{n} \left[R_i \left(t, qk, pk \right) \delta q_i + S_i \left(t, qk, pk \right) \delta p_i \right]$$

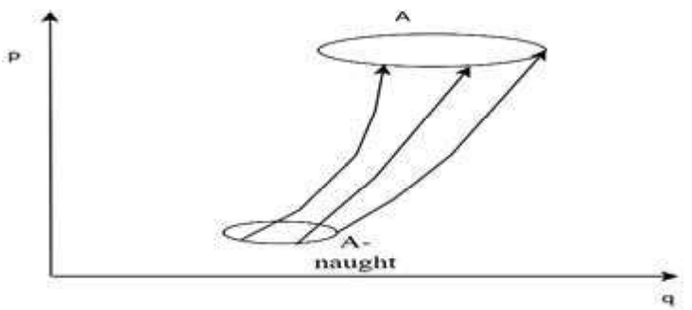

Fig. 4.1 *Position of Contour at Two Different Times in a Phase Plane*

It is an universal relative integral invariant, then

$$L' = cL_1 \tag{4.11}$$

Where L_1 is Poincare integral and c is any constant.

Solving this theorem for arbitrary $n(n=1)$, i.e., integral invariant of the first order.

Assume that for $n = 1$

$$L' = \oint \left[R(t, qk, pk)\delta q + S(t, qk, pk)\delta p \right] \tag{4.12}$$

Be the universal integral invariant, the integration will be performed in the phase plane (q, p) along a closed contour. Towards this direction, assume that there be some Hamiltonian system of differential equations with the function $H(t, q, p)$ is as follows:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{4.13}$$

The general solution of these equations is of the arrangement

$$q = q(t, q_0, p_0) \text{ and } \quad p = p(t, q_0, p_0) \tag{4.14}$$

Where q_0 and p_0 be the initial values of q and p at $t = t_0$

Further, let

$$\left\{ \begin{array}{l} q = q_0(\beta), p = p_0(\beta), 0 \le \beta \le \gamma \\ q_0(0) = q_0(\gamma), p_0(0) = p_0(\gamma) \end{array} \right\} \tag{4.15}$$

NOTES

Be the equations of closed contour A_0 in the phase plane. The points at which at were located on with form the contour A at some arbitrary instant of time *t*. This is depicted as shown in Figure 4.1.

From equations (4.14) and (4.15) the parametric equation of this contour A is given by,

$$q = q(t,\beta), \quad p = p(t,\beta), \quad 0 \le \beta \le \gamma$$

Putting these values of q and p in the integral given by equation (4.12),

We have

$$L = \oint \left[R(t, q(t,\beta), p(t,\beta))\delta q + S(t, q(t,\beta), p(t,\beta))\delta p \right]$$

So is invariant

$$\frac{dL}{dt} = 0$$

Hence, differentiating under the integral sign by parts,

$$0 = \frac{dL}{dt} = \oint \left[\frac{dR}{dt}\delta q + \frac{dS}{dt}\delta p + R\frac{d}{dt}\delta q + S\frac{d}{dt}\delta p \right]$$

$$= \oint \left[\frac{dR}{dt}\delta q + \frac{dS}{dt}\delta p + R\delta\left(\frac{dq}{dt}\right) + S\delta\left(\frac{dp}{dt}\right) \right]$$

By using the $-\oint u\delta v = -\oint v\delta u$ for a closed contour, integrating the last two terms by parts, then we get the equation is as follows:

$$0 = \oint \left[\frac{dR}{dt}\delta q - \delta R\frac{dq}{dt} + R\delta\left(\frac{dq}{dt}\right) + S\delta\left(\frac{dp}{dt}\right) \right]$$

$$* \frac{dR}{dt} = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial q}\frac{dq}{dt} + \frac{\partial R}{\partial p}\frac{dp}{dt}, \quad \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\frac{dq}{dt} + \frac{\partial S}{\partial p}\frac{dp}{dt}$$

$$\partial R = \frac{\partial R}{\partial q}\delta q + \frac{\partial R}{\partial p}\delta p, \quad \partial S = \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial p}\delta p$$

Replacing $\frac{dR}{dt}$, $\frac{dS}{dt}$, $\partial R$and, we have

NOTES

$$0 = \oint \left[\left(\frac{dR}{dp} - \frac{dS}{dq} \right) \frac{dp}{dt} + \frac{\partial R}{\partial t} \right] \delta q + \left[\left(\frac{dS}{dq} - \frac{dR}{dp} \right) \frac{dq}{dt} + \frac{\partial S}{\partial t} \right] \delta p$$

$$= \oint \left(-Z \frac{\partial H}{\partial q} + \frac{\partial R}{\partial t} \right) \delta q + \left(-Z \frac{\partial H}{\partial p} + \frac{\partial S}{\partial t} \right) \delta p$$

Now, from equation (4.13) and $Z = \frac{\partial R}{\partial p} - \frac{\partial S}{\partial q}$.

The equation under the sign of integral must be a perfect differential with regard to *q* and *p* because the integral is equal to zero for any value of the variable t taken as a parameter and for any arbitrary path of integration. Therefore, we will have

$$\frac{\partial}{\partial p} \left(-Z \frac{\partial H}{\partial q} + \frac{\partial R}{\partial t} \right) = \frac{\partial}{\partial q} \left(-Z \frac{\partial H}{\partial p} + \frac{\partial S}{\partial t} \right)$$

Hence

$$-\frac{\partial Z}{\partial p} \frac{\partial H}{\partial q} - Z \frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 R}{\partial p \partial t} = -\frac{\partial Z}{\partial q} \frac{\partial H}{\partial p} - Z \frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 S}{\partial q \partial t}$$

It implies that,

$$-\frac{\partial Z}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial Z}{\partial q} \frac{\partial H}{\partial p} + \frac{\partial}{\partial t} \left(\frac{\partial R}{\partial p} - \frac{\partial S}{\partial q} \right) = 0$$

Or

$$-\frac{\partial Z}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial Z}{\partial q} \frac{\partial H}{\partial p} + \frac{\partial Z}{\partial t} = 0$$

So, the choice of H is arbitrary, we can choose

$$\frac{\partial Z}{\partial p} = \frac{\partial Z}{\partial q} = \frac{\partial Z}{\partial t} = 0$$

Further,

$$Z = \frac{\partial R}{\partial p} - \frac{\partial S}{\partial q} = \text{any constant=c (say)}$$

Therefore,

$$\frac{\partial R}{\partial p} - \frac{\partial S}{\partial q} - \frac{\partial}{\partial p}(cp) = 0$$

Or $\frac{\partial}{\partial p}(R - cp) = \frac{\partial}{\partial q} S$

NOTES

Hence, it is clear that there exists a function $\phi(t, p, q)$ such that (t being a parameter)

$$(R - cp)\delta q + S\delta p = \frac{\partial \phi}{\partial q}\delta q + \frac{\partial \phi}{\partial p}\delta p = \delta \phi$$

So, $R\delta q + S\delta p = cp\delta q + \delta \phi$

Therefore

$$L' = \oint (R\delta q + S\delta p) = \oint (cp\delta q + \delta \phi)$$

$$= c\oint p\delta q$$

$$= cL_1$$

Since $\oint \delta \phi = 0$ as the contour is closed.

Hence the proof.

4.4 HAMILTON - JACOBI EQUATION

In mathematics, the Hamilton-Jacobi equation is a essential condition which explain extremal geometry in generalization of problems from the calculus of variation. It can be understood as special case of the Hamilton-Jacobi-Bellman equation from dynamic programming.

Let q, p be canonical variables. Their time evolution satisfies the Hamilton's equations,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$$

Let the initial data be $q(0) = Q$ and $p(0) = P$. Hamilton's equation induce a time-dependent mapping,

$$(Q, P) \mapsto (q(t), p(t))$$

such that,

$q = q(Q, P, t)$ and $p = p(Q, P, t)$

The Hamiltonian dynamics define a flow in phase space. We can also refer to the inverse mapping,

NOTES

$Q = Q(q, p, t)$ and $P = P(q, p, t)$

This mapping is canonical. We will find this mapping, i.e., we will express the coordinates and momenta at time t given their values at time zero. The aim of Hamilton-Jacobi theory is to find this mapping. Since this mapping is canonical, so it has a certain structure that its Jacobian must satisfy. Besides, the transformed variables satisfy Hamiltonian dynamics with a transformed Hamiltonian. In the present case, we want the new variables (Q, P) to be stationary in time, since we want

$q(t) = q(Q(t), P(t), t) = q(Q(0), P(0), t)$

A way to impose it is to require the transformed Hamiltonian K to be, say, zero (could be any function of time only). Suppose furthermore that we try to generate this transformation using a generating function of the form $F_2(q, P, t)$ (here again, the choice of generating function is somewhat arbitrary). Then,

$$K = H(q, p, t) + \frac{\partial F_2}{\partial t}(q, P, t) = 0$$

And

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial p_i}$$

Using the first of the canonical relations and substituting it into the Hamiltonian transformation, we obtain

$$\frac{\partial F_2}{\partial t}(q, P, t) + H\left(q, \frac{\partial F_2}{\partial q}(q, P, t), t\right) = 0 \tag{4.16}$$

This is the Hamilton-Jacobi equation. Fixing P, the initial data for the momentum, and defining

$S(q, t) = F_2(q, P, t)$

Equation (4.16) takes the form

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

It is a first order partial differential equation in $(n + 1)$ variables. Its solution is a generating function with P as a parameter which we can use in order to find the mapping between the coordinates at time 0 and time t.

Note: A partial differential equation in $n + 1$ variables is by no means simpler than a system of $2n$ ordinary differential equations. The transition from first-order partial differential equations and system of ordinary differential equations is standard in the analysis of hyperbolic partial differential equations.

Now, suppose we wish to solve the equation for S. Since it is a first-order equation in $n + 1$ variables, the solution involves $n + 1$ integration constant. Note however that S is only defined up to an additive constant, which will not affect the generating function anyways. Thus, without loss of generality, S depends on n integration constants α,

$S = S(q, \alpha, t)$

Since these constants are arbitrary, we are free to identify them with the momenta P, i.e., set

$F_2(q, P, t) = S(q, P, t)$

Note that the way of writing a solution with n integration constants is not unique. Now we can proceed,

$$p_i = \frac{\partial S}{\partial q_i}(q,P,t) \text{ and } Q_i = \frac{\partial S}{\partial P_i}(q,P,t)$$

Inverting these equations provides the required solution.

Let us take the example of a harmonic oscillator,

$$H(q,p) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

The Hamilton-Jacobi equation for $S(q, t)$ is,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = 0$$

We have to find a solution of the form,

$S(q, t) = A(q) - \alpha t$

Substituting we get,

$$\alpha = \frac{1}{2m}[(A'(q))^2 + m^2\omega^2 q^2]$$

i.e.,

$$S(q,t) = \int_0^q \sqrt{2m\alpha - m^2\omega^2 r^2}\, dr - \alpha t$$

We can now identify the constant of integration α with the initial momentum P. The transformation equations are,

$$p = \frac{\partial S}{\partial q} = \sqrt{2mP - m^2\omega^2 q^2}$$

And

NOTES

$$Q = \frac{\partial S}{\partial P} = -t + m\int_0^q \frac{1}{\sqrt{2mP - m^2\omega^2 r^2}}\,dr$$

The last integral can be easily solved,

$$Q = -t + \sqrt{\frac{m}{2P}}\int_0^q \frac{1}{\sqrt{1 - \frac{m\omega^2}{2P}r^2}}\,dr = -t + \frac{1}{\omega}\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2P}}q\right)$$

$$\Rightarrow\ q = \sqrt{\frac{2P}{m\omega^2}}\sin[\omega(t+Q)]$$

or,

$$p = \sqrt{2mP}\cos[\omega(t+Q)]$$

Q, P can be any pair of variables referring to time 0.

4.4.1 Jacobi's Theorem

Let $S(q_1, q_2 ... q_n, \alpha_1, \alpha_2, ..., \alpha_n, t)$, i.e., $S(q_i, \alpha_i, t), i = 1, 2, ..., n,$ be any integral of the equation,

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_i}, q_i, t\right) = 0$$

Then $\beta_i = \frac{\partial S}{\partial a_i}$ and $p_i = \frac{\partial S}{\partial q_i}$.

These 2n equations link p_i, q_i to α_i's and β_i's, and hence they provide the general solution to the original canonical equations. The crux of the first form of Jacobi's theorem is that α_i's and β_i's are constants of motion which we have to prove.

Proof: Fix $\alpha_1 = P_1, \alpha_2 - P_2, ...$ and consider the function, $F_2(q_1, ...q_n; P_1, ...P_n, t)$ $\equiv S(q_1, ...q_n, P_1, ..., P_1, ...P_n, t)$.

Such function generates a contact transformation. Hence,

$$Q_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial S}{\partial P_i}$$

$$P_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial S}{\partial q_i} \qquad ...(4.17)$$

And

$$K = H + \frac{\partial S}{\partial t} = 0$$

Since K is 0, the equations of motion are

$$P_i = -\frac{\partial K}{\partial Q_i} = 0 \text{ and } Q_i = -\frac{\partial K}{\partial P_i} = 0$$

Solving these equations, we get

$$\left.\begin{aligned} P_i &= \text{constant} = \alpha_i, \quad i = 1, 2, ..., n \\ Q_i &= \text{contant} = \beta_i, \quad i = 1, 2, ..., n \end{aligned}\right\} \quad ...(4.18)$$

Hence we have,

$$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}$$

So it is established that α_i and β_i are constants of motion and S is a solution of the equation, $K = H + \frac{\partial F_2}{\partial t}$ such that $K \equiv 0$

Here, *S* is known as Hamilton's principle or special function.

