B.Com., First Year
Accounting Group, Paper - II

BUSINESS MATHEMATICS

मध्यप्रदेश भोज (मुक्त) विश्वविद्यालय – भोपाल

MADHYA PRADESH BHOJ (OPEN) UNIVERSITY – BHOPAL

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SYLLABI-BOOK MAPPING TABLE

Business Mathematics

Syllabi	Mapping in Book
Unit 1 **Ratio** – Gaining and Sacrificing Ratio, Proportion, Percentage, Commission, Discount and Brokerage.	**Unit 1:** Ratio **(Pages 3 – 44)**
Unit 2 **Simultaneous Equations** – Meaning, Characteristics, Types and Calculations, Preparation of Invoice.	**Unit 2:** Simultaneous Equations **(Pages 45 – 96)**
Unit 3 **Elementary Matrices** – Definition and Calculations, Types of Matrices.	**Unit 3:** Elementary Matrices **(Pages 97 – 158)**
Unit– 4 **Logarithms and Antilogarithms** – Principles and Calculations, Simple and Compound Interest.	**Unit 4:** Logarithms and Antilogarithms **(Pages 159 – 196)**
Unit 5 **Averages** – Simple, Weighted and Statistical Averages, Arithmetic Mean, Harmonic Mean, Geometric Mean, Profit and Loss.	**Unit 5:** Averages **(Pages 197 – 260)**

Contents

INTRODUCTION

NOTES

Business mathematics is mathematics used by commercial enterprises to record and manage business operations. Commercial organizations use mathematics in accounting, inventory management, marketing, sales forecasting and financial analysis. The text book titled "Business Mathematics" comprises the topics like gaining and sacrificing ratio, proportion percentage, commission, discount and brokerage, stimulation equations, preparation of invoice, elementary matrices, logarithms and antilogarithms, averages, profit and loss, etc.

Ratio indicates how many times one number contains another. For example, if there are eight oranges and six lemons in a bowl of fruit, then the ratio of oranges to lemons is eight to six. The ratio is the number which can be used to express one quantity as a fraction of the other ones. The two numbers in a ratio can only be compared when they have the same unit. This unit includes various topics related to gaining and sacrificing ratio, proportion percentage, commission, discount and brokerage. Proportion is the equality between two or more ratios. If we express the fact that one ratio is equal to another ratio, then it forms a proportion. Quantities are said to be in continued proportion when the first is to the second, as the second is to the third, as the third is to the fourth and so on. The percentage difference between two values is calculated by dividing the absolute value of the difference between two numbers by the average of those two numbers. Percentage increase and decrease are calculated by computing the difference between two values and comparing that difference to the initial value. Mathematically, this involves using the absolute value of the difference between two values and dividing the result by the initial value, essentially calculating how much the initial value has changed.

The Simultaneous Equations includes the topics like meaning, characteristics, types and calculations of simultaneous equations and description about the preparation of invoice. An equation is a relation between two variables (two or more) and holds good only for certain values of the variables. Thus, it is clear that, in an equation, the equality holds for certain values of the variables. However, in case of the identities, the equality holds for any value of variables.

The equation of the form containing x^2 as the highest power of x is called an equation of the second degree in x or a quadratic equation. The quadratic equation has two and only two roots. These two roots may be equal or unequal. In Substitution method, the value of y (or x) is found in terms of x (or y) from an equation, and substituting this value in the other equation, we get a linear equation of one variable.

An elementary matrix is a matrix which differs from the identity matrix by one single elementary row operation. The elementary matrices generate the general linear group *GLn(F)* when *F* is a field. Left multiplication (pre-multiplication) by an elementary matrix represents elementary row operations, while right multiplication (post-multiplication) represents elementary column operations. Elementary row operations are used in Gaussian elimination to reduce a matrix to row echelon

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form. They are also used in Gauss-Jordan elimination to further reduce the matrix to reduced row echelon form. The authors explain in brief about meaning, definition and calculation of Elementary Matrices and types of matrices. A matrix in which the number of rows is not equal to the number of columns in called rectangle matrix. In a square matrix, the diagonal which starts from left hand top corner and ends at to right hand bottom corner is called “the principal diagonal or leading diagonal elements”. A diagonal matrix is a matrix in which all the elements except the elements in the principal diagonal are zeroes. Scalar matrix is a diagonal matrix in which all the elements in the principal diagonal elements are equal.

Logarithmic scales are useful for quantifying the relative change of a value as opposed to its absolute difference. The logarithmic function log(x) grows very slowly for large x. Logarithmic scales are used to compress large-scale scientific data. This unit includes the topics like principles and calculations of logarithms and antilogarithms, simple and compound interest calculations.

Interest is a charge paid for the use of borrowed money. It is paid by borrower to the lender. Hence, it is the compensation received by the lender of money from the borrower at a particular rate and for a specified period. If interest is calculated only on the principal, then it is called simple interest, i.e., principal alone produces interest. Simple interest depends upon three factors.

This unit includes the topics of meaning of averages, simple, weighted and statistical averages arithmetic mean, harmonic mean, geometric mean, profit and loss, etc. Descriptive statistics is the type of statistics that probably springs to most people’s minds when they hear the word “statistics.” Numerical measures are used to tell about features of a set of data. Measures of central tendency is a single value which can be considered as representative of a set of observations and around which the observations can be considered as centered is called an “Average” (or average value) or a center of location. An average is a single figure which sums up the characteristics of a whole group of figures. In the words of Clark, “average is an attempt to find one single figure to describe whole of figures”. An average is described as a measure of central tendency as it is more or less a central value around which various values cluster. In the words of Croxton and Cowden, “an average is a single value within the range of the data that is used to represent all of the values in the series.” Since an average is somewhere within the range of the data, it is called a measure of cultural value. In simple arithmetic mean, it is assumed that all the items are of equal importance. For finding simple average, the total values are divided by number of observations. The weight for such observations should be given on the basis of their relative importance.

Dr. Chandrashakara V.
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UNIT 1 RATIO

NOTES

Structure

1.0 INTRODUCTION

A ratio indicates how many times one number contains another. For example, if there are eight oranges and six lemons in a bowl of fruit, then the ratio of oranges to lemons is eight to six. The ratio is the number which can be used to express one quantity as a fraction of the other ones. The two numbers in a ratio can only be compared when they have the same unit. We make use of ratios to compare two things. The sign used to denote a ratio is : (colon).

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the concept of Ratio
- Discuss the Gaining and Sacrificing Ratio
- Describe the Proportion percentage
- Examine the Commission, Discount and Brokerage

1.2 RATIO

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A ratio is the relationship between two or more variables. The variable should be of same unit and same kind. It is always expressed by the fraction obtained by dividing the first number by the second number.

The ratio of two quantities '*a*' and '*b*' may be denoted as $a : b$, i.e., called *a* is to *b*.

1.3 ANTECEDENT AND CONSEQUENT

The ratio is expressed as a fraction $\frac{a}{b}$ where the first term '*a*' (numerator) is called antecedent and the second term '*b*' (denominator) is called the consequent.

Example 1:

Find the ratio compounded of 2 : 3, 4 : 7 and 6 : 15.

Solution:

Consider,

$$\frac{2}{3} \times \frac{4}{7} \times \frac{6}{15} = \frac{48}{315} = \frac{16}{105} = 16 : 105$$

Example 2:

Divide ₹ 28 between two men in the ratio of 2 : 5.

Solution:

Sum of the ratio = 2 + 5 = 7

First share is 2/7 of 28 = ₹ 8

Second share is 5/7 of 28 = ₹ 20

Example 3:

Two numbers are in the ratio 5 : 8. If the sum of the numbers in 195, then find the number.

Solution:

Let the number be $5x$ and $8x$

$$5x + 8x = 195$$

$$\Rightarrow 13x = 195$$

$$\Rightarrow x = \frac{195}{13} = 15$$

$$\therefore x = 15$$

So, numbers are (5 × 15) = 75 and (8 × 15) = 120.

1.4 GAINING AND SACRIFICING RATIO

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Gaining Ratio

Gaining ratio is a type of financial tool that helps in determining the proportion by which the remaining partners of a firm will share the profits of an existing partner in the event of his death or retirement. The ratio by which they share the profits is known as gaining ratio. It can also be defined as the difference between the old profit sharing ratio and the new profit sharing ratio.

For instance, Ajit, Dino and Gaurav are partners sharing profits in the ratio of 5 : 3 : 2. Dino retires. Ajit and Gaurav decided to share the profits of the new firm in the ratio of 3 : 2. The gaining ratio will be calculated as follows :

Gaining share of Continuing Partner = New share – Old share

Sacrificing Ratio

Sacrificing ratio is the ratio where the old partners give their consent to forego their share of gains into the new partner. The forego (sacrifice) by a partner is equivalent to: Old Share of Profit – New Share of Profit. Sacrificing ratio is computed during the time of addition or admission of a new associate partner.

Distinguish between Sacrificing Ratio and Gaining Ratio

Sacrificing Ratio	Gaining Ratio
1. It is calculated to ascertain the share of profit and loss given up by the existing partners in favour of new partners/partner.	1. It is calculated to ascertain the share of profit and loss given up by the existing partners in favour of new partners/partner.
2. It is the ratio in which old partners agree to sacrifice their share of profit in favour of new partners/partner.	2. It is the ratio in which old partners agree to sacrifice their share of profit in favour of new partners/partner.
3. It is calculated at the time of admission of new partners/partner.	3. It is calculated at the time of admission of new partners/partner.
4. Sacrificing ratio = Old ratio – New ratio.	4. Sacrificing ratio = Old ratio – New ratio.
5. It reduces the profit sharing ratio of the existing partners.	5. It reduces the profit sharing ratio of the existing partners.

1.5 PROPORTION

Proportion is the equality between two or more ratios. If we express the fact that one ratio is equal to another ratio, then it forms a proportion, i.e., the ratio $a : b$ be equal to the ratio $c : d$. The four terms a, b, c and d are said to be in proportion. when 'a' is as many times as b, and c is as many times as d, then a, b, c and d are called the terms of the proportion.

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For example, if we state that the ratio of 2 and 5 equal to the ratio of 4 to 10, then we form the proportion 2 : 5 : : 4 : 10. This equality of ratios can also be expressed in the following ways:

$$2 : 5 : : 4 : 10 \text{ or } \frac{2}{5} : \frac{4}{10} \text{ or } 2 \text{ to } 5 = 4 \text{ to } 10$$

In a proportion, the first and last terms (outside terms) are called the extremes, and the second and the third terms (inside terms) are called the means.

a : b = c : d

means

extremes

In every proportion, the product of the extreme is equal to the product of the means, i.e., $ad = bc$

Example 4:

A man can make 12 pencils in 15 minutes. How long will it take him to turn out 96 pencils?

