M.Sc. First Year

Zoology, Paper - I

# BIOSYSTEMATICS, TAXONOMY AND QUANTITATIVE BIOLOGY

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

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# SYLLABI-BOOK MAPPING TABLE

## Biosystematics, Taxonomy and Quantitative Biology

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| Uint-II | **Biosystematics and Taxonomy-II**<br>1. Importance and Applications of Biosystematics in Biology.<br>2. Trends in Biosystematics:<br>A. Chemotaxonomy B. Cytotaxonomy C. Molecular Taxonomy.<br>3. Molecular Perspective on the Conservation of Diversity: Diversity and eco-system Process, Theory, Achievements and Future Directions.<br>4. Wild Life and Its Conservation.<br>5. National Parks and Sanctuaries of India.<br>6. Geological and Zoogeographical Distribution of Animals.<br>7. Fossils and Paleozoology. | **Unit-2:** Biosystematics and Taxonomy - II<br>**(Pages 31-107)** |
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# INTRODUCTION

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Biosystematics is the science through which life forms are discovered, identified, described, named, classified and catalogued, with their diversity, life histories, living habits, roles in an ecosystem, and spatial and geographical distributions recorded. Biosystematics is the science that provides indispensable information to support many fields of research and beneficial applied programs. In biosystematics, systematics and taxonomy are two different concepts related to the study of diverse life forms and the relationships between them through a period of time. While taxonomy is involved in identification, nomenclature and classification, Systematics aims to determine the evolutionary relationship of organisms.

Taxonomic knowledge is critical as it underpins the understanding of complex biological interactions, for instance where different species fit within ecosystems, what they feed on, what feeds on them, what is their impact on others, and what might happen should they disappear. The branch of biology dealing with biodiversity in life, which includes microorganisms, plants and animals, is called classification/ taxonomy/systematic. Absence of naming and classifying living organisms leads to many problems and worst confusion is being confounded, because of a single animal will be called in different names in different countries.

Quantitative biology is an umbrella term encompassing the use of mathematical, statistical or computational techniques to study life and living organisms. It's helpful in quantitative modeling of biological processes, presenting modeling approaches, methodology, practical algorithms, software tools, and examples of current research. The quantitative modeling of biological processes promises to expand biological research from a science of observation and discovery to one of rigorous prediction and quantitative analysis. Quantitative Biology focuses on quantitative approaches and technologies to analyze and integrate biological systems, construct and model engineered life systems, and gain a deeper understanding of the life sciences.

Biostatistics comprises ideas and methods for quantifying the evidence in data to distinguish among competing hypotheses, for estimating unknown characteristics of populations, and for quantifying the uncertainty in those estimates. Biostatistics (or biometry) deals with the statistical processes and methods applied to the analysis of biological phenomena. The science of biostatistics incorporates the design of biological experiments and interpreting the collection, summarization, and analysis of data from those experiments.

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. A model may help to explain a system and to study the effects of different components, and to make predictions about behavior.

This book is divided into four units that provide the students an in-depth knowledge of basic concepts biosystematics and taxonomy, trends in biosystematics,

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biostatistics, basic mathematics and mathematical modelling. The book follows the Self-Instructional Mode or SIM format wherein each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The detailed content is then presented in a simple and structured manner interspersed with Answers to 'Check Your Progress' questions. A list of 'Key Terms', a 'Summary' and a set of 'Self-Assessment Questions and Exercises' is also provided at the end of each unit for effective recapitulation.

# UNIT 1 BIOSYSTEMATICS AND TAXONOMY - I

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**Structure**



## 1.0 INTRODUCTION

Biosystematics is the study of the diversification of living forms, both past and present, and the relationships among living things through time. The word systematics is derived from Latin word '*Systema',* which means systematic arrangement of organisms. Systematic biology is the field that provides scientific names for organisms, describes them, preserves collections of them, provides classifications for the organisms, keys for their identification, and data on their distributions, investigates their evolutionary histories, and considers their environmental adaptations. Systematics, in other words, is used to understand the evolutionary history of life on Earth.

Systematics uses taxonomy as a primary tool in understanding organism's relationships with other living things described in sufficient detail to identify and classify it correctly. Scientific classifications are aids in recording and reporting information to other scientists and to laymen. The Systematist, a scientist who specializes in systematics, must, therefore, be able to use existing classification systems, or at least know them well enough to skilfully justify not using them.

Taxonomy in biology encompasses the description, identification, nomenclature, and classification of organisms. In addition taxonomy is the practice and science of categorization or classification. The word finds its roots in the Greek language *'Taxis' (*meaning 'Order', 'Arrangement') and *'Nomos* ' ('Law' or 'science').

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A taxonomy (or taxonomical classification) is a scheme of classification, especially a hierarchical classification, in which things are organized into groups or types. Among other things, a taxonomy can be used to organize and index knowledge

Originally, taxonomy referred only to the categorisation of organisms or a particular categorisation of organisms. In a wider, more general sense, it may refer to a categorisation of things or concepts, as well as to the principles underlying such a categorisation. Taxonomy organizes taxonomic units known as "taxa" (singular 'Taxon').

In this unit, you will study about the history and classification of biosystematics, mechanism of Speciation, various taxonomic procedures, International Code of Zoological Nomenclature (ICZN), taxonomic categories and biodiversity indices.

## 1.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the history and classification of biosystematics
- Explain the mechanism of speciation
- Comprehend the various taxonomic procedures
- Interpret the 'International Code of Zoological Nomenclature' ( ICZN)
- Understand the various taxonomic categories
- Analyse the evolution of biodiversity indices

## 1.2 HISTORY AND CLASSIFICATION OF BIOSYSTEMATICS

Principally, biosystematics is the study of the variation and evolution of a population of organisms in relation to their taxonomic classification, as the term systematics refers to the science of systematic classification. Systematics is a branch of Biology that deals with categorisation of plants, animals and other organisms into categories that can be named, remembered, compared and studied. Study of only one organism of a group provides sufficient information about the remaining members of that group.

You must be aware that the Earth is currently experiencing rapid extinction of many species caused either directly or indirectly by human activities. Millions of species are still unknown to science. Systematics is the science dedicated to studying biodiversity, the important components of systematic biology and their relevance to the biodiversity crisis.

In biology, the term 'Biological Systematics' or 'Biosystematics' refers to the study of the diversification of living forms, both past and present, and the relationships among living things through time. Relationships are visualized as evolutionary trees, the synonyms are cladograms, phylogenetic trees, phylogenies.

Phylogenies include two components, namely the branching order (showing group relationships) and the branch length (showing amount of evolution). Phylogenetic trees of species and higher taxa are used to study the evolution of traits, such as the anatomical or molecular characteristics, and the distribution of organisms (biogeography). Systematics, in other words, is used to understand the evolutionary history of life on Earth.

The terms systematics, taxonomy and classification are often held as synonyms but theoretically they carry different meanings. Simpson, (1961) has defined systematics as the branch of biology that deals with the diversity of organism at every level of classification.

Taxonomy, systematics or classification of organisms is based on the study of their comparative morphology (form, external and internal structure), cytology, embryology, fossil relatives, biochemical analysis and ecological relationships.

**History of Systematics**

Initial classifications system were concerned exclusively for the identification of useful and harmful plants and animals. Hippocrates (460-377 BC, Father of Medicine) and Aristotle (384-322 BC, Father of Zoology), on the basis of their habitat organised the animals into aquatic, terrestrial and aerial animals.

Further the Greek scholars divided animals on the basis of single character into four major groups— insects, birds, fishes and whales. Theophrastus (Father of Botany, y10-285 BC) divided the plants on the basis of form, texture and habit into four groups — trees, shrubs, under-shrubs and herbs. He described 480 plants in his book '***Historia Plantarum***'.

Pliny the Elder (28-79 A.D.) was the first to introduce the first system of artificial classification. His book, '***Historia Naturalis***' (c75 AD), references over 1,000 economic plants. John Ray (1627-1705), the English Naturalist, described about 18600 plants in his three volumes of '***Historia Generalis Plantarum***' between the years 1686-1704. The term '**Species**' was introduced first time by the naturalist which is used for classification system till date. John Ray defined the term species as the grouping of individuals having analogous or similar parentage or paternity/maternity characteristics with the ability to pass these parental traits to the offspring's.

The Swedish naturalist Carolus Linnaeus developed the scientific system of naming the species, also known as the binomial system of nomenclature. Linnaeus described about 5900 species of plants in his book '***Species Plantarum***' (1753) and about 4326 species of animals in his book '***Systema Naturae***' (1758). The word '***Systematics***' is derived from Latin word '***Systema***' which means '*Systematic Arrangement of Organisms*'. Linnaeus used the term '***Syszterma Nature***' as the title of his book.

Beginning from Aristotle to Linnaeus, each and every naturalist and systematic used limited number of traits for classification of organisms. Consequently, the classification systems recommended by them is termed as artificial system of classification. Later with improved and thorough study of various biological domains, many more characters were taken into consideration by taxonomists for the classification of species. It brought out the natural affinities amongst the organisms. This characterised the specific phase/period of classical taxonomy which created or generated the today's natural systems of classification of species. Simultaneously the biologists and taxonomists started discovering the evolutionary and genetic relationships. This resulted in development of phylogenetic classification or cladistics, in Greek the term '***Klaclos***' means '***Branch***' while in Latin the term '***Dados***' means '***Branch***'. In cladistics, the organisms are arranged or organised in the specific historical order in which they evolved as branches of the parent tree. This particular phase or period is termed as the '**New Systematics**' or '**Biosystematics**'. Father of New Systematics or Biosystematics is Sir Julian Huxley (1940).

### Definitions of Biosystematics

1. The term 'Biosystematics' is the statistical analysis of data obtained from genetic, biochemical, and other studies to assess the taxonomic relationships of organisms or populations, especially within an evolutionary framework.
2. Principally, the term 'Biosystematics' refers to the study of the variation and evolution of a population of organisms in relation to their taxonomic classification, as the term systematics refers to the science of systematic classification.
3. The 'Biosystematics' is the study of morphological and other characteristics of taxonomic systems.
4. Systematics is the study of diversification and relationships of life forms of extinct extant.
5. According to Blackwelder and Boyden (1952), "Systematic is the entire field dealing with the kind of animals, their distinctions, classification, and evolution".
6. Simpson (1961) defines systematic as, "The scientific study which deals with kinds and diversity of organisms and any or all relationships among them".
7. According to Blackwelder (1967), "Systematics is that science which includes both taxonomy and classification, and all other aspects of dealing with kinds of organisms and the data accumulated about them".
8. Christofferson (1995) defined systematic as, "The theory, principles and practice of identifying (discovering) systems, i.e., of ordering the diversity of organism (parts) into more general systems of taxa (wholes) according to the most general causal processes".

9. According to Padian (1999), "Systematic can be seen as the philosophy of organization nature, taxonomy as the use of sets of organic data guided by systematic principles, and classification as the tabular or hierarchical end result of this activity".

**Basics of Systematic Study**

The systematic study includes the following features:

1. **Characterization:** The organism under the study is defined for all its morphological and other significant characteristics.
2. **Identification:** Based on the already studied characteristics, the identification of the organism is done to distinguish whether it is similar to any of the known group or taxa.
3. **Classification:** The organism is then classified on the basis of its resemblance to different taxa. It is also possible that the organism may not resemble any of the previous known taxa or groups. In this condition, the new group or taxon is added to accommodate it under the classification tree.
4. **Nomenclature:** When the organisms are placed and organized in various taxa, the accurate name is determined. If the organism is new to systematics, it is given a new name based on rules and conventions of nomenclature.

## 1.2.1 Taxonomy

Variation is the law of nature. The most remarkable feature of the world of life is its diversity and the uniqueness of its components. Neither any two sexually reproducing populations are identical or similar, nor any two populations, species or higher taxa. Approximately more than one million species of animals and half a million species of plants have been already defined. It is estimated that the number of undescribed living species ranges from about three to ten millions. Furthermore each species may exist in numerous different forms, such as sexes, age, seasonal forms, morphs, etc. Basically, it is not possible to deal with this enormous diversity if there is not an ordered and classified system. The term 'Systematic Zoology' attempts to systematize and order this diversity of the animal world and also to develop methods and principles to systematically classify the living animal species.

Taxonomy refers to the theory and practice of identifying plants and animals. In fact, taxonomy deals with the principals involved in the study of classification of organisms. It is the functional science which deals with identification, nomenclature and classification of different kinds of organisms all over the world. The word '**Taxonomy**' is derived from the Greek words '***Taxis***' meaning '***Arrangement***' and '***Nomos***' meaning '***Law***'. The term 'Taxonomy' was coined by A.P. de Candolle in 1813.

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## Definitions of Taxonomy

Different scientists have defined taxonomy differently from their view point. Following are some of the accepted definitions of taxonomy:

1. According to Mason (1950), taxonomy is the synthesis of all the facts about the organisms into a concept and expression of the interrelationship of organisms.
2. Harrison (1959) defined taxonomy as the study of principles and practices of classification, specifically the methods, the principles and the biological classification.
3. Simpson (1961) defines taxonomy as the theoretical study of classification, including its bases, principles, procedures, and rules.
4. Heywoods (1967) defined taxonomy as the way of arranging and interpreting information.
5. Blackwelder (1967) explains it as the day to day practice of handling different kinds of organisms. It includes collection and identification of specimens, the publication of data, the study of literature and the analysis of variations shown by the specimens.
6. According to Johnson (1979), taxonomy is the science of placing biological form in order.
7. Christoffersen (1995) defines taxonomy as the practice of recognizing, naming and ordering taxa into a system of words consistent with any kind of relationships among taxa that the investigator has discovered in nature.

## Classical Taxonomy

The classical taxonomy is based on the observable morphological characters with reference to the normal individuals and are considered that the expression will be analogous or similar while their variations can be imperfect expressions. Classical taxonomy originated with Plato followed by Aristotle (Father of Zoology), Theophrastus (Father of Botany) up to Linnaeus (Father of Taxonomy) and his contemporaries.

Typically,

1. Species are delimited on the basis of morphological characters.
2. Only some characters are used for classification.
3. Some individuals or their preserved specimens are used for study. It is termed as the typological concept.
4. Species are assumed to be either static or immutable.
5. Species is centre phase of the study. Its subunits are not significant.

## Modern Taxonomy (New Systematics)

The term new systematic was coined by Julian Huxley (1940). New systematic is the systematic study which includes all types of characters. In addition to classical

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morphology, it includes anatomy, cytology, physiology, biochemistry, ecology, genetics, embryology, behaviour, etc., of the whole population instead of a few typological specimens. In contrast, the classical systematics is mainly based on the study of morphological traits of one or a few specimens with supporting evidences from other fields. New systematics is also termed as population systematics and biosystematics. It tries to highlight the evolutionary relationship amongst organisms.

1. New systematic is based on the all types of variation in the species.
2. Along with morphological variations, other investigations are also carried out to know the variety of traits.
3. Delimitation of species is carried out on the basis of all types of biological traits. It is also called biological delimitation.
4. Traits indicating primitiveness and advancement are established.
5. Inter-relationships are defined.
6. Species are considered as dynamic unit.

### 1.2.2 Relationship between Taxonomy and the Systematics

The term 'Taxonomy' defines about the classification and nomenclature while the term 'Systematics' defines about both the taxonomy and the evolution. In simple terms, essentially there are two parts of systematic. The first part, taxonomy, is concerned with the naming of the different kinds of organisms, whether existing or extinct. The second part of systematics, evolution, states that how all these different animals are classified on the basis of characteristic features. Systematics uses taxonomy as a means to recognise organisms. Systematics explain the new methods and theories that can be used to classify species based on similarities of traits and possible mechanisms of evolution, and a change in gene pool of a population over time.

Following are the key differences between taxonomy and systematics:

1. Taxonomy is the most significant branch of systematics and thus systematics is a broader area than taxonomy.
2. Taxonomy is concerned with nomenclature, description, classification and identification of a species, but systematics is important to provide layout for all those taxonomic functions.
3. Evolutionary history of a species is studied under systematics but not in taxonomy.
4. The environmental factors are directly related with systematics but in taxonomy it is indirectly related.
5. Taxonomy is subjected to change in course of time, but systematics is not changed if it was properly done.

**Stages in Taxonomy**

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Following are the various stages of taxonomy:

- **Alpha Taxonomy:** In this stage species are identified and characterized on the basis of gross morphological features.
- **Beta Taxonomy:** In this stage species are arranged from lower to higher categories, i.e., hierarchic system of classification.
- **Gamma Taxonomy:** In this stage intraspecific differences and evolutionary history are studied.

## 1.3 MECHANISM OF DIFFERENT TYPES OF SPECIATION

**Speciation** is the evolutionary process by which populations evolve to become distinct species. Speciation is the splitting of one species into two different species. Rapid speciation of a repeated group of taxa usually occurs in response to niche differentiation or evolution of a novel feature. This is called adaptive radiation .

The different types of speciation are as follows:

- **Allopatric Speciation:** The formation of two or more species from a single species in geographically different locations is called allopatric speciation.

  During allopatric speciation, a population splits into two geographically isolated populations. The isolated populations then undergo genotypic or phenotypic divergence as:

  (a) They become subjected to dissimilar selective pressures;

  (b) They independently undergo genetic drift;

  (c) Different mutations arise in the two populations.

  When the populations come back into contact, they have evolved such that they are reproductively isolated and are no longer capable of exchanging genes.

- **Parapatric Speciation**: In parapatric speciation new species are formed in the absence of any specific extrinsic barrier to speciation.

  In parapatric speciation, there is only partial separation of the zones of two diverging populations afforded by geography; individuals of each species may come in contact or cross habitats from time to time, but reduced fitness of the heterozygote leads to selection for behaviours or mechanisms that prevent their interbreeding. Parapatric speciation is modelled on continuous variation within a 'Single', connected habitat acting as a source of natural selection rather than the effects of isolation of habitats produced in peripatric and allopatric speciation. Parapatric speciation may be associated with differential landscape-dependent selection.

- **Peripatric Speciation:** In peripatric type of speciation, two or more species are formed in peripheral, isolated, small populations. Peripatric speciation is thus a special type of allopatric speciation mode.

  In peripatric speciation new species are formed in isolated, smaller peripheral populations that are prevented from exchanging genes with the main population. It is related to the concept of a founder effect, since small populations often undergo bottlenecks. Genetic drift is often proposed to play a significant role in peripatric speciation.

- **Sympatric speciation:** It is the formation of two or more descendant species from a single ancestral species, all in the same geographic location. Sympatric speciation is distinctive as it occurs when two populations of the same species are found in the same territory but are able to split into two different groups and genetically develop so differently that they can no more interbreed and become different species. Sympatric speciation events are quite common in plants, which are prone to acquiring multiple homologous sets of chromosomes.

Figure 1.1 illustrate the different types of specication and their meachenism

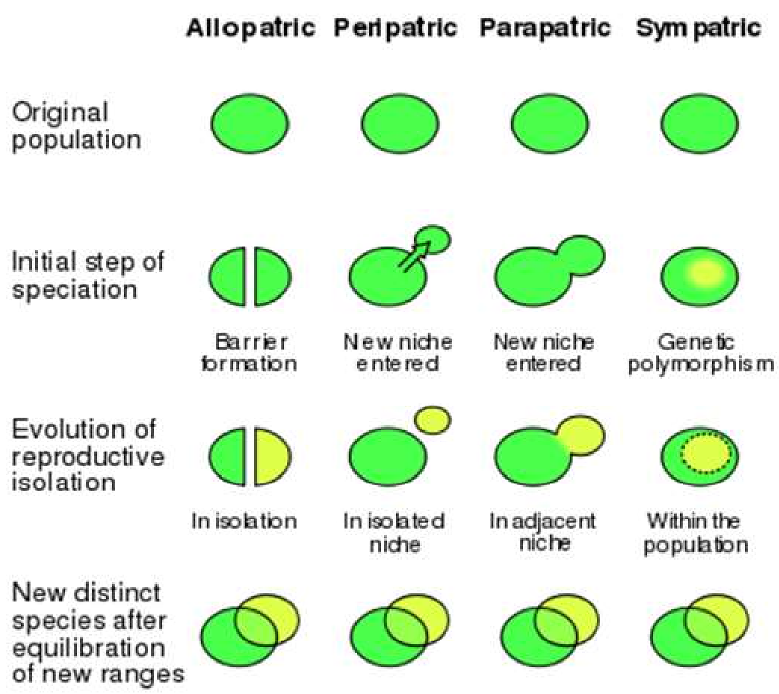


***Fig1.1*** *Comparison of Allopatric, Peripatric, Parapatric and Sympatric Speciation*

**Check Your Progress**

1. What is biosystematics?
2. Who gave the theory of binomial system of nomenclature?
3. What is taxonomy?
4. What is the difference between new systemic (morden) taxonomy and classical taxonomy?

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## 1.4 TAXONOMIC PROCEDURE

**Preservation of Speciman**

The biological collections are typically preserved animal specimens which are given specimen documentation, such as labels and notations. The biological collections are of two types, either dry collections or wet collections. The collections can also be preserved at low temperatures or microscopy collections. The term 'Dry Collections' refers to those specimens that are preserved in a dry state. Some biological specimens can be preserved naturally, such as the Starfish. Drying method is the best available option for preserving the natural colours of the biological specimens, for example the natural colours of Butterflies or distinguishing features, such as skeletal parts. Such biological specimens in a dry state may have great potential for further analysis and research.

The term 'Wet Collections' are used for the specimens that are kept or preserved in a liquid preservative to prevent their deterioration. The wet collection method is used when colour preservation is not essential.

The biological low-temperature collections includes the biological specimens that are preserved and maintained at low temperatures. The specimens that can be preserved at low temperatures include some Algae, Protozoa (especially parasitic strains), Viruses, Cloned Viral Genomes, Bacteria, Bacteriophages, Plasmids. Animal Tissues (such as, the dissected Organs, Muscles),Cell Lines - Blood and Blood Components (Whole Blood, Serum, Plasma, Antisera) - Semen, Venom, etc.

Biological microscopy collections includes the certain preserved specimens as microscope preparations to preserve whole or partial organisms to analysis and for different types of microscopic examinations, for example biochemical analyses, including extraction of DNA.

Most of the biological collections are extremely valuable due to the following reasons:

- Museums are only place where extinct species are preserved.
- The specimens has exceptional historical value as the specimens are rarely found in any collections.
- Many areas in world are geographically inaccessible hence specimen collected from such area are important and are preserved with extra care.
- A specimen is of unique value as it forms the basis of published research.
- The details of specimen can be used for the verification of original data or for advanced study using new techniques.

**Collection of Specimen**

1. Attracting the nocturnal insects using UV light. Many insects can perceive and attracted towards the ultraviolet light, which has shorter wavelengths as

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compared to the light visible to the human eye. Thus, a black light can attract different insects than a regular incandescent light. The black light is focused in front of a white sheet, so that the flying insects get a surface to land. This helps in collecting the insects as they are clearly seen on the white sheet.

2. Malaise traps is a huge tent-like structure used for trapping flies and wasps. Insects fly into the tent wall and are funnelled into a collecting vessel attached to it.
3. The specimens of organisms living in water can be collected by taking the water specimen form the identified area.
4. The Plankton net can be used for collecting the aquatic insects and other arthropods.

### 1.4.1 Curetting Process of Identification

**Identification**: Identification in biology is the process of assigning a pre-existing taxon name to an individual organism. Identification of organisms to individual scientific names (or codes) may be based on individualistic natural body features, experimentally created individual markers (e.g., color dot patterns), or natural individualistic molecular markers (similar to those used in maternity or paternity identification tests).

Identification of collections may be carried out based on published references and museum specimens.

Curation is the process of gathering information relevant to a particular topic or area of interest, Content curation is the process of gathering information relevant to a particular topic or area of interest, usually with the intention of adding value through the process of selecting, organizing, and looking after the items in a collection or exhibition. Services or people that implement content curation are called curators.

**Curating of Collections** Every taxonomist has to take the responsibility of curating collections. This requires a great deal of expertise, knowledge and clear understanding of the function of different collections. Following are some process of curating.

#### Preparation of Material

There are certain materials which are ready for study as soon as collected from the field e.g., bird and mammal skins. There are certain insects which should never be placed in alcohol or any other liquid preservative, whereas others are useless when dried. Most insects are pinned, and the wings are spread if they are taxonomically important as in butterflies, moths and some grasshoppers. Certain invertebrates are to be preserved in alcohol or formalin before their study.

**Housing:** Research collections are housed in fireproof and dustproof buildings. Most museums keep their collections in air conditioned buildings as rapid changes in temperature and humidity are harmful to museum cases and specimens. Storage cases built are insect proof. Photographs and films are stored in air-conditioned rooms. Improperly preserved or inadequately labelled specimens are eliminated by the curator. The most efficient method for the elimination of useless material is to ask specialists to pull out such specimens while scrutinizing the material during a revision.

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### Cataloguing

The method of cataloguing depends on the group of animals. All the specimens including vertebrates collected at a given locality or district or by one expedition are entered in the catalogue together. This greatly facilitates in knowing the distributional data and the preparation of faunistic analyses. Cataloguing is usually done after the specimens have been identified, at least up to the genus level. In groups where the collections consist of large numbers of specimens, it is customary to catalogue the specimens by lots. Each lot consists of a set of specimens from a given locality or region. It is also important to note whether a lot was received as a gift or by purchase or exchange. The names of the collector and donor are always given. When museums and their collections were small, curators had maintained card-files which provided all sorts of information such as collecting station, name of the collector etc.

### Arrangement of the Collection

The collection should be arranged in the same sequence as some generally adopted classification. The sequence of orders and families is usually standardized in many classes of animals. The contents of trays and cases should be clearly indicated on the outside which could serve as a check list. Where specimens are of large and unequal size, they have to be stored separately.

### Curating of Types

The names of species are based on type specimens. Many descriptions of classical authors are equally applicable to several related species. Types are usually deposited in large collections in public or private institutions which have come to be recognized as standard repositories of types. While conducting an authoritative revision of a given genus, a specialist should be able to see all the existing types. If many of them are in a single institution, the specialist should travel there and obtain scattered types. Modern curators are quite liberal in lending type specimens to qualified specialists. It is recommended that the type collections should be arranged alphabetically according to the given specific name. A type collection is a reference collection rather than a classification. Type specimens assume such an important role in the taxonomy of lesser-known groups that many workers believe that no individual should retain a type in his private collection after the study has been completed.

**Exchange of Material** The selecting of material for exchanges and keeping its record is time consuming, so the exchanges are not as popular as they used to be. Among private collectors this practice is common. A specialist doing a monograph on a certain genus or family can always borrow material from other institutions and return it after completing his work. Exchanges are not desirable in groups where series of unlimited size can be obtained and where the concerned areas are not easily accessible. Exchanges are sometimes necessary to build up complete

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identification collections. Many specialists give away excess specimens as open exchanges not expecting any return. Improperly preserved or inadequately labelled specimens should be eliminated by the curator. The most efficient method for the elimination of useless material is to ask specialists to pull out such specimens while scrutinizing the material during a revision.

**Need of Curating**

If new data sets are not curated into databases for long-term sustainability and integrated with pre-existing data, they may lose their accessibility and utility over time. If new, important data sets are not used, knowledge production and discovery rates will lag behind data production rates. In other words, data must be captured, standardized, organized, and made accessible to the scientific community if it is going to have a significant and lasting impact. In addition, a database is only as good as its data. If members of the scientific community do not find the data in their popular databases up-to-date, accurate, or transferable, then the database is of little use and will be obsolete soon. Likewise, if an online database's interface is not intuitive, few researchers will utilize the database. The role of a biocurator is therefore to provide up-to-date, accurate, and accessible information, and, through this critical activity, facilitate scientific discovery.

## 1.5 INTERNATIONAL CODE OF ZOOLOGICAL NOMENCLATURE

The need for a code to give a scientific name to every species was first realised by British Association for the Advancement of Science in 1842, when a set of rules were framed by it. This was also felt by American Association for the Advancement of Science in 1877. Then similar learned bodies in different countries like France, Germany and Soviet Union developed codes for their respective countries.

The object of the code is to promote stability and universality in the scientific name of animals, and to ensure that each name is unique and distinct.

The International Code of Zoological Nomenclature (ICZN) is a widely accepted convention in zoology that rules the formal scientific naming of organisms treated as animals. It is also informally known as the ICZN Code, for its publisher, the International Commission on Zoological Nomenclature (which shares the acronym 'ICZN'). The rules of the International Commission on Zoological Nomenclature principally regulate the following,

1. How names are correctly established in the frame of binominal nomenclature?
2. Which name must be used in case of confliction?
3. How scientific literature must cite names?

Zoological nomenclature is independent of any other systems of nomenclature, for example botanical nomenclature. The animals are given the specific genric names for identification.

The rules in the code determine what names are valid for any taxon in the family group, genus group, and species group. It has additional (but more limited) provisions on names in higher ranks. The code recognizes no case low. Any dispute is decided first by applying the code directly, and not by reference to precedent.

**NOTES**

**International Code of Zoological Nomenclature (1964)** is the system of rules and recommendations authorized by the International Congress of Zoology. The object of the code is to promote stability and universality in the scientific names of animals and to ensure that each name is unique and distinct. Code does not restrict the freedom of taxonomic thought and action.

At present the naming of the animal is governed by the International Code of Zoological Nomenclature. There are many rules (Articles) concerning the Zoological Nomenclature.

Of these rules, some important rules are given below:

1. Zoological nomenclature is independent of other system of nomenclature.
2. The scientific name of a species is to be binomial (Art. 5.1) and a subspecies to be trinomial (Art. 5.2).
3. The first part of a scientific name is generic (L. Genus = Race) and is a single word and the first alphabet or letter must be written in Capital letter. The genus must be a noun in the nominative singular.
4. The second part of a name is species (L. species = Particular kind) name and may be a single word or a group of words. The first alphabet or letter of the species name must be written in small letter.
5. If the species names are framed after any person's name, the endings of the species are *i, ii* and *ae,* or if the species name are framed after geographical place, the endings of the species are '*ensis', 'iensis'*, for example

Species name after person's name
- **Hooded cuttle fish** — *Sepia prashadi* (Prasad + i)
- **Tree frog** — *Rhacophorus jerdonii* (Jerdon + ii)
- **Antarctic flying squid** — *Todarodes filippovae* (Filippov + ae)

Species name after place
- **Common Indian monitor** — *Varanus bengalensis* (Bengal + ensis)
- **Cookiecutter shark** — *Isistius brasiliensis* (Brasil + iensis)
- **Butterfly fish** — *Chaetodon madagascariensis* (Madagascar + iensis)

6. First part of a compound species-group name is a Latin letter and denotes a character of the taxon, connected to the remaining part of the name by a hyphen (-), e.g., Sole (a kind of flat fish)—*Aseraggodes sinus-arabici.*
7. If a subgenus taxon is used, it is included within parenthesis in between genus and species part and is not included in binomial and trinomial nomenclature for example,

| Name | Genus | Subgenus | Species | Subspecies |
|---|---|---|---|---|
| **Fan shell** (Bivalvia) | *Atrina* | *(Servatrina)* | *pectinata* | *pectinata* |
| **Dussumieri's half beak** (Osteichthyes) | *Hemirhampus* | *(Reporhampus)* | *dussumieri* | |

8. The person who first publishes the scientific name of an animal, is the original author of a name, may be written after the species name along with the year of publication. The author's name may be in its abbreviated form.
9. Comma is only used between author's name and the year of publication (Art. 22. A. 2.1), for example, the scientific name of common octopus is *vulgaris Cuvier, 1797.*(specified in the year)
10. If the original generic name given by the first author who also reported the species name, transfers the species part from one genus to the other, the name of the original author is put within parenthesis, for example, Tiger: *Felis tigris Linnaeus, 1758.*
11. The scientific names must be either in Latin or Latinised or so constructed that they can be treated as a Latin word.
12. The scientific names must be italicised in printed form, or underlined in hand written or in typed forms,
13. Two species under a same genus should not have the same name.
14. Nomenclature of a hybrid/hybrids cannot be considered because the hybrids are normally individuals but not population. Hybrids are typically sterile and become synaptic failure during meiosis. Thus such names have no status in nomenclature.
15. A name published without satisfying the conditions of availability has no standing in zoological nomenclature and is best never recorded, even in synonymy.
16. A scientific valid name which is not used about 50 years in literature, then as per zoological code's provision the unused senior valid scientific name is treated as obliterated name and junior name which is used continuously in literature (at least by 10 authors in 25 publications) becomes the accepted official name.

## 1.6 TAXONOMIC CATEGORIES

The Earth is the home for millions of living forms which includes diverse species of animals and plants. Billions of years of evolution on Earth have resulted in an enormous variety of different categories of organisms. For more than two thousand years, humans have been trying to organize this great diversity of life. The classification system introduced by the Swedish botanist Carolus Linnaeus in the early 1700s has been the most widely used classification system for almost 300 years.

The taxonomic categories are also termed as the Linnaean hierarchy or taxonomic hierarchy or taxonomic classification. It was first proposed by Linnaeus. Hierarchy of categories is the classification of organism in a definite sequence of categories, the taxonomic categories, in a descending order starting from Kingdom. The number of similar characters of categories decreases from the lowest rank to the highest rank. The hierarchy includes seven distinct categories – Kingdom, Division or Phylum, Class, Order, Family, Genus and Species. These categories are typically arranged or organised in the descending sequence keeping the Kingdom at the top and the Species at the end. In order to make taxonomic position of species further precise, certain categories are specifically added to this list which are termed as the intermediate categories.

In the biological classification system, the taxonomic rank is defined as the relative level of a group of organisms (a taxon) in a taxonomic hierarchy. The examples of taxonomic ranks includes the Species, Genus, Family, Order, Class, Phylum, Kingdom, Domain, etc. A given rank includes the more specific descriptions of life forms. Each rank is classified within more general categories of organisms and groups of organisms related to each other through inheritance of traits or features from common ancestors. The rank of any species and the description of its genus is basic, for example consider a particular species, the red fox, *Vulpes vulpes*, the next rank above, the genus Vulpes, comprises of all the 'True Foxes'. Their closest relatives are in the immediately higher rank, the family Canidae, which includes dogs, wolves, jackals, and all foxes. The next higher rank, the order Carnivora, includes Caniforms (bears, seals, weasels, skunks, raccoons and all those mentioned above), and Feliforms (cats, civets, hyenas, mongooses). Carnivorans are one group of the hairy, warm-blooded, nursing members of the class Mammalia, which are classified among animals with backbones in the phylum Chordata, and with them among all animals in the kingdom Animalia. Finally, at the highest rank all of these are grouped together with all other organisms possessing cell nuclei in the domain Eukarya. Figure 1.2 illustrates the major ranks – Domain, Kingdom, Phylum, Class, Order, Family, Genus, and Species, for the Red Fox, *Vulpes vulpes*.

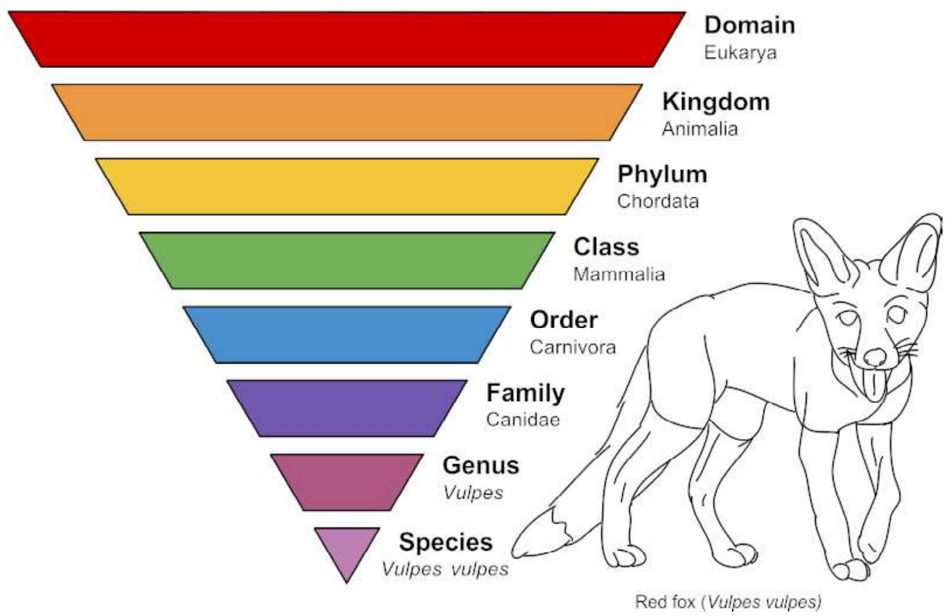


***Fig. 1.2*** *The Species of the Red Fox, Vulpes vulpes*

The International Code of Zoological Nomenclature (ICZN) defines rank as, 'The level, for nomenclatural purposes, of a taxon in a taxonomic hierarchy, for example all families are for nomenclatural purposes at the same rank, which lies between superfamily and subfamily).'

Carl Linnaeus in his landmark publications the '***Systema Naturae***' used a specific ranking scale which is limited to Kingdom, Class, Order, Genus, Species, and one rank below Species, i.e., Subspecies. At present, nomenclature is measured and structured by the nomenclature codes. There are seven main taxonomic ranks: Kingdom, Phylum or Division, Class, Order, Family, Genus, Species. In addition, Domain (proposed by Carl Woese) is now widely used as a fundamental rank, although it is not mentioned in any of the nomenclature codes, and is a synonym for Dominion (Latin Dominium), introduced by Moore in 1974. Table 1.1 shows the main taxonomic ranks in Latin and in English.

***Table 1.1*** *Main Taxonomic Ranks*

| Latin | English |
|---|---|
| *Vitae* | Life |
| *Regio* | Domain |
| *Regnum* | Kingdom |
| *Phylum* | Phylum (In Zoology) |
| *Classis* | Class |
| *Ordo* | Order |
| *Familia* | Family |
| *Genus* | Genus |
| *Species* | Species |

A taxon is typically assigned a rank when it is given its formal name. The basic ranks are Species and Genus. When an organism is given a Species name it is assigned to a Genus, and the Genus name is part of the Species name. The Species name is also called a Binomial, i.e., a two term name, for example the zoological name for the human species is *Homo sapiens*. Here, *Homo* is the generic name for Gene and it is capitalized whereas *sapiens* indicates the Species and it is not capitalized.

## 1.6.1 Types of Taxonomic Categories

Following are the 7 main taxonomic categories.

1. **Species:** Species (used both as singular and plural) is a natural population of individuals or group of population which resemble one another in all essential morphological and also reproductive characters so that they are able to interbreed freely and produce fertile offspring.

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2. **Genus:** It is a group or assemblage of related species which resemble one another in certain correlated characters. The 'Correlated Characters' have the similar or common features which are used in definition of a taxon above the rank of species. All the species of genus are presumed to have evolved from a common ancestor.
3. **Family:** It is a taxonomic category which contains one or more related genera. All the genera of a family have some common features or correlated characters. They are separable from genera of a related family by significant and characteristic differences in both vegetative and reproductive features. Thus the genera of Cats (Felis) and Leopard (Panther) are included in the Family Felidae.
4. **Order:** The category includes one or more related families. Thus the Family Solanaceae is placed in the Order Polemoniales along with four other related Families (Convolvulaceae, Boraginaceae, Hydrophyllaceae and Polemoniaceae). Similarly, the Families Fekidae and Canidae are included under the Order Carnivore along with Hyaenidae (Hyaenas) and Ursidae (Bears).
5. **Class:** A class is made of one or more relates order. For example, the Class Mammalian of animals includes all mammals which range from Bats (Order Chiroptera), Kangaroos (Order Marsupialia), Rodents (Order Rodentia), Whales (Order Cetacean), Carnivores (Order Cornivora) to Great Apes and Man (Order Primate).
6. **Division or Phylum:** It is a category higher than the Class. The term Phylum is used for animals while Division is commonly employed for plants. A Division or Phylum is formed of one or more classes. The Phylum Chordate of animals contains not only Class Mammalian but also Aves (Birds), Reptilian (Reptiles), Amphibian (Amphibians), Cyclostomata, Chondrichthyes, Osteichthyes (Fishes), etc.
7. **Kingdom:** It is the highest taxonomic category. All plants are included in Kingdom Plantae while all animals belong to Kingdom Animalia.

## 1.7 EVALUTION OF BIODIVERSITY INDICES

In biology, a diversity index is a quantitative measure that specifies how many different types (such as, species) there are in a dataset (a community), and simultaneously takes into account how evenly the basic entities (such as, individuals) are distributed among those types. When diversity indices are used in ecology, then the types studied are usually species, but the other categories can also be studied, such as Genera, Families, Functional Types or Haplotypes. The entities or objects of study are typically individual plants or animals, and the measure of abundance can be, for example, number of individuals, biomass or coverage. The most commonly used diversity indices are simple transformations of the effective number of types, also termed as 'True Diversity', but each diversity index can also

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be interpreted in its own right as a measure corresponding to some real phenomenon but a different one for each diversity index.

The types of diversity indices of biodiversity includes the Dominance Indices, and Information-Statistic Indices.

**Information-Statistic Indices**

Information-statistic indices includes the rare species in a community. The calculation of information-statistic indices are based on the rationale that diversity in a natural system can be measured in a way that is analogous to the way information contained in a code or message is measured.

By analogy, if we know how to calculate the uncertainty of the next letter in a coded message, then we can use the same technique to calculate the uncertainty of the next species to be found in a community.

## 1.7.1 Dominance Indices

Dominance indices are weighted toward the abundance of the commonest species. A widely used dominance index is Simpson's diversity index. It takes into account both richness and evenness.

**Simpson's Diversity Indices**

The term 'Simpson's Diversity Index' can actually refer to any one of 3 closely related indices.

**Simpson's Index (*D*):** Simpson's index measures the probability that any two individuals drawn at random from an infinitely large community will belong to same species. There are two versions of the formula for calculating *D*.

Both the following notations are acceptable but is to be consistent:

$$D = \sum (n/N)^2 \qquad D = \frac{\sum n(n-1)}{N(N-1)}$$

Where,

$n$ = Total Number of Individuals of Each Species.

$N$ = Total Number of Organisms of All Species.

The value of *D* ranges between 0 and 1.

With this index, '0' represents Infinite Diversity while '1' represents No Diversity, i.e., the bigger the value of *D*, the lower the diversity. This is not theoretically logical, hence to get over this problem, *D* is often subtracted from 1 or the reciprocal of the index is taken.

**Simpson's Index of Diversity 1-*D*:** This index represents the probability that two individuals randomly selected from a community will belong to different species. The value of this index also ranges between 0 and 1, but here, the greater the value, the greater the diversity.

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**Simpson's Reciprocal Index 1/*D*:** The value of this index starts with 1 as the lowest possible number. This number would represent a community containing only one species. The higher the value, the greater would be the diversity. The maximum value is the number of species in the sample. For example, if there are five species in the sample, then maximum value is 5.

The name Simpson's diversity index is often very loosely applied and all three related indices described above, namely the Simpson's index, Simpson's index of diversity and Simpson's reciprocal index depending on authors.

As an example, let us consider the following table showing the different number of species:

| Species | Number(n) | N(n – 1) |
|---|---|---|
| A | 2 | 2 |
| B | 8 | 56 |
| C | 1 | 0 |
| D | 1 | 0 |
| E | 3 | 6 |
| Total (N) | 15 | 64 |

Putting the values into the formula for Simpson's index, we obtain:

$$D = \frac{\sum n(n-1)}{N(N-1)} = \frac{64}{15 \times 14} = 0.3 \text{ (Simpson's index)}$$

Therefore, the Simpson's index of diversity is 1 **-** $D = 0.7$ and the Simpson's reciprocal index is $1/D = 3.3$.

All these three values represent the same biodiversity. It is, therefore, significant to establish which index has actually been used in any comparative studies of biodiversity. The disadvantage of Simpson's index is that it is comprehensively weighted toward the most abundant species, as are in all dominance indices.

The addition of rare species with one individual will fail to change the index. As a result, Simpson's index is of limited value in conservation biology if an area has many rare species with just one individual.

### 1.7.2 Shannon–Weiner Index

In biodiversity, the terms species richness and species diversity are sometimes used interchangeably. Though there the many species diversity indices used, but the 'Shannon Index' is perhaps the most commonly used. It is also sometimes called the 'Shannon–Wiener Index' and the 'Shannon–Weaver Index'.

Consequently, the Shannon–Wiener index and Simpson's index are considered as the two most used indices. The Shannon index is used for the variations in significance of the rarest species, while the Shannon–Wiener index is used for the variations in significance of the most abundant ones. These indices are often associated with evenness indices to interpret them correctly.

**Species Richness (*S*):** It refers to the total number of different organisms present. It does not take into account the proportion and distribution of each species within the local aquatic community.

**Simpson Index (*D*):** It is a measurement that accounts for the richness and the percent of each species from a biodiversity sample within a local aquatic community. The index assumes that the proportion of individuals in an area indicate their importance to diversity.

**Shannon–Wiener Index (*H*):** It is similar to the Simpson's index. This measurement takes into account species richness and proportion of each species within the local aquatic community. The index comes from information science. It has also been called the Shannon index and the Shannon–Weaver index in the ecological literature.

**Definition: The Shannon–Wiener Index of Diversity or Information Index** is a measure derived from information theories developed by Claude E. Shannon and Norbert Wiener and published in 1949 by Shannon and Warren Weaver, which is used by ecologists when a system contains too many individuals for each to be identified and examined. A small sample is used, the index (*D*) is the ratio of the number of species to their importance values (for example, biomass or productivity) within a trophic level or community. Consider that $D = -\sum_{i}^{s} p_i \log p_i$, where *S* is the total number of species in the sample, *i* is the total number of individuals in one species, $p_i$ (a decimal fraction) is the number of individuals of one species in relation to the number of individuals in the population, and the log is to base-2 or base-*e*.

**Shannon Index:** The widely used diversity index is the Shannon index.

The Shannon Index is given by,

$$H_s = \sum_{i=1}^{s} p, Inp_i$$

Where,

$p_i$ = Proportion of Individuals Found in the ***i***th Species.

*In* = Denotes Natural Logarithm.

The following table gives an example:

| Species | Abundance | $P_i$ | $P_i \text{In } p_i$ |
|---|---|---|---|
| A | 50 | 0.5 | – 0.347 |
| B | 30 | 0.3 | – 0.361 |
| C | 10 | 0.1 | – 0.217 |
| D | 9 | 0.09 | – 0.217 |
| E | 1 | 0.01 | – 0.046 |
| **Total 5** | 100 | 1.00 | – 1.201 |

Putting the values into the formula for Shannon index, $H_S = 1.201$.

Even the rare species with one individual (species E) contributes some value to the Shannon index, so if an area has many rare species, their contributions would accommodate. Shannon index has a minus sign in the calculation, so the index actually becomes 1.201, not -1.201. Values of Shannon index for real communities are often found to occur between 1.5 and 3.5. The value obtained from a sample is in itself of no significance. The index becomes useful only while comparing two or more sites.

**Brillouin Index:** The Brillouin index is a second information-statistic index, designed to reflect species abundance.

The Brillouin index is given by:

$$H_8 = \frac{1n(N!) - \sum 1n(ni!)}{n}$$

Where,

$N$ = Total Number of Individuals in the Community.

$ni$ = Number of Individuals in the ***i***th Species.

The following table gives an example:

| Species | No. of Individuals | In($n_i$!) |
|---|---|---|
| A | 5 | 4.79 |
| B | 5 | 4.79 |
| C | 5 | 4.79 |
| D | 5 | 4.79 |
| E | 5 | 4.79 |
| | N = 25 | $\sum$ In(n!) = 23.95 |

Putting the values into the formula for Brillouin index, we get:

$$H_B = \frac{1n(25!) - 23.95}{25} = \frac{58 - 23.95}{25} = 1.362$$

This index describes a known population. There is no scope for uncertainty while using this index. It gives additional emphasis on species richness and is discreetly sensitive to sample size.

A comparison of one or all of these measures of biodiversity can illustrate changes in water quality conditions within a local community. Water quality parameters like light penetration, dissolved oxygen and salinity can have dramatic impacts on levels of biodiversity.

## 1.7.3 Similarity and Dissimilarity Indices

Biological systems are organized on many different levels, such as the molecules, cells, organisms, populations, communities and ecosystems. Species diversity is a characteristic unique to the community level of biological organization. Higher species diversity is generally thought to indicate a more complex and improved community because a greater variety of species allows for more species interactions, hence greater system stability, and indicates good environmental conditions. A variety of diversity indices can be calculated to compare ecological communities. In addition, pairs of communities can be compared using **community similarity indices**.

Species diversity has two parts. Richness refers to the number of species found in a community and evenness refers to the relative abundance of each species. A community is said to have high species diversity if many nearly equally abundant species are present. If a community has only a few species or if only a few species are very abundant, then species diversity is low. Consider a community with 100 individuals distributed among 10 species. Then if there are 10 individuals in each of the 10 species in the community then it is more diverse than if there are 91 individuals in one species and one individual in each of the other nine species.

### Similarity Indices: Jaccard's Index

The 'Index of Similarity' uses the Jaccard's index which helps in measuring the similarity between two sample species. Principally, the Jaccard's index helps in analysing the diversity of not only a single location, but in comparing biodiversity levels across different locations. An intuitive measure of similarity between two samples can summarize the fraction of species they share.

Jaccard's index is represented as '***J***' in the simplest form and has the following notation:

$$J = \frac{S_c}{S_a + S_b + S_c}$$

Where, $S_a$ and $S_b$ are the numbers of species unique to samples *a* and *b*, respectively, and $S_c$ is the number of species common to the two samples.

**Interpretation:** Jaccard's index of similarity is very unique since it is simply the fraction of species shared between the samples. Remember, however, that Jaccard's index only utilizes the richness component of diversity, since it does not require any information on abundance. As a pairwise measure, it can be examined that how Jaccard's index varies with the distance or environmental differences between the different locations.

### Index of Dissimilarity

The 'Index of Dissimilarity' is a demographic measure of the evenness with which two groups or species are distributed across component geographic areas that make up a larger area. The index score can also be interpreted as the percentage of one of the two groups included in the calculation that would have to move to different geographic areas in order to produce a distribution that matches that of

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the larger area. The index of dissimilarity can be used as a measure of segregation.

The basic formula for the index of dissimilarity is:

$$\frac{1}{2}\sum_{i-1}^{N}\left|\frac{a_i}{A}-\frac{b_i}{B}\right|$$

Where,

$a_i$ = Population of Group $A$ in the $i$th Area.

$A$ = Population in Group $A$ in the Large Geographic Entity for which the Index of Dissimilarity is being Calculated.

$b_i$ = Population of Group $B$ in the $i$th Area.

$B$ = Total Population in Group $B$ in the Large Geographic Entity for which the Index of Dissimilarity is being Calculated.

The index of dissimilarity is applicable to any categorical variable, whether demographic or not, and because of its simple properties it is useful for input into multidimensional scaling and clustering programs. It has been used extensively in the study of social mobility to compare distributions of origin (or destination) occupational categories.

**Association Index**

Researchers often use association indices to convert observations into a measure of propensity for individuals to be seen together. At its simplest, this measure is just the probability of observing both individuals together given that one has been seen (the simple ratio index). However, this probability becomes more challenging to calculate if the detection rate for individuals is imperfect. We first evaluate the performance of existing association indices at estimating true association rates under scenarios where:

1. Only a proportion of all groups are observed (group location errors),
2. Not all individuals are observed despite being present (individual location errors), and
3. A combination of the two.

Commonly used methods aimed at dealing with incomplete observations perform poorly because they are based on arbitrary observation probabilities. Using calibration data is an important step when constructing animal social networks. Association indices are widely used to describe relationships between individuals. Indices include built-in assumptions about how animals are observed. These assumptions are rarely considered in studies of animal social networks.

If a is the' number of random samples of a given series in which species A occurs and h is the number of samples in which another species B occurs together with A, then the association index B/A = h/a.

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**Check Your Progress**

5. Define taxonomic rank.
6. How an organism is given a scientific name?
7. What is diversity index? What are its types?
8. What is species richness?

## 1.8 ANSWERS TO 'CHECK YOUR PROGRESS'

1. 'Biological Systematics' or 'Biosystematics' refers to the study of the diversification of living forms, both past and present, and the relationships among living things through time.
2. The Swedish naturalist Carolus Linnaeus developed the scientific system of naming the species, it is also known as the binomial system of nomenclature. Linnaeus described about 5900 species of plants in his book '*Species Plantarum*' (1753) and about 4326 species of animals in his book '*Systema Naturae*' (1758)
3. Taxonomy refers to the theory and practice of identifying plants and animals. In fact, taxonomy deals with the principals involved in the study of classification of organisms. It is the functional science which deals with identification, nomenclature and classification of different kinds of organisms
4. The classical taxonomy is based on the observable morphological characters with reference to the normal individuals and are considered that the expression will be analogous or similar while their variations can be imperfect expressions. While New systematic is the systematic study which includes all types of characters. In addition to classical morphology, it includes anatomy, cytology, physiology, biochemistry, ecology, genetics, embryology, behaviour, etc., of the whole population instead of a few typological specimens.
5. In the biological classification system, the taxonomic rank is defined as the relative level of a group of organisms (a taxon) in a taxonomic hierarchy. The examples of taxonomic ranks includes the Species, Genus, Family, Order, Class, Phylum, Kingdom, Domain, etc.
6. A taxon is typically assigned a rank when it is given its formal name. The basic ranks are species and genus. When an organism is given a species name it is assigned to a genus, and the genus name is part of the species name. The species name is also called a 'Binomial' i.e., a two term name, for example the zoological name for the human species is *Homo sapiens*. Here, *Homo* is the generic name for gene and it is capitalized whereas *sapiens* indicates the species and it is not capitalized.
7. A diversity index is a quantitative measure that specifies how many different types (such as, species) are there in a dataset (a community), and simultaneously takes into account how evenly the basic entities (such as, individuals) are distributed among those types.

8. Species richness refers to the total number of different organisms present. It does not take into account the proportion and distribution of each species within the local aquatic community.

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## 1.9 SUMMARY

- Biosystematics is the study of the variation and evolution of a population of organisms in relation to their taxonomic classification,
- Taxonomy, systematics or classification of organisms is based on the study of their comparative morphology (form, external and internal structure), cytology, embryology, fossil relatives, biochemical analysis and ecological relationships
- Systematics is the study of diversification and relationships of life forms of extinct extant.
- Taxonomy refers to the theory and practice of identifying plants and animals
- The classical taxonomy is based on the observable morphological characters with reference to the normal individuals in a dry state.
- New systematic is the systematic study which include all type of characters.
- The term 'Taxonomy' defines about the classification and nomenclature of the organisms.
- The term 'Wet Collections' are used for the specimens that are kept or preserved in a liquid preservative to prevent their deterioration.
- The term dry collection refers to those specimens that are preserved in dry state.
- The taxonomic categories are also termed as the Linnaean hierarchy.
- Information-stastic indices include the rare species in a community.
- Dominance indices are weighed towards the abundance of the commonest species.
- Simpson index is the measurement that account for the richness and the percent of each species from biodiversity sample within a local community.
- The 'Index of Similarity' uses the Jaccard's index which help in measuring the similarity between two samples.
- The 'Index of Dissimilarity' is a demographic measure of the evenness with which two groups or species are distributed across component geographic areas that make up a larger area.

## 1.10 KEY TERMS

- **Biosystematics**: Biosystematics is the study of the variation and evolution of a population of organisms.
- **Taxonomy:** Taxonomy refers to the theory and practice of identifying plants and animals.

- **Classical taxonomy:** The classical taxonomy is based on the observable morphological characters with reference to the normal individuals.
- **New systematic:** New systematic is the systematic study which includes all types of characters.
- **Dry collection**: The term dry collection refers to those specimens that are preserved in dry state.
- **Wet collections**: The term 'Wet Collections' are used for the specimens that are kept or preserved in a liquid preservative to prevent their deterioration.
- **Linnaean hierarchy**: The taxonomic categories are termed as the Linnaean hierarchy or taxonomic hierarchy or taxonomic classification.
- **Species**: Species (used both as singular and plural) is a natural population of individuals or group of population.
- **Genus:** Genus is a group or assemblage of related species which resemble one another in certain correlated characters.
- **Family:** Family is a taxonomic category which contains one or more related genera.
- **Order**: Order includes one or more related families.
- **Kingdom**: It is the highest taxonomic category.
- **Species richness:** 'Species Richness' refers to the total number of different organisms present.

## 1.11 SELF-ASSESSMENT QUESTIONS AND EXERCISES

### Short-Answer Questions

1. What are the basis of systematic studies?
2. Mention the differences between taxonomy and systematics.
3. Define the various stages of taxonomy.
4. Write a short note on taxonomic ranks.
5. What is speciation?
6. What do you understand by international code of zoological nomenclature?
7. Define the term spices diversity.
8. What are the limitations of Simpson's index?

### Long-answer Questions

1. Who introduced the hierarchy of categories? Explain the hierarchy of categories in detail giving examples.
2. Discuss the types of taxonomic categories in detail giving suitable examples.
3. Interpret briefly about the different types of speciation mechanism.

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4. Briefly describe the international code of zoological nomenclature giving relevant examples.
5. What is the index of dissimilarity? How it is calculated? Give examples.
6. Give a detail account of 'The Shannon-Wiener Index of Diversity'.
7. Briefly describe the biological collections and its types with the help of relevant examples.

## 1.12 FURTHER READING

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# UNIT 2 BIOSYSTEMATICS AND TAXONOMY - II

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**Structure**



## 2.0 INTRODUCTION

In biology, systematics is the study and classification of living things; in other words, grouping organisms based on a set of rules (or system). The science of classifying organisms. Systematics can be divided into two closely related and overlapping levels of classification taxonomic and phylogenetic.

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Systematics provides basic knowledge and information about the components of biodiversity which is essential for the effective decision-making about the preservation, protection and ecological use.

Biosystematics is the science through which life forms are discovered, identified, described, named, classified and catalogued, and we can assess their diversity, life histories and living habits. The roles of a species in an ecosystem and their geographical distributions could be understood with the help of biosystematics . The important objectives of biosystematics is to catalogue and preserve the biodiversity collected from different sources.

Biosystematics provides a vivid and rich picture of the existing organic biodiversity of the earth. There are three interrelated levels of Biodiversity: diversity within species (genetic diversity), between species (species diversity) and between ecosystems (ecosystem diversity)

It has been estimated that in the world more than 50 million species of plants, animals and micro-organisms are existing. All this diversity of life is confined to only about one-kilometre-thick layer of lithosphere hydrosphere and atmosphere which form biosphere. Biodiversity is sum of all the genes, species, population, varieties in different ecosystem and their relative abundance in a particular ecosystem.

Today, India's biodiversity is in jeopardy. Due to various reasons, many wild species are disappearing rapidly. An incalculable number of species are already gone forever, and a large percentage of the rest are threatened with extinction. In almost all cases, the threats to wildlife can be traced to human activities. Today, with the population explosion, more and more land is being cleared for agriculture, habitation and other developmental projects. Since the human beings are enjoying all the benefits from biodiversity, they should take proper care for the preservation of biodiversity in all its form and good health for the future generation, i.e., the human being should prevent the degradation and destruction of the habitats thereby maintaining the biodiversity at its optimum level.

In this unit, you will study about the importance and applications of biosystematics in biology, the various trends In biosystematics, molecular perspective on the conservation of diversity, about wildlife and its conservation methods, various national parks and sanctuaries of India, geological and zoographical distribution of animals, fossils and paleozoology.

## 2.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the importance and applications of biosystematics in biology
- Comprehend the various trends in biosystematics
- Understand the molecular perspective on the conservation of diversity
- Explain wildlife and its conservation
- Analyse about national park and sanctuaries of India
- Understand geological and zoographical distribution of animals

# 2.2 IMPORTANCE AND APPLICATIONS OF BIOSYSTEMATIC IN BIOLOGY

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Systematics, as already discussed, is the study of diversification and relationships of different life forms of extinct and present. Currently systematics extensively uses molecular biology and computer programs to study various microorganisms and macro-organisms.

Following are the important objectives of biosystematics.

1. To catalogue and preserve the biodiversity collected from different sources.
2. To differentiate the variations among organisms and arranged them on the basis of their relationships or associations.
3. To provide scientific name to the taxa, so that one can recorded, stored and retrieved when required.
4. To establish a set of rules to choice characters for arranging species into hierarchic classification.
5. To study the genetic and phylogenetic relationships among life forms.
6. To make extensive use of computer for analysing and differentiating the intra and interspecific relationships among organisms.

**Applications of Biosystematics**

Systematics is the key that helps in understanding the fascinating biodiversity everywhere in the world. Systematics benefits the human beings by providing the fundamental knowledge about the sustainable resource management, environmental protection, and landscape preservation to food security. Systematic biology provides the skill to make policies for successful implementation of preservation and management of our biodiversity, which is critical to have long term quality of life for us as well as to our nature.

The impact of systematic to biology can be studied under following two heads:

- Theoretical Biology
- Applied Biology

**1. Theoretical Biology**

Systematics play significant role in the field of theoretical biology, which can be summarized as follows:

1. It is accountable in creating conceptual contribution, such as population thinking.
2. It is accountable in solving the problems of multiplication of species. Basically, it illustrates the structure of the species and the evolutionary processes.
3. Simulation and other significant evolutionary areas can be easily explained through taxonomy.

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4. It plays significant role in the development of behavioural science.
5. Taxonomy is the key for studying ecology, because no ecological survey can be undertaken unless all the species of ecological significance are identified.

**2. Applied Biology**

Systematics provides basic knowledge and information about the components of biodiversity which is essential for the effective decision-making about the preservation, protection and ecological use. Following are the most essential and significant aspects of applied biology:

- **Agriculture and Forestry:** Currently the environment is encountered with the critical and challenging problem of saving the crops and trees from the attack of various kinds of pests. Therefore, it is essential to identify the correct names of such pests so that their proper control and eradication can be done. Taxonomists provide the accurate identification of pest species, which is vital for its effective control. Similarly, many of the plant diseases are caused by certain vectors. The precise and accurate identification of a particular vector is essential for the vector under control by killing its transmitters.
- **Biological Control:** Natural enemies of pests can be used for the biological control of pests. The biological control is considered more economical as compared to the chemical control.
- **Public Health:** Taxonomy plays significant role in public health programs. There are number of diseases, which are spread by many Arthropods. Therefore, the controlled measures must be effectively planned to attack the target species, for example, all the *Anopheles maculipennis* cannot transmit malaria. This species includes several sibling species of which only a few are responsible for transmitting malaria. An expert taxonomist can identify this specific sibling species that causes malaria. An accurate identification ensures a maximum of effective control at minimum cost.
- **Quarantine:** Many new pests and diseases of plants, animals and human beings can spread from one country to another through transportation. The Governments of respective countries are taking precautions and have established ‘Quarantine Laboratories’ at airports, harbours, other ports, etc., to check such transmissions. Taxonomists has key role in prompt identification of such pests and diseases.
- **Wild Life Management:** Currently the attention is being given to conserve, preserve and propagate wild life. The natural environment is significantly effected and disturbed due to the indiscriminate killing of animals and chopping or cutting of trees. Taxonomists support all environmental protectors by identifying the economically and ecologically essential and significant wild life. This is very important for the preservation and protection of the biodiversity.

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- **Mineral Prospecting:** The accurate identification of fauna and flora in sedimentary rocks provides a perfect representation of the sequence of geological events, which helps in searching for fuels and mineral deposits. The paleontologists helps in the identification of such fossil specimens of the sedimentary rocks by giving the accurate depiction of the correct sequence of geological events.
- **National Defense:** Information regarding disease vectors and parasites is an evident application of systematics to national defense. The use of biological means in the war is economical and requires fewer efforts in their operation. During World War II, the use of Japanese paper balloons created havoc in the forest of North East America. Eventually a balloons was recovered with sand that actually contained a great number of shells of microorganisms. How America did recognised that these paper balloons are made by Japan? By observing the shells of microorganisms, the taxonomists of America confirmed that this specific type of sand represent the mainland island of Japan. Additionally, the identification of potential disease vectors is essential to the health of both military and civilian populations all over the world.
- **Environmental Problem:** Taxonomists have significant and imperative role in detecting some of the environmental problems. When certain pesticides that have entered in the food chain of ecosystem and the bio-magnification of pesticide takes place at certain trophic level, then a taxonomist helps in detecting such problems and can take effective measures to control it. At present, the water pollution is considered as a major environmental problem. Certain planktons are considered as the reliable indicator for measuring the degree of water pollution. The taxonomists identify or detect and categorize such organism and then provide prompt information for detecting pollution and taking appropriate measures.
- **Soil Fertility:** Some organisms have vital role in increasing the fertility of soil. Hence, it is essential to distinguish such animals for their proper management in agriculture.
- **In Commerce:** Many animals and animal's products are extensively and commercially used by human beings, such as honey, bee wax, wool, milk and dairy products, meat, eggs, oil, leather, silk, lac, dyes, etc. Systematics, therefore, play significant role by providing vital information about the respective species which helps in increasing and improving the qualities of the animal products.

### Importance of Biosystematics

In biology, the importance of biosystematics can be summarized as follows:

1. Biosystematics provides a vivid and rich picture of the existing organic biodiversity of the earth.

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2. It provides ample information authorising a reconstruction of the phylogeny of the life.
3. It reveals abundant evolutionary phenomena making them available for spontaneous study by the other branches of biology.
4. It provides the exclusive information necessary for entire branches of biology.
5. It is essential and indispensable for studying the ecologically and medically important organisms.
6. It provides the standard classifications system which has experiential importance and explanatory value in most branches of biology, such as the evolutionary biology, biochemistry, immunology, ecology, ethology and historic.

## 2.3 TRENDS IN BIOSYSTEMATICS

Previous approaches of biosystematics were exclusively based on the observed characters without considering the infra specific differences. Many of the species are, therefore, known or specified on the basis of single or a few specimens. The old morphological concept now includes the ecological, genetics, biochemical and other significant characters.

Modern taxonomists consider that the gross morphological characters are not always appropriate for providing the means of differentiation to determine the genetically and evolutionary relationship between different taxa. To accomplish this, the taxonomical evidences from anatomy, embryology, palynology, cytology, palaeobotany, ecology, biochemistry, etc., are taken into consideration for accurate ordering of species.

Some current approaches in biosystematics include the chemotaxonomy, cytotaxonomy and molecular taxonomy.

### 2.3.1 Chemotaxonomy

As per the 'Merriam-Webster' dictionary, "The chemotaxonomy is the method of biological classification based on similarities in the structure of certain compounds among the organisms being classified".

The systematists suggest that because proteins are more closely controlled by genes and less subjected to natural selection as compared to the anatomical features, hence they are more reliable indicators of genetic relationships. The compounds studied include the proteins, amino acids, nucleic acids, peptides, etc.

Physiology refers to the study of functioning of organs in a living being. Since working of the organs involves chemicals of the body, therefore these compounds are termed as the biochemical evidences. The study of morphological change has revealed that there are changes in the structure of animals which result in evolution. When changes take place in the structure of a living organism, then they will be naturally accompanied by changes in the physiological or biochemical processes.

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John Griffith Vaughan was considered as the one of the pioneers of chemotaxonomy. The chemotaxonomy includes the following biochemical products.

### Biochemical Products

The body of any animal in the animal kingdom is made up of a number of chemicals. Of these, only the following few biochemical products are considered for developing the evidence for evolution.

**Protoplasm:** Every living cell, from a bacterium to an elephant, from grasses to the blue whale, has protoplasm in the cell. Though the complexity and constituents of the protoplasm increases from lower to higher living organism, the basic compound is always the protoplasm. The 'Evolutionary Significance' provides the evidence that all living things have a common origin point or a common ancestor, which in turn had protoplasm. Its complexity increased due to changes in the mode of life and habitat.

**Nucleic Acids:** DNA and RNA (Deoxyribo Nucleic Acid and Ribo Nucleic Acid)are the two types of nucleic acids present in all living organisms. They are present in the chromosomes. The structure of these acids has been found to be similar in all animals. DNA always has two chains forming a double helix, and each chain is made up of nucleotides. Each nucleotide has a pentose sugar, a phosphate group, and nitrogenous bases like Adenine (A), Guanine (G), Cytosine (C), and Thymine (T). RNA contains 'Uracil' instead of 'Thymine'. It has been scientifically proved in the laboratory that a single strand of DNA of one species can match with the other strand from another species. If the alleles of the strands of any two species are close, then it can be concluded that these two species are more closely related.

**Digestive Enzymes:** Enzymes are chemical compounds that help in digestion. Proteins are always digested by a particular type of enzymes like pepsin, trypsin, etc., in all animals from a single celled amoeba to a human being. The complexity in the composition of these enzymes increases from lower to higher organisms but are fundamentally the same. Similarly, the carbohydrates are always digested by amylase, and the fats by lipase.

**End Products of Digestion:** Irrespective of the type of animal, the end products of protein, carbohydrates and fats are amino acids, simple sugars, and fatty acids, respectively. It can thus be comfortably concluded that the similarity of the end products is due to common ancestry.

**Hormones:** Hormones are secretions from ductless glands called the endocrine glands, such as the thyroid, pituitary, adrenal, etc. Their chemical nature is the same in all animals, for example thyroxine is secreted from the thyroid gland, irrespective of the species of the animal. It is used to control metabolism in all animals. If a human being is deficient in thyroxine, it is not mandatory that this hormone should be supplemented from another human being. It can be extracted from any mammal and injected into humans for normal metabolism to happen. Similarly, insulin is secreted from the pancreas. If the thyroid gland from a tadpole

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is removed and replaced with a bovine thyroid gland, then normal metabolism will take place and the tadpole will metamorphose into a frog. Since there is a basic fundamental relationship among these animals, therefore such exchange of hormones or glands is possible.

**Nitrogenous Excretory Products:** Mainly three types of nitrogenous waste is excreted by living organisms, namely the ammonia which is a characteristics of aquatic life form, the urea which is formed by the land and water dwellers, and the uric acid which is excreted by terrestrial life forms. A frog, in its tadpole stage excretes ammonia similar to a fish. When the tadpole grows into an adult frog and moves to land, then it excretes urea instead of ammonia. Consequently, establishing an aquatic ancestry to the land animal. Another example can be taken from the chick development stages. The chick up to its 5th day of development excretes ammonia, from its 5th to 9th day it excretes urea, and afterwards it excretes uric acid. Based on these findings, Baldwin sought a biochemical recapitulation in the development of vertebrates with reference to nitrogenous excretory products.

**Phosphagens:** Phosphagens are energy reservoirs of animals. They are present in the muscles and supply energy for the synthesis of Adenosine TriPhosphate (ATP). In general, there are two types of phosphagens in animals, the PhosphoArginine (PA) in invertebrates and the PhosphoCreatine (PC) in vertebrates. Among the echinoderms and prochordates, some have PA while some have PC and only a few have both PA and PC. Biochemically, these two groups are related. This is the most basic proof that states the first chordate animals should have been derived only from echinoderm-like ancestors.

**Body Fluid of Animals:** When the body fluids of both aquatic and terrestrial animals are analysed, then on analysis it illustrates that they resemble sea water in their ionic composition. There is ample evidence that primitive members of most of the phyla lived in the sea in Paleozoic times. It is assumed that the first life appeared only in the sea and then evolved onto the land. Additionally, the body fluids of most animals contain less magnesium and more potassium as compared to the water of the present-day ocean. In the past, the ocean contained less magnesium and more potassium. The body of animals' accumulated more of these minerals due to the structure of DNA, and this characteristic feature is same even today. When the first life forms appeared in the sea, they acquired the composition of the contemporary sea water, and retained it even after their evolution onto land, as it was a favourable trait.

**Visual Pigments:** In the vertebrates, vision is controlled by the two very distinct types of visual pigments, porphyropsin and rhodopsin. These are present in the rods of the retina. Fresh water fishes have porphyropsin, while the marine ones and land vertebrates have rhodopsin. In amphibians, a tadpole living in fresh water has porphyropsin, and the adult frog which lives on land most of the time has rhodopsin. In catadromous fish, which migrate from fresh water to the sea, the porphyropsin is replaced by rhodopsin. In an anadromous fish, which migrates from the sea to freshwater, the rhodopsin is replaced by porphyropsin. These examples prove about the freshwater origin of vertebrates which then deviated into two lines, one leading to marine life and the other to terrestrial life.

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**Serological Evidence:** In recent years, experiments made in the composition of blood offer good evidence for evolution. It has been found that blood can be transmitted only between animals that are closely related. The degree of relationship between these animals is determined by the feature serological evidence. Though there are various methods of determining the relationship but the method mostly used is given by George Nuttall and is termed as the precipitation method. In this method, anti-serum of the involved animals has to be prepared. For human study, human blood is collected and allowed to clot. Then, the serum is separated from the erythrocytes. A rabbit is then injected with a small amount of serum at regular intervals, which is allowed to incubate for a few days. This forms antibodies in the body of rabbit. The blood of rabbit is then drawn and clotted. The serum separated from the Red Blood Cells (RBCs) is called the 'Anti-Human Serum'. When such a serum is treated with that of blood of monkeys or apes, a clear white precipitate is formed. When the serum is treated with the blood of any other animal like dogs, cats, or cows, then no precipitate appears. Therefore, it can be concluded that humans are more closely related to monkeys and apes. Consequently, on the basis of studies it has been determined that lizards are closely related to snakes, horses to donkeys, dogs to cats, etc.

The field of biochemistry has significantly advanced since Darwin's time, and this serological study provides as one of the most recent sections of evidence of evolution. A number of biochemical products, such as nucleic acids, enzymes, hormones and phosphagens evidently show the relationship of all life forms. The composition of body fluid establishes that the first life form was originated in the oceans. The presence of nitrogenous waste products reveal the aquatic ancestry of vertebrates, and the nature of visual pigments points out the fresh water ancestry of land vertebrates. Serological tests indicate relationships within these animal phyla.

## 2.3.2 Cytotaxonomy

Cytotaxonomy is the branch of biology that deals with the relationships and classification of organisms using comparative studies of chromosomes during meiosis.