4.4.2 Method of Separation of Variables in Hamilton-Jacobi Equation

The Hamilton-Jacobi Equation (HJE) is of most use when it can be solved via additive separation of variables, which directly identifies constants of motion. For example, the time *t* can be separated if the Hamiltonian does not depend on time explicitly. In that case, the time derivative $\partial S / \partial t$ in the HJE must be a constant, usually denoted by $-E$, providing the separated solution,

$$S = W(q_1, q_2 .... q_n) - Et$$

Where the time-independent function $W(\mathbf{q})$ is sometimes called **Hamilton's characteristic function**. The reduced Hamilton-Jacobi equation can now be expressed as,

$$H\left(q, \frac{\partial S}{\partial q}\right) = E$$

To illustrate separability for other variables, we assume that a certain generalized coordinate q_k and its derivative $\partial S / \partial q_k$ appear together as a single function,

$$\psi\left(q_k, \frac{\partial S}{\partial q_k}\right)$$

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NOTES

in the Hamiltonian,

$$H = H(q_1, q_2 \cdots q_{k-1}, q_{k+1} \cdots q_N; p_1, p_2 \cdots p_{k-1}, p_{k+1} \cdots p_N; \psi; t)$$

In that case, the function S can be partitioned into two functions, one that depends only on q_k and another that depends only on the remaining generalized coordinates

$$S = S_k(q_k) + S_{rem}(q_1 \cdots q_{k-1}, q_{k+1} \cdots q_N, t)$$

Substitution of these formulae into the Hamilton-Jacobi equation shows that the function ψ must be a constant (denoted here as $x\Gamma_k$), yielding a first-order ordinary differential equation for $S_k(q_k)$.

$$\psi\left(qk, \frac{dS_k}{dq_k}\right) = \Gamma_k$$

In many cases, the function S can be separated completely into N functions $S_m(q_m)$,

$$S = S_1(q_1) + S_2(q_2) + \cdots + S_N(q_N) - Et$$

In such a case, the problem devolves to N ordinary differential equations.

Check Your Progress

1. State the Hamilton's Principle.
2. What do you understand by Whittaker's equations?
3. Give the statement of Lee Hwa Chung's
4. What is the Hamilton-Jacobi equation?

4.5 LAGRANGE BRACKETS

Lagrange brackets are certain expressions closely related to Poisson brackets that were introduced by Joseph Louis Lagrange in 1808–1810 for the purposes of mathematical formulation of classical mechanics. Let $(q_1,...q_n,p_1,...p_n)$ be any functions of two variables (u, v). Then the expression given by,

$$[u,v] = \sum_{r=1}^{n}\left(\frac{\partial q_r}{\partial u}\frac{\partial p_r}{\partial v} - \frac{\partial p_r}{\partial u}\frac{\partial q_r}{\partial v}\right) \quad ...(4.19)$$

is called a Lagrange bracket.

The Lagrange brackets are anticommutative, i.e.,

$$[u_l, u_m] = -[u_m, u_l] \quad ...(4.20)$$

NOTES

If $(q_1,...q_n, p_1,...p_n)$ are any functions of variables $(Q_1,...Q_n, Q_1,...Q_n)$, then

$$\sum_{r=1}^{n}(d\ p_r\ \delta\ q_r - \delta\ p_r\ d\ q_r) = \sum_{kl}[u_k, u_l]\,(du_j\ \delta\ u_k - \delta uj\ du_k). \quad ...(4.21)$$

Here the summation on the right hand side is taken over all pairs of variables $(u_k,\ u_l)$ in the set $(Q_1,...Q_n, P_1,..., P_n)$. But if the transformation from $(q_1,...q_n,\ p_1,...p_n)$ to $(Q_1,...Q_n,\ P_1,...P_n)$ is a contact transformation, then

$$\sum_{r=1}^{n}(d\ p_r\ \delta\ q_r\ dq_r) = \sum_{r=1}^{n}(d\ P_r\ \delta Q_r - \delta\ P_r\ d\ Q_r) \quad ...(4.22)$$

that gives

$$[P_i, P_k] = 0 \text{ for } i,k = 1,2,...,n \quad (4.23)$$

$$[Q_i, Q_k] = 0 \text{ for } i,k = 1,2,...,n \quad (4.24)$$

$$[Q_i, P_k] = 0 \text{ for } i,k = 1,2,...,n, i \neq k \quad (4.25)$$

$$[Q_i, P_i] = 0 \text{ for } i = 1,2,...,n. \quad (4.26)$$

Additionally, these can be regarded as partial differential equations which must be satisfied by $(q_1,...q_n,\ p_1,...P_n)$, considered as function of $(Q_1,...Q_n,\ P_1,...P_n)$ in order that the transformation from one set of variables to the other may be a contact transformation.

Let $(u_1,...u_{2n})$ be 2_n independent functions of the variables $(q_1,...q_n,\ p_1,...p_n)$. Then the Poisson bracket (u_r, u_s) is connected with the Lagrange bracket (u_r, u_s) by the relation,

$$\sum_{t=1}^{2n}(u_t, u_r)\,[u_t, u_s] = \delta_{rs} \quad (4.27)$$

where δ_{rs} is the Kronecker delta.

4.5.1 Canonical Transformation in Terms of Lagrange's and Poisson Bracket

The canonical transformations are those transformations of canonical variables,

$$Qk^{(Q-1)} = F_{km}(qk^{(m-1)}, p_{k/m}t), \quad P_{k/m}G_{km}\,(qk^{(m-1)}, P_{k/m,}t),$$

NOTES

that maintain the shape of the generalized Hamilton's equations invariant and may be solved with respect to the old canonical variables. The necessary and sufficient condition for this transformation to be canonical is,

$$\sum_k \sum_m P_{k/m} dq_k^{(m-1)} - H\,dt = c\left(\sum_k \sum_m P_{k/m} d\,Qk^{(m-1)} - Kdt\right) + dG$$

where K is the new Hamiltonian and G is the corresponding generating function. This condition for the canonical transformation can also be expressed by primary variables as,

$$\sum_k \sum_m \left(p_{k|m} - c \sum_{k'} \sum_{m'} P_{k'|m'} \frac{\partial Q_{k'}^{(m'-1)}}{\partial q_k^{(m-1)}} \right) dq_k^{(m-1)} + \sum_k \sum_m \left(- c \sum_{k'} \sum_{m'} P_{k'|m'} \right.$$

$$\left. \times \frac{\partial Q_{k'}^{(m'-1}}{\partial p_{k|m}} \right) dp_{k|m} + \left(c\,K - H - c \sum_{k'} \sum_{m'} P_{k'|m'} \frac{\partial Q_{k'}^{(m'-1)}}{\partial t} \right) dt = dG.$$

Using generalized Lagrange bracket,

$$\{u, v\} = \sum_k \sum_m \left(\frac{\partial q_k^{(m-1)}}{\partial u} \frac{\partial p_{k|m}}{\partial v} - \frac{\partial q_k^{(m-1)}}{\partial v} \frac{\partial p_{k|m}}{\partial u} \right)$$

We get,

$$\left.\begin{aligned} \left\{q_k^{(m-1)}, q_l^{n-1}\right\}_{Q,P} = 0, \left\{p_{k|m}, p_{l|n}\right\}_{Q,P} = 0, \\ \{q_k^{(m-1)}, p_{l|n}\} Q, P = \frac{1}{c} \delta_{kl} \delta_{mn} \end{aligned}\right\} \quad (4.28)$$

The relation between Poisson and Lagrange brackets is,

$$\sum_{l=1}^{2rs} \{u_l, u_i\} \left[u_l, u_j\right] = \delta_{ij}$$

Here u_l are the arbitrary functions of the canonical variables. On the basis of this relation and Equation (4.28) can also be obtained for Poisson brackets as,

$$\left.\begin{aligned} [q_k^{(m-1)}, q_l^{(n-1)}]_{Q,P} = 0, [p_{k|m}, p_{l|n}]_{Q,P} = 0, \\ [q_k^{(m-1)}, pl \mid n] Q, P = c\,\delta_{kl}\,\delta_{mn} \end{aligned}\right\}$$

4.5.2 Invariance of Lagrange and Poisson Bracket under Canonical Transform

Lagrange Bracket

Lagrange brackets, with respect to variables *u* and *v*

There is, a sum of the form is,

$$\sum_{i=} 1^{n} \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} \right) \equiv [u, v]_{p,q}, \tag{4.29}$$

NOTES

Here is, $q=(q_1 \ldots q_n)$ and $p=(p_1 \ldots p_n)$ are certain functions of u and v.

If $q = (q_1 \ldots q_n)$ and $p = (p_1 \ldots p_n)$ are canonical variables and $Q = Q(q,p)$, $P = P(q,p)$ are canonical transformations, then the Lagrange bracket is an **invariant of this transformation**,

$$[u, v]_{q,p} = [u, v]_{Q,P}$$

For this purpose the indices q,p on the right-hand side of Equation (4.29) are often omitted. The Lagrange bracket is said to be fundamental when the variables u and v coincide with some pair of the $2n$ variables q,p. From them one can form three matrices:

$$[p, p] = \{[p_i, p_j]\}_{i,j=1}^{n}, \quad [q, q], \quad [q, p]$$

The first two of which are the zero, and the last one is the unit matrix. There is a definite connection between Lagrange brackets and Poisson brackets. Specifically, if the functions $u_i = u_i(q,p)$, $1 \le i \le n$, induce a diffeomorphism $\mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$, then the matrices formed from the elements $[u_i,u_j]$ and (u_i,u_j) are **inverse** to each other.

Poisson Bracket

The definition of Poisson bracket of two functions is,

$$\{\omega, \sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \tag{4.30}$$

Computing the Poisson bracket needs knowing of ω and σ as functions of the coordinate's q_i and momenta p_i in the particular coordinate system which we are use. On the other hand, we have seen that the Euler-Lagrange and Hamilton's equations are invariant under a canonical transformation and since the Poisson bracket is a fundamental quantity in classical mechanics, in specific because the time derivative of a function ω is the Poisson bracket $\{\omega,H\}$ with the Hamiltonian, it's natural to ask how the Poisson bracket of two functions transforms under a canonical transformation. The simplest way of finding out is to write the canonical transformation as:

$$\begin{aligned} \bar{q}_i &= \bar{q}_i(q,p) \\ \bar{p}_i &= \bar{p}(q,p) \end{aligned} \tag{4.31}$$

NOTES

We can then write the Poisson bracket in the new coordinates as follow

$$\{\omega,\sigma\}_{\bar{q},\bar{p}} = \sum_j \left(\frac{\partial \omega}{\partial \bar{q}_j}\frac{\partial \sigma}{\partial \bar{p}_j} - \frac{\partial \omega}{\partial \bar{p}_j}\frac{\partial \sigma}{\partial \bar{q}_j} \right) \tag{4.32}$$

Supposing that the transformation is invertible, we can use the chain rule to compute the derivatives with respect to the barred coordinates. This gives the following (we have used the summation convention in which any index repeated twice in a product is summed; thus in the following, there are implied sums over *i, j* and *k*):

$$\begin{aligned}\{\omega,\sigma\}_{\bar{q},\bar{p}} = &\left(\frac{\partial \omega}{\partial q_i}\frac{\partial q_i}{\partial \bar{q}_j} + \frac{\partial \omega}{\partial p_i}\frac{\partial p_i}{\partial \bar{q}_j} \right)\left(\frac{\partial \sigma}{\partial q_k}\frac{\partial q_k}{\partial \bar{p}_j} + \frac{\partial \sigma}{\partial p_k}\frac{\partial p_k}{\partial \bar{p}_j} \right) \\ &\left(\frac{\partial \omega}{\partial q_i}\frac{\partial q_i}{\partial \bar{p}_j} + \frac{\partial \omega}{\partial p_i}\frac{\partial p_i}{\partial \bar{p}_j} \right)\left(\frac{\partial \sigma}{\partial q_k}\frac{\partial q_k}{\partial \bar{q}_j} + \frac{\partial \sigma}{\partial p_k}\frac{\partial p_k}{\partial \bar{q}_j} \right)\end{aligned} \tag{4.33}$$

$$\begin{aligned}= &\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_k}\left(\frac{\partial q_i}{\partial \bar{q}_j}\frac{\partial p_k}{\partial \bar{p}_j} - \frac{\partial q_i}{\partial \bar{p}_j}\frac{\partial p_k}{\partial \bar{q}_j} \right) + \\ &\frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_k}\left(\frac{\partial p_i}{\partial \bar{q}_j}\frac{\partial q_k}{\partial \bar{p}_j} - \frac{\partial p_i}{\partial \bar{p}_j}\frac{\partial q_k}{\partial \bar{q}_j} \right) + \\ &\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial q_k}\left(\frac{\partial q_i}{\partial \bar{q}_j}\frac{\partial q_k}{\partial \bar{p}_j} - \frac{\partial q_i}{\partial \bar{p}_j}\frac{\partial q_k}{\partial \bar{q}_j} \right) + \\ &\frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial p_k}\left(\frac{\partial p_i}{\partial \bar{q}_j}\frac{\partial p_k}{\partial \bar{p}_j} - \frac{\partial p_i}{\partial \bar{p}_j}\frac{\partial p_k}{\partial \bar{q}_j} \right)\end{aligned} \tag{4.34}$$

$$\begin{aligned}= &\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial p_k}\{q_i,p_k\} + \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial q_k}\{p_i,q_k\} + \\ &\frac{\partial \omega}{\partial q_i}\frac{\partial \sigma}{\partial q_k}\{q_i,q_k\} + \frac{\partial \omega}{\partial p_i}\frac{\partial \sigma}{\partial p_k}\{p_i,p_k\}\end{aligned} \tag{4.35}$$

For a canonical transformation, the Poisson brackets in the last equation satisfy

$$\begin{aligned}\{q_i,p_k\} &= -\{p_i,q_k\} = \delta_{ik} \\ \{q_i,q_k\} &= \{p_i,p_k\} = 0\end{aligned} \tag{4.36}$$

Applying these conditions in Equation (4.35), we find that

$$\{\omega,\sigma\}_{\bar{q},\bar{p}} = \left(\frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_k} - \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_k}\right)\delta_{ik}$$
$$= \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_i} \qquad (4.37)$$
$$= \{\omega,\sigma\}_{q,p}$$

Thus the Poisson bracket is *invariant under a canonical transformation*.