Solution:

Let the time taken be x.

$$\therefore \text{ We have } 12 : 96 : : 15 : x$$

$$\text{or } 12x = 96 \times 15 \text{ or } x = \frac{96 \times 15}{12}$$

$$\therefore x = 120 \text{ minutes} = 2 \text{ hours}$$

1.5.1 Direct Proportion

Quantities are said to be in direct proportion when an increase (or decrease) in one kind is accompanied by an increase (or decrease) in the other.

Example 5:

If a labour earns wages at ₹ 12 per day, his earnings may be indicated as follows:

Number of days	2	5	6	8	10
Earnings (₹)	24	60	72	96	120

Here, we notice that as the number of days increases, wages earned by a man also increases proportionately, and if the number of days decreases, the earnings also decreases proportionately, i.e., the amount he earns is directly proportional to the number of days he works.

1.5.2 Inverse Proportion

Quantities are said to be in inverse proportion when an increase (or decrease) of kind is accompanied by a decrease (or an increase) in the order.

Consider the following example showing the speed of a train in kms per hour and distance of 150 kms covered in hours.

Speed of a train in kms per hour   10, 15, 20, 25, 30

Time taken for journey of 150 kms   15, 10, $7\frac{1}{7}$, 6, 5

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Here, we notice that as the speed of the train decreases, the time for the journey increases proportionately, and conversely, as the rate increases the time decreases proportionately, i.e., the speed is inversely proportional to the time in hours for journey.

If $a : b = c : d$ represents a direct proportion, then $a : b = d : c$ represents an inverse proportion. It may also be represented as $b : a = c : d$.

1.5.3 Continued Proportion

Quantities are said to be in continued proportion when the first is to the second, as the second is to the third, as the third is to the fourth and so on.

Thus, $a, b, c, d, \ldots$ are continued proportion when $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, etc.

If the three quantities a, b and c are in continued proportion, then

$a : b = b : c$

$\therefore ac = b^2$

Example 6:

A number is divided into three parts in ratio 2 : 3 : 4. If the third part is 20, then what are the others?

Solution:

Let the three parts be $2x$, $3x$ and $4x$.

Given third part, i.e., $4x = 20 \therefore x = 5$

Hence, first part = $2x = 2 \times 5 = 10$

Second part = $3x = 3 \times 5 = 15$

Example 7:

If a man earns ₹ 65 per week, how long must he work to earn ₹ 780?

Solution:

Let the number of weeks required to a man to earn ₹ 780 be x.

The ratio of earning is ₹ 65 : ₹ 780

The ratio of the weeks is $1 : x$

$\therefore 65 : 780 :: 1 : x$, i.e., $65x = 780$ (direct proportion)

$$\therefore x = \frac{780}{65} = 12 \text{ weeks}$$

Hence, a man works 12 weeks to earn ₹ 780.

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Example 8:

8 men or 16 boys can do a work in 39 days. In how many days will 4 men and 18 boys do it?

Solution:

It is given that 8 men's work = 16 boys' work

$\therefore$ 1 man's work = 2 boys' work

4 men's work + 18 boys' work = 8 boys' work + 18 boys' work

= 26 boys' work

Hence, the given problem in terms of proportion –

16 boys : 26 boys :: x : 39 (inverse proportion)

$$\therefore 26x = 16 \times 39 \qquad \Rightarrow x = \frac{16 \times 39}{26} = 24 \text{ days}$$

$\therefore$ The required number of days is 24.

Example 9:

20 pencils cost as much as 6 dot pens and 15 dot pens as much as 5 pens. If the cost of 3 pens is ₹ 30, find the cost of a pencil.

Solution:

Let x be the cost of a pencil.

₹ x	= 1 pencil (in cost)
20 pencils	= 6 dot pens (in cost)
3 pens	= ₹ 30
15 dot pens	= 5 pens (in cost)

It is a compound proportion, and hence, we have

$$x \times 20 \times 15 \times 3 = 1 \times 6 \times 5 \times 30$$

$$\therefore x = \frac{1 \times 6 \times 5 \times 30}{20 \times 15 \times 3} = ₹\ 1$$

$\therefore$ Cost of pencil is ₹ 1.

Example 10:

8 men and 16 boys can finish a job in 6 days, but 12 men and 24 boys can finish it in 8 days. How man days will 16 men and 20 boys take to finish the job?

Solution:

Problems of this type can be easily solved if equivalence is established between different categories of workmen.

8 men + 16 boys can do the job in 6 days

$\therefore$ 48 men + 96 boys can do it in 1 day

Next, 12 men + 24 boys can do the job in 8 days

$\therefore$ 96 men + 192 boys can do it in 1 day

Hence, in doing one day's work, it may be stated that

48 men + 96 boys = 96 men + 192 boys

Or 48 men = 96 boys

Or 1 man = 2 boys

$\therefore$ 8 men + 16 boys = 32 boys can do a job in 6 days

Similarly, 16 men + 20 boys = 52 boys can do a job in how many days?

This is a case of inverse proportion.

$\therefore$ Let the unknown quantity be 'x' days.

$$\Rightarrow 32 : 52 :: x : 6 \qquad \Rightarrow 52x = 32 \times 6$$

$$\therefore x = \frac{32 \times 6}{52} = 3.70 \text{ days}$$

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Example 11:

A contractor undertake a piece of work in 60 days and employs 300 men. But after 40 days, he finds that only 3/5th of the work has been completed. How many extra men to be engaged so that the work may be finished in time?

Solution:

Number of men employed = 300

Let the number of men required to finish the work be x.

Remaining days = 60 – 40 = 20 days

Work completed = 3/5

Work to be completed $= 1 - \frac{3}{5} = \frac{2}{5}$

The above values are the terms of the inverse proportion. Therefore, we have

$$\left.\begin{array}{l} 300 : x \text{ men} \\ 40 \text{ days} : 20 \text{ days} \end{array}\right\} :: \frac{3}{5} : \frac{2}{5} \text{ or } 3 : 2$$

i.e., $x \times 20 \times 3 = 300 \times 40 \times 2$

$$\therefore x = \frac{300 \times 40 \times 2}{20 \times 3} = 400 \text{ men}$$

$\therefore$ More men needed to finish the work is 400 men – 300 men = 100 men

Example 12:

A contractor undertook the work to make 15 kms of roadway in 40 weeks. In 10 weeks, 3 kms were completed by 180 men working 8 hours a day, then the men agreed to work 1 hour a day overtime and some boys were engaged to assist them. The work was finished in the stipulated time (i.e., 40 weeks). How many boys were employed if the work of 3 boys be equal to that of 2 men?

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Solution:

The given stipulated time = 40 weeks

Work done in 10 weeks = 3 kms of roadway

Remaining weeks = 40 weeks – 10 weeks = 30 weeks

Working hours = 8 hours per day in first 10 weeks

Working hours in remaining 30 weeks = (8 + 1) = 9 hours

Work done = 3 kms; work to be done 15 – 3 = 12 kms. The above values are the terms of respective proportion.

The relation,

$$\left.\begin{array}{l}10 \text{ weeks} : 30 \text{ weeks} \\ 8 \text{ hours} : 9 \text{ hours} \\ 180 \text{ men} : x \text{ men}\end{array}\right\} :: 3 : 12 \text{ or } 1 : 4 \text{ (All are direct proportion.)}$$

i.e., $30 \times 9x \times 1 = 10 \times 8 \times 180 \times 4$

$$x = \frac{10 \times 8 \times 180 \times 4}{30 \times 9} = \frac{640}{3}$$

Hence, more men needed $= \frac{640}{3} - 180 = \frac{100}{3}$

Given, that 3 boys' work = 2 men's work

$\therefore$ 1 men's work = $\frac{3}{2}$ boys' work

$\therefore$ Number of boys required $= \frac{100}{3} \times \frac{3}{2} = 50$ boys

Example 13:

Two taps can separately fill a tank in 12 and 15 minutes respectively. The tank, when full, can be emptied by a drain pipe in 20 minutes. When the tank was empty, all the three were opened simultaneously. In what time will the tank be filled up?

Solution:

Tap 1 can full $\frac{1}{12}$ of tank in one minute.

Tap 2 can fill $\frac{1}{20}$ of tank in one minute.

Pipe an empty $\frac{1}{20}$ of tank in one minute.

$\therefore$ In 1 minute, the tank will be $\left(\frac{1}{12} + \frac{1}{15} - \frac{1}{20}\right)$ full.

$$\left(\frac{20 + 16 - 12}{240}\right) = \frac{24}{240} = \frac{1}{10}$$

Hence, in one minute, $\frac{1}{10}$ of tank will be full.

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$\therefore$ The whole tank will get filled in 10 minutes.

Composition of Ratio

(i) Two or more ratios are multiplied together, then they are said to be compounded. $ac : bd$ is the compounded ratio of $a : b$ and $c : d$.

(ii) Two ratio, if antecedent of one is the consequent of the other and *vice versa*, then they are said to be reciprocals to each other.

(iii) If the ratio $a : b$ is compounded with itself, the result is the ratio of $a^2 : b^2$ which is duplicate ratio of $a : b$. Similarly, $a^3 : b^3$ is triplicate ratio $a^4 : b^4$ is quadruplicate ratio of $a : b$ and so on.

Properties of Proportion

If $\frac{a}{b} = \frac{c}{d}$, then their reciprocals are equal, i.e., $\frac{b}{a} = \frac{d}{c}$. This property is called INVERTENDO.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, *then* $\frac{a + c + e}{b + d + f}$ (*ADDENDO*)

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, *then* $\frac{a - c - e}{b - d - f}$ (*SUBTRACTENDO*)

If $\frac{a}{b} = \frac{c}{d}$, we have $ad = bc$ and $\frac{a}{c} = \frac{b}{d}$. This is called ALTERNANDO.

If $\frac{a}{b} = \frac{c}{d}$, adding one on both sides, we get $\frac{a}{b} + 1 = \frac{c}{d} + 1$ *or* $\frac{a + b}{b} = \frac{c + d}{d}$. This is called COMPONENDO.

When $\frac{a}{b} = \frac{c}{d}$, subtracting one on both sides, we get $\frac{a - b}{b} = \frac{c - d}{d}$. This is called DIVIDENDO.