As per the 'Merriam-Webster' dictionary, "The cytotaxonomy is the study of the relationships and classification of organisms using both classical systematic techniques and comparative studies of chromosomes. It includes the nuclear cytologic character of an organism".

Cytotaxonomy is the specific branch of taxonomy, which uses the characteristics of cellular structures, such as somatic chromosome to classify the organism. Enhancement in the cytological techniques authorize chromosomal studies. The study of primate chromosome provides sufficient information on relationships

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(Chiarelli, 1996). The number, structure and behaviour of chromosomes is of great significance in taxonomy, and is most widely used because the chromosome number provides the basic information and predefined established characters. Chromosome numbers are usually determined at mitosis and quoted as the diploid number (2n), unless dealing with a polyploid series where the base number or number of chromosomes in the genome of the original haploid is quoted. Another significant taxonomic character is the position of the centromere. Meiotic behaviour may show the heterozygosity of inversions. This may be constant for a taxon and provides further taxonomic evidences. Cytological data is regarded as having more significance as compared to other taxonomic evidences.

### 2.3.3 Molecular Taxonomy

Taxonomy is the branch of science concerned with the classification of organisms. A taxonomic designation is more than just a name. Ideally, it reflects evolutionary history and the relationship between organisms. Traditionally, taxonomic classification has relied upon morphological features and physiological characteristics.

Molecular taxonomy is the classification of organisms on the basis of the distribution and composition of chemical substances in them. Molecular genetic methods help taxonomy to identify or describe species, as these methods can be applied to all organisms and offer quantifiable characters. The molecular taxonomy includes the Molecular (DNA, RNA, proteins).

Molecular taxonomy is principally effective in combination with other methods, typically with morphology. The advent of DNA cloning and sequencing methods have contributed immensely to the development of molecular taxonomy and population genetics over the last two decades. These modern methods have revolutionised the field of molecular taxonomy and population genetics with improved analytical power and precision. In biology, the molecular techniques help in establishing the genetic relationship between the members of different taxonomic categories.

Molecular phylogenetic refers to the study of evolutionary relationships among biological entities, such as the individuals, populations, species, or higher taxa, by using a combination of molecular data, such as DNA and protein sequences, presence or absence of transposable elements and gene-order data, and statistical techniques.

Molecular phylogenetics is the branch of phylogeny that analyses genetic, hereditary molecular differences, predominately in DNA sequences, in order to obtain information on an organism's evolutionary relationships. From these analyses, it is possible to determine the processes by which diversity among species has been achieved. The result of a molecular phylogenetic analysis is expressed in a phylogenetic tree. Molecular phylogenetics is one aspect of molecular systematics. The term molecular systematics includes the use of molecular data in taxonomy and biogeography.

There is a correlation between the molecular phylogenetics and molecular evolution. Molecular evolution is the process of selective changes (mutations) at a molecular level (genes, proteins, etc.) throughout various branches in the tree of life (evolution). Molecular phylogenetics makes inferences of the evolutionary relationships that arise due to molecular evolution and results in the construction of a phylogenetic tree. The Figure 2.1 depicts the phylogenetic tree of life as one of the first detailed trees, according to information given by the Haeckel in 1870s.

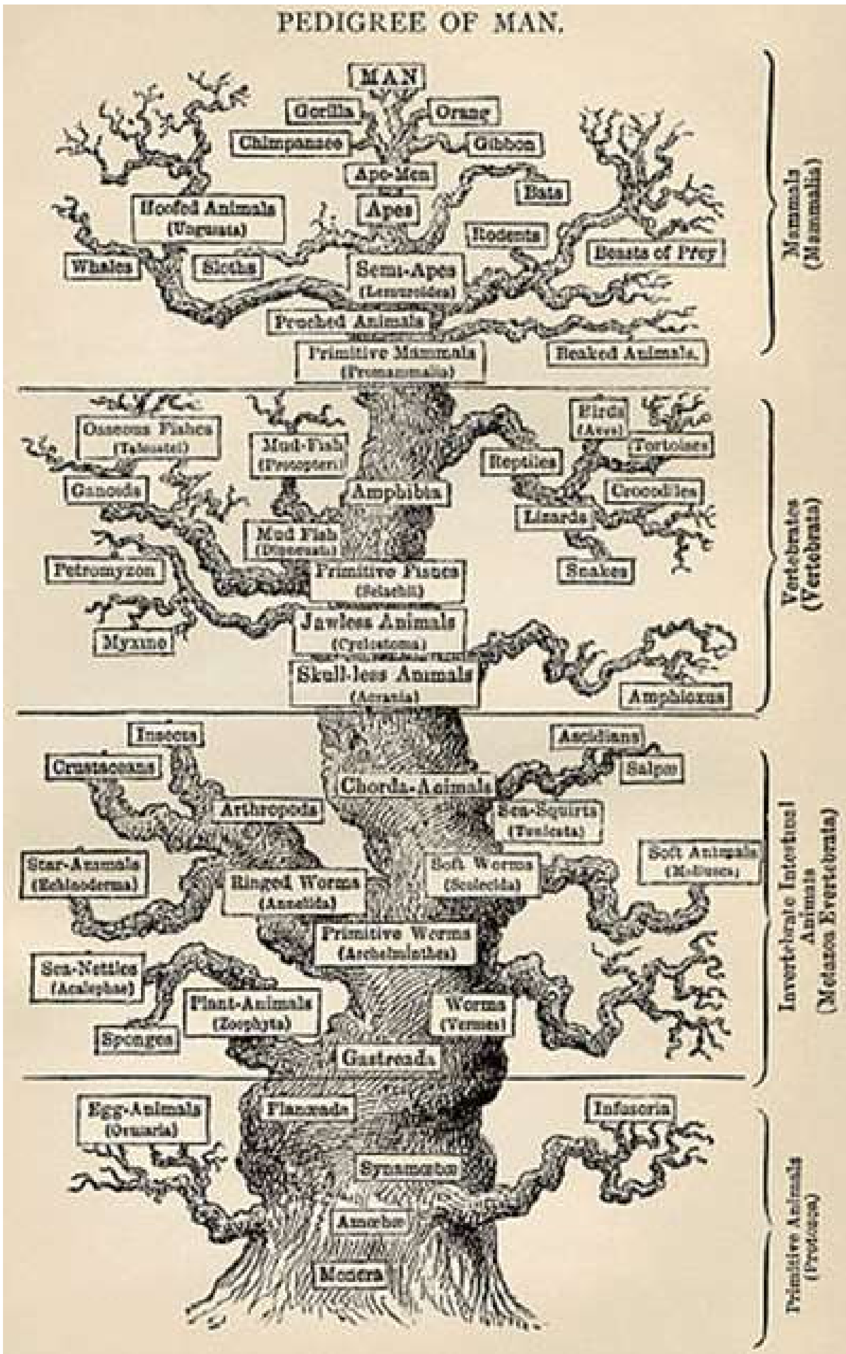


*Fig. 2.1 Phylogenetic Tree of Life based on the Information given by Haeckel*

Initial challenges at molecular systematics were termed as chemotaxonomy which made use of proteins, enzymes, carbohydrates, and other molecules that

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were separated and characterized using techniques, such as chromatography. These have been replaced in recent times largely by DNA sequencing, which produces the exact sequences of nucleotidesor bases in either DNA or RNA segments extracted using different techniques. In general, these are considered superior for evolutionary studies, since the actions of evolution are ultimately reflected in the genetic sequences. Presently, it is still a long and expensive process to sequence the entire DNA of an organism (its genome). However, it is quite feasible to determine the sequence of a defined area of a particular chromosome. Typical molecular systematic analyses require the sequencing of around 1000 base pairs. At any location within such a sequence, the bases found in a given position may vary between organisms. The particular sequence found in a given organism is referred to as its haplotype. In principle, since there are four base types, with 1000 base pairs, we could have 41000 distinct haplotypes. However, for organisms within a particular species or in a group of related species, it has been found empirically that only a minority of sites show any variation at all, and most of the variations that are found are correlated, so that the number of distinct haplotypes that are found is relatively small.

In a phylogenetic tree, numerous groupings (clades) exist. A clade may be defined as a group of organisms having a common ancestor throughout evolution. Figure 2.2 illustrates how a clade in a phylogenetic tree may be expressed.

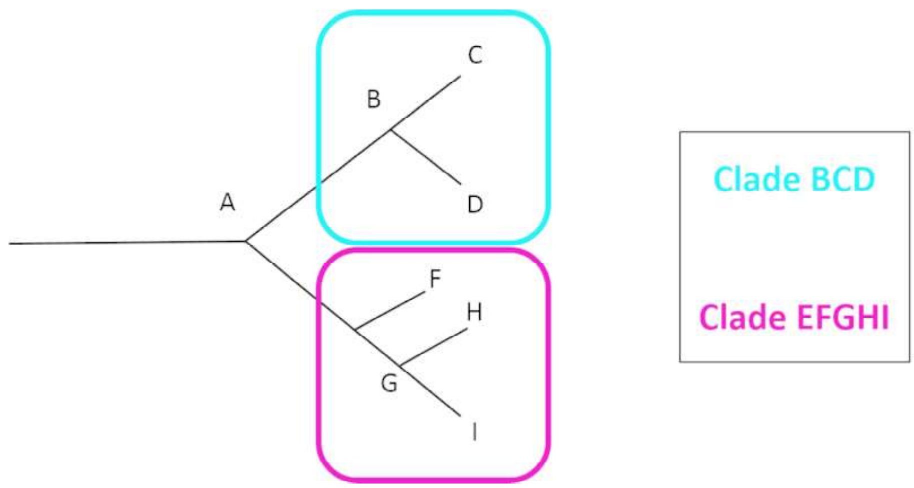


***Fig. 2.2** Clade in a Phylogenetic Tree*

In a molecular systematic analysis, the haplotypes are determined for a defined area of genetic material. For this, a substantial sample of individuals of the target species or other taxon is used, however, many current studies are also based on single individuals. Haplotypes of individuals of closely related, yet different, taxa are also determined. Finally, haplotypes from a smaller number of individuals from a definitely different taxon are determined which are referred to as an out group. The base sequences for the haplotypes are then compared. In the simplest case, the difference between two haplotypes is assessed by counting the number of locations where they have different bases referred to as the number of substitutions.

Some other types of differences between haplotypes can also occur, for example, the insertion of a section of nucleic acid in one haplotype that is not present in another. The difference between organisms is usually re-expressed as a percentage divergence, by dividing the number of substitutions by the number of base pairs analysed assuming that this measure will be independent of the location and length of the section of DNA that is being sequenced.

Once the divergences between all pairs of samples have been determined, then the resulting triangular matrix of differences is defined to some form of statistical cluster analysis. Any group of haplotypes that are all more similar to one another than any of them is to any other haplotype can constitute a clade, as shown in Figure 2.2.

**Check Your Progress**

1. What is the need of quarantine?
2. What is mineral propecting?
3. Write about the application of systematics in national defence?
4. What are digestive enzymes?
5. Define cytotaxonomy.
6. Define clade.

## 2.4 MOLECULAR PRESPECTIVE ON THE CONSERVATION OF DIVERSITY

Term biodiversity was introduced by Walter Rosen in 1986 and later E.O. Wilson in 1988 popularized the term biodiversity. Word biodiversity formed by combining two words 'Biological' and 'Diversity'. It refers to variety and variability of all life forms from all sources including terrestrial, marine, and other aquatic ecosystems present on the earth.

It has been estimated that in the world more than 50 million species of plants, animals and micro-organisms are existing. Among the animals, insects are the more species rich taxonomic group making up more than 70% of the total. Fungi species are more than the combined total of the species of the fishes, amphibians, reptiles and mammals. Each species is adapted to live in specific environment, from mountain peaks to the depth of seas, from polar ice caps to tropical rain forests and deserts. All this diversity of life is confined to only about one-kilometre-thick layer of lithosphere hydrosphere and atmosphere which form biosphere. Biodiversity is sum of all the genes, species, population, varieties in different ecosystem and their relative abundance in a particular ecosystem.

India is one of the 12 mega diverse countries of the world and shares 8.1% of global diversity. Nearly 45000 plants species from India have been recorded and animal species are twice the plant species.

Ecological concepts are our general understanding of ecosystems and how to manage them is summarized in:

**Levels of Biological Organization:** Levels of biological organization (genes, populations, species, communities, ecosystems, landscapes, regions): Plants and animals and their supporting natural systems are sustained by dynamic ecological patterns and processes at all levels of biological organization. These range from very small scale (processes shaping the life-cycle of leaves) to very large scale (climatic processes) and all are interdependent.

**Native Species:** Native plants, animals, fungi and microbes are those species which exists naturally at a given location or in a particular ecosystem, are the foundation of the natural systems that sustain biological diversity. Native species are not only displaced by human activities but also by the invasion of non-native species such as the Water Hyacinth in Yamuna River in Delhi. Non-native or alien species threaten ecosystems or species with economic or environmental harm.

**Keystone Species:** Keystone species is a species that has a disproportionately large effect on the communities in which it occurs. Such species help to maintain local biodiversity within a community either by controlling populations of other species that would otherwise dominate the community or by providing critical resources for a wide range of species. Some species like salmon and sea otters have effects on their biological communities disproportionate to their abundance and biomass. Keystone ecosystems (such as riparian areas) and keystone processes (such as wildfire and pollination by insects) are equally vital.

**Population Viability Thresholds:** Viability is the probability of survival of a population/species in the face of ecological processes, such as disturbance. Loss of habitat can reduce the survival viability of a population or species.

**Ecological Resilience:** It is the capacity of an ecosystem to cope with disturbance or stress and return to a stable state. Ecosystems can absorb disturbance or stress and remain within their natural variability. However, too much disturbance can lead to ecosystem collapse. Both functional diversity and response diversity are important to maintain ecological resilience.

**Disturbances:** Disturbances can be due to natural or human activities that cause a change in the existing condition of an ecological system. Natural events such as wildfire, flood, drought or disease outbreak or human-induced events such as urban development, deforestation, construction of dams, pollution change the existing condition of an ecosystem, and may put its survival at risk.

**Natural Range of Variability (NRV):** NRV is used to describe naturally occurring variation over time of the composition and structure found in a system, resulting in part from sequences of disturbances. The naturally occurring variation over time of the composition and structure found in an ecosystem represents the range of conditions occurring over hundreds of years prior to industrial-scale society.

**Connectivity/Fragmentation:** It is the degree to which ecosystem structure facilitates or impedes the movement of organisms between resource patches. The degree to which ecosystems are linked internally as well as to one another to form an integrated network is essential to support the movement and adaptation of species; breaks in these links through human activity can have adverse impacts on biodiversity. Humans can impact connectivity and cause fragmentation in ways that can adversely affect biodiversity. Connectivity and fragmentation are both important contributors to ecosystem function and processes.

**Principles of Biodiversity**

Ecological principles are basic assumptions (or beliefs) about ecosystems and how they function. The following ecological principles describe the assumptions needed to plan actions for conserving biodiversity:

- Protection of species and species subdivisions will support biodiversity.
- Maintaining habitat is fundamental to conserving species.
- Large areas usually contain more species than smaller areas with similar habitat.
- All things are connected but the nature and strength of the connection varies.
- Disturbances shape the characteristics of populations, communities and ecosystems.
- Climate change will increasingly influence terrestrial, freshwater and marine ecosystems.

## 2.4.1 Types of Biodiversity

There are three interrelated levels of Biodiversity: diversity within species (genetic diversity), between species (species diversity) and between ecosystems (ecosystem diversity) (Refer Figure 2.3).

**Genetic Diversity:** It is variation of genes within species. This variation could be of alleles or chromosomal structure. There is no similarity between No two individuals belonging to same species. For example, each human being show diversity from another human being.

While all species have descended from a single, common ancestor, species diverge and develop their own peculiar attributes with time, thus making their own contribution to biodiversity.

- Sources of genetic diversity
- Mutation (Point Mutations; Gene Number or Sequence; mutation rates)
- Sexual recombination
- One mutation in every 100,000 genes per generation in plants and animals but much higher in microorganisms and viruses.

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Genetic diversity has the following importance:

- It helps in speciation or evolution of new species;
- It is useful in adaptation to changes in environmental conditions;
- It is important for agricultural productivity and development.

**Species Diversity:** It is diversity of species within a community, i.e., species richness (It stands for the number and distribution of species per unit area). Species are the basic units of biological classification and thus the normal measure of biological diversity. Depending upon the varied environmental conditions the number of species in a region varies widely. This diversity is seen both in natural ecosystems and in agricultural ecosystems. Some areas are richer in species than others. The world total is estimated at five to 10 million species, though only 1.75 million have been named scientifically so far. Some areas are richer in species than others.

**Ecosystem or Community Diversity:** Diversity at the level of community or ecosystem. Ecosystem diversity can be described for a specific geographical region, or a political entity such as a country, a state or a taluka. It is the grouping and Communication of species living together and the physical environment in a given area. For example, the landscapes like grass lands, deserts, mountains etc. show ecosystem diversity.

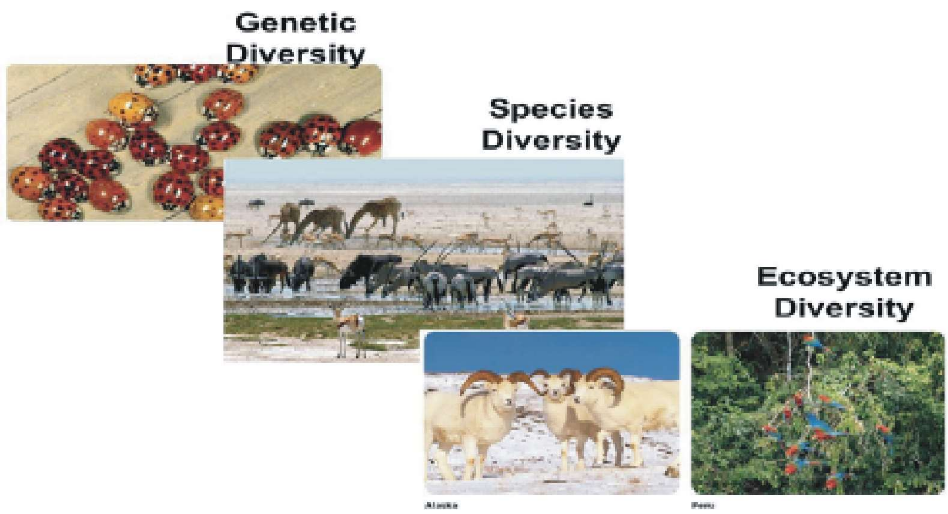


***Fig. 2.3*** *Three Types of Biodiversity*

## 2.4.2 Patterns of Biodiversity

**Latitudinal Gradients:** The diversity is not uniform, rather it is present in the form of uneven distribution. For majority of groups and animal, latitudinal gradient pattern can be observed. Usually, the diversity of species decreases as we move away from equator, towards the poles. Therefore, tropics harbor has more species

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as compared to polar or temperate areas. Colombia, located near equator, has around 1400 species, while New York at 41 degree north has 105 species, and Greenland, at 71 degree north has 56 species.

**Species Area Relationships:** Every area has different species of plants and animals. It is important to note that the relation between species richness with area of wide variety of taxa (birds, plants, freshwater fishes and bats) turns out to be rectangular hyperbola (Refer Figure 2.4).

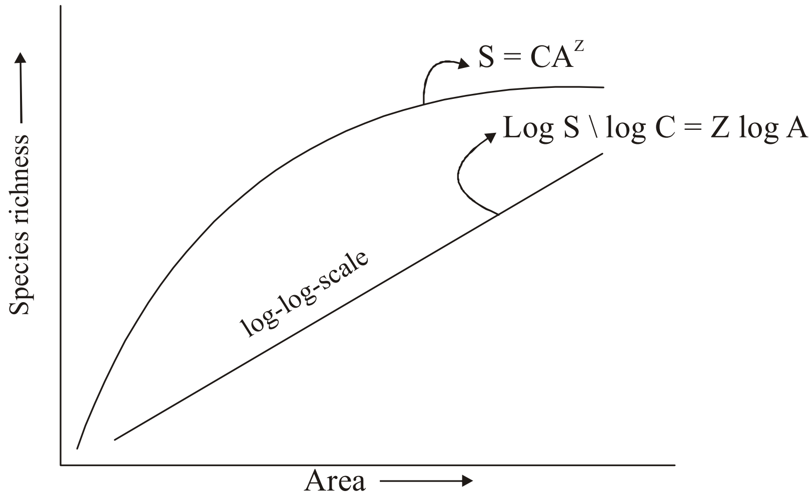


***Fig. 2.4*** *Species Area Relationship*

The above Figure 2.4 on the shows the species area relationship. It is important to note that on log scale the relationship becomes linear and is described by the following equation:

$\text{Log } S = \log C + Z \log A$

Where,

S = Species Richness

A = Area

Z = Slope of the Line (Regression Coefficient)

C = Y - Intercept

### 2.4.3 Measurement of Biodiversity

Biodiversity measured as an attribute that has two components — richness and evenness. Richness is the number of groups of genetically or functionally related individuals. Evenness is proportions of species or functional groups present on a site. Biodiversity is a measure that combines richness and evenness across species. It is often measured because high biodiversity is perceived a synonymous with ecosystem health (Refer Figure 2.5).

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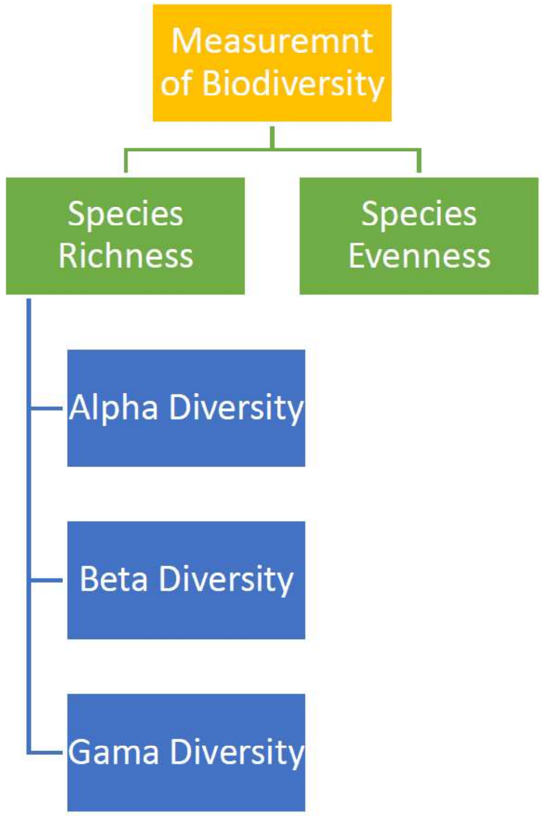


***Fig. 2.5** Biodiversity Measurement*

### Measurement of Diversity

Diversity is a single statistic in which the number of species and evenness are compounded. At its simplest level diversity can be measured by 'Species Richness'. Measures of species diversity can be divided into three categories (Magurran, 1988).

These are:

- Species Richness Indices
- Species Abundance Models
- Species Proportional Abundance Based Indices

**Species Richness**: It is the measure of number of species found in a community or that live in a certain location. In other words we can say that, species richness, as measure of diversity, has been used by ecologists. Species density or the number of species per $m^2$ is most commonly used to measure species richness. However, species richness increases with sample size.

**Alpha Diversity**: It refers to diversity within a particular area or ecosystem or community. It is usually expressed by the number of species in that ecosystem.

**Beta Diversity**: It is a comparison of diversity between ecosystems, usually measured as the change in amount of species between the ecosystems.

**Gamma Diversity:** It is a measure of the overall diversity for the different ecosystems within a region.

**Species Evenness**: It measures the proportion of species at a given site, for example low evenness indicates that a few species dominate the site.

**Rarefaction:** As the sample sizes are always unequal, Sanders technique called Rarefaction is used to cope with this difficulty**.**

Sanders's formula, as modified by Hurlbert (1971) is as follows:

$$E(s) = \sum \left\{ 1 - \left[ \binom{N - N_i}{n} \Big/ \binom{N}{n} \right] \right\}$$

Where,

*E(s)* = Expected Number of Species in the Rarefied Sample

*n* = Standardized Sample Size

*N* = Total Number of Individuals Recorded in the Sample to be Rarefied

$N_i$ = Number of Individuals in the *it*h Species in the Sample to be Rarefied

The simplest approach is to take the number of individuals in the smallest sample as the standardized sample size.

This may be explained with the help of the following example: If in one catch of fish we obtain 9 species with 23 individuals, and in another catch from the same area made for the same duration we obtained only 13 individuals belonging to 6 species, Hurlberts formula may be used to find out the number of species we would have expected in the first catch if it too had only 13 individuals. Thus, expected number of species for the first catch x is 6.6 species

***Table 2.1*** *Rarefaction with the help of Hurbert's formula (see text for details)*

| ***Species*** | ***Catch X*** | ***Catch Y*** |
|---|---|---|
| A | 9 | 1 |
| B | 3 | 0 |
| C | 0 | 1 |
| D | 4 | 0 |
| E | 2 | 0 |
| F | 1 | 0 |
| G | 1 | 1 |
| H | 0 | 2 |
| I | 1 | 0 |
| J | 0 | 5 |
| K | 1 | 3 |
| L | 1 | 0 |
| Total No. of species (S) | 9 | 6 |
| Total No. of individuals (N) | 23 | 13 |

The term $\binom{x}{y}$ is a 'combination' which is calculated as follows:

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}$$

NOTES

$x!$ is a factorial. For example $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Both these points in mind the computations can proceed

The first step is to take each species abundance from catch X and insert it in the formula

$$\left\{1-\left[\left(\frac{N-N}{n}\right)\Big/\left(\frac{N}{n}\right)\right]\right\}$$

Thus, for the species A which was represented by 9 individuals, the calculations are:

$$\left\{1-\left[\left(\frac{14!}{13!\times 1}\right)\Big/\left(\frac{23!}{13!\times 10!}\right)\right]\right\}$$

$$= \{1 - [14/1144066]\} = 1 - 0.00 = 1.00$$

The result for each species is listed and summed to give the expected species number for catch X.

| *Ni* | |
|---|---|
| 9 | 1.00 |
| 3 | 0.93 |
| 4 | 0.98 |
| 2 | 0.82 |
| 1 | 0.57 |
| 1 | 0.57 |
| 1 | 0.57 |
| 1 | 0.57 |
| 1 | 0.57 |
| Expected No. of Species for catch X | E(s) = 6.58 or 6.6 |

**Menhinick's Index (I Mn):** This index is based on the ratio of number of species (S) and the square root of the total number of individuals (N).

$$IMn = S/\sqrt{N} \text{ or } DMn = S/\sqrt{N}$$

It is claimed that this index may be used to compare samples of different sizes and that the effect of the number of individuals is reduced. However, some authors have shown that this index is not independent of sample size.

Using the data given in Table above, the value of IMn for catch x and catch y will be 1.88 and 1.66 respectively.

**Margalefs Index (IMg):** This index also relates the number of species to the number of individuals.

$$IMg = S - 1/loge\ N \text{ or } DMg= (S - 1)/\ln N$$

The index is influenced by sample size. However, some authors have demonstrated that both this and Manhinick's index are insensitive to changes in community structure.

Using the data given in Table above, the value of for sample x and sample y will be 2.55 and 1.95 respectively.

**Species Abundance Models:** No community has species of equal abundance. Some species are very abundant, others may have medium abundance and still others may be rare or represented by only a few individuals. This observation led to the development of species abundance models.

Species diversity data is frequently described by one or more patterns of distribution (Piclou, 1975), diversity is usually examined in relation to the following four models:

- The Geometric Series
- The Log Normal Distribution
- The Logarithmic Series
- The Broken Stick Model (the Random Niche Boundary Hypothesis)

When plotted on a rank abundance graph, the four models represent a progression ranging from the geometric series where a few species are dominant with the remaining fairly uncommon, through the log series and log normal distributions where species of intermediate abundance become more common and ending in the conditions represented by the broken stick model in which species are equally abundant as may be hardly observed**.**

**Species Proportional Abundance Based Indices:** These indices provide an alternative approach to the measurement of diversity. These indices are called heterogeneity indices (Peet 1974) as they take both species richness and evenness into consideration. South wood (1978) called them nonparametric indices in view of the fact that no assumptions are made about the shape of the underlying species abundance distribution.

**Diversity Indices**: It is a quantitative measure that reflects how many different types (such as species) there are in a dataset (community) and simultaneously takes into account how evenly the basic entities (individual) are distributed among those types.

The following indices are used.

- Dominance Indices / Simpson Index.
- Information – Statistic Indices / Shannon-Wiener Index.

**Simpson Index:** It takes into account both richness and evenness. It was introduced by Edward H. Simpson to measure the degree of concentration when individuals are classified into types.

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$$D = \sum (\text{n} / \text{N})2 \text{ and } D = \frac{\sum n(n-1)}{N(N-1)}$$

$n$ = Total Number of Organisms of a Particular Species

$N$ = Total Number of Organisms of All Species

The value of D ranges between 0 and 1

With this index, 0 represents infinite diversity and 1, no diversity. That is, the bigger the value of D, the lower the diversity.

**Simpson's Index of Diversity 1-D:** The value of this index starts with 1 as the lowest possible figure. This figure would represent a community containing only one species. The higher the value, the greater would be the diversity. The maximum value is the number of species in the sample. For example, if there are five species in the sample, then maximum value is 5.

**Simpson's Reciprocal Index 1/D:** The value of this index starts with 1 as the lowest possible figure. The higher the value, the greater would be the diversity.

**Shannon-Wiener Index:** This is commonly used to characterize species diversity in a community.

### Scales to Measure Biodiversity

**Alpha Diversity**: It refers to diversity within a particular area, community or ecosystem, and is usually measured by counting the number of taxa within the ecosystem (usually species level. For example in the figure below, Alpha Diversity of Site A = 7 species, Site B = 5 species, Site C = 7 species.

**Beta Diversity:** It is species diversity between ecosystems; this involves comparing the number of taxa that are unique to each of the ecosystems. In the example below, the greatest Beta Diversity is observed between Site A and C with 10 species that differ between them and only 2 species in common.

**Gamma Diversity:** It is a measure of the overall diversity for different ecosystems within a region. In this example, the gamma diversity is 3 habitats with 12 species total diversity.

**Biodiversity and Biodiversity Hotspots Biodiversity:** The variability among living organisms from all sources including terrestrial, marine and other aquatic ecosystems, and the ecological complexes.

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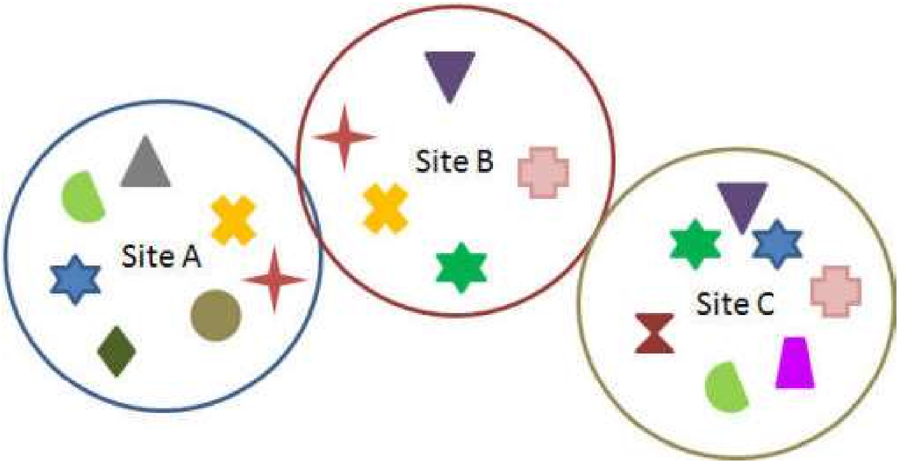


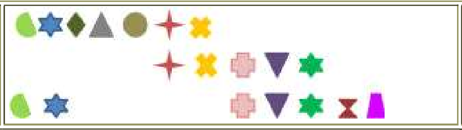

| | | |
|---|---|---|
| Site A = 7 Species<br>Site B = 5 Species<br>Site C = 7 Species |  | A vs B = 8 species<br>B vs C = 4 species<br>A vs C = 10 species |

***Fig. 2.6*** *Biodiversity and Biodiversity Hotspots Biodiversity*

## 2.4.4 Loss of Biodiversity

The Earth's biodiversity is in danger. Loss of biodiversity refers to the extinction of human, plant or animal species worldwide. It also includes the decrease in the number of a species in a certain habitat. Habitat destruction is a major cause for biodiversity loss. Habitat loss is caused by deforestation, overpopulation, pollution and global warming.

The major factors that contribute to the loss of biodiversity include the following:

**Destruction of Habitat:** The natural habitat of animals is destroyed by man for the purpose of settlement, agriculture, mining, industries, construction of highways, and so on. As a result of this, the species must either adapt to the changes in the environment or move to other places. If not, they become target to predation, starvation, disease and eventually die. Several rare butterfly species are facing extinction due to habitat destruction in the Western Ghats. Of the 370 butterfly species available in the Ghats, around 70 are at the brink of extinction.

**Hunting**: Hunting of wild animals is done for the commercial utilisation of their products. These include hides and skin, fur, meat, tusk, cosmetics, perfumes, pharmaceuticals, and decoration purposes. In recent years, 95% of the black rhino population in Africa has been exterminated by poachers for their horn.

In addition to this, over one-third of Africa's elephants have been killed in the last decade to collect 3,000 tonnes of ivory. Though the formulation of International laws and Indian regulations has reduced hunting in a large amount but poaching still continues to be a threat to biodiversity. In India, rhino is hunted for its horns, tiger for bones and skin, musk deer for musk (medicinal value), elephant for ivory, Gharial and crocodile for skin and jackal for fur trade in Kashmir. One of the most publicized commercial hunts is that on whale. Convention on International Trade in Endangered Species of Wild Fauna and Flora (CITES)

**NOTES**

listed 9 Indian animal species which have been severely depleted due to international trade.

**Exploitation of Selected Species:** Exploitation of medicinally important plants results in their disappearance from their natural habitat. Examples of the plants which are ruthlessly collected for laboratory and other works are the pitcher plant, *Nepenthes khasiana*, *Drosera* sp., *Psilotum* sp. *Isoetes* sp*,* etc.

**Habitat Fragmentation:** An 'unnatural separation of expansive tracts of habitats into spatially segregated fragments' that is too limited to maintain their different species for the future, is known as habitat fragmentation. The landmass is broken into smaller units which eventually lead to the extinction of species.

Habitat fragmentation is one of the most serious causes of erosion of biodiversity. Fragmentation leads to artificially created 'terrestrial islands'. The most serious effect of fragmentation is segregation of larger populations of a species into more than one smaller population.

**Collection for Zoo and Research:** Animals and plants are collected for zoos and biological laboratories. This is majorly done for research in science and medicine. Primates such as monkeys and chimpanzees are sacrificed for research because of their anatomical, genetic and physiological similarities to human beings.

**Introduction of Exotic Species:** A species which is not a natural inhabitant of the locality but is deliberately or accidentally introduced into the system is termed as an exotic species. Due to the introduction of exotic species, native species have to compete for food and space. There are many instances when introduction of exotic species has caused extensive damage to natural biotic community of the ecosystem. The introduction of Nile perch from north in Lake Victoria, Africa's largest lake, has driven almost half of the 400 original fish species of the lake to near extinction.

While economically useful plants are deliberately introduced a large number of exotic weeds are transferred from one locality to another accidentally. Both of these plants have spread throughout India as a pernicious weed in wheat fields. Parthenium was first observed growing on a rubbish heap in Pune in 1960. It is an aggressive plant which matures rapidly and produces thousands of seeds. The native grasses and other herbs are crowded out of existence. Water hyacinth, *Eichornia crassipes*, was introduced in 1914 in West Bengal.

**Pollution:** Pollution makes survival difficult for the species as it alters their natural habitat. Water pollution is injurious to the biotic components of coastal ecosystems. Toxic wastes entering the water bodies disturb the food chain. In addition, materials like insecticides, pesticides, sulphur and nitrogen oxides, and acid rain also adversely affect the plant and animal species. The impact of coastal pollution is also very important. It is seen that coral reefs are being threatened by pollution from industrialization, oil transport and offshore mining along the coastal areas.

Noise pollution is also the cause of wildlife extinction. This has been evidenced by the study by the Canadian Wildlife Protection Fund. According to a study, Arctic Whales are seen on the verge of extinction as a result of increasing noise

**Control of Pests and Predators:** Generally, non-target species that are a component of balanced ecosystem may also get killed in the predator and pest control measures.

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**Natural Calamities:** Floods, draught, forest fires, earth-quakes and other natural calamities sometimes take a heavy toll of plant and animal life. These trap a large number of animals while frittering away soil nutrients. Forest fires in densely wooded localities often reduce to ashes a large number of plant and animal species and so do earthquakes. Volcanic eruptions may at times completely destroy plant and animal life in its surrounding areas. Epidemics sometimes destroy large portions of a natural population.

Other Factors

Other ecological factors that contribute to the loss of biodiversity include:

- **Distribution Range**: The threat of extinction increases as the size of distribution range becomes smaller.
- **Degree of Specialization**: Specialized organisms are more vulnerable to extinction as compared to the non-specialized ones.
- **Position of the Organism in the Food Chain**: The higher the position of the organism in the food chain, the more susceptible it is.

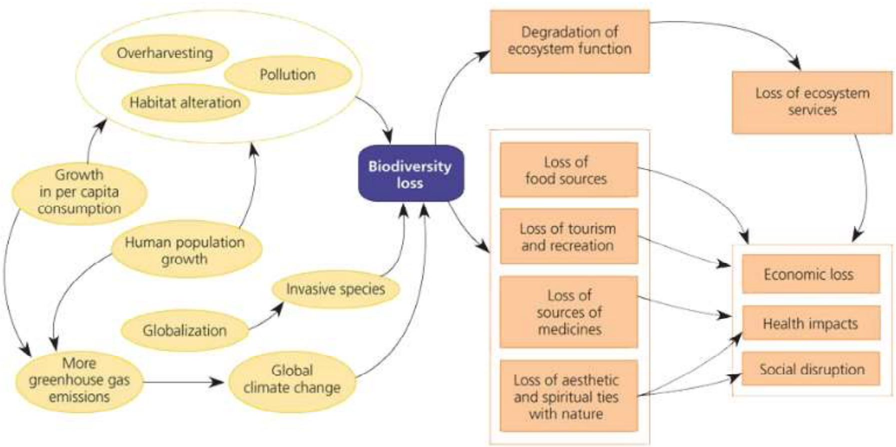


*Fig. 2.7 Biodiversity Loss: Causes and Consequences*

## 2.4.5 Effect of Biodiversity

The negative effects of the loss in biodiversity from a healthy stable state include dramatic influence on the food web and chain. Even reductions in only one species can adversely affect the entire food chain which further leads to an overall reduction in biodiversity. Reduced biodiversity leads to immediate danger for food security by reducing ecosystem services and for humankind also.

The effects of extinction of animal and plant species are widespread. Here are six significant problems caused by loss of biodiversity:

**Monetary Implication of Lost Biodiversity:** The economic cost of biodiversity around the world tops the list. We will have to pay for costs of pollination, irrigation, soil reclamation and other functions if nature is unable to take care of them. The estimated value of global biodiversity is in the trillions. Deforestation costs around $2-5 trillion annually worldwide.

**Threat to Existing Species:** The introduction of new species is happening on farms, too, where natives are pushed out because of imported foreign breeds of cattle. The effect of this is the narrowing of the world's livestock population. They are also becoming more susceptible to disease, drought, and changes in climate.

**Increased Contact with Diseases:** The loss of biodiversity has two major effects on human health and the spread of disease. Firstly, it increases the count of animals carrying disease in local populations. As habitats reduce in size, these animals become common, winning out the species that do not generally transmit disease.

**More Unpredictable Weather:** Indeed, unseasonable weather and extreme weather is a huge problem which leads to destruction and displacement. Research has shown that loss of species causes more unpredictable weather.

**Loss of Livelihoods:** Biodiversity is essential for maintaining livelihoods. Taking an example, when ocean ecosystems collapse, entire communities built on the plenty they provide lose their means of employment as well. The cause can be pollution, overfishing, or a combination of these. Humans are always affected by the downfall of the ecosystem surrounding them.

**Losing Sight of Nature:** The worth of nature to humanity is far beyond the utility of it. The physical deflation of nature certainly does affect humans. People always tend to find solace in nature. It also provides a recreation spots for us to take a break from our busy lives. But loss of biodiversity threatens to take away the value that man finds in nature.

## 2.4.6 Values of Biodiversity

Values of biodiversity can be categorized as follows: Biodiversity or biological diversity simply means the variety and variability among living organisms and the ecological complexes in which they occur. Such variety refers to the variety at the species, genetic and ecosystem level (Refer Figure 2.8).

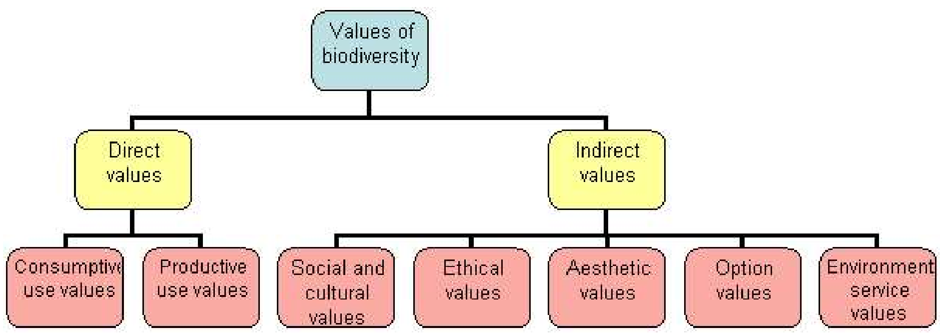


***Fig. 2.8*** *Values of Biodiversity*

**Direct Values**: The direct value include food resources like grains, vegetables, fruits which are obtained from plant resources and meat, fish, egg, milk and milk products from animal resources. These also include other values like medicine, fuel, timber, fiber, wool, wax, resin, rubber, silk and decorative items.

The direct values are of two types:

- Consumptive Use Value
- Productive Use Value

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**Consumptive Use Value:** These are the direct use values where the biodiversity products can be harvested and consumed directly. Example: Food, fuel and drugs. These goods are consumed locally and do no figure in national and international market.

**Food**

**Plants:** The most fundamental value of biological resources particularly plants is providing food. Basically three crops, i.e., wheat, maize and rice constitute more than two third of the food requirement all over the world.

**Fish:** Through the development of aquaculture, techniques, fish and fish products have become the largest source of protein in the world.

**Fuel:** Since ages forests have provided wood which is used as a fuel. Moreover fossil fuels like coal, petroleum, natural gas are also product of biodiversity which are directly consumed by humans.

**Drugs and Medicines:** The traditional medical practice like ayurveda utilizes plants or their extracts directly. In allopathy, the pharmaceutical industry is much more dependent on natural products. Many drugs are derived from plants like

- Quinine: The famous anti malaria drug is obtained from cinchona tree.
- Penicillin: A famous antibiotic is derived from pencillium, a fungus.
- Tetracycline: It is obtained from bacterium.
- Recently vinblastin and vincristine, two anti-cancer drugs have been obtained from catharanthus plant which has anti-cancer alkaloids.

**Productive Use Values:** These are the direct use values where the product is commercially sold in national and international market. Many industries are dependent upon these values. Example- Textile, leather, silk, paper and pulp industry etc. Although there is an international ban on trade of products from endangered species like tusks of elephants, wool from sheep, fur of many animals, etc. These are traded in market and fetch a booming business.

**Indirect Values**: Biodiversity provides indirect benefits to human beings which support the existence of biological life and other benefits which are difficult to quantify. These include social and cultural values, ethical values, aesthetic values, option values and environment service values.

**Social and Cultural Value:** Many plants and animals are considered holy and sacred in India and are worshipped like Tulsi, peepal, cow, snake, etc. In Indian society great cultural value is given to forest and as such tiger, peacock and lotus are named as the national animal, bird and flower respectively.

**Ethical:** These values are related to conservation of biodiversity where ethical issue of 'all life forms must be preserved' is laid down. There is an existence value which is attached to each species because biodiversity is valuable for the survival of human race. Moreover all species have a moral right to exist independent of our need for them.

**Aesthetic Value:** There is a great aesthetic value which is attached to biodiversity. Natural landscapes at undisturbed places are a delight to watch and also provide opportunities for recreational activities like bird watching, photography, etc. It promotes eco-tourism which further generates revenue by designing of zoological, botanical gardens, national parks, wild life conservation, etc.

**Option Values:** These values include the unexplored or unknown potentials of biodiversity.

**Environment Service Values:** The most important benefit of biodiversity is maintenance of environment services which includes

- Carbon dioxide fixation through photosynthesis.
- Maintaining of essential nutrients by Carbon (C), Oxygen (O), Nitrogen (N), Sulphur (S), Phosphorus (P) cycles.
- Maintaining water cycle and recharging of ground water.
- Soil formation and protection from erosion.
- Regulating climate by recycling moisture into the atmosphere.
- Detoxification and decomposition of waste.

### 2.4.7 Hotspots of Biodiversity

A biodiversity hotspot is a biogeographic region that is both a significant reservoir of biodiversity and is threatened with destruction. The term biodiversity hotspot specifically refers to 25 biologically rich areas around the world that have lost at least 70 percent of their original habitat.

**Criteria for Hotspot Selection:** To qualify as a biodiversity hotspot, a region must meet two strict criteria:

- It must have at least 1,500 vascular plants as endemics — which is to say, it must have a high percentage of plant life found nowhere else on the planet. A hotspot, in other words, is irreplaceable.
- It must have 30% or less of its original natural vegetation. In other words, it must be threatened.

Around the world, 36 areas qualify as hotspots. They represent just 2.4% of Earth's land surface, but they support more than half of the world's plant species as endemics, i.e., species found no place else — and nearly 43% of bird, mammal, reptile and amphibian species as endemics.

**Biodiversity Hotspots in India**

**I. Himalaya:** Includes the entire Indian Himalayan region (and that falling in Pakistan, Tibet, Nepal, Bhutan, China and Myanmar)

- The Eastern Himalayas is the region encompassing Bhutan, north-eastern India, and southern, central, and eastern Nepal.
- The abrupt rise of the Himalayan Mountains from less than 500 meters to more than 8,000 meters results in a diversity of ecosystems that range from alluvial grasslands and subtropical broad leaf forests along the foothills to temperate broad leaf forests in the mid hills, mixed conifer and conifer forests in the higher hills, and alpine meadows above the tree line.

**Biodiversity of the Eastern Himalayas**

- The Eastern Himalayan hotspot has nearly 163 globally threatened species (both flora and fauna) including the One-horned Rhinoceros [Vulnerable], the Wild Asian Water buffalo [Endangered].
- There are an estimated 10,000 species of plants in the Himalayas, of which one-third are endemic and found nowhere else in the world.
- A few threatened endemic bird species, such as the Himalayan Quail, Cheer pheasant, Western tragopan are found here, along with some of Asia's largest and most endangered birds, such as the Himalayan vulture and White-bellied heron.
- Mammals like the Golden langur, The Himalayan tahr, the pygmy hog, Langurs, Asiatic wild dogs, sloth bears, Gaurs, Muntjac, Sambar, Snow leopard, Black bear, Blue sheep, Takin, the Gangetic dolphin, wild water buffalo, swamp deer call the Himalayan ranged their home.

**II. Indo-Burma:** Includes entire North-eastern India, except Assam and Andaman group of Islands (and Myanmar, Thailand, Vietnam, Laos, Cambodia and southern China)

- The Indo-Burma region encompasses several countries.
- It is spread out from Eastern Bangladesh to Malaysia and includes North-Eastern India south of Brahmaputra River, Myanmar, the southern part of China's Yunnan province, Lao People's Democratic Republic, Cambodia, Vietnam and Thailand.
- The Indo-Burma region is spread over 2 million sq. km of tropical Asia.
- Since this hotspot is spread over such a large area and across several major landforms, there is a wide diversity of climate and habitat patterns in this region.

**Biodiversity of Indo-Burma Region**

- Much of this region has been deteriorating rapidly in the past few decades.
- This region is home to several primate species such as monkeys, langurs and gibbons with populations numbering only in the hundreds.
- Many of the species, especially some freshwater turtle species, are endemic.
- Almost 1,300 bird species exist in this region including the threatened white-eared night-heron [Endangered], the grey-crowned crocias (Endangered), and the orange-necked partridge (Near Threatened).
- It is estimated that there are about 13,500 plant species in this hotspot, with over half of them endemic. Ginger, for example, is native to this region.

**III. Sundalands:** Includes Nicobar group of Islands (and Indonesia, Malaysia, Singapore, Brunei, Philippines)

This region lies in South-East Asia and includes Thailand, Singapore, Indonesia, Brunei, and Malaysia. The Nicobar Islands represent India. These islands were declared as the world biosphere reserve in 2013 by United Nations. These islands have a rich terrestrial as well as marine ecosystem including mangroves, seagrass beds, and coral reefs. Species, such as dolphins, whales, turtles, crocodiles, fishes, prawns, lobsters and sea shells comprise the marine biodiversity. In case the marine resources are over-used, it can pose a serious threat to the biodiversity.

**IV. Western Ghats and Sri Lanka:** Includes entire Western Ghats and Sri Lanka.

- Western Ghats, also known as the 'Sahyadri Hills' encompasses the mountain forests in the southwestern parts of India and highlands of southwestern Sri Lanka.
- The entire extent of hotspot was originally about 1,82,500 square kms, but due to tremendous population pressure, now only 12,445 square Km or 6.8% is in pristine condition.
- The wide variation of rainfall patterns in the Western Ghats, coupled with the region's complex geography, produces a great variety of vegetation types.
- These include scrub forests in the low-lying rain shadow areas and the plains, deciduous and tropical rainforests up to about 1,500 meters, and a unique mosaic of montane forests and rolling grasslands above 1,500 meters.
- In Sri Lanka diversity includes dry evergreen forests to dipterocarpus dominated rainforests to tropical montane cloud forest.
- The important populations include Asian elephant, Niligiri tahr, Indian tigers, lion tailed macaque [All Endangered], Indian Giant squirrel [Least Concern], etc.

## 2.4.8 Importance of Biodiversity

The maintenance of biodiversity is important for the following reasons:

**Ecological Stability:** Each species performs a particular function within an ecosystem. They can capture and store energy, produce organic material, decompose organic material, help to cycle water and nutrients throughout the ecosystem, control erosion or pests, fix atmospheric gases, or help regulate climate.

Ecosystems provide support of production and services without which humans could not survive. These include soil fertility, pollinators of plants, predators, decomposition of wastes, purification of the air and water, stabilisation and moderation of the climate, decrease of flooding, drought and other environmental disasters.

Research show that the more diverse an ecosystem the better it can withstand environmental stress and the more productive it is. The loss of a species thus decreases the ability of the system to maintain itself or to recover in case of damage. There are very complex mechanisms underlying these ecological effects.

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**Economic Benefits to Humans:** For all humans, biodiversity is first a resource for daily life. Such 'Crop Diversity' is also called agro-biodiversity.

Most people see biodiversity as a reservoir of resources to be drawn upon for the manufacture of food, pharmaceutical, and cosmetic products. Thus resource shortages may be related to the erosion of the biodiversity.

Some of the important economic commodities that biodiversity supplies to humankind are:

**Food:** crops, livestock, forestry, and fish.

**Medication:** Wild plant species have been used for medicinal purposes since before the beginning of recorded history. For example, quinine (Used to treat malaria) comes from the bark of the Amazonian tree Cinchona tree; digitalis from the Foxglove plant (chronic heart trouble), and morphine from the Poppy plant (pain relief).

According the National Cancer Institute of the USA, over 70 % of the promising anti-cancer drugs come from plants in the tropical rainforests. Animal may also play a role, in particular in research. It is estimated that of the 250,000 known plant species, only 5,000 have been researched for possible medical applications.

**Industry:** Fibres for clothing, wood for shelter and warmth. Biodiversity may be a source of energy (such as biomass). Other industrial products are oils, lubricants, perfumes, fragrances, dyes, paper, waxes, rubber, latexes, resins, poisons and cork can all be derived from various plant species. Supplies from animal origin are wool, silk, fur, leather, lubricants, waxes. Animals may also be used as a mode of transportation.

**Tourism and Recreation:** Biodiversity is a source of economical wealth for many areas, such as many parks and forests, where wild nature and animals are a source of beauty and joy for many people. Ecotourism in particular, is a growing outdoor recreational activity.

### Ethical Reasons

The role of biodiversity is to be a mirror of our relationships with the other living species, an ethical view with rights, duties, and education. If humans consider species have a right to exist, they cannot cause voluntarily their extinction. Besides, biodiversity is also part of many cultures' spiritual heritage.

## 2.4.9 Conservation of Biodiversity

Biodiversity conservation is very essential, following are the objectives, advantages and measures taken to conserve the biodiversity (Refer Figure 2.9).

### Objectives and Advantages of Conservation

- Conservation of biological diversity leads to conservation of essential ecological diversity to preserve the continuity of food chains.

- The genetic diversity of plants and animals is preserved.
- It ensures the sustainable utilisation of life support systems on earth.
- It provides a vast knowledge of potential use to the scientific community.
- A reservoir of wild animals and plants is preserved, thus enabling them to be introduced, if need be, in the surrounding areas.
- Biological diversity provides immediate benefits to the society, such as recreation and tourism.
- Biodiversity conservation serves as an insurance policy for the future.

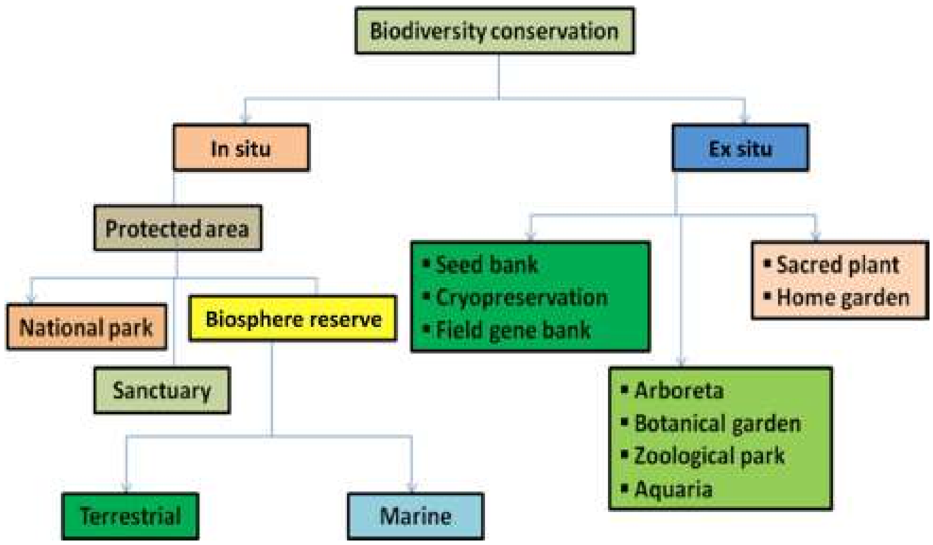


***Fig. 2.9*** *Biodiversity Conservation*

### Convention on Biological Diversity

The aim of the convention is to save species and plants from extinction and their habitats from destruction. The developed countries are looking for a sustainable supply of biological resources from the developing countries and easy access to them as well. The developing countries lacking the technology to exploit their resources are inviting the developed countries to do so. This has resulted in the developed nations channeling out the benefits of these natural resources. The developing countries are now demanding a higher share of the accrued economic benefits. The developed nations are also concerned by the unsustainable exploitation of natural wealth, particularly rainforests.

### Key Points from the Convention on Biological Diversity

The aim of the Convention on Biological Diversity is 'The conservation of biological diversity, the sustainable use of its components and the fair and equitable sharing of the benefits arising out of the utilization of genetic resources. The convention stipulates that Parties must:

- Develop national strategies for the conservation and sustainable use of biological resources.
- Establish protected areas, restore degraded ecosystems, control alien species, and establish ex-situ conservation facilities.

- Establish training and research programmes for the conservation and sustainable use of biodiversity and support such programmes in developing countries.
- Promote public education and awareness of the conservation and sustainable use of biodiversity.
- Recognize the right of governments to regulate access to their own genetic resources, and, wherever possible, grant other Parties access to genetic resources for environmentally sound uses.
- Encourage technology and biotechnology transfer particularly to developing countries.
- Establish an information exchange between the parties on all subjects relevant to biodiversity.
- Promote technical and scientific cooperation between parties (particularly to developing countries) to enable them to implement the convention.
- Ensure that countries that provide genetic resources have access to the benefits arising from them.
- Provide financial resources to developing countries/parties to enable them to carry out the requirements of the convention.

**Biodiversity of India**

As per available data, the varieties of species living on the earth are 1753739. Out of the above species, 134781 are residing in India although surface area of India is 2% of the earth's surface. Wild life Institute of India has divided it into ten biogeographical regions and twenty five biotic provinces.

**Biogeographical Regions are:**

- Trans Himalayas
- Gangetic Plain
- Desert
- Semiarid Zone
- Western Ghats
- Deccan Peninsula
- North Eastern Zone
- Coastal Lands
- Himalayas
- Islands

India is one of the twelve mega diversity nations of the world due to the following reasons:

- It has 7.3% of the global fauna and 10.88% of global flora as per the data collected by Ministry of Environment and forest.
- It has 350 different mammals, 1200 species of birds- 453 different reptiles, 182 amphibians and 45,000 plants spices.

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- It has 50,000 known species of insects which include 13,000 butterflies and moths.
- It has 10 different biogeographical regions and 25 biotic provinces having varieties of lands and species.
- In addition to geographical distribution, geological events in the land mass provide high level of biological diversity.
- Several crops arose in the country and spread throughout the world.
- There is wide variety of domestic animals like cows, buffaloes, goats, sheep, pigs, horses, etc.
- The marine biota includes sea weeds, fishes, crustaceans, molluscs, corals, reptiles, etc.
- There are a number of hot spots (namely Eastern Ghats, Western Ghats, North Eastern hills, etc.).

**Conservation of Biodiversity**

Biodiversity is being depleted by the loss of habitat, fragmentation of habitat, over exploitation of resources, human sponsored ecosystems, climatic changes, pollution invasive exotic spices, diseases, shifting cultivation, poaching of wild life, etc.

Since the human beings are enjoying all the benefits from biodiversity, they should take proper care for the preservation of biodiversity in all its form and good health for the future generation, i.e., the human being should prevent the degradation and destruction of the habitats thereby maintaining the biodiversity at its optimum level.

Conservation of biodiversity is protection, upliftment and scientific management of biodiversity so as to maintain it at its threshold level and derive sustainable benefits for the present and future generation. In other words, conservation of bio-diversity is the proper management of the biosphere by human beings in such a way that it gives maximum benefits for the present generation and also develops its potential so as to meet the needs of the future generations.

Mainly the conservation of biodiversity has three basic objectives:

- To maintain essential ecological processes and life supporting systems.
- To preserve the diversity of species.
- To make sustainable utilisation of species and ecosystems.

## 2.4.10 Strategies for Conservation of Biodiversity

The following strategies should be undertaken in order to conserve biodiversity:

- All the possible varieties (old or new) of food, forage and timber plants, livestock, agriculture animals and microbes should be conserved.
- All the economically important organisms in protected areas should be identified and conserved.
- Critical habitats for each species should be identified and safeguarded.
- Priority should be given to preserve unique ecosystems.

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- There should be sustainable utilisation of resources.
- International trade in wild life should be highly regulated.
- The poaching and hunting of wildlife should be prevented as far as practicable.
- Care should be taken for the development of reserves and protected areas.
- Efforts should be made to reduce the level of pollutants in the environment.
- Public awareness should be created regarding biodiversity and its importance for the living organisms.
- Priority should be given in wildlife conservation programme to endangered species over vulnerable species and to vulnerable species over rare species.
- The habitats of migratory birds should be protected by bilateral and multilateral agreement.
- The over exploitation of useful products of wild life should be prevented.
- The useful animals, plants and their wild relatives should be protected both in their natural habitat (in-situ) and in zoological botanical gardens (ex-situ)
- Efforts should be made for setting up of National parks and wild life sanctuaries to safeguard the genetic diversity and their continuing evolution.
- Environmental laws should be strictly followed.

### Conservation Methods

**In-Situ Biodiversity Conservation**: In-situ conservation means the conservation of species within their natural habitats, this way of conserving biodiversity is the most appropriate method for biodiversity conservation. In this strategy you have to find out the area with high biodiversity means the area in which number of plants and animals are present. After that this high biodiversity area should be covered in the form of natural park/ sanctuary/ biosphere reserve etc. In this way biodiversity can be conserve in their natural habitat from human activities.

The different advantages of in-situ conservation are as follows:

- It is a cheap and convenient way of conserving biological diversity.
- It offers a way to preserve a large number of organisms simultaneously, known or unknown to us.
- The existence in natural ecosystem provides opportunity to the living organisms to adjust to differed' environmental conditions and to evolve in to a better life form.

The only disadvantage of in-situ conservation is that it requires large space of earth which is often difficult because of growing demand for space. The protection and management of biodiversity through in situ conservation involve certain specific areas known as protected areas which include national parks, Sanctuaries and Biosphere reserves.

**Protected Areas:** The protected areas are biogeographical areas where biological diversity along with natural and cultural resources are protected, maintained and managed through legal and administrative measures. The demarcation of biodiversity in each area is determined on the basis of climatic and physiological conditions.

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In these areas, hunting, firewood collection, timber harvesting etc. are prohibited so that the wild plants and animals can grow and multiply freely without any hindrance. Some protected areas are: Cold desert (Ladakh and Spiti), Hot desert (Thar), Saline Swampy area (Sunderban and Rann of Kutch), Tropical moist deciduous forest (Western Ghats and north East) etc. Protected areas include national parks, sanctuaries and biosphere reserves. There are 37,000 protected areas throughout the world. As per World Conservation Monitoring Centre, India has 581 protected areas, national parks and sanctuaries.

**National Parks:** These are the small reserves meant for the protection of wild life and their natural habitats. These are maintained by government. The area of national parks ranges 0.04 to 3162 km. The boundaries are well demarcated and circumscribed. The activities like grazing forestry, cultivation and habitat manipulation are not permitted in these areas. There are about 89 national parks in India.

Some important national Parks of India are:

- Biological Park, Nandankanan, Orissa
- Corbett National Park Nainital, Uttarakhand (First National Park)
- Koziranga national Park, Jorhat, Assam
- Tudula National Park, Maharashtra
- Hazaribagh National Park, Hazaribagh, Bihar
- Band havgarh National Park, M.P.
- Bandipur National Park, Karnataka
- Kanha National Park, M.P.
- Reibul Lamjao National Park, Manipur
- Nawgaon National Park, Maharashtra

**Sanctuaries:** These are the areas where only wild animals (fauna) are present. The activities like harvesting of timbers, collection of forest products, cultivation of lands etc. are permitted as long as these do not interfere with the project. That is, controlled biotic interference is permitted in sanctuaries, which allows visiting of tourists for recreation. The area under a sanctuary remains in between 0.61 to 7818 km.

Some Important Sanctuaries of Orissa are as follows:

- Nandankanan Zoological Park
- Chandaka Elephant reserve
- Simlipal Tiger Reserve
- Bhitarkanika Wild life Sanctuary
- Gharial project at Tikarpada
- Chilika (Nalaban) Sanctuary

**Biosphere Reserves:** Biosphere reserves or natural reserves are multipurpose protected areas with boundaries circumscribed by legislation. The main aim of biosphere reserve is to preserve genetic diversity in representative ecosystems by protecting wild animals, traditional life style of inhabitant and domesticated plant/

animal genetic resources. These are scientifically managed allowing only the tourists to visit.

Some importance of biosphere reserves are as follows:

- These help in the restoration of degraded ecosystem.
- The main role of these reserves is to preserve genetic resources, species, ecosystems, and habitats without disturbing the habitants.
- These maintain cultural, social and ecologically sustainable economic developments.
- These support education and research in various ecological aspects.

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**Advantages of In-Situ Conservation**

- The flora and fauna live in natural habitats without human interference.
- The life cycles of the organisms and their evolution progresses in a natural way.
- In-situ conservation provides the required green cover and its associated benefits to our environment.
- It is less expensive and easy to manage.
- The interests of the indigenous people are also protected.

**Ex-Situ Conservation Methods**: Ex-situ conservation involves the conservation of biological diversity outside of their natural habitats. This involves conservation of genetic resources, as well as wild and cultivated or species, and draws on a diverse body of techniques and facilities.

**Some Important Areas under these Conservation are:**

- **Seed Gene Bank:** These are cold storages where seeds are kept under controlled temperature and humidity for storage and this is easiest way to store the germ plasma of plants at low temperature. Seeds preserved under controlled conditions (minus temperature) remain viable for long durations of time.
- **Field Gene Bank:** Genetic variability also is preserved by gene bank under normal growing conditions. These are cold storages where germ plam are kept under controlled temperature and humidity for storage; this is an important way of preserving the genetic resources.
- **Botanical Gardens**: A botanical garden is a place where flowers, fruits and vegetables are grown. The botanical gardens provide beauty and calm environment. Most of them have started keeping exotic plants for educational and research purposes.
- **Zoos:** In zoos wild animals are maintained in captivity and conservation of wild animals (rare, endangered species). The oldest zoo, the Schonbrumm zoo which exists today also, was established in VIENNA in 1759.

In India, the 1st zoo came into existence at Barrackpore in 1800. In world there are about 800 zoos. Such zoos have about 3000 species of vertebrates. Some zoos have undertaken captive breeding programmes.

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- **Cryopreservation:** This is the newest application of technology for preservation of biotic parts. This type of conservation is done at very low temperature (196°C) in liquid nitrogen. The metabolic activities of the organisms are suspended under low temperature, which are later used for research purposes.
- **Tissue Culture Bank:** Cryopreservation of disease free meristems is very helpful. Long term culture of excised roots and shoots are maintained. Meristem culture is very popular in plant propagation as it is a virus and disease free method of multiplication.
- **Long Term Captive Breeding:** The method involves capture, maintenance and captive breeding on long term basis of individuals of the endangered species which have lost their habitat permanently or certain highly unfavourable conditions are present in their habitat.
- **Animal Translocation:** Release of animals in a new locality which come from anywhere else.

Translocation is carried in following cases:

- When a species on which an animal is dependent becomes rare.
- When a species is endemic or restricted to a particular area.
- Due to habit destruction and unfavourable environment conditions.
- Increase in population in an area.

The strategies for ex-situ conservations are:

- Identification of species to be conserved.
- Adoption of Different ex-situ methods of conservation.
- Long-term captive breeding and propagation for the species which have lost their habitats permanently.
- Short-term propagation and release of the animals in their natural habitat
- Animal translocation
- Animal reintroduction
- Advanced technology in the service of endangered species.

The different advantages of ex-situ conservation are:

- It gives longer life time and breeding activity to animals.
- Genetic techniques can be utilised in the process.
- Captivity breed species can again be reintroduced in the wild.
- It is useful for declining population of species.
- Endangered animals on the verge of extinction are successfully breeded.
- Threatened species are breeded in captivity and then released in the natural habitats.
- Ex-situ centres offer the possibilities of observing wild animals, which is otherwise not possible.

- It is extremely useful for conducting research and scientific work on different species.

Some disadvantages of this method are:

- The favourable conditions may not be maintained always.
- Mew life forms cannot evolve.
- This technique involves only few species.

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**Man and Biosphere Programme (MAB Programme)**

- It was first started by UNESCO in 1971.
- Later introduced in India in 1986.

**Aim**

- Studying the effects of human interference and pollution on the biotic and abiotic components of ecosystems.
- Conservation the ecosystems for the present as well as future.

The main objects of MAB programme are to:

- Conserve representative samples of ecosystem.
- Provide long term in situ conservation of genetic diversity.
- Provide opportunities for education and training.
- Provide appropriate sustainable managements of the living resources.
- Promote infer national co-operation.

**Citizen Movements to Conserve Biodiversity**

Following are some of the citizen movements to conserve biodiversity:

**Chipko Movement**

- It is a social-ecological movement that practiced the Gandhian methods of satyagraha and non-violent resistance, through the act of hugging trees to protect them from falling.
- The modern Chipko movement started in the early 1970s in the Garhwal Himalayas of Uttarakhand, with growing awareness towards rapid deforestation.
- The landmark event in this struggle took place on March 26, 1974, when a group of peasant women in Reni village, Hemwalghati, in Chamoli district, Uttarakhand, India, acted to prevent the cutting of trees and reclaim their traditional forest rights that were threatened by the contractor system of the state Forest Department.
- Their actions inspired hundreds of such actions at the grassroots level throughout the region.
- By the 1980s the movement had spread throughout India and led to formulation of people-sensitive forest policies, which put a stop to the open felling of trees in regions as far reaching as Vindhyas and the Western Ghats.

- The first recorded event of Chipko however, took place in village Khejarli, Jodhpur district, in 1730 AD, when 363 Bishnois, led by Amrita Devi sacrificed their lives while protecting green Khejri trees, considered sacred by the community, by hugging them, and braved the axes of loggers sent by the local ruler, today it is seen an inspiration and a precursor for Chipko movement of Garhwal.

**Appiko Movement**

- Appiko movement was a revolutionary movement based on environmental conservation in India.
- The Chipko movement in Uttarakhand in the Himalayas inspired the villagers of the district of Karnataka province in southern India to launch a similar movement to save their forests.
- In September 1983, men, women and children of Salkani 'hugged the trees' in Kalase forest. (The local term for 'hugging' in Kannada is appiko.)
- Appiko movement gave birth to a new awareness all over southern India.

**Projects to Save Threatened Species**

**Project Tiger:** Project Tiger was initiated as a Central Sector Scheme in 1973 with 9 tiger reserves located in different habitat types in 9 different states. There are totally 18 Reserves in 13 states. At present tiger Conservation has been viewed in India not only as an effort to save an endangered species but, with equal importance, also as a means of preserving biotypes of sizeable magnitude.

**Crocodile Breeding Project:** The project was started in Orissa and then extended to several other states in April 1975 with UNDP assistance. The main objective was to protect the three endangered species of crocodiles namely - *Gavialis gangeticus, Crocodylus palustris* and the salt water crocodile, *Crocodylus porosus*.

**Lesser Cats Project:** The project was launched in 1976 with the assistance of WWF in India for conservation of four species of lesser cats, for example *Felis bengalensis Kerr, Felis marmorta Martin, Felis lemruinki Vigors Horsfield* and *Felis viverrina Bennet*, found in Sikkim and Northern part of West Bengal.

**The Manipur Brow-Antlered Deer Project**

This was launched in 1981 in Manipur to save the brow-antlered deer (*Cerevus eldi eldi*) which is on the verge of extinction. The habitat includes 35 sq.km. of park and sanctuary. The population of the deer has increased from 18 to 27.

**Project Elephant:** It was launched in 1991 to protect the Asiatic elephant which is also a highly endangered species because of large scale poaching.

**Project Rhino:** It was launched in 1987 in Kaziranga Wildlife Sanctuary in Assam to save the lesser one horned rhinoceros from extinction. It covers an area of 430 sq.km. and is the natural of the dwindling rhino.

**Himalayan Musk Deer Project:** This was launched in 1981 to save the endangered musk deer which is facing extinction. Captive breeding has yielded good results.

**Project Hangul:** This project was launched in 1970 in Kashmir valley to save the highly endangered Kashmir stag (*Cerevus elaphus hanglu*) which is facing extinction. As a result their population has increased levels.

**Check Your Progress**

7. What is a native species?
8. Define keystone species.
9. What is species richness?
10. Define habitat fragmentation.

## 2.5 WILDLIFE AND ITS CONSERVATION

Traditionally, the term 'Wildlife' refers to undomesticated animal species, but has come to include all organisms that grow or live wild in an area without being introduced by humans. Wildlife can be found in all ecosystems, such as the deserts, forests, rain forests, plains, grasslands and other areas including the most developed urban areas, all have distinct forms of wildlife. Though the term usually refers to animals that are untouched by human factors, but it is a fact that wildlife is affected by human activities. Some animals, however, have adapted to suburban environments, for example the domesticated cats, dogs, mice, and gerbils. The global wildlife population decreased by 52 percent between 1970 and 2014, according to a report by the World Wildlife Fund (WWF).

Some of the wildlife values cannot or can only just be quantified as aesthetic, educational, ecological or ethical values. The non-consumptive use of wildlife is mostly based on the aesthetic value of wildlife. Wildlife becomes the support of the tourism industry, as beaches are the support of the seaside tourism industry. Many nations have established their tourism sector around their natural wildlife. South Africa has, for example, many opportunities for tourists to see the country's wildlife in its national parks, such as the Kruger Park. In South India, the Periar Wildlife Sanctuary, Bandipur National Park and Mudumalai Wildlife Sanctuary are situated around and in forests. India is home to many national parks and wildlife sanctuaries showing the diversity of its wildlife, much of its unique fauna, and excels in the range. There are 89 national parks, 13 bio reserves and more than 400 wildlife sanctuaries across India which are the best places to go to see Bengal tigers, Asiatic lions, Indian elephants, Indian rhinoceroses, birds, and other wildlife which reflect the importance that the country places on nature and wildlife conservation. Figure 2.10 illustrates a tiger, *Panthera tigris*.

***Fig. 2.10*** *Tiger, Panthera tigris*

Many animal species have spiritual significance in different cultures around the world, and they and their products may be used as sacred objects in religious rituals. For example, eagles, hawks and their feathers have great cultural and spiritual value to Native Americans as religious objects. In Hinduism the cow is regarded sacred.

The loss of animals from ecological communities is termed as **defaunation**.

The rate of extinctions of entire species of plants and animals across the planet has been so high in the last few hundred years and so it is widely believed that we are in the sixth great extinction event on this planet, termed as the 'Holocene Mass Extinction'. The four most general reasons that lead to destruction of wildlife include overkill, habitat destruction and fragmentation, impact of introduced species and chains of extinction.