Check Your Progress

5. Define the Lagrange brackets.
6. What is Kronecker delta?
7. Define the poisson bracket of two function.

4.6 ANSWERS 'CHECK YOUR PROGRESS'

1. Hamilton's principle is an integral principle. This means that it considers the entire motion of a system between time t_1 and t_2 . The instantaneous configuration of the system is described by the values of n generalized coordinates $q_1,...,q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is called the configuration space.
2. Whittaker's equations are given by

$$\frac{\partial q_j}{dq_1} = \frac{\partial \bar{K}}{\partial p_j}, \frac{dp_j}{dq_1} = -\frac{\partial \bar{K}}{\partial q_j}, \qquad (j = 2,...,n)$$

3. Lee Hwa Chung stated that any other universal integral invariant differs from one of the enumerated integrals by a constant factor. This refers as Lee Hwa Chung theorem.
4. The Hamilton-Jacobi equation is given by

$$\frac{\partial F_2}{\partial t}(q,P,t) + H\left(q, \frac{\partial F_2}{\partial q}(q,P,t), t\right) = 0$$

5. Lagrange brackets are certain expressions closely related to Poisson brackets that were introduced by Joseph Louis Lagrange in 1808–1810 for the purposes of mathematical formulation of classical mechanics. Let $(q_1,...q_n, p_1,...p_n)$ be any functions of two variables (u, v).

$$[u,v] = \sum_{r=1}^{n}\left(\frac{\partial q_r}{\partial u}\frac{\partial p_r}{\partial v} \frac{\partial p_r}{\partial u}\frac{\partial q_r}{\partial v}\right)$$

is called a Lagrange bracket.

NOTES

6. Let $(u_1,...u_{2n})$ be 2_n independent functions of the variables $(q_1,...q_n, p_1,...p_n)$. Then the Poisson bracket (u_r,u_s) is connected with the Lagrange bracket (u_r,u_s) by the relation,

$$\sum_{t=1}^{2n}(u_t,u_r)\,[u_t,u_s]=\delta_{rs}$$

where δ_{rs} is the Kronecker delta.

7. The definition of Poisson bracket of two functions is,

$$\{\omega,\sigma\}=\sum_i\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_i}-\frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial q_i}\right)$$

Computing the Poisson bracket needs knowing of ω and σ as functions of the coordinate's q_i and momenta p_i in the particular coordinate system which we are use.

4.7 SUMMARY

- Hamilton's principle is an integral principle. This means that it considers the entire motion of a system between time t_1 and t_2. The instantaneous configuration of the system is described by the values of *n* generalized coordinates $q_1,..., q_n$, and corresponds to a particular point in a Cartesian hyperspace where the *q*'s form the *n* coordinate axes. This *n*-dimensional space is called the configuration space.
- The correct path of motion of the system is such that the action has a stationary value which means that the integral along the given path has the same value to within first-order infinitesimals as that along all neighbouring paths.
- The function $\eta(t)$ is arbitrary although it must satisfy the boundary values, $\eta(t_1)=\eta(t_2)=0$ and it must be twice differentiable.
- Henri Poincaré has shown that for any Hamiltonian system the form $w=\sum_i dp_i dq_i$ possesses the property that $\int_D{}^W$ over any two-dimensional manifold *D* in $(p_1,..., p_n, q_1,...q_n)$-space is independent of *t.* A differential form with such a property is called a Poincaré integral invariant.
- If $F(x_1,..., x_k, t, dk_1,..., dx_k)$ is a Poincaré integral invariant of a system in which derivatives are identified with functions of the variables and *t*, then $F(x_1,..., x_k, t, dx - A_1\,dt,..., dx - A_k\,dt)$ is an invariant form of the system expressible in terms of differentials and the functions A_k and is called the related Cartan form.

- The Lagrangian, as defined, is very general and in this form, applies even to quantum mechanics. For all known classical systems, the Lagrangian can be written as the difference between the total kinetic energy, T, of the system and the potential energy, U, i.e., for the classical systems, $L = T - U$.
- Lee Hwa Chung stated that any other universal integral invariant differs from one of the enumerated integrals by a constant factor. This refers as Lee Hwa Chung theorem.
- Be the universal integral invariant, the integration will be performed in the phase plane (q, p) along a closed contour.
- The equation under the sign of integral must be a perfect differential with regard to q and p because the integral is equal to zero for any value of the variable t taken as a parameter and for any arbitrary path of integration.
- In mathematics, the Hamilton-Jacobi equation is a essential condition which explain extremal geometry in generalization of problems from the calculus of variation. It can be understood as special case of the Hamilton-Jacobi-Bellman equation from dynamic programming.
- The aim of Hamilton-Jacobi theory is to find this mapping. Since this mapping is canonical, so it has a certain structure that its Jacobian must satisfy. Besides, the transformed variables satisfy Hamiltonian dynamics with a transformed Hamiltonian.
- A way to impose it is to require the transformed Hamiltonian K to be, say, zero (could be any function of time only). Suppose furthermore that we try to generate this transformation using a generating function of the form $F_2(q, P, t)$ (here again, the choice of generating function is somewhat arbitrary).
- A partial differential equation in $n + 1$ variables is by no means simpler than a system of $2n$ ordinary differential equations. The transition from first-order partial differential equations and system of ordinary differential equations is standard in the analysis of hyperbolic partial differential equations.
- The Hamilton-Jacobi Equation (HJE) is of most use when it can be solved via additive separation of variables, which directly identifies constants of motion. For example, the time t can be separated if the Hamiltonian does not depend on time explicitly.
- Where the time-independent function W(q) is sometimes called Hamilton's characteristic function.
- Lagrange brackets are certain expressions closely related to Poisson brackets that were introduced by Joseph Louis Lagrange in 1808–1810 for the purposes of mathematical formulation of classical mechanics. Let $(q_1,...q_n, p_1,...p_n)$ be any functions of two variables (u, v).

NOTES

- Additionally, these can be regarded as partial differential equations which must be satisfied by $(q_1,...q_n, p_1,...P_n)$, considered as function of $(Q_1,...Q_n, P_1,...P_n)$ in order that the transformation from one set of variables to the other may be a contact transformation.

4.8 KEY TERMS

- **Hamilton's principle:** Hamilton's principle is an integral principle. This means that it considers the entire motion of a system between time t_1 and t_2. The instantaneous configuration of the system is described by the values of n generalized coordinates $q_1,..., q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes.
- **Principle of least action:** The stationary-action principle is also known as the principle of least action is a variational principle that give state us when applied to the action of a mechanical system, yields the equations of motion for that system. The principle states that the trajectories (i.e. the solutions of the equations of motion) are stationary points of the system's action functional.
- **Lagrangian:** For every mechanical system, there exists a function of the generalized coordinates (and velocities) called the Lagrangian.
- **Hamilton-Jacobi Equation (HJE):** The Hamilton-Jacobi Equation (HJE) is of most use when it can be solved via additive separation of variables, which directly identifies constants of motion.

4.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. State Hamilton's principle.
2. Give the principle of least action.
3. What do you mean by the Lee Hwa Chung theorem?
4. Give statement of Jacobi's theorem.
5. What do you understand canonical transformation in terms of Poisson's brackets?
6. Define the Lagrange bracket.

Long-Answer Questions

1. Explain in detail about the Hamilton's principle with the help of examples.
2. Describe the Lee Hwa chung theorem with appropriate examples.

NOTES

3. Elaborate on the Jacobi theorem and method of separtion variables giving examples.
4. Discuss Lagrange's brackets and canonical transformation in terms of Lagrange's and Poisson brackets.
5. Analyse the invariance of Lagrange and Poisson bracket under canonical transformation giving examples.

4.10 FURTHER READING

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UNIT 5 GRAVITATION

NOTES

Structure

5.0 INTRODUCTION

Gravity, also called gravitation, in mechanics, the universal force of attraction acting between all matters. It is by far the weakest known force in nature and thus plays no role in determining the internal properties of everyday matter. The force of attraction between all masses in the universe; especially the attraction of the earth's mass for bodies near its surface. The energy stored in a body due to the gravitational force between the body and the earth is called the gravitational potential energy.

In mathematics, particularly multivariable calculus, a surface integral is a generalisation of multiple integrals to integration over surfaces. Come to the Gauss Divergence theorem, the surface integral of a vector field A over a closed surface is equal to the volume integral of the divergence of a vector field A over the Volume (V) enclosed by the closed surface. Surface Integrals are used to determine pressure and gravitational force otherwise the Gauss' Law of Electro statistics, it is used to compute the electric field.

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Laplace's Equation, which is the single most important equation in potential theory. Therefore the gravitational potential at any point outside a uniform shell is equivalent to that of a point source of the same mass located at the centre. On the other hand Poisson's Equation is states that the Laplacian of the electric potential field is equal to the volume charge density divided by the permittivity, with a change of sign.

The self-attracting systems exhibit gravitational or collapse-like transition. As the fermionic. Degeneracy or the softness radius increases, the gravitational phase transition crosses over to a normal first-order phase transition, becomes second-order at a critical point, and finally disappears. However the potential-distribution theorem is state that under the exact statistical mechanical. Expression for the excess chemical potential at a point in a fluid at equilibrium.

An equipotential surface is the collection of points in space that are all at the same potential. Equipotential lines are the two-dimensional representation of equipotential surfaces. Equipotential surfaces are always perpendicular to electric field lines. For an additionally in mathematics and physical science, spherical harmonics are special functions defined on the surface of a sphere. They are often employed in solving partial differential equations in many scientific fields.

In this unit, you will learn about the gravitation, attraction and potential of rod, disc, spherical shells and sphere, surface integral of normal attraction, Gauss theorem and their applications, Laplace and Poisson equation for attraction and potential, work done by self-attracting system, distribution for given potential, equipotential surface, solid harmonic surface and density in term of surface harmonic.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the gravitation
- Determine the attraction and potential of rod, disc, spherical shells and sphere
- Give the Gauss theorem and their applications
- Explain about the surface integral of normal attraction
- Elaborate on the Laplace equations for attraction and potential
- Discuss about the Poisson equation for attraction and potential
- Analyse the work done by self-attracting systems
- Learn about the distributions for a given potential
- Describe on the equipotential surfaces
- Interpret the density in term of surface harmonics
- Know about the harmonic surface and solid harmonic surface
- Explain the surface and solid harmonics
- Discuss the surface density in terms of surface harmonics

5.2 BASIC CONCEPT OF GRAVITATION

NOTES

Gravity is also known as 'Gravitation', according to mechanics, the universal force of attraction acting between all matters. It is by far the weakest known force in nature and thus plays no role in determining the internal properties of everyday matter. Alternatively, through its long reach and universal action, it controls the trajectories of bodies in the solar system and to another place in the universe and the structures and evolution of stars, galaxies, and the whole cosmos. On Earth all bodies have a weight, or downward force of gravity, proportional to their mass, which Earth's mass exerts on them. Gravity is measured by the acceleration that it gives to freely falling objects. At Earth's surface the acceleration of gravity is about 9.8 metres (32 feet) per second. Thus, for every second an object is in free fall, its speed increases by about 9.8 metres per second. At the surface of the Moon the acceleration of a freely falling body is about 1.6 metres per second per second.

Gravity word is also came from the Latin *gravitas* which means that is 'Weight'. The gravitation, is a natural occurrence by which all things with mass or energy, including planets, stars, galaxies and even light, are attracted to (or gravitate toward) one another. On Earth, gravity gives weight to physical objects, and the Moon's gravity causes the tides of the oceans. The gravitational attraction of the original gaseous matter present in the Universe caused it to begin coalescing and forming stars and caused the stars to group together into galaxies, so gravity is responsible for many of the large-scale structures in the universe. Gravity has an infinite range, while its effects become weaker as objects get farther away.

Gravity is most accurately described by the general 'Theory of Relativity' which proposed by Albert Einstein in 1915, the statement of theory is, gravity not as a force, but as the curvature of space-time, caused by the uneven distribution of mass, and causing masses to move along geodesic lines. The most extreme example of this curvature of space time is a black hole, from which nothing not even light can escape once past the black hole's event horizon. However, for most applications, gravity is well approximated by Newton's law of universal gravitation, which describes gravity as a force causing any two bodies to be attracted toward each other, with magnitude proportional to the product of their masses and inversely proportional to the square of the distance between them.

Current models of particle physics imply that the earliest instance of gravity in the universe, possibly in the form of quantum gravity, supergravity or a gravitational singularity, along with ordinary space and time, developed during the Planck epoch (up to 10^{-43} seconds after the birth of the universe), possibly from a primeval state, such as a false vacuum, quantum vacuum or virtual particle, in a currently unknown manner.

NOTES

Newton's Theory of Gravitation

According to Newton's law "any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them". We can represent by following formula,

$$F = G\frac{m_1 m_2}{r^2}$$

Here F is the force, m_1 and m_2 are the masses of the objects interacting, r is the distance between the centres of the masses and G is the gravitational constant.

General Relativity

In general relativity, the effects of gravitation are ascribed to space time curvature instead of a force. The starting point for general relativity is the equivalence principle, which equates free fall with inertial motion and describes free-falling inertial objects as being accelerated relative to non-inertial observers on the ground. Einstein proposed that space time is curved by matter, and that free-falling objects are moving along locally straight paths in curved space time. These straight paths are called geodesics. Like Newton's first law of motion, Einstein's theory states that if a force is applied on an object, it would deviate from a geodesic. For example, we are no longer following geodesics while standing because the mechanical resistance of the Earth exerts an upward force on us, and we are non-inertial on the ground as a result. Einstein discovered the field equations of general relativity, which relate the presence of matter and the curvature of space time and are named after him. The Einstein field equations are a set of 10 simultaneous, non-linear, differential equations. The solutions of the field equations are the components of the metric tensor of space time. A metric tensor describes a geometry of space time. The geodesic paths for a space time are calculated from the metric tensor.

Gravity and Quantum Mechanics

General relativity describes large-scale bulk properties whereas quantum mechanics is the framework to describe the smallest scale interactions of matter. One path is to describe gravity in the outline of quantum field theory, which has been successful to accurately describe the other fundamental interactions. The electromagnetic force arises from an exchange of virtual photons, where the QFT (Quantum Field Theory) explanation of gravity is that there is an exchange of virtual gravitons. This explanation reproduces general relativity in the classical limit. However, this approach fails at short distances of the order of the Planck length, where a more complete theory of quantum gravity.

Earth's Gravity

Every planetary body (including the Earth) is surrounded by its own gravitational field, which can be hypothesised with Newtonian physics as exerting an attractive

force on all objects. Assuming a spherically symmetrical planet, the strength of this field at any given point above the surface is proportional to the planetary body's mass and inversely proportional to the square of the distance from the centre of the body.