When $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a + b}{b} = \frac{c + d}{d}$ (componendo) and $\frac{a - b}{b} = \frac{c - d}{d}$ (dividendo).

$\therefore \frac{a + b}{a - b} = \frac{c + d}{c - d}$. This process is termed as Componendo et Dividendo.

Example 14:

Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=a$.

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Solution:

By Componendo et Dividendo,

$$\frac{\sqrt{a+x}+\sqrt{a-x}+\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}-\sqrt{a+x}+\sqrt{a-x}}=\frac{a+1}{a-1}\quad\left(\therefore a=\frac{a}{1}\right)$$

Example 15:

If $\dfrac{x}{a+2b+c}=\dfrac{y}{a-c}=\dfrac{z}{a-2b+c}$, prove that

$\dfrac{a}{x+2y+z}=\dfrac{b}{x-z}=\dfrac{c}{x-2y+z}$.

Solution:

Let the given fractions be equal to K.

Then $x=K(a+2b+c),\ y=K(a-c),\ z=K(a-2b+c)$

$$\frac{a}{x+2y+z}=\frac{a}{K(a+2b+c)+2K(a-c)+K(a-2b+c)}$$

$$\frac{a}{4Ka}=\frac{1}{4K}$$

$$\frac{b}{x-z}=\frac{b}{K(a+2b+c)-K(a-2b+c)}$$

$$=\frac{b}{K(a+2b+c-a+2b-c)}$$

$$\frac{b}{4Kb}=\frac{1}{4K}$$

$$\frac{c}{x-2y+z}=\frac{c}{K(a+2b+c)-2K(a-c)+K(a-2b+c)}$$

$$=\frac{c}{K(a+2b+c-2a+2c+a-2b+c)}$$

$$\frac{c}{4Kc}=\frac{1}{4K}$$

$$\therefore\ \frac{d}{x+2y+z}=\frac{b}{x-z}=\frac{c}{x-2y+z}$$

Example 16:

Solve $\dfrac{x + \sqrt{12a - x}}{x - \sqrt{12a - x}} = \dfrac{\sqrt{a} + 1}{\sqrt{a} - 1}$.

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Solution:

By Componendo et Dividendo, we get

$$\frac{\left(x + \sqrt{12a - x}\right) + \left(x - \sqrt{12a - x}\right)}{\left(x + \sqrt{12a - x}\right) - \left(x - \sqrt{12a - x}\right)} = \frac{\left(\sqrt{a} + 1\right) + \left(\sqrt{a} - 1\right)}{\left(\sqrt{a} + 1\right) - \left(\sqrt{a} - 1\right)}$$

$$\Rightarrow \frac{x}{\sqrt{12a - x}} = \frac{\sqrt{a}}{1} = \frac{x^2}{12a - x} = \frac{a}{1}$$

Squaring both sides, we get

$$\Rightarrow x^2 = 12a^2 - ax \quad \Rightarrow x^2 - 12a^2 + ax = 0$$

$$\Rightarrow x^2 + 4ax - 3ax - 12a^2 = 0$$

$$\Rightarrow x(x + 4a) - 3a(a + 4a) = 0$$

$$\Rightarrow (x + 4a)(x - 3a) = 0$$

$$\therefore x = 3a \text{ or } -4a$$

Example 17:

If $\dfrac{y+z}{b+c} = \dfrac{z+x}{c+a} = \dfrac{x+y}{a+c}$, show that

$$\frac{2(x+y+z)}{(a+b+c)} = \frac{(b+c)x+(c+a)y+(a+b)z}{bc+ca+ab}.$$

Solution:

We have, $\dfrac{y + z}{b + c} = \dfrac{z + x}{c + a} = \dfrac{x + y}{a + b} = \dfrac{x + y + z}{a + b + c}$...............$(i)$

Also,

$$\frac{a(y + z)}{a(b + c)} = \frac{b(z + x)}{b(c + a)} = \frac{c(x + y)}{c(a + b)}$$

$$= \frac{ay + az + bz + bx + cx + cy}{2(ab + bc + ca)} \text{..........}(ii)$$

By (i) and (ii), we get

$$\frac{2(x + y + z)}{a + b + c} = \frac{x(b + c) + y(c + a) + z(a + b)}{(ab + bc + ca)}$$

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Example 18:

If $\frac{x}{ax+by+cz}=\frac{y}{bx+cy+az}=\frac{z}{cx+ay+bz}$, show that each ratio is equal to $\frac{1}{a+b+c}$.

Solution:

$$\text{Each ratio} = \frac{\text{Sum of Numerators}}{\text{Sum of Denominators}}$$

$$= \frac{x+y+z}{ax+by+cz+bx+cy+az+cx+ay+bz}$$

$$= \frac{x+y+z}{a(x+y+z)+b(x+y+z)+c(x+y+z)}$$

$$= \frac{x+y+z}{(x+y+z)+(a+b+c)} = \frac{1}{a+b+c}$$

Example 19:

A bill for ₹ 42,000 was drawn on 1-4-2018 at 6 months' date. It was discounted on 11-5-2018 at 12% p.a. Calculate:

(i) B.D.

(ii) Present worth

(iii) T.D.

(iv) B.G.

Solution:

A = ₹ 42,000, R = 12%.

Bill was drawn on 1-4-2018.

Due date of the bill: 1-4-2018 + 6 months + 3 (grace days)

The bill was discounted on 11-5-2018.

Unexpired period = May (20) + June (30) + July (31) + August (31) + September (30) + October (4) = 146 days

1. $B.D. = \frac{ANR}{100} = \frac{42,000 \times 146 \times 12}{365 \times 100} = 2016$

2. Present worth $= \frac{100A}{100+NR} = \frac{100 \times 42,000}{100 + \frac{146 \times 12}{365}} = \frac{42,000}{100+4.8}$

$$= \frac{42,00000}{104.8} = 40076.34$$

3. $T.D. = \frac{ANR}{100} = \frac{42,000 \times 0.40 \times 12}{100 + 0.40 \times 12} = \frac{2,01,600}{100 + 4.8} = 1923.66$

$N = \frac{146}{365} = 0.40$

4. $B.G. = B.D. - T.D. = 2016 - 1923.66 = 92.34$

NOTES

Example 20:

Find the third proportion of 4 and 8.

Solution:

Let the third proportion be x.

$4 : 8 : x$ are in the third proportion.

$$8^2 = 4x \Rightarrow 4x = 64 \Rightarrow x = \frac{64}{4} \Rightarrow x = 16$$

$\therefore$ 16 is the third proportion to 4 : 8 : 16.

Example 21:

Find the fourth proportion of 6 : 8 : 9.

Solution:

Let the fourth proportion be x.

$$\Rightarrow \frac{6}{8} :: \frac{9}{x} \Rightarrow 6x = 72 \Rightarrow x = \frac{72}{6}$$

$\therefore x = 12$

Example 22:

Express 0.235 as a rational number.

Solution:

Rational number are expressed in the form of $\frac{x}{y}$, where x and y are integers.

Given 0.235

$$\therefore 0.235 = \frac{235}{1000} = \frac{47}{200}$$

Example 23:

Two numbers are in the ratio of 7 : 3. Their difference is 20. Find the number.

Solution:

Let the number be $7x : 3x$.

Their difference $7x - 3x = 20$.

$$\Rightarrow 4x = 20 \Rightarrow x = \frac{20}{4} : x = 5$$

Hence, the required number are $7x$ and $3x$

$$= 7(5) = 3(5)$$

$$= 35 = 15$$

NOTES

Example 24:

A number is divided into three parts in the ratio of 2 : 3 : 4. If the second part is 81, then find the other numbers.

Solution:

Let the numbers be $a + d$, a and $a + d$. Hence, $a = 81$.

Ratio between 1st and 2nd numbers = 2 : 3

$$\therefore \frac{a-d}{a} = \frac{2}{3} \Rightarrow \frac{81-d}{81} = \frac{2}{3} \Rightarrow 3(81-d) = 2(81)$$

$$\Rightarrow 243 - 3d = 162 \Rightarrow -3d = 162 - 243 \Rightarrow -3d = -81 \Rightarrow d = \frac{-81}{3}$$

$$\therefore d = 27$$

1st number = 81 – 27 = 54

Now, ratio of 2nd and 3rd number = 3 : 4

$$\therefore \frac{a}{a+e} = \frac{3}{4} \Rightarrow \frac{81}{81+e} = \frac{3}{4} \Rightarrow 3(81+e) = 81 \times 4$$

$$\Rightarrow 243 - 3e = 324 \Rightarrow 3e = 324 - 243 \Rightarrow 3e - 81 \Rightarrow e = \frac{81}{3}$$

$$\therefore \quad e = 27$$

$\therefore$ 3rd number = 81 + 27 = 108

Example 25:

A's monthly salary is ₹ 250 and *B*'s annual income from agriculture is ₹ 4,000. What is the ratio of their income?

Solution:

A's monthly salary	=	250
$\therefore$ A's monthly income	=	$250 \times 12 = 3,000$
B's annual income	=	4,000
The ratio of their income	=	3000 : 4000 = 3 : 4

Example 26:

Two numbers are in the ratio 5 : 8. If the sum of the numbers is 182, then find the numbers.

Solution:

Let the numbers be $5x$ and $8x$.

$$5x + 8x = 182$$

$$\Rightarrow \quad 13x = 182 \Rightarrow x = \frac{182}{13} \therefore x = 14$$

1^{st} number $= 5 \times 14 = 70$

2^{nd} number $= 8 \times 14 = 112$

NOTES

Example 27:

The ratio of price of two vehicles was 8 : 7. Three years later, when the price of first had increased by ₹ 880 and the second by 10%, the ratio of their prices increases. What were their original prices?

Solution:

Let the original prices of the two vehicles be $8x$ and $7x$.

Consider,

$$\frac{8x + 880}{7x + \frac{10}{100} of\ 7x} = \frac{13}{11}$$

$$\Rightarrow \frac{8x + 880}{7x + 0.7x} = \frac{13}{11}$$

$$\Rightarrow 11(8x + 880) = 13(7x + 0.7x)$$

$$\Rightarrow 88x + 9680 = 13(7.7x)$$

$$\Rightarrow 88x + 9680 = 100.1x$$

$$\Rightarrow 100.1x - 88x = 9680$$

$$\Rightarrow \quad 12.1x = 9680 \Rightarrow x = \frac{9680}{12.1}$$

$$\Rightarrow \quad x = 800$$

Original price of the two vehicles are:

1^{st} vehicle $= 8x = 8 \times 800 = 6400$

2^{nd} vehicle $= 7x = 7 \times 800 = 5600$

Example 28:

The ratio of incomes, expenses and savings of A and B are respectively 5 : 3 : 8 : 5 and 2 : 1. The joint savings of both them are ₹ 3,600 in a year. Find their monthly incomes.