### Habitat Destruction and Fragmentation

The habitat of any given species is considered its preferred area or territory. Many processes associated with human habitation of an area cause loss of this area and decrease the carrying capacity of the land for that species. In some conditions, these changes in land can cause a patchy break-up of the wild landscape. Agricultural land frequently displays this type of extremely fragmented habitat. Farms spread across the landscape with patches of uncleared woodland or forest dotted in-between occasional paddocks.

Examples of habitat destruction include grazing of bushland by farmed animals, changes to natural fire regimes, forest clearing for timber production and wetland draining for city expansion.

### Impact of Introduced Species

Mice, cats, rabbits, dandelions and poison ivy are all examples of species that have become invasive threats to wild species in various parts of the world. Habitually, species that are uncommon in their home range become out-of-control invasions in distant but similar climates. The reasons for this have not always been clear and Charles Darwin defined that it was unlikely that the exotic species would ever be able to grow abundantly in a place in which they had not evolved. The reality is that the vast majority of species exposed to a new habitat do not reproduce successfully. Occasionally, however, some populations do take hold and after a period of acclimation can increase in numbers significantly, having destructive effects on many elements of the native environment of which they have become part.

### Chains of Extinction

All wild populations of living things have many complex intertwining links with other living things that are in their surroundings. Large herbivorous animals, such as the hippopotamus have populations of insectivorous birds that feed off the many parasitic insects that grow on the hippo.

Should the hippo die out, so too will these groups of birds, leading to further destruction as other species dependent on the birds are affected. Also referred to as a domino effect, this series of chain reactions is by far the most destructive process that can occur in any ecological community.

Another example is the bird black drongo (Refer Figure 2.11) and the cattle egrets (Refer Figure 2.12) found in India. These birds feed on insects on the back of cattle, which helps to keep them disease-free. Destroying the nesting habitats of these birds would cause a decrease in the cattle population because of the spread of insect-borne diseases.

Figure 2.11 illustrates the Black Drongo (*Dicrurus macrocercus*), also known as the King Crow, is a small Asian passerine bird of the drongo family Dicruridae. Figure 2.12 illustrates Cattle Egret (*Bubulcus ibis*) is a cosmopolitan species of heron (family Ardeidae) found in the tropics, subtropics, and warm temperate zones. It is the only member of the monotypic genus Bubulcus, although two of its subspecies are refereed as full species, namely the Western Cattle Egret and the Eastern Cattle Egret.

***Fig. 2.11*** *The Black Drongo (Dicrurus macrocercus)*

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***Fig. 2.12*** *Cattle Egret (Bubulcus ibis)*

## 2.5.1 Present Status of Wildlife in India

India is home to a variety of animals. Apart from a handful of domesticated animals, such as cows, water buffaloes, goats, chickens, and both Bactrian and Dromedary camels, India has a wide variety of animals native to the country. It is home to Bengal and Indochinese tigers, Asiatic lions, Indian and Indochinese leopards, snow leopards, clouded leopards, various species of Deer, including Chital, Hangul, Barasingha, the Indian Elephant, the Great Indian Rhinoceros, and many others. The region's diverse wildlife is preserved in more than 120 National Parks, 18 Bio-Reserves and more than 500 Wildlife Sanctuaries across the country. India has some of the most biodiverse regions of the world and contains four of the world's 36 biodiversity hotspots – the Western Ghats, the Eastern Himalayas, Indo-Burma and Sunda Land. Wildlife management is essential to preserve the rare and endangered endemic species. India is one of the seventeen megadiverse countries. According to one study, India along with the other 16 megadiverse countries is home to about 60-70% of the world's biodiversity. India, lying within the Indomalaya Ecozone, is home to about 7.6% of all Mammalian, 12.6% of Avian (Bird), 6.2% of Reptilian, and 6.0% of Flowering Plant Species.

Many Indian species are descendants of taxa originating in Gondwana, of which India originally was a part. Peninsular India's subsequent movement towards, and collision with, the Laurasian landmass set off a mass exchange of species. However, volcanism and climatic change 20 million years ago caused the extinction of many endemic Indian forms. Soon thereafter, mammals entered India from Asia through two zoogeographical passes on either side of the emerging Himalaya. As a result, among Indian species, only 12.6% of Mammals and 4.5% of Birds are Endemic, contrasting with 45.8% of Reptiles and 55.8% of Amphibians. Notable endemics are the Nilgiri leaf monkey and the brown and carmine Beddome's toad of the Western Ghats. India contains 172, or 2.9%, of IUCN (International Union for Conservation of Nature) designated threatened species. These include the

Asian Elephant, The Asiatic Lion, Bengal Tiger, Indian Rhinoceros, Mugger Crocodile, and Indian White Rumped Vulture, which suffered a near-extinction from ingesting the carrion of diclofenac-treated cattle.

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In recent decades, human encroachment has posed a threat to India's wildlife, in response, the system of national parks and protected areas, first established in 1935, was substantially expanded. In 1972, India enacted the Wildlife Protection Act and Project Tiger to safeguard crucial habitat, further federal protections were promulgated in the 1980s. Along with over 515 wildlife sanctuaries, India now hosts 18 biosphere reserves, 10 of which are part of the World Network of Biosphere Reserves; 26 wetlands are registered under the Ramsar Convention.

### 2.5.2 Fauna of India

India is home to several well-known large mammals, including the Asian Elephants, Bengal and Indochinese Tigers, Asiatic Lions, Snow Leopards, Clouded Leopards, Indian Leopards, Indian Sloth Bear and Indian Rhinoceros. Some other well-known large Indian mammals are the rare Wild Asian Water Buffalo, common domestic Asian Water Buffalo, Gail, Gaur, and several species of Deer and Antelope. Some members of the dog family, such as the Indian Wolf, Bengal Fox and Golden Jackal, and the dhole or wild dogs are also widely distributed. However, the Dhole, also known as the Whistling Hunter, is the most endangered top Indian carnivore, and the Himalayan Wolf is now a critically endangered species endemic to India. It is also home to the striped Hyena, Macaques, Langur and Mongoose species. Figure 2.13 illustrates a female Indian Elephant in the National Park. India has the largest population of this subspecies of Asian elephants.

***Fig. 2.13*** *Female Indian Elephant in the National Park*

### 2.5.3 Conservation Efforts

The need for conservation of wildlife in India is often questioned because of the apparently incorrect priority in the face of direct poverty of the people. However, Article 48 of the Constitution of India specifies that, 'The state shall endeavour to protect and improve the environment and to safeguard the forests and wildlife of the country' and Article 51-A states that 'It shall be the duty of every citizen of

India to protect and improve the natural environment including forests, lakes, rivers, and wildlife and to have compassion for living creatures.' The committee in the Indian Board for Wildlife, in their report, defines wildlife as 'The entire natural uncultivated flora and fauna of the country' while the Wildlife (protection) Act 1972 defines it as 'Any Animal, Bees, Butterflies, Crustacea, Fish, Moths and Aquatic or Land Vegetation which forms part of any habitat.'

Despite the various environmental issues faced, the country still has a rich and varied wildlife compared to Europe. Large and charismatic mammals are important for wildlife tourism in India, and several national parks and wildlife sanctuaries cater to these needs. Project Tiger, started in 1972, is a major effort to conserve the tiger and its habitats. At the turn of the 20th century, one estimate of the tiger population in India placed the figure at 40,000, yet an Indian tiger census conducted in 2008 revealed the existence of only 1,411 tigers. 2010 tiger census revealed that there are 1700 tigers left in India. As per the latest tiger census (2015), there are around 2226 tigers in India. By far, there is an overall 30% increase in tiger population. Various pressures in the later part of the 20th century led to the progressive decline of wilderness resulting in the disturbance of viable tiger habitats. At the International Union for the Conservation of Nature and Natural Resources (IUCN) General Assembly meeting in Delhi in 1969, serious concern was voiced about the threat to several species of wildlife and the shrinkage of wilderness in India. In 1970, a national ban on tiger hunting was imposed, and in 1972 the Wildlife Protection Act came into force. The framework was then set up to formulate a project for tiger conservation with an ecological approach. However, there is not much optimism about this framework's ability to save the peacock, which is the national bird of India.

The exploitation of land and forest resources by humans along with capturing and trapping for food and sport has led to the extinction of many species in India in recent times. These species include mammals, such as the Asiatic Cheetah, Wild Zebu, Indian Javan Rhinoceros, and Northern Sumatran Rhinoceros. While some of these large mammal species are confirmed extinct, there have been many smaller animal and plant species whose status is harder to determine. Many species have not been seen since their description. Gir forest in India has the only surviving population of Asiatic lions in the world. In the late 1960s there were only about 180 Asiatic lions. There were 523 Asiatic lions in the Gir sanctuary in Gujarat state which in 2018 increased to more than 600.

Some species of birds have gone extinct in recent times, including the Pink-Headed Duck (*Rhodonessa caryophyllacea*) and the Himalayan Quail (*Ophrysia superciliosa*). A species of Warbler, *Acrocephalus orinus*, known earlier from a single specimen collected by Allan Octavian Hume from near Rampur in Himachal Pradesh, was rediscovered after 139 years in Thailand.

### 2.5.4 Bio-Reserves

The Indian government has established eighteen biosphere reserves of India which protect larger areas of natural habitat and often include one or more national parks and/or preserves, along buffer zones that are open to some economic uses. Protection is granted not only to the flora and fauna of the protected region, but also to the human communities who inhabit these regions, and their ways of life.

The bio-reserves are:

- Achanakmar-Amarkantak
- Agasthyamalai
- Dibru Saikhowa
- Dihang Dibang
- Great Nicobar
- Gulf of Mannar
- Kachchh
- Khangchendzonga
- Manas
- Nanda Devi
- The Nilgiris
- Nokrek
- Pachmarhi
- Simlipal
- Sundarbans
- Cold Desert
- Seshachalam hills
- Panna

Eleven of the eighteen biosphere reserves are a part of the World Network of Biosphere Reserves, based on the UNESCO Man and the Biosphere Programme (MAB) list.

- Gulf of Mannar Biosphere Reserve
- Nanda Devi Biosphere Reserve
- Nilgiri Biosphere Reserve
- Nokrek National Park
- Pachmarhi Biosphere Reserve
- Simlipal National Park
- Sundarbans Biosphere Reserve

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- Achanakmar-Amarkantak Biosphere Reserve
- Nicobar Islands
- Agasthyamala Biosphere Reserve
- Khangchendzonga

**Wildlife Protection Act**

Fast-track forest clearances that are an indicator of India's development also pose a question–what is the future of wildlife in India?

***Fig. 2.14*** *Rhinoceros*

Wildlife conservation in India has a long history, dating back to the colonial period when it was rather very restrictive to only targeted species and that too in a defined geographical area. Then, the formation of the Wildlife Board at the national level and enactment of Wildlife Act in 1972 laid the foundation of present day 'Wildlife Conservation' era in post-independent India. Henceforth, the Act has been amended several times and the National Wildlife Advisory Board has undergone various changes.

Project Tiger in the 1970s and the Project Elephant in 1992–both with flagship species–attracted global attention. India then also became a member of all major international conservation treaties related to habitat, species and environment (like Ramsar Convention, 1971; Convention on International Trade in Endangered Species of Wild Fauna and Flora, 1973; Convention on Migratory Species, 1979; Convention on Biological Diversity, 1992, among others). Today, a chain of 41 tiger reserves and 28 elephant reserves, besides a network of 668 Protected Areas, bear testimony to the efforts of Centre. The Environmental Protection Act, 1986, and notifications issued thereunder made serious efforts to protect wildlife habitats and wildlife corridors.

With the opening up of Indian market and process of globalization, the country has made significant progress in achieving higher Gross Domestic Product (GDP). But, on the other hand, disturbing developments about dilution of conservation efforts on the part of the system of governance on one side and a significant increase in the death toll of protected species, combined with intervention within Protected Areas came to fore. Take for instance, the restrictions by Environmental Impact Assessment (EIA) for Development Projects, covering more than 30 sectors as far back in 1994, surprisingly omitting all railway projects from its ambit. The history of last 20 years bears testimony to the sad fact, in attempts of the so called 'Development Lobby' to establish practices like 'Green Blockade'. EIA notification, for instance, puts special restriction for development projects in and around 'Protected Areas'- largely on the basis of requirement of 'Forest Clearance' or on the assessment of impacts on Wildlife Habitat or on well found apprehension of fragmentation of wildlife habitat or corridors.

It will be worthwhile to mention that in the 31st meeting of the Standing Committee for National Board for Wildlife (NBWL), held between August 12-13, 2014, as many as 173 projects were listed for clearance from 24 states of India. A total of 130 projects were cleared, but were eventually struck down by the Supreme Court of India on the grounds that the current constitution of NBWL is a violation of law (PA Update, 2014-15: 12-22). Again, in a single NBWL meeting, held on January 21, 2015, at least 34 project proposals, cutting across 12 states have been approved; including those for road, rail, oil drilling, pipeline, canal construction–all being within the declared boundary of 27 wildlife sanctuaries, four national parks, one tiger reserves and two bird sanctuaries, among others. All these projects involve diversion of forest land within 'Protected Area' for non-forestry purpose (PA Update, 2015. 21(2): 21-23).

Besides, at least 15 proposals from 10 states got clearance for diversion of forest land within 10 km radius of national parks and wildlife sanctuaries, which according to EIA norm should not have been given permission. The range of projects included construction of jetty in water ways and highway on land, storage facilities, irrigation, canal construction, road, mining, thermal power, hydrocarbon exploration.

The 48 projects recommended for clearance in January 2015, if undertaken, will convert 2,144 ha of forest land within the Protected Area. But in some cases, forest area has not been clearly defined and maneuvered in such language as 'Afforestation of boundary of Protected Area for exclusion of part of limestone bearing mineral zone' in Kamur Wildlife Sanctuary, Bihar. The title at least does not indicate 'What the limestone bearing area is' that is referred to within the sanctuary. In June 2015, NBWL had again cleared 18 new projects and deferred four projects without rejecting a single one. These include six projects within five tiger reserve areas (PA update, August, 2015). One can recall how years ago, dolomite mining was totally banned in Buxa Tiger Reserve, although mining history dates back 50 years before the tiger reserve was notified.

The forest cover in India has a target to reach 33 per cent of land area but forests within the Protected Areas have special significance in terms of biodiversity and wildlife conservation. Years back, a study by Zoological Survey of India on

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tiger reserves of India revealed how tiger reserves have contributed towards efforts of conservation of biological diversity in the country by protecting keystone species and forests. One has to remember that till date 70 per cent of biodiversity has been recorded from the forested area in the world.

## 2.6 NATIONAL PARK AND SANCTUARIES OF INDIA

A **national park** is a park in use for conservation purposes. Often it is a reserve of natural, semi-natural, or developed land that a sovereign state declares or owns. Although individual nations designate their own national parks differently, there is a common idea: the conservation of 'wild nature' for posterity and as a symbol of national pride. An international organization, the International Union for Conservation of Nature (IUCN), and its World Commission on Protected Areas (WCPA), has defined 'National Park' as its Category II type of protected areas.

While this type of national park had been proposed previously, the United States established the first 'Public park or pleasuring-ground for the benefit and enjoyment of the people', Yellowstone National Park, in 1872. Although Yellowstone was not officially termed a national park in its establishing law, it was always termed such in practice and is widely held to be the first and oldest national park in the world. However, the Tobago Main Ridge Forest Reserve(established in 1776), and the area surrounding Bogd Khan Uul Mountain (1778) are seen as the oldest legally protected areas, predating Yellowstone by nearly a century. The first area to use national park in its creation legislation was the U.S.'s Mackinac, in 1875. Australia's Royal National Park, established in 1879, was the world's third official national park. In 1895 ownership of Mackinac National Park was transferred to the State of Michigan as a state park and national park status was consequently lost. As a result, Australia's Royal National Park is by some considerations the second oldest national park now in existence. Canada established Parks Canada in 1911, becoming the world's first national service dedicated to protecting and presenting natural and historical treasures.

The largest national park in the world meeting the IUCN definition is the Northeast Greenland National Park, which was established in 1974. According to the IUCN, 6,555 national parks worldwide met its criteria in 2006. IUCN is still discussing the parameters of defining a national park.

National parks are almost always open to visitors. Most national parks provide outdoor recreation and camping opportunities as well as classes designed to educate the public on the importance of conservation and the natural wonders of the land in which the national park is located.

There are 104 existing national parks in India covering an area of 40501.13 km$^2$, which is 1.23% of the geographical area of the country (National Wildlife Database, May, 2019).

| State | S.No. | Name of State/ Protected Area | Year of Establishment | Area (km²) |
|---|---|---|---|---|
| Andaman & Nicobar Islands | 1 | Campbell Bay National Park | 1992 | 426.23 |
| Andaman & Nicobar Islands | 2 | Galathea Bay National Park | 1992 | 110 |
| Andaman & Nicobar Islands | 3 | Mahatama Gandhi Marine (Wandoor)National Park | 1983 | 281.5 |
| Andaman & Nicobar Islands | 4 | Middle Button Island National Park | 1987 | 0.44 |
| Andaman & Nicobar Islands | 5 | Mount Harriett National Park | 1987 | 46.62 |
| Andaman & Nicobar Islands | 6 | North Button Island National Park | 1987 | 0.44 |
| Andaman & Nicobar Islands | 7 | Rani Jhansi Marine National Park | 1996 | 256.14 |
| Andaman & Nicobar Islands | 8 | Saddle Peak National Park | 1987 | 32.54 |
| Andaman & Nicobar Islands | 9 | South Button Island National Park | 1987 | 0.03 |
| Andhra Pradesh | 1 | Papikonda National Park | 2008 | 1012.86 |
| Andhra Pradesh | 2 | Rajiv Gandhi (Rameswaram) National Park | 2005 | 2.4 |
| Andhra Pradesh | 3 | Sri Venkateswara National Park | 1989 | 353.62 |
| Arunachal Pradesh | 1 | Mouling National Park | 1986 | 483 |
| Arunachal Pradesh | 2 | Namdapha National Park | 1983 | 1807.82 |
| Assam | 1 | Dibru-Saikhowa National Park | 1999 | 340 |
| Assam | 2 | Kaziranga National Park | 1974 | 858.98 |
| Assam | 3 | Manas National Park | 1990 | 500 |
| Assam | 4 | Nameri National Park | 1998 | 200 |
| Assam | 5 | Rajiv Gandhi Orang National Park | 1999 | 78.81 |
| Bihar | 1 | Valmiki National Park | 1989 | 335.65 |
| Chhattisgarh | 1 | Guru Ghasidas (Sanjay) National Park | 1981 | 1440.705 |
| Chhattisgarh | 2 | Indravati (Kutru) National Park | 1982 | 1258.37 |
| Chhattisgarh | 3 | Kanger Valley National Park | 1982 | 200 |
| Goa | 1 | Mollem National Park | 1992 | 107 |
| Gujarat | 1 | Vansda National Park | 1979 | 23.99 |
| Gujarat | 2 | Blackbuck (Velavadar) National Park | 1976 | 34.53 |
| Gujarat | 3 | Gir National Park | 1975 | 258.71 |
| Gujarat | 4 | Marine (Gulf of Kachchh) National Park | 1982 | 162.89 |
| Haryana | 1 | Kalesar National Park | 2003 | 46.82 |
| Haryana | 2 | Sultanpur National Park | 1989 | 1.43 |
| Himachal Pradesh | 1 | Great Himalayan National Park | 1984 | 754.4 |
| Himachal Pradesh | 2 | Inderkilla National Park | 2010 | 104 |
| Himachal Pradesh | 3 | Khirganga National Park | 2010 | 710 |
| Himachal Pradesh | 4 | Pin Valley National Park | 1987 | 675 |
| Himachal Pradesh | 5 | Simbalbara National Park | 2010 | 27.88 |
| Jammu & Kashmir | 1 | City Forest (Salim Ali) National Park | 1992 | 9 |
| Jammu & Kashmir | 2 | Dachigam National Park | 1981 | 141 |
| Jammu & Kashmir | 3 | Hemis National Park | 1981 | 3350 |
| Jammu & Kashmir | 4 | Kishtwar National Park | 1981 | 425 |
| Jharkhand | 1 | Betla National Park | 1986 | 226.33 |
| Karnataka | 1 | Anshi National Park | 1987 | 417.34 |
| Karnataka | 2 | Bandipur National Park | 1974 | 874.2 |
| Karnataka | 3 | Bannerghatta National Park | 1974 | 260.51 |
| Karnataka | 4 | Kudremukh National Park | 1987 | 600.32 |
| Karnataka | 5 | Nagarahole (Rajiv Gandhi) National Park | 1988 | 643.39 |
| Kerala | 1 | Anamudi Shola National Park | 2003 | 7.5 |
| Kerala | 2 | Eravikulam National Park | 1978 | 97 |
| Kerala | 3 | Mathikettan Shola National Park | 2003 | 12.82 |
| Kerala | 4 | Pambadum Shola National Park | 2003 | 1.318 |
| Kerala | 5 | Periyar National Park | 1982 | 350 |

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|---|---|---|---|---|
| Kerala | 6 | Silent Valley National Park | 1984 | 89.52 |
| Madhya Pradesh | 1 | Bandhavgarh National Park | 1968 | 448.85 |
| Madhya Pradesh | 2 | Dinosaur Fossils National Park | 2011 | 0.8974 |
| Madhya Pradesh | 3 | Fossil National Park | 1983 | 0.27 |
| Madhya Pradesh | 4 | Indira Priyadarshini Pench National Park | 1975 | 292.85 |
| Madhya Pradesh | 5 | Kanha National Park | 1955 | 940 |
| Madhya Pradesh | 6 | Madhav National Park | 1959 | 375.22 |
| Madhya Pradesh | 7 | Panna National Park | 1981 | 542.67 |
| Madhya Pradesh | 8 | Sanjay National Park | 1981 | 466.88 |
| Madhya Pradesh | 9 | Satpura National Park | 1981 | 585.17 |
| Madhya Pradesh | 10 | Van Vihar National Park | 1979 | 4.45 |
| Maharashtra | 1 | Chandoli National Park | 2004 | 317.67 |
| Maharashtra | 2 | Gugamal National Park | 1975 | 361.28 |
| Maharashtra | 3 | Nawegaon National Park | 1975 | 133.88 |
| Maharashtra | 4 | Pench (Jawaharlal Nehru) National Park | 1975 | 257.26 |
| Maharashtra | 5 | Sanjay Gandhi (Borivilli) National Park | 1983 | 86.96 |
| Maharashtra | 6 | Tadoba National Park | 1955 | 116.55 |
| Manipur | 1 | Keibul-Lamjao National Park | 1977 | 40 |
| Meghalaya | 1 | Balphakram National Park | 1985 | 220 |
| Meghalaya | 2 | Nokrek Ridge National Park | 1986 | 47.48 |
| Mizoram | 1 | Murlen National Park | 1991 | 100 |
| Mizoram | 2 | Phawngpui Blue Mountain National Park | 1992 | 50 |
| Nagaland | 1 | Intanki National Park | 1993 | 202.02 |
| Odisha | 1 | Bhitarkanika National Park | 1988 | 145 |
| Odisha | 2 | Simlipal National Park | 1980 | 845.7 |
| Rajasthan | 1 | Desert National Park | 1992 | 3162 |
| Rajasthan | 2 | Keoladeo Ghana National Park | 1981 | 28.73 |
| Rajasthan | 3 | Mukundra Hills National Park | 2006 | 200.54 |
| Rajasthan | 4 | Ranthambhore National Park | 1980 | 282 |
| Rajasthan | 5 | Sariska National Park | 1992 | 273.8 |
| Sikkim | 1 | Khangchendzonga National Park | 1977 | 1784 |
| Tamil Nadu | 1 | Guindy National Park | 1976 | 2.82 |
| Tamil Nadu | 2 | Gulf of Mannar Marine National Park | 1980 | 6.23 |
| Tamil Nadu | 3 | Indira Gandhi (Annamalai) National Park | 1989 | 117.1 |
| Tamil Nadu | 4 | Mudumalai National Park | 1990 | 103.23 |
| Tamil Nadu | 5 | Mukurthi National Park | 1990 | 78.46 |
| Telangana | 1 | Kasu Brahmananda Reddy National Park | 1994 | 1.43 |
| Telangana | 2 | Mahaveer Harina Vanasthali National Park | 1994 | 14.59 |
| Telangana | 3 | Mrugavani National Park | 1994 | 3.6 |
| Tripura | 1 | Clouded Leopard National Park | 2007 | 5.08 |
| Tripura | 2 | Bison (Rajbari) National Park | 2007 | 31.63 |
| Uttar Pradesh | 1 | Dudhwa National Park | 1977 | 490 |
| Uttarakhand | 1 | Corbett National Park | 1936 | 520.82 |
| Uttarakhand | 2 | Gangotri National Park | 1989 | 2390.02 |
| Uttarakhand | 3 | Govind National Park | 1990 | 472.08 |
| Uttarakhand | 4 | Nanda Devi National Park | 1982 | 624.6 |
| Uttarakhand | 5 | Rajaji National Park | 1983 | 820 |
| Uttarakhand | 6 | Valley of Flowers National Park | 1982 | 87.5 |
| West Bengal | 1 | Buxa National Park | 1992 | 117.1 |
| West Bengal | 2 | Gorumara National Park | 1992 | 79.45 |
| West Bengal | 3 | Jaldapara National Park | 2014 | 216.51 |
| West Bengal | 4 | Neora Valley National Park | 1986 | 159.89 |
| West Bengal | 5 | Singalila National Park | 1986 | 78.6 |
| West Bengal | 6 | Sunderban National Park | 1984 | 1330.1 |

### Wildlife Sanctuary

Wildlife sanctuary, is a naturally occurring sanctuary, such as an island, that provides protection for wildlife species from hunting, predation, competition or poaching; it is a protected area, a geographic territory within which wildlife is protected. Refuges can preserve animals that are endangered.

Such wildlife refuges are generally officially designated territories. They are created by government legislation, publicly or privately owned. Unofficial sanctuaries can also occur as a result of human accidents; the Chernobyl Exclusion Zone has in practice become a wildlife refuge since very few people live in the area. Wildlife has flourished in the Zone since the Chernobyl nuclear accident in 1986.

In the United States, the U.S. Fish and Wildlife Service applies the term refuge to various categories of areas administered by the Secretary of the Interior for the conservation of fish and wildlife. The Refuge System includes areas administered for the protection and conservation of fish and wildlife that are threatened with extinction, as well as wildlife ranges, game ranges, wildlife management areas, and waterfowl production areas.

### History

In the 3rd century BC, King Devanampiya Tissa declared the area around Mihintale, Sri Lanka as a sanctuary for wildlife, probably the first of its kind in the ancient world. According to stone inscriptions found in the vicinity, the king commanded the people not to harm animals or destroy trees within the area.

The first North American wildlife refuge, Lake Merritt Wildlife Refuge at Lake Merritt, was established by Samuel Merritt and enacted in California state law in 1870 as the first government owned refuge. The first federally owned refuge in the United States is Pelican Island National Wildlife Refuge and was established by Theodore Roosevelt in 1903 as part of his Square Deal campaign to improve the country. At the time, setting aside land for wildlife was not a constitutional right of the president. More recently, a bi-partisan group of US House of Representatives members established the Congressional Wildlife Refuge Caucus to further the needs of the National Wildlife Refuge System in the US Congress.

Today there are several national and international organizations that have taken the responsibility of supervising numerous systems of non-profit animal sanctuaries and refuges in order to provide a general system for sanctuaries to follow. Among them, the American Sanctuary Association monitors and aids in various facilities to care for exotic wildlife. Their accredited facilities follow high standards and a rigid application processes to ensure that the animals under their care are avidly cared for and maintained. The number of sanctuaries has substantially increased over the past few years.

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List of some of the popular Wildlife Sanctuaries in India are:

| State | Wildlife Sanctuaries | Started | Area (km2) |
|---|---|---|---|
| Andhra Pradesh | Srisailam Sanctuary | | 506.94 |
| | Manjira Wildlife Sanctuary | 1978 | 20 |
| | Nagarjunasagar Wildlife Sanctuary | 1978 | 3568 |
| Gujarat | Sasangir Wildlife Sanctuary | 1965 | 1153.42 |
| | Wild Ass Wildlife Sanctuary | 1973 | 4953.7 |
| Haryana | Sultanpur Bird Sanctuary | 1989 | 1.43 |
| Kerala | Idukki Wildlife Sanctuary | 1976 | 77 |
| | Periyar Wildlife Sanctuary | 1950 | 472 |
| | Wayanad Wildlife Sanctuary | 1973 | 344.44 |
| | Aralam Wildlife Sanctuary | 1984 | 55 |
| | Neyyar Wildlife Sanctuary | 1958 | 128 |
| | Peppara Wildlife Sanctuary | 1983 | 55 |
| | Parambikulam Wildlife Sanctuary | 1973 | 285 |
| | Thattekad Bird Sanctuary | 1981 | 1258.37 |
| Karnataka | B R Hills Wildlife Sanctuary | 1987 | 539.52 |
| | Dandeli Wildlife Sanctuary | | 834.16 |
| | Ranganathittu Wildlife Sanctuary | 1984 | 119 |
| Madhya Pradesh | Karera Wildlife Sanctuary | 1981 | 202.21 |
| Maharashtra | Sanjay Gandhi Wildlife Sanctuary | 1983 | 86.96 |
| Odisha | Chilka Lake Bird Sanctuary | 1987 | 15.53 |
| | Bhitarkanika Wildlife Sanctuary | 1985 | 70 |
| | Satkosia Basipalli Sanctuary | | 1330 sq Kms |
| Rajasthan | Sambhar Wildlife Sanctuary | | |
| Sikkim | Kyongnosla Alpine Sanctuary | | 31 |
| | Singba Rhododendron Sanctuary | | 43 |
| | Barsey Rhododendron Sanctuary | | 104 |
| Tamil Nadu | Mudumalai Sanctuary | 1978 | 321.55 |
| Uttar Pradesh | Hastinapur Sanctuary | 1986 | 20.73 |
| | Kishanpur Sanctuary | 1972 | 227.12 |
| | National Chambal Sanctuary | 1992 | 635 |
| Uttarakhand | Assan Barrage Bird Sanctuary | | |

### Difference between Wildlife Sanctuary and National Park

We must make every effort to preserve, conserve and manage biodiversity. Protected areas, from large wilderness reserves to small sites for particular species, and reserves for controlled uses. The wildlife sanctuary and national park are designated places for protecting the wild plants, animals and natural habitats.

### National Park

It is a home to many species of birds and animals which is established by central and state government for the conservation.

### Characteristics of National Park

- Reserve area of land, owned by the government.
- Area is protected from human exploitation, industrialization and pollution.
- No cutting, Grazing allowed, Outside Species Allowed
- It came under the category called 'Protected Areas'. The Protected Areas are declared under Wildlife (Protection) Act, 1972.
- Conservation of 'Wild Nature' for posterity and as a symbol of national pride.

- International Union for Conservation of Nature (IUCN), and its World Commission on Protected Areas, has defined its Category II type of protected areas.

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### Wildlife Sanctuary

It is a consecrated place where sacred species are kept. It is not open for general public, unlike zoo. In other words, we say, it tries not to allow any activity that would place the animals in an unduly stressful situation. India has 543 wildlife sanctuaries.

### Characteristics of Wildlife Sanctuary

- It is natural area which is reserve by a governmental or private agency for the protection of particular species.
- Area is designated for the protection of wild animals.
- Only animals are conserved, Could be private property also, outside activities allowed
- It came under the category called 'Protected Areas'. The Protected Areas are declared under Wildlife (Protection) Act, 1972.
- International Union for Conservation of Nature (IUCN) has defined its Category IV type of protected areas

### List of top Wildlife Sanctuaries/National Parks in India

| S.No. | Name | Located at ( District, State) | Established Year | Area in Km | Attractions |
|---|---|---|---|---|---|
| 1 | Corbett National Park (Jim Corbett National Park) | Nainital, Uttarakhand | 1936 | 521 km2 | The imposing Bengal Tigers |
| 2 | Ranthambore National Park | Sawai Madhopur, Rajasthan | 1980 | 392 km2 | Majestic Tigers |
| 3 | Bandipur National Park | Gundlupet, Chamarajanagar District, Bandipur, Karnataka | 1974 | 874 km2 | Tiger, Asian elephant and many types of biomes |
| 4 | Keoladeo Ghana National Park | Bharatpur, Rajasthan | 1905 | 28.7 km2 | Avifauna Birds |
| 5 | Nagarhole National Park | Kodagu district and Mysore district, Karnataka | 1988 | 642.39 km2 | Tigers, Indian bison and elephants |
| 6 | Sariska National Park | Near Kraska, Alwar District, Sariska, Rajasthan | 1955 | 866 km2 | Bengal tigers |
| 7 | Kaziranga National Park | Kanchanjuri, Assam | 1908 | 430 $km^2$ | One horned Rhinos, Tigers and Wild Buffaloes |

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|---|---|---|---|---|---|
| 8 | Bhadra Wildlife Sanctuary | Chikkamagaluru town, Karnataka | 1951 | 492.46 $km^2$ | Tiger, |
| 9 | Kanha National Park | Madla/ Balaghat districts, Madhya Pradesh | 1955 | 940 $km^2$ | Tigers |
| 10 | Sunderbans National Park | Dayapur, Gosaba, West Bengal | 1984 | 1,330.12 $km^2$ | Bengal Tigers |
| 11 | Bandhavgarh National Park | Badhavgarh, District Umaria, Madhya Pradesh | 1968 | 1,536 $km^2$ | White Tiger |
| 12 | Gir National Park and Sasan Gir Sanctuary | Junagadh District, Gir Somnath District and Amreli District Gujarat, India | 1965 | 1,412 $km^2$ | Asiatic Lion |
| 13 | Periyar National Park | Idukki, Kottayam and Pathanamthitta, Kerala state, India | 1982 | 305 $km^2$ | Asian Elephants, Periyar Lake |
| 14 | Pench National Park | Turia, Seoni Dist, Kurai, Madhya Pradesh | 1983 | 758 $km^2$ | Inspired Rudyard Kiplingto write 'The Jungle Book', While Tigers |
| 15 | Manas National Park | Fatemabad - Mathanguri Road, Jyoti Gaon, Assam | 1990 | 490.3 $km^2$ | Wild water buffalo, Assam roofed turtle, hispid hare, golden langur and pygmy hog |
| 16 | Dudhwa National Park | Palia Kalan, Uttar Pradesh | 1977 | 542.67 $km^2$ | Tiger and Swamp Deer |
| 17 | Panna National Park | Panna and Chhatarpur districts, Madhya Pradesh, India | 1981 | 625.4 $km^2$ | Tiger |
| 18 | Tadoba Andhari Tiger Reserve | Chandrapur, Maharashtra, India | 1955 | 90.44 $km^2$ | Tiger |
| 19 | Chinnar Wildlife Sanctuary | Munnar - Udumalpet Road, Munnar, Kerala | 1984 | 1,171 $km^2$ | Endangered Grizzled Giant Squirrel |
| 20 | The Great Himalayan National Park | SaiRopa, Kullu, Himachal Pradesh | 1984 | 866.41 $km^2$ | Globally Threatened, Musk Deer and The Western Horned Tragopan |
| 21 | Dandeli Wildlife Sanctuary | Uttara Kannada District , Karnataka India | NA | 2.36 $km^2$ | Crocodiles, great hornbill and Malabar pied hornbill |
| 22 | Silent Valley National Park | Mannarkkad, Palakkad District, Kerala | 1905 | 820 $km^2$ | Lion Tailed Macaque |
| 23 | Rajaji National Park | Dehradun, Uttarakhand | 1983 | 107 $km^2$ | Asian elephants, Bengal tigers |

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|---|---|---|---|---|---|
| 24 | Bhagwan Mahavir Wildlife Sanctuary | Caranzol, Goa | 1978 | 4,400 $km^2$ | Snakes, particularly the king cobra and black panther |
| 25 | Hemis National Park | Leh, Jammu and Kashmir | 1981 | 101 $km^2$ | Snow leopards, gompas and holy chortens |
| 26 | Interview Island Wildlife Sanctuary | Andaman and Nicobar Islands, Bay of Bengal | NA | 7506.22 $km^2$ | Elephants |
| 27 | Kachchh/ Kutch Desert WLS | Great Rann of Kutch, Kutch district, Gujarat, India | 1986 | 5000 $km^2$ | Greater flamingo |
| 28 | Nelapattu Bird Sanctuary | Andhra Pradesh | 1976 | 4953.71 $km^2$ | Largest habitat for pelicans, also Babblers, Flamingos, Open Billed Stork, etc. |
| 29 | Karakoram Wildlife Sanctuary | Leh District, Jammu and Kashmir, India | 1987 | 4,149 $km^2$ | Chiru or Tibetan Antelope, ibetan gazalle, Himalayan ibex |
| 30 | Indian Wild Ass Sanctuary (Wild Ass WLS) | Little Rann of Kutch, Gujarat, India | 1973 | 3,568 $km^2$ | Endangered Indian Wild Ass/ ghudkhar |
| 31 | Dibang Wildlife Sanctuary | Dibang Valley , Arunachal Pradesh, India | 1991 | 6.14 $km^2$ | Rare species - mishmi takin, asiatic black bear, gongshan muntjac, red panda, red goral and musk deer |
| 32 | Nagarjuna Sagar-Srisailam WLS/ Rajiv Gandhi WLS | Nalgonda & Mahaboobnagar, Telangana | 1978 | 861.95 $km^2$ | Bengal tiger |
| 33 | Rollapadu Wildlife Sanctuary | Kurnool District, Rollapadu, Andhra Pradesh | 1988 | 783 $km^2$ | Great Indian Bustard and Lesser Florican |
| 34 | Papikonda Wildlife Sanctuary | Andhra Pradesh | 1978 | 282 $km^2$ | Tiger, Wild water buffalo was seen here till 1980s, but appears to be extinct in this region |
| 35 | Pakhui/ Pakke Tiger Reserve | Kameng district, Arunachal Pradesh, India | 1977 | 26.22 $km^2$ | Large cats - tiger, leopard and clouded leopard |
| 36 | Kamlang Wildlife Sanctuary | Lohit District, Arunachal Pradesh, India | 1989 | 6.05 $km^2$ | Elephant , Tiger, giant flying squirrel |
| 37 | Mehao Wildlife Sanctuary | Lower Dibang Valley district, Arunachal Pradesh | 1980 | 20.98.62 $km^2$ | Bengal tiger, hoolock gibbon, leopard and clouded leopard. |

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|---|---|---|---|---|---|
| 38 | Eaglenest Wildlife Sanctuary | Arunachal Pradesh | 1989 | 681.99 $km^2$ | Birdwatcher's Paradise. Main attraction is Bugun liocichla , Asian elephant, capped langur (endangered), red panda, gaur, Asiatic black bear, Arunachal macaque |
| 39 | Bornadi Wildlife Sanctuary | Udalguri District& Baksa District Assam, India | 1980 | 551.55 $km^2$ | Pygmy hog, hispid hare (both protected) |
| 40 | Garampani Wildlife Sanctuary | Karbi Anglong district, Assam | 1952 | 607.70 $km^2$ | Hoolock gibbons and golden langurs, hot springs |
| 41 | Hoollongapar Gibbon Sanctuary | Jorhat, Assam | 1997 | 608.55 $km^2$ | Western hoolock gibbon and 15 species of apes |
| 42 | Bhimbandh Wildlife Sanctuary | Munger district, Bihar | 1976 | 485.20 $km^2$ | Tigers, panthers, birds |
| 43 | Achanakmar Wildlife Sanctuary | Bilaspur, Chhattisgarh | 1975 | 1,027.53 $km^2$ | Tiger, leopard, bison |
| 44 | Shoolpaneshwar Wildlife Sanctuary | Narmada district of Gujarat | 1982 | 975 $km^2$ | Python, pangolin, Flying squirrels |
| 45 | Tamor Pingla Wildlife Sanctuary | Surajpur District, Chhattisgarh | 1978 | 2073 $km^2$ | Elephant |
| 46 | Shenbagathoppu Grizzled Squirrel Wildlife Sanctuary | Virudhunagar and Madurai districts, Tamil Nadu | 1988 | 423.55 $km^2$ | Grizzled giant squirrel (vulnerable species), Periyar Tiger |
| 47 | Cauvery Wildlife Sanctuary | Karnataka, India | 1987 | 219 $km^2$ | Popular for Mahseer fish, birds |
| 48 | Kedarnath Wildlife Sanctuary | Chamoli & Rudraprayag, Uttarakhand | 1972 | 975 $km^2$ | Endangered Himalayan musk deer |
| 49 | Hastinapur WLS | Meerut, Hapur, Bijnore and Amroha, in Uttar Pradesh, India | 1986 | 2073 $km^2$ | Twelve - Horned Deer or Baara Singha, the State Bird - Saaras [Crane], crocodiles, turtles, playful Ganga Dolphins, Birds |
| 50 | Koyna Wildlife Sanctuary | Satara, Maharashtra | 1985 | 423.55 $km^2$ | Royal Bengal Tiger and King Cobra |
| 51 | Ramgarh Vishdhari Wildlife Sanctuary | Alwar District, Rajasthan | 1955 | 219 $km^2$ | Bengal tigers, leopards |

## 2.7 GEOLOGICAL AND ZOOGEOGRAPHICAL DISTRIBUTION OF ANIMALS

NOTES

Biogeography studies the patterns of distribution of biological systems, processes or characteristics at several spatial and temporal scales. Spatial scales under analysis span over a wide range, including genes, organisms or groups of organisms and ecosystems or biomes.

Zoogeography is the branch of the science of biogeography that is concerned with geographic distribution (present and past) of animal species. Zoogeography incorporates methods of molecular biology, genetics, morphology, phylogenetics, and Geographic Information Systems (GIS) to delineate evolutionary events within defined regions of study around the globe.

Zoogeography is one of the various branches of biology. It deals primarily with the geological distribution of animals. It also studies the causes, effects, and interactions (in the past, the present, or the future) resulting in the geological distribution of particular species of animals. An expert in this filed is called a zoogeographer.

Modern-day zoogeography also places a reliance on GIS to integrate a more precise understanding and predictive model of the past, current, and future population dynamics of animal species both on land and in the ocean. Through employment of GIS technology, linkages between abiotic factors of habitat such as topography, latitude, longitude, temperatures, and sea level can serve to explain the distribution of species populations through geologic time. Understanding correlations of habitat formation and the migration patterns of organisms at an ecological level allows for explanations of speciation events that may have arisen due to physical geographic isolation events.

The world's zoogeographical regions were historically defined on an intuitive basis, with no or a limited amount of analytical testing Eleven vertebrate-rich (Nearctic, Caribbean, Neotropical, Andean, Palearctic, Afrotropical, Madagascan, Indo-Malaysian, Wallacean, New Guinean, Australian) and three vertebrate-poor (Arctic, Antarctic, Polynesian) zoogeographical regions were derived; the Neotropical,-Afrotropical, and Australian had the highest numbers of characteristic tetrapod genera.

The history of zoogeography includes the most prominent scientists of the last two-three centuries, scholars who gave an immense contribution to modern biology and ecology. Alfred Russel Wallace, the author in of 'The Geographical Distribution of Animals, With a Study of the Relations of Living and Extinct Faunas As Elucidating the Past Changes of the Earth's Surface, is regarded as the "father" of zoogeography.

Zoogeography has two major divisions: (1) ecological zoogeography and (2) historical zoogeography. Ecological zoogeography attempts to understand and determine the role of the present biotic and abiotic interactions that affect the distribution of a particular group of animals. Historical zoogeography is concerned

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with determining and understanding the origin, extinction, and dispersal of a particular taxon. It aims to understand the past distribution of animals that led to their present day pattern. Thus, it encompasses and makes use of geography, geological history, evolutionary theories, physiography, climate, etc. in their study.

Based on the proposal of Philip Sclater and Alfred Wallace, there are six main zoogeographic regions of the world. These are as follows:

- Ethiopian region
- Oriental region
- Palaearctic region
- Nearctic Region
- Neotropical region
- Australian regions

1. Ethiopian Region: Africa (except the northern corner), with part of southern Arabia.
2. Oriental Region: Tropical Asia, with associated continental islands.
3. Palearctic Region: Eurasia above the tropics, with the northern corner of Africa.
4. Nearctic Region: North America, except the tropical part of Mexico.
5. Neotropical Region: South and Central America with the tropical part of Mexico.
6. Australian Region: Australia, with New Guinea, etc.

## 2.7.1 Animal Distribution

The study of the geographical distribution of living and extinct organisms has recently become one of the most important branches of philosophical natural history, it throws the light on both, the former condition of the earth, and on the origin of species. The geographical distribution of animals has lately received much attention, the most important contribution to the subject being a large work by Mr. A.R.Wallace

Animals are not uniformly distributed on land and water. They are restricted to certain areas by several factors such as climate, food, shelter etc.

**Distribution in Space**

In this type of distribution the arrangement of different types of animal on the earth's surface is considered, i.e., animal's distribution both on land and water.

This can further be sub-divided into two types:

(i) Geographical or Horizontal Distribution over earth's surface,
(ii) Bathymetric or Vertical Distribution.

**(i) Geographical or Horizontal Distribution:** The distribution of animals on land and fresh water in different continents and on different islands is called geographical or horizontal distribution.

**(ii) Bathymetric or Vertical Distribution:** The distribution of animals on the vertical surface of land and water is called the bathymetric or

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vertical distribution. This type of distribution can be studied under three heads.

(a) Limnobiotic Distribution: It deals with the distribution of animals in fresh water sources.

(b) Holobiotic Distribution: Distribution of animals in sea.

(c) Geobiotic Distribution: Distribution of animals on land.

### Distribution in Time

The distribution of animals in the past of Earth's history is called the distribution in time. This can be studied by fossil evidences only. The description of this type of distribution can be made on the Geological time scale.

### Patterns of Distribution

It is observed that animals are distributed everywhere in the environment. But the pattern and type of distribution is different for different animals.

There are generally four types of pattern recognised for animal distribution:

(a) Arctic distribution

(b) Tropical distribution

(c) Continuous distribution, and

(d) Discontinuous distribution.

(a) **Arctic Distribution:** A number of species are found to inhabit only in the Arctic or Antarctic waters with no representatives in the intermediate oceans. This type of distribution is called bipolar distribution and this characteristic is called as bipolarity.

Examples:

(i) Coelenterats: *Botrynema, Grammaria, Lampta, Myriothela*

(ii) Molluscs: *Limacina helicina, Clione*

(iii) Arthropoda: *Peripatus*

(iv) Ascidian: *Didetnnus albidum*

(v) Fishes: *Oncorhynchus* (salmon), *Latnma cornubica*

(vi ) Birds: *Aptenodytes* (Penguin)

(vii) Mammals: *Balaenoptera musculus* (blue whale), *Callorhinus* (Seal), Caribou, etc.

(b) **Tropical Distribution:** The distribution of animals on the landmass between Tropic of Cancer and Tropic of Capricorn are called tropical distribution. This region is having the maximum biodiversity, i.e., the animals of all the phyla have developed enormously in this part.

(c) **Continuous Distribution**: This is also called cosmopolitan distribution. When an animal is found over a wide area, i.e., uninterrupted range of surface distribution, then it is called continuous distribution.

Examples:

(i) Mollusca: The green mussel (*Mytilus)*

(ii) Arthropoda: Brine shrimp (*Artemia salina*), cockroach (*Periplaneta)*

(iii) Birds: Hawks, cuckoo

(iv) Mammals: Rats, bats, etc.

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Those animals which can thrive over a large range of atmospheric change and have continuous distribution are called '**eurytopic**' animals, e.g., cockroach. On the other hand, the animals, which are restricted in a definite area are called 'Stenotopic', e.g., Peripatus present in arctic areas only.

(d) **Discontinuous Distribution**: When different species of same genus or different members of a species are living in different parts of the earth, then it is called discontinuous distribution.

| Animals | Distribution at present |
|---|---|
| 1. Annelida :<br>*Notodrilus* (a type of earthworm) | 1. New Zealand, Central and North West Australia, South America, South Africa and Central America. |
| 2. Arthropoda :<br>(a) *Peripatus* (living fossil) (Onychophora)<br>(b) *Belostoma* (Giant water bug) | 2. (a) Indo-Malaya, East Indies, Australia, New Zealand, Africa, Central America, Mexico, West Indies.<br>(b) America, Africa, Southern Asia, Australia. |
| 3. Lung Fish (Dipnoi) :<br>(a) *Lepidosiren*<br>(b) *Protopterus*<br>(c) *Neoceratodus* | 3.<br>(a) South America<br>(b) Africa<br>(c) Australia |
| 4. Teleost fish :<br>*Galaxias* | 4. South America, Africa, Australia and New Zealand. |
| 5. Amphibia :<br>*Gymnophiona* (limbless amphibia) | 5. Africa, South Asia, East Indies and Tropical America. |
| 6. Reptile :<br>(a) Plaurodian turtle<br>(b) (i) *Sphenodon* (living fossil)<br>(ii) *Alligator* | 6. (a) Confined to Southern hemisphere now, but fossils obtained from Egypt, India, New Zealand, North America.<br>(b) (i) New Zealand<br>(ii) China, North America |
| 7. Birds : Ratitae (Flightless birds)<br>(a) Ostrich (*Struthio*)<br>(b) *Rhea*<br>(c) Cassowary (*Casuarius*)<br>(d) Emu (*Dromiceius*)<br>(e) Kiwi (*Apteryx*) | 7.<br>(a) Desert of Africa and Arabia<br>(b) Plains of Argentina, South Brazil<br>(c) Jungles of New Guinea, North Queensland of Australia<br>(d) Grassy plain & open forest of Australian region<br>(e) Damp forest of New Zealand |
| 8. Mammals :<br>(a) Marsupials<br>(b) Camelidae (Fig. 4.36)<br>Camels – (i) *Camelus bactrianus* (two humped)<br>Camels – (ii) *C. dromidarius* (one humped)<br>Llamas<br>(c) Proboscidian<br>(i) *Elephas*<br>(ii) *Loxodonta*<br>(d) Tapiridae<br>(i) *Tapirus*<br>(ii) *Tapirella* | 8.<br>(a) Australia, North and South Africa<br>(b)<br>(i) Desert of Mongolia and China wild<br>(ii) Domestic camel of North Africa<br>South America<br>(c)<br>(i) Asia<br>(ii) Africa<br>(d)<br>(i) Malaya, Java, Sumatra<br>(ii) South and Central America |

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## 2.8 FOSSILS AND PALAEOZOOLOGY

Fossils are remnant, impression, or trace of an animal or plant of a past geologic age that has been preserved in Earth's crust. The complex of data recorded in fossils worldwide is known as the fossil record it is the primary source of information about the history of life on Earth. The study of the fossil record has provided important information for different purposes. The progressive changes observed within an animal group are used to describe the evolution of that group. Fossils also provide the geologist a quick and easy way of assigning a relative age to the strata in which they occur. Fossils are useful in the exploration for minerals and mineral fuels. For example, they serve to indicate the stratigraphic position of coal seams.

Palaeozoology is the branch of palaeontology concerned with the study of animals throughout geological time, as revealed by their fossil remains. It is concerned with all aspects of the biology of ancient life forms: their shape and structure, evolutionary patterns, taxonomic relationships with each other and with modern living species, geographic distribution, and interrelationships with the environment.

### 2.8.1 Fossils

A fossil is any preserved remains, impression, or trace of any once-living thing from a past geological age. Examples include bones, shells, exoskeletons, stones imprints of animals or microbes, objects preserved in amber, hair, petrified wood, oil, coal, and DNA remnants etc. The totality of fossils is known as the fossil record. Word fossil is obtain from the Latin word '*Fossilis',* that literally means 'Obtained by Digging'.

Paleontology is the study of fossils: their age, method of formation, and evolutionary significance. Specimens are usually considered to be fossils if they are over 10,000 years old. There are many processes that lead to fossilization, including permineralization, casts and molds, authigenic mineralization, replacement and recrystallization, adpression, carbonization, and bioimmuration.

Fossils can be very large or very small. Microfossils are only visible with a microscope. Bacteria and pollen are microfossils. Macrofossils can be several meters long and weigh several tons. Macrofossils can be petrified trees or dinosaur bones. Fossils vary in size from one-micrometre (1 μm) bacteria to dinosaurs and trees, many meters long and weighing many tons. A fossil normally preserves only a portion of the deceased organism, usually that portion that was partially mineralized during life, such as the bones and teeth of vertebrates, or the chitinous or calcareous exoskeletons of invertebrates.

Fossils may also consist of the marks left behind by the organism while it was alive, such as animal tracks or feces (coprolites). These types of fossil are called trace fossils or ichno fossils, as opposed to body fossils. Some fossils are biochemical and are called chemofossils or biosignatures.

Only a small fraction of ancient organisms are preserved as fossils, and usually only organisms that have a solid and resistant skeleton are readily preserved.

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**Body Fossils and Trace Fossils**

The fossils of bones, teeth, and shells are called body fossils. Most dinosaur fossils are collections of body fossils.

Trace fossils are rocks that have preserved evidence of biological activity. They are not fossilized remains, just the traces of organisms. The imprint of an ancient leaf or footprint is a trace fossil. Burrows can also create impressions in soft rocks or mud, leaving a trace fossil.

The process of a once living organism becoming a fossil is called fossilization. Fossilization is a very rare process, of all the organisms that have lived on Earth, only a tiny percentage of them ever become fossils. To see why, imagine an antelope that dies on the African plain. Most of its body is quickly eaten by scavengers, and the remaining flesh is soon eaten by insects and bacteria, leaving behind only scattered bones. As the years go by, the bones are scattered and fragmented into small pieces, eventually turning into dust and returning their nutrients to the soil. It would be rare for any of the antelope's remains to actually be preserved as a fossil. For animals that lack hard shells or bones, fossilization is even rarer. As a result, the fossil record contains many animals with shells, bones, or other hard parts, and few soft bodied organisms. There is virtually no fossil record of jellyfish, worms, or slugs. Insects, which are by far the most common land animals, are only rarely found as fossils.

## 2.8.2 Fossilization Processes

The process of fossilization varies according to tissue type and external conditions.

**Permineralization:** Permineralization is a process of fossilization that occurs when an organism is buried. The empty spaces within an organism (spaces filled with liquid or gas during life) become filled with mineral-rich groundwater. Minerals precipitate from the groundwater, occupying the empty spaces. This process can occur in very small spaces, such as within the cell wall of a plant cell. Small scale permineralization can produce very detailed fossils. Permineralized bryozoan from the Devonian of Wisconsin is shown in Figure 2.15


***Fig.2.15*** *Permineralized Bryozoan*

**Cast and Molds:** Sometimes the original remains of the organism completely dissolve or are otherwise destroyed. The remaining organism-shaped hole in the rock is called an external mold. If this void is later filled with sediment, the resulting cast resembles what the organism looked like.

External mold of a bivalve from the Logan Formation, Lower Carboniferous, Ohio is shown in Figure 2.16

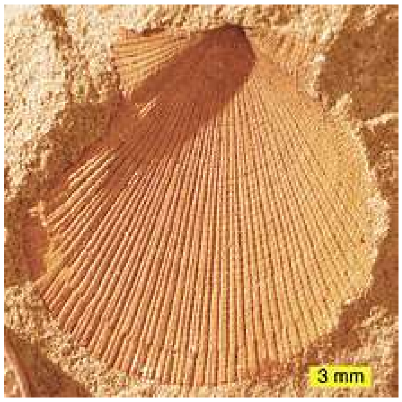


***Fig 2.16** External Mold of a Bivalve*

**Authigenic Mineralization:** This is a special form of cast and mold formation. The organism (or fragment of organism) can act as a nucleus for the precipitation of minerals such as siderite, resulting in a nodule forming around it. If this happens rapidly before significant decay to the organic tissue, very fine three-dimensional morphological detail can be preserved. Nodules from the Carboniferous Mazon Creek fossil beds of Illinois, USA, are among the best documented examples of such mineralization.

Silicified (replaced with silica) fossils from the Road Canyon Formationis is shown in Figure 2.17

***Fig 2.17** Silicified Fossils from the Road Canyon*

**Replacement and Recrystallization:** Replacement occurs when the shell, bone, or other tissue is replaced with another mineral. In some cases mineral replacement of the original shell occurs so gradually and at such fine scales that microstructural features are preserved despite the total loss of original material.

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Recrystallized Scleractinian coral (aragonite to calcite) is shown in Figure 2.18


***Fig 2.18*** *Recrystallized Scleractinian Coral*

**Adpression (Compression-Impression):** Compression fossils, such as those of fossil ferns, are the result of chemical reduction of the complex organic molecules composing the organism's tissues. . In this case the fossil consists of original material, albeit in a geochemically altered state in which case the fossil is known as a compression. Often, however, the phytoleim is lost and all that remains is an impression of the organism in the rock is called an impression fossil. In many cases, however, compressions and impressions occur together. For instance, when the rock is broken open, the phytoleim will often be attached to one part (compression), whereas the counterpart will just be an impression.

**Soft Tissue, Cell and Molecular Preservation:** Based on various experiments that studied the interaction of iron in haemoglobin with blood vessels and tissue by the discovery of soft tissue in dinosaur fossils, including blood vessels, and the isolation of proteins and evidence for DNA fragments. Scientist proposed that solution hypoxia coupled with iron chelation enhances the stability and preservation of soft tissue and provides the basis for an explanation for the unforeseen preservation of fossil soft tissues.

**Carbonization and Coalification:** Fossils that are carbonized or coalified consist of the organic remains which have been reduced primarily to the chemical element carbon. Carbonized fossils consist of a thin film which forms a silhouette of the original organism, and the original organic remains were typically soft tissues. Coalified fossils consist primarily of coal, and the original organic remains were typically woody in composition.

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Carbonized fossil of a possible leech is shown in Figure 2.19


***Fig 2.19*** *Carbonized Fossil*

**Bioimmuration:** Bioimmuration occurs when a skeletal organism overgrows or otherwise subsumes another organism, preserving the latter, or an impression of it, within the skeleton .Usually it is a sessile skeletal organism, such as a bryozoan or an oyster, which grows along a substrate, covering other sessile sclerobionts. The star-shaped holes (Catellocaula vallata) in this Upper Ordovician bryozoan represent a soft-bodied organism preserved by bioimmuration in the bryozoan skeleton is shown in Figure 2.20


***Fig 2.20*** *Bioimmuration in the Bryozoan Skeleton*

## 2.8.3 Types of Fossils

**Index Fossils:** Index fossils (also known as guide fossils, indicator fossils or zone fossils) are fossils used to define and identify geologic periods (or faunal stages). They work on the premise that, although different sediments may look different depending on the conditions under which they were deposited, they may include the remains of the same species of fossil**.** Figure 2.21 illustrate examples of index fossils.

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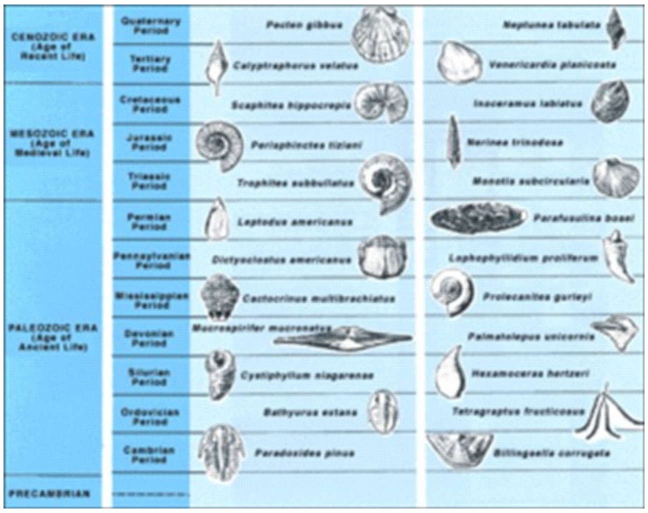


***Fig 2.21*** *Examples of Index Fossils*

**Trace Fossils:** Trace fossils consist mainly of tracks and burrows, but also include coprolites (fossil feces) and marks left by feeding. Trace fossils are particularly significant because they represent a data source that is not limited to animals with easily fossilized hard parts, and they reflect animal behaviours. A trace fossil is shown in Figure 2.22.

***Fig 2.22*** *Cambrian Trace Fossil*

**Transitional Fossil:** A transitional fossil is any fossilized remains of a life form that exhibits traits common to both an ancestral group and its derived descendant group. This is especially important where the descendant group is sharply differentiated by gross anatomy and mode of living from the ancestral group.

**Microfossils:** Microfossil is a descriptive term applied to fossilized plants and animals whose size is just at or below the level at which the fossil can be analyzed by the naked eye. Figure 2.23 illustrate a microfossil.

***Fig 2.23*** *Microfossil*

**Fossil Resin**: Fossil resin (colloquially called amber) is a natural polymer found in many types of strata throughout the world, even the Arctic. The oldest fossil resin dates to the Triassic, though most dates to the Cenozoic. The excretion of the resin by certain plants is thought to be an evolutionary adaptation for protection from insects and to seal wounds. Fossil resin often contains other fossils called inclusions that were captured by the sticky resin. These include bacteria, fungi, other plants, and animals.

**Derived, Or Reworked Fossil:** A derived, reworked or remanié fossil is a fossil found in rock that accumulated significantly later than when the fossilized animal or plant died.

**Fossil Wood:** Fossil wood is wood that is preserved in the fossil record. Wood is usually the part of a plant that is best preserved (and most easily found). Fossil wood may or may not be petrified.

**Subfossil**: The term subfossil can be used to refer to remains, such as bones, nests, or defecations, whose fossilization process is not complete, either because the length of time since the animal involved was living is too short (less than 10,000 years) or because the conditions in which the remains were buried were not optimal for fossilization.

**Chemical Fossils:** Chemical fossils, or chemofossils, are chemicals found in rocks and fossil fuels (petroleum, coal, and natural gas) that provide an organic signature for ancient life. Molecular fossils and isotope ratios represent two types of chemical fossils.

## 2.8.4 Palaeozoology

Palaeozoology, (also spelled as Paleozoology ,Greek: palaeon 'Old' and zoon 'Animal'), is the branch of paleontology, paleobiology, or zoology dealing with the recovery and identification of multicellular animal remains from geological or archaeological contexts, and the use of these fossils in the reconstruction of prehistoric environments and ancient ecosystems.

**Vertebrate Paleozoology**: Vertebrate paleozoology refers to the use of morphological, temporal, and stratigraphic data to map vertebrate history in evolutionary theory. Vertebrates are classified as a subphylum of Chordata, a

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phylum used to classify species adhering to a rod-shaped, flexible body type called a notochord. They differ from other phyla in that other phyla may have cartilage or cartilage-like tissues forming a sort of skeleton, but only vertebrates possess what we define as bone.

Evolutionary origins of vertebrates as well as the phylum Chordata have not been scientifically determined. Many believe vertebrates diverged from a common ancestor of chordates and echinoderms. This belief is well supported by the prehistoric marine creature Amphioxus. Amphioxus does not possess bone, making it an invertebrate, but it has common features with vertebrates including a segmented body and a notochord. This could imply that Amphioxus is a transitional form between an early chordate, echinoderm or common ancestor, and vertebrates.

**Quantitative Paleozoology**: Quantitative paleozoology is a process of taking a census of fossil types rather than inventory. They differ in that inventory refers to a detailed log of individual fossils, whereas census attempts to group individual fossils to tally the total number of a species. This information can be used to determine which species were most dominant and which had the largest population at a time period or in a geological region.

In the early 1930s, paleontologists Chester Stock and Hildegarde Howard devised special units for quantitative paleozoology and quantitative paleontology. The first unit used, Number of Identified Species (NISP), specified exact quantity of fossils from a specific species recorded. Stock and Howard determined this unit to be problematic for quantitative purposes as an excess of a small fossil—such as teeth—could exaggerate quantity of the species. There was also an amount of confusion as to whether bone fragments should be assembled and counted as one bone or tallied individually. Stock and Howard then devised the Minimum Number of Individuals (MNI), which estimated the minimum number of animals needed to produce the fossils recorded. For example, if five scapulae from a species were found, it might be difficult to determine whether some of them were paired right and left on one individual or whether each came from a different individual, which could alter census, but it could be said that there must be at least three individuals to produce five scapulae. Three would thus be the MNI. In rare cases where enough of a collection of fossils can be assembled into individuals as to provide an accurate number of individuals, the unit used is 'Actual Number of Individuals', or ANI.

Another unit commonly used in quantitative paleozoology is biomass. Biomass is defined as the amount of tissue in an area or from a species. It is calculated by estimating an average weight based on similar modern species and multiplying it by the MNI. This yields an estimate of how much the entire population of a species may have weighed. Problems with this measurement include the difference in weight between youngsters and adults, seasonal weight changes due to diet and hibernation, and the difficulty of accurately estimating the weight of a creature with only a skeletal reference. It is also difficult to determine exact age of fossilized matter within a year or a decade, so a biomass might be grossly exaggerated or under exaggerated if the estimated time frame in which the fossils were alive is incorrect.

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Palaeozoological data is used in research concerning conservation biology. Conservation biology refers to biological study used for conservation, control, and preservation of various species and ecosystems. In this context, the palaeozoological data used is obtained from recently deceased decomposing matter rather than prehistoric matter.

**Check Your Progress**

11. Define the term wildlife.
12. What is defaunation?
13. Define bio-reserves.
14. What is national park?
15. Which is the largest national park?
16. How many national parks are there in India?
17. Define wildlife sanctuary.
18. What is Zoogeography?
19. What is Palaeozoology?

## 2.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Many new pests and diseases of plants, animals and human beings can spread from one country to another through transportation. To prevent these disease quarantine is needed.
2. The accurate identification of fauna and flora in sedimentary rocks provides a perfect representation of the sequence of geological events, it helps in searching for fuels and mineral deposits. It is known as mineral Prospecting.
3. Information regarding disease vectors and parasites is an evident application of systematics to national defence. The use of biological means in the war is economical and requires fewer efforts in their operation.
4. Digestive enzymes are chemical compounds that help in digestion. For example, Proteins are always digested by a particular type of enzymes like pepsin, trypsin, etc., in all animals from a single celled amoeba to a human being.
5. Cytotaxonomy is the branch of biology that deals with the relationships and classification of organisms using comparative studies of chromosomes during meiosis.
6. A clade can be defined as a group of organisms having a common ancestor throughout evolution.
7. Native plants, animals, fungi and microbes are those species which exists naturally at a given location or in a particular ecosystem. They are the foundation of the natural systems that sustain biological diversity.

8. Keystone species is a species that has a disproportionately large effect on the communities in which it occurs. Such species help to maintain local biodiversity within a community either by controlling populations of other species that would otherwise dominate the community or by providing critical resources for a wide range of species.
9. Species richness is the measure of number of species found in a community or that live in a certain location. In other words we can say that, species richness has been used by ecologists as measure of diversity.
10. An 'unnatural separation of expansive tracts of habitats into spatially segregated fragments' that is too limited to maintain their different species for the future, is known as habitat fragmentation. The landmass is broken into smaller units which eventually lead to the extinction of species.
11. The term 'Wildlife' refers to undomesticated animal species, but has come to include all organisms that grow or live wild in an area without being introduced by humans. Wildlife can be found in all ecosystems, such as the deserts, forests, rain forests, plains, grasslands and other areas including the most developed urban areas, all have distinct forms of wildlife.
12. The loss of animals from ecological communities is termed as defaunation.
13. Biospheres are the larger areas which are established by government of India to protect the natural habitat and often include one or more national parks and/or preserves, along buffer zones that are open to some economic uses. Protection is granted not only to the flora and fauna of the protected region, but also to the human communities who inhabit these regions, and their ways of life.
14. A national park is a park in use for conservation purposes. Often it is a reserve of natural, semi-natural, or developed land that a sovereign state declares or owns.
15. The largest national park in the world meeting the IUCN definition is the Northeast Greenland National Park, which was established in 1974.
16. There are 104 existing national parks in India covering an area of 40501.13 km2, which is 1.23% of the geographical area of the country (National Wildlife.
17. Wildlife sanctuary, is a naturally occurring sanctuary, such as an island, that provides protection for wildlife species from hunting, predation, competition or poaching; it is a protected area, a geographic territory within which wildlife is protected.
18. Zoogeography is the branch of the science of biogeography that is concerned with geographic distribution (present and past) of animal species.
19. Palaeozoology is the branch of palaeontology concerned with the study of animals throughout geological time, as revealed by their fossil remains.

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## 2.10 SUMMARY

- Systematics is the key that helps in understanding the fascinating biodiversity everywhere in the world.
- Taxonomists have significant and imperative role in detecting some of the environmental problems.
- The chemotaxonomy is the method of biological classification based on similarities in the structure of certain compounds among the organisms being classified.
- Every living cell, from a bacterium to an elephant, from grasses to the blue whale, has protoplasm in the cell.
- Phosphagens are energy reservoirs of animals. They are present in the muscles and supply energy for the synthesis of Adenosine Tri Phosphate (ATP).
- Cytotaxonomy is the specific branch of taxonomy, which uses the characteristics of cellular structures, such as somatic chromosome to classify the organism.
- Molecular taxonomy is principally effective in combination with other methods, typically with morphology.
- Molecular evolution is the process of selective changes (mutations) at a molecular level (genes, proteins, etc.)
- Term biodiversity was introduced by Walter Rosen in 1986 and later E.O. Wilson in 1988 popularized the term biodiversity.
- Depending upon the varied environmental conditions the number of species in a region varies widely.
- Biodiversity measured as an attribute that has two components — richness and evenness.
- Loss of biodiversity refers to the extinction of human, plant or animal species worldwide.
- A biodiversity hotspot is a biogeographic region that is both a significant reservoir of biodiversity and is threatened with destruction.
- Conservation of biodiversity is protection, upliftment and scientific management of biodiversity so as to maintain it at its threshold level.
- The global wildlife population decreased by 52 percent between 1970 and 2014, according to a report by the World Wildlife Fund (WWF).
- Wildlife sanctuary, is a naturally occurring sanctuary, such as an island, that provides protection for wildlife species from hunting, predation, competition or poaching.
- Paleontology is the study of fossils, their age, method of formation, and evolutionary significance of fossils.

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- Fossils are remnant, impression, or trace of an animal or plant of a past geologic age that has been preserved in Earth's crust.
- The fossils of bones, teeth, and shells are called body fossils.
- Trace fossils are rocks that have preserved evidence of biological activity.

## 2.11 KEY TERMS

- **Systematics:** It is the study of diversification and relationships of different life forms of extinct and present.
- **Chemotaxonomy**: The chemotaxonomy is the method of biological classification based on similarities in the structure of certain compounds among the organisms.
- **Hormones:** Hormones are secretions from ductless glands called the endocrine glands, such as the thyroid, pituitary, adrenal.
- **Phosphagens:** Phosphagens are energy reservoirs of animals.
- **Cytotaxonomy**: Cytotaxonomy is the branch of biology that deals with the relationships and classification of organisms using comparative studies of chromosomes.
- **Molecular taxonomy**: Molecular taxonomy is the classification of organisms on the basis of the distribution and composition of chemical substances in them.
- **Molecular phylogenetics**: Molecular phylogenetics is the branch of phylogeny that analyses genetic, hereditary molecular differences, predominately in DNA sequences, in order to obtain information on an organism's evolutionary relationships.
- **Ecological resilience:** Ecological resilience is the capacity of an ecosystem to cope with disturbance or stress and return to a stable state.
- **Genetic diversity:** Genetic diversity is variation of genes within species.
- **Species diversity:** Species diversity is diversity of species within a community.
- **Ecosystem or community diversity:** Diversity at the level of community or ecosystem.
- **Alpha diversity**: Alpha diversity refers to diversity within a particular area, community or ecosystem.
- **Beta diversity:** Beta diversity is species diversity between ecosystems; this involves comparing the number of taxa that are unique to each of the ecosystems.
- **Gamma diversity:** Gamma diversity is a measure of the overall diversity for different ecosystems within a region.
- **Hotspots of biodiversity:** A biodiversity hotspot is a biogeographic region that is both a significant reservoir of biodiversity and is threatened with destruction.

- **Protected areas:** The protected areas are biogeographical areas where biological diversity along with natural and cultural resources are protected, maintained and managed through legal and administrative measures.
- **National parks:** National parks are the small reserves meant for the protection of wild life and their natural habitats.
- **Sanctuaries:** sanctuaries are the areas where only wild animals (fauna) are present.
- **Endemic:** When a species is found only in a particular geographical region because of its isolation, soil and climatic conditions it is said to be endemic.
- **Endangered:** A species in danger of becoming extinct is known to be an endangered species.
- **Vulnerable:** A species experiencing a decline in the number of its population is a vulnerable species.
- **Extinct:** If a species permanently disappear from the wild, after repeated searches of known or likely areas where they may occur. That species is said to be an extinct species.
- **IUCN:** International Union for Conservation of Nature and Natural Resources.
- **Fossils**: A fossil is any preserved remains, impression, or trace of any once-living thing from a past geological age.
- **Permineralization:** Permineralization is a process of fossilization that occurs when an organism is buried.
- **Transitional fossil:** A transitional fossil is any fossilized remains of a life form that exhibits traits common to both an ancestral group and its derived descendant group.
- **Fossil wood:** Fossil wood is wood that is preserved in the fossil record.
- **Quantitative paleozoology**: Quantitative paleozoology is a process of taking a census of fossil types rather than inventory.
- **Vertebrate paleozoology**: Vertebrate paleozoology refers to the use of morphological, temporal, and stratigraphic data to map vertebrate history in evolutionary theory.

## 2.12 SELF-ASSESSMENT QUESTIONS AND EXERCISES

### Short-Answer Questions

1. What are main objectives of biosystematics?
2. How does a palaeontologist help in mineral prospecting?
3. What is chemotaxonomy?
4. Write a short note on phosphagens.
5. What are the different scales to measure biodiversity?