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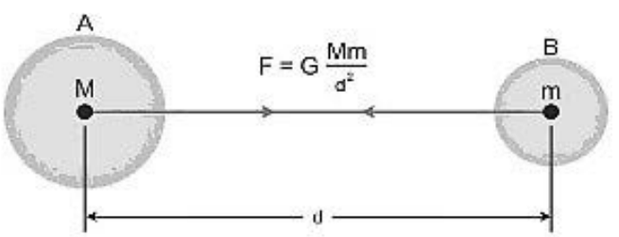
The strength of the gravitational field is numerically equal to the acceleration of objects under its influence. The rate of acceleration of falling objects near the Earth's surface varies very slightly depending on latitude, surface features such as mountains and ridges, and perhaps unusually high or low sub-surface densities. For purposes of weights and measures, a standard gravity value is defined by the International Bureau of Weights and Measures, under the International System of Units (SI).

That value, denoted g, is $g = 9.80665\ m/s^2$ ($32.1740\ ft/s^2$).

The standard value of $9.80665\ m/s^2$ is the one originally adopted by the International Committee on Weights and Measures in 1901 for 45° latitude, even though it has been shown to be too high by about five parts in ten thousand. This value has persisted in meteorology and in some standard atmospheres as the value for 45° latitude even though it applies more precisely to latitude of 45°32'33".

Every object in the universe attract every other object with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres. The direction of the force is along the line joining the centre of two object.



$$F \propto M \times m \qquad ...(5.1)$$

$$F \propto 1/d^2 \qquad ...(5.2)$$

For Equations (5.1) and (5.2)

$$F \propto \frac{M \times m}{d^2}$$

$$F = G\frac{M \times m}{d^2}$$

Whereas G is gravitation constant.

Consider a mass distribution with density $\rho(x)$. There is a corresponding gravitational field $F(x)$ which we may express in terms of a gravitational potential

NOTES

$\Phi(x)$. Consider an arbitrary fixed volume V with surface S containing a total mass $M_V = \iiint_V \rho(x)dV$.

Gauss showed that the flux of the gravitational field through S is equal to $-4\pi GM_V$.

Hence,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -4\pi G M_V$$

$$\Rightarrow \quad -\iint_S \nabla\Phi \cdot \mathbf{n}\, dS = -4\pi G \iiint_V \rho(\mathbf{x})\, dV$$

$$\Rightarrow \quad \iiint_V \nabla \cdot (\nabla\Phi)\, dV = 4\pi G \iiint_V \rho(\mathbf{x})\, dV.$$

This is true for all volumes V, so we must have

$$\boxed{\nabla^2\Phi = \nabla.(\nabla\Phi) = 4\pi G\rho}$$

5.3 ATTRACTION AND POTENTIAL

Attraction is a force between two or more dissimilar or unlike charges. Two charges of dissimilar characteristics pull towards each other. On the other hand in classical mechanics, the gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location. It is analogous to the electric potential with mass playing the role of charge. The reference location, where the potential is zero, is by convention infinitely far away from any mass, resulting in a negative potential at any finite distance. In mathematics, the gravitational potential is also known as the Newtonian potential and is fundamental in the study of potential theory. It may also be used for solving the electrostatic and magneto static fields generated by uniformly charged or polarized ellipsoidal bodies. The scalar quantity characteristic of a point in a gravitational field whose gradient equals the intensity of the field and equal to the work required to move a body of unit mass from given point to a point infinitely remote.

5.3.1 Attraction and Potential of Rod

Attraction of Rod

AB be a rod

$\angle APB = \alpha - \beta$ Show in Figure 5.1 as follow

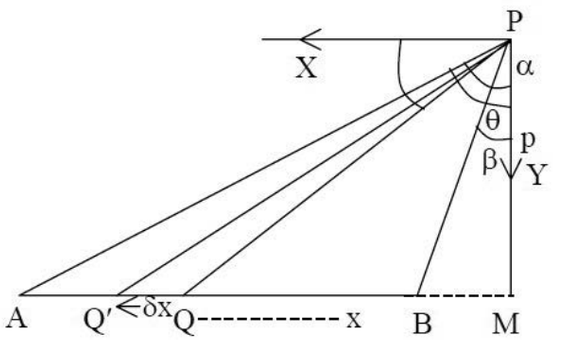


Fig. 5.1 *Mathematical Representation of Rod*

Let us consider m is the mass per unit length of a uniform rod AB.

It is essential to find the components of attraction of the rod AB at an external point P.

$MP = p$

Consider an element QQ′ of the rod where

$MQ = x$

$QQ' = dx$

$\angle MPQ = \theta$

In ΔMPQ,

$$\tan\theta = \frac{MQ}{MP} = \frac{x}{p} \Rightarrow x = p\tan\theta \qquad ...(5.3)$$

$$\cos\theta = \frac{MP}{PQ} = \frac{p}{pQ}$$

$$\Rightarrow \quad PQ = \frac{p}{\cos\theta} \qquad \Rightarrow PQ = p\sec\theta \qquad ...(5.4)$$

Mass of element QQ' of rod $= mdx$

$= mp\sec^2\theta\, d\theta$

The attraction at P of the element QQ' is,

$$= \frac{\text{mass}}{(\text{distance})^2} = \frac{mp\sec^2 d\theta}{(PQ)^2} \text{along } PQ$$

So that, force of attraction at P of the element QQ′ is,

$$= \frac{mp\sec^2\theta d\theta}{p^2\sec^2 g\theta}$$

$$= \frac{m}{p}d\theta \qquad \text{along } PQ \qquad ...(5.5)$$

NOTES

Let assuming that $\angle MPA = \alpha$ and $\angle MPB = \alpha$

$$f = \int_{\beta}^{\alpha} \frac{m}{p} d\theta$$

Let consider *X* and *Y* be the components of attraction of the rod parallel and perpendicular to rod, then

$$X = \int_{\beta}^{\alpha} \frac{m}{p} \sin\theta \, d\theta$$

And,

$$Y = \int_{\beta}^{\alpha} \frac{m}{p} \cos\theta \, d\theta$$

$$X = \frac{m}{p}\left[-\cos\theta\right]_{\beta}^{\alpha} = \frac{m}{p}\left[\cos\beta - \cos\alpha\right]$$

$$= \frac{m}{p}\left[2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\beta}{2}\right] \quad ...(5.6)$$

Therefore,

$$Y = \frac{m}{p}\left[\sin\theta\right]_{\beta}^{\alpha} = \frac{m}{p}\left[\sin\alpha - \sin\beta\right]$$

$$= \frac{m}{p}\left[2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right] \quad ...(5.7)$$

Subsequent force of Attraction R is given by,

$$R = \sqrt{X^2 + Y^2}$$

$$R = \sqrt{X^2 + Y^2}$$

By Equations (5.6) and (5.7))

$$R = \frac{2m}{p}\sin\frac{\alpha-\beta}{2}$$

$$= \frac{2m}{p}\sin\angle\frac{APB}{2}$$

Resultant R makes angle $\tan^{-1}\frac{X}{Y}$.

Or,

$$\frac{1}{2}(\alpha+\beta)\text{with PM} \qquad \left[\Theta\ \tan^{-1}\left(\frac{X}{Y}\right) = \left[\tan^{-1}\left(\tan\frac{\alpha+\beta}{2}\right)\right]\right]$$

i.e., it acts along bisector of angle $\angle$APB.

Therefore,

$$X=\frac{m}{PB}-\frac{m}{PA}$$

NOTES

$$\Theta\cos\beta=\frac{p}{PB},\cos\alpha=\frac{p}{PA}\ \text{(By Using Equation 5.6)}$$

Corollary: If the rod is infinitely long, the angle APB is two right angles and resultant attraction is $=\frac{2m}{p}\perp_r$ to the rod.

Potential of Rod

By definition, the potential at P is given by,

$$V=\int\frac{m}{PQ}dx$$

$$V=\int_{\beta}^{\alpha}\frac{mp\sec^2\theta}{p\sec\theta}d\theta$$

$$=\int_{\beta}^{\alpha}m\sec\theta\,d\theta$$

$$=m\left[\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]_{\beta}^{\alpha}$$

$$=m\left[\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)-\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right]$$

$$=m\log\left[\frac{\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\beta}{2}+\frac{\pi}{4}\right)}\right]$$

5.3.2 Attraction and Potential of Disc

Attraction of Disc

Here radius of disc = a

So that,

$OP = r, \qquad PQ = \sqrt{x^2+r^2}$

$OQ = x$

We consider two element of masses dm at the two opposite position Q and Q′ as Shown in Figure 5.2. Now element dm at Q causes attraction on unit mass at P in the direction PQ. Similarly other mass dm at Q′ causes attraction on same unit mass at P in the direction PQ′ and the force of attraction is same in magnitude. These two attraction forces when resolved into two direction one along the axes

NOTES

PO and other at right angle PO. Components along PO are additive and component along perpendicular to PO cancelling each other

Mass of Ring = $2\pi x dx\rho$

$$df = \frac{(\sum dm)\cos\theta}{(PQ)^2}$$

$$df = \frac{\cos Q.2\pi x\, dx\rho}{(PQ)^2} = \frac{r.2\pi x dx\,\rho}{(PQ)^3} \quad \text{along PO} \qquad [\Theta \cos\theta = \frac{r}{PQ} \text{in} \Delta OPQ\ ]$$

$$= \frac{2\pi\rho.rx\, dx}{\left(r^2 + x^2\right)^{\frac{3}{2}}}$$

Therefore, the resultant attraction at P due to the whole disc along PO is given by,

$$f = \pi\rho r\int_0^a (2x)\left(r^2 + x^2\right)^{-\frac{3}{2}} dx$$

$$f = \pi\rho r\int_0^a (2x)\left(r^2 + x^2\right)^{-\frac{3}{2}} dx$$

$$= \pi\rho r\left[-2\left(x^2 + r^2\right)^{-\frac{1}{2}}\right]_0^a$$

$$= 2\pi\rho r\left[\frac{1}{r} - \frac{1}{\sqrt{a^2 + r^2}}\right] \text{along PO}$$

Assume that M = mass of disc of radius a

$=\Pi a^2$

$$\Pi\rho = \frac{M}{a^2}\rho$$

$$\rho = \frac{2M}{a^2}\left[1 - \frac{r}{\sqrt{a^2 + r^2}}\right]$$

$$= \frac{2M}{a^2}\left[1 - \cos\alpha\right]$$

Where α is the angle which any radius of disc subtends at P.

Example 5.1: If radius of disc becomes infinite.

Solution: Let assume that $\alpha = \pi/2$

Therefore,

$$\rho = \frac{2M}{a^2}\left[1 - \cos\frac{\pi}{2}\right]$$

$$= \frac{2M}{a^2} = \text{constant [here, it is independent of position of P]}$$

Example 5.2: when P is at a very large distance from the disc.

Solution: Therefore then $\alpha \to 0$

So that,

$$\rho = \frac{2M}{a^2}(1 - \cos 0)$$

$$= 0.$$

Potential of Disc

Let us the uniform circular disc of radius 'a' and P is a plate. on the axis of disc and plate. P is at a distance r from the centre 0, i.e., OP = r, OQ = x, PQ = $\sqrt{r^2 + x^2}$ let us divide the disc into a number of concentric rings and assuming a ring which have radius 'x' and width dx

Now, Mass of ring is = $2\pi x dx \rho$

Here ρ is density of material of disc ρ = Mass/Area

So that,

Potential at P due to this ring is given by,

$$dv = \frac{2\pi x dx \rho}{\sqrt{r^2 + x^2}}$$

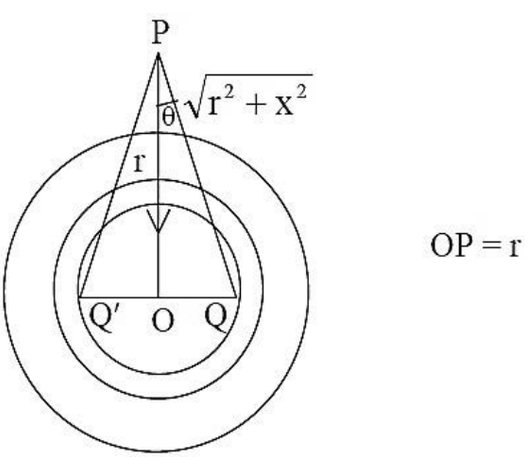


Fig. 5.2 *Uniform Circular Disc*

Therefore, the potential at P due to the whole disc is represented by,

$$V = 2\Pi \rho \int_0^a \frac{x\,dx}{\sqrt{x^2 + r^2}}$$

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$$V = \frac{2\pi\rho}{2}\int_0^a 2x\left(x^2 + r^2\right)^{-\frac{1}{2}}\,dx$$

$$V = 2\Pi\,\rho\left[\sqrt{a^2 + r^2} - r\right]$$

Let Mass of disc = M

$$= \Pi a^2\,\rho$$

$$\Rightarrow \qquad \Pi\,\rho = \frac{M}{a^2}$$

So that,

$$V = \frac{2m}{a^2}\left[\sqrt{a^2 + r^2} - r\right]$$

is required potential at any plate P which lies on the axis of disc.

5.3.3 Attraction and Potential of a Spherical Shell

Attraction of a Spherical Shell

Let suppose for that consider a slice BB′C′C at point P, the attraction due to this slice is,

$$df = \frac{2\pi a^2 \rho \sin\theta\, d\theta}{x^2} \text{ along PB}$$

The resultant attraction directed along PO is given by,

$$df = \frac{2\pi a^2 \rho \sin\theta\, d\theta}{x^2}\cos\alpha$$

We know that

$$\sin\theta\, d\theta = \frac{x dx}{ar}$$

In ΔBDP,

$$\cos\alpha = \frac{PD}{PB} = \frac{r - a\cos\theta}{x}$$

$$df = \frac{2\pi a^2 \rho x dx}{ar.x^2}\left(\frac{r - a\cos\theta}{x}\right)$$

We distinguish that,

$$x^2 = a^2 + r^2 - 2ar\cos\theta$$

$$x^2 - a^2 + r^2 = 2r^2 - 2ar\cos\theta$$

$$\frac{x^2 - a^2 + r^2}{2r} = r - a\cos\theta$$

So that,

$$df = \frac{2\pi a^2 \rho x dx}{ar.x^2}\frac{\left(x^2 - a^2 + r^2\right)}{2r,x}$$

$$= \frac{\pi a\rho}{r^2}\left(\frac{x^2 - a^2 + r^2}{x^2}\right)dx$$

$$= \frac{a\pi\rho}{r^2}\left(1 + \frac{r^2 + a^2}{x^2}\right)dx$$

Hence the attraction for the whole spherical shell is obtained by integration Subsequently,

$$f = \frac{\pi a\rho}{r^2}\int\left[1 + \frac{r^2 - a^2}{x^2}\right]dx$$

Now we consider the following cases depending upon the position of P.