NOTES

Solution:

Given, the ratio of their incomes = 5 : 3

Let A's Income $= 5x$

B's Income $= 3x$

Again, given, the ratio of their saving = 2 : 1

Sum of the values = 2 + 1 = 3

The joint savings = 3,600

A's savings $= \frac{2}{3} \times 3600 = \frac{7200}{3} = 2,400$

B's savings $= \frac{1}{3} \times 3600 = \frac{3600}{3} = 1,200$

A's expenditure = $5x - 2400$

B's expenditure = $3x - 1200$

But given that ratio of expenditure = 8 : 5

i.e., $(5x - 2400) : (3x - 1200) = 8 : 5$

$\Rightarrow 8(3x - 1200) = 5(5x - 2400)$

$\Rightarrow \quad 24x - 9600 = 25x - 12000 \Rightarrow 25x - 24x = 12000 - 9600$

$\Rightarrow \quad x = 2400$

Thus,

A's income $= 5 \times 2400 = 12,000$

B's income $= 3 \times 2400 = 7,200$

Example 29:

If $A : B : = 2 : 3$, $B : C = 4 : 5$ and $C : D = 6 : 7$, then find the ratio between A and B.

Solution:

Let $A : B = 2 : 3$, $B : C = 4 : 5$ and $C : D = 6 : 7$

Consider,

$$\frac{A}{D} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \Rightarrow \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \Rightarrow \frac{16}{35}$$

$A : D = 16 : 35$

NOTES

Check Your Progress

1. The ratio of price of two vehicles was 8 : 7. Three years later, when the price of first had increased by ₹ 880 and the second by 10%, the ratio of their prices increases. What were their original prices?
2. A number is divided into three parts in the ratio of 2 : 3 : 4. If the second part is 81, then find the other numbers.

1.6 PERCENTAGE

Per cent comes from the Latin word "*Per Centum*". The Latin word *Centum* means 100. For example, a Century is 100 years.

My Dictionary says "Percentage" is "the result obtained by multiplying a quantity by a per cent". So, 10% of 50 apples is 5 apples. Therefore, 5 apples is the percentage.

But in practice, people use both words the same way.

1.6.1 Meaning of Percentage

In mathematics, a percentage is a number or ratio that represents a fraction of 100. It is often denoted by the symbol "%" or simply as "per cent" or "pct". For example, 35% is equivalent to the decimal 0.35, or the fraction $\frac{35}{100}$.

1.6.2 Percentage Formula

Although the percentage formula can be written in different forms, it is essentially an algebraic equation involving three values:

$$P \times V_1 = V_2$$

where P is the percentage, V_1 is the first value that the percentage will modify, and V_2 is the result of the percentage operating on V_1. The calculator provided automatically converts the input percentage into a decimal to compute the solution. However, if solving for the percentage, the value written will be the actual percentage, not its decimal representation.

Example: $P \times 30 = 1.5$

$$P = \frac{1.5}{30} = 0.05 \times 100 = 5\%$$

1.6.3 Percentage Difference Formula

The percentage difference between two values is calculated by dividing the absolute value of the difference between two numbers by the average of those two numbers. Multiplying the result by 100 will yield the solution in per cent, rather than decimal form.

$$\text{Percentage Difference} = \frac{|V_1 - V_2|}{(V_1 + V_2)/2} \times 100$$

NOTES

Example: $\dfrac{|12 - 8|}{(12 + 8)/2} = \dfrac{4}{10} = 0.4 = 40\%$

1.6.4 Percentage Change Formula

Percentage increase and decrease are calculated by computing the difference between two values and comparing that difference to the initial value. Mathematically, this involves using the absolute value of the difference between two values, and dividing the result by the initial value, essentially calculating how much the initial value has changed.

$$\text{Percentage change} = \frac{|\text{New value} - \text{Initial value}|}{\text{Initial value}} \times 100$$

The percentage increase calculator above computes an increase or decrease of a specific percentage of the input number. It basically involves converting a percent into its decimal equivalent, and either subtracting or adding the decimal equivalent from and to 1, respectively. Multiplying the original number by this value will result in either an increase or decrease of the number by the given per cent.

Example: 600 increased by 10% (0.1)

$600 \times (1 + 0.1) = 660$

600 decreased by 10%

$600 \times (1 - 0.1) = 540$

Example 30:

Calculate 25% of 90.

Solution:

$25\% = \dfrac{25}{100}$

and $\dfrac{25}{100} \times 90 = 22.5$

So, 25% of 90 is 22.5.

Example 31:

15% of 300 oranges are bad. How many orange are bad?

Solution:

$15\% = \dfrac{15}{100}$

and $\frac{15}{100} \times 300 = 45$

So, 45 oranges are bad.

NOTES

Example 32:

A whiteboard is reduced 20% in price in a sale. The old price was ₹ 150. Find the new price.

Solution:

First, find 20% of ₹ 150:

$$20\% = \frac{20}{100}$$

and $\frac{20}{100} \times 150 = 30$

20% of 150 is 30.

So, the reduction is 30 from original price, i.e., (150 – 30) = 120.

1.7 COMMISSION

A commission is the amount of money paid to an employee for selling something. It is usually a percentage. Payment for some jobs include an amount per hour as well as a commission on total sales. The commission is a motivation or reward for the employee to sell products. So, the company has more sales and can make more money.

The formula for calculating commission is:

Total Commission = Total Sales × Commission Percentage

Salespeople often receive a commission, or per cent of total sales, for their sales. Their income may be just the commission they earn, or it may be their commission added to their hourly wages or salary. The commission they earn is calculated as a certain percent of the price of each item they sell. That per cent is called the rate of commission.

To find the commission on a sale, multiply the rate of commission by the total sales. Just as we did for computing sales tax, remember to first convert the rate of commission from a per cent to a decimal.

The Gupta family's house was sold for ₹ 300,000. How much money will they have after they pay their real estate agent a 5% commission?

Commission is paid to an employee or company as an incentive to sell more. A commission is generally a percentage of sales. The real estate agent was hired by the Gupta family to sell their house for a 5% sales commission.

NOTES

Example 33:

An agent got 2% commission for selling a plot for ₹ 1,000,000. Find his commission.

Solution:

Let Sale Price = ₹ 1,000,000

Commission Rate = 2%

Amount of Commission = 2% × 1,000,000

= 2/100 × 1,000,000

= ₹ 20,000

Example 34:

An agent received ₹ 6,000 as commission for selling a house. If his commission rate is 2%, what is the selling price?

Solution:

Let Selling Price = x

Commission Rate = 2%

Commission = (Selling Price) (Rate)

6000 = x × 2/100

x = ₹ 3,00,000

Check Your Progress

3. Calculate 25% of 90.
4. An agent got 2% commission for selling a plot for ₹ 1,000,000. Find his commission.

1.8 DISCOUNT AND BROKERAGE

Trade discount is a discount which is referred to as discount given by the seller to the buyer at the time of purchase of goods. It is given as a deduction in the list price or retail price of the quantity sold. This is usually allowed by the sellers to attract more customers and receive the order in bulk, i.e., to increase the number of sales. No record is to be maintained in the books of accounts of both the buyer and seller for this discount.

Features of Trade Discount

1. It is usually allowed with the aim of facilitating bulk sales.
2. It can be generally allowed to all customers who want to purchase in bulk.
3. No entry is made in the books of accounts of both the buyer and seller in case of trade discount.

4. It is always deducted before any type of an exchange takes place. So, it does not form part of the books of accounts of the business.
5. It is usually allowed at the time of purchase.
6. It usually differs from the number of goods purchased and the number of purchases.

NOTES

Cash Discount

A cash discount, also called a purchase discount or sales discount, is a reduction in the purchase price of a good because of early cash payment. In other words, the seller of goods is willing to reduce the price of the goods if the buyer is willing to pay for the good earlier.

Cash Discount = Purchase Price × Discount Rate

The discount rate may be expressed as either a percentage or a decimal number. For example, the discount rate can be expressed as either 2% or .02. The formula can be expressed algebraically as:

$CD = P \times R$

where CD = the cash discount, P = the price and R = the discount rate expressed as a decimal.

Differences between Trade Discount and Cash Discount

Trade Discount	Cash Discount
1. It is given during sale/purchase of goods.	1. It is given during time of payment.
2. It is directly proportional to the magnitude of the sales/purchases.	2. It is inversely proportional to the amount of time taken before making a payment.
3. It is not recorded in the cash book. It is recorded on the sales/ purchases books.	3. It is recorded in the cash book as discount allowed on the debit side.
4. The deduction of this discount is made on the invoice value of the goods purchased.	4. The deduction is not made on the invoice value of the goods purchased.
5. It can be found in the purchases book or the sales book as a deduction from the sale/purchase price.	5. It can be found on the debit side of the seller's cash book as discount allowed and on the profit and loss account.
6. Its allowed as a temporary business or product promotion measure as per the announced business strategy.	6. Its allowed in cash when payments are made promptly as per the policy the business has declared although there is no set provision for cash discount.

Bills Discounting Procedure

A bills of exchange have different options to exchange the bill.

(i) Bill can be hold till the date of maturity and received the cash.

(ii) Bill can transfer to any one of the creditors by endorsing the bill.

(iii) Bill can be discounted with the banker before maturity date on behalf of discount.

Discount Date

In bills of exchange, the drawer holds the bill up to maturity date. If before the maturity date, the bill is discounted with banks on which date, that period is called discount date.

Discount Period

When the bill is discounted from that date to maturity date of bills, it is called discount period.

Discount Rate

The rate charged by the banker after the discounting the bill is called discount rate. Generally, discount rate is calculated by simply interest for discounting the bill.

The Present Worth and Discount

The present worth of a sum of money due to the end of any period is its actual value at the present time.

For example, if ₹ 500 is to be paid at the end of 2 years, interest being 5% per annum, it is clear that the sum of money is now less than ₹ 500. The sum of money which should be deducted on account of immediate payment is called the discount of the sum due, sometime hence. Thus, we get the following relations:

1. Discount = Given sum – Its present worth
2. Present worth = Given sum – Its discount

The present worth of a sum of money + the interest on the present worth = the given sum = its present worth + its discount.