6. Which characteristics are used in cytotaxonomy for classification of organisms?
7. On what basis the classification is done in molecular taxonomy?
8. Differentiate between two haplotype assessed.
9. Define the term species area relationship.
10. What is the criteria for hotspot selection?
11. State on the principles of biodiversity.
12. Give the importance of biodiversity for humans.
13. Give the examples of some animals which are in threat of extinction because of human activities.
14. State the characteristic features of national parks.
15. What are the characteristics of wildlife sanctuary?
16. Define six main zoogeographic regions of the world.
17. What do you understand by the term animal distribution?

**Long-Answer Questions**

1. Briefly explain the importance of biosystematics.
2. Give a detail account of molecular phylogenetics.
3. Explain the different types of biodiversity with the help of relevant examples.
4. What do you understand by the term loss of biodiversity? What are the major factors that contribute to loss of biodiversity? Explain giving appropriate examples.
5. Give a detailed account on the values of biodiversity.
6. Briefly discuss the various strategies used for the conservation of biodiversity.
7. Write about the present status of wild life in India in detail.
8. Write a detail note on fossils and their types giving examples.

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# UNIT 3 BIOSTATICS

NOTES

**Structure**



## 3.0 INTRODUCTION

Biostatistics (also known as biometry) is the development and application of statistical methods to a wide range of topics in biology. It encompasses the design of biological experiments, the collection and analysis of data from those experiments and the interpretation of the results. Biostatistical modelling forms an important part of numerous modern biological theories. Genetics studies, since its beginning, used statistical concepts to understand observed experimental results.

Some genetics scientists even contributed with statistical advances with the development of methods and tools. Gregor Mendel started the genetics studies

Investigating genetics segregation patterns in families of peas and used statistics to explain the collected data.

The word 'Statistics' can be referred to in two ways. In a common way, it refers simply to numerical statements of facts such as the number of children in a family, the number of books on statistics in the college library. The second meaning of statistics refers to the field of study rather than simply to numerical statements.

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As an area of study, it is primarily concerned with making scientific and rational decisions about various properties and characteristics of some population of interest, such as stock market trends, interest rates, demographic shifts, inflation rates over the years.

In addition 'Biostatistics' help us to understand our environment better, and further help us in formulating specific policies and attitudes to address and solve issues of interest.

In this unit you will study about general concept and significance of biostatics to biosciences, probability theory, probability distribution and their applications, correlation and regression, sampling theory and experimental designing, uses and applications of 'Chi-Square' test and *t*-test.

## 3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the general concept and significance of biostatics to bioscience
- Explain the probability theory
- Describe about the probability distribution and their applications
- Understand the correlation and regression analyses
- Comprehend on the sampling theory and experimental designing
- Elaborate on the uses and applications of Chi-Square test
- Understand the significance of *t*-test

## 3.2 GENERAL CONCEPT AND SIGNIFICANCE OF BIOSTATICS TO BIOSCIENCE

**Biostatistics** (also known as **biometry**) is the development and application of statistical methods to a wide range of topics in biology. It is includes the design of biological experiments, the collection and analysis of data from those experiments and the interpretation of the results.

### Biostatistics and Genetics

Biostatistical modelling forms an important part of numerous modern biological theories. Genetics studies, since its beginning, used statistical concepts to understand observed experimental results. Some genetics scientists even contributed with statistical advances with the development of methods and tools. Gregor Mendel started the genetics studies investigating genetics segregation patterns in families of peas and used statistics to explain the collected data. In the early 1900s, after the rediscovery of Mendel's Mendelian inheritance work, there were gaps in understanding between genetics and evolutionary Darwinism. Francis Galton tried to expand Mendel's discoveries with human data and proposed a different model with fractions of the heredity coming from each ancestral composing an infinite series. He called this the theory of 'Law of Ancestral Heredity'. His ideas were strongly disagreed by William Bateson, who followed Mendel's conclusions that genetic inheritance were exclusively from the parents, half from each of them.

### Definition

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Most business decisions are made today on the basis of relevant information and statistical analysis of such information. Quantitative analysis has replaced intuition and experienced guess work in solving most business problems. One of the tools to understand information is statistics.

In general, business statistics can be defined as 'a body of methods for obtaining, organizing, summarizing, presenting, interpreting, analysing and acting upon numerical facts related to an activity of interest. Numerical facts are usually subjected to statistical analysis with a view to helping a decision-maker make wise decisions in the face of uncertainty'.

The word 'Statistics' can be referred to in two ways. In a common way, it refers simply to numerical statements of facts such as the number of children in a family, the number of books on statistics in the college library, the number of students enrolled in the department of economics in Delhi University, and so on. The following statements indicate the use of statistics as referring to numbers.

- Around 20 million Americans have a serious drinking problem.
- Nearly 52,000 Americans died in automobile accidents last year.
- More than 76 per cent voters turned out to vote during elections in Punjab in February 2007.
- Majority of Americans consider Japanese cars superior in quality than American cars.

All these statements represent statistical conclusions in some form. These conclusions help us in formulating specific policies and attitudes with respect to diverse areas of interest.

The second meaning of statistics refers to the field of study rather than simply to numerical statements. As an area of study, it is primarily concerned with making scientific and rational decisions about various properties and characteristics of some population of interest, such as stock market trends, interest rates, demographic shifts, inflation rates over the years, and so on. Consider the following statistical statements:

- The crime rate in the city has gone up by 15 per cent over what it was last year. (This statistical conclusion could help us in making decisions regarding our safety and security in the city).
- The rate of inflation is expected to remain less than 5 per cent per year over the next five years. (This could help us in making more educated judgements about the general economic health of the country in the near future).
- Less than 20 per cent of all high school graduates enter colleges for higher education and less than 40 per cent of those who do enter colleges actually graduate. (This statement gives us a good indication of the educational philosophy of the country and the community and the reasons for such low rates of admission into colleges and graduation could be investigated).

All these statements represent statistical conclusions in some form, which help us understand our environment better, and further help us in formulating specific policies and attitudes to address and solve issues of interest.

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### 3.2.1 Descriptive Statistics

As the name suggests, descriptive statistics merely describe the data and consist of methods and techniques used in collection, organization, presentation and analysis of data in order to describe the various features and characteristics of such data. These methods can either be graphical or computational. Thus data can be presented in the form of a chart or a table in order to show certain trends, proportions, maximum and minimum values, and so on. For example, if we simply describe the number of workers in different types of industries in America, then that would constitute descriptive statistics. In addition to the organization of data, the field of descriptive statistics is concerned with the analysis of data so that the data can be easily understood. Averages, proportions and other measures that describe the spread of data around the average are also some of the measures used to describe the data. By using these measures, we summarize the data and even though we may lose the detail, we gain clarity and compactness. For example, the following statistics, in their most summarized presentation describe in some way the characteristics of the population from which they were drawn.

- The ages of students in my statistics class range from 19 to 45 years.
- The average IQ of students at our college is 140.
- 20 per cent of the students in my class are married.

All these examples simply summarize and describe the data. Not much can be inferred from them, nor can definite decisions be made or conclusions drawn.

For a proper appreciation of the various descriptive statistics involved, it is necessary to note that most of the statistical distribution have some common features. Though the size of the variables varies from item to item, most of the items are distributed in such a manner that if we move from the lowest value to the highest value of the variable, the number of items at each successive stage increases with a certain amount of regularity till we reach a maximum; and then as we proceed further, they decrease with the similar regularity. If we plot the percentage frequency density, i.e., the percentage of cases in an interval of unit variable width, we get frequency curves of the type shown in Figure 3.1. (Note that the area under each curve should be equal to 100, the total percentage points).

There are various 'Gross' ways in which frequency curves can differ from one another. Even when the 'General' shapes of the curves are the same (the area under them already made equal by the strategy of plotting the per cent density), the details of the shape may change. Thus the curve *B* has a smaller spread than *A*, the curve *C* is more peaky and curve *E* is less symmetrical. Even when the curves have almost the same shape (i.e., same spread, peakness, symmetry, etc.) as in curves *A* and *D*, the two may differ in location along the variable axis. Thus the items of distribution *D* are generally larger than those of *A*. So are those of *B* compared to *A*. Thus, a kind of an 'Average' location of the distribution along the variable axis is an important descriptive statistics. These statistics are collectively known as measures of location or of central tendency.

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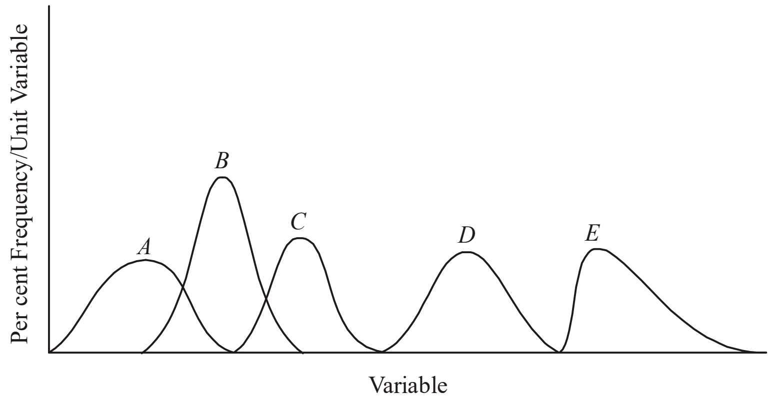


***Fig. 3.1*** *Representation of Measures of Central Tendency*

### 3.2.2 Inferential Statistics

Inferential statistics can be defined as those methods that are used to estimate a characteristic of a population or making of a decision concerning a population on the basis of the results obtained from a sample taken from the same population. The measured characteristics of the sample are known as sample statistics, while the measured characteristics of the population are known as population parameters. A major portion of statistics deals with making decisions, inferences, predictions and forecasts about the population based on the results obtained from samples taken from such populations.

The need for inferential statistical methods derives from the need for sampling. As the population becomes large, it is usually too costly, too time consuming and too cumbersome to take the entire population into consideration in order to obtain our information of interest. Of course, the results obtained from the entire population are the most accurate and if the population indeed is small, then it is advisable to consider the entire population. However, when the population is large, sometimes considered infinite, then sampling method is used.

The question is, How do these sample statistics relate to population parameters? Can we state that the conclusions drawn from the analysis of the sample are exactly the same as the conclusions that would be drawn from the entire population from which the representative sample was taken? The answer is unlikely. How close is the sample characteristics to the population characteristics would depend upon the randomness of the sample as well as the size of the sample? The more random the sample is and larger the sample is, the more closely its characteristics would be with the population characteristics. This link, in terms of the degree of closeness is provided by probability theory. Probability theory provides the link by ascertaining the likelihood that the results from the sample reflect the results from the population.

Our interest is not in finding the characteristics of a sample but our to find the characteristics of the population. Sampling is simply a means to the end. For example, if we want to know the salary of university professors, we mean the salary of all university professors and not simply of the sample we have taken. Only then can observations and decisions be made in this regard. Similarly, if we want to know what percentage of eligible voters will vote for Congress in the next general elections in India, a sample in itself would not indicate that, and we cannot ask the entire population. Our decisions and projections would be based on the inclination of the entire population. A sample in itself would not mean much, if any thing. How ever, if the sample truly represents the population, then we can draw conclusions about the population on the basis of sample results. Appended to these conclusions will be a probability statement specifying the likelihood or confidence that the results from the sample reflect the voting behaviour of the population. Usually, the margin of error is stated as plus or minus three to five per cent.

Statistical inference deals with methods of inferring or drawing conclusions about the characteristics of the population based upon the results of the sample taken from the same population. The measured characteristics of the sample are called sample statistics and the measured characteristics of the population are known as population parameters. The question is: How do these sample statistics relate to population parameters? Can we state that the conclusions drawn from the analysis of the sample are exactly the same as the conclusions that would be drawn from the entire population from which the representative sample was taken?

Following are some of the situations that the field of inferential statistics deals with.

(a) Between 35 per cent and 40 per cent of graduate students in the universities are married. These statistics refer to the entire population of graduate students. It would be reasonable to assume that these percentages were calculated on the basis of samples taken from the population of all graduate students. The students in these samples were asked in order to know as to how many of these students were married. The answers formed the basis for drawing conclusions about the entire population of the graduate students.

(b) There is a definitive association between smoking and lung cancer. This statement is the result of endless research on many samples taken and studied in order to find out if there was any correlation between smoking and lung cancer and based upon the results thus obtained from sample studies, a valid statement about the association of smoking with lung cancer in the whole population can be made.

(c) 30 per cent of all television viewers watched the show 20/20 last night. This statement can be compared with the following statement: 30 per cent of those who were interviewed watched the show 20/20 last night. The latter statement is descriptive statistics since it is only presenting the data in a summarized form. However, if we infer from the second statement to reach at the first statement, then the first statement is an example of statistical inference.

(d) Suppose that the Chancellor of Punjabi University wanted to conduct a survey to learn about student perceptions concerning the quality of life on campus. The population will be all the students enrolled in the university, while a sample will consist of only the students who have been randomly selected to be included in the sample to participate in the survey. The goal is to determine various attitudes and characteristics of interest relating to quality of student life in the entire university using the sample statistics to draw conclusions about the similar population characteristics

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(e) Between 35 per cent to 40 per cent of graduate students in the universities are married. These statistics refer to the entire population of graduate students. It would be reasonable to presume that these percentages were calculated on the basis of samples taken from the population of all graduate students. The students in these samples were asked in order to know how many of these students were married. The answers formed the basis for drawing conclusions about the entire population of graduate students.

(f) There is a definite association between smoking and lung cancer. This statement is the result of endless research on many samples taken and studied in order to find out if there was any correlation between smoking and lung cancer and based upon the results thus obtained from these sample studies, a valid statement about the association of smoking with lung cancer in the whole population could be made.

## 3.3 PROBABILITY THEORY

The probability theory helps a decision-maker to analyse a situation and decide accordingly. The following are few examples of such situations:

- What is the *chance* that sales will increase if the price of the product is decreased?
- What is the *likelihood* that a new machine will increase productivity?
- How *likely* is it that a given project will be completed in time?
- What are the possibilities that a competitor will introduce a cheaper substitute in the market?

Probability theory is also called the theory of chance and can be mathematically derived using the standard formulas. A probability is expressed as a real number, $p \in [0, 1]$ and the probability number is expressed as a percentage (0 per cent to 100 per cent) and not as a decimal. For example, a probability of 0.55 is expressed as 55 per cent. When we say that the probability is 100 per cent, it means that the event is certain while the 0 per cent probability means that the event is impossible. We can also express probability of an outcome in the ratio format. For example, we have two probabilities, i.e., 'Chance of Winning' (1/4) and 'Chance of Not Winning' (3/4), then using the mathematical formula of odds, we can say,

'Chance of Winning' : 'Chance of Not Winning' = 1/4 : 3/4 = 1 : 3 or 1/3

We are using the probability in vague terms when we predict something for future. For example, we might say it will probably rain tomorrow or it will probably be a holiday the day after. This is subjective probability to the person predicting, but implies that the person believes the probability is greater than 50 per cent.

Different types of probability theories are as follows:

(i) Axiomatic Probability Theory

(ii) Classical Theory of Probability

(iii) Empirical Probability Theory

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**(i) Axiomatic Probability Theory**

The axiomatic probability theory is the most general approach to probability, and is used for more difficult problems in probability. We start with a set of axioms, which serve to define a probability space. These axioms are not immediately intuitive and are developed using the classical probability theory.

**(ii) Classical Theory of Probability**

The classical theory of probability is the theory based on the number of favourable outcomes and the number of total outcomes. The probability is expressed as a ratio of these two numbers. The term 'Favourable' is not the subjective value given to the outcomes, but is rather the classical terminology used to indicate that an outcome belongs to a given event of interest.

### 3.3.1 Classical Definition of Probability

If the number of outcomes belonging to an event $E$ is $N_E$, and the total number of outcomes is $N$, then the probability of event $E$ is defined as $p_E = \frac{N_E}{N}$.

For example, a standard pack of cards (without jokers) has 52 cards. If we randomly draw a card from the pack, we can imagine about each card as a possible outcome. Therefore, there are 52 total outcomes. Calculating all the outcome events and their probabilities, we have the following possibilities:

- Out of the 52 cards, there are 13 clubs. Therefore, if the event of interest is drawing a club, there are 13 favourable outcomes, and the probability of this event becomes $\frac{13}{52} = \frac{1}{4}$.
- There are 4 kings (one of each suit). The probability of drawing a king is $\frac{4}{52} = \frac{1}{13}$.
- What is the probability of drawing a king or a club? This example is slightly more complicated. We cannot simply add together the number of outcomes for each event separately (4 + 13 = 17) as this inadvertently counts one of the outcomes twice (the king of clubs). The correct answer is $\frac{16}{52}$ from $\frac{13}{52} + \frac{4}{52} - \frac{1}{52}$.

  We have this from the probability equation, $P$(club) + $P$(king) – $P$(king of clubs).
- Classical probability has limitations, because this definition of probability implicitly defines all outcomes to be equiprobable and this can be only used for conditions such as drawing cards, rolling dice, or pulling balls from urns. We cannot calculate the probability where the outcomes are unequal probabilities.

It is not that the classical theory of probability is not useful because of the described limitations. We can use this as an important guiding factor to calculate the probability of uncertain situations as just mentioned and to calculate the axiomatic approach to probability.

### 3.3.2 Frequency of Occurrence

This approach to probability is used for a wide range of scientific disciplines. It is based on the idea that the underlying probability of an event can be measured by repeated trials.

**Probability as a Measure of Frequency:** Let $n_A$ be the number of times event $A$ occurs after $n$ trials. We define the probability of event $A$ as,

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}$$

It is not possible to conduct an infinite number of trials. However, it usually suffices to conduct a large number of trials, where the standard of large depends on the probability being measured and how accurate a measurement we need.

***Definition of Orobability***

To understand whether the sequence $\frac{n_A}{n}$ in the limit will converge to the same result every time, or it will not converge at all let us consider an experiment consisting of flipping a coin an infinite number of times. We want that the probability of heads must come up. The result may appear as the following sequence:

*HTHHTTHHHHTTTTHHHHHHHHTTTTTTTTHHHHHHHHHHHHH HHHTTTTTTTTTTTTTTTT...*

This shows that each run of $k$ heads and $k$ tails are being followed by another run of the same probability. For this example, the sequence $\frac{n_A}{n}$ oscillates between, $\frac{1}{3}$ and $\frac{2}{3}$ which does not converge. These sequences may be unlikely, and can be right. The definition given above does not express convergence in the required way, but it shows some kind of convergence in probability. The problem of exact formulation can be solved using the axiomatic probability theory.

### 3.3.3 Empirical Probability Theory

The empirical approach to determine probabilities relies on data from actual experiments to determine approximate probabilities instead of the assumption of equal likeliness. Probabilities in these experiments are defined as the ratio of the frequency of the possibility of an event, $f(E)$, to the number of trials in the experiment, $n$, written symbolically as $P(E) = f(E)/n$. For example, while flipping a coin, the empirical probability of heads is the number of heads divided by the total number of flips.

The relationship between these empirical probabilities and the theoretical probabilities is suggested by the 'Law of Large Numbers'. The law states that as the number of trials of an experiment increases, the empirical probability approaches the theoretical probability. Hence, if we roll a die a number of times, each number would come up approximately 1/6 of the time. The study of empirical probabilities is known as *statistics*.

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### Sample Space

A sample space is the collection of all possible events or outcomes of an experiment. For example, there are two possible outcomes of a toss of a fair coin: a head and a tail. Then, the sample space for this experiment denoted by $S$ would be,

$$S = [H, T]$$

So that the probability of the sample space equals 1, or

$$P[S] = P[H, T] = 1$$

This is so because in the toss of the coin, either a head or a tail, must occur. Similarly, when we roll a die, any of the six faces can come as a result of the roll since there are a total of six faces. Hence, the sample space is $S = [1, 2, 3, 4, 5, 6]$, and $P[S] = 1$, since one of the six faces must occur.

### Events

An event is an outcome or a set of outcomes of an activity or a result of a trial. For example, getting two heads in the trial of tossing three fair coins simultaneously would be an event. The following are the types of events:

- **Elementary Event:** An elementary event, also known as a simple event, is a single possible outcome of an experiment. For example, if we toss a fair coin, then the event of a head coming up is an elementary event. If the symbol for an elementary event is ($E$), then the probability of the event ($E$) is written as $P[E]$.
- **Joint Event:** A joint event, also known as a compound event, has two or more elementary events in it. For example, drawing a black ace from a pack of cards would be a joint event, since it contains two elementary events of black and ace.
- **Simple Probability:** Simple probability refers to a phenomenon where only a simple or elementary event occurs. For example, assume that event ($E$), the drawing of a diamond card from a pack of 52 cards, is a simple event. Since there are 13 diamond cards in the pack and each card is equally likely to be drawn, the probability of event ($E$) or $P[E] = 13/52$ or 1/4.
- **Joint Probability:** The joint probability refers to the phenomenon of occurrence of two or more simple events. For example, assume that event ($E$) is a joint event (or compound event) of drawing a black ace from a pack of cards. There are two simple events involved in the compound event, which are, the card being black and the card being an ace. Hence, $P$[Black ace] or $P[E] = 2/52$ since there are two black aces in the pack.
- **Complement of an Event:** The complement of any event $A$ is the collection of outcomes that are not contained in $A$. This complement of $A$ is denoted as $A¢$ ($A$ prime). This means that the outcomes contained in $A$ and the outcomes contained in $A¢$ must equal the total sample space. Therefore,

$$P[A] + P[A'] = P[S] = 1$$

Or, $$P[A] = 1 - P[A']$$

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For example, if a passenger airliner has 300 seats and it is nearly full, but not totally full, then event $A$ would be the number of occupied seats and $A'$ would be the number of unoccupied seats. Suppose there are 287 seats occupied by passengers and only 13 seats are empty. Typically, the stewardess will count the number of empty seats which are only 13 and report that 287 people are aboard. This is much simpler than counting 287 occupied seats. Accordingly, in such a situation, knowing event $A'$ is much more efficient than knowing event $A$.

- **Mutually-Exclusive Events:** Two events are said to be mutually exclusive, if both events cannot occur at the same time as outcome of a single experiment. For example, if we toss a coin, then either event head or event tail would occur, but not both. Hence, these are mutually exclusive events.

### 3.3.4 Venn Diagrams

We can visualize the concept of events, their relationships and sample space using Venn diagrams. The sample space is represented by a rectangular region and the events and the relationships among these events are represented by circular regions within the rectangle.

For example, two mutually exclusive events $A$ and $B$ are represented in the Venn diagram in Figure 3.2.

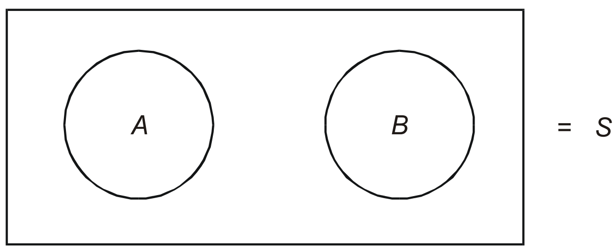


***Fig. 3.2*** *Venn Diagram of two Mutually Exclusive Events A and B*

Event $P[A \cup B]$ is represented in the Venn diagram in Figure 3.3.

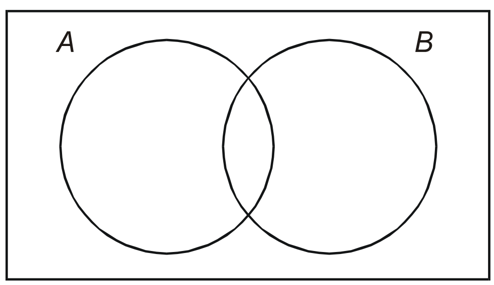


***Fig. 3.3*** *Venn Diagram Showing Event* $P[A \cup B]$

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Event [*AB*] is represented in Figure 3.4.

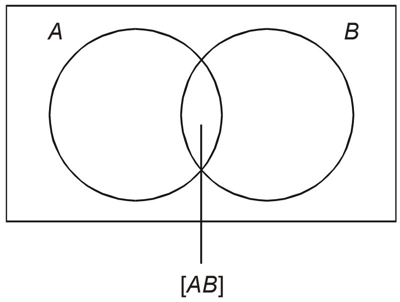


***Fig. 3.4*** *Venn Diagram Showing Event [A B ]*

**Union of Three Events**

The process of combining two events to form the union can be extended to three events so that $P[A \cup B \cup C]$ would be the union of events *A*, *B*, and *C*. This union can be represented in a Venn diagram as in Figure 3.5. Example 3.1 explains the union of three events better:

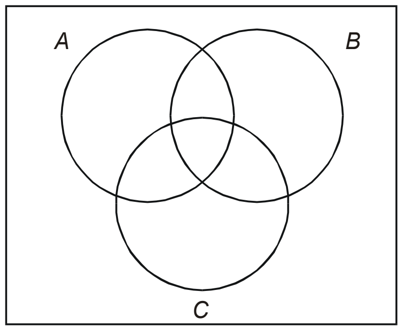


***Fig. 3.5*** *Venn Diagram Showing Union of Three Events $P[A \cup B \cup C]$*

**Example 3.1:** A sample of 50 students is taken and a survey is made on the reading habits of the sample selected. The survey results are shown as follows:

| *Event* | *Number of students* | *Magazine they read* |
|---|---|---|
| [*A*] | 20 | Time |
| [*B*] | 15 | Newsweek |
| [*C*] | 10 | Filmfare |
| [*AB*] | 8 | Time and Newsweek |
| [*AC*] | 6 | Time and Filmfare |
| [*BC*] | 4 | Newsweek and Filmfare |
| [*ABC*] | 2 | Time and Newsweek and Filmfare |

Find out the probability that a student picked up at random from this sample of 50 students does not read any of these three magazines.

**Solution:**

The problem can be solved by a Venn diagram as follows:

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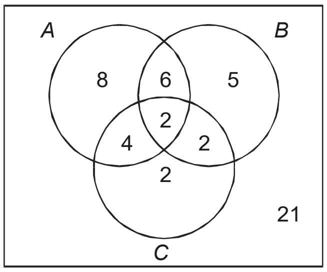


Since there are 21 students who do not read any of the three magazines, the probability that a student picked up at random among this sample of 50 students who does not read any of these three magazines is 21/50.

The problem can also be solved by the formula for probability for union of three events, given as follows:

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[AB] - P[AC] - P[BC] + P[ABC]$$
$$= 20/50 + 15/50 + 10/50 - 8/50 - 6/50 - 4/50 + 2/50$$
$$= 29/50$$

The above is the probability that a student picked up at random among the sample of 50 reads either *Time* or *Newsweek* or *Filmfare* or any combination of the two or all the three. Hence, the probability that such a student does not read any of these three magazines is 21/50 which is [1 – 29/50].

### 3.3.5 Addition and Multiplication Theorems on Probability

**Law of Addition**

When two events are mutually exclusive, then the probability that either of the events will occur is the sum of their separate probabilities. For example, if you roll a single die, then the probability that it will come up with a face 5 or face 6, where event *A* refers to face 5 and event *B* refers to face 6 and both events being mutually exclusive events, is given by,

$$P[A \text{ or } B] = P[A] + P[B]$$

Or,

$$P[5 \text{ or } 6] = P[5] + P[6] = 1/6 + 1/6 = 2/6 = 1/3$$

*P* [*A* or *B*] is written as $P[A \cup B]$ and is known as *P* [*A* union *B*].

However, if events *A* and *B* are not mutually exclusive, then the probability of occurrence of either event *A* or event *B* or both is equal to the probability that event *A* occurs plus the probability that event *B* occurs minus the probability that events common to both *A* and *B* occur.

Symbolically, it can be written as,

$$P[A \cup B] = P[A] + P[B] - P[A \text{ and } B]$$

*P*[*A* and *B*] can also be written as $P[A \cap B]$, known as *P* [*A* intersection *B*] or simply *P*[*AB*].

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Events [$A$ and $B$] consist of all those events which are contained in both $A$ and $B$ simultaneously. For example, in an experiment of taking cards out of a pack of 52 playing cards, assume the following:

Event $A$ = An ace is drawn.
Event $B$ = A spade is drawn.
Event [$AB$] = An ace of spade is drawn.

Hence, $P[A \cup B] = P[A] + P[B] - P[AB]$
$= 4/52 + 13/52 - 1/52$
$= 16/52$
$= 4/13$

This is so, because there are 4 aces, 13 cards of spades, including 1 ace of spades out of a total of 52 cards in the pack. The logic behind subtracting $P[AB]$ is that the ace of spades is counted twice—once in event $A$ (4 aces) and once again in event $B$ (13 cards of spade including the ace).

Another example for $P[A \cup B]$, where event $A$ and event $B$ are not mutually exclusive is as follows:

Suppose a survey of 100 persons revealed that 50 persons read *India Today* and 30 persons read *Time* magazine and 10 of these 100 persons read both *India Today* and *Time.* Then,

Event [$A$] = 50
Event [$B$] = 30
Event [$AB$] = 10

Since event [$AB$] of 10 is included twice, both in event $A$ as well as in event $B$, event [$AB$] must be subtracted once in order to determine the event $[A \cup B]$ which means that a person reads *India Today* or *Time* or both. Hence,

$$P[A \cup B] = P[A] + P[B] - P[AB]$$
$$= 50/100 + 30/100 - 10/100$$
$$= 70/100 = 0.7$$

**Law of Multiplication**

Multiplication rule is applied when it is necessary to compute the probability if both events $A$ and $B$ will occur at the same time. The multiplication rule is different if the two events are independent as against the two events being not independent.

If events $A$ and $B$ are independent events, then the probability that they both will occur is the product of their separate probabilities. This is a strict condition so that events $A$ and $B$ are independent if and only if,

$$P[AB] = P[A] \times P[B]$$

Or $\quad = P[A]\,P[B]$

For example, if we toss a coin twice, then the probability that the first toss results in a head and the second toss results in a tail is given by,

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$P[HT] = P[H] \times P[T]$

$= 1/2 \times 1/2 = 1/4$

However, if events $A$ and $B$ are not independent, meaning that the probability of occurrence of an event is dependent or conditional upon the occurrence or non-occurrence of the other event, then the probability that they will both occur is given by,

$P[AB] = P[A] \times P[B/\text{Given outcome of } A]$

This relationship is written as,

$P[AB] = P[A] \times P[B/A] = P[A]\, P[B/A]$

Where, $P[B/A]$ means the probability of event $B$ on the condition that event $A$ has occurred. As an example, assume that a bowl has 6 black balls and 4 white balls. A ball is drawn at random from the bowl. Then a second ball is drawn without replacement of the first ball back in the bowl. The probability of the second ball being black or white would depend upon the result of the first draw as to whether the first ball was black or white. The probability that both these balls are black is given by,

$P$ [Two black balls] = $P$ [Black on 1st draw] × $P$ [Black on 2nd draw/ Black on 1st draw]

$= 6/10 \times 5/9 = 30/90 = 1/3$

This is so, because there are 6 black balls out of a total of 10, but if the first ball drawn is black then we are left with 5 black balls out of a total of 9 balls.

### Independent Events

Two events $A$ and $B$ are said to be independent events, if the occurrence of one event is not influenced at all by the occurrence of the other. For example, if two fair coins are tossed, then the result of one toss is totally independent of the result of the other toss. The probability that a head will be the outcome of any one toss will always be 1/2, irrespective of whatever the outcome is of the other toss. Hence, these two events are independent.

Let us assume that one fair coin is tossed 10 times and it happens that the first nine tosses resulted in heads. What is the probability that the outcome of the tenth toss will also be a head? There is always a psychological tendency to think that a tail would be more likely in the tenth toss since the first nine tosses resulted in heads. However, since the events of tossing a coin 10 times are all independent events, the earlier outcomes have no influence whatsoever on the result of the tenth toss. Hence, the probability that the outcome will be a head on the tenth toss is still 1/2.

On the other hand, consider drawing two cards from a pack of 52 playing cards. The probability that the second card will be an ace would depend upon whether the first card was an ace or not. Hence, these two events are not independent events.

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**Conditional Probability**

In many situations, a manager may know the outcome of an event that has already occurred and may want to know the chances of a second event occurring based upon the knowledge of the outcome of the earlier event. We are interested in finding out as to how additional information obtained as a result of the knowledge about the outcome of an event affects the probability of the occurrence of the second event. For example, let us assume that a new brand of toothpaste is being introduced in the market. Based on the study of competitive markets, the manufacturer has some idea about the chances of its success. Now, he introduces the product in a few selected stores in a few selected areas before marketing it nationally. A highly positive response from the test-market area will improve his confidence about the success of his brand nationally. Accordingly, the manufacturer's assessment of high probability of sales for his brand would be conditional upon the positive response from the test-market.

Let there be two events $A$ and $B$. Then the probability that event $A$ occurs, given that event $B$ has occured. The notation is given by,

$$P[A/B] = \frac{P[AB]}{P[B]}$$

Where $P[A/B]$ is interpreted as the probability of event $A$ on the condition that event $B$ has occurred and $P\ [AB]$ is the joint probability of event $A$ and event $B$, and $P[B]$ is not equal to zero.

As an example, let us suppose that we roll a die and we know that the number that came up is larger than 4. We want to find out the probability that the outcome is an even number given that it is larger than 4.

Let, Event $A$ = Even

And Event $B$ = Larger than 4

Then, $$\text{P[Even / Larger than 4]} = \frac{P\text{[Even and larger than 4]}}{P\text{[Larger than 4]}}$$

Or $$P[A/B] = \frac{P[AB]}{P[B]} = (1/6)/(2/6) = 1/2$$

For, however independent events, $P[AB] = P[A]\ P[B]$. Thus, substituting this relationship in the formula for conditional probability, we get,

$$P[A/B] = \frac{P[AB]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

This means that $P[A]$ will remain the same no matter what the outcome of event $B$ is. For example, if we want to find out the probability of a head on the second toss of a fair coin, given that the outcome of the first toss was a head, this probability would still be 1/2 because the two events are independent events and the outcome of the first toss does not affect the outcome of the second toss.

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### 3.3.6 Bayes' Theorem

Reverend Thomas Bayes (1702–1761), introduced his theorem on probability, which is concerned with a method for estimating the probability of causes which are responsible for the outcome of an observed effect. Being a religious preacher himself as well as a mathematician, his motivation for the theorem came from his desire to prove the existence of God by looking at the evidence of the world that God created. He was interested in drawing conclusions about the causes by observing the consequences. The theorem contributes to the statistical decision theory in revising prior probabilities of outcomes of events based upon the observation and analysis of additional information.

Bayes' theorem makes use of conditional probability formula where the condition can be described in terms of the additional information which would result in the revised probability of the outcome of an event.

Suppose that, there are 50 students in our statistics class out of which 20 are male students and 30 are female students. Out of the 30 females, 20 are Indian students and 10 are foreign students. Out of the 20 male students, 15 are Indians and 5 are foreigners, so that out of all the 50 students, 35 are Indians and 15 are foreigners. This data can be presented in a tabular form as follows:

| | ***Indian*** | ***Foreigner*** | ***Total*** |
|---|---|---|---|
| Male | 15 | 5 | 20 |
| Female | 20 | 10 | 30 |
| Total | 35 | 15 | 50 |

Based upon this information, the probability that a student picked up at random will be female is 30/50 or 0.6, since there are 30 females in the total class of 50 students. Now suppose that we are given additional information that the person picked up at random is Indian, then what is the probability that this person is a female? This additional information will result in revised probability or posterior probability in the sense that it is assigned to the outcome of the event after this additional information is made available.

Since we are interested in the revised probability of picking a female student at random provided that we know that the student is Indian. Let $A_1$ be the event female, $A_2$ be the event male and B be the event Indian. Then based upon our knowledge of conditional probability, the Bayes' theorem can be stated as,

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1) + P(A_2)(P(B/A_2)}$$

In the example discussed, there are two basic events which are $A_1$ (female) and $A_2$ (male). However, if there are $n$ basic events, $A_1, A_2, .....A_n$, then Bayes' theorem can be generalized as,

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1) + P(A_2)(P(B/A_2) + ... + P(A_n)P(B/A_n)}$$

Solving the case of two events we have,

$$P(A_1 / B) = \frac{(30/50)(20/30)}{(30/50)(20/30)+(20/50)(15/20)} = 20/35 = 4/7 = 0.57$$

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This example shows that while the prior probability of picking up a female student is 0.6, the posterior probability becomes 0.57 after the additional information that the student is an American is incorporated in the problem.

Refer Example 3.2 to understand the theorem better.

**Example 3.2:** A businessman wants to construct a hotel in New Delhi. He generally builds three types of hotels. These are hotels with 50 rooms, 100 rooms and 150 rooms, depending upon the demand for rooms, which is a function of the area in which the hotel is located, and the traffic flow. The demand can be categorized as low, medium or high. Depending upon these various demands, the businessman has made some preliminary assessment of his net profits and possible losses (in thousands of dollars) for these various types of hotels. These pay-offs are shown in the following table:

| | Demand for Rooms | | | |
|---|---|---|---|---|
| | *Low* ($A_1$) | *Medium* ($A_2$) | *High* ($A_3$) | |
| | 0.2 | 0.5 | 0.3 | Demand Probability |
| $R_1$=(50) | 25 | 35 | 50 | Number of Rooms |
| $R_2$=(100) | –10 | 40 | 70 | |
| $R_3$=(150) | –30 | 20 | 100 | |

**Solution:**

The businessman has also assigned 'Prior Probabilities' to the demand structure or rooms. These probabilities reflect the initial judgement of the businessman based upon his intuition and his degree of belief regarding the outcomes of the states of nature.

| *Demand for Rooms* | *Probability of Demand* |
|---|---|
| Low ($A_1$) | 0.2 |
| Medium ($A_2$) | 0.5 |
| High ($A_3$) | 0.3 |

Based upon these values, the expected pay-offs for various rooms can be computed as,

$$EV\,(50) = (25 \times 0.2) + (35 \times 0.5) + (50 \times 0.3) = 37.50$$

$$EV\,(100) = (-10 \times 0.2) + (40 \times 0.5) + (70 \times 0.3) = 39.00$$

$$EV\,(150) = (-30 \times 0.2) + (20 \times 0.5) + (100 \times 0.3) = 34.00$$

This gives us the maximum pay-off of $39,000 for building a 100 rooms hotel.

Now, the hotelier must decide whether to gather additional information regarding the states of nature, so that these states can be predicted more accurately than the preliminary assessment. The basis of such a decision would be the cost of

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obtaining additional information. If this cost is less than the increase in maximum expected profit, then such additional information is justified.

Suppose that the businessman asks a consultant to study the market and predict the states of nature more accurately. This study is going to cost the businessman \$10,000. This cost would be justified if the maximum expected profit with the new states of nature is at least \$10,000 more than the expected pay-off with the prior probabilities. The consultant made some studies and came up with the estimates of low demand ($X_1$), medium demand ($X_2$), and high demand ($X_3$) with a degree of reliability in these estimates. This degree of reliability is expressed as conditional probability, which is the probability that the consultant's estimate of low demand will be correct and the demand will be actually low. Similarly, there will be a conditional probability of the consultant's estimate of medium demand, when the demand is actually low and, so on. These conditional probabilities are expressed in the following table:

*Conditional Probabilities*

| | | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|---|
| States of | ($A_1$) | 0.5 | 0.3 | 0.2 |
| Nature | ($A_2$) | 0.2 | 0.6 | 0.2 |
| (Demand) | ($A_3$) | 0.1 | 0.3 | 0.6 |

The values in the preceding table are conditional probabilities and are interpreted as follows:

The first value of 0.5 is the probability that the consultant's prediction will be for low demand ($X_1$) when the demand is actually low. Similarly, the probability is 0.3 that the consultant's estimate will be for medium demand ($X_2$) when in fact the demand is low and so on. In other words, $P(X_1/A_1) = 0.5$ and $P(X_2/A_1) = 0.3$. Similarly, $P(X_1/A_2) = 0.2$ and $P(X_2/A_2) = 0.6$, and so on.

Our objective is to obtain posteriors which are computed by taking the additional information into consideration. One way to reach this objective is to first compute the joint probability, which is the product of prior probability and conditional probability for each state of nature. Joint probabilities as computed is given as,

*Joint Probabilities*

| *State of Nature* | *Prior Probability* | $P(A_1X_1)$ | *Joint Probabilities* $P(A_1X_2)$ | $P(A_1X_3)$ |
|---|---|---|---|---|
| $A_1$ | 0.2 | $0.2 \times 0.5 = 0.10$ | $0.2 \times 0.3 = 0.06$ | $0.2 \times 0.2 = 0.04$ |
| $A_2$ | 0.5 | $0.5 \times 0.2 = 0.10$ | $0.5 \times 0.6 = 0.30$ | $0.5 \times 0.2 = 0.10$ |
| $A_3$ | 0.3 | $0.3 \times 0.1 = 0.03$ | $0.3 \times 0.3 = 0.09$ | $0.3 \times 0.6 = 0.18$ |
| Total Marginal Probabilities. | | $= 0.23$ | $= 0.45$ | $= 0.32$ |

Now, the posterior probabilities for each state of nature $A_i$ are calculated as,

$$P(A_i / X_j) = \frac{\text{Joint probability of } A_i \text{ and } X_j}{\text{Marginal probability of } X_j}$$

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By using this formula, the joint probabilities are converted into posterior probabilities and the computed table for these posterior probabilities is given as,

| *States of Nature* | *Posterior Probabilities* | | |
|---|---|---|---|
| | $P(A_1/X_1)$ | $P(A_1/X_2)$ | $P(A_1/X_3)$ |
| $A_1$ | 0.1/0.23 = 0.435 | 0.06/0.45 = 0.133 | 0.04/0.32 = 0.125 |
| $A_2$ | 0.1/0.23 = 0.435 | 0.30/0.45 = 0.667 | 0.1/0.32 = 0.312 |
| $A_3$ | 0.03/0.23 = 0.130 | 0.09/0.45 = 0.200 | 0.18/0.32 = 0.563 |
| Total | = 1.0 | = 1.0 | = 1.0 |

Now, we have to compute the expected pay-offs for each course of action with the new posterior probabilities assigned to each state of nature. The net profits for each course of action for a given state of nature is the same as before and is restated. These net profits are expressed in thousands of dollars.

| | | *Low* ($A_1$) | *Medium* ($A_2$) | *High* ($A_3$) |
|---|---|---|---|---|
| Number of Rooms | ($R_1$) | 25 | 35 | 50 |
| | ($R_2$) | – 10 | 40 | 70 |
| | ($R_3$) | – 30 | 20 | 100 |

Let $O_{ij}$ be the monetary outcome of course of action $i$ when $j$ is the corresponding state of nature, so that in the above case $O_{i1}$ will be the outcome of course of action $R_1$ and state of nature $A_1$, which in our case is \$25,000. Similarly, $O_{i2}$ will be the outcome of action $R_2$ and state of nature $A_2$, which in our case is \$10,000, and so on. The expected value $EV$ (in thousands of dollars) is calculated on the basis of the actual state of nature that prevails as well as the estimate of the state of nature as provided by the consultant. These expected values are calculated as,

Course of action $= R_i$

Estimate of consultant $= X_i$

Actual state of nature $= A_i$

Where, $i = 1, 2, 3$

Then,

(i) Course of action $= R_1 =$ Build 50 rooms hotel

$$EV\left(\frac{R_1}{X_1}\right) = \Sigma P\left(\frac{A_i}{X_1}\right)O_{i1}$$

$$= 0.435(25) + 0.435\ (-10) + 0.130\ (-30)$$

$$= 10.875 - 4.35 - 3.9 = 2.625$$

$$EV\left(\frac{R_1}{X_2}\right) = \Sigma P\left(\frac{A_i}{X_2}\right)O_{i1}$$

$$= 0.133(25) + 0.667\ (-10) + 0.200\ (-30)$$

$$= 3.325 - 6.67 - 6.0 = -9.345$$

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$$EV\left(\frac{R_1}{X_3}\right) = \Sigma P\left(\frac{A_i}{X_3}\right)O_{i1}$$

$= 0.125(25) + 0.312(–10) + 0.563(–30)$

$= 3.125 – 3.12 – 16.89$

$= –16.885$

(ii) Course of action $= R_2 =$ Build 100 rooms hotel

$$EV\left(\frac{R_2}{X_1}\right) = \Sigma P\left(\frac{A_i}{X_1}\right)O_{i2}$$

$= 0.435(35) + 0.435 (40) + 0.130 (20)$

$= 15.225 + 17.4 + 2.6 = 35.225$

$$EV\left(\frac{R_2}{X_2}\right) = \Sigma P\left(\frac{A_i}{X_1}\right)O_{i2}$$

$= 0.133(35) + 0.667 (40) + 0.200 (20)$

$= 4.655 + 26.68 + 4.0 = 35.335$

$$EV\left(\frac{R_2}{X_3}\right) = \Sigma P\left(\frac{A_i}{X_3}\right)O_{i2}$$

$= 0.125(35) + 0.312(40) + 0.563(20)$

$= 4.375 + 12.48 + 11.26 = 28.115$

(iii) Course of action $= R_3 =$ Build 150 rooms hotel

$$EV\left(\frac{R_3}{X_1}\right) = \Sigma P\left(\frac{A_i}{X_1}\right)O_{i3}$$

$= 0.435(50) + 0.435(70) + 0.130 (100)$

$= 21.75 + 30.45 + 13 = 65.2$

$$EV\left(\frac{R_3}{X_2}\right) = \Sigma P\left(\frac{A_i}{X_2}\right)O_{i3}$$

$= 0.133(50) + 0.667 (70) + 0.200 (100)$

$= 6.65 + 46.69 + 20 = 73.34$

$$EV\left(\frac{R_3}{X_3}\right) = \Sigma P\left(\frac{A_i}{X_3}\right)O_{i3}$$

$= 0.125(50) + 0.312(70) + 0.563(100)$

$= 6.25 + 21.84 + 56.3 = 84.39$

The expected values in thousands of dollars, as calculated, are presented as follows in a tabular form.

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*Expected Posterior Pay-offs*

| Outcome | $EV(R_1/X_i)$ | $EV(R_2/X_i)$ | $EV(R_3/X_i)$ |
|---|---|---|---|
| $X_1$ | 2.625 | 35.225 | 65.2 |
| $X_2$ | –9.345 | 35.335 | 73.34 |
| $X_3$ | –16.885 | 28.115 | 84.39 |

This table can now be analysed. If the outcome is $X_1$, it is desirable to build 150 rooms hotel, since the expected pay-off for this course of action is maximum of \$65,200. Similarly, if the outcome is $X_2$, the course of action should again be $R_3$ since the maximum pay-off is \$73,34. Finally, if the outcome is $X_3$, the maximum payoff is \$84,390 for course of action $R_3$.

Accordingly, given these conditions and the pay-off, it would be advisable to build a 150 rooms hotel.

**Check Your Progress**

1. Define the term biostatics.
2. What is descriptive statistics?
3. Give the definition of inferential statistics.
4. What do you know about the law of large numbers?
5. When do we use probability theory?
6. Define sample space.
7. What is an elementary event?
8. Define the term joint probability.

## 3.4 PROBABILITY DISTRIBUTION AND THEIR APPLICATIONS

In probability theory and statistics, a probability distribution is the mathematical function that gives the probabilities of occurrence of different possible outcomes for an experiment. It is a mathematical description of a random phenomenon in terms of its sample space and the probabilities of events (subsets of the sample space). Examples of random phenomena include the weather condition in a future date, the height of a randomly selected person, the fraction of male students in a school, the results of a survey to be conducted, etc.

### Probability Distribution: Normal, Binomial and Poisson Distributions

In probability theory and statistics, a probability distribution is a mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment. In more technical terms, the probability distribution is a description of a random phenomenon in terms of the probabilities of events. For example, if the random variable $X$ is used to denote the outcome of a coin toss of the experiment, then the probability distribution of $X$ would take the value 0.5 for $X$ = Heads, and 0.5 for $X$ = Tails assuming that the coin is unbiased. Examples of random phenomena can include the results of an experiment or survey.

NOTES

Fundamentally, a probability distribution specifies the probability of getting an observation in a particular range of values. Distribution is a significant measure of analysing data sets which indicates all the potential outcomes of the data, and how frequently they occur. The 'Normal Distribution' describes continuous data which have a symmetric distribution, with a characteristic 'Bell-Shaped' curve. The 'Binomial Distribution' describes the distribution of binary data from a finite sample. The 'Poisson Distribution' describes the distribution of binary data from an infinite sample.

### 3.4.1 Normal Distribution

Normal distribution is often termed as a bell curve and is generally utilized in statistics, business settings, and government entities.

Normal distribution holds the following characteristics:

- It occurs naturally in numerous situations.
- Data points are similar and occur within a small range.
- The mean, mode and median are all equal.
- The curve is symmetric at the centre, i.e., around the mean, $\mu$.
- The curve of the distribution is bell-shaped and symmetrical about the line $x = \mu$.
- The total area under the curve is 1.
- Exactly half of the values are to the left of the centre and the other half to the right.
- Can be utilized to model risks following the distribution of likely outcomes for certain events.
- The formula for calculating the Normal Distribution is,

$$Z = \frac{X - \mu}{\sigma}$$

Where,

$X$ = Value that is being Consistent

$\mu$ = Mean of the Distribution

$\sigma$ = Standard Deviation of the Distribution

A standard normal distribution is defined as the distribution with mean 0 and standard deviation 1 for the PDF (Partial Differential Equation) such that it becomes:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad for -\infty < x < \infty$$

Figure 3.6 illustrates the curve for standard normal distribution.

NOTES

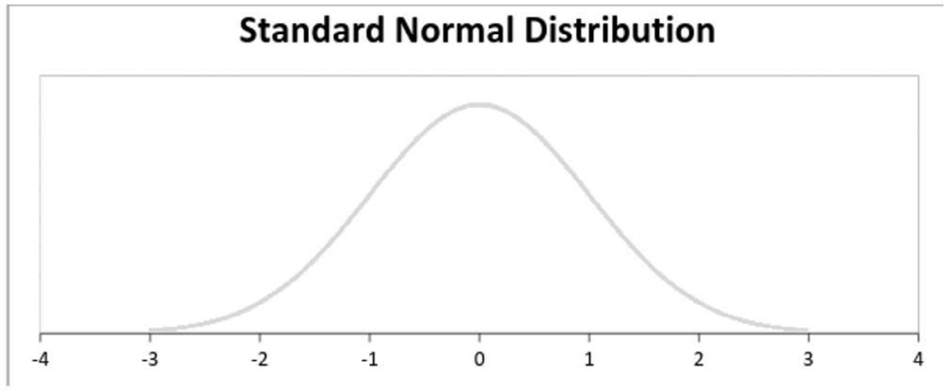


***Fig. 3.6*** *Curve for Standard Normal Distribution*

The 'Normal Distribution' can be represented by the histogram of a continuous variable obtained from a single measurement on different subjects will have a characteristic 'Bell Shaped' distribution curve termed as the Normal distribution. The normal distribution can be represented as histogram, for example the curve shown in Figure 3.7 represents the birth weight of the 3,226 new born babies (in kilograms).

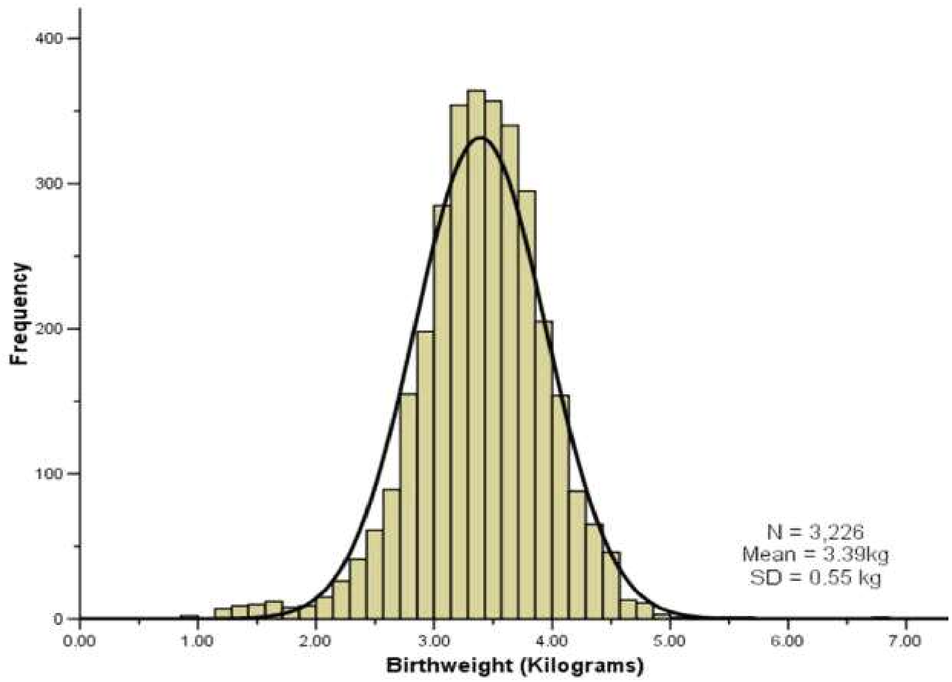


***Fig. 3.7*** *Histogram showing the Distribution Curve for the Birth Weight of 3,226 New Born Babies (Data from O' Cathain et al. 2002)*

Figure 3.7 shows the histogram of the sample data for an estimate of the population distribution of birth weights in new born babies. This population distribution can be estimated by the superimposed smooth 'Bell Shaped' curve or 'Normal Distribution'. Considering the entire population of new born babies and plotting the histogram of the distribution of birth weight would have exactly the 'Normal Shape'.

NOTES

The Normal distribution is described by two parameters μ and σ, where μ represents the population mean, or centre of the distribution, and σ the population standard deviation. It is symmetrically distributed around the mean. Populations with small values of the standard deviation σ have a distribution concentrated close to the centre μ, those with large standard deviation have a distribution widely spread along the measurement axis. One mathematical property of the Normal distribution is that exactly 95% of the distribution lies between,

$\mu - (1.96x\sigma)$ and $\mu + (1.96x\sigma)$

Altering the multiplier 1.96 to 2.58, exactly 99% of the Normal distribution lies in the corresponding interval.

In practice the two parameters of the Normal distribution, μ and σ, must be estimated from the sample data. For this a random sample from the population is taken. The sample mean  and the sample standard deviation, are then calculated. If a sample is taken from such a Normal distribution, and provided the sample is not too small, then approximately 95% of the sample lie within the interval:

$$\bar{x} - \left[1.96 \times SD(\bar{x})\right] \text{ to } \bar{x} + \left[1.96 \times SD(\bar{x})\right]$$

This is calculated by merely replacing the population parameters μ and σ by the sample estimates  and $S$ in the previous expression. In the appropriate situations this interval may estimate the reference interval for any required specific laboratory test which can be used for analysis and diagnostic determinations.

To calculate the reference range, consider that the sample birth weight data look normally distributed. As already mentioned that about 95% of the observations from a Normal distribution lie within ±1.96 SDs of the mean. Therefore a reference range for our sample of new born babies, using the values as represented in the histogram of Figure 3.7, is:

$= 3.39 - [1.96 \times 0.55]$ to $3.39 + [1.96 \times 0.55]$

$= 2.31$ kg to 4.47 kg

### 3.4.2 Binomial Distribution

Binomial Distribution is considered as the likelihood of a pass or fail outcome in a survey or experiment that is replicated numerous times. There are only two potential outcomes for this type of distribution, such as a True or False, or Heads or Tails. For example, assume that on flipping the coin you won the toss, i.e., Head is appeared, then this indicates a successful event. There are only two possible outcomes. Head denoting success and tail denoting failure. Therefore, probability of getting a Head = 0.5 and the probability of failure, i.e., getting a Tail = 0.5. A distribution where only two outcomes are possible, such as success or failure, gain or loss, win or lose, true or false, and where the probability of success and failure is same for all the trials then it is termed as a Binomial Distribution.

Each trial is independent since the outcome of the previous toss does not determine or affect the outcome of the current toss. An experiment with only two possible outcomes repeated **n** number of times is called binomial. The parameters of a binomial distribution are **n** and **p** where **n** is the total number of trials and **p** is the probability of success in each trial.

NOTES

The characteristic properties of a Binomial Distribution are:

1. Each trial is independent.
2. There are only two possible outcomes in a trial - either a success or a failure.
3. A total number of **n** identical trials are conducted.
4. The probability of success and failure is same for all trials. Trials are identical.

The mathematical representation of binomial distribution is given by:

$$P(x)=\frac{n!}{(n-x)!x!}p^x q^{n-x}$$

A binomial distribution graph where the probability of success does not equal the probability of failure looks as shown in Figure 3.8.

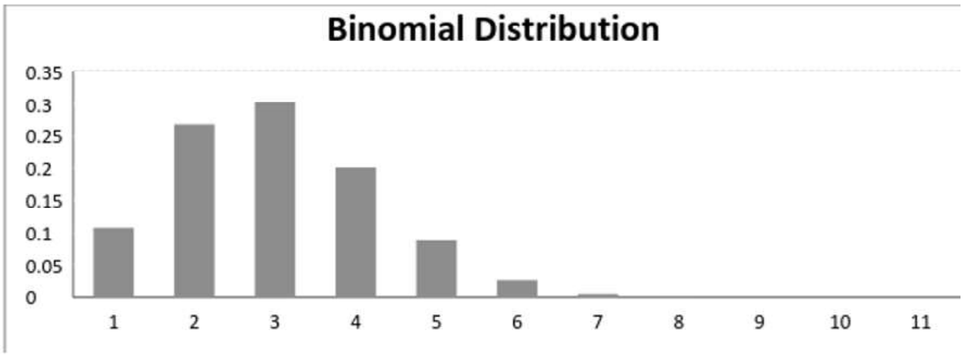


***Fig. 3.8*** *Binomial Distribution Graph*

Further, when

Probability of Success = Probability of Failure

Then in such a condition the graph of binomial distribution appears as shown in Figure 3.9.

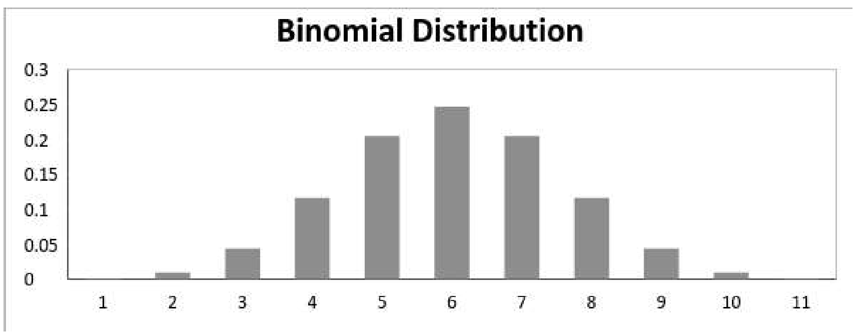


***Fig. 3.9*** *Binomial Distribution Graph for Probability of Success = Probability of Failure*

The mean and variance of a binomial distribution are given by:

Mean $\rightarrow \mu = n*p$

Variance $\rightarrow Var(X) = n*p*q$

NOTES

### 3.4.3 Poisson Distribution

The Poisson distribution is named after the French mathematician Siméon Denis Poisson. It is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant rate and independently of the time since the last event. The Poisson distribution can also be used for the number of events in other specified intervals, such as distance, area or volume. The Poisson distribution is used to describe discrete quantitative data, such as counts in which the population size **n** is large, the probability of an individual event is small, but the expected number of events, **n**, is moderate (say five or more). Typical examples are the number of deaths in a town from a particular disease per day, or the number of admissions to a particular hospital. Poisson distribution is applicable in situations where events occur at random points of time and space, but we will only consider the number of occurrences of the event.

A distribution is called **Poisson distribution** when the following assumptions are valid:

1. Any successful event should not influence the outcome of another successful event.
2. The probability of success over a short interval must equal the probability of success over a longer interval.
3. The probability of success in an interval approaches zero as the interval becomes smaller.

Now, if any distribution validates the above assumptions then it is a Poisson distribution. The notations used in Poisson distribution are:

- $\lambda$ = Rate at which an event occurs.
- t = Length of a time interval.
- $X$ = Number of events in that time interval.

Here, $X$ is called a 'Poisson Random Variable' and the probability distribution of $X$ is called Poisson distribution. For example, if $\mu$ denote the mean number of events in an interval of length $t$. Then, $\mu = \lambda*t$.

The probability of $X = x$ following a Poisson distribution is given by:

$$P(X = x) = e^{-\mu}\frac{\mu^x}{x!} \qquad \text{for } x = 0, 1, 2, ...,$$

The mean $\mu$ is the parameter of this distribution. In addition, the $\mu$ is also defined as the $\lambda$ times length of that interval. The graph of a Poisson distribution is shown in Figure 3.10.

NOTES

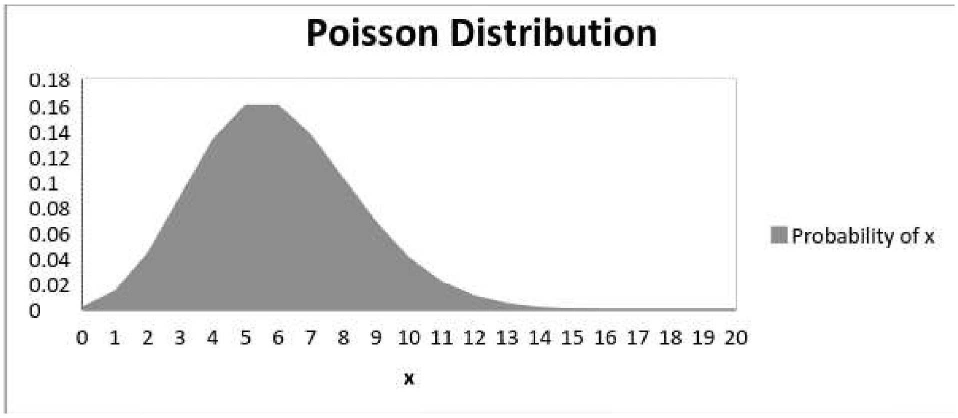


*Fig. 3.10 Poisson Distribution for Probability of* x

The graph shown in Figure 3.11 illustrates the shift in the curve due to increase in the mean.

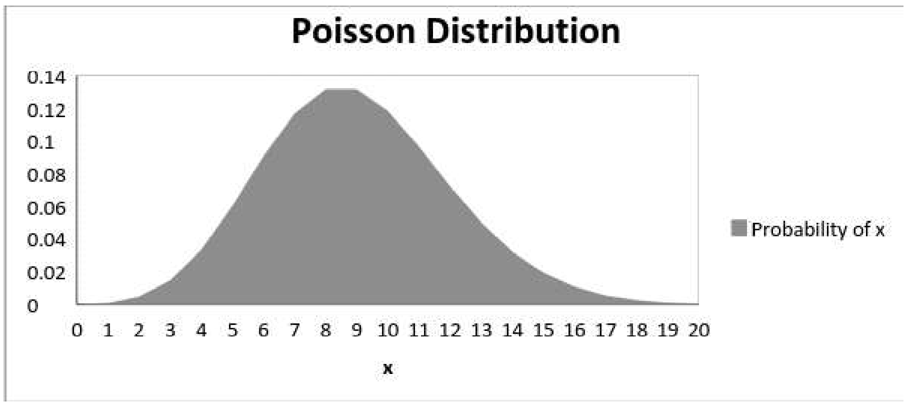


*Fig. 3.11 Graph for the Shift in the Curve due to Increase in Mean*

It is observed that as the mean increases, the curve shifts to the right.

The mean and variance of $X$ following a Poisson distribution is given as:

Mean $\rightarrow E(X) = \mu$

Variance $\rightarrow Var(X) = \mu$

## 3.5 CORRELATION AND REGRESSION

Correlation is a statistical measure that expresses the extent to which two variables are linearly related, i.e., they change together at a constant rate. It is a common tool for describing simple relationships without making a statement about cause and effect. Fundamentally, the correlation is a bivariate analysis that measures the strength of association between two variables and the direction of the relationship. In terms of the strength of relationship, the value of the correlation coefficient varies between +1 and –1. A value of ± 1 indicates a perfect degree of association

NOTES

between the two variables. As the correlation coefficient value goes towards 0, the relationship between the two variables will be weaker. The direction of the relationship is indicated by the sign of the coefficient. A '+' sign indicates a positive relationship and a '–' sign indicates a negative relationship. Correlation is, therefore, a statistical technique that can show whether and how strongly pairs of variables are related, for example, height and weight of an individual, fatty and skinny individual, taller and shorter people, etc. The relationship can be correlated as, people of the same height vary in weight, after analysis you can find that which two people of the population with shorter height is heavier than the taller one. Correlation can define that how much of the variation in peoples' weights is related to their heights.

A perfect positive correlation means that the correlation coefficient is exactly 1 while a perfect negative correlation means that two assets move in opposite directions, while a zero correlation implies no relationship at all.

In correlation analysis, a sample correlation coefficient is estimated which is denoted as '$r$'. This ranges between –1 and +1 and quantifies the direction and strength of the linear association between the two variables. The correlation between two variables can be positive, i.e., higher levels of one variable are associated with higher levels of the other or negative, i.e., higher levels of one variable are associated with lower levels of the other.

The sign of the correlation coefficient indicates the direction of the association while the magnitude of the correlation coefficient indicates the strength of the association. For example, a correlation of $r = 0.9$ recommends a strong, positive association between two variables, whereas a correlation of $r = -0.2$ recommends a weak, negative association. A correlation close to zero suggests no linear association between two continuous variables.

The Formula for Correlation is,

$$r = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sqrt{\sum (X - \bar{X})^2}\sqrt{(Y - \bar{Y})^2}}$$

Where,

$r$ = Correlation Coefficient

$\bar{X}$ = Average of Observations of Variable $X$

$\bar{Y}$ = Average of Observations of Variable $Y$

A correlation between variables indicates that as one variable changes in value, the other variable tends to change in a specific direction. The value of one variable can be used to predict the value of the other variable. For example, height and weight are correlated, hence as height increases the weight also tends to increase.

**NOTES**

Scatterplot diagram helps to check for relationships between pairs of continuous data. The scatterplot shown in Figure 3.12 displays the height and weight of teenage girls. Each dot on the graph represents an individual girl and her height and weight on the representative axis.

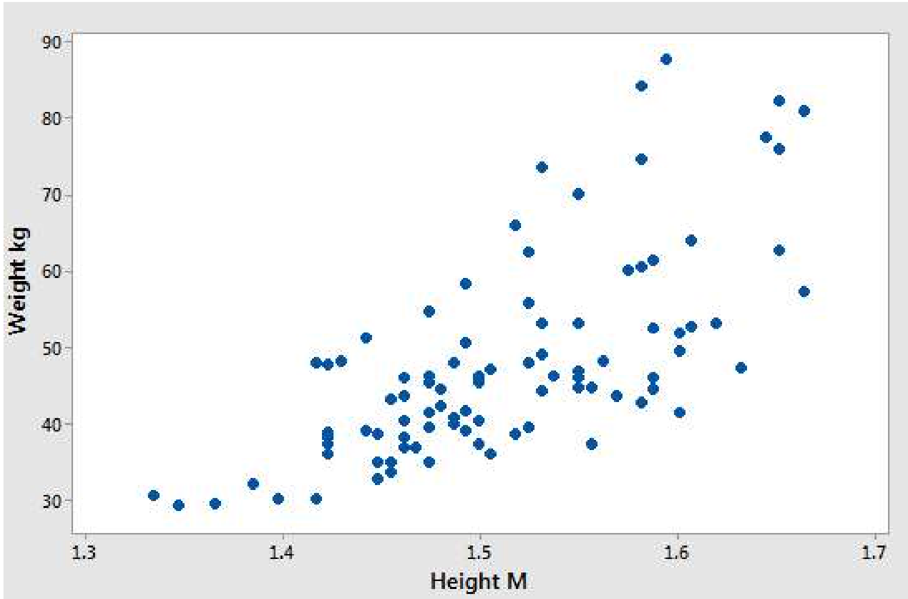


***Fig. 3.12** Scatterplot of the Height and Weight of Teenage Girls*

Figure 3.12 shows that there is a relationship between height and weight. As height increases, the weight may also increase. However, this is not a perfect relationship, for example when a specific height, say 1.5 meters, is considered then there is a range of weights associated with this specific height. However, the common tendency that height and weight increase together is unquestionably observed. Pearson's correlation takes all of the data points on this graph and represents them as a single number.

The following example data in Table 3.1 illustrates the correlation between the Gestational Age and Birth Weight of 17 Infants.

**Experimental Example for Studying the Correlation between the Gestational Age and Birth Weight of 17 Infants**

An experimental study was conducted on 17 infants to investigate the association or correlation between the gestational age at birth which was measured in weeks and the birth weight which was measured in grams. The observed data was tabulated as shown in Table 3.1.

*Table 3.1 Correlation between the Gestational Age and Birth Weight of 17 Infants*

| Infant ID # | Gestational Age (wks) | Birth Weight (gm) |
|---|---|---|
| 1 | 34.7 | 1895 |
| 2 | 36.0 | 2030 |
| 3 | 29.3 | 1440 |
| 4 | 40.1 | 2835 |
| 5 | 35.7 | 3090 |
| 6 | 42.4 | 3827 |
| 7 | 40.3 | 3260 |
| 8 | 37.3 | 2690 |
| 9 | 40.9 | 3285 |
| 10 | 38.3 | 2920 |
| 11 | 38.5 | 3430 |
| 12 | 41.4 | 3657 |
| 13 | 39.7 | 3685 |
| 14 | 39.7 | 3345 |
| 15 | 41.1 | 3260 |
| 16 | 38.0 | 2680 |
| 17 | 38.7 | 2005 |

NOTES

Now we will estimate the association or correlation between the gestational age and infant birth weight from the obtained experimental data. In this example, birth weight is the dependent variable and gestational age is the independent variable. Thus $y$ = Birth Weight and $x$ = Gestational Age. This data is plotted on a graph and is shown in a scatter diagram form as illustrated in the Figure 3.13.

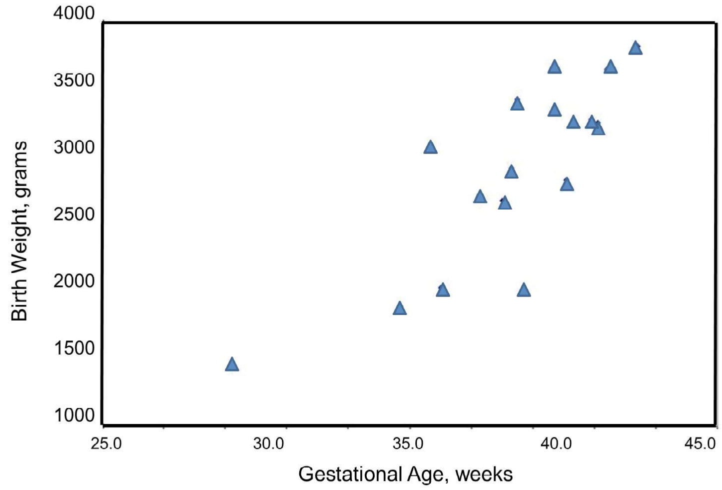


***Fig. 3.13** Scatter Diagram for Gestational Age and Birth Weight of 17 Infants*

In the scatter diagram shown in Figure 3.13, each point represents an $(x, y)$ pair, i.e., the gestational age measured in weeks and the birth weight measured in grams. The independent variable 'Gestational Age' is on the horizontal axis (or X-axis) and the dependent variable 'birth weight' is on the vertical axis (or Y-axis). The scatter plot displays a positive or direct association/correlation between the gestational age and the birth weight. The probability is that the infants with shorter gestational ages are more likely to be born with lower weights while the infants with longer gestational ages are more likely to be born with higher weights.

**NOTES**

The formula for the sample correlation coefficient is,

$$r = \frac{\text{Cov}(x, y)}{\sqrt{s_x^2 * s_y^2}}$$

Where Cov(x,y) is the covariance of x and y defined as,

$$\text{Cov}(x, y) = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{n-1}$$

$s_x^2$ and $s_y^2$ are the sample variances of x and y, defined as

$$s_x^2 = \frac{\sum (X - \bar{X})^2}{n-1} \text{ and } s_y^2 = \frac{\sum (Y - \bar{Y})^2}{n-1}$$

The variances of x and y measure the variability of the x scores and y scores around their respective sample means ($\bar{X}$ and $\bar{Y}$, considered separately). The covariance measures the variability of the $(x, y)$ pairs around the mean of $x$ and mean of $y$, considered simultaneously.

To calculate the sample correlation coefficient, we need to compute the variance of gestational age, the variance of birth weight and also the covariance of gestational age and birth weight.

We first summarize the gestational age data as shown below. The mean gestational age is calculated as:

$$\bar{X} = \frac{\sum X}{n} = \frac{652.1}{17} = 38.4$$

To calculate the variance of gestational age, we need to sum the squared deviations (or differences) between each observed gestational age and the mean gestational age. The calculations are summarized in Table 3.2.

***Table 3.2*** *Gestational Age and the Mean Gestational Age*

| Infant ID # | Gestational Age | $(X - \bar{X})$ | $(X - \bar{X})^2$ |
|---|---|---|---|
| 1 | 34.7 | -3.7 | 13.69 |
| 2 | 36.0 | -2.4 | 5.76 |
| 3 | 29.3 | -9.1 | 82.81 |
| 4 | 40.1 | 1.7 | 2.89 |
| 5 | 35.7 | -2.7 | 7.29 |
| 6 | 42.4 | 4.0 | 16.00 |
| 7 | 40.3 | 1.9 | 3.61 |
| 8 | 37.3 | -1.1 | 1.21 |
| 9 | 40.9 | 2.5 | 6.25 |
| 10 | 38.3 | -0.1 | 0.01 |
| 11 | 38.5 | 0.1 | 0.01 |
| 12 | 41.4 | 3.0 | 9.00 |
| 13 | 39.7 | 1.3 | 1.69 |
| 14 | 39.7 | 1.3 | 1.69 |
| 15 | 41.1 | 2.7 | 7.29 |
| 16 | 38.0 | -0.4 | 0.16 |
| 17 | 38.7 | 0.3 | 0.09 |
| | $\Sigma X = 652.1$ | $\Sigma (X - \bar{X}) = 0$ | $\Sigma(X - \bar{X})^2 = 159.45$ |

The variance of gestational age is given as:

$$s_x^2 = \frac{\sum (X - \bar{X})}{n-1} = \frac{159.45}{16} = 10.0$$

NOTES

Subsequently, we summarize the birth weight data as follows. The mean birth weight is calculated as:

$$\bar{Y} = \frac{\sum Y}{n} = \frac{49,334}{17} = 290.2$$

The variance of birth weight is calculated in the similar method as we have done for the gestational age. The calculation of birth weight and the mean birth weight is shown in Table 3.3.