Example 5.3: When point P is inside the shell.

Solution: The limits of integration are x = (r – a) to(r + a)

$$\vec{f} = \frac{\pi a\rho}{r^2}\int_{r-a}^{r+a}\left(1 + \frac{r^2 - a^2}{x^2}\right)dx$$

$$\vec{f} = \frac{\pi a\rho}{r^2}\left[x + \left(r^2 - a^2\right)\left(\frac{-1}{x}\right)\right]_{r-a}^{r+a}$$

$$= \frac{4\pi a^2\rho}{r^2} = \frac{M}{r^2}$$

Example 5.4: When plate P is on the shell.

Solution: Let consider the limit of integration are x = 0 to 2a

$$f = \frac{\pi a\rho}{r^2}\int_0^{2a}\left(1 + \frac{r^2 - a^2}{x^2}\right)dx$$

Here integration is not possible (due to second term is becoming indeterminant), because when P is on the shell, so that

r = a; x = 0

Hence to evaluate the integral, we consider that plate P is situated not on the surface but very near to the surface

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Let r = a + δ, where δ is very small

Then attraction is,

$$f = \frac{a\pi\rho}{r^2}\left[\int_{\delta}^{2a+\delta} dx + \int_{\delta}^{2a+\delta}\left\{\frac{(a+\delta)^2 - a^2}{x^2}\right\}dx\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a + \int_{\delta}^{2a+\delta}\frac{2a\delta}{x^2}dx\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a + 2a\delta\left(\frac{-1}{x}\right)_{\delta}^{2a+\delta}\right]$$

$$= \frac{\pi a\rho}{r^2}\left[2a - \frac{2a\delta}{2a+\delta} + \frac{2a\delta}{\delta}\right]$$

$$= \frac{2\pi a^2\delta}{r^2}\left[2 - \frac{\delta}{2a+\delta}\right] \text{ as } \delta \to o, \text{ then } r = a$$

$$= \frac{4\pi a^2\rho}{a^2} = \frac{M}{a^2}$$

Example 5.5: Point P is inside the shell.

Solution: Supposing that limits are x = a – r to a + r

$$f = \frac{\pi a\rho}{r^2}\int_{a-r}^{a+r}\left[1 + \frac{r^2 + a^2}{x^2}\right]dx$$

$$= \frac{\pi a\rho}{r^2}\left[x - \left(r^2 - a^2\right)\left(\frac{1}{x}\right)\right]_{a-r}^{a+r} = 0$$

So, there is no resultant attraction inside the shell.

Potential of a Spherical Shell

We suppose that a thin spherical shell of radius 'a' and surface density 'ρ' let P be a point at a distance 'r' from the centre O of the shell. We consider a slice show in Figure 5.3 BB′ C′ C in the form of ring with two planes close to each other and perpendicular to OP.

Area of Ring (slice) BB′C′ C is,

= 2ΠBD × BB′

Here is radius of ring, BD = a sinθ

Width of ring, BB′= a dθ

Consequently, Mass of slice (ring) is,

= 2Π a sinθ adθ ρ

= 2Πa2 ρ sinθ dθ

Therefore, Potential at P due to slice (ring) is,

$$dV = \frac{2\pi a^2 \rho \sin\theta d\theta}{x} \quad ...(5.8)$$

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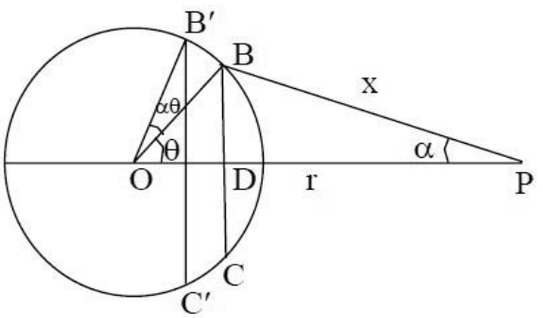

Fig. 5.3 *Thin Spherical Shell*

Now, from ΔBOP,

$BP^2 = OP^2 + OB^2 - 2OP \cdot OB \cos\theta$

$x^2 = r^2 + a^2 - 2ar\cos\theta$

Differentiating,

$$2x\,dx = 2ar\sin\theta\,d\theta$$

$$\frac{x}{ar}dx = \sin\theta d\theta$$

Above equation putting in Equation 5.8 so we will get,

$$dV = \frac{2\pi a^2 \rho x\,dx}{x.ar}$$

$$= \frac{2\pi a\rho dx}{r} \quad ...(5.9)$$

Therefore, Potential for the whole spherical shell is obtained by integrating Equation (5.9), we get

$$V = \int \frac{2\pi a\rho}{r} dx$$

$$= \frac{2\pi a\rho}{r}\int dx$$

NOTES

Example 5.6: The point P is outside the shell.

Solution: In this case, the limit of integration extends from is,

$x = (r - a)$ to $x = (r + a)$

Therefore,

$$V = \frac{2\pi a\rho}{r}\int_{r=a}^{r+a} dx$$

$$V = \frac{4\pi a^2\rho}{r}$$

Whereas, Mass of spherical shall = $4\Pi a^2\rho$

Hence,

$$V = \frac{M}{r}$$

Example 5.7: When P is on the spherical shell, then limits are from x = 0 to x = 2a. (Given r = a)

Solution: we know that,

$$V = \frac{2\pi a\rho}{a}\int_{0}^{2a} dx$$

So that,

$$= \frac{4\pi a^2\rho}{a} = \frac{M}{a}$$

Example 5.8: When P is inside the spherical shell.

Solution: Supposing that limit are from $x = (a - r)$ to $(a + r)$.

$$V = 4\Pi a\ \rho = \frac{M}{a}$$

5.3.4 Attraction and Potential of a Sphere

Attraction for a Sphere

Condition I: At an external point

$$F = \frac{m_1}{r^2} + \frac{m_2}{r^2} + ...$$

$$F = \frac{M}{r^2},\ M = m_1 + m_2 + ...$$

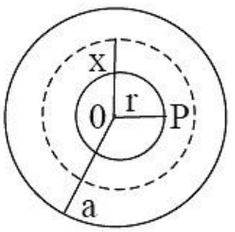

Fig. 5.4 *Uniform Solid Sphere*

M = Mass of sphere and m_1, m_2…. are masses of concentric spherical shells.

Condition II: At a point on the sphere,

Here we put r = a in above result

We found that,

$$F = \frac{M}{a^2}$$

Condition III: At a point inside the sphere.

The point P is external to the solid sphere of radius r and it is internal to thick spherical shell of radii r and a. And we know that attraction (forces of attraction) at an internal point in case of spherical shell is zero. Hence the resultant attraction at P is only due to solid sphere of radius r and is given by

$$F = \frac{\text{mass of sphere of radius r}}{r^2}$$

$$= \frac{4}{3}\frac{\pi r^3 \rho}{r^2} = \frac{4}{3}\pi r\rho$$

iff and only iff,

$$M = \frac{4}{3}\pi a^3 \rho \;\Rightarrow\; \pi\rho = \frac{3M}{4a^3}$$

Hence,

$$F = \frac{Mr}{a^3}.$$

Potential of a Sphere

A uniform solid sphere may be supposed to be made up of a number of thin uniform concentric spherical shells. The masses of spherical shells may be supposed to be concentric at centre O.

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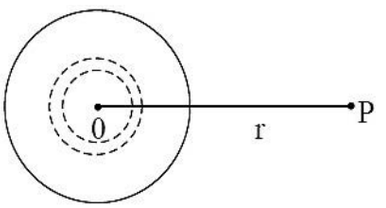
Example 5.9: Show that P is an external point.



Solution: Therefore the potential due to all such shells at an external point P is given by,

$$V = \frac{m_1}{r} + \frac{m_2}{r} + \ldots$$

Where m_1, m_2 … etc. are the masses of shells.

$$V = \frac{1}{r}(m_1 + m_2 + \ldots) = \frac{M}{r}$$

Here M is the mass of solid sphere.

Example 5.10: The point P is on the sphere.

Solution: In Example 5.9, put r = a

Then,

$$V = \frac{M}{a}$$

Where a = radius of sphere.

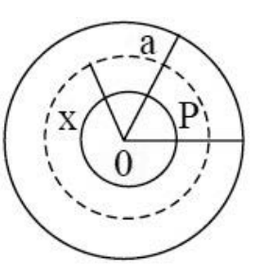
Example 5.11: When P at an internal point.

Solution: Here point P is considered to be external to solid sphere of radius r and internal to the shell of internal radius r, external radius = a in following figure.

Let V_1 = potential due to solid sphere of radius r

V_2 = potential due to thick shell of internal radius r and external radius a

So that,



Calculation of V_2

We consider a thin concentration shell of radius 'x' and thickness dx. The potential at P due to thin spherical shell under consideration is given by,

$$\frac{4\pi x^2 dx\rho}{x} = 4\pi x\, dx\, \rho$$

Hence for the thick shell of radius r and a, the potential is given by,

$$V_2 = 4\pi\rho \int_r^a x\, dx$$

$$= 4\pi\rho\left(\frac{a^2 - r^2}{2}\right) = 2\pi\rho(a^2 - r^2)$$

So, the potential at P due to given solid sphere.

$$V = V_1 + V_2 = \frac{2}{3}\pi\rho(3a^2 - r^2)$$

Where M = Mass of given solid sphere = $\frac{4}{3}\pi a^3\rho$.

$$\Rightarrow \qquad \pi\rho = \frac{3M}{4\pi a^3}$$

$$\text{Hence } V = \frac{2}{3}.\frac{3M}{4\pi a^3}(3a^2 - r^2) = \frac{M}{2a^3}(3a^2 - r^2).$$

NOTES

Check Your Progress

1. What is gravitational potential?
2. State the Newton's theory of gravitation.
3. Define the attraction.
4. What do you understand by potential?

5.4 SURFACE INTEGRAL OF NORMAL ATTRACTION

In mathematics, particularly multivariable calculus, a surface integral is a generalization of multiple integrals to integration over surfaces. It can be thought of as the double integral analogue of the line integral. Given a surface, one may integrate a scalar field (that is, a function of position which returns a scalar as a value) over the surface, or a vector field (that is, a function which returns a vector as value). If a region R is not flat, then it is called a surface. Surface integrals have applications in physics, particularly with the theories of classical electromagnetism.

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In vector calculus, the divergence theorem, also known as *Gauss's theorem* or *Ostrogradsky's theorem*, is a theorem which relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed. Moreover, the divergence theorem states that the surface integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the 'Volume Integral' of the divergence over the region inside the surface. Naturally, it states that the sum of all sources of the field in a region (with sinks regarded as negative sources) gives the net flux out of the region. The divergence theorem is an important result for the mathematics of physics and engineering, particularly in electrostatics and fluid dynamics. In these fields, it is usually applied in three dimensions. However, it generalises to any number of dimensions. In one dimension, it is equivalent to integration by parts. In two dimensions, it is equivalent to Green's theorem.

Suppose for that m be a mass at a point (x_0, y_0, z_0) outside the surface $\boldsymbol{S}$ show in Figure 5.5.

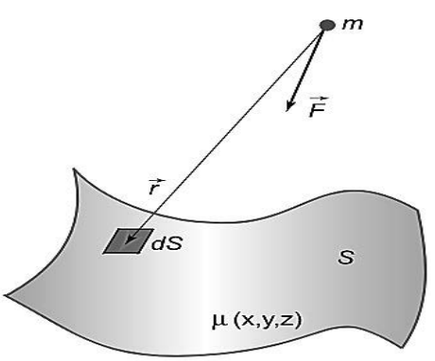

Fig. 5.5 *Applied Gravitational Force*

Then the *force of attraction* between the surface $\boldsymbol{S}$ and the mass m is given by,

$$\mathbf{F} = Gm \iint_S \mu(x, y, z) \frac{\mathbf{r}}{r^3} dS,$$

Here is,

$$\mathbf{r} = (x - x_0, y - y_0, z - z_0)$$

G is gravitational constant and $\mu(x, y, z)$ is the density function.

5.4.1 Gauss Theorem

The surface integral of the normal component of a vector function $\vec{F}$ over a simple closed surface S enclosing a volume V is equal to the volume integral of the divergence of $\vec{F}$ taken throughout V.

i.e., $$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

Where $\hat{n}$ is the unit normal drawn outwards to the surface S.

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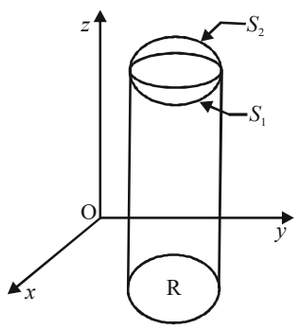


Fig 5.6 *Volume Integral and Divergence*

Let S be a simple closed surface. Any line parallel to the axes will cut the surface only at two points. Let R be the projection of S on the xy plane. The cylinder on R with generators parallel to the z-axis will touch S along a curve and thus divide S into two parts S_1 and S_2, S_2 being the upper part.