It can be stated that the discount of a sum of money is the interest on the present worth of the sum for the given sum.

Types of Discount

(i) Banker's Discount

(ii) True Discount

Banker's Discount (BD)

The difference between the amount of the bill and the amount received by the creditor is called the " Banker's discount". Banker's discount is usually the simple interest on the bill itself (Present Worth + Interest).

$BD = F.t.r.$

where, F = Face value

t = Discount period

r = Discount rate

NOTES

True Discount (TD)

True Discount is the difference between the face value of the bill and the true present value. It is different from Banker's Discount.

$TD = p.t.r$

where, p = True present value

t = Discount period

r = Discount rate

So, Banker's Discount is always greater than the True Discount. The difference between the Banker's Discount (BD) and the True Discount (TD) is called the "Banker's Gain" (BG).

Mathematically, BG = BD – TD

If P is the principal (present worth), N is the number of year, R is the rate of interest and A is the given sum (sum due), then we have the following formulae:

A = PW + Its discount

$$= p + \frac{PNR}{100} = P\left(1 + \frac{NR}{100}\right) \therefore A = P\left(1 + \frac{NR}{100}\right)$$

$$\therefore P = \frac{A}{1 + \frac{NR}{100}} = \frac{100A}{100 + NR}$$

Therefore, we have $P = \frac{100A}{100 + NR}$ ....(i)

True discount = TD = Given sum – Its present worth

$$= A - P = A - \frac{100A}{100 + NR} = A\left(1 - \frac{100}{100 + NR}\right)$$

$$= A\left(\frac{100 + NR - 100}{100 + NR}\right) \therefore = \frac{ANR}{100 + NR} \text{..............}(ii)$$

Banker's Discount is the simple interest on the face value (sum due) of the bill.

$$BD = \frac{ANR}{100} \text{..............}(iii)$$

Thus, Banker's Gain is the difference between BD and TD.

Thus, $BG = BD - TD = \frac{ANR}{100} - \frac{ANR}{100 + NR}$

$$= ANR\left[\frac{1}{100} - \frac{1}{100 + NR}\right] = ANR\left[\frac{100 + NR - 100}{100(100 + NR)}\right]$$

NOTES

$$= ANR\left[\frac{NR}{100(100 + NR)}\right] \quad \therefore BG = \left[\frac{AN^2R^2}{100(100 + NR)}\right] \text{..........}(iv)$$

Note that:

BD = Interest on PW + Interest on TD

$$= \frac{100A}{100 + NR}\left(\frac{NR}{100}\right) + \textit{Interest on TD}$$

$$= \frac{100A}{100 + NR} + \textit{Interest on TD}$$

$= TD + \textit{Interest on TD}$

$\therefore BD - TD = \textit{Interest on TD}$

We know that $BD - TD = BG$

$\therefore BG = \textit{Interest on TD}$

Consider $BG = \textit{Interest on TD}$

$$= \left[\frac{ANR}{100 + NR}\right]\left[\frac{NR}{100}\right] = \left[\frac{ANR}{100 + NR}\right]\left[\frac{NR}{100}\right]\frac{A}{A}$$

$$= \frac{1}{A}\left[\frac{ANR}{100 + NR}\right]\left[\frac{ANR}{100}\right] = \frac{1}{A}[TD][BD]$$

$\therefore \quad A\ (BG) = (BD)\ (TD)$ . . . . (v)

Consider $BG = \dfrac{AN^2R^2}{100(100 + NR)}$

$$BG(100)(100 + NR) = AN^2R^2$$

$$\therefore \quad A = \frac{BG(100)(100 + NR)}{AN^2R^2} = \frac{A(BG)(100)(100 + NR)}{AN^2R^2}$$

$$= A(BG).\frac{1}{BG} = \frac{A(BG)}{BD - TD}$$

$$\therefore \quad A(BG) = BD.TD$$

$$\therefore \quad A = \frac{BD \times TD}{BD - TD} \text{ . . . . } (vi)$$

Note the following:

(i) Discount is always calculated on the amount.

(ii) Discount = Amount – Present worth.

(iii) Present worth = Amount – Discount.

(iv) Banker's discount is the simple interest on the face value or amount or sum due.

(v) Banker's Gain = $BD - TD$.

(vi) Interest on $TD = BD - TD$.

NOTES

Brokerage

A brokerage fee compensates a broker for executing a transaction. It is usually, but not always, a percentage of the transaction value. In finance, stockbrokers most often come to mind, but real estate agents and business brokers frequently charge brokerage fees.

Problems on Trade Discount and Cash Discount

Example 35:

The list price of an article is ₹ 10,000. The producer offers 20% trade discount to the customer. A cash discount of 3% is offered for immediate payment. Assuming that the customer makes immediate payment for the purchase, compute the selling price.

Solution:

	₹
List Price	10,000
Less: Trade Discount @ 20% (10,000 × 20%)	2,000
Invoice Price	8,000
Less: Cash Discount @ 3% (8000 × 3%)	240
Selling Price	**7,760**

Example 36:

The list price of TV set sold by a seller is ₹ 50,000. The seller offers 25% trade discount, and in addition, the seller also offers 2% cash discount, if the account is settled within 15 days. Assume that the payment is made within 15 days by the customer. Find out the selling price of the book.

Solution:

	₹
List Price of books	50,000
Less: Trade Discount @ 25% (50,000 × 25%)	12,500
Invoice Price	37,500
Less: Cash Discount @ 2% (37,500 × 2%)	750
Hence, Selling Price	**36,750**

NOTES

Example 37:

The catalogue price of an article sold is ₹ 89,000. The trade discount and cash discount are 10% and 5% respectively. Find out the net cash price of the article sold.

Solution:

	₹
Catalogue Price of the article	89,000
Less: Trade Discount @ 10% (₹ 89,000 × 10%)	8,900
Invoice Price	80,100
Less: Cash Discount @ 5% (₹ 80,100 × 5%)	4,005
Hence, Selling Price	**76,095**

Example 38:

A manufacturer offers his goods to a wholesaler at a trade discount of 20% of the list price. He also allows a cash discount of 2% for immediate payment. If he makes a profit of 10%, at what percent the list price is higher than the cost price?

Solution:

	₹
Let us assume list price as	100
Less: Trade Discount @ 20%	20
Invoice Price	80
Less: Cash Discount @ 2%	4
Selling Price	**76**

Percentage of Profit = 10%

Selling Price = Cost + Profit

= 100 + 10

= 110

If the selling price is ₹ 110, cost = 100

If the sale price ₹ 76, cost = 76 × 100 ÷ 110

Therefore, cost price = 69.09

Amount of profit = 100 – 69.09

= ₹ 30.91

Percentage by which list price isgreater than cost price

= 30.91 × 100 ÷ 69.09

= 44.74%

NOTES

Example 39:

A retailer buys a product at 20% trade discount on the catalogue price of ₹ 8,000 and also a cash discount of 2%. He sells it at the catalogue price. What is the amount of profit and percentage of profit made by the retailer?

Solution:

	₹
Catalogue Price of the product	8,000
Less: Trade Discount @ 20% (8,000 × 20%)	1,600
Invoice Price	6,400
Less: Cash Discount @ 2% (6400 × 2%)	128
Hence, Selling Price	**6,272**

Price at which the product is sold by the retailer = ₹ 8,000

Therefore, profit made by him = ₹ 8,000 – ₹ 6272

= ₹ 1728

Percentage of profit = 1728 × 100 ÷ 8000

= 21.6%

Example 40:

A product is sold by the producer to a retailer by allowing 20% discount on the invoice price. The retailer sells it at 10% below the marked price. If a customer pays ₹ 900 for the product, what percentage of profit made by the retailer?

Solution:

	₹
Price paid by the customer (90% of marked price)	900
Thus, invoice price (₹ 900 × 100 ÷ 90)	1,000
Cost to the retailer (₹ 1,000 × 20%)	800
Profit made by the retailer (₹ 900 – ₹ 800)	100

Percentage of profit (Profit × 100 ÷ Cost) = ₹ 100 × 100 ÷ ₹ 800)

= 12.5%

Example 41:

A producer sells 10 chairs to a retailer at a list price of ₹ 2,000 each. The trade discount applicable is 20% and cash discount is 2%, if the payment is made within 10 days. The retailer has paid the worth of 5 chairs on 4th day and the remaining payment is made after 15 days. Find out the total amount to be paid to the producer by the retailer.

NOTES

Solution:

	₹
Catalogue Price of 10 chairs @ ₹ 2,000 each	20,000
Less: Trade Discount @ 20% (20,000 × 20%)	4,000
Invoice Price	16,000
Less: Cash Discount @ 2% for 5 chairs (16,000 × 2% × 50%)	160
Hence, Selling Price	**₹ 15,840**

Example 42:

The catalogue price of an article is ₹ 25,000. Trade discount applicable is 20%. Find out the net cash price if:

(a) the cash discount is 5%

(b) the cash discount is 2%.

Solution:

	₹
Catalogue Price of the article	25,000
Less: Trade Discount @ 20% (25,000 × 20%)	5,000
Invoice Price	20,000
(a) If the cash discount is 5%	
Less: Cash Discount @ 5% (20,000 × 5%)	1,000
Hence, Selling Price	**₹ 19,000**
(b) If the cash discount is 2%	
Less: Cash Discount @ 2% (20,000 × 2%)	400
Hence, Selling Price	**₹ 18,600**

Example 43:

A retailer offers 20% trade discount and also 5% cash discount for the table fans sold by him. State Bank of India has bought 20 such table fans by paying ₹ 1,67,200. What is the catalogue price of each table fan?

Solution:

	₹
Price paid by SBI for 20 fans	₹ 1,67,200
Invoice Price of 20 fans (1,67,200 × 100 ÷ 95)	₹ 1,76,000
Therefore, catalogue price of 20 fans (₹ 1,76,000 × 100 ÷ 80)	2,20,000
Thus, catalogue price of each fan (₹ 2,20,000 ÷ 20)	11,000

NOTES

Example 44:

The wholesaler offer a product at catalogue price of ₹ 1,00,000 with 20% cash discount and 5% cash discount to the retailer. Retailer sells it at 10% lesser than the catalogue price by offering 2% cash discount to the final customer. Find out:

(a) Profit made by the retailer

(b) Price to be paid by the customer

Solution:

	₹
Catalogue Price of the product	1,00,000
Less: Trade Discount @ 20% (1,00,000 × 20%)	20,000
Invoice Price	80,000
Less: Cash Discount @ 5% (80,000 × 5%)	4,000
Hence, Selling Price to retailer	76,000
Price at which the product is sold by the retailer (1,00,000 – 10% of 1,00,000)	90,000
Less: Cash Discount to the customer @ 2% (90,000 × 2%)	1,800
Hence, Selling Price to customer	**88,200**
(a) Profit made by the retailer (88,200 – 76,000)	= 12,200
Percentage of profit (12,200 × 100 ÷ 76,000)	= 16.05%
(b) Price to be paid by the customer	= 76,000

Example 45:

After allowing a discount of $7\frac{1}{2}\%$ on the marked price of an article, an article is sold for ₹ 555. Find its market price.