***Table 3.3*** *Birth Weight and the Mean Birth Weight*

| Infant ID # | Birth Weight | $(Y - \bar{Y})$ | $(Y - \bar{Y})^2$ |
|---|---|---|---|
| 1 | 1895 | -1007 | 1,014,049 |
| 2 | 2030 | -872 | 760,384 |
| 3 | 1440 | -1462 | 2,137,444 |
| 4 | 2835 | -67 | 4,489 |
| 5 | 3090 | 188 | 35,344 |
| 6 | 3827 | 925 | 855,625 |
| 7 | 3260 | 358 | 128,164 |
| 8 | 2690 | -212 | 44,944 |
| 9 | 3285 | 383 | 146,689 |
| 10 | 2920 | 18 | 324 |
| 11 | 3430 | 528 | 278,784 |
| 12 | 3657 | 755 | 570,025 |
| 13 | 3685 | 783 | 613,089 |
| 14 | 3345 | 443 | 196,249 |
| 15 | 3260 | 358 | 128,164 |
| 16 | 2680 | -222 | 49,284 |
| 17 | 2005 | -897 | 804,609 |
| | $\Sigma$ Y = 49,334 | $\Sigma\ (Y - \bar{Y})$=0 | $\Sigma(Y - \bar{Y})^2$ = 7,767,660 |

The variance of birth weight is given as:

$$s_y^2 = \frac{\sum (Y - \bar{Y})^2}{n-1} = \frac{7,767,660}{16} = 485,578.8.$$

The covariance is calculated as follows,

$$\text{Cov}(x, y) = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{n-1}$$

To calculate the covariance of gestational age and birth weight, multiply the deviation from the mean gestational age by the deviation from the mean birth weight for each participant (i.e., $(X - \bar{X})(Y - \bar{Y})$).

The calculations are summarized in Table 3.4. In this Table 3.4, we have copied the deviations from the mean gestational age and mean birth weight from the two Tables 3.2 and 3.3 and then multiplied.

*Table 3.4 Covariance of Mean Gestational Age and Mean Birth Weight*

NOTES

| Infant Identification Number | $(X - \overline{X})$ | $(Y - \overline{Y})$ | $(X - \overline{X})(Y - \overline{Y})$ |
|---|---|---|---|
| 1 | -3.7 | -1007 | 3725.9 |
| 2 | -2.4 | -872 | 2092.8 |
| 3 | -9.1 | -1462 | 13,304.2 |
| 4 | 1.7 | -67 | -113.9 |
| 5 | -2.7 | 188 | -507.6 |
| 6 | 4.0 | 925 | 3700.0 |
| 7 | 1.9 | 358 | 680.2 |
| 8 | -1.1 | -212 | 233.2 |
| 9 | 2.5 | 383 | 957.5 |
| 10 | -0.1 | 18 | -1.8 |
| 11 | 0.1 | 528 | 52.8 |
| 12 | 3.0 | 755 | 2265.0 |
| 13 | 1.3 | 783 | 1017.9 |
| 14 | 1.3 | 443 | 575.9 |
| 15 | 2.7 | 358 | 966.6 |
| 16 | -0.4 | -222 | 88.8 |
| 17 | 0.3 | -897 | -269.1 |
| | | | $\Sigma (X - \overline{X})(Y - \overline{Y})$= 28,768.4 |

The covariance of gestational age and birth weight is given as:

$$s_y^2 = \frac{\sum (Y - \overline{Y})^2}{n-1} = \frac{7,767,660}{16} = 485,578.8.$$

We now calculate the sample correlation coefficient as follows:

$$r = \frac{\text{Cov}(x, y)}{\sqrt{s_x^2 * s_y^2}} = \frac{1798.0}{\sqrt{10.0 * 485,578.8}} = \frac{1798.0}{2199.4} = 0.82.$$

The sample correlation coefficient specifies a strong positive correlation.

The sample correlation coefficients range from –1 to +1. In fact, meaningful correlations (i.e., correlations that are clinically or practically significant) can be as small as 0.4 (or –0.4) for positive (or negative) associations. There are also other statistical tests which help to determine whether an observed correlation is statistically significant or not (i.e., statistically significantly different from zero).

### 3.5.1 Correlation Coefficient

A **correlation coefficient** is a numerical measure of some type of correlation, meaning a statistical relationship between two variables. The variables may be two columns of a given data set of observations, often called a **sample**, or two components of a multivariate random variable with a known **distribution**.

Several types of correlation coefficient exist, each with their own definition and own range of usability and characteristics. They all assume values in the range from –1 to +1, where ±1 indicates the strongest possible agreement and 0 the strongest possible disagreement.

NOTES

The correlation coefficient is a statistical measure that calculates the strength of the relationship between the relative movements of two variables. The values range between –1.0 and 1.0. A calculated number greater than 1.0 or less than –1.0 means that there was an error in the correlation measurement. A correlation of –1.0 shows a *perfect negative correlation*, while a correlation of 1.0 shows a *perfect positive correlation*. A correlation of 0.0 shows *no relationship* between the movements of the two variables. The correlation coefficient, $r$, can be calculated as shown below.

**Correlation Coefficient, *r***

The quantity $r$, called the *linear correlation coefficient*, measures the strength and the direction of a linear relationship between two variables. The linear correlation coefficient is sometimes referred to as the *Pearson product moment correlation coefficient* in honour of its developer Karl Pearson.

The mathematical formula for computing ***r*** is:

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right) - \left(\sum x\right)^2}\,\sqrt{n\left(\sum y^2\right) - \left(\sum y\right)^2}}$$

Where $n$ is the number of pairs of data.

The value of $r$ is such that $-1 \le r \le +1$. The '+' and '–' signs are used for positive linear correlations and negative linear correlations, respectively.

**Positive Correlation:** If $x$ and $y$ have a strong positive linear correlation, $r$ is close to +1. An $r$ value of exactly +1 indicates a perfect positive fit. Positive values indicate a relationship between $x$ and $y$ variables such that as values for $x$ increases, values for $y$ also increase.

**Negative Correlation:** If $x$ and $y$ have a strong negative linear correlation, $r$ is close to –1. An $r$ value of exactly –1 indicates a perfect negative fit. Negative values indicate a relationship between $x$ and $y$ such that as values for $x$ increase, values for $y$ decrease.

**No Correlation:** If there is no linear correlation or a weak linear correlation, then $r$ is close to 0. A value near zero means that there is a random, nonlinear relationship between the two variables

Remember that $r$ is a dimensionless quantity, i.e., it does not depend on the units employed.

**Perfect Correlation:** A perfect correlation of ± 1 occurs only when the data points all lie exactly on a straight line. If $r = +1$, the slope of this line is positive. If $r = -1$, then the slope of this line is negative.

A correlation greater than 0.8 is generally described as *strong*, whereas a correlation less than 0.5 is generally described as *weak*. These values can vary based upon the 'type' of data being examined. A study utilizing scientific data may require a stronger correlation than a study using social science data.

NOTES

### 3.5.2 Regression Analysis

Regression analysis is a set of statistical processes for estimating the relationships among variables. It includes many techniques for modelling and analysing several variables, when the focus is on the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps one understand how the typical value of the dependent variable changes when any one of the independent variables is varied, while the other independent variables are held fixed.

**Definition:** Regression is a statistical measurement used in finance, investing and other disciplines that attempts to determine the strength of the relationship between one dependent variable (usually denoted by Y) and a series of other changing variables (known as independent variables).

Regression analysis is a set of statistical methods used for the estimation of relationships between a dependent variable and one or more independent variables. It can be utilized to assess the strength of the relationship between variables and for modelling the future relationship between them. The regression analysis includes several variations, such as linear, multiple linear, and nonlinear. The most common models are simple linear and multiple linear. Nonlinear regression analysis is commonly used for more complicated data sets in which the dependent and independent variables show a nonlinear relationship.

**Linear Model Assumptions for Regression Analysis**

Linear regression analysis is based on following six fundamental assumptions:

1. The dependent and independent variables show a linear relationship between the slope and the intercept.
2. The independent variable is not random.
3. The value of the residual (error) is zero.
4. The value of the residual (error) is constant across all observations.
5. The value of the residual (error) is not correlated across all observations.
6. The residual (error) values follow the normal distribution.

**Simple Linear Regression**

Simple linear regression is a model that assesses the relationship between a dependent variable and one independent variable. The simple linear model is expressed using the following equation:

$$Y = a + bX + \in$$

Where:

Y = Dependent Variable

X = Independent (Explanatory) Variable

a = Intercept

b = Slope

$\in$ = Residual (Error)

### Multiple Linear Regression



Multiple linear regression analysis is essentially similar to the simple linear model, with the exception that multiple independent variables are used in the model. The mathematical representation of multiple linear regression is:

$$Y = a + bX_1 + cX_2 + dX_3 + \in$$

Where:

Y = Dependent Variable

$X_1, X_2, X_3$ = Independent (Explanatory) Variables

a = Intercept

b, c, d = Slopes

ε = Residual (Error)

Multiple linear regression follows the same conditions as the simple linear model. However, since there are several independent variables in multiple linear analysis, there is another mandatory condition for the model named as Non-Collinearity.

**Non-Collinearity:** Independent variables should show a minimum of correlation with each other. If the independent variables are highly correlated with each other, it will be difficult to assess the true relationships between the dependent and independent variables.

Regression takes a group of random variables, thought to be predicting Y, and tries to find a mathematical relationship between them. This relationship is typically in the form of a **straight line (linear regression)** that best approximates all the individual data points. In multiple regression, the separate variables are differentiated by using numbers with subscripts.

**Check Your Progress**

9. What is probability distribution?
10. Define the term binomial distribution.
11. How is scatterplot helpful?
12. What do you understand by the term correlation?
13. What is regression analysis?

## 3.6 SAMPLING THEORY AND EXPERIMENTAL DESIGNING

Under census or complete enumeration survey method, data is collected for each and every unit (e.g., person, consumer, employee, household, organization) of the population or universe which are the complete set of entities and which are of interest in any particular situation. In spite of the benefits of such an all-inclusive approach, it is infeasible in most of the situations. Besides the time and resource constraints of the researcher, infinite or huge population, the incidental destruction of the population unit during the evaluation process (as in the case of bullets, explosives, etc), cases of data obsolescence (by the time census ends) do not permit this mode of data collection.

Sampling is simply a process of learning about the population on the basis of a sample drawn from it. Thus, in any sampling technique, instead of every unit of the universe, only a part of the universe is studied and the conclusions are drawn on that basis for the entire population. The process of sampling involves selection of a sample based on a set of rules, collection of information and making an inference about the population. It should be clear to the researcher that a sample is studied not for its own sake, but the basic objective of its study is to draw inference about the population. In other words, sampling is a tool which helps us know the characteristics of the universe or the population by examining only a small part of it. The values obtained from the study of a sample, such as the average and dispersion are known as 'statistics' and the corresponding such values for the population are called 'parameters'.

Although diversity is a universal quality of mass data, every population has characteristic properties with limited variation. The following two laws of statistics are very important in this regard.

1. The law of statistical regularity states that a moderately large number of items chosen at random from a large group are almost sure on the average to possess the characteristics of the large group. By random selection, we mean a selection where each and every item of the population has an equal chance of being selected.
2. The law of inertia of large numbers states that, other things being equal, larger the size of the sample, more accurate the results are likely to be.

Hence, a sound sampling procedure should result in a representative, adequate and homogeneous sample while ensuring that the selection of items should occur independently of one another.

**Random Sampling**

It refers to that sampling technique in which each and every unit of the population has an equal chance of being selected in the sample. One should not mistake the term 'Arbitrary' for 'Random'. To ensure randomness, one may adopt either the lottery method or consult the table of random numbers, preferably the latter. Being a random method, it is independent of personal bias creeping into the analysis besides enhancing the representativeness of the sample. Furthermore, it is easy to assess the accuracy of the sampling estimates because sampling errors follow the principles of chance. However, a completely catalogued universe is a prerequisite for this method. The sample size requirements would be usually larger under random sampling than under stratified random sampling, to ensure statistical reliability. It may escalate the cost of collecting data as the cases selected by random sampling tend to be too widely dispersed geographically.

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### Stratified Random Sampling

In this method, the universe to be sampled is subdivided (stratified) into groups which are mutually exclusive but collectively exhaustive based on a variable known to be correlated with the variable of interest. Then, a simple random sample is chosen independently from each group. This method differs from simple random sampling in that in the latter the sample items are chosen at random from the entire universe. In stratified random sampling, the sampling is designed in such a way that a designated number of items is chosen from each stratum. If the ratio of items between various strata in the population matches with the ratio of corresponding items between various strata in the sample, it is called proportionate stratified sampling; otherwise, it is known as disproportionate stratified sampling. Ideally, we should assign greater representation to a stratum with a larger dispersion and smaller representation to one with small variation. Hence, it results in a more representative sample than simple random sampling.

## 3.7 USES AND APPLICATIONS OF CHI-SQUARE TEST

A **Chi-Squared Test**, also written as $\chi^2$ **test**, is any statistical hypothesis test where the sampling distribution of the test statistic is a Chi-squared distribution when the null hypothesis is true. Without other prerequisite, the 'Chi-Squared Test' often is used as short for Pearson's Chi-squared test. The Chi-squared test is used to determine whether there is a significant difference between the expected frequencies and the observed frequencies in one or more groups/categories.

In the standard applications of this test, the observations are classified into mutually exclusive classes, and there is some theory, or say null hypothesis, which gives the probability that any observation falls into the corresponding class. The purpose of the test is to evaluate how likely the observations that are made would be, assuming the null hypothesis is true.

Chi-squared tests are often constructed from a sum of squared errors, or through the sample variance. Test statistics that follow a chi-squared distribution arise from an assumption of independent normally distributed data, which is valid in many cases due to the Central Limit Theorem (CLT). A Chi-squared test can be used to attempt rejection of the null hypothesis that the data are independent.

Also considered a chi-squared test is a test in which this is asymptotically true, meaning that the sampling distribution (if the null hypothesis is true) can be made to approximate a Chi-squared distribution as closely as desired by making the sample size large enough. Figure 3.14 illustrates the Chi-squared distribution, showing $\chi^2$ on the $x$-axis and $p$-value on the $y$-axis.

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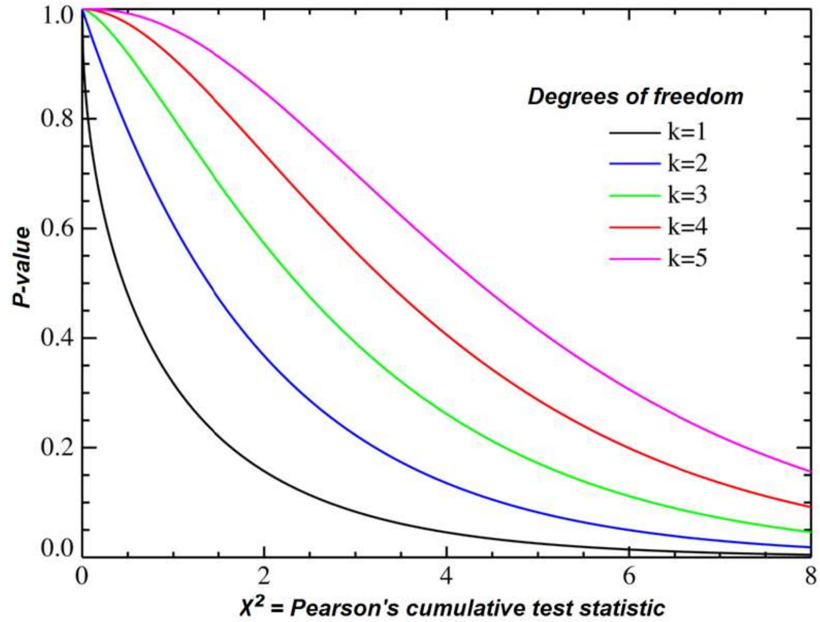


***Fig. 3.14*** *Chi-Squared Distribution*

In the 19th century, statistical analytical methods were mainly applied in biological data analysis and it was customary for researchers to assume that observations followed a normal distribution. Until the end of 19th century, Pearson noticed the existence of significant skewness within some biological observations. In order to model the observations regardless of being normal or skewed, Pearson formulated the Pearson distribution, a family of continuous probability distributions, which includes the normal distribution and many skewed distributions, and proposed a method of statistical analysis consisting of using the Pearson distribution to model the observation and performing the test of goodness of fit to determine how well the model and the observation really fit.

The Chi-squared distribution is continuous probability distribution whose shape is defined by the number of degrees of freedom. It is a right-skew distribution, but as the number of degrees of freedom increases it approximates the Normal distribution, as shown in Figure 3.15. The Chi-squared distribution is important for its use in Chi-squared tests. These are often used to test deviations between observed and expected frequencies, or to determine the independence between categorical variables. When conducting a Chi-squared test, the probability values derived from Chi-squared distributions can be looked up in a statistical table. Figure 3.15 illustrates the Chi-squared distribution for various degrees of freedom (df). The distribution becomes less right-skew as the number of degrees of freedom increases.

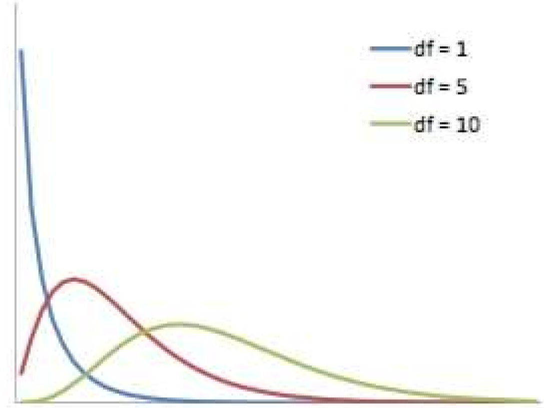


*Fig. 3.15 Chi-Squared Distribution for Various Degrees of Freedom*

There are two types of Chi-square tests. Both use the Chi-square statistic and distribution for different purposes:

1. A Chi-square goodness of fit test determines if a sample data matches a population.
2. A Chi-square test for independence compares two variables in a contingency table to see if they are related. In a more general sense, it tests to see whether distributions of categorical variables differ from each another.
   - A very small Chi-square test statistic means that your observed data fits your expected data extremely well. In other words, there is a relationship.
   - A very large Chi-square test statistic means that the data does not fit very well. In other words, there is not a relationship.

A Chi-square test will give a *p*-value. The *p*-value defines that the test results are significant or not. In order to perform a Chi-square test and get the *p*-value, the following information is required:

1. Degrees of Freedom. That is just the number of categories minus 1.
2. The alpha level ($\alpha$). This is chosen by the experimenter/researcher. The usual alpha level is 0.05 (5%), but you could also have other levels like 0.01 or 0.10.

## 3.8 *t*-TEST

The *t*-test is any statistical hypothesis test in which the test statistic follows a Student's *t*-distribution under the null hypothesis. The *t*-statistic was introduced in 1908 by William Sealy Gosset, a chemist working for the Guinness brewery in Dublin, Ireland. 'Student' was his pen name, so this test is also termed as 'Student *t*-Test'.

A *t*-test is most commonly applied when the test statistic would follow a normal distribution if the value of a scaling term in the test statistic were known. When the scaling term is unknown and is replaced by an estimate based on the

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data, the test statistics (under certain conditions) follow a Student's $t$-distribution. The $t$-test can be used, for example, to determine if the means of two sets of data are significantly different from each other.

Among the most frequently used $t$-tests are:

- A one-sample location test of whether the mean of a population has a value specified in a null hypothesis.
- A two-sample location test of the null hypothesis such that the means of two populations are equal. All such tests are usually called Student's $t$-tests, though strictly speaking that name should only be used if the variances of the two populations are also assumed to be equal, the form of the test used when this assumption is dropped is sometimes called Welch's $t$-test. These tests are often referred to as 'unpaired' or 'Independent Samples' $t$-tests, as they are typically applied when the statistical units underlying the two samples being compared are non-overlapping.

The test statistics have the form $t = Z/s$, where $Z$ and $s$ are functions of the data.

$Z$ may be sensitive to the alternative hypothesis, i.e., its magnitude tends to be larger when the alternative hypothesis is true, whereas $s$ is a scaling parameter that allows the distribution of $t$ to be determined.

As an example, in the one-sample $t$-test,

$$t = \frac{Z}{s} = \frac{\overline{X} - \mu}{\hat{\sigma}/\sqrt{n}}$$

Where,

- $X$ is the sample mean from a sample $X_1, X_2, \ldots, X_n$, of size $n$.
- $s$ is the standard error of the mean.
- $\hat{\sigma}$ is the estimate of the standard deviation of the population.
- $\mu$ is the population mean.

The assumptions underlying a $t$-test in its simplest form are that:

- $X$ follows a normal distribution with mean $\mu$ and variance $\sigma^2/n$.
- $s^2$ follows a $\chi^2$ distribution with $n - 1$ degrees of freedom.
- $Z$ and $s$ are independent.

### *t*-Distribution

Student's $t$-distribution is a continuous probability distribution with a similar shape to the Normal distribution but with wider tails. The $t$-distributions are used to describe samples which have been drawn from a population, and the exact shape of the distribution varies with the sample size. The smaller the sample size, the more spread out the tails, and the larger the sample size, the closer the $t$-distribution is to the Normal distribution, as shown in Figure 3.16. Whilst in general the Normal distribution is used as an approximation when estimating means of samples from a Normally distribution population, when the same size is small (say $n < 30$), then the *t*-distribution should be used in preference. Figure 3.16 illustrates the *t*-distribution for various sample sizes. As the sample size increases, then the *t*-distribution more closely approximates the Normal.

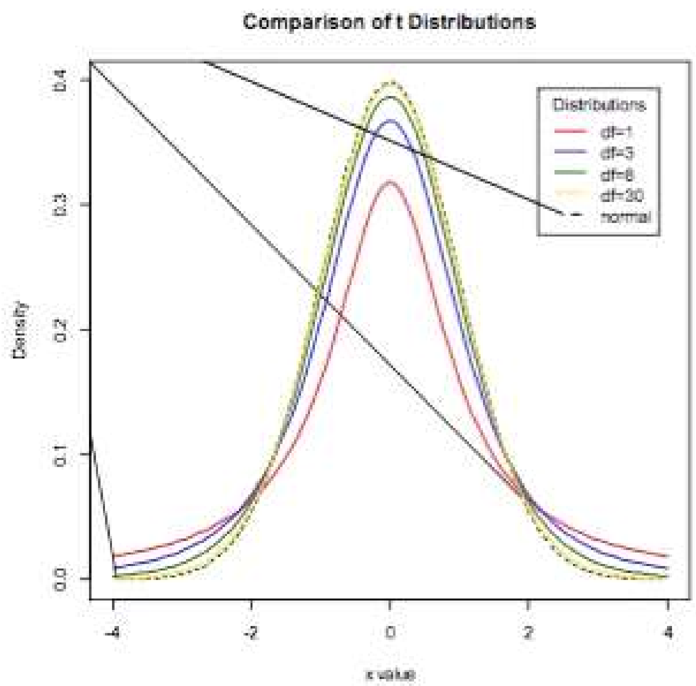


***Fig. 3.16*** *Comparison of the t-Distribution for Various Sample Sizes*

**Check Your Progress**

14. Define sampling.
15. State the law of statistical regularity.
16. What do you understand by *p*-value?
17. State the information needed to perform a Chi-Square test.

## 3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Biostatistics (also known as biometry) is the development and application of statistical methods to a wide range of topics in biology. It is includes the design of biological experiments, the collection and analysis of data from those experiments and the interpretation of the results
2. Descriptive statistics describes the data and is consist of methods and techniques used in collection, organization, presentation and analysis of data in order to describe the various features and characteristics of such data.
3. Inferential statistics can be defined as those methods that are used to estimate a characteristic of a population or making of a decision concerning a

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population on the basis of the results obtained from a sample taken from the same population

4. The 'Law of Large Numbers' states that as the number of trials of an experiment increases, the empirical probability approaches the theoretical probability. Hence, if we roll a die a number of times, each number would come up approximately 1/6 of the time. The study of empirical probabilities is known as statistics.
5. We use the probability in vague terms when we predict something for future. For example, we might say it will probably rain tomorrow or it will probably be a holiday the day after. This is subjective probability to the person predicting, but implies that the person believes the probability is greater than 50 percent.
6. A sample space is the collection of all possible events or outcomes of an experiment. For example, there are two possible outcomes of a toss of a fair coin: a head and a tail. Then, the sample space for this experiment denoted by $S$ would be,

   $S = [H, T]$
7. An elementary event, also known as a simple event, is a single possible outcome of an experiment. For example, if we toss a fair coin, then the event of a head coming up is an elementary event.
8. The joint probability refers to the phenomenon of occurrence of two or more simple events. For example, assume that event ($E$) is a joint event (or compound event) of drawing a black ace from a pack of cards. There are two simple events involved in the compound event, which are, the card being black and the card being an ace. Hence, $P$[Black ace] or $P[E]$ = 2/52 since there are two black aces in the pack.
9. A probability distribution is the mathematical function that gives the probabilities of occurrence of different possible outcomes for an experiment. It is a mathematical description of a random phenomenon in terms of its sample space and the probabilities of events (subsets of the sample space).
10. In binomial distribution there are only two potential outcomes for this type of distribution, such as a True or False, or Heads or Tails. For example, assume that on flipping the coin you won the toss, i.e., Head is appeared, then this indicates a successful event. There are only two possible outcomes. Head denoting success and tail denoting failure.
11. Scatterplot diagram helps to check for relationships between pairs of continuous data.
12. Correlation is a statistical measure that expresses the extent to which two variables are linearly related, i.e., they change together at a constant rate. It is a common tool for describing simple relationships
13. Regression analysis is a set of statistical processes for estimating the relationships among variables. It includes many techniques for modelling and analysing the variables.

14. Sampling is a process of learning about the population on the basis of a sample drawn from it. Thus, in any sampling technique, instead of every unit of the universe, only a part of the universe is studied and the conclusions are drawn on that basis for the entire population. Sampling is a tool which helps us know the characteristics of the universe or the population by examining only a small part of it.
15. The law of statistical regularity states that a moderately large number of items chosen at random from a large group are almost sure on the average to possess the characteristics of the large group. By random selection, we mean a selection where each and every item of the population has an equal chance of being selected.
16. A Chi-Square test gives a p-value. The *p*-value defines that the test results are significant or not.

## 3.10 SUMMARY

- Biostatistics (also known as biometry) is the development and application of statistical methods to a wide range of topics in biology.
- Descriptive statistics describes the data and consist of methods and techniques used in collection, organization, presentation and analysis of data.
- Inferential statistics can be defined as those methods that are used to estimate a characteristic of a population.
- The probability theory helps a decision-maker to analyse a situation and decide accordingly.
- The classical theory of probability is the theory based on the number of favourable outcomes and the number of total outcomes
- A sample space is the collection of all possible events or outcomes of an experiment.
- An event is an outcome or a set of outcomes of an activity or a result of a trial.
- Simple probability refers to a phenomenon where only a simple or elementary event occurs.
- The joint probability refers to the phenomenon of occurrence of two or more simple events.
- When two events are mutually exclusive, then the probability that either of the events will occur is the sum of their separate probabilities.
- Multiplication rule is applied when it is necessary to compute the probability if both events $A$ and $B$ will occur at the same time. Two events $A$ and $B$ are said to be independent events, if the occurrence of one event is not influenced at all by the occurrence of the other.
- A probability distribution specifies the probability of getting an observation in a particular range of values.

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- The Poisson distribution is named after the French mathematician Siméon Denis Poisson.
- Correlation is a statistical measure that expresses the extent to which two variables are linearly related, i.e., they change together at a constant rate.
- A correlation coefficient is a numerical measure of some type of correlation, meaning a statistical relationship between two variables.
- Simple linear regression is a model that assesses the relationship between a dependent variable and one independent variable.
- A Chi-Squared Test, also written as $\chi^2$ test, is any statistical hypothesis test where the sampling distribution of the test statistic is a Chi-squared distribution when the null hypothesis is true. Without other prerequisite, the 'Chi-Squared Test' often is used as short for Pearson's Chi-squared test. The Chi-squared test is used to determine whether there is a significant difference between the expected frequencies and the observed frequencies in one or more groups/categories.
- A *t*-distribution is a continuous probability distribution with a similar shape to the Normal distribution but with wider tails.

## 3.11 KEY TERMS

- **Population parameters**: the measured characteristics of the population are known as population parameters.
- **Sample statistics**: The measured characteristics of the sample are called sample statistics.
- **Sample space**: It is the collection of all possible events or outcomes of an experiment.
- **Elementary event**: An elementary event is a single possible outcome of an experiment.
- **Simple probability:** Simple probability refers to a phenomenon where only a simple or an elementary event occurs.
- **Poisson distribution**: It is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space.
- **Correlation coefficient**: A correlation coefficient is a numerical measure of some type of correlation, meaning a statistical relationship between two variables.
- **Sampling:** Sampling is a process of learning about the population on the basis of a sample drawn from it.
- **Random sampling**: It refers to that sampling technique in which each and every unit of the population has an equal chance of being selected in the sample.

- ***t*-distribution:** It is a continuous probability distribution with a similar shape to the Normal distribution but with wider tails.

## 3.12 SELF-ASSESSMENT QUESTIONS AND EXERCISES

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**Short-Answer Questions**

1. What are the limitations of classical probability theory?
2. Define the term independent events.
3. What do you understand by conditional probability?
4. Give the characteristics features of normal distribution.
5. Write a short note on correlation and correlation coefficient.
6. What are the various linear model assumptions for regression analysis?
7. Define the laws of statistics.
8. What precautions should be taken while sampling?
9. Give the difference between simple random samples and stratified random samples.
10. What is *t*-distribution?

**Long-Answer Questions**

1. Explain in detail about the different probability theories with the help of examples.
2. What is an event? Explain the different types of events.
3. Define Poisson distribution. When a distribution is called Poisson distribution? Explain giving examples.
4. Discuss about the Bayes' theorem with the help of examples.
5. Give a detail account of various linear regression giving suitable examples.
6. Briefly describe the uses and applications of Chi-Square test.
7. Explain about the *t*-test in detail.

## 3.13 FURTHER READING

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# UNIT 4 BASIC MATHEMATICS AND MATHEMATICAL MODELING

NOTES

**Structure**



## 4.0 INTRODUCTION

In mathematics, a matrix is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to characterise a mathematical object or a property of such an object.

Vector, in mathematics, a quantity that has both magnitude and direction but not position, i.e., a vector is a quantity or phenomenon that has two independent properties: magnitude and direction.

Exponential function is of prime importance in mathematics and finds its wide application in calculus and many branches of science and engineering. An exponential function of $x$ is written as exp $(x)$ or $e^x$. Here $e$ is a constant and an irrational number. Where a periodic function is a function that repeats its values at regular intervals.

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In mathematics, a differential equation is an equation that relates one or more functions and their derivatives. In mathematics, an integral assigns numbers to functions in a way that describes displacement, area, volume, and other concepts that arise by combining infinitesimal data. The process of finding integrals is called integration.

Thermodynamics is a branch of physics which deals with heat, work, and temperature, and their relation to energy, radiation, and physical properties of matter.

Mathematical modelling has been used for decades to help scientists understand the mechanisms and dynamics behind their experimental observations. A nutrient cycle (or ecological recycling) is the movement and exchange of organic and inorganic matter back into the production of matter. Energy flow is a unidirectional and noncyclic pathway, whereas the movement of mineral nutrients is cyclic

Clutch size refers to the number of eggs laid in a single brood by a nesting pair of birds. Morphogenesis is the biological process that causes a cell, tissue or organism to develop its shape. Genetic drift is the change in the frequency of an existing gene variant in a population due to random sampling of organisms.

In this unit, you will study about the matrices, vectors, exponential and periodic function, differential equations integration, thermodynamics and their applications in biological field, mathematical modelling-treatment of selected specific models from different areas of biology, cycling of nutrients in ecosystem/eutrophication model, optimal clutch size in birds, morphogenesis, andgenetic drift.

## 4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the general concept of matrices
- Analyse the vectors
- Discuss about the exponential function
- Explain the periodic function
- Comprehend on the differential equations
- Define what integration?
- State the laws of thermodynamics
- Understand the applications of thermodynamics in biological field
- Explain the mathematical modelling-treatment of selected specific models
- Elaborate on the cycling of nutrients in ecosystem/eutrophication model
- Define the morphogenesis
- Analyse the genetic drift

## 4.2 MATRICES

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A matrix is an arrangement of elements, which are numbers (real or complex), variables and/or functions in a rectangular array comprising of rows and columns and enclosed inside a square bracket or parenthesis. Position of an element in a matrix is described by specifying row number and column number. Following arrangement of numbers shows a matrix with three rows and three columns.

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 7 \\ 3 & 7 & 8 \end{bmatrix} \text{ or } \begin{pmatrix} 2 & 4 & 5 \\ 4 & 6 & 7 \\ 3 & 7 & 8 \end{pmatrix}$$

Conventionally, a matrix is represented by a capital letter, such as *A, B, C* or *X, Y, Z.* Elements of the matrix is represented by lowercase letters, such as *a, b, c,* etc., with subscript showing the row number and column number of the element. An element $a_{ij}$ shows an element of matrix having *i*th row and *j*th column. Thus, a matrix *A* is denoted as $a_{ij}$ where $i = 1, 2, 3,..., m$ represents rows and $j = 1, 2, 3,\ldots, n$ denotes columns. Order of a matrix is denoted by number of rows and columns it has.

In the given example, there are 3 rows and 3 columns. Hence, this matrix is said to be of the order 3 × 3. If there are *m* rows and *n* columns in a matrix then it is known as $m \times n$ matrix and numbers of elements are given by *mn*. Also, *aij* denotes the element of the matrix lying in *i*th row and *j*th column and we call this element as the $(i, j)$th element of the matrix. In this example, the element at first row and first column is $a_{11} = 2$. Other elements are $a_{12} = 4$, $a_{13} = 5$, $a_{21} = 4$, $a_{22} = 6$, $a_{23} = 7$ and so on.

A matrix with *m* rows and *n* columns are written as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}$$

In symbolic language same is written as $A = a_{ij}$ where $i = 1, 2, 3, .. m$ and $j = 1, 2, 3, \ldots n$.

***Notes:*** (1) Matrices will contain numbers which may be real or complex.

(2) A matrix is simply an arrangement of elements and has no numerical value.

### 4.2.1 Types of Matrices

Depending on the arrangement of elements in rows and columns there are various types of matrices. These are given below.

1. ***Row Matrix.*** A matrix which has exactly one row is called a *row matrix.* For example, (1 2 3 4) is a row matrix.
2. ***Column Matrix.*** A matrix which has exactly one column is called a *column matrix*.

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For example, $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3. ***Square Matrix.*** A matrix in which the number of rows is equal to the number of columns is called a *square matrix*.

   For example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2 × 2 square matrix.

4. ***Null or Zero Matrix.*** A matrix each of whose elements is zero is called a *null matrix* or *zero matrix.*

   For example, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 2 × 3 Null matrix.

5. ***Diagonal Matrix.*** The elements $a_{ij}$ are called diagonal elements of a square matrix $(a_{ij})$ if $i = j$. For example, in matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

   the diagonal elements are $a_{11} = 1, a_{22} = 5, a_{33} = 9$

   A square matrix whose every element other than diagonal elements is zero, is called a *diagonal matrix*. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ is a diagonal matrix.}$$

   Note that, the diagonal elements in a diagonal matrix may also be zero. For example,

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ are also diagonal matrices.}$$

6. ***Scalar Matrix.*** A diagonal matrix whose diagonal elements are equal, is called a *scalar matrix*. For example,

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ are scalar matrices.}$$

7. ***Identity Matrix.*** A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is an identity matrix.}$$

8. ***Triangular Matrix.*** A square matrix whose elements $a_{ij} = 0$ when either $i < j$ or $i > j$, is known as trangular matrix. A square matrix $(a_{ij})$, whose elements $a_{ij} = 0$ when $i < j$ is called a *lower triangular matrix.*

   Similarly, a square matrix $(a_{ij})$ whose elements $a_{ij} = 0$ whenever $i > j$ is called an *upper triangular matrix.*

   For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \text{ are lower triangular matrices}$$

And $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ are upper triangular matrices.

NOTES

## 4.2.2 Operations on Matrices

Here, we will learn various elementary operations on matrix as discussed below:

**Equality of Matrices**

Two matrices A and B are said to be equal if,

(*i*) *A* and *B* are of same order.

(*ii*) Corresponding elements in *A* and *B* are same.

For example, the following two matrices are equal.

$$\begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix}$$

But the following two matrices are not equal.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

As matrix on left is of order $2 \times 3$, while on right, it is of order $3 \times 3$.

The following two matrices are also not equal.

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 9 \end{pmatrix}$$

As (2, 1)th element in LHS matrix is 7, while in RHS matrix it is 4.

**Addition of Matrices**

If *A* and *B* are two matrices of the same order, then addition of *A* and *B* is defined to be the matrix obtained by adding the corresponding elements of *A* and *B*.

For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

Then, $A + B = \begin{pmatrix} 1+2 & 2+3 & 3+4 \\ 4+5 & 5+6 & 6+7 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \end{pmatrix}$

Also, $A - B = \begin{pmatrix} 1-2 & 2-3 & 3-4 \\ 4-5 & 5-6 & 6-7 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$

Note that addition (or subtraction) of two matrices is defined only when *A* and *B* are of the same order.

**Properties of Matrix Addition**

(*i*) Matrix addition is commutative.

i.e., $A + B = B + A$

NOTES

For, $(i, j)$th element of $A + B$ is $(a_{ij} + b_{ij})$ and of $B + A$ is $(b_{ij} + a_{ij})$, and they are same even if $a_{ij}$ and $b_{ij}$ are complex numbers.

(*ii*) Matrix addition is associative.

i.e., $A + (B + C) = (A + B) + C$

For, $(i, j)$th element of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$ and of $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij}$ which are same.

(*iii*) If $O$ denotes null matrix of the same order as that of $A$ then,

$A + O = A = O + A$

For $(i, j)$th element of $A + O$ is $a_{ij} + O$ and $O + A$ is $O + a_{ij}$, which is same as $(i, j)$th element of $A$.

(*iv*) To each matrix $A$, there corresponds a matrix $B$ such that,

$A + B = O = B + A$.

For, let $(i, j)$th element of $B$ be $-a_{ij}$. Then $(i, j)$th element of $A + B$ is,

$a_{ij} - a_{ij} = 0$.

Thus, the set of $m \times n$ matrices forms an abelian group under the composition of matrix addition.

**Multiplication of Matrix by Scalar**

If $k$ is any real or complex number and $A$, a given matrix, then $kA$ is the matrix obtained from $A$ by multiplying each element of $A$ by $k$. The number $k$ is called *scalar*.

For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } k = 2$$

Then, $$kA = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

It can be easily shown that,

(*i*) $k(A + B) = kA + kB$ (*ii*) $(k_1 + k_2)A = k_1A + k_2A$

(*iii*) $1A = A$ (*iv*) $(k_1k_2)A = k_1(k_2A)$

**Example 4.1:** If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

Verify $A + B = B + A$.

**Solution:** $A + B = \begin{pmatrix} 1+0 & 2+1 & 3+2 \\ 4+3 & 5+4 & 6+5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$

$$B + A = \begin{pmatrix} 0+1 & 1+2 & 2+3 \\ 3+4 & 4+5 & 5+6 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$$

So, $A + B = B + A$

**Example 4.2:** If $A$ and $B$ are matrices as in Example 4.1

And $C = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, verify $(A + B) + C = A + (B + C)$

**NOTES**

**Solution:** Now $A+B=\begin{pmatrix}1 & 3 & 5\\ 7 & 9 & 11\end{pmatrix}$

So, $(A+B)+C=\begin{pmatrix}1-1 & 3+0 & 5+1\\ 7+1 & 9+2 & 11+3\end{pmatrix}=\begin{pmatrix}0 & 3 & 6\\ 8 & 11 & 14\end{pmatrix}$

Again, $B+C=\begin{pmatrix}0-1 & 1+0 & 2+1\\ 3+1 & 4+2 & 5+3\end{pmatrix}=\begin{pmatrix}-1 & 1 & 3\\ 4 & 6 & 8\end{pmatrix}$

So, $A+(B+C)=\begin{pmatrix}1-1 & 2+1 & 3+3\\ 4+4 & 5+6 & 6+8\end{pmatrix}=\begin{pmatrix}0 & 3 & 6\\ 8 & 11 & 14\end{pmatrix}$

Therefore, $(A+B)+C=A+(B+C)$

**Example 4.3:** If $A=\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}$, find a matrix $B$ such that $A+B=0$

**Solution:** Let $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}$

Then, $A+B=\begin{pmatrix}1+b_{11} & 2+b_{12}\\ 3+b_{21} & 4+b_{22}\\ 5+b_{31} & 6+b_{32}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}$

It implies, $b_{11}=-1, b_{12}=-2, b_{21}=-3, b_{22}=-4,$
$b_{31}=-5, b_{32}=-6$

Therefore, required $B=\begin{pmatrix}-1 & -2\\ -3 & -4\\ -5 & -6\end{pmatrix}$

**Example 4.4:** (*i*) If $A=\begin{pmatrix}0 & 1 & 2\\ 2 & 3 & 4\\ 4 & 5 & 6\end{pmatrix}$ and $k_1=i, k_2=2$, verify,

$$(k_1+k_2)A=k_1A+k_2A$$

(*ii*) If $A=\begin{pmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{pmatrix}$, $B=\begin{pmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{pmatrix}$, find the value of $2A+3B$.

**Solution:** (*i*) Now $k_1A=\begin{pmatrix}0 & i & 2i\\ 2i & 3i & 4i\\ 4i & 5i & 6i\end{pmatrix}$ and $k_2A=\begin{pmatrix}0 & 2 & 4\\ 4 & 6 & 8\\ 8 & 10 & 12\end{pmatrix}$

So, $k_1A+k_2A=\begin{pmatrix}0 & 2+i & 4+2i\\ 4+2i & 6+3i & 8+4i\\ 8+4i & 10+5i & 12+6i\end{pmatrix}$

Also, $(k_1+k_2)A=\begin{pmatrix}0 & 2+i & 4+2i\\ 4+2i & 6+3i & 8+4i\\ 8+4i & 10+5i & 12+6i\end{pmatrix}$

Therefore, $(k_1+k_2)A=k_1A+k_2A$

NOTES

(*ii*) $$2A = \begin{pmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{pmatrix}$$

$$3B = \begin{pmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{pmatrix}$$

So, $$2A + 3B = \begin{pmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{pmatrix}$$

**Example 4.5:** If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 1 & 2 \end{pmatrix}$ find $a_{11}, a_{22}, a_{33}, a_{31}$ and $a_{41}$.

**Solution:** $a_{11}$ = Element of $A$ in first row and first column = 1
$a_{22}$ = Element of $A$ in second row and second column = 5
$a_{33}$ = Element of $A$ in third row and third column = 9
$a_{31}$ = Element of $A$ in third row and first column = 7
$a_{41}$ = Element of $A$ in fourth row and first column = 0

**Example 4.6:** In an examination of Mathematics, 20 students from college $A$, 30 students from college $B$ and 40 students from college $C$ appeared. Only 15 students from each college could get through the examination. Out of them, 10 students from college $A$ and 5 students from college $B$ and 10 students from college $C$ secured full marks. Write down the above data in matrix form.

**Solution:** Consider the matrix,

$$\begin{pmatrix} 20 & 30 & 40 \\ 15 & 15 & 15 \\ 10 & 5 & 10 \end{pmatrix}$$

First row represents the number of students in college $A$, college $B$, college $C$, respectively.

Second row represents the number of students who got through the examination in three colleges respectively.

Third row represents the number of students who got full marks in the three colleges respectively.

**Example 4.7:** A publishing house has two branches. In each branch, there are three offices. In each office, there are 3 peons, 4 clerks and 5 typists. In one office of a branch, 6 salesmen are also working. In each office of other branch 2 head clerks are also working. Using matrix notation find (*i*) the total number of posts of each kind in all the offices taken together in each branch, (*ii*) the total number of posts of each kind in all the offices taken together from both the branches.

**Solution:** (*i*) Consider the following row matrices,

$$A_1 = (3\ \ 4\ \ 5\ \ 6\ \ 0), \quad A_2 = (3\ \ 4\ \ 5\ \ 0\ \ 0), \quad A_3 = (3\ \ 4\ \ 5\ \ 0\ \ 0)$$

These matrices represent the three offices of the branch (say $A$) where elements appearing in the row represent the number of peons, clerks, typists, salesmen and head-clerks taken in that order working in the three offices.

NOTES

Then, $A_1 + A_2 + A_3 = (3+3+3 \;\; 4+4+4 \;\; 5+5+5 \;\; 6+0+0 \;\; 0+0+0)$
$= (9 \;\; 12 \;\; 15 \;\; 6 \;\; 0)$

Thus, total number of posts of each kind in all the offices of branch $A$ are the elements of matrix $A_1 + A_2 + A_3 = (9 \;\; 12 \;\; 15 \;\; 6 \;\; 0)$

Now consider the following row matrices,

$$B_1 = (3 \;\; 4 \;\; 5 \;\; 0 \;\; 2), \quad B_2 = (3 \;\; 4 \;\; 5 \;\; 0 \;\; 2), \quad B_3 = (3 \;\; 4 \;\; 5 \;\; 0 \;\; 2)$$

Then $B_1, B_2, B_3$ represent three offices of other branch (say $B$) where the elements in the row represents number of peons, clerks, typists, salesmen and head clerks respectively.

Thus, total number of posts of each kind in all the offices of branch $B$ are the elements of the matrix $B_1 + B_2 + B_3 = (9 \;\; 12 \;\; 15 \;\; 0 \;\; 6)$

(*ii*) The total number of posts of each kind in all the offices taken together from both branches are the elements of matrix,

$$(A_1 + A_2 + A_3) + (B_1 + B_2 + B_3) = (18 \;\; 24 \;\; 30 \;\; 6 \;\; 6)$$

**Example 4.8:** Let $A = \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}$ where first row represents the number of table fans and second row represents the number of ceiling fans which two manufacturing units I and II make in one day. The first and second column represent the manufacturing units I and II. Compute 5$A$ and state what it represents.

**Solution:** $5A = \begin{pmatrix} 50 & 100 \\ 150 & 200 \end{pmatrix}$

It represents the number of table fans and ceiling fans that the manufacturing units I and II produce in five days.

**Example 4.9:** Let $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ where rows represent the number of items of type I, II, III, respectively. The four columns represents the four shops $A_1$, $A_2$, $A_3$, $A_4$, respectively.

Let, $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \end{pmatrix}$, $\quad C = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 \end{pmatrix}$

Where elements in $B$ represent the number of items of different types delivered at the beginning of a week and matrix $C$ represent the sales during that week. Find,

(*i*) The number of items immediately after delivery of items.

(*ii*) The number of items at the end of the week.

(*iii*) The number of items needed to bring stocks of all items in all shops to 6.

**Solution:** (*i*) $A + B = \begin{pmatrix} 3 & 5 & 7 & 9 \\ 5 & 5 & 7 & 9 \\ 7 & 7 & 7 & 9 \end{pmatrix}$

Represent the number of items immediately after delivery of items.

(*ii*) $(A + B) - C = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 4 & 3 & 4 & 5 \\ 5 & 4 & 3 & 5 \end{pmatrix}$

Represent the number of items at the end of the week.

NOTES

(*iii*) We want that all elements in $(A + B) - C$ should be 6.

$$\text{Let } D = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

Then, $(A + B) - C + D$ is a matrix in which all elements are 6. So, $D$ represents the number of items needed to bring stocks of all items of all shops to 6.

**Example 4.10:** The following matrix represents the results of the examination of B. Com. class:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

The rows represent the three sections of the class. The first three columns represent the number of students securing 1st, 2nd, 3rd divisions respectively in that order and fourth column represents the number of students who failed in the examination.

(*i*) How many students passed in three sections respectively?

(*ii*) How many students failed in three sections respectively?

(*iii*) Write down the matrix in which number of successful students is shown.

(*iv*) Write down the column matrix where only failed students are shown.

(*v*) Write down the column matrix showing students in 1st division from three sections.

**Solution:** (*i*) The number of students who passed in three sections respectively are $1 + 2 + 3 = 6$, $5 + 6 + 7 = 18$, $9 + 10 + 11 = 30$.

(*ii*) The number of students who failed from three sections respectively are 4, 8, 12.

(*iii*) $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{pmatrix}$

(*iv*) $\begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$ represents column matrix where only failed students are shown.

(*v*) $\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$ represents column matrix of students securing 1st division.

**Multiplication of Matrices**

The product $AB$ of two matrices $A$ and $B$ is defined only when the number of columns of $A$ is same as the number of rows in $B$ and by definition the product $AB$ is a matrix $G$ of order $m \times p$ if $A$ and $B$ were of order $m \times n$ and $n \times p$ respectively. The following example will give the rule to multiply two matrices:

$$\text{Let, } \quad A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \quad B = \begin{pmatrix} d_1 & e_1 \\ d_2 & e_2 \\ d_3 & e_3 \end{pmatrix}$$

Order of $A = 2 \times 3$, Order of $B = 3 \times 2$

NOTES

So, $AB$ is defined as,

$$G = AB = \begin{pmatrix} a_1d_1 + b_1d_2 + c_1d_3 & a_1e_1 + b_1e_2 + c_1e_3 \\ a_2d_1 + b_2d_2 + c_2d_3 & a_2e_1 + b_2e_2 + c_2e_3 \end{pmatrix}$$

$$= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$g_{11}$ : Multiply elements of the first row of $A$ with corresponding elements of the first column of $B$ and add.

$g_{12}$ : Multiply elements of the first row of $A$ with corresponding elements of the second column of $B$ and add.

$g_{21}$ : Multiply elements of the second row of $A$ with corresponding elements of the first column of $B$ and add.

$g_{22}$ : Multiply elements of the second row of $A$ with corresponding elements of the second column of $B$ and add.

**Notes:** 1. In general, if $A$ and $B$ are two matrices, then $AB$ may not be equal to $BA$. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Then } AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

And $BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. So, $AB \neq BA$

2. If product $AB$ is defined, then it is not necessary that $BA$ must also be defined. For example, if $A$ is of order $2 \times 3$ and $B$ is of order $3 \times 1$, then $AB$ can be defined but $BA$ cannot be defined (as the number of columns of $B \neq$ the number of rows of $A$).

It can be easily verified that,

(*i*) $A(BC) = (AB)C$

(*ii*) $A(B + C) = AB + AC$

$(A + B)C = AC + BC$.

**Example 4.11:** If $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 0 \\ -2 & -3 \end{pmatrix}$, write down $AB$.

**Solution:**

$$AB = \begin{pmatrix} 2 \times 7 + (-1) \times (-2) & 2 \times 0 + (-1) \times (-3) \\ 0 \times 7 + 3 \times (-2) & 0 \times 0 + 3 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 3 \\ -6 & -9 \end{pmatrix}$$

**Example 4.12:** Verify the associative law $A(BC) = (AB)C$ for the following matrices.

$$A = \begin{bmatrix} -1 & 0 \\ 7 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$$

**Solution:**

$$AB = \begin{bmatrix} -1 & 0 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & -5+0 \\ -7-14 & 35+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 \\ -21 & 35 \end{bmatrix}$$

NOTES

$$BC = \begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+10 & 1+0 \\ -7+0 & -7+0 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ -7 & -7 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} -1 & 0 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 11 & 1 \\ -7 & -7 \end{bmatrix} = \begin{bmatrix} -11+0 & -1+0 \\ 77+14 & 7+14 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -1 \\ 91 & 21 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 1 & -5 \\ -21 & 35 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1-10 & -1+0 \\ 21+70 & 21+0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -1 \\ 91 & 21 \end{bmatrix}$$

Therefore, $A(BC) = (AB)C$

**Example 4.13:** If $A$ is a square matrix, then $A$ can be multiplied by itself. Define $A^2 = A.\ A$ (called power of a matrix). Compute $A^2$ for the following matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

**Solution:** $$A^2 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 15 & 16 \end{pmatrix}$$

(Similarly, we can define $A^3, A^4, A^5, \ldots$ for any square matrix $A$.)

**Example 4.14:** If $A = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$ find $A^2 + 3A + I$ where $I$ is unit matrix of order 2.

**Solution:** $$A^2 = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ -3 & -6 \end{pmatrix}$$

$$3A = \begin{pmatrix} 3 & 6 \\ -9 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $$A^2 + 3A + I = \begin{pmatrix} -5 & 2 \\ -3 & -6 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ -9 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 8 \\ -12 & -5 \end{pmatrix}$$

**Example 4.15:** If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, show that $AB = -BA$ and $A^2 = B^2 = I$

**Solution:** Now, $$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

NOTES

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

So, $AB = -BA$

Also, $$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This proves the result.

**Transpose of a Matrix**

Let $A$ be a matrix. The matrix obtained from $A$ by interchanging of its rows and columns, is called the *transpose* of $A$. For example,

If, $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$ then transpose of $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$

Transpose of $A$ is denoted by $A'$.

It can be easily verified that,

(*i*) $(A)' = A$

(*ii*) $(A + B)' = A' + B'$

(*iii*) $(AB)' = B'A'$

**Example 4.16:** For the following matrices $A$ and $B$ verify $(A + B)' = A' + B'$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 8 & 6 \end{pmatrix}$$

**Solution:** $A' = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ $B' = \begin{pmatrix} 2 & 1 \\ 3 & 8 \\ 4 & 6 \end{pmatrix}$

So, $A' + B' = \begin{pmatrix} 3 & 5 \\ 5 & 13 \\ 7 & 12 \end{pmatrix}$

Again, $A + B = \begin{pmatrix} 3 & 5 & 7 \\ 5 & 13 & 12 \end{pmatrix}$

So, $(A + B)' = \begin{pmatrix} 3 & 5 \\ 5 & 13 \\ 7 & 12 \end{pmatrix}$

Therefore, $(A + B)' = A' + B'$

## 4.2.3 Elementary Operations

Consider the matrices,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

Matrix $B$ is obtained from $A$ by interchange of first and second row.

Consider $C = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 5 \end{pmatrix}$, $D = \begin{pmatrix} 6 & 3 & 9 \\ 3 & 4 & 5 \end{pmatrix}$

Matrix $D$ is obtained from $C$ by multiplying first row by 3.

Consider $E = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 2 \end{pmatrix}$, $F = \begin{pmatrix} 2 & 3 & 4 \\ 7 & 12 & 14 \end{pmatrix}$

**NOTES**

Matrix $F$ is obtained from $E$ by multiplying first row of $E$ by 3 and adding it to second row.

Such operations on rows of a matrix as described above are called *elementary row operations*.

Similarly, we define *elementary column operations*.

An elementary operation is either elementary row operation or elementary column operation and is of the following three types:

***Type I.*** The interchange of any two rows (or column).

***Type II.*** The multiplication of any row (or column) by a non-zero number.

***Type III.*** The addition of multiple of one row (or column) to another row (or column).

We shall use the following notations for three types of Elementary operations.

The interchange of $i$th and $j$th rows (columns) will be denoted by $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$).

The multiplication of $i$th row (column) by non-zero number $k$ will be denoted by $R_i \to k\,R_i$ ($C_i \to k\,C_i$)

The addition of $k$ times the $j$th row (column) to $i$th row (column) will be denoted by $R_i \to R_i + kR_j$ ($C_i \to C_i + kC_i$).

**Elementary Matrices**

Matrix obtained from identity matrix by a single elementary operation is called ***Elementary matrix***.

For example, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

These are elementary matrices, the first is obtained by $R_1 \leftrightarrow R_2$ and the second by $C_1 \to 2C_1$ on the identity matrix.

We state the following result without proof:

'An elementary row operation on product of two matrices is equivalent to elementary row operation on prefactor.'

It means that if we make elementary row operation in the product $AB$, then it is equivalent to making same elementary row operation in $A$ and then multiplying it with $B$.

Let, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$

Then, $AB = \begin{pmatrix} 9 & 13 \\ 13 & 19 \end{pmatrix}$

Suppose we interchange first and second row of $AB$.

NOTES

Then, the matrix we get is,

$$C = \begin{pmatrix} 13 & 19 \\ 9 & 13 \end{pmatrix}$$

Now, interchange first and second row of $A$ and get new matrix,

$$D = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

Multiply $D$ with $B$ to get $DB$,

Where, $$DB = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 19 \\ 9 & 13 \end{pmatrix}$$

Hence, $$DB = C$$

### 4.2.4 Reduction of a Matrix to Echelon Form

Consider, $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Apply the following elementary row operations on $A$.

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$$

We obtain a new matrix.

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -3 & -6 \\ 0 & -5 & -7 & -8 \end{pmatrix}$$

Apply, $R_2 \to -\frac{1}{3} R_2$ on $B$ to get,

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -7 & -8 \end{pmatrix}$$

Apply, $R_3 \to R_3 + 5R_2$ on $C$ to get,

$$D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

The matrix $D$ is in *Echelon form*, i.e., elements below the diagonal are zero.

We, thus, find elementary row operations reduce Matrix $A$ to Echelon form.

In fact, *any matrix can be reduced to Echelon form by elementary row operations.* The procedure is as follows:

***Step I.*** Reduce the element in (1, 1)th place to unity by some suitable elementary row operation.

***Step II.*** Reduce all the elements in 1st column below 1st row to zero with the help of unity obtained in first step.

***Step III.*** Reduce the element in (2, 2)th place to unity by suitable elementary row operations.

***Step IV.*** Reduce all the elements in 2nd column below 2nd row to zero with the help of unity obtained in Step *III*.

NOTES

Proceeding in this way, any matrix can be reduced to the Echelon form.

**Example 4.17:** Reduce $A = \begin{pmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{pmatrix}$ to Echelon form.

**Solution:**

*Step I.* Apply $R_1 \leftrightarrow R_3$ to get,

$$\begin{pmatrix} 1 & -5 & 2 \\ -1 & 12 & -2 \\ 3 & -10 & 5 \end{pmatrix}$$

*Step II.* Apply $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 3R_1$ to get,

$$\begin{pmatrix} 1 & -5 & 2 \\ 0 & 7 & 0 \\ 0 & 5 & -1 \end{pmatrix}$$

*Step III.* Apply $R_2 \to \frac{1}{7} R_2$ to get,

$$\begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{pmatrix}$$

*Step IV.* Apply $R_3 \to R_3 - 5R_2$ to get,

$\begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ which is a matrix in Echelon form.

**Example 4.18:** Reduce $A = \begin{pmatrix} 2 & 2 & 4 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ to Echelon form.

**Solution:**

*Step I.* Apply $R_1 \to \frac{1}{2} R_1$ to get,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

*Step II.* Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 4R_1$ to get,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

*Step III.* (2, 2)th place is already unity.

*Step IV.* Apply $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ to get,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

*Step V.* Apply $R_3 \to (-1)\, R_3$ to get,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

*Step VI.* Apply $R_4 \to R_4 + 2R_3$ to get,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which is a matrix in Echelon form.

**Example 4.19:** Reduce $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 6 & 4 \\ 0 & 3 & 9 & 3 \\ 0 & 4 & 13 & 4 \end{pmatrix}$ to Echelon form.

**Solution:**

*Step I.* Since all the elements in 1st column are zero Step *I* and Step *II* are not needed.

*Step III.* Apply $R_2 \to \frac{1}{2} R_2$ to get,

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 3 \\ 0 & 4 & 13 & 4 \end{pmatrix}$$

*Step IV.* Apply $R_3 \to R_3 - 3R_2, R_4 \to R_4 - 4R_2$ to get,

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

*Step V.* Apply $R_3 \leftrightarrow R_4$ to get,

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

*Step VI.* Since elements below (3, 3)rd place are zero, Step *VI* is not needed.

Hence, *A* is reduced to Echelon form.

NOTES

**Matrix Method of Solution of Equations**

In this section, we will discuss a method to solve a system of linear equations with the help of elementary operations.

Consider the equations,

$$a_{11}x + a_{12}y + a_{13}z = b_{11}$$
$$a_{21}x + a_{22}y + a_{23}z = b_{12}$$
$$a_{31}x + a_{32}y + a_{33}z = b_{13}$$

These equations can be also expressed as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} \text{ i.e., } AX = B$$

Where $A$ is the matrix obtained by writing the coefficients of $x, y, z$ in three rows respectively, and $B$ is the column matrix consisting of constants in RHS of given equations.

We reduce the matrix $A$ to Echelon form and write the equations in the form stated earlier, and then solve. This will be made clear in the following examples. A system of equations is called *consistent* if and only if there exists a common solution to all of them, otherwise it is called *inconsistent.*

**Example 4.20:** Solve the system of equations,

$$x - 3y + z = -1$$
$$2x + y - 4z = -1$$
$$6x - 7y + 8z = 7$$

**Solution:** Let, $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$

Assume that there exists a matrix,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

such that, given system of equations becomes,

$$AX = B$$

Then, $$\begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 6R_1$

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 7 & -6 \\ 0 & 11 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 13 \end{pmatrix}$$

NOTES

Applying $R_2 \to \frac{1}{7}R_2$

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -\frac{6}{7} \\ 0 & 11 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{1}{7} \\ 13 \end{pmatrix}$$

Applying $R_3 \to R_3 - 11R_2$

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -\frac{6}{7} \\ 0 & 0 & \frac{80}{7} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{1}{7} \\ \frac{80}{7} \end{pmatrix}$$

Thus, we have reduced coefficient matrix $A$ to Echelon form. Note that each elementary row operation that we applied on $A$, was also applied on $B$ simultaneously.

From, the last matrix equation we have,

$$x - 3y + z = -1$$

$$y - \frac{6}{7}z = \frac{1}{7}$$

$$\frac{80}{7}z = \frac{80}{7}$$

So, $z = 1, y = 1, x = 1$

Hence, the given system of equations has a solution, $x = 1, y = 1, z = 1$

**Example 4.21:** Solve the system of equations,

$$2x - 5y + 7z = 6$$

$$x - 3y + 4z = 3$$

$$3x - 8y + 11z = 11, \text{ if consistent.}$$

**Solution:** Let $A = \begin{pmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 3 \\ 11 \end{pmatrix}$

So, given system of equations can be written as,

$$\begin{pmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 11 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix}$$ (Interchanging row 1 with row 2)

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

NOTES

Applying $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x - 3y + 4z &= 3 \\ y - z &= 0 \\ 0 &= 2 \end{aligned}$$

Since, $0 = 2$ is false, the given system of equations has no solution. So, the given system of equations is inconsistent.

**Example 4.22:** Solve the system of equations,

$$\begin{aligned} x + y + z &= 7 \\ x + 2y + 3z &= 16 \\ x + 3y + 4z &= 22 \end{aligned}$$

**Solution:** Given system of equations in matrix form,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 16 \\ 22 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 15 \end{pmatrix}$$

Applying $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + y + z &= 7 \\ y + 2z &= 9 \\ z &= 3 \end{aligned}$$

So, $z = 3, y = 3, x = 1$

The given system of equations has a solution, 1, 3, 3

**Example 4.23:** Solve the system of equations,

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + 3z &= 5 \\ x + 3y + 6z &= 11 \\ x + 4y + 10z &= 21 \end{aligned}$$

**Solution:** We have,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 11 \\ 21 \end{pmatrix}$$

NOTES

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$

Then,
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \\ 0 & 3 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 9 \\ 19 \end{pmatrix}$$

Applying $R_3 \to R_3 - 2R_2, R_4 \to R_4 - 3R_2$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 10 \end{pmatrix}$$

Applying $R_4 \to R_4 - 3R_3$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $x + y + z = 2, \; y + 2z = 3, \; z = 3, \; 0 = 1$

This is absurd. So, the given system is inconsistent.

**Example 4.24:** Solve the system of equations,

$$x - 3y - 8z = -10$$
$$3x + y - 4z = 0$$
$$2x + 5y + 6z = 13$$

**Solution:** We have,
$$\begin{pmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ 13 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$
$$\begin{pmatrix} 1 & -3 & -8 \\ 0 & 10 & 20 \\ 0 & 11 & 22 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \\ 33 \end{pmatrix}$$

Applying $R_2 \to \frac{1}{10} R_2$
$$\begin{pmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 11 & 22 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \\ 33 \end{pmatrix}$$

Applying $R_3 \to R_3 - 11R_2$
$$\begin{pmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \\ 0 \end{pmatrix}$$

$\Rightarrow$
$$x - 3y - 8z = -10$$
$$y + 2z = 3$$

Let, $z = k \Rightarrow y = 3 - 2k$

and $x = 9 - 6k + 8k - 10 = 2k - 1$

So, the given system has infinite number of solutions of the form $x = 2k - 1$, $y = 3 - 2k$, $z = k$ where $k$ is any number.

**Example 4.25:** Solve the system of equations,

NOTES

$$x + 2y + 3z + 4w = 0$$
$$8x + 5y + z + 4w = 0$$
$$5x + 6y + 8z + w = 0$$
$$8x + 3y + 7z + 2w = 0$$

**Solution:** We have,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 5 & 1 & 4 \\ 5 & 6 & 8 & 1 \\ 8 & 3 & 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - 8R_1, R_3 \to R_3 - 5R_1, R_4 \to R_4 - 8R_1$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -11 & -23 & -28 \\ 0 & -4 & -7 & -19 \\ 0 & -13 & -17 & -30 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to -\frac{1}{11} R_2$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{23}{11} & \frac{28}{11} \\ 0 & -4 & -7 & -19 \\ 0 & -13 & -17 & -30 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 + 4R_2, R_4 \to R_4 + 13R_2$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{23}{11} & \frac{28}{11} \\ 0 & 0 & \frac{15}{11} & -\frac{97}{11} \\ 0 & 0 & \frac{112}{11} & \frac{34}{11} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to \frac{11}{15} R_3$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{23}{11} & \frac{28}{11} \\ 0 & 0 & 1 & -\frac{97}{15} \\ 0 & 0 & \frac{112}{11} & \frac{34}{11} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

NOTES

Applying $R_4 \to R_4 - \frac{112}{11} R_3$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{23}{11} & \frac{28}{11} \\ 0 & 0 & 1 & -\frac{97}{15} \\ 0 & 0 & 0 & \frac{11374}{165} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x + 2y + 3z + 4w = 0$$

$$y + \frac{23}{11} z + \frac{28}{11} w = 0$$

$$z - \frac{97}{15} w = 0$$

$$w = 0$$

$$\Rightarrow \quad x = 0, y = 0, z = 0, w = 0$$

Thus, system has only one solution, namely $x = y = z = w = 0$

### 4.2.5 Square Matrix

According to caylex-Hamilton theorem,

*Every square matrix satisfies its own characteristic equation.*

The characteristic equation of a square matrix $A$ is $|A - \lambda I| = 0$. This can be written as

$$p_0\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + ... + p_n = 0 \quad (4.1)$$

We have to prove that,

$$p_0A^n + p_1A^{n-1} + p_2A^{n-2} + ... + p_n I = 0 \quad (4.2)$$

Consider the matrix $B = Adj(A - \lambda I)$. The elements of $B$ are polynomials in $\lambda$ of degree $(n - 1)$ or less. Therefore, $B$ can be written in the form,

$$B = B_0\lambda^{n-1} + B_1\lambda^{n-2} + ... + B_{n-2}\lambda + B_{n-1} \quad (4.3)$$

Where, $B_0$, $B_1$, $B_2$,...., $B_{n-1}$ are matrices of order $n$ and whose elements are polynomials of the elements of $A$. It is known that for any matrix $A$,

$$A \; Adj \; A = |A|I$$

Using this property for the matrix $(A - \lambda I)$, we get

$$(A - \lambda I) \; Adj \; (A - \lambda I) = |A - \lambda I|I$$

$$(A - \lambda I) \; B = |A - \lambda I|I$$

Using Equations (4.2) and (4.3), we get

$$(A - \lambda I)\,(B_0\lambda^{n-1} + B_1\lambda^{n-2} + .... + B_{n-2}\lambda + B_{n-1}) = (p_0\lambda^n + p_1\lambda^{n-1} + ... + p_n)I$$

On equating the coefficients of $\lambda^n$, $\lambda^{n-1}$, we get

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$$-B_0 = p_0 I$$
$$AB_0 - B_1 = p_1 I$$
$$AB_1 - B_2 = p_2 I$$
$$\vdots$$
$$AB_{n-1} = p_n I$$

Premultiplying these equations by $A^n, A^{n-1}......A, I$, we get

$$-A^n B_0 = p_0 A^n$$
$$A^n B_0 - A^{n-1} B_1 = p_1 A^{n-1}$$
$$A^{n-1} B_1 - A^{n-2} B_2 = p_2 A^{n-2}$$
$$\vdots$$
$$A^2 B_{n-2} - AB_{n-1} = p_{n-1} I$$
$$AB_{n-1} = p_n I$$

Adding these equations, we get

$0 = p_0 A^n + p_1 A^{n-1} + ... + p_n I$, which proves the theorem.

**Example 4.26:** Verify Cayley – Hamilton theorem for the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ and hence find $A^{-1}$ and $A^4$.

**Solution:** $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

This square matrix is of the Order 3.

Characteristic equation is $\lambda^3 - \beta_1 \lambda^2 + \beta_2 \lambda - \beta_3 = 0$

$\beta_1$ = Sum of leading deagonal elements

$= a_{11} + a_{22} + a_{33}$

$\beta_2$ = Sum of the minors of leading diagonal elements of A

And $\beta_3 = |A|$

Hence, $\beta_1 = 6$, $\beta_2 = 11$, $\beta_3 = |A| = 6$

Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

We have to prove that $A^3 - 6A^2 + 11A - 6I = 0$ (i)

NOTES

$$A^2 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} \text{ and}$$

$$A^3 = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$$

Using them in Equation (i), we find that the equation is satisfied.

$\therefore \quad A^3 - 6A^2 + 11A - 6I = 0$

Premultiplying each term by $A^{-1}$, we get $A^2 - 6A + 11I - 6A^{-1} = 0$

$$\therefore \quad A^{-1} = \frac{1}{6}[A^2 - 6A + 11I] = \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Premultiplying each term of Equation (i) by $A$, we get

$$A^4 = 6A^3 - 11A^2 + 6A = \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$$

## 4.3 VECTORS

Vectors can be multiplied in two different ways, namely the scalar product and the vector product. The scalar product of two vectors results in a scalar quantity, while a vector product results in a vector quantity. Therefore, the scalar products and vector products are two ways of multiplying two different vectors. The scalar product of two vectors is defined as the product of the magnitudes of the two vectors and the cosine of the angles between them. Scalar products can be found by taking the component of one vector in the direction of the other vector and multiplying it with the magnitude of the other vector. It can be defined as, 'Scalar product or dot product is an algebraic operation that takes two equal-length sequences of numbers and returns a single number'. The magnitude vector product of two given vectors can be found by taking the product of the magnitudes of the vectors times the sine of the angle between them. It can be defined as, 'Vector product or cross product is a binary operation on two vectors in three-dimensional space'.

### Scalar Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ in this order and is written as $\left(\vec{a} \times \vec{b}\right).\vec{c}$ or $\left[\vec{a}\vec{b}\vec{c}\right]$ or $\left[\vec{a}, \vec{b}, \vec{c}\right]$.

NOTES

### Geometrical Representation of Scalar Triple Product

Geometrically, the scalar product represents the volume of parallelepiped having $\vec{a}, \vec{b}$ and $\vec{c}$ as its coterminous edges.

Consider a parallelepiped with $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ as coterminous edges (Refer Figure 4.1).

$\vec{a} \times \vec{b}$ is the vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Let $\hat{u}$ be a unit vector along $\vec{a} \times \vec{b}$ and θ be the angle between $\hat{u}$ and $\vec{c}$.

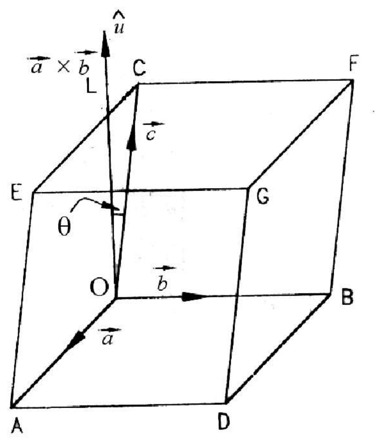


***Fig. 4.1*** *Parallelepiped*

Now, $\left(\vec{a} \times \vec{b}\right).\vec{c}$ = (Area of Parallelogram OADB) $\hat{u}.\vec{c}$

= (Area of Parallelogram OADB) $|\hat{u}|\,|\vec{c}|\cos\theta$

= (Area of Parallelogram OADB) OC cosθ

= (Area of Parallelogram OADB) OL

= (Area of Parallelogram OADB) × Height

= Volume of Parallelepiped

### Properties of Scalar Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors and $m$ be a scalar.

(i) The cyclic permutation of three vectors does not change the value of scalar product.

$$\left(\vec{a} \times \vec{b}\right).\vec{c} = \left(\vec{b} \times \vec{c}\right).\vec{a} = \left(\vec{c} \times \vec{a}\right).\vec{b} \text{ or } \left[\vec{a}\vec{b}\vec{c}\right] = \left[\vec{b}\vec{c}\vec{a}\right] = \left[\vec{c}\vec{a}\vec{b}\right]$$

(ii) The change in the cyclic order of three vectors changes the sign of the scalar triple product but not the magnitude.

$$\left[\vec{a}\vec{b}\vec{c}\right] = -\left[\vec{b}\vec{a}\vec{c}\right] = -\left[\vec{c}\vec{b}\vec{a}\right] = -\left[\vec{a}\vec{c}\vec{b}\right]$$

NOTES

(iii) In a scalar triple product, the dot and cross can be interchanged provided that the cyclic order of the vectors remains the same.