Let the equation of S_1 and S_2 be,

Let, $$z = f_1(x, y) \text{ and } f_2(x, y)$$

$$\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$\iiint_V (\nabla \cdot \vec{F})dV = \iiint_V \sum\left(\frac{\partial F_1}{\partial x}\right)dV = \iiint_V \sum\left(\frac{\partial F_1}{\partial x}\right)dxdydz$$

Consider, $$\iiint_V \frac{\partial F_3}{\partial z}dV = \iiint_V \left(\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_R (F_3)_{z_1}^{z_2}\,dx\,dy$$

Where z_1 and z_2 are the coordinates of the points of intersection of the line parallel to z- axis through $(x, y, 0)$ on xy plane with surfaces S_1 and S_2.

$$= \iint_R F_3(x, y, z_2) - F_3(x, y, z_1)dxdy$$

$$= \iint_R F_3(x, y, f_2) - F_3(x, y, f_1)dxdy$$

Consider, $$\iint_S F_3(x, y, z)\bar{k} \cdot \hat{n}dS$$

$$= \iint_{S_2} F_3(x, y, f_2)\bar{k} \cdot \hat{n}_2 dS_2 + \iint_{S_1} F_3(x, y, f_1)\bar{k} \cdot \hat{n}_1 dS_1$$

$\bar{k} \cdot \hat{n}_2 dS_2$ is the projection of the area element dS_2 on the surface S_2 on the xy plane and $\bar{k} \cdot \hat{n}_1 dS_1$ is the negative of the projection of the area element dS_1 on the surface S_1 on the xy plane.

$$\iint_S F_3(x, y, z)\bar{k} \cdot \hat{n}dS$$

$$= \iint_R F_3(x,y,f_2) - F_3(x,y,f_1)dxdy = \iiint_V \frac{\partial F_3}{\partial z} dV$$

NOTES

Similarly, $\iiint_V \frac{\partial F_1}{\partial x} dV = \iint_S F_1(x,y,z)\bar{i} \cdot \hat{n} dS$

$$\iiint_V \frac{\partial F_2}{\partial y} dV = \iint_S F_2(x,y,z)\bar{j} \cdot \hat{n} dS$$

$$\therefore \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV = \iint_S (F_1\bar{i} + F_2\bar{j} + F_3\bar{k}) \cdot \hat{n} dS$$

i.e., $\iiint_V (\nabla \cdot \vec{F}) dV = \iint_S \vec{F} \cdot \hat{n} dS$

Corollary: $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$

Green's Identities

Let, $\vec{F} = \psi\nabla\varphi$, where ψ and φ are scalar point functions.

$\nabla \cdot \vec{F} = \nabla(\psi\nabla\varphi) = \nabla\psi \cdot \nabla\varphi + \psi\nabla^2\varphi$

Using this in Gauss Theorem, we get

$$\iint_S (\psi\nabla\varphi) \cdot \hat{n} dS = \iiint_V \left(\nabla\psi . \nabla\varphi + \psi\nabla^2\varphi \right) dV \qquad (5.10)$$

Interchanging ψ and φ we get,

$$\iint_S (\varphi\nabla\psi) \cdot \hat{n} dS = \iiint_V \left(\nabla\varphi . \nabla\psi + \varphi\nabla^2\psi \right) dV \qquad (5.11)$$

From Equations (5.10) and (5.11) we get,

$$\iint_S (\psi\nabla\varphi - \varphi\nabla\psi) \cdot \hat{n} dS = \iiint_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi) dV \qquad (5.12)$$

The Equations (5.10), (5.11) and (5.12) are called Green's identities.

Note: $\nabla\phi \cdot \bar{n} = \frac{\partial\phi}{\partial n}$. Using this notation, Equation (5.12) can be written as,

$$\iint_S \left[\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right] ds = \iiint_V \left(\psi\nabla^2\psi - \phi\nabla^2\psi \right) dV$$

5.4.2 Applications

- Surface Integrals are used to determine pressure and gravitational force. In Gauss' Law of Electro statistics, it is used to compute the electric field. It is

also use the find mass of the shell. It analysed the how to calculate the moment of inertia and the centre of mass of the shell.

- One of the most common applications of the divergence theorem is to *electrostatic fields*. An important result in this subject is Gauss' law. This law states that if S is a closed surface in electrostatic field $\boldsymbol{E}$, then the flux of $\boldsymbol{E}$ across S is the total charge enclosed by S (divided by an electric constant).
- As per the conclusion of the divergence theorem, a host of physical laws can be written in both a differential form (where one quantity is the divergence of another) and an integral form (where the flux of one quantity through a closed surface is equal to another quantity). Three examples are Gauss's law (in electrostatics), Gauss's law for magnetism, and Gauss's law for gravity.
- *Continuity equations* offer more examples of laws with both differential and integral forms, related to each other by the divergence theorem. In fluid dynamics, electromagnetism, quantum mechanics, relativity theory, and a number of other fields, there are continuity equations that describe the conservation of mass, momentum, energy, probability, or other quantities. Generically, these equations state that the divergence of the flow of the conserved quantity is equal to the distribution of sources or sinks of that quantity. The divergence theorem states that any such continuity equation can be written in a differential form (in terms of a divergence) and an integral form (in terms of a flux).
- Any *inverse-square law* can instead be written in a Gauss's law-type form (with a differential and integral form). Two examples are Gauss's law (in electrostatics), which follows from the inverse-square Coulomb's law, and Gauss's law for gravity, which follows from the inverse-square Newton's law of universal gravitation.
- The divergence theorem has many uses in physics and engineering; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Example 5.12: Evaluate $\iint_S \bar{a} \cdot \hat{n} dS$, where $\bar{a} = lx\bar{i} + my\bar{j} + nz\bar{k}$ and S is a closed surface whose volume is V.

Solution: $\iint_S \bar{a} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \bar{a}) dV$

$$= \iiint_V (l + m + n) dV = (l + m + n)V$$

Where V is the volume enclosed by S.

NOTES

Example 5.13: Verify Gauss divergence theorem for $\vec{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

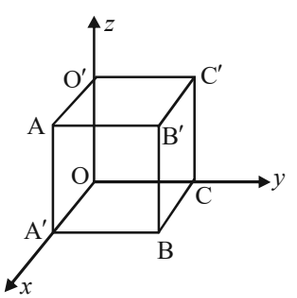
Solution: $\nabla \cdot \vec{F} = (4z - 2y + y) = (4z - y)$

$$\iiint_V (\nabla \cdot \vec{F})dV = \int_0^1\int_0^1\int_0^1 (4z - y)dx\,dy\,dz = \int_0^1\int_0^1 (4zx - xy)_0^1\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z - y)dy\,dz = \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right)dz = \left(2z^2 - \frac{z}{2}\right)_0^1 = \frac{3}{2}$$

Surface of the cube consists of six square faces. Unit normals and surface elements on these faces are tabulated below to facilitate evaluation of the surface integral $\int_S \vec{F} \cdot \hat{n}dS$



Surface	***Equation***	***$\hat{n}$ dS***
$OA'BC(S_1)$	$z = 0$	$-\bar{k}\ dxdy$
$O'AB'C'(S_2)$	$z = 1$	$\bar{k}\ dxdy$
$O'OA'A(S_3)$	$y = 0$	$-\bar{j}\ dxdz$
$BCC'B'(S_4)$	$y = 1$	$\bar{j}\ dxdz$
$OCC'O'(S_5)$	$x = 0$	$-\bar{i}\ dydz$
$AB'BA'(S_6)$	$x = 1$	$\bar{i}\ dydz$

$$\iint_{S_1} \vec{F} \cdot \hat{n}dS = \int_0^1\int_0^1 (-y^2\bar{j}) \cdot (-\bar{k})dx\,dy = 0$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}dS = \int_0^1\int_0^1 (4x\bar{i} - y^2\bar{j} + y\bar{k}) \cdot \bar{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = 1/2$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} dS = \int_0^1 \int_0^1 (4x\bar{i}) \cdot (-\bar{j})\, dx\, dz = 0$$

$$\iint_{S_4} \vec{F} \cdot \hat{n} dS = \int_0^1 \int_0^1 (4xz\bar{i} - \bar{j} + z\bar{k}) \cdot \bar{j}\, dx\, dz = \int_0^1 \int_0^1 -dx\, dz = -1$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} dS = \int_0^1 \int_0^1 (-y^2\bar{j} + yz\bar{k}) \cdot (\bar{i})\, dy\, dz = 0$$

$$\iint_{S_6} \vec{F}.\hat{n} dS = \int_0^1 \int_0^1 (4z\bar{i} - y^2\bar{j} + yz\bar{k}) \cdot (-\bar{i}) dy dz = \int_0^1 \int_0^1 4z\, dy\, dz = 2$$

Adding we get,

$$\iint_S \vec{F} \cdot \hat{n} dS = 3/2 = \iiint_V (\nabla . \bar{F}) dV$$

Hence, Gauss divergence theorem is verified.

Example 5.14: If $\vec{F} = \bar{g} \times \bar{a}$, where $\bar{a}$ is an arbitrary constant vector, prove that $\iint_S \hat{n} \times \bar{g} dS = \iiint_V (\nabla \times \bar{g}) dV$.

Solution: We know that,

$$\nabla \cdot (\bar{U} \times \bar{V}) = \bar{V} . \text{curl}\, \bar{U} - \bar{U} \cdot \text{curl}\, \bar{V}$$

$$\therefore \quad \nabla \cdot (\bar{g} \times \bar{a}) = \bar{a} \cdot (\nabla \times \bar{g}) - \bar{g} \cdot (\nabla \times \bar{a})$$

$$= \bar{a} \cdot (\nabla \times \bar{g}) \text{ (as } \bar{a} \text{ is a constant vector, } \nabla \times \bar{a} = 0)$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla . \bar{F} dV \text{ gives,}$$

$$\iint_S \hat{n} \cdot (\bar{g} \times \bar{a}) dS = \iiint_V \nabla . (\bar{g} \times \bar{a}) dV$$

$$\iint_S \bar{a} \cdot (\hat{n} \times \bar{g}) dS = \bar{a} \cdot \iiint_V (\nabla \times \bar{g}) dV$$

$$\text{or,} \quad \bar{a} \cdot \left[\iint_S (\hat{n} \times \bar{g}) dS - \iiint_V (\nabla \times \bar{g}) dV \right] = 0$$

As $\bar{a}$ is arbitrary vector $\bar{a} \neq \bar{0}$ we get, $\iint_S (\hat{n} \times \bar{g}) dS = \iiint_V (\nabla \times \bar{g}) dV$

NOTES

NOTES

5.5 LAPLACE EQUATIONS

In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as,

$$\nabla^2 \varphi = 0$$

Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables. Laplace equation and Poisson equation are examples of elliptic partial differential equations. The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great important in many fields of science.

Twice differentiable real-valued functions *f* of real variables *x*, *y* and *z* are found using the following notations.

In **Cartesian coordinates**:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

In **Cylindrical coordinates**:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

In **Spherical coordinates**:

$$\frac{1}{r^2}\frac{\partial}{\partial r} + \left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin \varphi}\frac{\partial}{\partial \varphi}\left(\sin \varphi \frac{\partial f}{\partial \varphi}\right) + \frac{1}{r^2 \sin 2\,\varphi}\frac{\partial^2 f}{\partial \theta^2} = 0$$

The Laplace equation $\nabla^2 \varphi = 0$ can also be written as $\nabla \cdot \nabla \varphi = 0$.

It is also sometimes written using the notation $\nabla \varphi = 0$, where Δ is also the Laplace operator.

Solutions of Laplace equation are harmonic functions. If the right-hand side is specified as a given function, *f*(*x*, *y*, *z*) then the whole equation can be written as,

$$\nabla \varphi = f$$

This is the Poisson equation. The Laplace equation is also considered as a special type of the Helmholtz equation.

5.5.1 Laplace Equation for Attraction and Potential

Let V be the potential of the system of attracting particles at a point P(x, y, z) (Refer Figure 5.7) not in contact with the particles so that

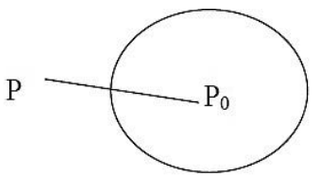


***Fig. 5.7** System Potential of Attracting Particles*

NOTES

$$V = \sum \frac{m}{r} \qquad ...(5.13)$$

Where m is the mass of particle at position of P_0 (a,b,c) r = distance of P from the $P_{0,}$

So that,

$$r^2 = (X - a)^2 + (y - b)^2 + (Z - c)^2 \qquad ...(5.14)$$

$$V = \sum \frac{m}{r} \qquad \Rightarrow \frac{\partial V}{\partial x} = -\sum \frac{m}{r^2}\frac{\partial r}{\partial x} = -\sum \frac{m}{r^2}\frac{(r-a)}{r^2}$$

$$\left[\Theta\ (2) \Rightarrow 2r\frac{\partial r}{\partial x} = 2(x-a) \Rightarrow \frac{\partial r}{\partial x} = \frac{x-a}{r}\right]$$

Therefore,

$$\frac{\partial V}{\partial y} = -\sum \frac{m(y-b)}{r^3}$$

$$\frac{\partial V}{\partial z} = -\sum \frac{m(z-c)}{r^3}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}[-\Sigma m\,(x-a)\,r^{-3}]$$

$$= -\Sigma m(x-a)\,(-3\,r^{-4})\,\frac{\partial r}{\partial x}$$

$$-\,\Sigma m\,r^{-3}\,(1)$$

$$= \Sigma m\,3\frac{(x-a)^2}{r^5} \;--\sum \frac{m}{r^3}$$

$$\frac{\partial^2 V}{\partial y^2} = \sum \frac{m(y-b)^2}{r^5} - \sum \frac{m}{r^3}$$

$$\frac{\partial^2 V}{\partial z^2} = 3\sum \frac{m(z-c)^2}{r^5} - \sum \frac{m}{r^3}\,ss$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

NOTES

Which is, Laplace equation.

$V \rightarrow$ potential

dv = small volume

$\therefore \quad dm = \rho\, dv$

Consequently,

$$V = \int \frac{\rho dv}{r}$$

$$\frac{\partial V}{\partial x} = \int \left(\frac{-1}{r^2}\right) \frac{\partial r}{\partial x}\ \rho\, dv$$

5.6 POISSON EQUATIONS

In mathematics, Poisson equation is a partial differential equation. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson. The Poisson equation is,

$$\Delta\varphi = f$$

Here Δ is the Laplace operator and *f* and φ are real or complex-valued functions on a manifold. If the manifold is Euclidean space, then the Laplace operator is denoted as Δ^2 and hence Poisson equation can be written as,

$$\nabla^2\varphi = f$$

In three dimensional Cartesian coordinates, it takes the form:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \varphi(x, y, z) = f(x, y, z).$$

For disappearing *f*, this equation becomes Laplace equation and is denoted as,

$$\Delta\varphi = 0.$$

The Poisson equation may be solved using a Green's function; a general exposition of the Green's function for the Poisson equation is given in the article on the screened Poisson equation. There are various methods for numerical solution. The relaxation method, an iterative algorithm, is one example.