Solution:

Let the market price be x.

Discount rate – Discount of market price = Price sold

$$\Rightarrow \quad x - 7.5\%x = 555 \Rightarrow x = \frac{7.5x}{100} = 555$$

$$\Rightarrow \quad \frac{92.5x}{100} = 555 \Rightarrow x = \frac{55,500}{92.5}$$

$$\therefore \quad x = 600$$

Hence, market price is ₹ 600.

NOTES

Example 46:

A machine listed at ₹ 625 was sold at a discount of 15%. Find the discount and the net price.

Solution:

Discount $= 625 \times 0.15 =$ ₹ 93.75

$\therefore$ Net price $=$ ₹ 625 – ₹ 93.75 = ₹ 531.25

Example 47:

What is the face value of the bill discounted at 5% p.a. 73 days earlier than the date of maturity, the Banker's Gain being ₹ 10 only?

Solution:

Let the face value be F.

Rate of interest $R = 5\%$, Time Period $T = 73$ days and Banker's gain = ₹ 10

Banker Gain = True Discount $\times T \times R$

$$\Rightarrow 10 = TD \times \frac{73}{365} \times \frac{5}{100} \qquad \Rightarrow 10 = \frac{365 \times TD}{365 \times 100}$$

$$\Rightarrow TD = 1{,}000$$

Again,

Banker's Gain = Banker Discount – TD

10 = BD – 1,000

$$\Rightarrow BD - 1000 - 10 = 990$$

Face value of the bill: $BD = F \times T \times R$

$$\Rightarrow 990 = F \times \frac{73}{365} \times \frac{5}{100}$$

$$\Rightarrow 990 = \frac{F}{365} \Rightarrow F = 990 \times 100$$

$\therefore F = 99{,}000$

Hence, face value of the bill is ₹ 99,000.

Example 48:

Find Banker's Gain on a certain bill due 6 months hence is ₹ 40. The rate of interest being 20% p.a. Find the face value of the bill.

Solution:

Here, $N = 6/12 = 1/2$, $BG =$ ₹ 40 and $R = 20\%$

$$BG = \frac{AN^2R^2}{100(100 + NR)}$$

NOTES

$$\Rightarrow 40 = \frac{A\left[\frac{1}{2}\right]^2 (20)^2}{100\left(100 + \frac{1}{2} \times 20\right)} = \frac{A.\frac{1}{4} \times 400}{100(100 + 10)}$$

$$\therefore \quad 40 = \frac{100A}{11,000} = \quad \therefore A = ₹\ 4,400$$

Hence, the face value of the bill is ₹ 4,400.

Example 49:

A bill for ₹ 8,500 on 25th April for 5 months was discounted on 17 July at the rate of 5 p.a. Find the discounted value of the bill.

Solution:

Let F = Face value, R = Rate of interest and T = Time period

Now,

Time: July – 14 days, August – 31 days, September – 28 days

Total = 73 days

Discounted value $= F(1 - TR)$

$$= 8500\left(1 + \frac{73}{365} \times \frac{5}{100}\right)$$

$$= 8500\left(1 + \frac{1}{100}\right) = 8500 \times 0.99$$

$$= ₹\ 8,415$$

Example 50:

A man sold two ratios at ₹ 924 each. On one, he gains 20% and on another, he loses 20%. How much does he gain or lose on the whole transaction?

Solution:

Selling price of 1st ratio = ₹ 924

Gain = 20%

Let the cost price by x.

$$\therefore x + \frac{20}{100}x = 924 \qquad \Rightarrow \frac{6x}{5} = 924$$

$$x = 770$$

Gain = 924 – 770 = ₹ 154

Selling price of 2nd ratio = ₹ 154

Loss = 20%

NOTES

Let the cost price be y.

$$y - 20/100 = 924$$

$$4y/5 = 924$$

$$y = ₹\ 1,155$$

Loss = 1,155 – 924 = ₹ 231

Since loss is greater than gain, the whole transaction resulted in a loss of (₹ 231 – ₹ 154) = ₹ 77.

Example 51:

The difference between *TD* and *BD* on a bill due 6 months at 4% is ₹ 24. Find the *TD*, *BD* and bill amount.

Solution:

Here, $BG = BD - TD = ₹\ 24$

$$N = \frac{6}{12} = \frac{1}{2} \text{ years,}$$

$$R = 4\%.$$

We know,

$$TD = \frac{ANR}{100 + NR} = \frac{A \times \frac{1}{2} \times 4}{100 + \frac{1}{2} \times 4} = \frac{2A}{102} = \frac{A}{51}$$

$$BD = \frac{ANR}{100} = \frac{A \times \frac{1}{2} \times 4}{100} = \frac{2A}{100} = \frac{A}{50}$$

Consider, $BG = BD - TD$. We have $BG = 24$

$$i.e., \quad 24 = \frac{A}{50} - \frac{A}{51} \Rightarrow 24 = \frac{51A - 50A}{2550}$$

$$\Rightarrow \quad 24 = \frac{A}{2,550} \quad \therefore A \Rightarrow 61,200$$

Face value of the bill = ₹ 61,200

$$\text{Now, TD} = \frac{A}{51} = \frac{61,200}{51} = ₹\ 1,200$$

$$\text{and } BD = \frac{A}{50} = \frac{61,200}{50} = ₹\ 1,224.$$

Example 52:

Calculate the present value of annuity at ₹ 5,000 p.a. for 12 years' interest being 4% p.a. compound annually.

Solution:

Given, A = ₹ 5,000, r = 4%, = 0.04 and n = 12 years.

Consider, Present value of an annuity =

$$\frac{a}{r}\left[1-(1+r)^{-n}\right] \Rightarrow \frac{5000}{0.04}\left[1-(1+0.04)^{-12}\right]$$

$$= \frac{5,00,000}{4}\left[1-(1.04)^{-12}\right]$$

$$= 1,25,000\left[1-\frac{1}{(1.04)^{12}}\right]$$

$$= 1,25,000\left(1-\frac{1}{1.601}\right) = 1,25,000(1-0.6245) = 1,25,500(0.3754)$$

$$= 46,925$$

Example 53:

A bill for ₹ 12,750 drawn on 27th May for 4 months was discounted on 19th July at 4% p.a. Find:

(i) Banker's discount

(ii) True discount

(iii) The Banker's gain and

(iv) How much the holder of bill received?

Solution:

A = ₹ 12750	the date on which the bill is drawn = May 27
R = 4%	the date on which the bill is due = September 30
n = ?	the date on which the bill is discounted = July 19

Unexpired period July 19 to September 30

July + August + September

12 + 31 + 30 = 73 days

$$TD = \frac{ANR}{100+NR}$$

$$= \frac{12750 \times \frac{1}{5} \times 4}{100+\left(\frac{1}{5}\times 4\right)}$$

$$= \frac{\frac{51000}{5}}{100+\frac{4}{5}}$$

$$= \frac{10,200}{\frac{500 + 4}{5}}$$

NOTES

$$= \frac{10,200}{\frac{504}{5}}$$

$$= \frac{10,200}{100.8}$$

$$= ₹\ 102$$

$$TD = \frac{ANR}{100 + NR}$$

$$= \frac{12750 \times \frac{1}{5} \times 4}{100 + \left(\frac{1}{5} \times 4\right)} = \frac{\frac{51000}{5}}{100 + \frac{4}{5}}$$

$$= \frac{10,200}{\frac{500 + 4}{5}} = \frac{10,200}{\frac{504}{5}} = \frac{10,200}{100.8}$$

$$= 101.19$$

$$BG = BD - TD = 102 - 101.19 = 0.81$$

Amount received by the holder of the bill = 12,750 – 102

= ₹ 12,548

Example 54:

The Banker's gain on a bill due in 4 months discounted @ 15% p.a. is ₹ 360. Find:

(i) True Discount.

(ii) Banker's Discount.

(iii) Face value of the bill.

Solution:

Let, T = 4 Months, R =15% pa, Banker's Gain = ₹ 360

(i) True discount:

Banker's Gain = $TD \times T \times R$

$$\Rightarrow 360 = TD \times \frac{4}{12} \times \frac{15}{100}$$

$$\Rightarrow TD = \frac{360 \times 12 \times 100}{4 \times 15} \Rightarrow TD = 7,200$$

(ii) Banker's discount:

Banker's gain = $BD - TD$

$360 = BD - 7{,}200$

$BD = 7{,}200 + 360$

$BD = 7{,}560$

(iii) Face value of the bill:

$BD = F \times T \times R$

$$\Rightarrow 7{,}560 = F \times \frac{4}{12} \times \frac{15}{100}$$

$$\Rightarrow F = \frac{7{,}560 \times 12 \times 100}{4 \times 15}$$

$$\Rightarrow F = \frac{9072000}{60}$$

$$\Rightarrow F = 1{,}51{,}200$$

NOTES

Check Your Progress

5. A manufacturer offers his goods to a wholesaler at a trade discount of 20% of the list price. He also allows a cash discount of 2% for immediate payment. If he makes a profit of 10%, then at what per cent, the list price is higher than the cost price?
6. The difference between *TD* and *BD* on a bill due 6 months at 4% is ₹ 24. Find the *TD*, *BD* and bill amount.

1.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The ratio of price of 2 vehicles was 8 : 7. Three years later, when the price of first had increased by ₹ 880 and the second by 10%, the ratio of their prices became bill. What were the original prices?

   Solution:

   Let the original of the 2 vehicles be $8x$ and $7x$.