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\vec{a}.\left(\vec{b}\times\vec{c}\right)$$

(iv) The scalar triple product of three vectors is zero if any two of them are equal.

$$\left[\vec{a}\vec{b}\vec{c}\right]=0 \text{ if } \vec{a}=\vec{b} \text{ or } \vec{b}=\vec{c} \text{ or } \vec{c}=\vec{a}$$

(v) $\left[m\vec{a}\vec{b}\vec{c}\right]=m\left[\vec{a}\vec{b}\vec{c}\right]$

(vi) The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

$$\left[\vec{a}\vec{b}\vec{c}\right]=0 \text{ if } \vec{a}=m\vec{b} \text{ or } \vec{b}=m\vec{c} \text{ or } \vec{c}=m\vec{a}$$

**Note:** $\left[\hat{i}\hat{j}\hat{k}\right]=\left(\hat{i}\times\hat{j}\right).\hat{k}=\hat{k}.\hat{k}=1$, where $\hat{i}, \hat{j}$ and $\hat{k}$ stands for the unit vectors along the axes. Similarly, $\left[\hat{j}\hat{k}\hat{i}\right]=\left[\hat{k}\hat{i}\hat{j}\right]=1$ and thus $\left[\hat{i}\hat{j}\hat{k}\right]=\left[\hat{j}\hat{k}\hat{i}\right]=\left[\hat{k}\hat{i}\hat{j}\right]=1$.

**Coplanarity of Three Vectors**

The necessary and sufficient condition for three non-zero non-collinear vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is that $\left[\vec{a}\vec{b}\vec{c}\right]=0$, i.e,

$$\vec{a}, \vec{b}, \vec{c} \text{ are coplanar} \Leftrightarrow \left[\vec{a}\vec{b}\vec{c}\right]=0$$

**Proof: Condition is Necessary:** Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$. This means that,

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=0 \text{ or } \left[\vec{a}\vec{b}\vec{c}\right]=0$$

**Condition is Sufficient:** Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero non-collinear vectors such that $\left[\vec{a}\vec{b}\vec{c}\right]=0$ or $\left(\vec{a}\times\vec{b}\right).\vec{c}=0$.

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero non-collinear vectors; thus, $\vec{a}\times\vec{b}\neq 0$ and $\vec{c}\neq 0$. This means that $\left(\vec{a}\times\vec{b}\right)\perp\vec{c}$. But $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ lies in the same plane, i.e., they are coplanar.

**Scalar Triple Product in Terms of Components**

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three vectors, then,

NOTES

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Volume of a Tetrahedron**

Let ABCD be a tetrahedron, and its three edges AB, AC and AD represent three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively (Refer Figure 4.2).

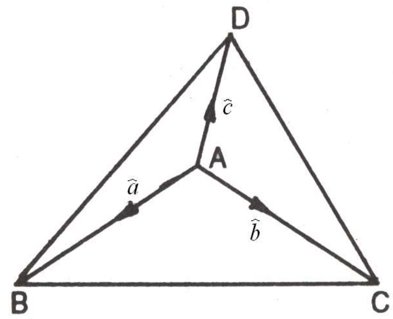


***Fig. 4.2*** *Tetrahedron*

Volume of Tetrahedron = $\frac{1}{3}$ (Area of $\Delta$ABC) × (Height of Vertex D Above the Plane ABC)

$= \frac{1}{3}\left(\frac{1}{2}\text{ Area of Parallelogram whose Adjacent Edges AB and AC}\right) \times$ (Height of Vertex D Above the Plane ABC)

$= \frac{1}{6}$ (Volume of Parallelepiped having AB, AC and AD as Coterminous Edges)

$$= \frac{1}{6}\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right] = \frac{1}{6}\left[\vec{a}\vec{b}\vec{c}\right]$$

Thus, the volume of tetrahedron with three edges AB, AC and AD representing three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $\frac{1}{6}\left[\vec{a}\vec{b}\vec{c}\right]$.

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the vertices A, B, C and D of the tetrahedron ABCD, then $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$ and $\overrightarrow{AD} = \vec{d} - \vec{a}$.

$$\therefore \text{ Volume of Tetrahedron} = \frac{1}{6}\left[\vec{b} - \vec{a} \quad \vec{c} - \vec{a} \quad \vec{d} - \vec{a}\right]$$

NOTES

Four points A, B, C and D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if the volume of tetrahedron ABCD is 0, i.e.,

$$\frac{1}{6}\left[\vec{b}-\vec{a} \quad \vec{c}-\vec{a} \quad \vec{d}-\vec{a}\right] = 0 \Rightarrow \left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right].\left(\vec{d}-\vec{a}\right) = 0$$

$$\Rightarrow \left[\vec{b}\vec{c}\vec{d}\right] - \left[\vec{b}\vec{c}\vec{a}\right] + \left[\vec{b}\vec{d}\vec{a}\right] - \left[\vec{c}\vec{d}\vec{a}\right] = 0$$

**Example 4.27:** Find the volume of a parallelepiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

**Solution:** Let $\vec{a}=-3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b}=-5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c}=7\hat{i}-5\hat{j}-3\hat{k}$. We know that the volume of a parallelepiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is equal to $\left\|\left[\vec{a}\vec{b}\vec{c}\right]\right\|$.

We have,

$$\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= -3(-21-15) - 7(15+21) + 5(25-49)$$
$$= 108 - 252 - 120 = -264$$

So, required volume of the parallelepiped $= \left\|\left[\vec{a}\vec{b}\vec{c}\right]\right\| = |-264|$

$= 264$ cubic units

**Example 4.28:** Find the value of $\lambda$ for which the four points with position vectors $3\hat{i}-2\hat{j}-\hat{k}, 2\hat{i}+3\hat{j}-4\hat{k}, -\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$ are coplanar.

**Solution:** Let A, B, C, D be the given points. Then,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= \left(2\hat{i}+3\hat{j}-4\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right) = -\hat{i}+5\hat{j}-3\hat{k}$$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$$= \left(-\hat{i}+\hat{j}+2\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right) = -4\hat{i}+3\hat{j}+3\hat{k}$$

$\overrightarrow{AD}$ = Position vector of D – Position vector of A

$$= \left(4\hat{i}+5\hat{j}+\lambda\hat{k}\right)-\left(3\hat{i}-2\hat{j}-\hat{k}\right) = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$$

The given points are coplanar iff vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar.

$\therefore$ Points A, B, C, D are coplanar.

$\Leftrightarrow$ $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar,

$$\Leftrightarrow \left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = 0$$

$$\Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

NOTES

$\Leftrightarrow -1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-28 - 3) = 0$

$\Leftrightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Leftrightarrow 17\lambda + 146 = 0$

$\Rightarrow \lambda = -\dfrac{146}{17}$

**Example 4.29:** Find the altitude of a parallelepiped determined by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, if the base is taken to the parallelogram determined by $\vec{a}$ and $\vec{b}$, and if $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + 3\hat{k}$.

**Solution:** Volume of the Parallelepiped $= \left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix}$

$= (12 + 1) - (6 + 1) + (2 - 4)$

$= 4$ cubic units ...(i)

Area of the Base $= \left|\vec{a} \times \vec{b}\right|$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & -1 \end{vmatrix} = -5\hat{i} + 3\hat{j} + 2\hat{k}$

$\therefore \left|\vec{a} \times \vec{b}\right| = \sqrt{25 + 9 + 4} = \sqrt{38}$

Also,

Volume of the Parallelepiped = Area of the Base × Altitude

$= \left(\sqrt{38}\right) \times$ Altitude ...(ii)

From Equations (i) and (ii), we have,

$\sqrt{38} \times$ Altitude $= 4$

$\Rightarrow$ Altitude $= \dfrac{4}{\sqrt{38}}$ units.

**Vector Triple Product**

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the vector products of $\vec{a} \times \vec{b}$ with $\vec{c}$ and $\vec{a}$ with $\vec{b} \times \vec{c}$ are called the **vector triple products** of $\vec{a}, \vec{b}$ and $\vec{c}$. These products are written as $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$.

NOTES

**Expansion Formula**

For three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

Let us prove this formula.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\hat{j} +$$
$$[a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\hat{k}$$

Now, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) -$

$$(a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$[\because \vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \text{ and } \vec{a}.\vec{c} = a_1c_1 + a_2c_2 + a_3c_3 \ ]$$

$$= (a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1)\hat{i} +$$
$$(a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2)\hat{j} +$$
$$(a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3)\hat{k}$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} +$$
$$[a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\hat{j} +$$
$$[a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\hat{k}$$

Thus, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Similarly, it can be shown that $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

**Note:** (i) The vector triple product $\vec{a} \times (\vec{b} \times \vec{c})$ is a linear combination of those two vectors which are within brackets.

(ii) $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and $\vec{b}\times\vec{c}$.

(iii) $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$, i.e., vectors, triple product is not associative.

NOTES

$\vec{a}\times(\vec{b}\times\vec{c})$ is a vector which lies in the plane of $\vec{b}$ and $\vec{c}$ whereas $(\vec{a}\times\vec{b})\times\vec{c}$ is a vector which lies in the plane of $\vec{b}$ and $\vec{a}$.

$(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) =$

$$-\left[(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}\right]=(\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{a}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$$

**Example 4.30:** If $\vec{a},\vec{b},\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then find the angles, which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.

**Solution :** We have, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$\Rightarrow (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$\Rightarrow \left(\vec{a}.\vec{c}-\frac{1}{\sqrt{2}}\right)\vec{b}-\left(\vec{a}.\vec{b}+\frac{1}{\sqrt{2}}\right)\vec{c}=\vec{0}$

$\Rightarrow \vec{a}.\vec{c}-\frac{1}{\sqrt{2}}=0$ and $\vec{a}.\vec{b}+\frac{1}{\sqrt{2}}=0$ [$\because$ $\vec{b}$ and $\vec{c}$ are non-collinear vectors]

$\Rightarrow \vec{a}.\vec{c}=\frac{1}{\sqrt{2}}$ and $\vec{a}.\vec{b}=-\frac{1}{\sqrt{2}}$

$\Rightarrow \cos\alpha=\frac{1}{\sqrt{2}}$

and $\cos\beta=-\frac{1}{\sqrt{2}}$, where $\alpha$ and $\beta$ are the angles made by $\vec{a}$ with $\vec{b}$ and $\vec{c}$, respectively.

$\Rightarrow \alpha=\frac{\pi}{4}$ and $\beta=\frac{3\pi}{4}$.

**Scalar Product of Four Vectors**

If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are any four vectors, then the scalar product of $\vec{a}\times\vec{b}$ with $\vec{c}\times\vec{d}$ is called the **scalar product** of $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$. This product is written as, $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$.

$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \left[(\vec{a}\times\vec{b})\times\vec{c}\right].\vec{d}$ [interchanging the Dot and Cross Product]

$$= \left[(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}\right].\vec{d}$$

NOTES

$$= \left(\vec{a}.\vec{c}\right)\left(\vec{b}.\vec{d}\right) - \left(\vec{a}.\vec{d}\right)\left(\vec{b}.\vec{c}\right)$$

$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \left(\vec{a}.\vec{c}\right)\left(\vec{b}.\vec{d}\right) - \left(\vec{a}.\vec{d}\right)\left(\vec{b}.\vec{c}\right)$ is known as Lagrange's identity. It can be expressed in the form of determinant as,

$$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}$$

**Vector Product of Four Vectors**

If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ be any four vectors, then the vector product of $\vec{a}\times\vec{b}$ with $\vec{c}\times\vec{d}$ is called the **vector product** of $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$. This product is written as, $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$.

Vector product $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ being a vector perpendicular to $\vec{a}\times\vec{b}$ is coplanar with $\vec{a}$ and $\vec{b}$. Also, it being a vector perpendicular to $\vec{c}\times\vec{d}$ is coplanar with $\vec{c}$ and $\vec{d}$. Thus, we can look upon $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ as a vector triple product in two ways by putting $\vec{a}\times\vec{b} = \vec{q}$ and $\vec{c}\times\vec{d} = \vec{p}$.

(i) Expressing $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of $\vec{a}$ and $\vec{b}$.

$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \left(\vec{a}\times\vec{b}\right)\times\vec{p}$, where $\vec{c}\times\vec{d} = \vec{p}$

$$= \left(\vec{a}.\vec{p}\right)\vec{b} - \left(\vec{b}.\vec{p}\right)\vec{a}$$

$$= \left(\vec{a}.\vec{c}\times\vec{d}\right)\vec{b} - \left(\vec{b}.\vec{c}\times\vec{d}\right)\vec{a}$$

$$= \left[\vec{a}\vec{c}\vec{d}\right]\vec{b} - \left[\vec{b}\vec{c}\vec{d}\right]\vec{a}$$

Here, the vector product appears as the linear combination of $\vec{a}$ and $\vec{b}$.

(ii) Expressing $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of $\vec{c}$ and $\vec{d}$.

$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \vec{q}\times\left(\vec{c}\times\vec{d}\right)$, where $\vec{a}\times\vec{b} = \vec{q}$

$$= \left(\vec{q}.\vec{d}\right)\vec{c} - \left(\vec{q}.\vec{c}\right)\vec{d}$$

$$= \left(\vec{a}\times\vec{b}.\vec{d}\right)\vec{c} - \left(\vec{a}\times\vec{b}.\vec{c}\right)\vec{d}$$

$$= \left[\vec{a}\vec{b}\vec{d}\right]\vec{c} - \left[\vec{a}\vec{b}\vec{c}\right]\vec{d}$$

Here, the vector product appears as the linear combination of $\vec{c}$ and $\vec{d}$.

Thus, $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \left[\vec{a}\vec{c}\vec{d}\right]\vec{b} - \left[\vec{b}\vec{c}\vec{d}\right]\vec{a} = \left[\vec{a}\vec{b}\vec{d}\right]\vec{c} - \left[\vec{a}\vec{b}\vec{c}\right]\vec{d}$

NOTES

**Example 4.31:** Prove that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0$.

**Solution:** We know that,

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})=(\vec{b}.\vec{a})(\vec{c}.\vec{d})-(\vec{b}.\vec{d})(\vec{c}.\vec{a})$$

$$(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})=(\vec{c}.\vec{b})(\vec{a}.\vec{d})-(\vec{c}.\vec{d})(\vec{a}.\vec{b})$$

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

Adding we get,

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0$$

**Reciprocal System of Vectors**

If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, i.e., $[\vec{a}\vec{b}\vec{c}]\neq 0$ and if $\vec{a'},\vec{b'}$ and $\vec{c'}$ are three other vectors such that,

$$\vec{a'}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\vec{b'}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]} \text{ and } \vec{c'}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

then $\vec{a'},\vec{b'}$ and $\vec{c'}$ are called the **reciprocal system** to the vectors $\vec{a},\vec{b}$ and $\vec{c}$.

**Theorem 4.1:** If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a'},\vec{b'}$ and $\vec{c'}$ form a reciprocal system of vectors, then $\vec{a}.\vec{a'}=\vec{b}.\vec{b'}=\vec{c}.\vec{c'}=1$

**Theorem 4.2:** If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a'},\vec{b'}$ and $\vec{c'}$ form a reciprocal system of vectors, then $\vec{a}.\vec{b'}=\vec{a}.\vec{c'}=\vec{b}.\vec{a'}=\vec{b}.\vec{c'}=\vec{c}.\vec{a'}=\vec{c}.\vec{b'}=0$

**Theorem 4.3:** If $\vec{a},\vec{b}$ and $\vec{c}$ and $\vec{a'},\vec{b'}$ and $\vec{c'}$ form a reciprocal system of vectors, then $[\vec{a'}\vec{b'}\vec{c'}]=\frac{1}{[\vec{a}\vec{b}\vec{c}]}$

**Proof:** $[\vec{a'}\vec{b'}\vec{c'}]=(\vec{a'}\times\vec{b'}).\vec{c'}=\left[\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}\times\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}\right].\vec{c'}$

$$=\left[\lambda(\vec{b}\times\vec{c})\times\lambda(\vec{c}\times\vec{a})\right].\vec{c'} \qquad \left(\text{Let } \lambda=\frac{1}{[\vec{a}\vec{b}\vec{c}]}\right)$$

$$=\lambda^2\left[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\right].\vec{c'}$$

$$=\lambda^2\left[\{(\vec{b}\times\vec{c}).\vec{a}\}\vec{c}-\{(\vec{b}\times\vec{c}).\vec{c}\}\vec{a}\right].\vec{c'}$$

$$=\lambda^2\left[[\vec{b}\vec{c}\vec{a}]\vec{c}-[\vec{b}\vec{c}\vec{c}]\vec{a}\right].\vec{c'}$$

$$=\lambda^2\left[[\vec{b}\vec{c}\vec{a}]\vec{c}\right].\vec{c'} \qquad (\because [\vec{b}\vec{c}\vec{c}]=0)$$

$$= \lambda^2 \left[ \left[ \vec{b}\vec{c}\vec{a} \right] \left( \vec{c}.\vec{c'} \right) \right]$$

$$= \lambda^2 \left[ \vec{b}\vec{c}\vec{a} \right] \qquad \left( \because \vec{c}.\vec{c'} = 1 \right)$$

**NOTES**

$$= \frac{1}{\left[ \vec{a}\vec{b}\vec{c} \right]^2} \left[ \vec{b}\vec{c}\vec{a} \right]$$

$$= \frac{1}{\left[ \vec{a}\vec{b}\vec{c} \right]}$$

**Theorem 4.4:** The orthonormal vector triads $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a self-reciprocal system, i.e.,

$$\hat{i'} = \hat{i}, \ \hat{j'} = \hat{j} \ \text{ and } \hat{k'} = \hat{k}$$

**Proof:** Let $\hat{i'}$, $\hat{j'}$ and $\hat{k'}$ be the system of vectors parallel to the system of $\hat{i}$, $\hat{j}$ and $\hat{k}$. Then, $\hat{i'} = \dfrac{\hat{j} \times \hat{k}}{\left[ \hat{i}\hat{j}\hat{k} \right]} = \hat{i}$.

**Theorem 4.5:** If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, i.e., $\left[ \vec{a}\vec{b}\vec{c} \right] \neq 0$ and $\vec{a'}, \vec{b'}$ and $\vec{c'}$ constitute the reciprocal system of vectors, then any vector $\vec{r}$ can be expressed as $\vec{r} = \left( \vec{r}.\vec{a'} \right) \vec{a} + \left( \vec{r}.\vec{b'} \right) \vec{b} + \left( \vec{r}.\vec{c'} \right) \vec{c}$

**Proof:** Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, $\vec{r}$ can be expressed as the linear combination in the form

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}, \text{ where } x, y \text{ and } z \text{ are scalars} \quad \ldots\text{(i)}$$

Multiplying both sides by $\vec{b} \times \vec{c}$,

$$\vec{r}.\left( \vec{b} \times \vec{c} \right) = x\vec{a}.\left( \vec{b} \times \vec{c} \right) + y\vec{b}.\left( \vec{b} \times \vec{c} \right) + z\vec{c}.\left( \vec{b} \times \vec{c} \right)$$

$$\Rightarrow \vec{r}.\left( \vec{b} \times \vec{c} \right) = x \left[ \vec{a}\vec{b}\vec{c} \right] \Rightarrow x = \frac{\vec{r}.\left( \vec{b} \times \vec{c} \right)}{\left[ \vec{a}\vec{b}\vec{c} \right]} = \vec{r}.\vec{a'}$$

Similarly, $y = \vec{r}.\vec{b'}$ and $z = \vec{r}.\vec{c'}$

Substituting the values of $x$, $y$ and $z$ in Equation (i), we get

$$\vec{r} = \left( \vec{r}.\vec{a'} \right) \vec{a} + \left( \vec{r}.\vec{b'} \right) \vec{b} + \left( \vec{r}.\vec{c'} \right) \vec{c}$$

**Example 4.32:** Given $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{c} = \hat{i} + 3\hat{j} - k$, find the reciprocal triads $\vec{a'}, \vec{b'}, \vec{c'}$ and verify that $\left[ \vec{a}, \vec{b}, \vec{c} \right] \left[ \vec{a'}, \vec{b'}, \vec{c'} \right] = 1$.

NOTES

**Solution :** $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}2 & -1 & 3\\ 2 & 1 & -1\\ 1 & 3 & -1\end{vmatrix}$ = 2(– 1 + 3) + 1(– 2 + 1) + 3(6 – 1)

= 4 – 1 + 15 = 18

Now, $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}=\dfrac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & 3 & -1\end{vmatrix}}{18}=\dfrac{1}{18}\left[2\hat{i}+\hat{j}+5\hat{k}\right]$

$\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}=\dfrac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 3 & -1\\ 2 & -1 & 3\end{vmatrix}}{18}=\dfrac{1}{18}\left[8\hat{i}-5\hat{j}-7\hat{k}\right]$

$\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}=\dfrac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 3\\ 2 & 1 & -1\end{vmatrix}}{18}=\dfrac{1}{18}\left[-2\hat{i}+8\hat{j}+4\hat{k}\right]$

Now, $\left[\vec{a}',\vec{b}',\vec{c}'\right]=\dfrac{1}{(18)^3}\begin{vmatrix}2 & 1 & 5\\ 8 & -5 & -7\\ -2 & 8 & 4\end{vmatrix}$

$=\dfrac{1}{(18)^3}\left[2(-20+56)-(32-14)+5(64-10)\right]$

$=\dfrac{1}{18\times18\times18}\left[72-18+270\right]$

$=\dfrac{324}{18\times18\times18}=\dfrac{1}{18}$

$\therefore\ \left[\vec{a}\,\vec{b}\,\vec{c}\right]\left[a'\,b'\,c'\right]=18\times\dfrac{1}{18}=1$

### 4.3.1 Vector Differentiation

Vectors differentiation refers to the differentiation of vector functions. In vector analysis we compute derivatives of vector functions of a real variable. Fundamentally, vector calculus or vector analysis, is concerned with differentiation and integration of vector fields, primarily in 3-dimensional Euclidean space $R^3$.

**Scalar Function and Vector Function**

If $t$ is a scalar variable, then a rule denoted by '$f$', which associates to each $t$ in an interval say $a \le t \le b$, a unique scalar $f(t)$, is called a **scalar function** of the scalar variable $t$. The scalar $f(t)$ means the value of $f$ at $t$.

NOTES

If $t$ is a scalar variable, then a rule denoted by '$\vec{f}$' which associates to each $t$ in an interval say $a \le t \le b$, a unique vector $\vec{f}(t)$, is called a **vector function** of the scalar variable $t$. The vector $\vec{f}(t)$ means the value of $\vec{f}$ at $t$.

We know that in rectangular co-ordinate system, every vector can be uniquely expressed as a linear combination of three fixed non-coplanar unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ along the axes OX, OY and OZ, respectively. Therefore, we can write $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$.

The scalar functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are called the components of the vector $\vec{f}(t)$ along the co-ordinate axes.

### Limit and Continuity of a Vector Function

The definitions of limit and continuity for vectors functions bear a strong resemblance to the corresponding definitions for scalar functions.

### Limit of a Vector Function

Let a vector function $\vec{f}(t)$ be defined for all values of $t$ in $a \le t \le b$ and $t_0$ be a point in this interval. Then, a vector $\vec{a}$ is called the **limit** of $\vec{f}(t)$ as $t \to t_0$ and is written as $\lim_{t \to t_0} \vec{f}(t) = \vec{a}$ if for any given number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that,

$$\left|\vec{f}(t) - \vec{a}\right| < \varepsilon, \text{ whenever } 0 < |t - t_0| < \delta.$$

In terms of components, if $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $\lim_{t \to t_0} f(t) = \vec{a}$ iff $\lim_{t \to t_0} f_i(t) = a_i$

Where $i = 1, 2, 3$

### Continuity of a Vector Function

A vector function $\vec{f}(t)$ is said to be **continuous** at a point $t_0$ in its interval of definition iff $\lim_{t \to t_0} \vec{f}(t) = \vec{f}(t_0)$

In other words, a vector function $\vec{f}(t)$ is said to be continuous at $t = t_0$ iff $\vec{f}(t_0)$ is defined and for any given number $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that,

$$\left|\vec{f}(t) - \vec{f}(t_0)\right| < \varepsilon, \text{ whenever } 0 < |t - t_0| < \delta.$$

In terms of components, if $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then $\vec{f}(t)$ is continuous iff $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous. A vector function is continuous in an interval iff its components are continuous in that interval.

NOTES

## Differentiation of Vectors

If a vector $\vec{r}$ varies continuously as a scalar variable *t* changes, then $\vec{r}$ is said to be a function of *t* and is written as $\vec{r} = \vec{f}(t)$.

Let $\delta t$ be a small increment in *t* and $\delta\vec{r}$ be the corresponding increment in $\vec{r}$. Then, $\vec{r} + \delta\vec{r} = \vec{f}(t+\delta t)$

$$\Rightarrow \delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t) \Rightarrow \frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

If $\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$ exists, then the value of this limit is called the **derivative** of $\vec{r}$ with respect to *t* and is denoted by $\frac{d\vec{r}}{dt}$ or $\frac{d\vec{f}}{dt}$ or $\vec{f}'(t)$ .

Thus, $\frac{d\vec{r}}{dt} = \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$

Since $\frac{d\vec{r}}{dt}$ is itself a vector function of *t*, its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$ and is called the second derivative of $\vec{r}$ with respect to *t*. Similarly, we can define the high order derivatives of $\vec{r}$.

In terms of components, if vector function $\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then we can write,

$$\Rightarrow \frac{d\vec{r}}{dt} = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}$$

Thus, a vector function $\vec{f}(t)$ is differentiable in an interval iff its all components are differentiable in that interval.

Like scalar function, a vector function which is differentiable is necessarily continuous but the converse is not true.

### Rules for Differentiating Vectors

The rules for differentiating vectors are similar to those for scalar functions. If $\vec{a}, \vec{b}$ and $\vec{c}$ are vector functions of a scalar *t*, and $\phi$ is a scalar function of *t*, then:

(i) $\frac{d}{dt}\left(\vec{a} \pm \vec{b}\right) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}\left(\vec{a}.\vec{b}\right) = \vec{a}.\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}.\vec{b}$

(iii) $\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$

NOTES

(iv) $\frac{d}{dt}\left(\phi\vec{a}\right)=\phi\frac{d\vec{a}}{dt}+\frac{d\phi}{dt}\vec{a}$

(v) $\frac{d}{dt}\left[\vec{a}\vec{b}\vec{c}\right]=\left[\frac{d\vec{a}}{dt}\vec{b}\vec{c}\right]+\left[\vec{a}\frac{d\vec{b}}{dt}\vec{c}\right]+\left[\vec{a}\vec{b}\frac{d\vec{c}}{dt}\right]$

(vi) $\frac{d}{dt}\left\{\vec{a}\times\left(\vec{b}\times\vec{c}\right)\right\}=\frac{d\vec{a}}{dt}\times\left(\vec{b}\times\vec{c}\right)+\vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right)+\vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right)$

**Chain Rule**

Let $\vec{r}$ be a differentiable vector function of a scalar variable $s$, and $s$ be a differentiable function of another scalar variable $t$. Then $\vec{r}$ is the differential vector function of $t$.

If $\delta t$ is a small increment in $t$, then $\delta s$ and $\delta\vec{r}$ are the corresponding increments in $s$ and $\vec{r}$, respectively. Then,

$$\frac{\delta\vec{r}}{\delta t}=\frac{\delta\vec{r}}{\delta s}.\frac{\delta s}{\delta t}$$

Taking limits as $\delta t\to 0$ and hence $\delta s\to 0$, we have $\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}=\lim_{\delta s\to 0}\frac{\delta\vec{r}}{\delta s}.\lim_{\delta t\to 0}\frac{\delta s}{\delta t}$

Thus, $\frac{d\vec{r}}{dt}=\frac{d\vec{r}}{ds}.\frac{ds}{dt}$

**Example 4.33:** Given $\vec{r}=\sin t\,\hat{i}+\cos t\,\hat{j}+t\hat{k}$, find:

(i) $\frac{d\vec{r}}{dt}$ (ii) $\frac{d^2\vec{r}}{dt^2}$ (iii) $\left|\frac{d\vec{r}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

**Solution:**

(i) $\frac{d\vec{r}}{dt}=\frac{d}{dt}(\sin t)\hat{i}+\frac{d}{dt}(\cos t)\hat{j}+\frac{d}{dt}(t)\hat{k}$

$=\cos t\hat{i}-\sin t\hat{j}+\hat{k}$

(ii) $\frac{d^2\vec{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\frac{d}{dt}(\cos t)\hat{i}-\frac{d}{dt}(\sin t)\hat{j}+\frac{d}{dt}(1)\hat{k}$

$=-\sin t\,\hat{i}-\cos t\,\hat{\text{j}}$

(iii) $\left|\frac{d\vec{r}}{dt}\right|=\sqrt{(\cos t)^2+(-\sin t)^2+(1)^2}=\sqrt{\cos^2 t+\sin^2 t+1^2}=\sqrt{1+1}=\sqrt{2}$

(iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|=\sqrt{(-\sin t)^2+(-\cos t)^2}=\sqrt{\sin^2 t+\cos^2 t}=\sqrt{1}=1$

NOTES

**Example 4.34:** If $\vec{a} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\hat{k}$, $\vec{b} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\,\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} - 3\hat{k}$, find $\frac{d}{d\theta}\left\{\vec{a}\times(\vec{b}\times\vec{c})\right\}$ at $\theta = \frac{\pi}{2}$.

**Solution:** $\vec{a} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}$, $\vec{b} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\,\hat{k}$, $\vec{c} = 2\,\hat{i} + 3\,\hat{j} - 3\,\hat{k}$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix} = (3\sin\theta + 9)\,\hat{i} - (-3\cos\theta + 6)\,\hat{j} + (3\cos\theta + 2\sin\theta)\,\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & 3\cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$$

$$= (3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)\hat{i}$$
$$- (3\cos\theta\sin\theta + 2\sin^2\theta - 3\theta\sin\theta - 9\theta)\hat{j}$$
$$+ (3\cos\theta\sin\theta - 6\sin\theta - 3\sin\theta\cos\theta - 9\cos\theta)\hat{k}$$
$$= (3\cos^2\theta + \sin 2\theta - 3\theta\cos\theta + 6\theta)\hat{i}$$
$$-\left(\frac{3}{2}\sin 2\theta + 2\sin^2\theta - 3\theta\sin\theta - 9\theta\right)\hat{j}$$
$$+ (-6\sin\theta - 9\cos\theta)\hat{k}$$

$$\frac{d}{d\theta}\left(\vec{a}\times(\vec{b}\times\vec{c})\right) = (-6\sin\theta\cos\theta + 2\cos 2\theta - 3\cos\theta + 3\theta\sin\theta + 6)\hat{\text{i}}$$
$$- (3\cos 2\theta + 4\sin\theta\cos\theta - 3\theta\cos\theta - 3\sin\theta - 9)\hat{\text{j}}$$
$$+ (-6\cos\theta + 9\sin\theta)\hat{\text{k}}$$

$$\frac{d}{d\theta}\left\{\vec{a}\times(\vec{b}\times\vec{c})\right\} \text{at } \theta = \frac{\pi}{2} = \left\{0 + 2(-1) + 3\frac{\pi}{2} + 6\right\}\hat{i} - (-3 - 3 - 9)\hat{j} + 9\hat{k}$$

$$= \left(\frac{3\pi}{2} + 4\right)\hat{i} + 15\hat{j} + 9\hat{k}$$

### Constant Vector and Its Derivative

A **constant vector** is a vector whose both magnitude and direction are fixed, i.e., do not change. Let $\vec{r}$ be a constant vector function of the scalar $t$ and $\vec{r} = \vec{f}(t)$.

Since $\delta\vec{r} = 0$, thus $\vec{r} = \vec{f}(t + \delta t)$ and $\vec{f}(t + \delta t) - \vec{f}(t) = \vec{0}$

Now, $\dfrac{d\vec{r}}{dt} = \lim\limits_{\delta t \to 0} \dfrac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t} = \lim\limits_{\delta t \to 0} \vec{0} = \vec{0}$

Thus, the derivative of a constant vector is the null vector or the zero vector.

**Theorem 4.6:** If $\vec{f}(t)$ has a constant magnitude, then $\vec{f}.\dfrac{d\vec{f}}{dt} = 0$.

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**Proof:** $\vec{f}(t)$ has a constant magnitude, i.e.,

$$\Rightarrow \left|\vec{f}(t)\right| = \text{Constant}$$

$$\text{For } \vec{f}(t).\vec{f}(t) = \left|\vec{f}(t)\right|^2 = \text{Constant},$$

$$\Rightarrow \frac{d}{dt}\left(\vec{f}.\vec{f}\right) = 0$$

$$\Rightarrow \vec{f}.\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}.\vec{f} = 0 \Rightarrow 2\vec{f}.\frac{d\vec{f}}{dt} = 0$$

$$\Rightarrow \vec{f}.\frac{d\vec{f}}{dt} = 0$$

**Note:** $\vec{f}.\frac{d\vec{f}}{dt} = 0$ implies that $\frac{d\vec{f}}{dt} \perp \vec{f}$ provided that $\frac{d\vec{f}}{dt} \neq 0$.

**Theorem 4.7:** If $\vec{f}(t)$ has a constant direction, then $\vec{f} \times \frac{d\vec{f}}{dt} = 0$.

**Proof:** Let $\hat{g}(t)$ be a unit vector in the direction of $\vec{f}(t)$ such that $\vec{f}(t) = f(t)\,\hat{g}(t)$ and $f(t) = \left|\vec{f}(t)\right|$.

$$\therefore \frac{d\vec{f}}{dt} = f\frac{d\hat{g}}{dt} + \frac{df}{dt}\hat{g} \qquad \text{...(4.4)}$$

Since $\vec{f}(t)$ has a constant direction, $\hat{g}(t)$ will also have the constant direction. Thus, $\hat{g}(t)$ is a constant vector and $\frac{d\hat{g}}{dt} = 0$. So, Equation (4.4) reduces to

$$\frac{d\vec{f}}{dt} = \frac{df}{dt}\hat{g}.$$

Now, $$\vec{f} \times \frac{d\vec{f}}{dt} = f\hat{g} \times \left(\frac{df}{dt}\hat{g}\right) = f\frac{df}{dt}\hat{g} \times \hat{g} = 0$$

**Theorem 4.8:** If $\vec{f}$ is a differentiable vector of scalar variable $t$ and $\left|\vec{f}\right| = f$, then

(i) $\frac{d}{dt}\left(\overline{f^2}\right) = 2f\frac{df}{dt}$ (ii) $\vec{f}.\frac{d\vec{f}}{dt} = f\frac{df}{dt}$

**Proof:** (i) Since $\left|\vec{f}\right| = f$, then

$$\overline{f^2} = \left|\vec{f}\right|^2 = f^2$$

Thus, $\frac{d}{dt}\left(\overline{f^2}\right)=\frac{d}{dt}\left(f^2\right)=2f\frac{df}{dt}$

NOTES

(ii) $\frac{d}{dt}\left(\overline{f^2}\right)=\frac{d}{dt}\left(\vec{f}.\vec{f}\right)=\vec{f}.\frac{d\vec{f}}{dt}+\frac{d\vec{f}}{dt}.\vec{f}=2\vec{f}.\frac{d\vec{f}}{dt}$ ...(4.5)

But $\frac{d}{dt}\left(\overline{f^2}\right)=2f\frac{df}{dt}$

Substituting the value of $\frac{d}{dt}\left(\overline{f^2}\right)$ in Equation (4.5),

$$2f\frac{df}{dt}=2\vec{f}.\frac{d\vec{f}}{dt} \Rightarrow \vec{f}.\frac{d\vec{f}}{dt}=f\frac{df}{dt}$$

**Theorem 4.9:** If $\vec{f}$ is a differentiable vector of scalar variable $t$, then

$$\frac{d}{dt}\left(\vec{f}\times\frac{d\vec{f}}{dt}\right)=\vec{f}\times\frac{d^2\vec{f}}{dt^2}$$

**Proof:** $\frac{d}{dt}\left(\vec{f}\times\frac{d\vec{f}}{dt}\right)=\vec{f}\frac{d^2\vec{f}}{dt^2}+\frac{d\vec{f}}{dt}\times\frac{d\vec{f}}{dt}=\vec{f}\frac{d^2\vec{f}}{dt^2} \quad \left(\because \frac{d\vec{f}}{dt}\times\frac{d\vec{f}}{dt}=0\right)$

**Example 4.35:** Show that if $\vec{r}=\vec{a}\sin\omega t+\vec{b}\cos\omega t$, where $\vec{a},\vec{b},\omega$ are constants, then $\frac{d^2\vec{r}}{dt^2}=-\omega^2\vec{r}$ and $\vec{r}\times\frac{d\vec{r}}{dt}=-\omega\vec{a}\times\vec{b}$.

**Solution:** We know that if $\vec{r}=\phi\vec{f}$, where $\phi$ is a scalar function of $t$, then

$$\frac{d\vec{r}}{dt}=\phi\frac{d\vec{f}}{dt}+\frac{d\phi}{dt}\vec{f}.$$

But if $\vec{f}$ is a constant vector, then $\frac{d\vec{f}}{dt}=\vec{0}$

$$\therefore \frac{d\vec{r}}{dt}=\frac{d\phi}{dt}\vec{f}$$

Now $\vec{r}=\vec{a}\sin\omega t+\vec{b}\cos\omega t$

$$\Rightarrow \frac{d\vec{r}}{dt}=\vec{a}\,\omega\cos\omega t-\vec{b}\,\omega\sin\omega t$$

$$\therefore \frac{d^2\vec{r}}{dt^2}=-\vec{a}\,\omega^2\sin\omega t-\vec{b}\,\omega^2\cos\omega t$$

$$=-\omega^2\left(\vec{a}\sin\omega t+\vec{b}\cos\omega t\right)=-\omega^2\vec{r}$$

Also $\vec{r}\times\frac{d\vec{r}}{dt}=(\vec{a}\sin\omega t+\vec{b}\cos\omega t)\times(\vec{a}\,\omega\cos\omega t-\vec{b}\,\omega\sin\omega t)$

$$= \omega(-\vec{a}\times\vec{b}\sin^2\omega t + \vec{b}\times\vec{a}\cos^2\omega t) \qquad \left[\because \quad \vec{a}\times\vec{a} = \vec{b}\times\vec{b} = \vec{0}\right]$$

$$= \omega(-\vec{a}\times\vec{b}\sin^2\omega t - \vec{a}\times\vec{b}\cos^2\omega t) = -\omega\vec{a}\times\vec{b}$$

NOTES

**Example 4.36:** If $\vec{r}$ is a vector function of scalar $t$ and $\vec{a}$, $\vec{b}$ are constant vectors, $|\vec{r}| = r$; differentiate the following with respect to $t$.

(i) $\vec{r}.\vec{a}$ (ii) $\vec{r}\times\vec{a}$ (iii) $\vec{r}.\dfrac{d\vec{r}}{dt}$

(iv) $\vec{r}\times\dfrac{d\vec{r}}{dt}$ (v) $r^3\vec{r}+\vec{a}\times\dfrac{d\vec{r}}{dt}$

**Solution:** (i) $\dfrac{d}{dt}\left(\vec{r}.\vec{a}\right) = \dfrac{d\vec{r}}{dt}.\vec{a} + \vec{r}.\dfrac{d\vec{a}}{dt}$

$$= \frac{d\vec{r}}{dt}.\vec{a} \qquad \left[\vec{a}\text{ is a constant vector. Thus, } \frac{d\vec{a}}{dt} = 0\right]$$

(ii) $\dfrac{d}{dt}\left(\vec{r}\times\vec{a}\right) = \dfrac{d\vec{r}}{dt}\times\vec{a} + \vec{r}\times\dfrac{d\vec{a}}{dt}$

$$= \frac{d\vec{r}}{dt}\times\vec{a} \qquad \left[\because \frac{d\vec{a}}{dt} = 0\right]$$

(iii) $\dfrac{d}{dt}\left(\vec{r}.\dfrac{d\vec{r}}{dt}\right) = \dfrac{d\vec{r}}{dt}.\dfrac{d\vec{r}}{dt} + \vec{r}.\dfrac{d^2\vec{r}}{dt^2}$

$$= \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}.\frac{d^2\vec{r}}{dt^2}$$

(iv) $\dfrac{d}{dt}\left(\vec{r}\times\dfrac{d\vec{r}}{dt}\right) = \dfrac{d\vec{r}}{dt}\times\dfrac{d\vec{r}}{dt} + \vec{r}\times\dfrac{d^2\vec{r}}{dt^2}$

$$= \vec{r}\times\frac{d^2\vec{r}}{dt^2} \qquad \left[\because \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} = 0\right]$$

(v) $\dfrac{d}{dt}\left(r^3\vec{r} + \vec{a}\times\dfrac{d\vec{r}}{dt}\right) = \dfrac{d}{dt}(r^3\vec{r}) + \dfrac{d}{dt}\left(\vec{a}\times\dfrac{d\vec{r}}{dt}\right)$

$$= \left(r^3\frac{d\vec{r}}{dt} + 3r^2\frac{dr}{dt}\vec{r}\right) + \frac{d\vec{a}}{dt}\times\frac{d\vec{r}}{dt} + \vec{a}\times\frac{d^2\vec{r}}{dt^2}$$

$$\left[\because \frac{d}{dt}\phi\vec{f} = \phi\frac{d\vec{f}}{dt} + \frac{d\phi}{dt}\vec{f}\right]$$

$$= r^3\frac{d\vec{r}}{dt} + 3r^2\frac{dr}{dt}\vec{r} + \vec{a}\times\frac{d^2\vec{r}}{dt^2}$$

NOTES

**Partial Derivatives of Vector Functions**

When a vector $\vec{r}$ depends on scalar variables, say $x$, $y$ and $z$, we write $\vec{r} = \vec{f}(x, y, z)$.

Then $\lim_{\delta x \to 0} \frac{\vec{f}(x+\delta x, y, z) - \vec{f}(x, y, z)}{\delta x}$ is called the **partial derivative** of $\vec{r}$ with respect to $x$ and is denoted by $\frac{\partial \vec{r}}{\partial x}$.

Similarly, we define the partial derivatives

$$\frac{\partial \vec{r}}{\partial y} = \lim_{\delta y \to 0} \frac{\vec{f}(x, y+\delta y, z) - \vec{f}(x, y, z)}{\delta y}$$

$$\frac{\partial \vec{r}}{\partial z} = \lim_{\delta z \to 0} \frac{\vec{f}(x, y, z+\delta z) - \vec{f}(x, y, z)}{\delta z}$$

Higher order derivatives can also be defined in the similar way. For example,

$$\frac{\partial^2 \vec{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^2 \vec{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{r}}{\partial z}\right),$$

$$\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^3 \vec{r}}{\partial x \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \vec{r}}{\partial z^2}\right)$$

**Note:** If $\vec{r}$ has continuous partial derivatives of second order, then $\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial^2 \vec{r}}{\partial y \partial x}$, i.e., the order of differentiation does not matter.

**Rules for Partial Differentiation of Vectors**

If $\vec{a}$ and $\vec{b}$ are vector functions of scalars $x, y, z$ and $\phi$ is a scalar function of $x, y, z$, then some important rules of partial differentiation of vectors are stated as follows:

(i) $\frac{\partial}{\partial x}\left(\vec{a} + \vec{b}\right) = \frac{\partial \vec{a}}{\partial x} + \frac{\partial \vec{b}}{\partial x}$

(ii) $\frac{\partial}{\partial x}\left(\vec{a}.\vec{b}\right) = \vec{a}.\frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x}.\vec{b}$

(iii) $\frac{\partial}{\partial x}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x} \times \vec{b}$

(iv) $\frac{\partial}{\partial x}\left(\phi \vec{a}\right) = \phi \frac{\partial \vec{a}}{\partial x} + \frac{\partial \phi}{\partial x}\vec{a}$

(v) $\frac{\partial^2}{\partial y \partial x}\left(\vec{a}.\vec{b}\right) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left(\vec{a}.\vec{b}\right)\right\} = \frac{\partial}{\partial y}\left\{\vec{a}.\frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x}.\vec{b}\right\}$

$$= \vec{a}.\frac{\partial^2 \vec{b}}{\partial y \partial x} + \frac{\partial \vec{a}}{\partial y}.\frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x}.\frac{\partial \vec{b}}{\partial y} + \frac{\partial^2 \vec{a}}{\partial y \partial x}.\vec{b}$$

NOTES

(vi) $\frac{\partial^2}{\partial x \partial y}\left(\vec{a}\times\vec{b}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(\vec{a}\times\vec{b}\right)\right\} = \frac{\partial}{\partial x}\left\{\vec{a}\times\frac{\partial \vec{b}}{\partial y} + \frac{\partial \vec{a}}{\partial y}\times\vec{b}\right\}$

$$= \vec{a}\times\frac{\partial^2 \vec{b}}{\partial x \partial y} + \frac{\partial \vec{a}}{\partial x}\times\frac{\partial \vec{b}}{\partial y} + \frac{\partial \vec{a}}{\partial y}\times\frac{\partial \vec{b}}{\partial x} + \frac{\partial^2 \vec{a}}{\partial x \partial y}\times\vec{b}$$

(vii) If $\vec{r}(x, y, z) = f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k}$, then

$$\frac{\partial \vec{r}}{\partial x} = \frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_2}{\partial x}\hat{j} + \frac{\partial f_3}{\partial x}\hat{k}$$

**Total Differential**

Differential of vectors follow the similar rules to those of elementary calculus.

(i) If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, then $d\vec{f} = df_1\hat{i} + df_2\hat{j} + df_3\hat{k}$

(ii) $d\left(\vec{a}.\vec{b}\right) = \vec{a}.d\vec{b} + d\vec{a}.\vec{b}$

(iii) $d\left(\vec{a}\times\vec{b}\right) = \vec{a}\times d\vec{b} + d\vec{a}\times\vec{b}$

(iv) If $\vec{r} = \vec{f}(x, y, z)$, then $d\vec{r} = \frac{\partial \vec{f}}{\partial x}dx + \frac{\partial \vec{f}}{\partial y}dy + \frac{\partial \vec{f}}{\partial z}dz$

**Example 4.37:** If $\vec{a} = (2x^2 y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2 \cos y)\hat{k}$, find:

$$\frac{\partial \vec{a}}{\partial x}, \frac{\partial \vec{a}}{\partial y}, \frac{\partial^2 \vec{a}}{\partial x^2}, \frac{\partial^2 \vec{a}}{\partial y^2}, \frac{\partial^2 \vec{a}}{\partial x \partial y}, \frac{\partial^2 \vec{a}}{\partial y \partial x}$$

**Solution:** $\frac{\partial \vec{a}}{\partial x} = \frac{\partial}{\partial x}(2x^2 y - x^4)\hat{i} + \frac{\partial}{\partial x}(e^{xy} - y\sin x)\hat{j} + \frac{\partial}{\partial x}(x^2 \cos y)\hat{k}$

$$= (4xy - 4x^3)\hat{i} + (ye^{xy} - y\cos x)\hat{j} + (2x\cos y)\hat{k}$$

$$\frac{\partial \vec{a}}{\partial y} = \frac{\partial}{\partial y}(2x^2 y - x^4)\hat{i} + \frac{\partial}{\partial y}(e^{xy} - y\sin x)\hat{j} + \frac{\partial}{\partial y}(x^2 \cos y)\hat{k}$$

$$= 2x^2\hat{i} + (xe^{xy} - \sin x)\hat{j} - x^2 \sin y\hat{k}$$

$$\frac{\partial^2 \vec{a}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{a}}{\partial x}\right) = \frac{\partial}{\partial x}(4xy - 4x^3)\hat{i} + \frac{\partial}{\partial x}(ye^{xy} - y\cos x)\hat{j} + \frac{\partial}{\partial x}(2x\cos y)\hat{k}$$

$$= (4y - 12x^2)\hat{i} + (y^2 e^{xy} + y\sin x)\hat{j} + 2\cos y\hat{k}$$

$$\frac{\partial^2 \vec{a}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{a}}{\partial y}\right) = \frac{\partial}{\partial y}(2x^2)\hat{i} + \frac{\partial}{\partial y}(xe^{xy} - \sin x)\hat{j} - \frac{\partial}{\partial y}(x^2 \sin y)\hat{k}$$

$$= 0 + x^2 e^{xy}\hat{j} - x^2 \cos y\hat{k}$$

$$= x^2 e^{xy}\hat{j} - x^2 \cos y\hat{k}$$

NOTES

$$\frac{\partial^2 \vec{a}}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right) = \frac{\partial}{\partial x}(2x^2)\hat{i} + \frac{\partial}{\partial x}(xe^{xy} - \sin x)\hat{j} - \frac{\partial}{\partial x}(x^2 \sin y)\hat{k}$$

$$= 4x\hat{i} + (xye^{xy} + e^{xy} - \cos x)\hat{j} - 2x\sin y\hat{k}$$

$$\frac{\partial^2 \vec{a}}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{a}}{\partial y}\right) = \frac{\partial}{\partial x}(2x^2)\hat{i} + \frac{\partial}{\partial x}(xe^{xy} - \sin x)\hat{j} - \frac{\partial}{\partial x}(x^2 \sin y)\hat{k}$$

$$= 4x\hat{i} + (xye^{xy} + e^{xy} - \cos x)\hat{j} - 2x\sin y\hat{k}$$

**Example 4.38:** If $\phi(x, y, z) = xy^2 z$ and $\vec{a} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}\left(\phi\vec{a}\right)$ at the point (2, –1, 1).

**Solution:** $\phi\vec{a} = \left(xy^2 z\right)\left(xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}\right) = \left(x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k}\right)$

$$\frac{\partial}{\partial z}\left(\phi\vec{a}\right) = \frac{\partial}{\partial z}\left(x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k}\right) = 2x^2y^2z\hat{i} - x^2y^4\hat{j} + 3xy^3z^2\hat{k}$$

$$\frac{\partial^2}{\partial x\,\partial z}\left(\phi\vec{a}\right) = \frac{\partial}{\partial x}\left(2x^2y^2z\hat{i} - x^2y^4\hat{j} + 3xy^3z^2\hat{k}\right) = 4xy^2z\hat{i} - 2xy^4\hat{j} + 3y^3z^2\hat{k}$$

$$\frac{\partial^3}{\partial x^2\partial z}\left(\phi\vec{a}\right) = \frac{\partial}{\partial x}\left(4xy^2z\hat{i} - 2xy^4\hat{j} + 3y^3z^2\hat{k}\right) = 4y^2z\hat{i} - 2y^4\hat{j}$$

If $x = 2, y = -1, z = 1$ this becomes $4(-1)^2\,(1)\,\hat{i} - 2(-1)^4\hat{j} = 4\hat{i} - 2\hat{j}$.

## Curves in Space

To obtain a vector equation for the curve, we consider the position vector $\vec{r}(t)$ of each point on the curve corresponding to the parameter *t*.

Since the components of $\vec{r}(t)$ are precisely the coordinates of the point, it follows that $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$.

This equation is called the **vector equation of a space curve**. This equation is a vector function of a scalar variable *t* such that by assigning different values of *t*, we can obtain position vectors of the points on the curve.

## Tangent

Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the position vector of a point P. Then as *t* varies continuously, P traces out a curve C. Let the neighbouring point Q on this curve corresponds to $t + \delta t$, as shown in Figure 4.3.

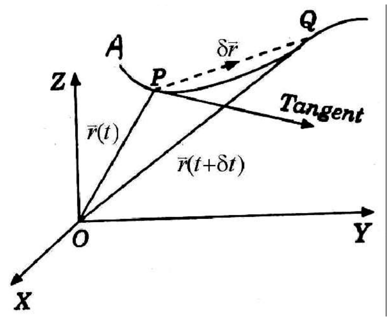



*Fig. 4.3 Tangent to the Curve*

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\Rightarrow \delta\vec{r} = \vec{r}(t+\delta t) - \vec{r}(t)$$

So, $\dfrac{\delta\vec{r}}{\delta t}$ is directed along the chord PQ.

As $\delta t \to 0$, Q $\to$ P, Chord PQ $\to$ Tangent to the Curve at P,

$\therefore \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{r}}{\delta t} = \dfrac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at P.

If the scalar parameter $t$ is replaced by $s$, where $s$ denotes the arc length from any convenient point A on the curve upto P, then,

Arc AP = $s$, Arc AQ = $s + \delta s$ such that $\delta s$ = Arc PQ.

Here, $\dfrac{d\vec{r}}{ds}$ will be a vector along the tangent at P.

Also, $\left|\dfrac{d\vec{r}}{ds}\right| = \lim\limits_{\delta s \to 0}\left|\dfrac{\delta\vec{r}}{\delta s}\right| = \lim\limits_{Q \to P} \dfrac{\text{chord PQ}}{\text{arc PQ}} = 1$

Thus, $\dfrac{d\vec{r}}{ds}$ is the unit vector $\hat{t}$ along the tangent at P, i.e.,

$$\hat{t} = \frac{d\vec{r}}{ds}$$

**Example 4.39:** Find the unit tangent vector at any point on the curve $x = t^2 + 2, y = 4t - 5,\ z = 2t^2 - 6t$, where $t$ is any variable. Also determine the unit tangent vector at the point $t = 2$.

**Solution:** If $\vec{r}$ is the position vector of any point $(x, y, z)$ on the given curve, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (t^2 + 2)\hat{i} + (4t - 5)\hat{j} + (2t^2 - 6t)\hat{k}$$

The vector $\dfrac{d\vec{r}}{dt}$ is along the tangent at the point $(x, y, z)$ to the given curve.

NOTES

Now $\frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$

And $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2} = \sqrt{20t^2 - 48t + 52} = 2\sqrt{5t^2 - 12t + 13}$

$\therefore$ The unit tangent vector $\hat{t} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{t\hat{i} + 2\hat{j} + (2t-3)\hat{k}}{\sqrt{5t^2 - 12t + 13}}$

Also the unit tangent vector at the point $t = 2$ is

$$\frac{2\hat{i} + 2\hat{j} + (2\times 2 - 3)\hat{k}}{\sqrt{5\times 4 - 12\times 2 + 13}} = \frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$

**Check Your Progress**

1. Define matrices.
2. What do you understand by scalar matrix?
3. What is elementary matrix?
4. What is scalar product?
5. Define magnitude vector.
6. What is the constant vector?

## 4.4 EXPONENTIAL FUNCTION

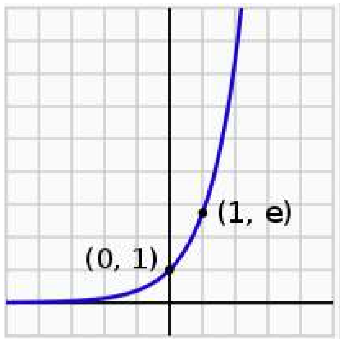
### Exponential

This function is of prime importance in mathematics and finds its wide application in calculus and many branches of science and engineering. An exponential function of $x$ is written as exp($x$) or $e^x$. Here $e$ is a constant and an irrational number. It has been estimated as 2.718281828 by Euler and bears his name. It is called 'Euler's number' and is also the base of natural logarithm. An exponential function is the inverse of a logarithmic function and is sometimes, called **anti logarithm**. Inverse of an exponential function is a logarithmic function.

The exponential function rises slowly and is almost flat for $x < 0$, but increases rapidly for values $x > 0$ and its value is 1 for $x = 0$. Its ordinate value is the slope of its curve at that point. That is why an exponential function with negative value of $x$ is known as exponential decay and those with positive value it is called exponential growth. Also, when growth is very fast we call it exponential growth, example, population growth.

The exponential function is almost flat, rising slowly, for negative values of $x$, and increases fast for positive values of $x$, and equals 1 when $x$ is equal to 0. Its $y$ value always equals the slope at that point.

NOTES




The graph of an exponential function always lies above the abscissa, since $e^x$ is always positive. It is increasing on the positive side of X-axis. In the negative side of X-axis it is decreasing but never touches the X- axis.

The exponential function $e^x$ may be expanded into an infinite series, called power series given below:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

This function can be defined as a limit which is given below:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \text{ or } e^x = \lim_{n\to\infty}(1+nx)^{\frac{1}{n}} \cdot$$

Exponential functions in mathematics, engineering and various science streams are predominantly because of the characteristic an exponential function with respect to its derivative, which is:

$$\frac{d}{dx}e^x = e^x$$

- The slope of the graph of $e^x$ at any point, $x = e^x$.
- The rate of increase of the function with respect to $x$, at a point $= e^x$.
- Since $y' = y$, this function is a solution of the differential equation $y' – y = 0$.

In higher mathematical applications there are great numbers of differential equations whose solution are exponential functions. Laplace's equation and equation of simple harmonic motion are examples. Equations for simple harmonic motion also give exponential functions.

There are exponential functions with other bases, like one given below for a function $y = a^x$:

$$\frac{d}{dx}a^x = (\text{In } a)a^x.$$

**Proof.**

$$y = a^x$$

$$1\text{n}y = 1\text{n}a^x$$

$$1\text{n}y = x\ 1\text{n}a$$

$$\frac{1}{y}\frac{dy}{dx} = 1\text{n}\ a$$

$$\frac{dy}{dx} = (1\text{n}\ a)y = (1\text{n}\ a)\ a^x$$

NOTES

This shows that 'Derivative' of an exponential function is a constant multiple of its own. If rate of change of a variable is proportional to the variable itself, the solution results in an exponential function. Population growth, radioactive decay, continuously compounded interest, etc., are examples of exponential function in practical life. In all these cases the variable is proportional to exponential function of time. For a differentiable function $f(x)$, as per chain rule:

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

### Exponential Function on the Complex Plane

As in case of real numbers, the exponential function can be defined in for complex quantities too. Some of these definitions are identical to those given for real valued exponential functions. The definition of power series can be used and for this real value replaced by a complex one, as given below:

$$e^z = \sum_{n=0}^{\infty}\frac{n^n}{n!}$$

The derivative, like that of real quantities also holds for complex quantities and this can be stated as below:

$$\frac{d}{dz}e^z = e^z \text{ holds in the complex plane.}$$

We can now extends the concept for real exponential function to complex one as below by writing as $e^{x+iy} = e^x e^{iy}$. The real part is $e^x$ and $e^{iy} = \cos(y) + i\sin(y)$. Thus we use the real definition without ignoring it.

We can now write,

$e^{a+bi} = e^a(\cos b + i \sin b)$

Here $a$ and $b$ are real values.

**Example 4.40:** Looking at the functions below, find the function(s) which is/are not exponential.

(*i*) $f(x) = 3e^{-2x}$

(*ii*) $g(x) = 2^{x/2}$

(*iii*) $h(x) = x^{3/2}$

(*iv*) $g(x) = 15/7^x$

(*v*) $p(x) = x^e$

**Solution:** Here, $h(x)$ and $p(x)$ are not exponential functions. For the function to be exponential, the independent variable should be the exponent.

**Example 4.41:** Find the domain and range of function defined as $f(x) = kb^x$. Discuss the nature of graph of this function. How $f(x)$ changes when (*i*) $x$ tends to infinity and (*ii*) $x$ tends to negative infinity? Are there any horizontal asymptotes? Tell about its horizontal asymptote.

**Solution:** Domain of this function is the set of real numbers, but the range is the set of all positive real numbers.

When $b > 1$, the function $f(x)$ is increasing; the graph rises in the right proton. ($i$) When $x$ tends to infinity $f(x)$ increases. ($ii$) When $x$ decreases tending to negative side of infinity the function, $f(x)$ goes on decreasing and tends to zero. The line given by $y = 0$, which is the $x$-axis, is the horizontal asymptote.

For $b < 1$, the condition is opposite to it. It decreases with increasing value of $x$ and decreases with the increasing value of $x$. It goes from high in the left to low in the right portion of the graph.

**Example 4.42:** The Bacteria grow exponentially in a culture. It was observed that number of bacteria at 2:00 p.m. was 80 and at 6:00 p.m. it was 500. The growth is given by a function $f(t) = k.e^{at}$. Find the population of bacteria at 10:00 p.m.

**Solution:** The growth is given by $f(t) = 80e^{0.4581\,t}$ at any time $t$. Number of bacteria at 10:00 p.m. will be 3125.

**Example 4.43:** A European country conducted the nuclear test on an island in the Pacific Ocean in 1990. Just after the explosion, the level of Strontium-90 on the island was noted as 100 times the 'safe level' for human habitation. Taking half-life of Strontium-90 as 28 years, find the number of years after which the island will once again be habitable.

**Solution:** The Island will be habitable after 186 years approximately which is the year 2176.

**Utility**

If $U(x, y)$ denotes the satisfaction obtained by an individual when he buys quantities $x$ and $y$ of two commodities $X$ and $Y$, then $U(x, y)$, the function of two variables $x$ and $y$ is called the utility function or utility index of the individual.

e.g., $U = (x + 3)\ (y + 1)$

$U = (x - 1)^{0.5}\ (y - 2)^{0.5}$

**Notes**

1. Still there are other functions such as Marginal Revenue Function and Marginal cost function, which are based on the (complete) derivatives or partial derivatives. They are dealt with in the respective chapters of differential/integral calculus.
2. Break-Even Analysis entails finding out the minimum quantum of production (and sales) that a firm has to achieve in its attempt to recover its investment (total fixed cost) whereafter profits start accruing.

   At Break-even point, profit = Loss = 0

   or Total Revenue = Total Cost

   i.e., $R(x) = C(x)$

   Or, $p.x. = (\text{TFC} + \text{AVC}.x)$

   $\Rightarrow\ x\,(P - \text{AVC}) = \text{TFC}$, where $p = P$ = unit Price or $x_B = \frac{\text{TFC}}{(P - \text{AVC})}$ units (Break-even output) ($Q_B$)

NOTES

Break-even Sales (Revenue)

$$s_B = p.x_B = p.Q_B = \frac{P(\text{TFC})}{(P - AVC)}$$

Or $s_B = \frac{(\text{TFC})}{\left(1 - \frac{\text{AVC}}{\text{P}}\right)}$ or $\frac{\text{TFC}}{\left(1 - \frac{\text{TFC}}{\text{TR}}\right)}$ (Break-even Sales)

## 4.5 PERIODIC FUNCTION

A function $f$ is said to be periodic if $f(x+ c,c) = f(x)$ for all values of $x$. The constant $c$ is called the period, and is required to be positive. A function with period $c$ will repeat on intervals of length $c$, and these intervals are sometimes also referred to as periods.

If c is the period of $f(x)$, then

$$f(x) = f(x + c) = f(x + 2c) = \ldots. = f(x + nc) = \ldots\ldots\ldots$$

$$f(x) = f(x - c) = f(x - 2c) = \ldots. = f(x - nc) = \ldots\ldots\ldots$$

$f(x) = f(x \pm c)$, where $n$ is a positive integer.

Let a periodic function $f(x)$ be defined in $(c, c + 2\pi)$ and if it can be expanded as the infinite trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nxdx \text{ and}$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nxdx$$

**Proof**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (4.6)$$

Integrating both sides of Equation (4.6) with respect to $x$ in the interval $(c, c + 2\pi)$, you get

$$\int_c^{c+2\pi} f(x)dx = \frac{a_0}{2} \int_c^{c+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nxdx + \sum_{n=1}^{\infty} b_n \sin nxdx$$

$$= \frac{a_0}{2} [x]_c^{c+2\pi} + \sum_{n=1}^{\infty} a_n \times 0 + \sum_{n=1}^{\infty} b_n \times 0$$

NOTES

Since $\int_c^{c+2\pi} \cos nx dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0$ and

$\int_0^{c+2\pi} \sin nx dx = \frac{-1}{n}[\cos nc - \cos nc] = 0$

$$\int_c^{c+2\pi} f(x) dx = \frac{a_0}{2}[c + 2\pi - c]$$

$$\therefore \qquad a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x) dx \qquad (4.7)$$

Multiplying Equation (4.7) by cos $mx$, where $m$ is a fixed integer and integrating from $c$ to $c + 2\pi$, you get

$$\int_c^{c+2\pi} f(x)\cos mx dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos mx \cos nx dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \cos mx \sin nx dx = a_m \int_c^{c+2\pi} \cos^2 mx dx$$

$$\int_c^{c+2\pi} f(x)\cos mx dx = a_m \,.\, \pi$$

Since, $\int_c^{c+2\pi} \cos^2 nx dx = \frac{1}{2}\int_c^{c+2\pi} (1 + \cos 2nx) dx = \pi$

$$\therefore\ a_m = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos mx dx$$

Changing $m$ to $n$ $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\, dx$

Multiplying Equation (4.6) again by sin $mx$, where $m$ is any fixed positive integer, and integrating from $c$ to $c + 2\pi$,

$$\int_c^{c+2\pi} f(x)\sin mx\, dx = \int_c^{c+2\pi} \frac{a_0}{2}\sin mx\, dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \sin mx \cos nx\, ds$$

$$+ \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin mx \sin nx\, dx$$

$$\int_c^{c+2\pi} f(x)\sin mx\, dx = b_m \int_c^{c+2\pi} \sin^2 mx\, dx$$

$$\therefore \qquad \int_c^{c+2\pi} f(x)\sin mx\, dx = b_m \,.\, \pi,$$

Since, $\int_c^{c+2\pi} \sin^2 nx dx = \frac{1}{2}\int_c^{c+2\pi} (1 - \cos 2nx) dx = \pi$

$$\therefore \qquad b_m = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin mx\, dx$$

Changing $m$ to $n$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\, dx \quad n \geq 1$$

The values of the coefficients $a_0$, $a_n$ and $b_n$ are called Euler's formula.

***Note:*** In the $a_0$, $a_n$ and $b_n$ integrations, we assume that term by term integration of the series is allowed.

### 4.5.1 Conditions for a Fourier Expansion

NOTES

**Drichlet's Conditions**

Let the function $f(x)$ be defined in the interval $(c, c+2l)$. This function can be expanded as an infinite trigonometric series of the form,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x),$$

if the following conditions called Drichlet's conditions are satisfied:

1. $f(x)$ is single valued, periodic with period $2l$ and finite in $(c, c+2l)$.
2. $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c+2l)$.
3. $f(x)$ can have finite number of maxima and minima in the given range.

The following are the values of certain definite integrals which you require in deriving Fourier series.

If $m$ and $n$ are positve integers or zeros,

1. $$\int_c^{c+2\pi} \cos nx dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \qquad \ldots(4.8)$$

2. $$\int_c^{c+2\pi} \sin nx dx = \left[\frac{-\cos nx}{n}\right]_c^{c+2\pi}$$
$$= -\frac{1}{n}\left[\cos(2n\pi+nc)-\cos nc\right]$$
$$= -\frac{1}{n}\left[\cos nc-\cos nc\right] = 0 \qquad \ldots(4.9)$$

3. $$\int_c^{c+2\pi} \cos mx \cos nx dx = \frac{1}{2}\left[\int_c^{c+2\pi}(\cos(m+n)x+\cos(m-n)x)dx\right] = 0$$
by Equations (4.8) ...(4.10)

4. $$\int_c^{c+2\pi} \sin mx \cos nx dx = \frac{1}{2}\left[\int_c^{c+2\pi}(\sin(m+n)x+\sin(m-n)x)dx\right] = 0$$
by Equations (4.9) ...(4.11)

5. $$\int_c^{c+2\pi} \sin mx \sin nx dx = \frac{1}{2}\left[\int_c^{c+2\pi}(\cos(m-n)x-\cos(m+n)x)dx\right] = 0$$
by Equations (4.8) ...(4.12)

   Results of Equations (4.10), (4.11) and (4.12) are for $m \neq n$.

6. If $m = n$ in result Equation (4.10) and $n \neq 0$, then you have

$$\int_c^{c+2\pi} \cos^2 nx dx = \frac{1}{2}\int_c^{c+2\pi}(1+\cos 2nx)dx = \pi \qquad \ldots(4.13)$$

7. If $m = n$ in result Equation (4.12) and $n \neq 0$, then you have

$$\int_c^{c+2\pi} \sin^2 nx dx = \frac{1}{2}\int_c^{c+2\pi} (1 - \cos 2nx)dx = \pi \qquad \ldots(4.14)$$

## Change of Interval

You have seen the Fourier expansion of $f(x)$ defined in the interval $(c, c + 2\pi)$. In practice, we often require to find a Fourier series expansion of a function $f(x)$ which is not of length $2\pi$ but some other interval, say $2l$.

You know that Fourier series expansion of $f(x)$ in $(c, c + 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

Where $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\, dx$, for $n = 0, 1, 2, 3, \ldots$ and

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\, dx, \text{ for } n = 1, 2, 3,\ldots$$

To expand $f(x)$ as Fourier series in the interval $(-l, l)$, let us define a new variable $y = \frac{\pi x}{l}$ or $x = \frac{ly}{\pi}$ and limits,

When $x = -l$, $y = -\pi$

And when, $x = l$, $y = \pi$

Hence, the function $f\left(\frac{yl}{\pi}\right)$ is defined in the interval $(-\pi, \pi)$ and corresponding Fourier series is,

$$f\left(\frac{yl}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny) \qquad (4.15)$$

Where $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right)\cos ny\, dy$$

And $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right)\sin ny\, dy$$

To find the Fourier series of $f(x)$ in $(-l, l)$

you revert to the variable $x$

i.e., $y = \frac{\pi x}{l} \quad \therefore\ x = \frac{yl}{\pi}$ and $dy = \frac{\pi}{l}dx$

then Equation (4.15) becomes,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

NOTES

With

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx \text{ for } n = 0, 1, 2, ...., \text{ and}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx$$

**Example 4.44:** Find the Fourier series of period $2l$ for the function $f(x) = x(2l - x)$ in $(0, 2l)$

Deduce the sum of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ....$

**Solution:**

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{(i)}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\cos\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi}{l}x}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[2(l - x)\cos\frac{n\pi}{l}x\right]_0^{2l}$$

$$= \frac{2l}{n^2\pi^2}\left[-2l\cos\ 2n\pi - 2l\right] = -\frac{4l^2}{n^2\pi^2}$$

$a_0$ cannot be deduced from $a_n$. So

$$a_0 = \frac{1}{l}\int_0^{2l} x(2l - x)dx = \frac{1}{l}\left[\frac{2l\,x^2}{2} - \frac{x^3}{3}\right]_0^{2l} = \frac{4}{3}l^2$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\int_0^{2l} x(2l - x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$$= 0 - 2\left(\frac{l^3}{n^3\pi^3}\right)(1 - 1) = 0$$

NOTES

Using the values of $a_0$, $a_n$, $b_n$ in Equation (i), we get

$$f(x) = \frac{2}{3}l^2 - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{l}x \quad \text{(ii)}$$

To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ......\infty$, put $x = l$ in Equation (ii)

$$l(2l - l) = \frac{2}{3}l^2 - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$l^2 - \frac{2}{3}l^2 = \frac{-4l^2}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + .........\infty\right]$$

$$\frac{l^2}{3} \times \left(\frac{-\pi^2}{4l^2}\right) = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + .........\infty$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + ....\infty$$

### Functions Having Points of Discontinuity

A point at which a function is discontinued is called point of discontinuity. The function ( $f$ ) is called continuous at a point $a$ if,

$$\lim_{x \to a} f(x) = f(a)$$

Discontinuities are of different types.

(*i*) ***Removable Discontinuity***: When the value of $f(a)$ is wrong or it is not defined, then this type of discontinuity is called *removable type*. This discontinuity can be removed by changing or defining $f(x)$ at point $a$.

(*ii*) ***Jump Discontinuity***: This type of discontinuity occurs when the right hand side limit is not equal to left hand side limit, or the limits of left hand and right hand side are not equal. The size of jump is the difference between the limits of right hand and left hand side and is not common for simple function formulas. This phenomenon can be seen when there is suddden discharge of a capacitor.

(*iii*) ***Infinite Discontinuity***: In this type of discontinuity, the limit on one side is given, whereas the limit on other side is $\pm\infty$.

(*iv*) ***Essential Discontinuity***: This type of discontinuity occurs when one hand side limit is given and the other hand side does not exist, not even as $\infty$ or $-\infty$.

### Odd and Even Functions

Functions defined on $(-l, l)$, $(-\pi, \pi)$ or $(-\infty, \infty)$ can be classified as even functions or odd functions.

### Even Functions

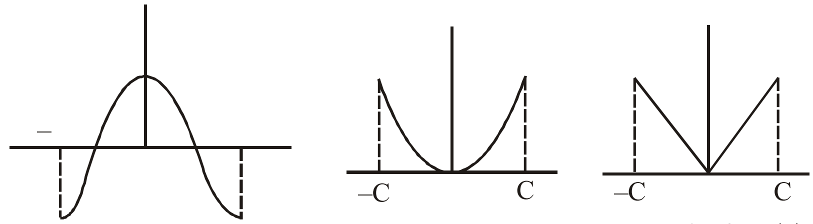
A function $f(x)$ is an even function on $(-l, l)$ if $f(-x) = f(x)$ in $-l < x < l$

The functions $\cos x$, $x^{2n}$, $|x|$ are even functions on $(-l, l)$ since

$$f(-x) = \cos(-x) = \cos x = f(x)$$
$$f(-x) = (-x)^{2n} = (-1)^{2n}x^{2n} = x^{2n} = f(x)$$
$$f(-x) = |-x| = |x| = f(x)$$

**NOTES**

Graphs of even functions $\cos x, x^2, |x|$ are given in the following Figures 4.4. It can be seen from the figures that the graphs are symmetrical about the $y$-axis.




Graph of $y = \cos x$ Graph of $y = x^2$ Graph of $y = |x|$

***Fig. 4.4*** *Graph of Even Functions*

If $f(x) = \begin{bmatrix} f_1(x) & \text{in} & (-l, 0) \\ f_2(x) & \text{in} & (0, l) \end{bmatrix}$

such that $f_1(-x) = f_2(x)$ or $f_2(-x) = f_1(x)$ then $f(x)$ is an even function of $x$ in $(-l, l)$ (Refer graph of function $y = |x|$).