A second order partial differential equation is of the form, $\nabla^2\Psi = -4\pi\rho$. If $\rho = 0$, then it reduces to Laplace equation. It can also be considered as Helmholtz differential equation of the form,

$$\nabla^2\Psi + k^2\Psi = 0$$

5.6.1 Poisson Equation for Attraction and Potential

Let the point P(x, y, z) be in contact (inside) the attracting matter (Refer Figure 5.8). We describe a sphere of small radius R and centre (a, b, c) contains the point P.

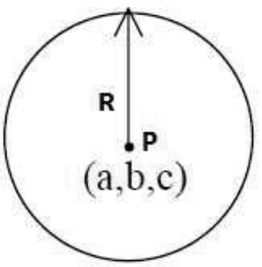


Fig 5.8 *Sphere with Small Radius*

ρ = density of material (sphere)

Since the sphere we describe is very small, therefore we consider the matter inside this sphere is of uniform density ρ.

So potential at P may be due to

(i) The matter inside the sphere

(ii) The matter outside the sphere.

V_1 = contribution towards potential at P by the matter outside the sphere

V_2 = contribution towards potential at P by the matter inside the sphere.

Since the point P is not in contact with the matter outside the sphere. Therefore by Laplace equation $\nabla^2 V_1 = 0$.

Here V_2 = potential at P(x, y, z) inside the sphere of radius R.

$$V_2 = \frac{2}{3}\pi\rho\,(3R^2 - r^2)$$

Whereas,

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

$$\frac{\partial V_2}{\partial x} = \frac{2}{3}\pi\rho\left(-2r\frac{\partial r}{\partial x}\right) = \frac{2}{3}\pi\rho(-2)\frac{r(x-a)}{r}$$

$$= \frac{-4}{3}\pi\rho(x-a)$$

$$\frac{\partial^2 V_2}{\partial x^2} = \frac{-4}{3}\pi\,\rho$$

Correspondingly,

$$\frac{\partial^2 V_2}{\partial y^2} = \frac{-4}{3}\pi\rho, \qquad \frac{\partial^2 V}{\partial z^2} = \frac{-4}{3}\pi\rho$$

NOTES

$$\therefore \quad \frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}+\frac{\partial^2 V_2}{\partial z^2}=-4\pi\rho$$

$$\Rightarrow \quad \nabla^2 V_2=-4\pi\rho$$

Since total potential is

$$V = V_1 + V_2$$

$$\therefore \quad \nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\boxed{\nabla^2 V=-4\pi\rho}$$

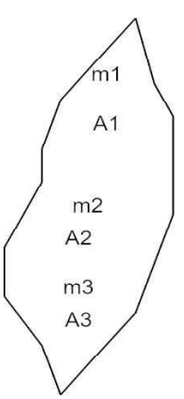
This equation is known as Poisson's equation.

CHECK YOUR PROGRESS

5. What is surface integral?
6. State the Gauss divergence theorem.
7. Give the uses of surface integral.
8. What do you mean by Laplace equation?
9. Define the Poisson equation.

5.7 WORK DONE BY SELF-ATTRACTING SYSTEMS

Calculating the work done by the mutual attractive forces of particles in a self-attracting system as they are transported from an unlimited distance to the places they occupy in the supplied system consists of particles of masses m_1, m_2 …. at A_1, A_2 …. etc., in the given system A.



NOTES

Start with from infinity to the position . Then the work done in this process is zero. Since there is no particle in the system to exact attraction on it next is brought from infinity to its position. Therefore, the work done on it by m_1 is = potential of m_2 at

$$A_2 \times m_2 = \frac{m_1}{r_{12}} m_2 = \frac{m_1 m_2}{r_{12}}$$

where = distances between m_1 and $m_2 \left(r_{12} = r_{21}\right)$

Then these two particles and attracts the third particle. Work done on by m_1 and m_2 is,

$$= \frac{m_1 m_3}{r_{13}} + \frac{m_1 m_2}{r_{12}}$$

when m_4 is brought from infinity to its position

$$A_4 = \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}$$

As a result, the total work process of collecting all the particles from infinity to their places in system A.

$$= \frac{m_1 m_2}{r_{12}} + \left(\frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \right) \cdots\cdots\cdots$$

$= \sum \frac{m_s m_t}{r_{st}}$, where summation extends to every pair of particles.

Let $V_1 - \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} + \cdots$

V_1 = Potential at A_1 of $m_2, m_3 \ldots s_1$

V_2 = Potential at A_2 of $m_1, m_3, m_4, \ldots.$

$$= \frac{m_1}{r_{21}} + \frac{m_3}{r_{23}} + \ldots.$$

$$V_3 = \frac{m_1}{r_{13}} + \frac{m_2}{r_{23}} + \frac{m_4}{r_{23}}$$

$$\Sigma = \frac{m_s m_1}{r_{st}} = \frac{1}{2}[V_1 m_1 + V_2 m_2 + \ldots + V_3 m_3]$$

Total work done = $\frac{1}{2} \sum mv$

NOTES

This depicts the work done by the system of particles' mutual attraction. If the system from a contable body, then work done will be

$$= \frac{1}{2}\int v dm$$

Conversely (if particle is scattered) the work done by the mutual attraction forces of the system as its particles are scattered at infinite distance from configuration A,

$$\text{then work done} = -\frac{1}{2}\sum mv = -\frac{1}{2}\int v dm$$

As the body shifts from one configuration A to another, we can see the work that has been done. the work done to change it from A to infinite state + work done to collect particles in infinity state to configuration B

$$= -\frac{1}{2}\int V dm + \frac{1}{2}\int_B V' dm'$$

$A \to \infty \to B$

Example 5.15: A self-attracting sphere of uniform density ρ and radius 'a' change to one of uniform density and radius 'b'. Demonstrate that the work carried out by its mutually attracting forces is provided by

$$\frac{3}{5}M^2\left(\frac{1}{b} - \frac{1}{a}\right)$$

where M is mass of sphere.

Solution: Here the work done by mutual attractive forces of the system. As the particle which constitute the sphere of radius 'a' are scattered to infinite distance

$$W_1 = -\frac{1}{2}\int v dm$$

We consider a point within the system at a distance x. The potential at this point within the spheres

$$V = \frac{2}{3}\pi\rho\left(3a^2 - x^2\right)$$

Let us now consider at this point, a spherical shell of radius x & thickness dx, then $dm = 4\pi x^2 dx\rho$

$$V dm = \frac{2}{3}\pi\rho\left(3a^2 - x^2\right)4\pi x^2 dx\rho$$

$$=\frac{8}{3}\pi^2\rho^2x^2\left(3a^2-x^2\right)\mathrm{dx}$$

$$\int \mathrm{Vdm}=\frac{8}{3}\pi^2\rho^2\int_0^a x^2\left(3a^2-x^2\right)\mathrm{dx}$$

$$=\frac{8}{3}\pi^2\rho^2\left[3a^2\frac{x^3}{3}-\frac{x^5}{5}\right]_0^a$$

$$=\frac{8}{3}\pi^2\rho^2\left[a^5-\frac{a^5}{5}\right]=\frac{8}{3}\pi^2\rho^2\frac{4a^5}{5}$$

$$=\frac{32}{15}\pi^2\rho^2a^5$$

M = Mass of sphere of radius a

$$=\frac{4}{3}\pi a^3\rho$$

$$\rho=\frac{3m}{4\pi a^3}$$

$$\int \mathrm{Vdm}=\frac{6}{5}\frac{m^2}{a}$$

$$\mathrm{W_1}=-\frac{1}{2}\int \mathrm{vdm}=-\frac{3}{5}\frac{m^2}{a}$$

Similarly, if is work done in bringing the particle ∞ to the second configuration (a sphere of radius b)

Then $\mathrm{W_2}=\frac{1}{2}\int \mathrm{v'dm'}=\frac{3}{5}\frac{m^2}{a}$

Total work done is given by

$$\mathrm{W=W_1+W_2}=\frac{3}{5}m^2\left(\frac{1}{b}-\frac{1}{a}\right)$$

NOTES

5.8 DISTRIBUTIONS FOR A GIVEN POTENTIAL

NOTES

We can find the corresponding distribution if the potential of a distribution is supplied over all space. If V be known, we can find $\nabla^2 V$ for every point of the space and when $\nabla^2 V = 0$ the corresponding density of the distribution is zero by Poisson's equation which in other words mean that there is no attracting mass at all such points. But when $\nabla^2 V$ is not equal to zero then we know by Gauss theorem that $\nabla^2 V = -4\pi\gamma$ density

So, the density is $-\frac{1}{4\pi\gamma}\nabla^2 V$

If the potential function inside any surface S differs from the potential function outside and if there be an abrupt charge in the value of $\frac{\partial V}{\partial n}$ as we pass across the surface, then the surface distribution σ on S is given by

$$\frac{\partial V_1}{\partial n_1}\delta s + \frac{\partial V_2}{\partial n_2}\delta s = -4\pi\gamma(\delta s.\sigma)$$

$$\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2} = -4\pi\gamma\sigma$$

Where the δn_1 and δn_2 are the small lengths of the normals drawn outward from each of the two faces δs.

Thus δn_2 is in the direction of the outward normal to the given surface S and δn_1 in the opposite direction.

Hence, we have

$$-\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2} = -4\pi\gamma\sigma$$

If δn be an element of outward normal at any point of the given surface S, then above equation becomes in limit

$$-\frac{\partial V_1}{\partial n} + \frac{\partial V_2}{\partial n} = -4\pi\gamma\sigma$$

$$\sigma = \frac{1}{4\pi\gamma}\left(\frac{\partial V_1}{\partial n} + \frac{\partial V_2}{\partial n}\right)$$

As a result, we get the formula for surface density of attracting matter.

5.9 EQUIPOTENTIAL SURFACES

NOTES

The potential V of a given attracting system is a function of coordinates x, y, z.

The equation $V(x, y, z) =$ constant

Represents a surface over which the potential is constant in nature. Such surfaces are known as **equipotential surfaces**. Condition that a family of given surfaces is a possible family of equipotential surfaces in a free space. To obtain the condition that the equation

$f(x, y, z) =$ constant

may represent the family of equipotential surface.

If the potential V is constant whenever $f(x, y, z)$ is constant, then there must be a functional relation between V and $f(x, y, z)$ say

$$V=\phi\{f(x,y,z)\}$$

$$V=\phi\{f\}$$

$$\frac{\partial V}{\partial x}=\phi'(f)\frac{\partial f}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2}=\phi''\left(\frac{\partial f}{\partial x}\right)^2+\phi'(f)\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial^2 V}{\partial y^2}=\phi''\left(\frac{\partial f}{\partial y}\right)^2+\phi'(f)\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 V}{\partial z^2}=\phi''\left(\frac{\partial f}{\partial z}\right)^2+\phi'(f)\frac{\partial^2 f}{\partial z^2}$$

On adding

$$\nabla^2 V=\phi'(f)\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)+\phi''(f)\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right\}$$

But in free space, $\nabla^2 V=0$

NOTES

$$\frac{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}}{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}=-\frac{\varphi'(f)}{\varphi'(f)}=\text{ a function of f}$$

$$=\psi(f) \text{ (say)} \tag{5.15}$$

This is the necessary condition and when it is satisfied, the potential V can be stated in terms of $f(x, y, z)$.

Then $V=\phi\{f\}$, where $\frac{\varphi'(f)}{\varphi'(f)}+\psi(f)=0$

Integrating, $\log\phi'(f)=\log A-\int\psi(f)\,df$

$$\log\left(\frac{\varphi'(f)}{A}\right)=-\int\psi(f)\,df$$

$$\phi'(f)=Ae^{-\int\psi(f)\,df}$$

Again integrating,

$$V=\phi(f)=Ae^{\int\psi(f)\,df}\,df+B \tag{5.16}$$

which is required expression in terms of $f(x, y, z)$ for V.

Example 5.16: Show that a family of right circular cones with a common axis & vertex is a possible family of equipotential surfaces & find the potential function.