   Consider,

$$\frac{8x + 880}{7x + \frac{10}{100} of\ 7x} = \frac{13}{11}$$

$$\Rightarrow \frac{8x + 880}{7x + 0.7x} = \frac{13}{11}$$

$$\Rightarrow 11(8x + 880) = 13(7x + 0.7x)$$

$$\Rightarrow 88x + 9680 = 13(7.7x)$$
$$\Rightarrow 88x + 9680 = 100.1x$$
$$\Rightarrow 100.1x - 88x = 9680$$

NOTES

$$\Rightarrow 12.1x = 9680 \Rightarrow x = \frac{9680}{12.1}$$

$$\Rightarrow x = 800$$

Original price of the two vehicles were:

1st vehicle = $8x = 8 \times 800 = 6{,}400$

2nd vehicle = $7x = 7 \times 800 = 5{,}400$

2. A number is divided into three parts in the ratio of 2 : 3 : 4. If the second part is 81, find the other numbers.

Solution:

Let the numbers be $a + d$, a and $a + d$. Hence, $a = 81$.

Ratio between 1st and 2nd numbers = 2 : 3

$$\therefore \frac{a - d}{a} = \frac{2}{3} \Rightarrow \frac{81 - d}{81} = \frac{2}{3} \Rightarrow 3(81 - d) = 2(81)$$

$$\Rightarrow 243 - 3d = 162 \Rightarrow -3d = 162 - 243 \Rightarrow -3d = -81 \Rightarrow d = \frac{-81}{3}$$

$$\therefore d = 27$$

1st number = 81 – 27 = 54

Now, ratio of 2nd and 3rd number = 3 : 4

$$\therefore \frac{a}{a + e} = \frac{3}{4} \Rightarrow \frac{81}{81 + e} = \frac{3}{4} \Rightarrow 3(81 + e) = 81 \times 4$$

$$\Rightarrow 243 - 3e = 324 \Rightarrow 3e = 324 - 243 \Rightarrow 3e - 81 \Rightarrow e = \frac{81}{3}$$

$$\therefore \quad e = 27$$

$\therefore$ 3rd number = 81 + 27 = 108

3. Calculate 25% of 90.

Solution:

$$25\% = \frac{25}{100}$$

and $\frac{25}{100} \times 90$

= 22.5

So, 25% of 90 is 22.5.

4. An agent got 2% commission for selling a plot for ₹ 10,00,000. Find his commission.

Solution:

Let Sale Price = ₹ 10,00,000

Commission Rate = 2%

Amount of Commission = 2% × ₹ 10,00,000

= 2/100 × ₹ 10,00,000

= ₹ 20,000

5. A manufacturer offers his goods to a wholesaler at a trade discount of 20% of the list price. He also allows a cash discount of 2% for immediate payment. If he makes a profit of 10%, then at what per cent, the list price is higher than the cost price?

Solution:

	₹
Let us assume list price as	100
Less: Trade Discount @ 20%	20
Invoice Price	80
Less: Cash Discount @2%	4
Selling Price	**76**
Percentage of Profit	10%
Selling price	= Cost + Profit
	= 100 + 10
	= 110
If the selling price is ₹ 110, cost	= 100
If the sale price ₹ 76, cost	= 76 × 100 ÷ 110
Therefore, Cost price	= 69.09
Amount of profit	= 100 – 69.09
	= ₹ 30.91
Percentage by which list price is greater than cost price	= 30.91 × 100 ÷ 69.09
	= 44.74%

NOTES

6. The difference between *TD* and *BD* on a bill due 6 months at 4% is ₹ 24. Find the *TD*, *BD* and bill amount.

Solution:

Here, $BG = BD - TD = ₹\ 24$, $N = \frac{6}{12} = \frac{1}{2}$ years, $R = 4\%$.

We know,

$$TD = \frac{ANR}{100 + NR} = \frac{A \times \frac{1}{2} \times 4}{100 + \frac{1}{2} \times 4} = \frac{2A}{102} = \frac{A}{51}$$

$$BD = \frac{ANR}{100} = \frac{A \times \frac{1}{2} \times 4}{100} = \frac{2A}{100} = \frac{A}{50}$$

Consider, $BG = BD - TD$. We have $BG = 24$

$$i.e., \quad 24 = \frac{A}{50} - \frac{A}{51} \Rightarrow 24 = \frac{51A - 50A}{2550}$$

$$\Rightarrow \quad 24 = \frac{A}{2,550} \quad \therefore A \Rightarrow 61,200$$

Face value of the bill = ₹ 61,200

Now, $TD = \frac{A}{51} = \frac{61,200}{51} = ₹\ 1,200$

and $BD = \frac{A}{50} = \frac{61,200}{50} = ₹\ 1,224.$

1.10 SUMMARY

- A ratio is the relationship between two or more variables. The variable should be of same unit and same kind. It is always expressed by the fraction obtained by dividing the first number by the second number.
- Proportion is the equality between two or more ratios. If we express the fact that one ratio is equal to another ratio, then it forms a proportion. That is the ratio $a : b$ be equal to the ratio $c : d$. The four terms a, b, c and d are said to be in proportion. When 'a' is as many times as b, and c is as many times as d, then a, b, c and d are called the terms of the proportion.
- Quantities are said to be in continued proportion when the first is to the second, as the second is to the third, as the third is to the fourth and so on.
- The percentage difference between two values is calculated by dividing the absolute value of the difference between two numbers by the average of

those two numbers. Multiplying the result by 100 will yield the solution in percent, rather than decimal form.

- Percentage increase and decrease are calculated by computing the difference between two values and comparing that difference to the initial value. Mathematically, this involves using the absolute value of the difference between two values, and dividing the result by the initial value, essentially calculating how much the initial value has changed.

NOTES

1.11 KEY TERMS

- **Ratio:** A ratio is the relationship between two or more variables. The variable should be of same unit and same kind.
- **Proportion:** Proportion is the equality between two or more ratios. If we express the fact that one ratio is equal to another ratio, then it forms a proportion.
- **Inverse Proportion:** Quantities are said to be in inverse proportion when an increase (or decrease) of kind is accompanied by a decrease (or an increase) in the order.
- **Continued Proportion :** Quantities are said to be in continued proportion when the first is to the second, as the second is to the third, as the third is to the fourth and so on.
- **Commission:** A commission is the amount of money paid to an employee for selling something. It is usually a percentage.
- **Brokerage:** A brokerage fee compensates a broker for executing a transaction. It is usually, but not always, a percentage of the transaction value.
- **Cash Discount:** A cash discount, also called a purchase discount or sales discount, is a reduction in the purchase price of a good because of early cash payment.
- **Discount Date:** In bills of exchange, the drawer holding the bill up to maturity date, if before the maturity date, the bill is discounted with banks on which date, that period is called discount date.
- **Discount Period:** When the bill is discounted from that date to maturity date of bills is called discount period.
- **Discount Rate:** The rate is charged by the banker after discounting the bill is called discount rate. Generally, discount rate calculated by simply interest for discounting the bill.

1.12 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. Define ratio.
2. What is common ratio?
3. What is duplicate ratio?
4. What is operating ratio?
5. What do you mean by proportion?
6. What is direct proportion?
7. What do you mean by inverse proportion?
8. What do you mean by continued proportion?
9. What is banker discount?
10. What is true discount?

Long Answer Questions

1. State the difference between common ratio and common difference.
2. Difference between ratio and proportion.
3. The ratio of price of two vehicles was 8 : 7. Three years later, when the price of first had increased by ₹ 880 and the second by 10%, the ratio of their prices became bill. What were the original prices?
4. The ratio of incomes, expenses and savings A and B are respectively 5 : 3 : 8 : 5 and 2 : 1. The joint savings of both them are ₹ 3,600 in a year. Find their monthly incomes.
5. If $A : B = 2 : 3$, $B : C = 4 : 5$ and $C : D = 6 : 7$, find the ratio between A and D.
6. The monthly income of Suresh and Mahesh are in the ratio of 7 : 5 and their monthly expenditure are in the ratio of 5 : 3. If each save ₹ 120 per month, then find their monthly income and expenditure.
7. The banker's gain on a bill due in 4 months discounted at @ 15% p.a. is ₹ 360. Find:
   (i) True discount
   (ii) Banker's discount
   (iii) Face value of the bill.
8. Find the third proportion 4 and 8.
9. Find the fourth proportion of 6 : 8 : 9.
10. Express 0.235 as a rational number.

NOTES

11. Two numbers are in the ratio of 7 : 3. Their difference is 20. Find the number.
12. A number is divided into three parts in the ratio of 2 : 3 : 4. If the second part is 81, find the other numbers.
13. *A*' s monthly salary is ₹ 250 and *B*'s annual income from agriculture is ₹ 4,000. What is the ratio of their income?
14. Two numbers are in the ratio 5 : 8. If the sum of the numbers is 182, find the numbers.
15. The difference between *TD* and *BD* on a bill due 6 months at 4% is ₹ 24. Find *TD*, *BD* and bill amount.
16. Calculate the present value of annuity at ₹ 5,000 p.a. for 12 years' interest being 4% p.a. compound annually.
17. A bill for ₹ 12,750 drawn on 27^{th} May for 4 months was discounted on 19^{th} July at 4% p.a. Find:
    (i) Banker's discount
    (ii) True discount
    (iii) Banker's gain
    (iv) How much the holder of bill received?
18. On a bill of ₹ 20,750 due after 8 months at P.A. find the:
    (i) Present value
    (ii) True discount
    (iii) Banker's discount
    (iv) Bankers gain.
19. The banker's gain on a bill due in 4 months discounted at @ 15% p.a. is ₹ 360. Find:
    (i) True discount
    (ii) Banker's discount
    (iii) Face value of the bill.

1.13 FURTHER READING

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UNIT 2 SIMULTANEOUS EQUATIONS

NOTES

Structure

2.0 INTRODUCTION

Equality between two expressions in 'x' gives rise to an equation in 'x'. When terms are brought to one side, the other side comes to zero. Consider the relation $7x = 4x + 3x$ or $(x + y)^2 = x^2 + 2xy + y^2$. These relations are valid for all values of 'x' and 'y'. Such relation is called equation. Here, 'x' is variable.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the concept of Simultaneous Equations
- Discuss the characteristics of Equations
- Describe the types of Equations
- Examine the preparation of Invoice

2.2 CONCEPT OF EQUATION

An equation is a relation between two variables (two or more) and holds good only for certain values of the variables. Thus, it is clear that in an equation, the equality holds for certain values of the variables. However, in case of the identities, the equality holds for any value of variables.

2.2.1 Polynomial Equation

NOTES

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots\dots\dots + a_n$ where $a_0, a_1, a_2, a_3, \dots\dots\dots$ are all real numbers and n is a positive integer, is called a polynomial of n^{th} degree in the variable x. The polynomial may be 1st degree, 2nd degree, 3rd degree, etc. Here, $a_0, a_1, a_2, \dots\dots\dots$ are coefficient of $x^n, x^{n-1}, \dots\dots\dots$ respectively and a_n is called the constant term.