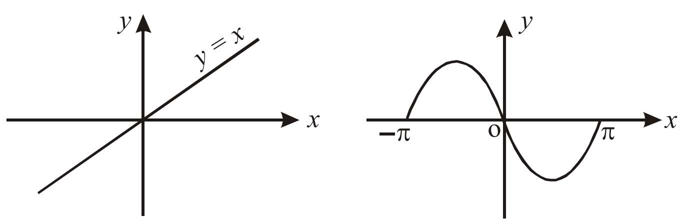
### Odd Functions

A function $f(x)$ is an odd function on $(-l, l)$ if $f(-x) = -f(x)$ in $-l < x < l$

The functions $x, x^3, \sin x$ are odd functions on $(-l, l)$ since

$$f(-x) = (-x) = -x = -f(x)$$
$$f(-x) = (-x)^3 = -x^3 = -f(x)$$
$$f(-x) = \sin(-x) = -\sin x = -f(x)$$

Graphs of some of the odd functions are given in the following Figure 4.5 and the graphs are symmetrical about the origin.




Graph of $y = x$ Graph of $y = \sin x$

***Fig. 4.5*** *Graph of Odd Functions*

If $f(x) = \begin{bmatrix} f_1(x) & \text{in} & (-l, 0) \\ f_2(x) & \text{in} & (0, l) \end{bmatrix}$

such that $f_1(-x) = -f_2(x)$ or $f_2(-x) = -f_1(x)$ then $f(x)$ is an odd function.

If $f(x)$ is an even function on $(-l, l)$ then

$$\int_{-l}^{l} f(x)dx = 2\int_{-l}^{l} f(x)dx$$

If $f(x)$ is an odd function on $(-l, l)$ then $\int_{-l}^{l} f(x)dx = 0$

The following results can easily be proved from the definition

(even function) (even function) = even function

(even function) (odd function) = odd function

(odd function) (odd function) = even function

If $f(x)$ is an even function, then

$$f(x)\cos\frac{n\pi}{l}x \text{ is an even function}$$

And

$$f(x)\sin\frac{n\pi}{l}x \text{ is an odd function}$$

If $f(x)$ is an odd function, then

$$f(x)\cos\frac{n\pi}{l}x \text{ is an odd function and}$$

$$f(x)\sin\frac{n\pi}{l}x \text{ is an even function}$$

Hence, if $f(x)$ is an even function on $(-\pi, \pi)$, then Fourier series expansion of $f(x)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

And $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

If $f(x)$ is an odd function on $(-\pi, \pi)$

Then $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

In other words, if $f(x)$ is an even function

$$b_n = 0, \text{ for } n = 1, 2, 3, \ldots$$

If $f(x)$ is an odd function

$$a_n = 0, \text{ for } n = 1, 2, 3, \ldots$$

Consider the equation,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

**NOTES**

With

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx \text{ for } n = 0, 1, 2, .... \text{ and}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\,dx$$

If $f(x)$ is an even function $b_n = 0$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$$

Where $$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi}{l}x\,dx \text{ for } n = 0, 1, 2,....$$

If $f(x)$ is an odd function $a_n = 0$.

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

Where $$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi}{l}x\,dx \text{ for } n = 1, 2, 3,....$$

**Example 4.45:** Find the Fourier series expansion of $f(x) = |x|, -1 < x < 1$.

**Solution:**

Here, $2l = 2 \Rightarrow l = 1$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$f(x) = |x|$ is an even function in the interval $(-1, 1)$.

So $b_n = 0$.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x \qquad (1)$$

$a_0 = \frac{1}{l}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)\,dx,$ since $f(x)$ is an even function and $l = 1$

$$= 2\int_{0}^{1} x\,dx = 1$$

$$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos\frac{n\pi}{l}x\,dx$$

$= 2\int_{-1}^{1} f(x)\cos n\pi x\,dx$ (Since $f(x)$ is an even function and $l = 1$)

NOTES

$$= 2\int_0^1 x\cos n\pi x\, dx$$

$$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[\cos n\pi x\right]_0^1 = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$= 0$ when $n$ is even.

$$= \frac{-4}{n^2\pi^2} \text{ when } n \text{ is odd.}$$

Substituting $a_0, a_n$ in Equation (i)

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos n\pi x$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + ...\right]$$

**Example 4.46:** Find the Fourier series for the function

$$f(x) = \begin{bmatrix} kx & 0 < x < l \\ 0 & l < x < 2l \end{bmatrix} \text{ repeating itself}$$

at intervals of $2l$, $k$ being a constant.

**Solution:**

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right) \quad \text{(i)}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi}{l}x\, dx$$

$$= \frac{1}{l}\int_0^{l} kx\cos\frac{n\pi}{l}x\, dx + \int_l^{2l} 0.\cos\frac{n\pi}{l}x\, dx$$

$$= \frac{k}{l}\left[x\left(\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) + \left(\frac{\cos\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{kl^2}{l\times n^2\pi^2}\left[\cos\frac{n\pi}{l}x\right]_0^l$$

$$= \frac{kl}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

$a_n$ cannot be deduced from $a_n$.

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l kx\, dx = \frac{k}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{kl}{2}$$

NOTES

$$b_n = \frac{1}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\int_0^l kx\sin\frac{n\pi}{l}x\,dx + \int_l^{2l} 0\,\sin\frac{n\pi}{l}dx$$

$$= \frac{k}{l}\left[x\left(\frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right)-\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{-k}{l}\times\frac{l}{n\pi}\left[x\cos\frac{n\pi}{l}x\right]_0^l$$

$$= \frac{-k}{n\pi}\left[l(-1)^n\right] = -\frac{kl}{n\pi}(-1)^n$$

Substituting the coefficients in Equation (i)

$$f(x) = \frac{kl}{4}+\frac{kl}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2}\right)\cos\frac{n\pi}{l}x - \frac{kl}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{l}x$$

**Example 4.47:** Find Fourier series with period 3 to represent $f(x) = 2x - x^2$ in the range (0, 3).

**Solution:**

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right) \quad \text{(i)}$$

Here, $2l = 3 \Rightarrow l = \frac{3}{2}$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (2x-x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^3 = \frac{2}{3}[9-9] = 0$$

$$a_n = \frac{1}{l}\int_0^3 (2x-x^2)\cos\frac{2n\pi}{3}x\,dx$$

$$= \frac{2}{3}\left[\left(2x-x^2\right)\left(\frac{\sin\frac{2n\pi}{3}x}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\cos\frac{2n\pi}{3}x}{\left(\frac{2n\pi}{3}\right)^2}\right)+(-2)\left(\frac{-\sin\frac{2n\pi}{3}x}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\times\frac{9\times 2}{4n^2\pi^2}\left[(1-x)\cos\frac{2n\pi}{3}x\right]_0^3$$

$$= \frac{3}{n^2\pi^2}[-2-1] = \frac{-9}{n^2\pi^2}$$

NOTES

$$b_n = \frac{1}{l}\int_0^3 f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi}{3}x\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi}{3}x}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\sin\frac{2n\pi}{3}x}{\left(\frac{2n\pi}{3}\right)^2}\right)+(-2)\left(\frac{\cos\frac{2n\pi}{3}x}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= -\frac{2}{3}\times\frac{3}{2n\pi}\left[(2x-x^2)\cos\frac{2n\pi}{3}x\right]_0^3 - \frac{2}{3}\times\frac{2\times 27}{4n^3\pi^3}\left[\cos\frac{2n\pi}{3}\right]_0^3$$

$$= -\frac{1}{n\pi}(-3)-\frac{9}{n^3\pi^3}[1-1] = \frac{3}{n\pi}$$

Substituting $a_0, a_n, b_n$ in Equation (i)

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{9}{n^2\pi^2}\right)\cos\frac{2n\pi}{3}x + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi}{3}x$$

$$= -\frac{9}{\pi^2}\left[\cos\frac{2\pi}{3}x+\frac{1}{2^2}\cos\frac{4\pi}{3}x+\frac{1}{3^2}\cos\frac{6\pi}{3}x+....\right]$$

$$= +\frac{3}{\pi}\left[\sin\frac{2\pi}{3}x+\frac{1}{2}\sin\frac{4\pi}{3}x+....\right]$$

**Example 4.48:** Find the Fourier expansion for the function

$f(x) = x - x^2, -1 < x < 1$

**Solution:**

Proceeding like the previous problem with $l = 1$,

$$a_0 = \int_{-1}^{1}(x-x^2)\,dx = \frac{-2}{3}$$

$$a_n = \int_{-1}^{1}(x-x^2)\cos n\pi\,dx = \frac{-4}{n^2\pi^2}\cos n\pi$$

$$= \frac{-4}{n^2\pi^2}(-1)^n \text{ and}$$

$$b_n = \int_{-1}^{1}(x-x^2)\sin n\pi x\,dx = \frac{-2(-1)^n}{n\pi}$$

$$\therefore\ f(x) = -\frac{1}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^2\cos n\pi x}{n^2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi}{n}$$

$$= -\frac{1}{3}-\frac{4}{\pi^2}\left[\frac{-\cos\pi x}{1^2}+\frac{\cos 2\pi x}{2^2}-\frac{\cos 3\pi x}{3^2}+....\right]$$

$$-\frac{2}{\pi}\left[-\frac{\sin\pi x}{1}+\frac{\sin 2\pi x}{2}-\frac{\sin 3\pi x}{3}+...\right]$$

$$= -\frac{1}{3}+\frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2}-\frac{\cos 2\pi x}{2^2}-\frac{\cos 3\pi x}{3^2}-...\right]$$

$$+\ \frac{2}{\pi}\left[\frac{\sin\pi x}{1}-\frac{\sin 2\pi x}{2}+\frac{\sin 3\pi x}{3}-...\right]$$

**NOTES**

**Example 4.49:** Find the Fourier series to represent,

$f(x) = x^2 - 2$, when $-2 \le x \le 2$.

**Solution:**

Let,

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right) \quad \text{(i)}$$

Here, $2l = 4$ or $l = 2$ and $f(x)$ is an even function.

$\therefore \quad b_n = 0,$

Where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{2}\int_{0}^{2}(x^2-2)\cos\frac{n\pi}{2}x\,dx \qquad \text{(Since } f(x) \text{ is an even function)}$$

$$= \left[(x^2-2)\left(\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right)-2\left(\frac{-\cos\frac{n\pi}{2}x}{\left(\frac{n\pi}{2}\right)^2}\right)+\frac{2\sin\frac{n\pi}{2}x}{\left(\frac{n\pi}{3}\right)^3}\right]_0^2$$

$$= \frac{8}{n^2\pi^2}\left[x\cos\frac{n\pi}{2}x\right]_0^2$$

$$= \frac{8\times 2}{n^2\pi^2}\cos n\pi = \frac{16(-1)^n}{n^2\pi^2}$$

$a_0$ cannot be deduced from $a_n$.

$$\therefore \quad a_0 = \frac{1}{2}\int_{-2}^{2}(x^2-2)dx$$

$$= \int_0^2 (x^2-2)dx \qquad (\because f(x) \text{ is an Even Function})$$

$$= \left(\frac{x^2}{3}-2x\right)_0^2 = \frac{8}{3}-4 = \frac{-4}{3}$$

Substituting $a_0$, $a_n$, $b_n$ in Equation (i)

$$f(x) = \frac{-2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}x$$

NOTES

**Example 4.50:** Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.

**Solution:**

Let,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right) \quad \text{(i)}$$

Let us find $a_n + ib_n$ and then separate the real and imaginary parts.

$$a_n + ib_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\left(\cos\frac{n\pi}{l}x + i\sin\frac{n\pi}{l}x\right)dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}.e^{\frac{in\pi}{l}x}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-\left(1-\frac{in\pi}{l}\right)x}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-\left(1-\frac{in\pi}{l}\right)x}}{-\left(1-\frac{in\pi}{l}\right)}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{-\left(1-\frac{in\pi}{l}\right)l} - e^{\left(1-\frac{in\pi}{l}\right)l}}{-\left(1-\frac{in\pi}{l}\right)}\right]$$

$$= -\frac{1}{l}\left[\frac{e^{-l}\,e^{in\pi} - e^{l}\,e^{-in\pi}}{\left(1-\frac{in\pi}{l}\right)}\right]$$

Writing the expansion of $e^{in\pi}$ and $e^{-in\pi}$,

$$= -\frac{1}{l}\left[\frac{e^{-l}(\cos n\pi + i\sin n\pi) - e^{l}(\cos n\pi - i\sin n\pi)}{1-\frac{in\pi}{l}}\right]$$

Multiply and divide by the conjugate of denominator.

$$= -\frac{1}{l}\left[\frac{(e^{-l} - e^{l})(-1)^n}{\left(1-\frac{in\pi}{l}\right)} \times \frac{\left(1+\frac{in\pi}{l}\right)}{\left(1+\frac{in\pi}{l}\right)}\right]$$

NOTES

$$= \frac{(-1)^n 2\sinh l\left(1+\frac{in\pi}{l}\right)}{l\left(1+\frac{n^2\pi^2}{l^2}\right)}$$

$$a_n + ib_n = \frac{(-1)^n 2l\sinh l\left(1+\frac{in\pi}{l}\right)}{\left(l^2+n^2\pi^2\right)}$$

Comparing the real and imaginary parts,

$$a_n = \frac{(-1)^n 2l\sinh l}{l^2+n^2\pi^2}, \qquad b_n = \frac{(-1)^n 2n\pi\sinh l}{l^2+n^2\pi^2}$$

Put $n = 0$ in $a_n$ and $a_0$ is deduced as,

$$a_0 = \frac{2l\sinh l}{l^2} = \frac{2l\sinh l}{l}$$

Substituting the values of the coefficients $a_0, a_n, b_n$ in Equation (i)

$$f(x) = \frac{\sinh l}{l} + 2l\sinh l\sum_{n=1}^{\infty}\frac{(-1)^n \cos\frac{n\pi}{l}x}{l^2+n^2\pi^2}$$

$$+\ 2\pi\sinh l\sum_{n=1}^{\infty}\frac{n(-1)^n \sin\frac{n\pi}{l}x}{l^2+n^2\pi^2}$$

**Example 4.51:** Obtain the Fourier series for the function

$$f(x) = \pi x, \qquad 0 \le x \le 1$$
$$= \pi(2-x), \quad 1 \le x \le 2$$

**Solution:**

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right) \qquad \text{(i)}$$

Here $2l = 2 \Rightarrow l = 1$

Where

$$a_n = \frac{1}{l}\int_0^{nl} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \int_0^{l}\pi\, x\cos n\pi x\,dx + \int_1^{2}\pi\,(2-x)\cos n\pi\, x\,dx$$

$$= \pi\left[x\left(\frac{\sin n\pi\, x}{n\pi}\right) - \left(\frac{-\cos n\pi\, x}{(n\pi)^2}\right)\right]_0^1$$

$$+\ \pi\left[(2-x)\left(\frac{\sin n\pi\, x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi\, x}{n^2\pi^2}\right)\right]_1^2$$

NOTES

$$= \frac{1}{n^2\pi}\left[\cos n\pi\, x\right]_0^1 - \frac{1}{n^2\pi}\left[\cos n\pi\, x\right]_1^2$$

$$= \frac{1}{\pi\, n^2}\left[(-1)^n - 1\right] - \frac{1}{n^2\pi}\left[1-(-1)^n\right]$$

$$= \frac{2\left[(-1)^n - 1\right]}{\pi n^2}$$

$= 0$, if $n$ is even.

$= -\frac{4}{\pi n^2}$, if $n$ is odd.

$a_0$ cannot be deduced from $a_n$.

$$\therefore \qquad a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$$

$$= \int_0^1 \pi\, x\, dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{\pi}{2} + \pi(2 - 3/2) = \pi$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi}{l}x\,dx \quad \text{But, } l = 1$$

$$= \int_0^1 \pi\, x \sin n\pi\, x\, dx + \int_1^2 \pi\,(2-x)\sin n\pi\, x\, dx$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{(n\pi)^2}\right)\right]_0^1$$

$$+\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \frac{-1}{n}\left[x\cos n\pi\, x\right]_0^1 - \frac{1}{n}\left[(2-x)\cos n\pi x\right]_1^2$$

$$= -\frac{1}{n}\left[\cos n\pi\right] - \frac{1}{n}\left[0 - \cos n\pi\right] = 0$$

$$\therefore \qquad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5}\frac{\cos n\pi x}{n^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

**Example 4.52:** Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when

$f(x) = 0, \qquad -2 < x < 0$

$\quad\;\; = 1, \qquad 0 < x < 2$

**Solution:**

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right) \qquad \text{(i)}$$

NOTES

Here, $2l = 4 \quad \therefore \quad l = 2$

Where

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{0}^{2}\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left[\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{1}{2}\times\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}x\right]_0^2 = \frac{1}{n\pi}\left[\sin n\pi - \sin 0\right]$$

Though $a_n = 0$, $a_0$ may exist

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{2} 1dx = 0 + \frac{1}{2}[x]_0^2 = 1$$

$$b_n = \frac{1}{2}\int_{-2}^{0} f(x)\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{-2}^{0}\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2 = \frac{-1}{n\pi}\left[\cos\frac{n\pi}{2}x\right]_0^2$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = 0 \text{ if } n \text{ is even}$$

$$= \frac{2}{n\pi} \text{ if } n \text{ is odd}$$

$\therefore$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}x$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + ...\right]$$

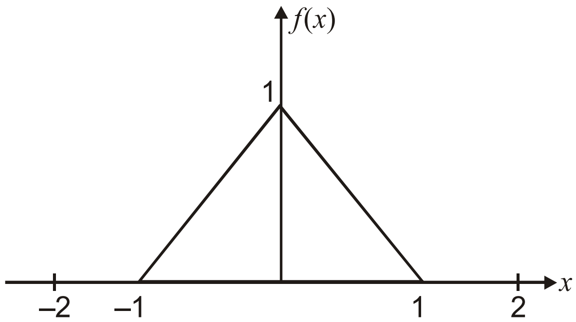
NOTES

**Example 4.53:** Express $f(x)$ as a Fourier series

$$f(x)=\begin{cases}0 & -2\le x\le -1\\ 1+x & -1\le x\le 0\\ 1-x & 0\le x\le 1\\ 0 & 1\le x\le 2\end{cases}$$

**Solution:**

Let us draw the graph of $f(x)$




The curve of $f(x)$ is symmetrical about $y$-axis. So $f(x)$ is an even function.

Let $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\left(\cos\dfrac{n\pi}{l}x+b_n\sin\dfrac{n\pi}{l}x\right)$

Since $f(x)$ is an even function $b_n=0$

$2l=4$ and hence $l=2$

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{2}x \qquad \text{(i)}$$

Let

$$a_n=\frac{1}{l}\int_{-l}^{l} lf(x)\cos\frac{n\pi}{l}x\,dx$$

$$=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$=\frac{2}{2}\int_{0}^{2} f(x)\cos\frac{n\pi}{2}dx \quad (\because \text{ Integrand is an Even Function})$$

$$=\int_{0}^{1}(1-x)\cos\frac{n\pi}{2}x\,dx$$

NOTES

$$= \left[ \frac{(1-x)\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}} - (-1)\left( \frac{-\cos\frac{n\pi}{2}x}{\left(\frac{n\pi}{2}\right)^2} \right) \right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[ \cos\frac{n\pi}{2} - 1 \right]$$

$a_0$ cannot be deduced from $a_n$

$$\therefore\ a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$$

$$= \int_0^2 f(x)dx \quad (\because f(x) \text{ is an even function})$$

$$= \int_0^1 (1-x)dx + \int_1^2 0\ dx = \left[ x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}$$

Substituting $a_0$, $a_n$ in Equation (i)

$$f(x) = \frac{1}{4} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left( \cos\frac{n\pi}{2} - 1 \right)\cos\frac{n\pi}{2}x$$

**Example 4.54:** Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

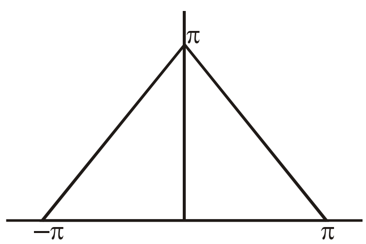
$$f(x) = \pi + x \quad -\pi < x < 0$$
$$= \pi - x \quad 0 < x < \pi$$

Hence deduce that a,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

**Solution:**

Let us draw the graph of $f(x)$




The curve is symmetrical about $y$-axis.

$\therefore f(x)$ is an even function.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Since $f(x)$ is even function, $b_n = 0$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{(i)}$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx \text{ since } f(x) \text{ is an even function}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{x^2}{2}\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx \quad (\because f(x) \text{ is an even function})$$

$$= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-2}{\pi n^2}[\cos x]_0^{\pi} = \frac{-2}{\pi n^2}[(-1)^n - 1]$$

$= 0$ if $n$ is even.

$$= \frac{4}{\pi n^2} \text{ if } n \text{ is odd.}$$

Substituting $a_0$, $a_n$ in Equation (i)

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty} \frac{4}{\pi n^2}\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] \qquad \text{(ii)}$$

To deduce

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \text{ put } x = 0 \text{ in Equation (ii)}$$

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

NOTES

$$\frac{\pi}{2} \times \frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

i.e., $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

**Example 4.55:** Find the Fourier series for $f(x)$ when,

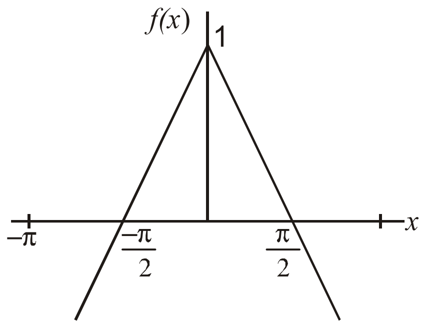
$$f(x) = 1 + \frac{2x}{\pi} \quad -\pi \le x \le 0$$

$$= 1 - \frac{2x}{\pi} \quad 0 \le x \le \pi$$

Hence show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

**Solution:**

Let us draw the graph of the function $f(x)$,




The curve is symmetrical about $y$-axis.

$\therefore f(x)$ is an even function.

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Since $f(x)$ is an even function, $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{(i)}$$

Where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx$$

**NOTES**

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \quad (\because f(x) \text{ is an even function})$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2}{\pi}x\right)\cos nx dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2}{\pi}x\right)\left(\frac{\sin nx}{n}\right)-\left(\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-4}{n^2\pi^2}[(-1)^n - 1]$$

$= 0$, if $n$ is even.

$$= \frac{8}{n^2\pi^2} \text{ if } n \text{ is odd.}$$

$a_0$ cannot be deduced from $a_n$. So let us get $a_0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

($f(x)$ is an even function)

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2}{\pi}x\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = 0$$

Substituting $a_0$, $a_n$ in Equation (i)

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos nx$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right] \qquad \text{(ii)}$$

To show

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8} \text{ put } x = 0 \text{ in Equation (ii)}$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

NOTES

i.e., $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

i.e., $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

**Example 4.56:** Expand $f(x) = |\sin x| - \pi < x < \pi$ as a Fourier series.

**Solution:**

$f(x) = |\sin x|$ is an even function.

$\therefore \quad b_n = 0$

So the Fourier series expansion of $f(x)$ is,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad \text{(i)}$$

Where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx \quad (\because f(x)\text{ is an even function})$$

$$=\frac{2}{\pi}\int_{0}^{\pi}|\sin x|\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{0}^{\pi}2\sin x\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{0}^{\pi}(\sin(1+n)x+\sin(1-n)dx$$

$$=\frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n}-\frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$=-\frac{1}{\pi}\left[\frac{(-1)^{n+1}-1)}{1+n}-\frac{(-1)^{n+1}-1)}{1-n}\right]$$

$$=\frac{-[(-1)^{n+1}-1]}{\pi}\left[\frac{1}{1+n}-\frac{1}{1-n}\right]$$

$$=\frac{((-1)^{n+1}-1)}{\pi}\left[\frac{2}{1-n^2}\right]$$

$$a_n=\frac{2((-1)^{n+1}-1)}{\pi(1-n^2)}$$

$= 0$ if $n$ is odd

$$= \frac{-4}{\pi(n^2 - 1)} \text{ if } n \text{ is even.}$$

$a_0$ can be deduced from $a_n$

$$a_0 = \frac{4}{\pi}$$

$$\therefore\ f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=2,4,6}^{\infty} \frac{\cos nx}{1 - n^2}$$

**Check Your Progress**

7. What do you understand by exponential function?
8. What is Euler number?
9. What is the periodic function?
10. Define the term infinite discontinuity.

## 4.6 DIFFERENTIAL EQUATIONS

In **mathematics**, a Differential Equation (DE) is defined as an equation of the form that interconnects certain function with its derivatives, where usually the function represents the physical quantity while the derivatives denote their rates of change and the relationship between the two is defined by the equation. Fundamentally, the 'Solutions of Differential Equations' are **functions** which precisely 'Represent the relationship or correlation between a continuously varying or fluctuating quantity and its rate of change'.

A Differential Equation (DE) comprises of one or more expressions including derivatives of one dependent variable '***y***' with reference to another independent variable '***x***', such as

$$\frac{dy}{dx} = 2x$$

An **ordinary differential equation** is a differential equation that includes a function of a single variable and some of its derivatives, such as

$$\frac{dy}{dx} = 2x^2 + 3x + 5$$

Principally, a differential equation is an equation for a function that relates the values of the function to the values of its derivatives. Therefore, a differential equation is an equation between specified derivative of an unknown function, its values and known quantities and functions or a Differential Equation (DE) is an equation that comprises of a function and its derivatives. Differential equations are categorized as Partial Differential Equations (PDE) or Ordinary Differential Equations (ODE) in accordance with whether or not they hold partial derivatives, while the order of a differential equation is defined on the basis of the highest order derivative that occurs in the equation.

NOTES

In mathematics, a **linear differential equation** is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form,

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} + b(x) = 0$$

where $a_0(x), ..., a_n(x)$ and $b(x)$ are arbitrary differentiable functions that do not essential to be linear, and $y', \ldots, y^{(n)}$ are considered as the successive derivatives of an unknown function $y$ of the variable $x$.

This is an Ordinary Differential Equation (ODE). A linear differential equation may also be a Linear Partial Differential Equation (Linear PDE), if the unknown function depends on several variables, and the derivatives that appear in the equation are partial derivatives.

If a linear differential equation or a system of linear equations are such that the associated homogeneous equations have constant coefficients then these may be solved by means of quadrature mathematics, i.e., the solutions may be expressed in terms of integrals. This is also true for a linear equation of order one with non-constant coefficients.

### Geometrical Meaning of a Differential Equation

Let a differential equation be $\frac{dy}{dx} = f(x, y)$. Then the slope of the tangent to the curve at the point $(x, y)$ is given by $\frac{dy}{dx}$.

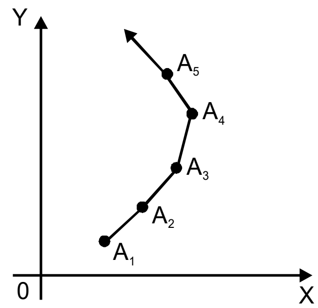



***Fig. 4.6*** *Geometrical Differential Equation*

As shown in Figure 4.6, take a point $A_1(x_1, y_1)$ in the $xy$-plane. If $m_1$ is the slope of the tangent to the curve at the point $(x_1, y_1)$, then at this point $\frac{dy}{dx}$ is equal to $m_1$. Suppose the point moves in the direction of $m_1$ from $(x_1, y_1)$ to $A_2(x_2, y_2)$ for an infinitesimal distance and the slope of the tangent at this new point is given by $m_2$. Again, the point moves from $A_2(x_2, y_2)$ for an infinitesimal distance in the direction of $m_2$ to the point $A_3(x_3, y_3)$ and the slope of the tangent at this new point is given by $m_3$.

Choosing the successive points $A_1, A_2, A_3, \ldots$ near to the one another makes the broken curve approximate to a smooth curve $y = \phi(x)$ which is associated with the initial point $A_1(x_1, y_1)$. The slope of the tangent to the curve at a point and the co-ordinate of that point always satisfy the equation $\frac{dy}{dx} = f(x, y)$.

On choosing different initial points, different curves with the same property will be obtained. Thus, we can say that the differential equation $\frac{dy}{dx} = f(x, y)$ represents a family of curves such that through each point of the $xy$-plane, there passes one curve of the family.

### Solving Linear Differential Equations

The linear differential equation is a differential equation in which the dependent variable and all its derivatives appear only in the first degree and are not multiplied together.

A linear differential equation of order $n$ is of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ....... + P_n y = X \qquad ...(4.16)$$

Where, $P_0, P_1, P_2, ....., P_n$ and X are either the functions of $x$ or constants.

When in Equation (4.16), $P_0, P_1, P_2, ..., P_n$ are all constants and not the function of $x$ while X is the some function of $x$, the equation is called a linear differential equation with constant co-efficients of $n^{th}$ order.

### Differential Operator 'D'

Sometimes, it is easier to write a linear differential equation in the simple form by replacing the part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ with D. This D or $\frac{d}{dx}$ is regarded as differential operator.

So, Equation (1) can be written in terms of differential operator D as

$$\left[P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + ....... + P_n\right] y = X$$

Or $f(D)y = X$ where $f(D) = P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + ....... + P_n$.

### Solution of Linear Differential Equations

**Theorem 4.10:** If $y = y_1, y = y_2, ..., y = y_n$ be the linear independent solutions of $\left(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ....... + a_n\right) y = 0$, then $y = c_1 y_1 + c_2 y_2 + .... + c_n y_n$ where $c_1, c_2, c_3, ..., c_n$ are arbitrary constants, is the general or complete solution of the differential Equation (4.16).

**Proof:** Given, $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ....... + a_n y = 0 \qquad ...(4.17)$

As $y = y_1, y = y_2, ..., y = y_n$ are solutions of Equation (4.17),

NOTES

$$\left.\begin{array}{l}\therefore a_0 D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + ....... + a_n y_1 = 0 \\ a_0 D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + ....... + a_n y_2 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_0 D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + ....... + a_n y_n = 0\end{array}\right\} \quad \ldots(4.18)$$

$$y = c_1 y_1 + c_2 y_2 + .... + c_n y_n$$

Now, by putting $y = c_1 y_1 + c_2 y_2 + .... + c_n y_n$ in Equation (4.17), we get

$$a_0 D^n (c_1 y_1 + c_2 y_2 + ... + c_n y_n) + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + ... + c_n y_n)$$
$$+ .................................. + a_n (c_1 y_1 + c_2 y_2 + ... + c_n y_n) = 0$$

$$\Rightarrow c_1 \left(a_0 D^n y_1 + a_1 D^{n-1} y_1 + ..... + a_n y_1\right) + c_2 \left(a_0 D^n y_2 + a_1 D^{n-1} y_2 + ..... + a_n y_2\right)$$
$$+ .................................. + c_n \left(a_0 D^n y_n + a_1 D^{n-1} y_n + ..... + a_n y_n\right) = 0$$

Substituting the values from Equation (4.18) in the above equation, we get

$c_1.0 + c_2.0 + ... + c_n.0 = 0 \Rightarrow 0 = 0$

Thus, $y = c_1 y_1 + c_2 y_2 + .... + c_n y_n$ satisfies the Equation (4.17), which is therefore its solution.

**Auxiliary Equation**

Consider $a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ....... + a_n y = 0$ … (4.19)

where, $a_0, a_1, a_2, ..., a_n$ are all constants.

Now, let one of the solution of Equation (4.19) be $y = e^{mx}$.

Then, $Dy = me^{mx}$, $D^2 y = m^2 e^{mx}$, …, $D^n y = m^n e^{mx}$

By putting all these values including $y = e^{mx}$ in Equation (4.19), we get

$$\left(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ....... + a_n\right) e^{mx} = 0$$

Since, $e^{mx} \neq 0$ for any $m$, therefore,

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ....... + a_n = 0 \quad \ldots(4.20)$$

This equation in $m$ is known as auxiliary equation of the given differential Equation (4.20).

If $m$ is a solution of Equation (4.20), then $e^{mx}$ will be a solution of Equation (4.19). To solve the Equation (4.19), write its auxiliary equation and then solve it first.

**Note:** The values of D obtained from

$$f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ....... + a_n = 0 \quad \ldots(4.21)$$

is same as the values of $m$ obtained from Equation (4.20). Hence, Equation (4.21), in general can be regarded as the auxiliary equation which is obtained by equating the symbolic coefficient of $y$ in Equation (4.19) to zero. Thus, in practice we do not replace D by $m$ to form an auxiliary equation.

**NOTES**

The roots of the auxiliary equation can be as follows:

- Real and Different
- Real and Repeated
- Complex

**Case I:** When all the roots of Equation (4.20) are real and different.

Let $m_1, m_2, m_3, .......m_n$ be the $n$ different and real roots of Equation (4.20). Then

$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, ..., e^{m_n x}$ will be the $n$ distinct solutions of Equation (4.19).

Thus, the general solution of Equation (4.19) is,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ..... + c_n e^{m_n x} \qquad \ldots (4.22)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

**Case II:** When all the roots of Equation (4.20) are real, two of them are repeated and all others are different.

Let $m_1$ and $m_2$ be the two equal roots of the auxiliary equation.

Then, Equation (4.22) becomes $y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + ..... + c_n e^{m_n x}$

which has only $(n - 1)$ arbitrary constants and so it is no longer a general solution.

Consider a differential equation of second order

$$(\mathrm{D} - m_1)(\mathrm{D} - m_1) y = 0 \qquad \ldots (4.23)$$

whose auxiliary equation $(\mathrm{D} - m_1)^2 = 0$ has equal roots.

Now, putting $(\mathrm{D} - m_1) y = v$ in the Equation (4.23), we get

$$(\mathrm{D} - m_1) v = 0 \Rightarrow \frac{dv}{dx} = m_1 v \Rightarrow \frac{dv}{v} = m_1 dx$$

On integrating both sides, we get

$$\log v = m_1 x + \log c$$

$$\Rightarrow v = ce^{m_1 x} \Rightarrow (\mathrm{D} - m_1) y = ce^{m_1 x} \Rightarrow \frac{dy}{dx} - m_1 y = ce^{m_1 x}$$

This is a linear equation of first order having I.F. as $e^{\int -m_1 dx} = e^{-m_1 x}$

$\therefore$ Its solution is $y.e^{-m_1 x} = \int ce^{m_1 x}.e^{-mx} dx + c$

$$\Rightarrow\ y.e^{-m_1x} = \int c\,dx + c'$$

$$\Rightarrow\ y.e^{-m_1x} = cx + c' \Rightarrow\ y = (cx + c').e^{m_1x} \Rightarrow\ y = (c_1 + c_2x)e^{m_1x}$$

**NOTES**

Thus, the general or complete solution of Equation (4.22) is

$$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

Similarly, we can prove that if $r$ roots of auxiliary equation are equal, then the general solution would be $y = \left(c_1 + c_2x + c_3x^2 + ..... + c_rx^{r-1}\right)e^{m_1x} + ..... + c_ne^{m_nx}$

**Case III:** When two roots of Equation (4.20) are complex and others are real and different.

Let a pair of complex roots of the auxiliary Equation (4.20) be $\alpha \pm i\beta$ and other real and different roots be $m_3, m_4, m_5, .......m_n$. Then, the complete solution of Equation (4.19) is,

$$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

$$= e^{\alpha x}\left(c_1e^{i\beta x} + c_2e^{-i\beta x}\right) + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right] + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

$\left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$

$$= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right] + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

$$= e^{\alpha x}\left[\mathrm{A}\cos\beta x + \mathrm{B}\sin\beta x\right] + c_3e^{m_3x} + ..... + c_ne^{m_nx}$$

**Corollary:** If the complex roots pair $\alpha \pm i\beta$ occur twice, the above equation reduces to

$$y = e^{\alpha x}\left[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\right] + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

**Example 4.57:** Solve the differential equation $\dfrac{d^4y}{dx^4} + a^4y = 0$.

**Solution:** The given differential equation is $\dfrac{d^4y}{dx^4} + a^4y = 0$.

Symbolic form of the equation is $\left(\mathrm{D}^4 + a^4\right)y = 0$

Its auxiliary equation is $\mathrm{D}^4 + a^4 = 0$

$$\Rightarrow \left(\mathrm{D}^2 + a^2\right)^2 - 2\mathrm{D}^2a^2 = 0$$

$$\Rightarrow \left(\mathrm{D}^2 + a^2\right)^2 - \left(\sqrt{2}\mathrm{D}a\right)^2 = 0$$

NOTES

$$\Rightarrow \left(D^2 - \sqrt{2}Da + a^2\right)\left(D^2 + \sqrt{2}Da + a^2\right) = 0$$

$$\Rightarrow D = \frac{-a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$$

Since, there are two pairs of complex roots, thus the general solution is

$$y = e^{-\frac{a}{\sqrt{2}}x}\left(c_1 \cos\frac{a}{\sqrt{2}}x + c_2 \sin\frac{a}{\sqrt{2}}x\right) + e^{\frac{a}{\sqrt{2}}x}\left(c_3 \cos\frac{a}{\sqrt{2}}x + c_4 \sin\frac{a}{\sqrt{2}}x\right)$$

$$\Rightarrow y = e^{-\frac{a}{\sqrt{2}}x} c_1 \cos\left(\frac{a}{\sqrt{2}}x + c_2\right) + e^{\frac{a}{\sqrt{2}}x} c_3 \cos\left(\frac{a}{\sqrt{2}}x + c_4\right)$$

**Example 4.58:** If $\lambda_1, \lambda_2$ are real and distinct roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ and $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$, are its solution, then prove that $y = c_1 y_1 + c_2 y_2$ is a solution of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$. Is this solution general?

**Solution:** The given auxiliary equation is $\lambda^2 + a_1\lambda + a_2 = 0$

$$\Rightarrow D^2 + a_1 D + a_2 = 0 \qquad ...(i)$$

[On replacing $\lambda$ by D]

Since the real and distinct roots of the above equation are $\lambda_1$ and $\lambda_2$.

$\therefore y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two independent solution of

$$\left(D^2 + a_1 D + a_2\right)y = 0$$

$$\Rightarrow D^2 y + a_1 Dy + a_2 y = 0 \qquad ...(ii)$$

$$\Rightarrow \left.\begin{array}{l} D^2 y_1 + a_1 Dy_1 + a_2 y_1 = 0 \\ D^2 y_2 + a_1 Dy_2 + a_2 y_2 = 0 \end{array}\right\} \qquad ...(iii)$$

On putting $y = c_1 y_1 + c_2 y_2$ in Equation (ii), we have

$$D^2\left(c_1 y_1 + c_2 y_2\right) + a_1 D\left(c_1 y_1 + c_2 y_2\right) + a_2\left(c_1 y_1 + c_2 y_2\right) = 0$$

$$\Rightarrow c_1\left(D^2 y_1 + a_1 Dy_1 + a_2 y_1\right) + c_2\left(D^2 y_2 + a_1 Dy_2 + a_2 y_2\right) = 0$$

$\Rightarrow c_1(0) + c_2(0) = 0$ which is true. [Using Equation (iii)]

$\therefore y = c_1 y_1 + c_2 y_2$ is the solution of Equation (ii) or $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$.

As the solution has two arbitrary constants and the order of the given equation is also 2, thus this solution is general.

NOTES

## Complementary Function and Particular Integral

**Theorem 4.11:** If the complete solution of the equation

$$\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)y = 0 \qquad \dots (4.24)$$

is $y = Y$ and the particular solution (containing no arbitrary constants) of the equation

$$\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)y = X \qquad \dots (4.25)$$

where X is a function of $x$ is $y = u$, then $y = Y + u$ will be the complete solution of the Equation (4.25).

**Proof:** By putting $y = Y + u$ in Equation (4.25), we get

$$\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)\left(Y + u\right) = X$$

$$\Rightarrow\ P_0D^n\left(Y + u\right) + P_1D^{n-1}\left(Y + u\right) + P_2D^{n-2}\left(Y + u\right) + ....... + P_n\left(Y + u\right) = X$$

$$\Rightarrow\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)Y + \left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)u = X \quad ...(4.26)$$

As $y = Y$ is the solution of Equation (4.24), we have

$$\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)Y = 0 \qquad \dots (4.27)$$

Also, since $y = u$ is the solution of Equation (4.25), we have

$$\left(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + ....... + P_n\right)u = X \qquad \dots (4.28)$$

By putting the values from Equations (4.27) and (4.28) into equation (4.26), we get

$0 + X = X$

$\Rightarrow\ X = X$, which is true.

Thus, $y = Y + u$ is the complete or general solution of Equation (4.25).

The expression Y is called Complementary Function (C.F.) and $u$ is known as Particular Integral (P.I.).

Thus, Complete Solution = Complementary Function + Particular Integral

## Inverse Operator

The operator $\dfrac{1}{f(D)}$ is known as the inverse operator of the operator $f(D)$, if $y = \dfrac{1}{f(D)}X$ gives the solution of the equation $f(D)y = X$.

If $y = \dfrac{1}{f(D)}X$ does not contain any arbitrary constant, then it is called the particular solution of the equation $f(D)y = X$. Thus, $\dfrac{1}{f(D)}X$, a function of $x$, free from constants, when operated on $f(D)$ gives X.

NOTES

### Some Theorems for Finding Particular Integrals

- $\frac{1}{f(\mathrm{D})}\mathrm{X}$ is the particular integral of $f(\mathrm{D})y = \mathrm{X}$.
- $\frac{1}{\mathrm{D}-a}\mathrm{X} = e^{ax}\int\left(e^{-ax}.\mathrm{X}\right)dx$, no arbitrary constant being added.
- $\frac{1}{\left(\mathrm{D}-a\right)^n}.e^{ax} = \frac{x^n}{n!}e^{ax}$
- The particular integral of $f(\mathrm{D})y = \mathrm{X}$ is

$$\mathrm{A}_1.e^{\alpha_1 x}\int e^{-\alpha_1 x}.\mathrm{X}\,dx + \mathrm{A}_2 e^{\alpha_2 x}\int e^{-\alpha_2 x}.\mathrm{X}\,dx + ....... + \mathrm{A}_n e^{\alpha_n x}\int e^{-\alpha_n x}.\mathrm{X}\,dx.$$

### Some Special Cases in Particular Integral

Case I: Prove that $\frac{1}{f(\mathrm{D})}.e^{ax} = \frac{1}{f(a)}.e^{ax}$ provided that $f(a) \neq 0$, where $a$ is a constant.

### Case of Failure

$$\text{If } f(a) = 0, \text{ then } \frac{1}{f(\mathrm{D})}e^{ax} = x.\frac{1}{\frac{d}{d\mathrm{D}}f(\mathrm{D})}.e^{ax}$$

**Example 4.59:** Solve the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}$.

**Solution:** The given differential equation is $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}$.

Symbolic form of the equation is $\left(\mathrm{D}^2 + 6\mathrm{D} + 25\right)y = 104e^{3x}$

Its auxiliary equation is $\mathrm{D}^2 + 6\mathrm{D} + 25 = 0$

$$\therefore \quad \mathrm{D} = \frac{-6 \pm \sqrt{36-100}}{2}$$

$$\Rightarrow \quad \mathrm{D} = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$\therefore \quad \text{C.F. } = e^{-3x}\left[c_1\cos 4x + c_2\sin 4x\right]$$

And
$$\text{P.I. } = \frac{1}{\mathrm{D}^2 + 6\mathrm{D} + 25}.104e^{3x} = \frac{1}{9+18+25}.104e^{3x} \qquad [\because \mathrm{D} = 3]$$

$$= \frac{104e^{3x}}{52} = 2e^{3x}$$

Thus, the complete solution is $y = \text{C.F. + P.I.}$

$$\Rightarrow \quad y = e^{-3x}\left(c_1\cos 4x + c_2\sin 4x\right) + 2e^{3x}$$

**Example 4.60:** Solve the differential equation $\frac{d^2y}{dx^2} + 16y = \sec 4x$

NOTES

**Solution:** The given differential equation is $\frac{d^2y}{dx^2} + 16y = \sec 4x$

Symbolic form of the equation is $\left(D^2 + 16\right)y = \sec 4x$

Its auxiliary equation is $D^2 + 16 = 0$

$\Rightarrow \qquad D = \pm 4i$

$\therefore \qquad \text{C.F.} = c_1 \cos 4x + c_2 \sin 4x$

And $\qquad \text{P.I.} = \frac{1}{D^2 + 16} \sec 4x$

$$= \frac{1}{8i}\left\{\frac{1}{D - 4i} - \frac{1}{D + 4i}\right\} \sec 4x$$

$$= \frac{1}{8i}\left\{e^{(4i)x}\int e^{-4ix} \sec\ 4x dx - e^{-4ix}\int e^{4ix} \sec 4x\, dx\right\}$$

$$= \frac{1}{8i}\left\{e^{4ix}\int (1 - i\tan 4x)\,dx - e^{-4ix}\int (1 + i\tan 4x)dx\right\}$$

$$= \frac{1}{8i}\left\{\left(e^{4ix} - e^{-4ix}\right).x - i\left(e^{4ix} + e^{-4ix}\right)\frac{1}{4}\log\sec 4x\right\}$$

$$= \frac{1}{4}\left\{x\sin 4x - \frac{1}{4}\cos 4x \log\sec 4x\right\}$$

$$= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x \log\cos 4x$$

Thus, the required solution is

$$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x \log\cos 4x$$

**Case II:** Prove that $\frac{1}{f\left(D^2\right)}\sin ax = \frac{1}{f\left(-a^2\right)}\sin ax$ if $f\left(-a^2\right) \neq 0.$

**Case of Failure:** If $f\left(-a^2\right) = 0$, then

(i) $$\frac{1}{f\left(D^2\right)}\sin ax = x.\frac{1}{\frac{d}{dD}f\left(D^2\right)}\sin ax$$

(ii) $$\frac{1}{f\left(D^2\right)}\cos ax = x.\frac{1}{\frac{d}{dD}f\left(D^2\right)}\cos ax$$

**Example 4.61:** Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$ .

NOTES

**Solution**: The given differential equation is $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$

Symbolic form of the equation is $(\mathrm{D}^2+\mathrm{D}+1)y=\sin 2x$

Its auxiliary equation is $\mathrm{D}^2+\mathrm{D}+1=0$

$$\Rightarrow \qquad \mathrm{D}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\therefore \qquad \text{C.F.} =e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]$$

And $\quad$ P.I. $=\frac{1}{\mathrm{D}^2+\mathrm{D}+1}\sin 2x$

$$=\frac{1}{-2^2+\mathrm{D}+1}\sin 2x \qquad [\because \mathrm{D}^2=-2^2]$$

$$=\frac{1}{\mathrm{D}-3}\sin 2x$$

$$=\frac{\mathrm{D}+3}{(\mathrm{D}-3)(\mathrm{D}+3)}\sin 2x=\frac{\mathrm{D}+3}{\mathrm{D}^2-9}\sin 2x$$

$$=\frac{\mathrm{D}+3}{-2^2-9}\sin 2x \qquad [\because \mathrm{D}^2=-2^2]$$

$$=-\frac{1}{13}(\mathrm{D}+3)\sin 2x$$

$$=-\frac{1}{13}[\mathrm{D}\sin 2x+3\sin 2x]$$

$$=-\frac{1}{13}\left[\frac{d}{dx}\sin 2x+3\sin 2x\right]=-\frac{1}{13}[2\cos 2x+3\sin 2x]$$

Thus, the required solution is

$$y=e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{13}(2\cos 2x+3\sin 2x)$$

**Case III:** To calculate $\frac{1}{f(\mathrm{D})}x^m$, where $m$ is a positive integer.

To calculate the P.I. $\frac{1}{f(\mathrm{D})}x^m$, follow the steps given below:

**Step 1:** Take out the lowest degree term from $f$(D) such that the remaining factor will be of the form $[1\pm\phi(\mathrm{D})]$.

**Step 2:** Take $[1\pm\phi(\mathrm{D})]$ in the numerator with negative index and expand it in ascending powers of D by Binomial Theorem up to the terms $\mathrm{D}^m$.

**Step 3:** Operate on $x^m$ by each term of the expansion.

**Note:** Do not write the terms of the expansion which contains powers of D greater than $m$ because $D^{m+1}$ when multiplied by $x^m$ will be zero.

**Example 4.62:** Solve the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$

**Solution**: The given differential equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$

Symbolic form of the equation is $\left(D^2 - 4D + 4\right)y = x^2 + e^x + \cos 2x$

Its auxiliary equation is $D^2 - 4D + 4 = 0$

$\Rightarrow \quad (D-2)^2 = 0 \Rightarrow D = 2, 2$

$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^{2x}$

And $\quad \text{P.I.} = \frac{1}{D^2 - 4D + 4}\left(x^2 + e^x + \cos 2x\right)$

$$= \frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4}e^x + \frac{1}{D^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4\left(1 - D + \frac{D^2}{4}\right)}x^2 + \frac{1}{1^2 - 4.1 + 4}e^x + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4}\left[1 - \left(D - \frac{D^2}{4}\right)\right]^{-1}.x^2 + e^x - \frac{1}{4}\left(\frac{1}{D}\cos 2x\right)$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + \left(D - \frac{D^2}{4}\right)^2 + \ldots\right]x^2 + e^x - \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + D^2 + \ldots\right]x^2 + e^x - \frac{1}{4}.\frac{\sin 2x}{2}$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\right]x^2 + e^x - \frac{1}{8}.\sin 2x$$

$$= \frac{1}{4}\left[x^2 + D\left(x^2\right) + \frac{3}{4}D^2\left(x^2\right)\right] + e^x - \frac{1}{8}.\sin 2x$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] + e^x - \frac{1}{8}\sin 2x$$

Thus, the required solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x.$$

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**Case IV:** $\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$, where V is a function of $x$.

**Example 4.63:** Solve the differential equation $\frac{d^2y}{dx^2} + 2y = x^2.e^{3x} + e^x.\cos 2x$.

**Solution:** The given differential equation is $\frac{d^2y}{dx^2} + 2y = x^2.e^{3x} + e^x.\cos 2x$

Symbolic form of the equation is $\left(D^2+2\right)y = x^2e^{3x} + e^x\cos 2x$

Its auxiliary equation is $D^2 + 2 = 0$

$\Rightarrow \quad D = \pm i\sqrt{2} = 0 \pm i\sqrt{2}$

$\therefore \quad$ C.F. $= e^{0x}\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$

And P.I. $= \frac{1}{D^2+2}\left(x^2e^{3x} + e^x\cos 2x\right)$

$$= \frac{1}{D^2+2}\left(e^{3x}x^2\right) + \frac{1}{D^2+2}\left(e^x\cos 2x\right)$$

$$= e^{3x}\frac{1}{(D+3)^2+2}.x^2 + e^x.\frac{1}{(D+1)^2+2}\cos 2x$$

$$= e^{3x}\frac{1}{D^2+6D+11}.x^2 + e^x.\frac{1}{D^2+2D+3}\cos 2x$$

$$= e^{3x}\frac{1}{11\left(1+\frac{6}{11}D+\frac{D^2}{11}\right)}.x^2 + e^x\frac{1}{-4+2D+3}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[1+\frac{6}{11}D+\frac{D^2}{11}\right]^{-1}.x^2 + e^x\frac{1}{2D-1}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[1-\left(\frac{6}{11}D+\frac{D^2}{11}\right)+\left(\frac{6}{11}D+\frac{D^2}{11}\right)^2 .....\right].x^2 + e^x.\frac{2D+1}{(2D-1)(2D+1)}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[1-\frac{6}{11}D-\frac{D^2}{11}+\frac{36}{121}D^2+.....\right]x^2 + e^x.\frac{2D+1}{4D^2-1}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[1-\frac{6}{11}D+\frac{25}{121}D^2+.....\right]x^2 + e^x.\frac{(2D+1)}{4(-4)-1}.\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[x^2-\frac{6}{11}D\left(x^2\right)+\frac{25}{121}D^2\left(x^2\right)\right] - \frac{e^x}{17}(2D+1)\cos 2x$$

$$= \frac{1}{11}e^{3x}\left[x^2-\frac{6}{11}.2x+\frac{25}{121}.2\right] - \frac{e^x}{17}\left[2D(\cos 2x)+\cos 2x\right]$$

$$= \frac{1}{11}e^{3x}\left[x^2-\frac{12}{11}x+\frac{50}{121}\right] - \frac{1.e^x}{17}\left[2(-2\sin 2x)+\cos 2x\right]$$

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$$= \frac{1}{11}e^{3x}\left[x^2 - \frac{12}{11}x + \frac{50}{121}\right] - \frac{1}{17}e^x\left[-4\sin 2x + \cos 2x\right]$$

$$= \frac{1}{11}e^{3x}\left[x^2 - \frac{12}{11}x + \frac{50}{121}\right] + \frac{1}{17}e^x\left[4\sin 2x - \cos 2x\right]$$

Thus, the required solution is:

$$y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{1}{11}e^{3x}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{1}{17}e^x\left(4\sin 2x - \cos 2x\right)$$

**Case V:** To calculate $\frac{1}{f(\mathrm{D})}.(x\mathrm{V})$, where V is any function of $x$.

Or prove that $\frac{1}{f(\mathrm{D})}.(x\mathrm{V}) = x.\frac{1}{f(\mathrm{D})}.\mathrm{V} + \frac{d}{d\mathrm{D}}\left[\frac{1}{f(\mathrm{D})}\right].\mathrm{V}$

**Example 4.64:** Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x.$

**Solution**: The given differential equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x.$

Symbolic form of the given equation is $\left(\mathrm{D}^2 - 2\mathrm{D} + 1\right)y = xe^x\sin x$

Its auxiliary equation is $\mathrm{D}^2 - 2\mathrm{D} + 1 = 0$

$\Rightarrow \quad (\mathrm{D}-1)^2 = 0 \Rightarrow \mathrm{D} = 1,1$

$\therefore$ C.F. $= (c_1 + c_2x)e^x$

And P.I. $= \frac{1}{\mathrm{D}^2 - 2\mathrm{D} + 1}\left(xe^x\sin x\right)$

$$= \frac{1}{(\mathrm{D}-1)^2}\left(xe^x\sin x\right)$$

$$= e^x.\frac{1}{(\mathrm{D}+1-1)^2}(x\sin x)$$

$$= e^x.\frac{1}{\mathrm{D}^2}(x\sin x)$$

$$= e^x\left[x.\frac{1}{\mathrm{D}^2}\sin x + \frac{d}{d\mathrm{D}}\left(\frac{1}{\mathrm{D}^2}\right)\sin x\right]$$

$$= e^x\left[x.\frac{1}{\mathrm{D}^2}\sin x - \frac{2}{\mathrm{D}^3}.\sin x\right]$$

$$= e^x\left[x.\frac{1}{-1}\sin x - \frac{2}{\mathrm{D}}\left(\frac{1}{\mathrm{D}^2}.\sin x\right)\right]$$

$$= e^x\left[-x\sin x - \frac{2}{\mathrm{D}}.\frac{1}{-1}\sin x\right]$$

$$= e^x\left[-x\sin x + \frac{2}{D}\sin x\right]$$

$$= e\left[-x\sin x + 2\int \sin x dx\right]$$

$$= e^x\left[-x\sin x + 2(-\cos x)\right] = -e^x\left[x\sin x + 2\cos x\right]$$

Thus, the required solution is $y = (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x)$.

NOTES

## 4.7 INTEGRATION

**Definitions**

After learning differentiation, we now come to the 'Reverse' process of it, namely integration. To give a precise shape to the definition of integration, we observe: If $g(x)$ is a function of $x$ such that,

$$\frac{d}{dx}\, g(x) = f(x)$$

then we define integral of $f(x)$ with respect to $x$, to be the function $g(x)$. This is put in the notational form as,

$$\int f(x)\, dx = g(x)$$

The function $f(x)$ is called the Integrand. Presence of $dx$ is there just to remind us that integration is being done with respect to $x$.

For example, since $\frac{d}{dx}\sin x = \cos x$

$$\int \cos x\, dx = \sin x$$

We get many such results as a direct consequence of the definition of integration, and can treat them as 'Formulas'. A list of such standard results are given:

(1) $\int 1\, dx = x$ because $\frac{d}{dx}(x) = 1$

(2) $\int x^n\, dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$ because $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$

(3) $\int \frac{1}{x}\, dx = \log x$ because $\frac{d}{dx}(\log x) = \frac{1}{x}$

(4) $\int e^x dx = e^x$ because $\frac{d}{dx}\,(e^x) = e^x$

(5) $\int \sin x\, dx = -\cos x$ because $\frac{d}{dx}\,(-\cos x) = \sin x$

(6) $\int \cos x\, dx = \sin x$ because $\frac{d}{dx}\,(\sin x) = \cos x$

(7) $\int \sec^2 x dx = \tan x$ because $\frac{d}{dx}\,(\tan x) = \sec^2 x$

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(8) $\int \text{cosec}^2 x\,dx = -\cot x$ because $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$

(9) $\int \sec x \tan x\,dx = \sec x$ because $\frac{d}{dx}(\sec x) = \sec x \tan x$

(10) $\int \text{cosec}\,x \cot x\,dx = -\text{cosec}\,x$ because $\frac{d}{dx}(-\text{cosec}\,x) = \text{cosec}\,x \cot x$

(11) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$ because $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(12) $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$ because $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(13) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x$ because $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(14) $\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a}$ because $\frac{d}{dx}\left[\frac{\log(ax+b)}{a}\right] = \frac{1}{ax+b}$

(15) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{n+1} \cdot \frac{1}{a} \quad (n \neq -1)$

because $\frac{d}{dx}\frac{(ax+b)^{n+1}}{a(n+1)} = (ax+b)^n,\ n \neq -1$

(16) $\int a^x\,dx = \frac{a^x}{\log a}$ because $\frac{d}{dx}a^x = a^x \log a$

One might wonder at this stage that since

$$\frac{d}{dx}(\sin x + 4) = \cos x$$

Then, by definition, why $\int \cos x\,dx$ is not $(\sin x + 4)$? In fact, there is nothing very sacred about number 4 and it could very well have been any constant. This suggests perhaps a small alteration in the definition.

We now define integration as:

If $\quad \frac{d}{dx}g(x) = f(x)$

Then, $\int f(x)\,dx = g(x) + c$

Where $c$ is **some** constant, called the *constant of integration.* Obviously, $c$ could have any value and thus, integral of a function is not unique! But, we could say one thing here, that *any two integrals of the same function differ by a constant.* Since $c$ could also have the value zero, $g(x)$ is one of the values of $\int f(x)dx$. By convention, we will not write the constant of integration (although it is there), and thus, $\int f(x)dx = g(x)$, and our definition stands.

The above is also referred to as **Indefinite Integral** (indefinite, because we are not really giving a definite value to the integral by not writing the constant of integration). We will give the definition of a definite integral later.

NOTES

**Some Properties of Integration**

($i$) *Differentiation and integration cancel each other*.

The result is clear by the definition of integration.

Let $\frac{d}{dx}\ g(x) = f(x)$

Then, $\int f(x)\ dx = g(x)$ [by definition]

$$\Rightarrow \quad \frac{d}{dx}\,[f(x)\,dx] = \frac{d}{dx}\ [g(x)] = f(x)$$

Which proves the result.

($ii$) *For any constant a*, $\int a\ f(x)\,dx = a\int f(x)\,dx$

Since $\frac{d}{dx}\ (a\int f(x)dx) = a\ \frac{d}{dx}\ \int f(x)\ dx$

$= a\,f(x)$

By definition, $\int\ a\ f(x)\ dx = a\ \int f(x)\ dx$

($iii$) *For any two functions* $f(x)$ and $g(x)$,

$$\int [\ f(x)\ \pm g(x)]\ dx = \int f(x)\ dx\ \pm\ \int g(x)\ dx$$

As $\frac{d}{dx}\left[\int f(x)\,dx \pm \int g(x)dx\right] = \frac{d}{dx}\ \int f(x)\ dx \pm \frac{d}{dx}\ \int g(x)\ dx$

$= f(x) \pm g(x)$

It follows by definition that,

$$\int [\ f(x) \pm g(x)]dx\ = \int\ f(x)\,dx \pm \int g(x)dx$$

This result could be extended to a finite number of functions, i.e., in general,

$$\int [\ f_1\ (x) \pm f_2\ (x)\ \pm \ldots \pm f_n\ (x)]dx = \int\ f_1\ (x)\ dx \pm \int f_2(x)dx\ \pm \ldots\ \pm \int\ f_n\ (x)\ dx$$

Let us now solve some problems to illustrate the use of these results.

**Example 4.65:** Find $\int (2x-3)^2\ dx$.

**Solution:** We have,

$$\int (2x-3)^2\ dx = \int (4x^2 + 9 - 12x)\ dx$$

$$= \int 4x^2\ dx + \int 9\ dx - \int 12x\ dx$$

$$= 4\int x^2 dx + 9\int dx - 12\ \int x dx$$

$$= 4\frac{x^3}{3} + 9x - \frac{12x^2}{2}$$

$$= \frac{4}{3}\ x^3 - 6x^2 + 9x$$

**NOTES**

**Example 4.66:** Find $\int (2x+1)^{1/3}\, dx$.

**Solution:** We have,

$$\int (2x+1)^{1/3}\, dx = \frac{(2x+1)^{1/3+1}}{\frac{1}{3}+1} \times \frac{1}{2}$$

$$= \frac{3}{8}(2x+1)^{4/3}$$

**Example 4.67:** Solve $\int \frac{x^3}{x+1}\, dx$.

**Solution:** By division, we note

$$\frac{x^3}{x+1} = (x^2 - x + 1) - \frac{1}{x+1}$$

Thus,

$$\int \frac{x^3}{x+1}\, dx = \int (x^2 - x + 1)\, dx - \int \frac{1}{x+1}\, dx$$

$$= \int x^2 dx - \int x dx + \int dx - \int \frac{1}{x+1}\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log (x+1)$$

**Example 4.68:** Find $\int \sqrt{1+\cos 2x}\, dx$

**Solution:** We observe,

$$\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int \cos x\, dx$$

$$= \sqrt{2}\sin x$$

**Example 4.69:** Evaluate $\int \left[\sqrt{x} - \frac{1}{\sqrt{x}}\right] dx$

**Solution:**

$$\int \left[\sqrt{x} - \frac{1}{\sqrt{x}}\right] dx = \int \sqrt{x} dx - \int \frac{1}{\sqrt{x}}\, dx$$

$$= \frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{-1/2+1}}{-\frac{1}{2}+1}$$

$$= \frac{2}{3}x^{3/2} - 2x^{½}$$

$$= \frac{2\sqrt{x}}{3}(x-3)$$

**Example 4.70:** Evaluate,

(*i*) $\int \left(\frac{ax^3 + bx^2 + cx + d}{x}\right) dx$ (*ii*) $\int \sin^3 x d\, x$

**Solution:** We have,

(*i*) $$\int \left(\frac{ax^3 + bx^2 + cx + d}{x}\right) dx = \int \left(ax^2 + bx + c + \frac{d}{x}\right) dx$$

$$= a\int x^2\, dx + b\int x\, dx + c\int dx + d\int \frac{1}{x}\, dx$$

$$= a\frac{x^3}{3} + \frac{bx^2}{2} + cx + d\log x$$

NOTES

(*ii*) We have,

$$\int \sin^3 x\, dx = \frac{1}{4}\int (3\sin x - \sin 3x)\ dx$$

$$= \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3\, x dx$$

$$= -\frac{3}{4}\cos x + \frac{\cos 3x}{12}$$

$$= \frac{1}{12}(\cos 3x - 9\cos x)$$

***Note:*** Remember that as $\frac{1}{a}\frac{d}{dx}$ $\cos ax = -\sin ax$

$$\int \sin ax\, dx = -\frac{\cos ax}{a}$$

Similarly, $\int \cos ax\, dx = \frac{\sin ax}{a}$ and $\int e^{bx}\, dx = \frac{e^{bx}}{b}$.

## To Evaluate $\int \frac{f'(x)}{f(x)}$ dx where $f'(x)$ is the derivative of $f(x)$

Put $f(x) = t$, then $f'(x)dx = dt$

Thus, $\int \frac{f'(x)}{f(x)}\ dx = \int \frac{dt}{t} = \log t = \log f(x)$

## To Evaluate $\int [f(x)]^n\ f'(x)dx,\ n \neq -1$

Put $f(x) = t$, then $f'(x)dx = dt$

Thus, $\int [f(x)^n\ f'(x)dx \quad \int t^n dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$

## To Evaluate $\int f'(ax+b)dx$

Put $ax + b = t$, then, $adx = dt$

$$\int f'(ax+b)dx = \int f'(t)\frac{dt}{a} = \frac{1}{a}\int f'(t)\, dt = \frac{f(t)}{a} = \frac{f(ax+b)}{a}$$

**Example 4.71:** Evaluate (*i*) $\int \tan x dx$ (*ii*) $\int \sec x dx$

**Solution:** (*i*) $\int \tan x dx = \int \frac{\sec x \tan x}{\sec x} dx = \log \sec x$

(*ii*) $\int \sec x dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x} dx = \log(\sec x + \tan x)$

**Example 4.72:** Find $\int x\sqrt{x^2+1}\, dx$

**Solution:** We have,

$$\int x\sqrt{x^2+1}\, dx = \frac{1}{2}\int (2x)\,(x^2+1)^{\frac{1}{2}}\ dx$$

NOTES

$$= \frac{1}{2} \frac{(x^2 + 1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \quad \text{[Using 7.2.4]}$$

$$= \frac{1}{3}(x^2 + 1)^{3/2}$$

**Example 4.73:** Evaluate $\int \frac{x+1}{x^2+2x+3}\, dx$

**Solution:** We have, $\int \frac{x+1}{x^2+2x+3}\, dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\, dx$

$$= \frac{1}{2}\log(x^2 + 2x + 3)$$

**Example 4.74:** Evaluate (*i*) $\int \cos x \cos 2x\, dx$

(*ii*) $\int \sin 4x \cos 2x\, dx$

**Solution:** (*i*) $\int \cos x \cos 2x dx = \frac{1}{2}\int 2\cos x \cos 2x\, dx$

$$= \frac{1}{2}\int (\cos 3x + \cos x)\, dx$$

$$= \frac{1}{2}\int \cos 3x\, dx + \frac{1}{2}\int \cos x\, dx = \frac{1}{2}\frac{\sin 3x}{3} + \frac{1}{2}\sin x$$

(*ii*) We have,

$$\sin 4x \cos 2x = \frac{1}{2} 2\sin 4x \cos 2x$$

$$= \frac{1}{2}[\sin(4x + 2x) + \sin(4x - 2x)]$$

$$= \frac{1}{2}[\sin 6x + \sin 2x]$$

Thus,

$$\int \sin 4x \cos 2\, x dx = \frac{1}{2}\int (\sin 6x + \sin 2x) dx$$

$$= \frac{1}{2}\int \sin 6\, x dx + \frac{1}{2}\int \sin 2x dx$$

$$= \frac{-\cos 6x}{12} - \frac{\cos 2x}{4}$$

$$= -\frac{1}{4}\left[\frac{\cos 6x}{3} + \cos 2x\right]$$

**Six Important Integrals**

We will now evaluate the following six integrals:

(*i*) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ (*ii*) $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$ (*iii*) $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

(*iv*) $\int \sqrt{a^2 - x^2}\, dx$ (*v*) $\int \sqrt{a^2 + x^2}\, dx$ (*vi*) $\int \sqrt{x^2 - a^2}\, dx$

(*i*) To evaluate $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$

Put $x = a \sin\theta$, then, $dx = a\cos\theta\, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \int 1.\, d\theta = \theta$$

$$= \sin^{-1}\left(\frac{x}{a}\right)$$

(*ii*) To evaluate $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$

Put $x = a \sin h\, \theta$, then $dx = a \cosh\theta\, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \int \frac{a\cos h\,\theta\, d\theta}{\sqrt{a^2 + a^2\sin h^2\theta}} = \int \frac{a\cos h\,\theta}{a\cos h\,\theta}\, d\theta$$

as $\cos h^2\theta - \sin h^2\theta = 1$

$$= \int d\theta = \theta = \sin h^{-1}\left(\frac{x}{a}\right)$$

***Aliter:*** Put $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2 + a^2\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta$$

$$= \log(\sec\theta + \tan\theta)$$

$$= \log\left[\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right]$$

$$= \log\left[\frac{x + \sqrt{x^2 + a^2}}{a}\right]$$

(*iii*) To evaluate $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$

Put $x = a\cos h\,\theta$, then $dx = a\sin h\,\theta\, d\theta$.

Thus,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{a\sin h\,\theta\, d\theta}{\sqrt{a^2\cos h^2\theta - a^2}} = \int \frac{a\sin h\,\theta}{a\sin h\,\theta}\, d\theta = \int d\theta$$

$$= \theta = \cos h^{-1}\frac{x}{a}$$

***Aliter:*** Put $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\, d\theta$.

Thus,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \sec\theta\, d\theta$$

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$$= \log (\sec \theta + \tan \theta)$$

$$= \log \left[ \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right]$$

$$= \log \left[ \frac{x + \sqrt{x^2 - a^2}}{a} \right]$$

(*iv*) To evaluate $\int \sqrt{a^2 - x^2}\ dx$

Put $x = a \sin \theta$, then $dx = a \cos \theta\, d\theta$.

Thus,

$$\int \sqrt{a^2 - x^2}\ dx = \int \sqrt{a^2 - a^2 \sin^2 \theta}\ .\ a \cos \theta\, d\theta = \int a^2 \cos^2 \theta\, d\theta$$

$$= a^2 \int \left( \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^2}{2} \left( \theta + \frac{\sin 2\theta}{2} \right)$$

$$= \frac{a^2}{2} (\theta + \sin \theta \cos \theta)$$

$$= \frac{a^2}{2} \left( \theta + \sin \theta \sqrt{1 - \sin^2 \theta} \right)$$

$$= \frac{a^2}{2} \left[ \sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} \right]$$

And hence,

$$\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

(*v*) To evaluate $\int \sqrt{a^2 + x^2}\ dx$

Put $x = a \sin h\, \theta$, then $dx = a \cos h\, \theta\, d\theta$

Thus,

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 + a^2 \sin h^2 \theta}\ .\ a \cos h\, \theta\, d\theta$$

$$= \int a^2 \cos h^2 \theta\, d\theta$$

$$= a^2 \int \frac{(\cos h\, 2\theta + 1)}{2}\, d\theta$$

(As $2 \cos h^2\theta = 1 + \cos h\, 2\theta$)

$$= \frac{a^2}{2} \left( \frac{\sin h\, 2\theta}{2} + \theta \right)$$

$$= \frac{a^2}{2} [\sin h\, \theta \cos h\, \theta + \theta]$$

(As $\sin h\, 2\theta = 2 \sin h\, \theta \cos h\, \theta$)



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$$= \frac{a^2}{2}\left[\sin h\, \theta \sqrt{1+\sin h^2\theta} + \theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \sin h^{-1} \frac{x}{a}\right]$$

And hence,

$$\int \sqrt{a^2+x^2}\ dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a}$$

***Aliter:*** Put $x = a \tan \theta$, then $dx = a \sec^2 \theta\, d\theta$

Thus,

$$\int \sqrt{a^2+x^2}\, dx = \int \sqrt{a^2+a^2 \tan^2\theta}\ .\ a \sec^2\theta\, d\theta$$

$$= \int a^2 \sec^3 \theta\, d\theta$$

$$= \frac{a^2}{2}[\sec\theta \tan\theta + \log(\sec\theta + \tan\theta)]$$

$$= \frac{a^2}{2}\sqrt{1+\frac{x^2}{a^2}}\ .\ \frac{x}{a} + \frac{a^2}{2}\log\left[\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right]$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2+a^2}}{a}\right]$$

(*vi*) To evaluate $\int \sqrt{x^2-a^2}\ dx$

Put $x = a \cos h\, \theta$, then $dx = a \sin h\, \theta\, d\theta$

Thus,

$$\int \sqrt{x^2-a^2}\ dx = \int \sqrt{a^2 \cos h^2\, \theta - a^2}\ \ a \sin h\, \theta\, d\theta$$

$$= a^2 \int \sin h^2\, \theta\, d\theta$$

$$= a^2 \int \frac{(\cos h\, 2\theta - 1)}{2} d\theta$$

$$= \frac{a^2}{2}\left(\frac{\sin h\, 2\theta}{2} - \theta\right)$$

$$= \frac{a^2}{2}(\sin h\, \theta \cos h\, \theta - \theta)$$

$$= \frac{a^2}{2}[\sqrt{\cos h^2\theta - 1}\ .\ \cos h\, \theta - \theta]$$

$$= \frac{a^2}{2}\left[\sqrt{\frac{x^2}{a^2}-1}.\ \frac{x}{a} - \cos h^{-1} x/a\right]$$

And hence,

$$\int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cos h^{-1} x/a.$$

**NOTES**

***Aliter:*** Put $x = a \sec\theta$, then $dx = a \sec\theta \tan\theta \, d\theta$

Thus

$$\int \sqrt{x^2 - a^2}\, dx = \int \sqrt{a^2 \sec^2\theta - a^2}\; a \sec\theta \tan\theta \, d\theta$$

$$= \int a^2 \sec\theta \,.\, \tan^2\theta \, d\theta$$

$$= a^2 \int \sec\theta \, (\sec^2\theta - 1)\, d\theta$$

$$= a^2 \int \sec^3\theta \, d\theta - a^2 \int \sec\theta \, d\theta$$

$$= \frac{a^2}{2}\left[\sec\theta \tan\theta + \log(\sec\theta + \tan\theta)\right]$$

$$-a^2 \log(\sec\theta + \tan\theta) \text{ [as in the previous case]}$$

$$= \frac{a^2}{2} \sec\theta \tan\theta - \frac{a^2}{2} \log(\sec\theta + \tan\theta)$$

$$= \frac{a^2}{2}\, \frac{x}{a} \sqrt{\frac{x^2}{a^2} - 1} - \frac{a^2}{2} \log\left[\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right]$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left[\frac{x + \sqrt{x^2 - a^2}}{a}\right]$$

**Thus, we get six results to remember:**

(*i*) $\displaystyle \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}$

(*ii*) $\displaystyle \int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \sin h^{-1}\frac{x}{a}$

$$= \log\left[\frac{x + \sqrt{x^2 + a^2}}{a}\right]$$

(*iii*) $\displaystyle \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \cos h^{-1}\frac{x}{a}$

$$= \log\left[\frac{x + \sqrt{x^2 - a^2}}{a}\right]$$

(*iv*) $\displaystyle \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(*v*) $\displaystyle \int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a}$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left[\frac{x + \sqrt{a^2 + x^2}}{a}\right]$$

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$$(vi) \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cos h^{-1}\frac{x}{a}$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[\frac{x + \sqrt{x^2 - a^2}}{a}\right]$$

**Example 4.75:** Solve $\int \frac{1}{\sqrt{x^2 + x + 1}}\, dx.$

**Solution:** We have,

$$I = \int \frac{1}{\sqrt{x^2 + x + 1}}\ dx = \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}\ dx$$

Put $x + \frac{1}{2} = t$, then $dx = dt$

Thus,

$$I = \int \frac{dt}{\sqrt{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} = \sin h^{-1}\frac{t}{\sqrt{3}/2}$$

(By the second integral evaluated above)

$$= \sin h^{-1}\frac{x + 1/2}{\sqrt{3}/2} = \sin h^{-1}\frac{2x + 1}{\sqrt{3}}$$

The above result could, of course, be written directly without actually making the substitution $x + \frac{1}{2} = t$, by taking $x$ as $x + \frac{1}{2}$ in the formula.