Solution: Taking z axis as common axis. The equation of family of cones is

$$f(x, y, z)=\frac{x^2+y^2}{z^2}=\text{ constant} \tag{5.17}$$

$$\frac{\partial f}{\partial x}=\frac{2x}{z^2},\qquad \frac{\partial^2 f}{\partial x^2}=\frac{2}{z^2}$$

$$\frac{\partial f}{\partial y}=\frac{2y}{z^2},\qquad \frac{\partial^2 f}{\partial y^2}=\frac{2}{z^2}$$

$$\frac{\partial f}{\partial z}=\left(x^2+y^2\right)(-2)\left(z^{-3}\right)$$

$$\frac{\partial^2 f}{\partial z^2}=6\left(x^2+y^2\right)z^{-4}$$

Therefore Equation (1) becomes

$$-\frac{\varphi''(f)}{\varphi'(f)}=\frac{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}}{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}$$

$$=\frac{\frac{2}{z^2}+\frac{2}{z^2}+\frac{6\left(x^2+y^2\right)}{z^4}}{\frac{4x^2}{z^4}+\frac{4y^2}{z^4}\left(\frac{\partial f}{\partial x}\right)^2+\frac{4\left(x^2+y^2\right)^2}{z^6}}$$

$$=\frac{2z^2+2z^2+6\left(x^2+y^2\right)}{z^4}\frac{z^6}{4z^2\left(x^2+y^2\right)4\left(x^2+y^2\right)^2}$$

$$=\frac{4z^2+6\left(x^2+y^2\right)}{\left\{4z^2\left(x^2+y^2\right)+\left(x^2+y^2\right)^2\right\}}$$

$$=\frac{2z\left[2+\frac{3\left(x^2+y^2\right)}{z^2}\right]}{4z^2\left[\frac{\left(x^2+y^2\right)}{z^2}+\left(\frac{x^2+y^2}{z^2}\right)^2\right]}$$

$$\frac{2+3f}{2\left(f+f^2\right)}=\frac{2+3f}{2f\,(1+f)}=\text{ function of f}$$

$$-\frac{\varphi''(f)}{\varphi'(f)}=\frac{2+3f}{2f\,(1+f)}\ [\text{It}\, f\to 0 \text{ Equation (1) is satisfied}]$$

NOTES

$$\frac{\varphi''(f)}{\varphi'(f)}+\frac{2+3f}{2f(1+f)}=0$$

$$\frac{\varphi''(f)}{\varphi'(f)}+\frac{1}{f}+\frac{1}{2(1+f)}=0$$

Integrating,

$$\log\phi'(f)+\log f+\frac{1}{2}\log(1+f)=\log C$$

$$\log\phi'(f)=\log\frac{C}{f\sqrt{1+f}}$$

$$\phi'(f)=\frac{C}{f\sqrt{1+f}}$$

$$\frac{d\varphi}{df}=\frac{C}{f\sqrt{1+f}}$$

$$\int d\phi=\int\frac{C}{f\sqrt{1+f}}df+C'$$

Putting $f=\tan^2\theta$

$df=2\tan\theta\sec^2\theta d\theta$

$$\phi=C\int\frac{2\tan\theta\sec^2\theta d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}+C'$$

$$=C\int\frac{2}{\tan\theta}\frac{\sec^2\theta}{\sec\theta}d\theta+C'$$

$$=2C\int\frac{\sec\theta}{\tan\theta}d\theta+C'$$

$$=2C\int\cos ec\theta d\theta+C'$$

Therefore,

$$V=\phi(f)=2C\log(\cos ec\theta-\cot\theta)+C'$$

$V=\phi(f)=2C\log\left(\tan\frac{\theta}{2}\right)+C'$ is the required potential function. So, V is constant when θ is constant.

5.10 HARMONIC FUNCTIONS

NOTES

Any solution of Laplace equation $\nabla^2 V=0$ in x, y, z is called Harmonic function or spherical harmonic.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

If V is a Harmonic function of degree *n*, then

$\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}\frac{\partial^t}{\partial z^t}$ is a harmonic function of degree $n-p-q-t$.

Now $\nabla^2 V=0$ [Laplace equation]

p times w.r.t. x

q times w.r.t. y

t times w.r.t. z

So, $\nabla^2\left[\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial y^q}\frac{\partial^t}{\partial z^t}\right]=0$

5.10.1 Surface and Solid Harmonics

In spherical polar coordinates (r, θ, φ) Laplace equation is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0 \qquad (5.18)$$

Let $V=r^n S_n$ where S_n is independent of r or $S_n(\theta,\varphi)$.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(r^n S_n\right)\right]$$

$$=\frac{\partial}{\partial r}\left[r^2 S_n n r^{n-1}\right]$$

$$=\frac{\partial}{\partial r}\left[S_n n r^{n+1}\right]=nS_n\frac{\partial}{\partial r}\left(r^{n+1}\right)$$

$$=n(n+1)r^n S_n$$

$$n(n+1)r^n S_n+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta.r^n\frac{\partial S_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}r^n\frac{\partial^2 S_n}{\partial\theta^2}=0$$

NOTES

$$n(n+1)r^n S_n + \frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial S_n}{\partial\theta}\right)+\frac{r^n}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2}=0$$

$$n(n+1)S_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial S_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2}=0$$

$$n(n+1)S_n + \cot\theta\frac{\partial S_n}{\partial\theta}+\frac{\partial^2 S_n}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2}=0 \qquad (5.19)$$

If $\cos\theta = \mu$

$$N(n+1)S_n + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial S_n}{\partial\mu}\right]+\frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial\varphi^2}=0 \qquad (5.20)$$

A solution S_n of equation (5.19) is called a Laplace function or a **surface harmonic** of order n. Since n(n+1) remains unaffected when we write –(n +1) for n. So, there are two solutions of Equation (5.19) of which is a factor namely r^nS_n and $r^{-n-1}S_n$. These are known as solid Harmonic of degree n and –(n +1) respectively.

Remark 1: If U is a Harmonic function of degree n, then $\frac{U}{r^{2n+1}}$ is also Harmonic function. $U=r^n S_n$

So that $\frac{U}{r^{2n+1}}=\frac{r^n S_n}{r^{2n+1}}=\frac{S_n}{r^{n+1}}=S_n r^{-(n+1)}$

which is Harmonic.

Let $xyz \rightarrow$ 3rd degree is a solution of Laplace equation, then $\frac{xyz}{r^7}$ is also Harmonic.

2. If is a Harmonic function of degree –(n +1), then U_r^{2n+1} is also a Harmonic function, may write

$U = r^{-n-1} S_n$

so that $r^{2n+1}U=r^{2n+1}r^{-n-1}S_n = r^n S_n$ which is Harmonic.

5.10.2 Surface Density in Terms of Surface Harmonics

The potential at any point P due to several particles situated on the surface of sphere of radius 'a' can be ut in the form

$$V_1 = \sum_{n=0}^{\infty}\frac{r^n}{a^{n+1}}U_n, \text{ When } r < a \qquad (5.21)$$

$$V_1 = \sum \frac{r^n}{a^{n+1}} U_n \quad , \text{When } r > a \tag{5.22}$$

where U_n denotes the sum of several surface Harmonics (for each particle) and therefore itself a surface harmonic. We assume Equations (5.21) and (5.22) to represent potential of a certain distribution of mass and want to find it (density) on the surface.

NOTES

Here U_1 is Harmonic

$\Rightarrow \quad \nabla^2 V_1 = 0 , \quad \nabla^2 V_2 = 0$

Here on the surface of sphere, density is given by

$$-4\pi\sigma = \left[\frac{\partial V_2}{\partial r} - \frac{\partial V_1}{\partial r} \right]_{r=a}$$

$$\Rightarrow \qquad \sigma = \frac{1}{4\pi} \left[\frac{\partial V_1}{\partial r} - \frac{\partial V_2}{\partial r} \right]_{r=a}$$

$$= \frac{1}{4\pi} \left[\sum \frac{U_n n a^{n-1}}{a^{n+1}} + \sum \frac{U_n a^n (n+1)}{r^{n+2}} \right]_{r=a}$$

$$= \frac{1}{4\pi} \left[\sum \frac{U_n n a^{n-1}}{a^{n+1}} + \sum \frac{U_n a^n (n+1)}{a^{n+2}} \right]$$

$$= \frac{1}{4\pi} \left[\sum U_n \frac{n}{a^2} + \sum U_n \frac{(n+1)}{a^2} \right]$$

$$\sigma = \sum \frac{(2n+1) U_n}{4\pi a^2} \tag{5.21}$$

if potential is given by Equations (5.20) and (5.21), then surface density is given by Equation (5.22).

Check Your Progress

10. What happens when m_4 is brought from infinity to its position?
11. When potential function is inside any surface S.
12. Define the harmonic function.

NOTES

5.11 ANSWERS 'CHECK YOUR PROGRESS'

1. Gravity is also known as 'Gravitation', according to mechanics, the universal force of attraction acting between all matters. It is by far the weakest known force in nature and thus plays no role in determining the internal properties of everyday matter. Alternatively, through its long reach and universal action, it controls the trajectories of bodies in the solar system and to another place in the universe and the structures and evolution of stars, galaxies, and the whole cosmos.
2. According to Newton's law "any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them". We can represent by following formula,

$$F = G\frac{m_1 m_2}{r^2}$$

Here F is the force, m_1 and m_2 are the masses of the objects interacting, r is the distance between the centres of the masses and G is the gravitational constant.

3. Attraction is a force between two or more dissimilar or unlike charges. Two charges of dissimilar characteristics pull towards each other.
4. In classical mechanics, the gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location. It is analogous to the electric potential with mass playing the role of charge.
5. In mathematics, particularly multivariable calculus, a surface integral is a generalization of multiple integrals to integration over surfaces. It can be thought of as the double integral analogue of the line integral.
6. The surface integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the 'Volume Integral' of the divergence over the region inside the surface.
7. Surface Integrals are used to determine pressure and gravitational force. In Gauss' Law of Electro statistics, it is used to compute the electric field. It is also use the find mass of the shell. It analysed the how to calculate the moment of inertia and the centre of mass of the shell.
8. In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as,

$$\nabla^2\varphi = 0$$

Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables.

9. In mathematics, Poisson equation is a partial differential equation. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson. The Poisson equation is,

$$\Delta\varphi = f$$

Here Δ is the Laplace operator and f and φ are real or complex-valued functions on a manifold.

10. When m_4 is brought from infinity to its position

$$A_4 = \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}$$

As a result, the total work process of collecting all the particles from infinity to their places in system A.

11. If the potential function inside any surface S differs from the potential function outside and if there be an abrupt charge in the value of $\frac{\partial V}{\partial n}$ as we pass across the surface, then the surface distribution σ on S is given by

$$\frac{\partial V_1}{\partial n_1}\delta s + \frac{\partial V_2}{\partial n_2}\delta s = -4\pi\gamma\left(\delta s.\sigma\right)$$

$$\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2} = -4\pi\gamma\sigma$$

Where the δn_1 and δn_2 are the small lengths of the normals drawn outward from each of the two faces δs.

12. Any solution of Laplace equation $\nabla^2 V=0$ in x, y, z is called Harmonic function or spherical harmonic.

5.12 SUMMARY

- Gravity word is also came from the Latin *gravitas* which means that is 'Weight'. The gravitation, is a natural occurrence by which all things with mass or energy, including planets, stars, galaxies and even light, are attracted to (or gravitate toward) one another.
- In general relativity, the effects of gravitation are ascribed to space time curvature instead of a force. The starting point for general relativity is the equivalence principle, which equates free fall with inertial motion and describes free-falling inertial objects as being accelerated relative to non-inertial observers on the ground.

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- General relativity describes large-scale bulk properties whereas quantum mechanics is the framework to describe the smallest scale interactions of matter. One path is to describe gravity in the outline of quantum field theory, which has been successful to accurately describe the other fundamental interactions.
- Every planetary body (including the Earth) is surrounded by its own gravitational field, which can be hypothesised with Newtonian physics as exerting an attractive force on all objects. Assuming a spherically symmetrical planet, the strength of this field at any given point above the surface is proportional to the planetary body's mass and inversely proportional to the square of the distance from the centre of the body.
- Attraction is a force between two or more dissimilar or unlike charges. Two charges of dissimilar characteristics pull towards each other.
- On the other hand in classical mechanics, the gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location.
- In mathematics, the gravitational potential is also known as the Newtonian potential and is fundamental in the study of potential theory. It may also be used for solving the electrostatic and magneto static fields generated by uniformly charged or polarized ellipsoidal bodies.
- The scalar quantity characteristic of a point in a gravitational field whose gradient equals the intensity of the field and equal to the work required to move a body of unit mass from given point to a point infinitely remote.
- If the rod is infinitely long, the angle APB is two right angles and resultant attraction is $= \frac{2m}{p} \perp_r$ to the rod.
- A uniform solid sphere may be supposed to be made up of a number of thin uniform concentric spherical shells. The masses of spherical shells may be supposed to be concentric at centre O.
- In mathematics, particularly multivariable calculus, a surface integral is a generalization of multiple integrals to integration over surfaces. It can be thought of as the double integral analogue of the line integral. Given a surface, one may integrate a scalar field (that is, a function of position which returns a scalar as a value) over the surface, or a vector field (that is, a function which returns a vector as value).
- In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a theorem which relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.

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- The sum of all sources of the field in a region (with sinks regarded as negative sources) gives the net flux out of the region.
- The divergence theorem is an important result for the mathematics of physics and engineering, particularly in electrostatics and fluid dynamics. In these fields, it is usually applied in three dimensions. However, it generalises to any number of dimensions. In one dimension, it is equivalent to integration by parts. In two dimensions, it is equivalent to Green's theorem.
- Laplace equation and Poisson equation are examples of elliptic partial differential equations. The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great important in many fields of science.
- Conversely (if particle is scattered) the work done by the mutual attraction forces of the system as its particles are scattered at infinite distance from configuration A, then work done $= -\frac{1}{2}\sum \text{mv} = -\frac{1}{2}\int \text{vdm}$.
- Thus δn_2 is in the direction of the outward normal to the given surface S and δn_1 in the opposite direction.
- U_n denotes the sum of several surface Harmonics (for each particle) & therefore itself a surface harmonic.

5.13 KEY TERMS

- **Gravitation:** Gravity is also known as 'Gravitation', according to mechanics, the universal force of attraction acting between all matters.
- **Attraction:** Attraction is a force between two or more dissimilar or unlike charges. Two charges of dissimilar characteristics pull towards each other.
- **Gravitational potential:** In classical mechanics, the gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location. It is analogous to the electric potential with mass playing the role of charge.
- **Gauss divergence theorem:** The Gauss divergence theorem states that the surface integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the 'Volume Integral' of the divergence over the region inside the surface.
- **Poisson equation:** The Poisson equation may be solved using a Green's function; a general exposition of the Green's function for the Poisson equation.
- **Spherical harmonic:** Any solution of Laplace equation $\nabla^2 V = 0$ in x, y, z is called Harmonic function or spherical harmonic.

5.14 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Define gravitation.
2. What do you understand by attraction and potential?
3. State the Gauss divergence theorem.
4. What is a Laplace equation?
5. Give the Poisson equation.
6. What is distributions potential?
7. Give a short note on equipotential surfaces.
8. Define harmonic function.

Long-Answer Questions

1. Elaborate on the gravitation with relevent examples.
2. Calculate the attraction and potential of rod, disc, spherical shell and sphere with examples.
3. Discuss briefly about the Gauss divergence theorem with the help of examples.
4. Explain briefly about the Laplace equations for attraction and potential with the help of examples.
5. Briefly explain the Poisson equation for attraction and potential. Give appropriate examples.
6. Analyse the work done by self-attracting system with the help of examples.
7. What do you understand by the distribution potential for a given potential? Give appropriate examples.
8. Describe the equipotential surfaces with the help of examples.
9. Explain the surface and solid harmonic surface density in terms of surface harmonic with relevent examples.

5.15 FURTHER READING

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