**Example 4.76:** Evaluate $\int \sqrt{-3x^2 + x + 2}\ dx.$

**Solution:** $\int \sqrt{-3x^2 + x + 2}\ dx = \int \sqrt{3}\sqrt{-x^2 + \frac{x}{3} + \frac{2}{3}}\ dx$

$$= \sqrt{3}\int \sqrt{\frac{2}{3} - \left(x^2 - \frac{x}{3}\right)}\ dx$$

$$= \sqrt{3}\int \sqrt{\left(\frac{2}{3} + \frac{1}{36}\right) - \left(x - \frac{1}{6}\right)^2}\ dx$$

$$= \sqrt{3}\int \sqrt{\left(\frac{5}{6}\right)^2 - \left(x - \frac{1}{6}\right)^2}\, dx$$

$$= \sqrt{3}\left[\frac{x - \frac{1}{6}}{2}\sqrt{\left(\frac{5}{6}\right)^2 - \left(x - \frac{1}{6}\right)^2} + \frac{\left(\frac{5}{6}\right)^2}{2}\sin^{-1}\frac{x - \frac{1}{6}}{5/6}\right]$$

$$= \frac{1}{12}(6x - 1)\sqrt{-3x^2 + x + 2} + \frac{25}{72}\sqrt{3}\sin^{-1}\left(\frac{6x - 1}{5}\right).$$

**Example 4.77:** Evaluate $\int \frac{4x-3}{\sqrt{x^2+1}}\, dx$.

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**Solution:** We have,

$$\int \frac{4x-3}{\sqrt{x^2+1}}\, dx = \int \frac{4x}{\sqrt{x^2+1}}\, dx - 3\int \frac{1}{\sqrt{x^2+1}}\, dx$$

$$= 4 \int x(x^2+1)^{-1/2}\, dx - 3 \sin h^{-1} x$$

$$= 2 \int (2x)\,(x^2+1)^{-1/2} dx - 3 \sin h^{-1} x$$

$$= 2\, \frac{(x^2+1)^{-1/2+1}}{-\frac{1}{2}+1} - 3 \sin h^{-1} x$$

$$= 4\sqrt{x^2+1} - 3 \sin h^{-1} x$$

**Example 4.78:** Evaluate $\int (4x+5)\sqrt{x^2+4x+9}\, dx$

**Solution:** We have, $\int (4x+5)\sqrt{x^2+4x+9}\, dx = \int 4x\sqrt{x^2+4x+9}\, dx +$

$$5\int \sqrt{x^2+4x+9}\, dx$$

$$= 2\int (2x+4-4)\sqrt{x^2+4x+9}\, dx + 5\int \sqrt{x^2+4x+9}\, dx$$

$$= 2\int (2x+4)\sqrt{x^2+4x+9}\, dx - 3\int \sqrt{x^2+4x+9}\, dx$$

$$= 2\int (2x+4)\,(x^2+4x+9)^{1/2}\, dx - 3\int \sqrt{(x+2)^2+5}\, dx$$

$$= 2\,\frac{(x^2+4x+9)^{1/2+1}}{\frac{1}{2}+1} - 3\left[\frac{x+2}{2}\sqrt{x^2+4x+9} + \frac{5}{2}\sin h^{-1}\frac{x+2}{\sqrt{5}}\right]$$

$$= \frac{4}{3}(x^2+4x+9)^{3/2} - 3\left[\frac{x+2}{2}\sqrt{x^2+4x+9} + \frac{5}{2}\sin h^{-1}\frac{x+2}{\sqrt{5}}\right]$$

**Example 4.79:** Evaluate $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\, dx$.

**Solution:** Let us determine three constants $\lambda$, $\mu$, $\nu$, such that

$$x^2 + 2x + 3 = \lambda\,(x^2 + x + 1) + \mu(2x + 1) + \nu$$

Where, $2x + 1$ = derivative of $x^2 + x + 1$

Comparing coefficients of $x^2$, $x$ and the constant terms,

We get, $\lambda = 1,\ \lambda + 2\mu = 2,\ \lambda + \mu + \nu = 3$

$\Rightarrow\ \lambda = 1,\ \mu = \frac{1}{2},\ \nu = \frac{3}{2}$

Thus,

$$x^2 + 2x + 3 = (x^2 + x + 1) + \frac{1}{2}\,(2x + 1) + \frac{3}{2}$$

$$\Rightarrow \quad \frac{x^2+2x+3}{\sqrt{x^2+x+1}} = \frac{x^2+x+1}{\sqrt{x^2+x+1}} + \frac{1}{2}\,\frac{(2x+1)}{\sqrt{x^2+x+1}} + \frac{3}{2}\,\frac{1}{\sqrt{x^2+x+1}}$$

$$\Rightarrow \int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}dx = \int \sqrt{x^2+x+1}\, dx + \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\, dx + \frac{3}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx + \frac{1}{2}\int (2x+1)\,(x^2+x+1)^{-1/2} dx$$

$$+ \frac{3}{2}\int \sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}\, dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{2}\sin h^{-1}\frac{x+\frac{1}{2}}{\sqrt{3}/2} + \frac{1}{2}\,\frac{(x^2+x+1)^{-1/2+1}}{-\frac{1}{2}+1} + \frac{3}{2}\sin h^{-1}\frac{x+\frac{1}{2}}{\sqrt{3}/2}$$

$$= \frac{2x+5}{4}\sqrt{x^2+x+1} + \frac{15}{8}\sin h^{-1}\frac{2x+1}{\sqrt{3}}$$

**Methods of Integration**

In the previous sections, we have seen that many integrals can be evaluated by using definition of integration. Still a good number of problems cannot be solved in this manner. We will try to list a few methods by which integration is made possible.

**Methods of Substitution**

In this method, we express the given integral $\int f(x)\,dx$ in terms of another integral in which the independent variable $x$ is changed to another variable $t$ through some suitable relation $x = \phi(t)$.

Let $\quad I = \int f(x)\, dx$

$$\frac{dI}{dx} = f(x)$$

$$\Rightarrow \quad \frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = f(x)\,\frac{dx}{dt}$$

Thus, $\quad I = \int f(x)\frac{dx}{dt}\,.\,dt = \int f\,[\varphi(t)]\,\varphi'(t)\,dt$

Note that we replace $dx$ by $\varphi'(t)\,dt$, which we get from the relation $\frac{dx}{dt} = \varphi'(t)$ by *assuming* that $dx$ and $dt$ can be separated.

In fact, this is done only for convenience.

The following example will make the process clear.

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**Example 4.80:** Integrate $x(x^2 + 1)^3$.

**Solution:** Put $x^2 + 1 = t \Rightarrow 2x\frac{dx}{dt} = 1$

Thus, $\Rightarrow 2xdx = dt$

$$\int x(x^2+1)^3\,dx = \int \frac{1}{2}t^3\ dt = \frac{1}{2}\int t^3\ dt = \frac{1}{2}\frac{t^4}{4} = \frac{t^4}{8} = \frac{(x^2+1)^4}{8}$$

**Example 4.81:** Find $\int e^{\tan\theta}\ \sec^2\theta\ d\theta$.

**Solution:** Put $\tan\theta = t$, then $\sec^2\theta\ d\theta = dt$

Thus, $\int e^{\tan\theta}\ \sec^2\theta\ d\theta = \int e^t\ dt = e^t = e^{\tan\theta}$

**Example 4.82:** Integrate $\frac{\cos x}{1+\sin^2 x}$

**Solution:** Put $\sin x = t$, then $\cos x\ dx = dt$

Thus, $\int \frac{\cos x}{1+\sin^2 x}\ dx = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1}(\sin x)$.

**Example 4.83:** Integrate $\frac{4x^2}{\sqrt{1-x^6}}$

**Solution:** Put $x^3 = t$ then $3x^2\,dx = dt$

Thus, $\int \frac{4x^2}{\sqrt{1-x^6}}\ dx = \frac{4}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{4}{3}\sin^{-1} t = \frac{4}{3}\sin^{-1}(x^3)$

## 4.8 LAWS OF THERMODYNAMICS AND THEIR APPLICATIONS IN BIOLOGICAL SYSTEM

The first law of thermodynamics has certain limitations as illustrated below:

The first law establishes definite relationship between the heat absorbed and the work performed by a system in a given process. But it pits no restriction on the direction of the flow of heat. According to the first law, for example, it is not impossible to extract heat from ice by cooling it to a low temperature and then use it for wanning water. But it is known from experience that such a transfer of heat from a lower to a higher temperature is not possible without expenditure of energy that is without doing some external work. It's known, on the other hand, that heat flows spontaneously, that is, of its own accord, from a higher to a lower temperature.

According to the first law, the total energy of an isolated system remains constant during a specified change of state. But it does not tell whether a specified change of or a process including a chemical reaction can occur spontaneously, whether it is feasible?

The first law states that energy of one form can be converted into an equivalent amount of energy of another form. But it does not tell that heat energy cannot be

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completely converted into an equivalent amount of work. There is thus need for another law, i.e., the second law of thermodynamics.

The second law of thermodynamics helps us to determine the direction in which energy can be transferred. It also helps us to predict whether a given process or a chemical reaction can occur spontaneously, that is, its own accord. It also helps us to know the equilibrium conditions. The law is therefore, of great importance in chemistry.

It is known from experience that although various forms of energy can be completely transformed into one another, yet heat is a typical form of energy which cannot be transformed into work. The second law helps us to calculate the maximum fraction of heat that can be converted into work in a given process.

The second law of thermodynamics is needed because the first law of thermodynamics does not define the energy conversion process completely. The first law is used to relate and to evaluate the various energies involved in a process. However, no information about the direction of the process can be obtained by the application of the first law. Early in the development of the science of thermodynamics, investigators noted that while work could be converted completely into heat, the converse was never true for a cyclic process. Certain natural processes were also observed always to proceed in a certain direction (for example, heat transfer occurs from a hot to a cold body). The second law was developed as an explanation of these natural phenomena.

The second law of thermodynamics is used to determine the maximum efficiency of any process. A comparison can then be made between the maximum possible efficiency and the actual efficiency obtained.

With the second law of thermodynamics, the limitations imposed on any process can be studied to determine the maximum possible efficiencies of such a process and then a comparison can be made between the maximum possible efficiency and the actual efficiency achieved. One of the areas of application of the second law is the study of energy-conversion systems. For example, it is not possible to convert all the energy obtained from a nuclear reactor into electrical energy. There must be losses in the conversion process. The second law can be used to derive an expression for the maximum possible energy conversion efficiency taking those losses into account. Therefore, the second law denies the possibility of completely converting into work all of the heat supplied to a system operating in a cycle, no matter how perfectly designed the system may be.

The second law of thermodynamics can be stated in a number of ways. The original statements were concerned with the French engineer Sadi Carnot's analysis of performance of steam engines. There are two important statements of the second law; one is due to Lord Kelvin and other is due to Clausius. The statement due to Kelvin may be expressed as:

It is impossible for a cyclic process to take heat from a cold reservoir and convert it into work without at the same time transferring heat from a hot to a cold reservoir.

The statement due to Clausius is:

It is impossible to construct a machine, which is able to convey heat by a cyclic process from one reservoir at a lower temperature to another at higher temperature unless work is done on the machine by some outside agency.

A cycle is a process in which a system returns to its original state after a succession of steps. A more useful statement of the second law can be given in terms of a quantity which is also a function of state called, *entropy*.

### Principal and Applications of Thermodynamics

The laws of thermodynamics define a group of physical quantities, such as temperature, energy, and entropy that characterise thermodynamic systems in thermodynamic equilibrium. The laws also use various parameters for thermodynamic processes, such as thermodynamic work and heat, and establish relationships between them. They state empirical facts that form a basis of precluding the possibility of certain phenomena, such as perpetual motion. In addition to their use in thermodynamics, they are important fundamental laws of physics in general, and are applicable in other natural sciences.

Traditionally, thermodynamics has recognised three fundamental laws, simply named by an ordinal identification, the first law, the second law, and the third law. A more fundamental statement was later labelled as the zeroth law, after the first three laws had been established.

The ***zeroth law of thermodynamics*** defines thermal equilibrium and forms a basis for the definition of temperature: If two systems are each in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.

The ***first law of thermodynamics*** states that, when energy passes into or out of a system (as work, heat, or matter), the system's internal energy changes in according with the law of conservation of energy.

The ***second law of thermodynamics*** states that in a natural thermodynamic process, the sum of the entropies of the interacting thermodynamic systems never decreases. Another form of the statement is that heat does not spontaneously pass from a colder body to a warmer body.

The ***third law of thermodynamics states*** that a system's entropy approaches a constant value as the temperature approaches absolute zero. With the exception of non-crystalline solids (glasses) the entropy of a system at absolute zero is typically close to zero.

The first and second law prohibit two kinds of perpetual motion machines, respectively: the perpetual motion machine of the first kind which produces work with no energy input, and the perpetual motion machine of the second kind which spontaneously converts thermal energy into mechanical work.

### Zeroth Law of Thermodynamics

The zeroth law of thermodynamics provides for the foundation of temperature as an empirical parameter in thermodynamic systems and establishes the transitive relation between the temperatures of multiple bodies in thermal equilibrium. The law may be stated in the following form:

- If two systems are both in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.
- Though this version of the law is one of the most commonly stated versions, it is only one of a diversity of statements that are labelled as the 'Zeroth Law'. Some statements go further, so as to supply the important physical fact that temperature is one-dimensional and that one can conceptually arrange bodies in a real number sequence from colder to hotter.
- These concepts of temperature and of thermal equilibrium are fundamental to thermodynamics and were clearly stated in the nineteenth century. The name 'Zeroth Law' was invented by Ralph H. Fowler in the 1930s, long after the first, second, and third laws were widely recognised. The law allows the definition of temperature in a non-circular way without reference to entropy, its conjugate variable. Such a temperature definition is said to be 'Empirical'.

**First Law of Thermodynamics**

The first law of thermodynamics is a related to the law of conservation of energy, adapted for thermodynamic processes. In general, the conservation law states that the total energy of an isolated system is constant; energy can be transformed from one form to another, but can be neither created nor destroyed.

In a closed system (i.e. there is no transfer of matter into or out of the system), the first law states that the change in internal energy of the system ($\Delta U_{\text{system}}$) is equal to the difference between the heat supplied to the system *(Q)* and the work *(W)* done by the system on its surroundings.

$$\Delta U_{system} = Q - W$$

When two initially isolated systems are combined into a new system, then the total internal energy of the new system, $U_{system}$, will be equal to the sum of the internal energies of the two initial systems, $U_1$ and $U_2$:

$$U_{system} = U_1 + U_2$$

**The First Law Encompasses Several Principles:**

- The Conservation of energy, which says that energy can be neither created nor destroyed, but can only change one form to another form. A particular consequence of this is that the total energy of an isolated system does not change.
- The concept of internal energy and its relationship to temperature. If a system has a definite temperature, then its total energy has three distinguishable components, termed kinetic energy (energy due to the motion of the system as a whole), potential energy (energy resulting from an externally imposed force field), and internal energy. The establishment of the concept of internal energy distinguishes the first law of thermodynamics from the more general law of conservation of energy.

$$E_{\text{total}} = \text{KE}_{\text{system}} + \text{PE}_{\text{system}} + U_{\text{system}}$$

- Work is a process of transferring energy to or from a system in ways that can be described by macroscopic mechanical forces acting between the system and its surroundings. The work done by the system can come from its overall kinetic energy, from its overall potential energy, or from its internal energy.

  For example, when a machine (not a part of the system) lifts a system upwards, some energy is transferred from the machine to the system. The system's energy increases as work is done on the system and in this particular case, the energy increase of the system is manifested as an increase in the system's gravitational potential energy. Work added to the system increases the potential energy of the system:

  When matter is transferred into a system that masses associated internal energy and potential energy are transferred with it.

  $(u \Delta M)_{\text{in}} = \Delta U_{system}$

  Where $u$ denotes the internal energy per unit mass of the transferred matter, as measured while in the surroundings; and $\Delta M$ denotes the amount of transferred mass.

- The flow of heat is a form of energy transfer. Heating is the natural process of moving energy to or from a system other than by work or the transfer of matter. In a diathermic system, the internal energy can only be changed by the transfer of energy as heat:

  $\Delta U_{system} = Q$

  Combining these principles leads to one traditional statement of the first law of thermodynamics: it is not possible to construct a machine which will perpetually output work without an equal amount of energy input to that machine. Or more briefly, a perpetual motion machine of the first kind is impossible.

**Second Law of Thermodynamics**

The second law of thermodynamics indicates the irreversibility of natural processes, and, in many cases, the tendency of natural processes to lead towards spatial homogeneity of matter and energy, and especially of temperature. It can be formulated in a variety of interesting and important ways. One of the simplest is the Clausius statement that heat does not spontaneously pass from a colder to a hotter body.

It implies the existence of a quantity called the entropy of a thermodynamic system. In terms of this quantity it implies that when two initially isolated systems in separate but nearby regions of space, each in thermodynamic equilibrium with itself but not necessarily with each other, are then allowed to interact, they will eventually reach a mutual thermodynamic equilibrium. The sum of the entropies of the initially isolated systems is less than or equal to the total entropy of the final combination. Equality occurs just when the two original systems have all their respective intensive variables (temperature, pressure) equal; then the final system also has the same values.

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The second law is applicable to a wide variety of processes, both *reversible* and *irreversible*. According to the second law, in a reversible heat transfer, an element of heat transferred, $\delta Q$, is the product of the temperature *(T),* both of the system and of the sources or destination of the heat, with the increment *(dS)* of the system's conjugate variable, its entropy *(S):*

$$\delta Q = T\,dS$$

While reversible processes are a useful and convenient theoretical limiting case, all natural processes are irreversible. A prime example of this irreversibility is the transfer of heat by conduction or radiation. It was known long before the discovery of the notion of entropy that when two bodies, initially of different temperatures, come into direct thermal connection, then heat immediately and spontaneously flows from the hotter body to the colder one.

Entropy may also be viewed as a physical measure concerning the microscopic details of the motion and configuration of a system, when only the macroscopic states are known. Such details are often referred to as disorder on a microscopic or molecular scale, and less often as dispersal of energy. For two given macroscopically specified states of a system, there is a mathematically defined quantity called the difference of information entropy between them. This defines how much additional microscopic physical information is needed to specify one of the macroscopically specified states, given the macroscopic specification of the other often a conveniently chosen reference state which may be presupposed to exist rather than explicitly stated. A final condition of a natural process always contains microscopically specifiable effects which are not fully and exactly predictable from the macroscopic specification of the initial condition of the process. This is why entropy increases in natural processes the increase tells how much extra microscopic information is needed to distinguish the initial macroscopically specified state from the final macroscopically specified state. Equivalently, in a thermodynamic process, energy spreads.

### Third Law of Thermodynamics

The third law of thermodynamics can be stated as: A system's entropy approaches a constant value as its temperature approaches absolute zero.

At zero temperature, the system must be in the state with the minimum thermal energy, the ground state. The constant value (not necessarily zero) of entropy at this point is called the *residual entropy* of the system. Note that, with the exception of non-crystalline solids (e.g. glasses) the residual entropy of a system is typically close to zero. However, it reaches zero only when the system has a unique ground state (i.e. the state with the minimum thermal energy has only one configuration, or microstate). Microstates are used here to describe the probability of a system being in a specific state, as each microstate is assumed to have the same probability of occurring, so macroscopic states with fewer microstates are less probable. In general, entropy is related to the number of possible microstates according to the Boltzmann principle:

$$S = k_B \ln \Omega$$

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Where $S$ is the entropy of the system, $k_B$ Boltzmann's constant, and Ω the number of microstates. At absolute zero there is only 1 microstate possible ($\Omega=1$ as all the atoms are identical for a pure substance and as a result all orders are identical as there is only one combination) and $ln(1)=0$

**Applications of Thermodynamics**

- Sweating in a Crowded Room: In a crowded room, everybody (every person) starts sweating. The body starts cooling down by transferring the body heat to the sweat. Sweat evaporates adding heat to the room. Again, this happens due to the first and second law of thermodynamics in action. One thing to keep in mind, heat is not lost but transferred attaining equilibrium with maximum entropy.
- The first law of thermodynamics proclaims constancy of the total energy of isolated system for all changes, taking place in this system: energy cannot be created or destroyed. According to the second law of thermodynamics in isolated system entropy is always increasing or remaining constant. All processes in the Universe are oriented to the equilibrium state. Nevertheless, biological systems, and, consequently, ecological systems create order from disorder, they create and support chemical and physical non-equilibrium state the basis they live on.
- According to the second law, heat always flows from a body at a higher temperature to a body at the lower temperature. This law is applicable to all types of heat engine cycles including Otto, Diesel, etc., for all types of working fluids used in the engines. This law has led to the progress of present-day vehicles.
- Another application of second law is refrigerators and heat pumps based on the Reversed Carnot Cycle. If you want to move heat from a body at a lower temperature to a body at a higher temperature, then you have to supply external work. In the original Carnot cycle, heat produces work while in the Reversed Carnot cycle work is provided to transfer heat from lower temperature reservoir to a higher temperature reservoir.
- Removing heat from the food items in the refrigerator and throwing it away to the higher temperature atmosphere does not happen automatically. We need to supply external work via the compressor to make this happen in the refrigerator.
- Air conditioner and heat pump follow the similar law of thermodynamics. The air conditioner removes heat from the room and maintains it at a lower temperature by throwing the absorbed heat into the atmosphere. The heat pump absorbs heat from the atmosphere and supplies it to the room which is cooler in winters.
- Melting of Ice Cube: Ice cubes in a drink absorb heat from the drink making the drink cooler. If we forget to drink it, after some time, it again attains room temperature by absorbing the atmospheric heat. All this happens as per the first and second law of thermodynamics.

**Check Your Progress**

11. What do you mean by differential equation?
12. Define ordinary differential equation.
13. What is integration?
14. Define integration constant.
15. What is the first law of thermodynamics?
16. State the second law of thermodynamics.
17. What is third law of thermodynamics?

## 4.9 MATHEMATICAL MODELLING-TREATMENT OF SELECTED SPECIFIC MODELS FROM DIFFERENT AREAS OF BIOLOGY

Mathematical modelling has been used for decades to help scientists understand the mechanisms and dynamics behind their experimental observations. In developmental biology, one of the most cited models is Turing's reaction-diffusion differential equations. Mathematical and computational models are increasingly used to help interpret biomedical data produced by high-throughput genomics and proteomics projects. The application of advanced computer models enabling the simulation of complex biological processes generates hypotheses and suggests experiments. Experimental data on a given biological phenomenon serve to derive a mathematical model that leads to hypotheses regarding the effects of perturbation of the system. This allows researchers to investigate novel scenarios and to develop hypotheses to guide the design of new and promising experiments. Mathematical biology aims at the mathematical representation and modeling of biological processes, using techniques and tools of applied mathematics. It can be useful in both theoretical and practical research.

Mathematical and theoretical biology or, biomathematics, is a branch of biology which employs theoretical analysis, mathematical models and abstractions of the living organisms to investigate the principles that govern the structure, development and behaviour of the systems, as opposed to experimental biology which deals with the conduction of experiments to prove and validate the scientific theories. The field is sometimes called **mathematical biology** or **biomathematics** to stress the mathematical side, or theoretical biology to stress the biological side. Theoretical biology focuses more on the development of theoretical principles for biology while mathematical biology focuses on the use of mathematical tools to study biological systems, even though the two terms are sometimes interchanged. Mathematical biology objectives at the mathematical representation and modeling of biological processes, using techniques and tools of applied mathematics. It can be useful in both theoretical and practical research. Describing systems in a quantitative manner means their behaviour can be better simulated, and hence properties can be predicted that might not be evident to the experimenter. This requires precise mathematical models. Because of the complexity of the living systems, theoretical biology employs several fields of mathematics, and has contributed to the development of new techniques (Refer Figure 4.6). Yellow chamomile head showing the Fibonacci numbers in spirals consisting of 21 (blue) and 13 (aqua). Such arrangements have been noticed since the middle Ages and can be used to make mathematical models of a wide variety of plants.

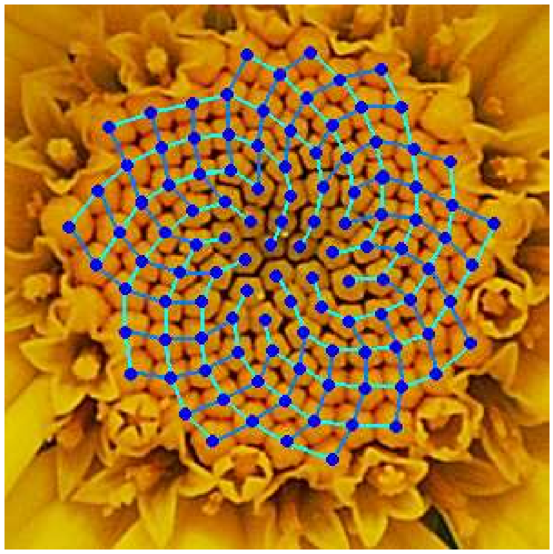

***Fig. 4.7*** *Mathematical Models of a Wide Variety of Plants*

### Modeling Cell and Molecular Biology

This area has received a boost due to the growing importance of molecular biology.

- Mechanics of biological tissues
- Theoretical enzymology and enzyme kinetics
- Cancer modelling and simulation
- Modelling the movement of interacting cell populations
- Mathematical modelling of scar tissue formation
- Mathematical modelling of intracellular dynamics
- Mathematical modelling of the cell cycle
- Mathematical modelling of apoptosis

**Mathematical Modelling of Nitrogen Cycle:** A model was developed using contemporary wetland theory to predict the fate of nitrogen runoff in a constructed wetland. The model utilises nitrogen concentrations of influent water as system inputs. The model is three-dimensional, one dimensional in time, and two dimensional in space. The physical domain of the model incorporates a flat emergent marsh and deep pool and includes the water body and underlying sediment. Solutions for concentration of sediment-bound organic nitrogen are obtained for the water body and the sediment-water interface, while solutions for concentration of ammonium and nitrate are obtained for the entire physical domain. Physical conditions are considered along the system boundaries, and a jump condition is modeled for nutrient diffusion through the sediment-water interface. A hyperbolic advection-settling equation models the transport and deposition of sediment-bound organic nitrogen; mineralisation of deposited nitrogen is modeled. A parabolic advection-diffusion equation is used to model the movement of dissolved ammonium and nitrate through the wetland water body; the equation is modified for both ammonium and nitrate to model diffusion and transformation in the sediment layer. Spatial variation of sediment layer aerobic and anaerobic regions is considered, as are temperature and pH effects on transformation rates. Numerical solutions are obtained using divided differences.

**Mathematical Modelling of Optimal Clutch Size in Birds:** To correct for phylogenetic effects, we obtained 1,000 phylogenetic trees for each data set and repeated the comparative analysis on each of these trees. These 1,000 trees are samples from a Bayesian estimate of the phylogeny of all birds and serve to represent our certainty and uncertainty about the relationships among the species in our analysis. By repeating our analyses across all 1,000 trees, we ensure that our biological inferences do not rely on the assumption that a single phylogeny is correct. Rather, our inferences account for the fact that certain parts of the phylogeny of birds are known with more certainty than other parts. This has been shown to be preferable to using a single phylogeny in a wide range of situations. We used **Phylogenetic Generalized Least Squares (PGLS)** regression models to account for phylogenetic relationships between species. We report the 95% highest posterior density intervals for p-value and â estimate for each predictor. Models were run using the Caper package in R (Orme ***et al.*** 2012). We ran a model for each type of parasite virulence (e.g., one for highly virulent parasites and one for nonvirulent parasites). We did not combine the data sets because it is difficult to quantify how much more costly it is to raise a highly virulent parasite compared to a nonvirulent one. Each model included the relative size of the parasite to the host (parasitism cost), mean latitude, body weight, and nest type as explanatory variables and mean clutch size as the response variable. We ran models with and without including non-hosts. When non-hosts were included, we considered non-host's costs to be zero, given that raising no parasite does not impose any additional cost on the species. Latitude was converted to absolute values and thus represents distance from the equator; weight was log transformed, given that it was non-normally distributed.

**Mathematical Modelling of Morphogenesis:** Morphogenesis is the ensemble of phenomena that generates the form and shape of organisms. Organisms are classified according to some of its structural characteristics, to its metabolism and to its form. In particular, the empirical classification associated with the ***phylum concept*** is related with the form and shape of organisms. Let be first introduce the class of mathematical models associated the ***Turing approach*** to pattern formation. In the Turing approach, morphogenesis models are described by reaction-diffusion parabolic partial differential equations. Based on this formalism, we present a mathematical model describing the first two hours of development of the fruit fly Drosophila. In the second part, we present results on Pareto optimality to calibrate and validate mathematical models.

**Mathematical Modelling of Genetic Drift:** Mathematical models of genetic drift can be designed using either branching processes or a diffusion equation describing changes in allele frequency in an idealised population.

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### Wright–Fisher Model

Consider a gene with two alleles, **A** or **B**. In diploid populations consisting of **N** individuals there are **2N** copies of each gene. An individual can have two copies of the same allele or two different alleles. We can call the frequency of one allele ***p*** and the frequency of the other ***q***. The Wright–Fisher model (named after Sewall Wright and Ronald Fisher) assumes that generations do not overlap (for example, annual plants have exactly one generation per year) and that each copy of the gene found in the new generation is drawn independently at random from all copies of the gene in the old generation. The formula to calculate the probability of obtaining k copies of an allele that had frequency ***p*** in the last generation is then.

$$\frac{(2N)!}{k!(2N-k)!}p^k q^{2N-k}$$

Where the symbol **"!"** signifies the factorial function. This expression can also be formulated using the binomial coefficient,

$$\binom{2N}{k}p^k q^{2N-k}$$

### Moran Model

The Moran model assumes overlapping generations. At each time step, one individual is chosen to reproduce and one individual is chosen to die. So in each time step, the number of copies of a given allele can go up by one, go down by one, or can stay the same. This means that the transition matrix is tridiagonal, which means that mathematical solutions are easier for the Moran model than for the ***Wright–Fisher model***. On the other hand, computer simulations are usually easier to perform using the Wright–Fisher model, because fewer time steps need to be calculated. In the Moran model, it takes **N** time steps to get through one generation, where **N** is the effective population size. In the Wright–Fisher model, it takes just one. In practice, the Moran and Wright–Fisher models give qualitatively similar results, but genetic drift runs twice as fast in the Moran model.

### Other Models of Drift

If the variance in the number of offspring is much greater than that given by the binomial distribution assumed by the ***Wright–Fisher model***, then given the same overall speed of genetic drift (the variance effective population size), genetic drift is a less powerful force compared to selection. Even for the same variance, if higher moments of the offspring number distribution exceed those of the binomial distribution then again the force of genetic drift is substantially weakened.

### Random Effects Other Than Sampling Error

Random changes in allele frequencies can also be caused by effects other than sampling error, for example random changes in selection pressure. One important alternative source of ***stochasticity***, perhaps more important than genetic drift, is genetic draft. Genetic draft is the effect on a locus by selection on linked loci. The mathematical properties of genetic draft are different from those of genetic drift. The direction of the random change in allele frequency is auto correlated across generations.

### 4.9.1 Cycling of Nutrients in Ecosystem/Eutrophication Model

A biogeochemical cycle is one of several natural cycles, in which conserved matter moves through the biotic and abiotic parts of an ecosystem. In biology conserved matter refers to the finite amount of matter, in the form of atoms that is present within the Earth. Since, according to the law of conservation of Mass, matter cannot be created or destroyed, all atoms of matter are cycled through Earth's systems albeit in various forms.

The main chemical elements that are cycled are: Carbon (C), Hydrogen (H), Nitrogen (N) Oxygen (O), Phosphorus (P) and Sulfur (S). These are building blocks of the life, and are used for essential processes, such as metabolism, the formation of amino acids, cell respiration and the building of tissues.

#### Nitrogen Cycle

The nitrogen cycle is the biogeochemical cycle by which nitrogen is converted into multiple chemical forms as it circulates among atmosphere, terrestrial, and marine ecosystems. The conversion of nitrogen can be carried out through both biological and physical processes. Nitrogen is both the most abundant element is the atmosphere and as a building block of proteins and Nucleic acid. The conversion of the Nitrogen can be carried out through both biological and physical processes. The conversion of nitrogen can be carried out through both biological and physical processes. Important processes in nitrogen cycle include fixation, ammonification, nitrification and denitrification (Refer Figure 4.7).

- Nitrogen Fixation
- Nitrification
- Assimilation
- Ammonification
- Denitrification

**Nitrogen Fixation**: Atmospheric nitrogen is fixed through biological processes. First nitrogen is deposited from the atmosphere into soils and surface water as its two nitrogen atom separate and combine with hydrogen to form ($NH_4^+$) ammonia. This is done by bacteria. Living in symbiotic relationships with certain plants, free anaerobic bacteria and algae.

Nitrogen fixation is the process in which Nitrogen ($N_2$) from the atmosphere is converted to Ammonia ($NH_3$) and then to Ammonium Ion ($NH_4^+$).

- Ammonia is an organic form of nitrogen, so it cannot be absorbed by plants.
- Ammonium is created when ammonia combines with a Hydrogen Ion ($H^{+-}$).

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- Natural or industrial processes, such as lightning and bacteria cause free nitrogen to combine with other elements to form these nitrogen compounds.

**Nitrification**: Ammonia can be used by some plants most of the nitrogen taken up by plants is converted by bacteria from ammonia into Nitrite ($NO_2$) and into Nitrate ($NO_3$). This process is called nitrification.

$$2NH_3 + 3O_2\ 2NO_2 + 2H_2O + 2H^+$$

**Steps**

- Nitrification is the two-step process in which ammonia is converted to Nitrites ($NO_2^-$) and then to Nitrates ($NO_3^-$).
- Two different species of bacteria that are present in the soil oxidize the ammonia into inorganic forms of nitrogen.
- The rate of nitrification is determined by these factors:
    - o **Temperature Dependency:** Rapid changes in temperature do not produce rapid changes in growth.
    - o **Oxygen Intake:** Nitrifying bacteria are sensitive to low oxygen concentrations.
    - o **pH Dependency:** Nitrification occurs the fastest when the pH is between 8 and 9.
    - o **Prevention Substances:** Many substances can prevent nitrification reactions, such as metals.

**Assimilation**: Nitrogen compounds in various forms, such as nitrate, nitrite, ammonia and ammonium are taken up from soil.

- Assimilation is the is the process by which living organisms incorporate $NO_3^-$ and $NH_4^+$ ammonium formed through nitrogen fixation and nitrification.
- Plants take in this form of nitrogen via the roots and incorporate them into nucleic acids and plant protein.
- Animals are then able to receive and utilize the nitrogen from plant tissues through consumption.

**Ammonification**: When Plants and animals die or when animals emit wastes, the nitrogen in the organic matter re-enters the soil where it is decompose by microorganism.

- Ammonification occurs when a plant or animal dies or excretes waste.
- Decomposers, such as bacteria and fungi, first break down the proteins in the organic matter.
- This releases ammonia, which dissolves with the water in the soil.
- Ammonia then combines with a hydrogen ion to create ammonium.

**Denitrification**

Nitrogen makes its way back into the atmosphere through a process called denitrification, in which Nitrate ($NO_3$) is converted in nitrogen. Denitrification is the process in which microorganisms, such as bacteria, breakdown nitrates to metabolize oxygen. This releases nitrogen gas back into the atmosphere, completing the cycle (Refer Figure 4.8).

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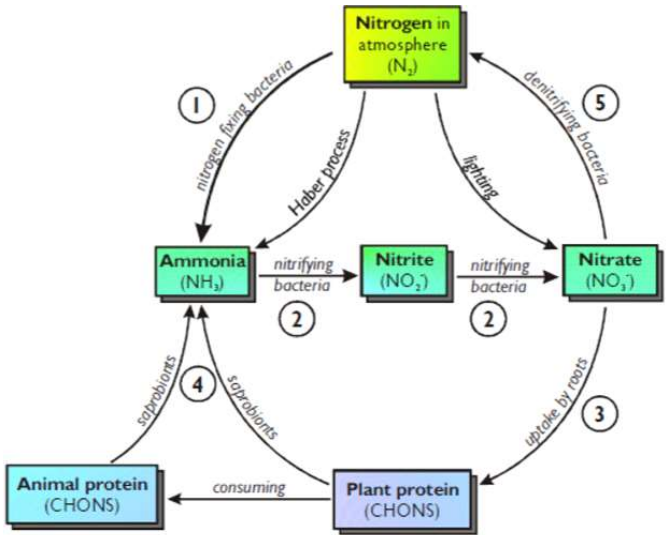


***Fig. 4.8*** *Nitrogen Cycle*

## Sulphur Cycle

Sulphur is one of the components that make up protein and vitamins. Protein consists of amino acids that contain sulphur atoms. Sulphur is important for the functioning of proteins and enzymes in plants and in animal that depend upon plants for sulphur.

Sulphur is transferred into biosphere then into ground or from ground to atmosphere. Microorganisms turn it into $H_2S$ gas. Oxidized in atmosphere to $SO_2$ and then to $H_2SO_4$ (an acid) with water contact. Mined over released to atmosphere in factories as $H_2S$ and $SO_2$.

Through precipitation, dry deposition, leaching $SO_4^{-2}$ leaches from soil into ocean as sediment and $H_2SO_4$ falls into ocean. Dimethyl sulfide, carbonyl sulfide (biogenic gases) released by plankton returns back into atmosphere (turn into $SO_2$). Either re-evaporated, left as sediment for long time or deposited on land. When back on land the cycle repeats.

The sulfur cycle contains both atmospheric and terrestrial processes. Within the terrestrial portion, the cycle begins with the weathering of rocks, releasing the stored sulfur. The sulfur then comes into contact with air where it is converted into Sulfate ($SO_4$). The sulfate is taken up by plants and microorganisms and is converted into organic forms; animals then consume these organic forms through foods they eat, thereby moving the sulfur through the food chain. As organisms die and decompose, some of the sulfur is again released as a sulfate and some enters the tissues of microorganisms. There are also a variety of natural sources that emit sulfur directly into the atmosphere, including volcanic eruptions, the breakdown of organic matter in swamps and tidal flats, and the evaporation of water. Sulfur eventually settles back into the Earth or comes down within rainfall. A continuous loss of sulfur from terrestrial ecosystem runoff occurs through drainage into lakes and streams, and eventually oceans. Sulfur also enters the ocean through fallout from the Earth's atmosphere. Within the ocean, some sulfur cycles through marine

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communities, moving through the food chain. A portion of this sulfur is emitted back into the atmosphere from sea spray. The remaining sulfur is lost to the ocean depths, combining with iron to form ferrous sulfide which is responsible for the black color of most marine sediments (Refer Figure 4.9).

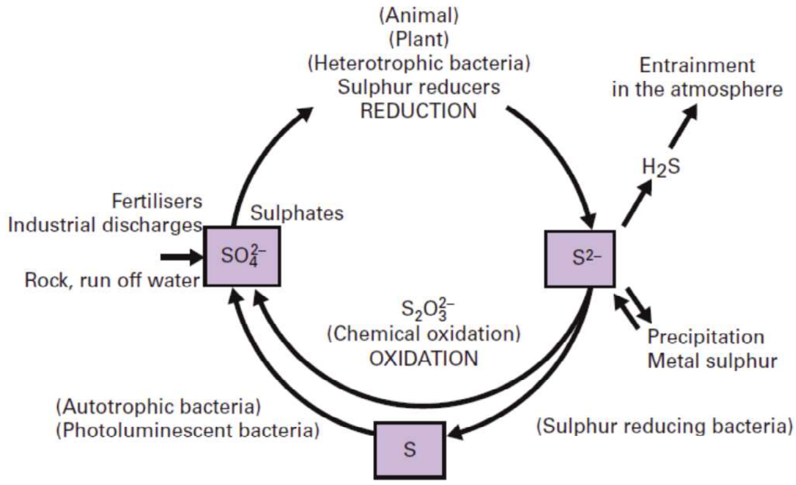


***Fig. 4.9*** *Sulphur Cycle*

## Impact of Human on Sulphur Cycle

Human activities have contributed to the amount of sulfur that enters the atmosphere, primarily through the burning of fossil fuels and the processing of metals. One-third of all sulfur that reaches the atmosphere—including 90% of sulfur dioxide—stems from human activities. Emissions from these activities, along with nitrogen emissions, react with other chemicals in the atmosphere to produce tiny particles of sulfate salts which fall as **acid rain**, causing a variety of damage to both the natural environment as well as to man-made environments, such as the chemical weathering of buildings. However, as particles and tiny airborne droplets, sulfur also acts as a regulator of global climate. Sulfur dioxide and sulfate aerosols absorb ultraviolet radiation, creating cloud cover that cools cities and may offset global warming caused by the greenhouse effect.

## Phosphorus Cycle

Phosphorus moves in a cycle in our atmosphere via rock, sediment, soil, water and living organisms. Over a long period of time weathering of rock leads to phosphate ions and minerals being released into the soil and water. This is absorbed by living organisms who need phosphorus to build nucleic acid, such as Deoxyribo Nucleic Acid DNA. Then when these living organisms die, phosphorus are released back into the soil. Phosphorus moves in a cycle through rocks, water, soil and sediments and organisms.

Here are the key steps of the phosphorus cycle:

- Over time, rain and weathering cause rocks to release phosphate ions and other minerals. This inorganic phosphate is then distributed in soils and water.
- Plants take up inorganic phosphate from the soil. The plants may then be consumed by animals. Once in the plant or animal, the phosphate is

incorporated into organic molecules, such as DNA. When the plant or animal dies, it decays, and the organic phosphate is returned to the soil.

- Within the soil, organic forms of phosphate can be made available to plants by bacteria that break down organic matter to inorganic forms of phosphorus. This process is known as mineralisation.
- Phosphorus in soil can end up in waterways and eventually oceans. Once there, it can be incorporated into sediments over time.

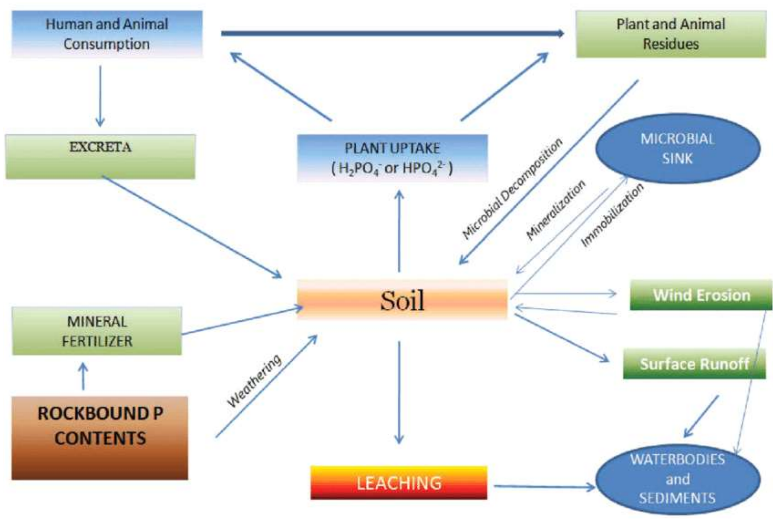


***Fig. 4.10*** *Phosphorous Cycle*

### Impact of Humans on Phosphorus Cycle

Humans have had a significant impact on the phosphorus cycle due to a variety of human activities, such as the use of fertilizer, the distribution of food products, and artificial eutrophication. Fertilizers containing phosphorus add to the phosphorus levels in the soil and are particularly detrimental when such products are washed into local aquatic ecosystems. When phosphorus is added to waters at a rate typically achieved by natural processes, it is referred to as natural eutrophication. A natural supply of phosphorus over time provides nutrients to the water and serves to increase the productivity of that particular ecosystem. However, when foods are shipped from farms to cities, the substantial levels of Phosphorus that is drained into the water systems is called artificial or anthropogenic eutrophication. When levels of phosphorus are too high, the overabundance of plant nutrients serves to drive the excessive growth of algae. However, these algae die or form algae blooms, which are toxic to the plants and animals in the ecosystem. Thus, human activities serve to harm aquatic ecosystems, whenever excess amounts of phosphorus are leached into the water.

## 4.9.2 Optimal Clutch Size in Birds

Clutch size denotes to the number of eggs laid in a single brood by a nesting pair of birds. The numbers laid by a particular species in a given location are usually well defined by evolutionary trade-offs with many factors involved, including resource availability and energetic constraints. Several patterns of variation have

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been noted and the relationship between latitude and clutch size has been a topic of interest in avian reproduction and evolution. David Lack and R.E. Moreau were among the first to investigate the effect of latitude on the number of eggs per nest. Since Lack's first paper in the mid-1940s there has been extensive research on the pattern of increasing clutch size with increasing latitude. The proximate and ultimate causes for this pattern have been a subject of intense debate involving the development of ideas on group, individual, and gene-centric views of selection.

***Fig. 4.11*** *Small Clutch*

## Food Limitation and Nest Predation Hypotheses

David Lack observed a direct relationship between latitude and avian clutch size. Comparable bird species near the equator laid approximately half as many eggs as those that resided in northern temperate habitats. He observed an increasing clutch-size from the equator towards the poles (something he referred to as the 'Latitude Trend') for many passerine (perching) birds, near-passerine (tree-dwelling) birds and in various other groups: Strigiformes, Falconiformes, Ciconiiformes, Laridae, Ralliformes, Galliformes, Podicipedifomes, and Glareolidae, and in some Limicolae.

### Skutch's Hypothesis

Skutch's Hypothesis is similar to the Nest Predation Hypothesis as it states that higher nest predation decreases the rate at which birds can deliver food to their offspring and thus limits clutch size. Few field studies have been published on this hypothesis.They analyzed whether Skutch's Hypothesis explained clutch size differences within or between latitudes. The study analyzed bird populations in large intact forests in Arizona, USA (with 7,284 nests) and subtropical Argentina where they monitored 1,331 nests. They found that clutches were larger in Arizona (4.61 eggs/nest) than in Argentina (2.58 eggs/nest) and that Skutch's Hypothesis explained the variation in clutch size within each, North and South America, but did not explain the latitudinal difference in clutch size.

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### Environmental Seasonality

As latitude increases, clutch size and seasonality also increase. The clutch size of birds occupying environments with low seasonal variations are smaller than those of birds residing in habitats that depict greater seasonality. Highly seasonal environments force birds to survive periods of low temperatures and reduced food availability during the non-reproductive season which causes increases in parental mortality.

### Day Length

As day length increases the size of the clutch also increase. Generally, a longer day enables parents to find more food per day and thus, sustain more offspring at one time. Several studies have found convincing evidence to support this observation. However, the energy requirement of the brood and parents could generate the latitudinal clutch size pattern in Ficedula hypoleuca as long as temperature and working day effects were considered. If photoperiod were a factor in the determination of clutch size in every avian species then nocturnal owls should show the opposite trend.

### Egg-Viability Hypothesis

Temperature is a possible factor that could explain the pattern in latitude and seasonal trends in clutch size. Since temperature varies with geographical location and time of year, it is possible that seasonal patterns in clutch size are affected by physiological processes that are temperature dependent. The egg-viability hypothesis states that high temperatures favour small clutches because of a reduction in egg viability. This explains the reason why when Red-winged Blackbirds – which are open nesters - lay large eggs at low latitudes, female birds initiate clutch incubation before the clutch completion.

### Nest Type

A strong intrinsic determinant of clutch size is nest type. Open nesters tend to be exposed to higher rates of nest predation in comparison to cavity nesters (e.g. woodpeckers). Thus, open nesters tend to have smaller clutch sizes. Half-open nests have clutch sizes that lie in between open nesters and cavity nesters. There are two major hypotheses that attempt to explain variation in clutch size among cavity-nesting bird species. The nest site limitation hypothesis states that weak excavators invest more energy in each breeding attempt and thus lay larger clutches because their nesting opportunities are more limited. The other competing hypothesis is that clutch size among cavity – nesting birds could be determined by diet. The clutch size of strong excavators may be larger because they are able to specialise on a more seasonally stable food source. Annual stability of food resources tends to have a larger impact on the variation of clutch size in excavators.

## 4.9.3 Morphogenesis

During gastrulation, cells from one region of embryo move to another to take up their future fateful position. Two terms, emboly and epiboly which are quite opposite in their meanings, are generally applied to explain the process of movement. Emboly means the throwing in or insertion of cells and epiboly signifies its extending. The movement of cells establishes a particular form and is involved in organ formation in embryo, so this movement is designated as the morphogenetic movement. Formation of primitive streak and head process is due to emboly.

**Formation of Primitive Streak**

Various prospective mesodermal and endodermal cells forming notochord of the epiblast converge toward the posterior edge of the area pellucida and form a conical thickening in the midline, called the initial primitive streak. It appears after 6 to 7 hours of incubation. The primitive streak grows anteriorly because of proliferation of its own cells as well as of the addition of cells that migrate to it from anterior and lateral parts of area pellucida. The elongated axis of the primitive streak marks the antero-posterior axis of the future embryo. It, thus, eventually extends to, about three fifths of the entire length of area pellucida. This is the fully developed definitive primitive streak and it is usually completed after 18 to 19 hours of incubation. The area pellucida also becomes pear-shaped. Along the middle of the primitive streak, when it is fully developed runs a narrow furrow, the primitive groove. At the anterior end of the primitive streak there is a thickening, the primitive knot or Hensen's node. The centre of Hensen's node is excavated to form a funnel-shaped depression. The movements in the blastoderm leading to the final placement of cells in the hypoblast and to the formation of the primitive streak in the epiblast may be called pregastrular movements (Refer Figure 4.12).

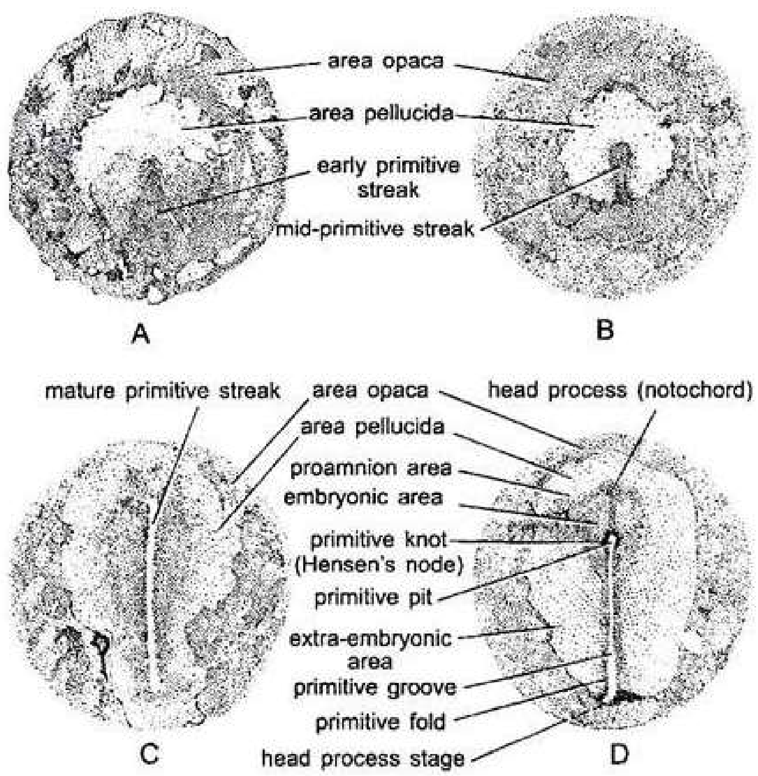


***Fig. 4.12*** *Surface View of Chick Blastodem showing Development of Primitive Streak during Gastrulation*

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### Invagination and Involution

At the stage of short primitive streak, the cells of the blastoderm already begin to migrate (invaginate and involute) into the blastocoel cavity between epiblast and hypoblast. Immigrating cells are replaced by more epiblast cells converging toward the streak area. The inward migrating cells also spread out sideways and forward from the anterior end of primitive streak. The notochordal cells immigrate through primitive pit. Endodermal cells invaginate through that part of the streak which lies just behind primitive pit. The mesodermal cells of somites just follow the path of endodermal cells. Whereas the lateral plate mesoderm cells invaginate through the middle section of primitive streak, but only after the disappearance of endoderm from the area pellucida. The extra-exbryonic mesoderm (of the yolk sac) immigrates through the posterior part of primitive streak. Meanwhile, some hypoblast cells expand into the area opaca to become extra-embryonic endoderm (the lining of yolk sac), while other hypoblast cells attach to mesodermal and notochordal cells are carried along by the latter's migration.

### Formation and Development of Head Process

Prospective notochoral cells converge on the node, sinks through it and then passes directly forward as a tongue of tissue known as head process or notochord process. The midline area of notochordal tissue develops into a rigid rod, anterior to the receding primitive streak. As the streak regresses posteriorly, the embryo develops anterior to it. The head process consists of a thick central mass of cells and more diffused lateral wings. In the beginning it is also blended in the midline with the hypoblast. The thicker central portion forms the definitive notochord, whereas the lateral wings form the paraxial (somitic) mesoderm. With its differentiation, the notochord becomes detached from the hypoblast below, except at the extreme end. Thus, the head process stage is completed at about 20 to 25 hours of incubation. Gastrulation is also completed at this stage.

### Disappearance of Primitive Streak

With the gradual disappearance of endodermal, notochordal and mesodermal cells from the primitive streak, it begins to shrink from anterior towards posterior side and its remains are partly included in the tail bud and partly into the cloacal region of the embryo.

### Formation of Endoderm

The first cells that migrate through the anterior part of streak form the endoderm. As the Hensen's node recedes backward and the notochordal process elongates, the presumptive endoderm of the middle and posterior part of the gut, located just behind the node, migrate inside as an endodermal strip beneath the notochord. The original hypoblast at the floor of the blastocoel contribute a very less amount to the gut, the upper migrated endodermal cells form the major part of the gut. In chick no archenteron is formed during gastrulation.

### Fully Formed Gastrula

Gastrula is fully formed when primitive streak completely disappears. The fully formed gastrula consists of three germ layers-ectoderm, chorda-mesoderm and endoderm. The ectoderm and chorda-mesoderm remain in continuity along the axis of primitive streak. The endoderm is also united with the mesoderm and ectoderm at the anterior and posterior end of streak, (Refer Figure 4.13).

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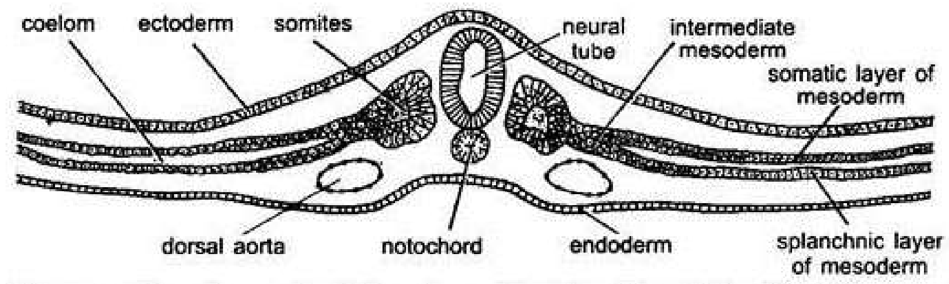


***Fig. 4.13*** *A Cross Section of an Early Chick Embryo*

## 4.9.4 Genetic Drift

All the individuals of a species constitute a population. The genetical studies for the inheritance of phenotypic traits in a given population is called population genetics. The population genetics is a quantitative science. To calculate the results of the mode of inheritance of genes in a given population various statistical and mathematical models are employed in it. Certain fundamental aspects of population genetics are the following:

**Mendelian Population**

A population of a particular species includes many inbreeding groups. The inbreeding groups may form a community within defined geographical boundaries and are called 'Mendelian Population'. A Mendelian population, thus, is a group of sexually reproducing organisms with a relatively close degree of genetic relationship (such as, species, subspecies breed, variety, strain, etc.) residing within defined geographical boundaries where interbreeding occurs.

**Gene Pool and Gene Frequency**

To get a $F_2$ 3:1 phenotypic ratio of a monohybrid cross, we began with two homozygous parental strains, such as AA and aa: that is, we introduced the alleles A and a in equal frequency (A frequency is the ratio of the actual number of a individuals falling in a single class to the total number of individuals. But in a Mendelian population, frequencies of alleles may vary considerably. For example in Menedelian population old man the gene for polydactyly is dominant, yet the polydactylous phenotype is fairly infrequent among infants. This indicates that frequency of the dominant allele in population is lower than that of its recessive allele and that both alleles do not exist in population in the 1:1 ratio like the individuals of monohybrid cross.

Further, if all the gametes produced by a Mendelian population are considered as a hypothetical mixture of genetic units from which the next generation will arise, we have the concept of a gene or gamete pool. The percentages of gametes in the gene pool for a pair of alleles (A and a) depend upon the genotypic frequencies of the parental generation whose gametes form the pool. Thus, if a population is of dominant genotype AA, then the frequency of dominant alleles in the gene pool will be relatively high and the percentage of gametes bearing the recessive (a) allele will be correspondingly low.

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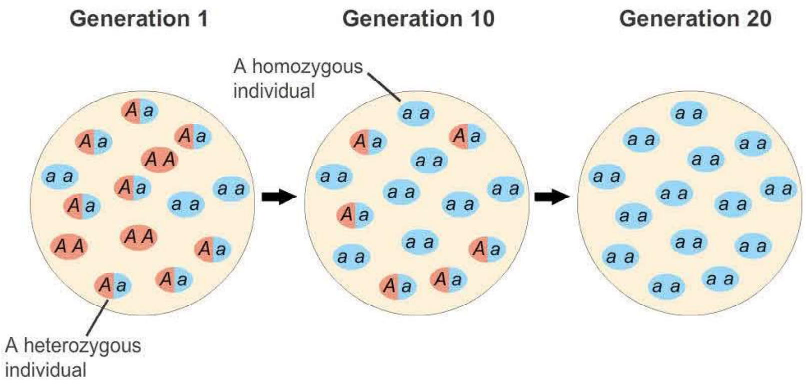


***Fig. 4.14*** *Showing Concept of Gene Pool*

### Hardy-Weinberg Law

The formula $(p + q)^2 = p^2 + q^2 + 2pq$ is expressing the genotypic expectations of progeny in terms of gametic or allelic frequencies of the parental gene pool and is originally formulated by a British mathematician Hardy and a German physician Weinberg (1908) independently Both forwarded the idea called Hardy-Weinberg law equilibrium after their names, that 'Both gene frequencies and genotype frequencies will remain constant from generation to generation in an infinitely large interbreeding population in which mating is at random and no selection, migration or mutation occur'. Should population initially be in disequilibrium, one generation of random mating is sufficient to bring it into genetic equilibrium and thereafter the population will remain in equilibrium (unchanged in gametic and zygotic frequencies) as long as Hardy-Weinberg condition persists.

Hardy-Weinberg law depends upon the following kinds of genetic equilibriums for its full attainment:

- The population is infinitely large and mate at random.
- No selection is operative.
- The population is closed, i.e., no immigration or emigration occurs.
- No mutation is operative in alleles.
- Meiosis is normal so that chance is the only factor operative in gametogenesis.

### Genetic Drift

Genetic drift (also known as allelic drift or the Sewall Wright effect) is the change in the frequency of an existing gene variant allele) in a population due to random sampling of organisms. The alleles in the offspring are a sample of those in the parents, and chance has a role in determining whether a given individual survives and reproduces. A population's allele frequency is the fraction of the copies of one gene that share a particular form. Genetic drift may cause gene variants to disappear completely and thereby reduce genetic variation (Refer Figure 4.15). It

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can also cause initially rare alleles to become much more frequent and even fixed. Genetic different have following characteristics:

- Genetic drift is a mechanism of evolution in which allele frequencies of a population change over generations due to chance (sampling error).
- Genetic drift occurs in all populations of non-infinite size, but its effects are strongest in small populations.
- Genetic drift may result in the loss of some alleles (including beneficial ones) and the fixation, or rise to 100% frequency, of other alleles.
- Genetic drift can have major effects when a population is sharply reduced in size by a natural disaster (bottleneck effect) or when a small group splits off from the main population to found a colony (founder effect).

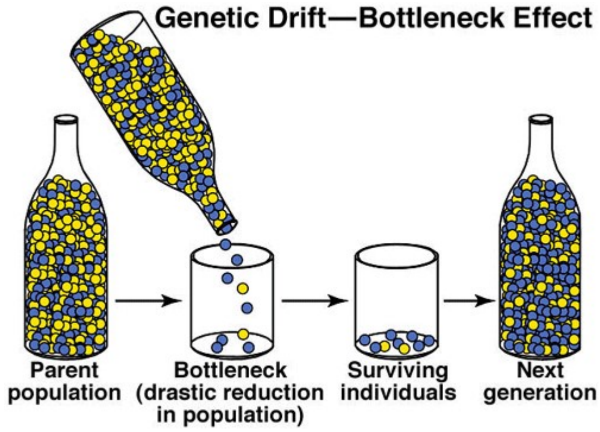


***Fig. 4.15*** *Concept of Genetic Drift: Bottleneck Effect*

**Check Your Progress**

18. What is mathematical Modelling in biology?
19. How mathematical models are used in biology?
20. What do you understand by biochemical cycle?
21. Define clutch size in birds.
22. What is morphogenesis?
23. Define the term population genetics.
24. What is genetic drift?

## 4.10 ANSWERS TO 'CHECK YOUR PROGRESS'

1. A matrix is an arrangement of elements, which are numbers (real or complex), variables and/or functions in a rectangular array comprising of rows and columns and enclosed inside a square bracket or parenthesis.
2. Scalar Matrix: A diagonal matrix whose diagonal elements are equal, is called a scalar matrix. For example,

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ are scalar matrices.}$$

3. Matrix obtained from identity matrix by a single elementary operation is called Elementary matrix.
4. The scalar product of two vectors is defined as the product of the magnitudes of the two vectors and the cosine of the angles between them.
5. The magnitude vector product of two given vectors can be found by taking the product of the magnitudes of the vectors times the sine of the angle between them.
6. A constant vector is a vector whose both magnitude and direction are fixed, i.e., do not change.
7. This function is of prime importance in mathematics and finds its wide application in calculus and many branches of science and engineering.
8. An exponential function of $x$ is written as exp($x$) or $e^x$. Here $e$ is a constant and an irrational number. It has been estimated as 2.718281828 by Euler and bears his name. It is called 'Euler's number' and is also the base of natural logarithm.
9. A function $f$ is said to be periodic if $f(x+ c,c) = f(x)$ for all values of $x$. The constant $c$ is called the period, and is required to be positive. A function with period $c$ will repeat on intervals of length $c$, and these intervals are sometimes also referred to as periods.
10. In this type of discontinuity, the limit on one side is given, whereas the limit on other side is $\pm \infty$.
11. In mathematics, a Differential Equation (DE) is defined as an equation of the form that interconnects certain function with its derivatives, where usually the function represents the physical quantity while the derivatives denote their rates of change and the relationship between the two is defined by the equation.
12. An ordinary differential equation is a differential equation that includes a function of a single variable and some of its derivatives, such as

$$\frac{dy}{dx} = 2x^2 + 3x + 5$$

13. After learning differentiation, we now come to the 'Reverse' process of it, namely integration. To give a precise shape to the definition of integration, we observe: If $g$ ($x$) is a function of $x$ such that,

$$\frac{d}{dx}\ g\ (x) = f\,(x)$$

then we define integral of $f$ ($x$) with respect to $x$, to be the function $g$ ($x$). This is put in the notational form as,

$$\int f\,(x)\,dx\ = g\ (x)$$

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The function $f(x)$ is called the Integrand. Presence of $dx$ is there just to remind us that integration is being done with respect to $x$.

14. $\int f(x)\,dx = g(x) + c$

    Where $c$ is some constant, called the *constant of integration.* Obviously, $c$ could have any value and thus, integral of a function is not unique! But, we could say one thing here, that any two integrals of the same function differ by a constant.

15. The first law of thermodynamics states that, when energy passes into or out of a system (as work, heat, or matter), the system's internal energy changes in according with the law of conservation of energy.
16. The second law of thermodynamics states that in a natural thermodynamic process, the sum of the entropies of the interacting thermodynamic systems never decreases. Another form of the statement is that heat does not spontaneously pass from a colder body to a warmer body.
17. The third law of thermodynamics states that a system's entropy approaches a constant value as the temperature approaches absolute zero. With the exception of non-crystalline solids (glasses) the entropy of a system at absolute zero is typically close to zero.
18. Mathematical modelling has been used for decades to help scientists understand the mechanisms and dynamics behind their experimental observations. In developmental biology, one of the most cited models is Turing's reaction-diffusion differential equations.
19. Mathematical and computational models are increasingly used to help interpret biomedical data produced by high-throughput genomics and proteomics projects. The application of advanced computer models enabling the simulation of complex biological processes generates hypotheses and suggests experiments. Experimental data on a given biological phenomenon serve to derive a mathematical model that leads to hypotheses regarding the effects of perturbation of the system. This allows researchers to investigate novel scenarios and to develop hypotheses to guide the design of new and promising experiments.
20. A biogeochemical cycle is one of several natural cycles, in which conserved matter moves through the biotic and abiotic parts of an ecosystem. In biology conserved matter refers to the finite amount of matter, in the form of atoms that is present within the Earth. Since, according to the law of conservation of Mass, matter cannot be created or destroyed, all atoms of matter are cycled through Earth's systems albeit in various forms.
21. Clutch size denotes to the number of eggs laid in a single brood by a nesting pair of birds. The numbers laid by a particular species in a given location are usually well defined by evolutionary trade-offs with many factors involved, including resource availability and energetic constraints. Several patterns of variation have been noted and the relationship between latitude and clutch size has been a topic of interest in avian reproduction and evolution.

22. During gastrulation, cells from one region of embryo move to another to take up their future fateful position. Two terms, emboly and epiboly which are quite opposite in their meanings, are generally applied to explain the process of movement. Emboly means the throwing in or insertion of cells and epiboly signifies its extending. The movement of cells establishes a particular form and is involved in organ formation in embryo, so this movement is designated as the morphogenetic movement. Formation of primitive streak and head process is due to emboly.
23. All the individuals of a species constitute a population. The genetical studies for the inheritance of phenotypic traits in a given population is called population genetics.
24. Genetic drift (also known as allelic drift or the Sewall Wright effect) is the change in the frequency of an existing gene variant allele) in a population due to random sampling of organisms.

## 4.11 SUMMARY

- Position of an element in a matrix is described by specifying row number and column number.
- Row Matrix. A matrix which has exactly one row is called a *row matrix.*
- A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix).
- 'An elementary row operation on product of two matrices is equivalent to elementary row operation on prefactor.'
- Vectors can be multiplied in two different ways, namely the scalar product and the vector product. The scalar product of two vectors results in a scalar quantity, while a vector product results in a vector quantity.
- Vectors differentiation refers to the differentiation of vector functions. In vector analysis we compute derivatives of vector functions of a real variable.
- The definitions of limit and continuity for vectors functions bear a strong resemblance to the corresponding definitions for scalar functions.
- An exponential function is the inverse of a logarithmic function and is sometimes, called anti logarithm. Inverse of an exponential function is a logarithmic function.
- An exponential function with negative value of $x$ is known as exponential decay and those with positive value it is called exponential growth.
- As in case of real numbers, the exponential function can be defined in for complex quantities too. Some of these definitions are identical to those given for real valued exponential functions.
- When the value of $f(a)$ is wrong or it is not defined, then this type of discontinuity is called removable type. This discontinuity can be removed by changing or defining $f(x)$ at point $a$.

- Fundamentally, the 'Solutions of Differential Equations' are functions which precisely 'Represent the relationship or correlation between a continuously varying or fluctuating quantity and its rate of change'.
- The linear differential equation is a differential equation in which the dependent variable and all its derivatives appear only in the first degree and are not multiplied together.
- The first law establishes definite relationship between the heat absorbed and the work performed by a system in a given process.
- The second law of thermodynamics helps us to determine the direction in which energy can be transferred.
- The first law of thermodynamics is a related to the law of conservation of energy, adapted for thermodynamic processes. In general, the conservation law states that the total energy of an isolated system is constant; energy can be transformed from one form to another, but can be neither created nor destroyed.
- Mathematical biology aims at the mathematical representation and modeling of biological processes, using techniques and tools of applied mathematics. It can be useful in both theoretical and practical research.
- Mathematical and theoretical biology or, biomathematics, is a branch of biology which employs theoretical analysis, mathematical models and abstractions of the living organisms to investigate the principles that govern the structure, development and behaviour of the systems, as opposed to experimental biology which deals with the conduction of experiments to prove and validate the scientific theories.
- Morphogenesis is the ensemble of phenomena that generates the form and shape of organisms. Organisms are classified according to some of its structural characteristics, to its metabolism and to its form.
- The main chemical elements that are cycled are: Carbon (C), Hydrogen (H), Nitrogen (N) Oxygen (O), Phosphorus (P) and Sulfur (S). These are building blocks of the life, and are used for essential processes, such as metabolism, the formation of amino acids, cell respiration and the building of tissues.
- Atmospheric nitrogen is fixed through biological processes. First nitrogen is deposited from the atmosphere into soils and surface water as its two nitrogen atom separate and combine with hydrogen to form ($NH_4^+$) ammonia. This is done by bacteria. Living in symbiotic relationships with certain plants, free anaerobic bacteria and algae.
- When Plants and animals die or when animals emit wastes, the nitrogen in the organic matter re-enters the soil where it is decompose by microorganism.
- Sulphur is one of the components that make up protein and vitamins. Protein consists of amino acids that contain sulphur atoms. Sulphur is important for the functioning of proteins and enzymes in plants and in animal that depend upon plants for sulphur.

- Immigrating cells are replaced by more epiblast cells converging toward the streak area. The inward migrating cells also spread out sideways and forward from the anterior end of primitive streak. The notochordal cells immigrate through primitive pit.
- Genetic drift is a mechanism of evolution in which allele frequencies of a population change over generations due to chance (sampling error).

## 4.12 KEY TERMS

- **Matrix:** A matrix is an arrangement of elements, which are numbers (real or complex), variables and/or functions in a rectangular array comprising of rows and columns and enclosed inside a square bracket or parenthesis.
- **Column matrix:** A matrix which has exactly one column is called a column matrix.
- **Vector product:** Vector product or cross product is a binary operation on two vectors in three-dimensional space.
- **Constant vector:** A constant vector is a vector whose both magnitude and direction are fixed, i.e., do not change.
- **Anti-logarithm:** An exponential function is the inverse of a logarithmic function and is sometimes, called anti logarithm.
- **Thermodynamics:** It is the branch of physical science that deals with the relationships between heat and other forms of energy.
- **Mathematical modelling:** Mathematical modelling has been used for decades to help scientists understand the mechanisms and dynamics behind their experimental observations.

## 4.13 SELF-ASSESSMENT QUESTIONS AND EXERCISES

### Short-Answer Questions

1. What is matrix?
2. What do you understand by multiplication of matrix by scalar?
3. Define the basic elementary operations.
4. What is square matrix?
5. What is scalar product?
6. Distinguish between the scalar and vector triple product.
7. Define the term vector differentiation.
8. State chain rule.
9. What are the properties of scalar triple vector?
10. Define exponential function.

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11. What do you mean by exponential function on the complex plane?
12. What is periodic function?
13. Give the conditions for Fourier expansion.
14. Differentiate between even and odd functions.
15. What do you understand by differential equations?
16. Define the term auxiliary equation.
17. What is integration?
18. Write any five formulas obtained as a result of direct consequence of integration.
19. State the second law of thermodynamics.
20. Name the applications of thermodynamics for biological field.
21. Why do biological scientists need mathematical models in biology?
22. What is mathematical biology?
23. Give the importance of nitrogen cycle.
24. State the formation of primitive streak.
25. What is optimal clutch size in birds?
26. What do you mean by invagination and involution?
27. What is genetic drift?

**Long-Answer Questions**

1. Explain in detail about the matrices and its types giving appropriate examples.
2. Describe the operations on matrices with the help of examples.
3. Discuss about the vector and scalar products with appropriate examples.
4. Analyse the vector differentiation with the help of theorems and examples.
5. Briefly explain about the exponential function with the help of examples.
6. What is periodic function? Explain the change interval with various types of examples.
7. Describe the differential equation with the help of theorems and examples.
8. Discuss about the integration giving relevant examples.
9. Briefly explain the first and second laws of thermodynamics giving appropriate examples.
10. Explain in detail about the principle and applications of thermodynamics.
11. Discuss about the mathematical modelling-treatment of selected specific models with appropriate examples and properties.
12. Briefly discuss the different types of nutrients cycles in ecosystem and eutrophication model.
13. Analyse the morphogenesis and genetic drift giving examples.

## 4.14 FURTHER READING

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