Examples:

(a) $5x + 3 = 0$ is polynomial equation of degree 1

(b) $3x^2 + 5x - 8 = 0$ is polynomial equation of degree 2

(c) $x^3 - 7x^2 + 6x + 12 = 0$ is polynomial equation of degree 3

2.2.2 Roots of an Equation

If $f(x) = 0$ is a polynomial equation in x, then a real number 'a' is said to be a root of the equation $f(x) = 0$, if $f(a) = 0$.

Example

$x = 2$ is the root of the equation $3x - 6 = 0$, for $3(2) - 6 = 0$. Similarly, $x = 2$ and $x = 3$ are called the roots of the quadratic equation $x^2 - 5x + 6 = 0$, because when $x = 2$ or $x = 3$ is substituted for x in the equation $x^2 - 5x + 6 = 0$, the equation is satisfied.

2.2.3 Linear Equations

The equations of degree '*one*' are called linear equations. For example, $x = 1$, $5x + 8 = 0$, $10x + 13 = 0$, etc. are all examples of Linear Equations in one variable x. The equation $ax + b = 0$ $(b \neq 0)$ is called the General Linear Equation in a single variable.

$5x + 8y + 6 = 0$ is a linear equation in two variables and $ax + by + c = 0$ $(a \neq 0)$ and $(b \neq 0)$ is the general form of a Linear Equation in two variables.

Example 1:

Form an equation whose roots are 5 and –8.

Solution:

Roots of the equations are 5 and –8

Sum of the roots $= 5 + (-8) = 5 - 8$

Product $= 5 \times (-8) = -40$

From the linear equation, $ax + b = 0$.

Therefore, $ax = -b \quad \Rightarrow x = \frac{-b}{a}$, which is the root.

$$\frac{-b}{a} = -3 \qquad (a \neq 0, b \neq 0)$$

$$\frac{c}{a} = -40$$

$\therefore$ The equation is $x^2 - 3x - 40 = 0$.

NOTES

Example 2:

Form an equation whose roots are 2 and –3.

Solution:

Let the equation be $x^2 + x(a+b) + ab = 0$.

Consider $x^2 + x(a+b) + ab = 0$

$$\Rightarrow x^2 + x(2-3) + \{2 \times (-3)\} = 0$$

$$\Rightarrow x^2 + x(-1) + (-6) = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$\therefore$ The equation is $x^2 - x - 6 = 0$ whose roots are 2 and –3.

2.3 METHOD OF SOLVING LINEAR (SIMPLE) EQUATIONS

The solutions of the Linear Equations can be obtained by taking the constant term to the right side and dividing by the coefficient of x.

Consider the General Linear Equation $ax + b = 0$

$\therefore ax = -b$

Then, $x = \frac{-b}{a}$ is the root of the equation $ax + b = 0$.

Example 3:

If $10(x - 4) - 15(x - 3) = 3(3 + x) - 20$, find x.

Solution:

Consider $10(x - 4) - 15(x - 3) = 3(3 + x) - 20$

$$10x - 40 - 15x + 45 = 9 + 3x - 20$$

$$\Rightarrow -5x + 5 = 3x - 11$$

$$\Rightarrow -5x - 3x = -11 - 5$$

$$\Rightarrow -8x = -16$$

$$\Rightarrow x = \frac{-16}{-8} \qquad \therefore x = 2$$

Example 4:

Solve for x: $3(4x + 1) - (4x - 1) = 2(x + 5) - 20$.

NOTES

Solution:

Consider $3(4x+1)-(4x-1)=2(x+5)$

$$12x+3-4x+1=2x+10$$

$$\Rightarrow 8x+4=2x+10$$

$$\Rightarrow 8x-2x=10-4$$

$$\Rightarrow 6x=6$$

$$\Rightarrow x=\frac{6}{6}=1$$

$$\therefore \quad x=1$$

Example 5:

If $[(x+2)+(x+3)]^2-[(x+2)-(x+3)]^2=4x^2+64$, find x.

Solution:

Consider $[(x+2)+(x+3)]^2-[(x+2)-(x+3)]^2=4x^2+64$

$$(x+2+x+3)^2-(x+2-x-3)]^2=4x^2+64$$

$$\Rightarrow (2x+5)^2-(-1)^2=4x^2+64$$

$$\Rightarrow 4x^2+20x+25-1=4x^2+64$$

$$\Rightarrow 4x^2-4x^2+20x=-24+64$$

$$\Rightarrow 20x=40 \Rightarrow x=\frac{40}{20}$$

$$\therefore \quad x=2$$

Example 6:

Solve for x : $b(b+x)=a^2-ax$.

Solution:

Consider $b(b+x)=a^2-ax$

$$b^2+bx=a^2-ax$$

$$\Rightarrow ax+bx=a^2-b^2$$

$$\Rightarrow x(a+b)=(a-b)(a+b)$$

$$\Rightarrow x=\frac{(a-b)(a+b)}{(a+b)}$$

$$\therefore \quad x=(a-b)$$

Example 7:

Solve for x : $\frac{3x-1}{2}+\frac{x+2}{3}=\frac{9x+12}{5}-2$.

NOTES

Solution:

Consider $\frac{3x-1}{2}+\frac{x+2}{3}=\frac{9x+12}{5}-2$

$$\frac{3(3x-1)+2(x+2)}{6}=\frac{9x+12-5(2)}{5}$$

$$\Rightarrow \frac{9x-3+2x+4}{6}=\frac{9x+12-10}{5}$$

$$\Rightarrow \frac{11x+1}{6}=\frac{9x+2}{5}$$

Cross multiplying, we get

$$5(11x+1)=6(9x+2)$$

$$\Rightarrow 55x+5=54x+12$$

$$\Rightarrow 55x-54x=12-5$$

$$\therefore\ x=7$$

Example 8:

Solve $\frac{x+1}{3}=2\frac{1}{2}+2x-1$.

Solution:

Consider $\frac{x+1}{3}=2\frac{1}{2}+2x-1$

$$\Rightarrow \frac{x+1}{3}=\frac{5}{2}+2x-1$$

$$\Rightarrow \frac{x+1}{3}=\frac{5+4x-2}{2}$$

Cross multiplying, we get

$$2(x+1)=3(5+4x-2)$$

$$\Rightarrow 2x+2=15+12x-6$$

$$\Rightarrow 2x+2=12x+9$$

$$\Rightarrow 2x-12x=9-2$$

$$\Rightarrow -10x = 7$$

$$\therefore \quad x = \frac{-7}{10}$$

NOTES

Example 9:

Solve $\dfrac{4x+5}{6x-1} = \dfrac{3+2x}{3x-4}$.

Solution:

Consider $\dfrac{4x+5}{6x-1} = \dfrac{3+2x}{3x-4}$

$$= (4x+5)(3x-4) = (3+2x)(6x-1)$$

$$\Rightarrow 12x^2 + 15x - 16x - 20 = 18x - 3 + 12x^2 - 2x$$

$$\Rightarrow 12x^2 - x - 20 = 12x^2 + 16x - 3$$

$$\Rightarrow -x - 20 = 16x - 3$$

$$\Rightarrow -20 + 3 = 16x + x$$

$$\Rightarrow -17 = 17x$$

$$\Rightarrow x = \frac{-17}{17}$$

$$\therefore x = -1$$

Example 10:

Solve $\dfrac{x}{ab} + \dfrac{x}{bc} + \dfrac{x}{ca} = a + b + c.$

Solution:

Consider $\dfrac{x}{ab} + \dfrac{x}{bc} + \dfrac{x}{ca} = a + b + c$

$$\frac{x}{ab} - c + \frac{x}{bc} - a + \frac{x}{ca} - b = 0$$

$$\frac{x-abc}{ab} + \frac{x-abc}{bc} + \frac{x-abc}{ca} = 0$$

$$\therefore \qquad (x-abc)\left[\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right] = 0$$

$$\Rightarrow x - abc = 0$$

$$\therefore \qquad x = abc$$

Example 11:

If $\frac{x+3}{x+1} = \frac{x^2+9x+20}{x^2-x-2}$, find x.

NOTES

Solution:

Consider $\frac{x+3}{x+1} = \frac{x^2+9x+20}{x^2-x-2}$

$$\frac{x+3}{x+1} = \frac{x^2+5x+4x+20}{x^2-2x+x-2}$$

$$\Rightarrow \frac{x+3}{x+1} = \frac{x(x+5)+4(x+5)}{x(x-2)+1(x-2)}$$

$$\Rightarrow \frac{x+3}{x+1} = \frac{(x+4)(x+5)}{(x+1)(x-2)}$$

Cancelling $(x+1)$ both sides, we get

$$\therefore\ (x+3)(x-2) = (x+4)(x+5)$$

$$\Rightarrow x^2 - 2x + 3x - 6 = x^2 + 5x + 4x + 20$$

$$\Rightarrow x^2 + x - 6 = x^2 + 9x + 20$$

$$\Rightarrow x - 9x = 20 + 6$$

$$\Rightarrow -8x = 26 \quad \Rightarrow x = \frac{26}{-8}$$

$$\therefore \quad x = -\frac{13}{4}$$

Example 12:

A man divided his property among his three sons in the following manner: He gave ₹ 2,500 to the eldest son, $\frac{5}{12}^{\text{th}}$ of the whole property to the second son, and to the youngest as much as to the first and the second together. How much did the youngest get?

Solution:

Let the total worth of the property be x.

To the first son, he has given ₹ 2,500.

To the second son, he has given $\frac{5}{12}x$.

NOTES

To the youngest son, he has given ₹ $2{,}500 + \frac{5}{12}x$

Hence, $2500 + \frac{5}{12}x + 2500 + \frac{5}{12}x = x$

$$\Rightarrow 5000 + \frac{5}{6}x = x$$

$$\Rightarrow 5000 = x - \frac{5}{6}x$$

$$\therefore \frac{x}{6} = 5000 \Rightarrow x = 30{,}000$$

$$\Rightarrow 5000 = \frac{6x - 5x}{6} = \frac{x}{6}$$

$\therefore$ Youngest son's share will be $\frac{1}{2}$ of the whole property.

$\therefore$ Youngest son will get ₹ 15,000.

Check Your Progress

1. Form an equation whose roots are 2 and –3.
2. Solve for x: $\frac{3x-1}{2} + \frac{x+2}{3} = \frac{9x+12}{5} - 2$.
3. If $\frac{x+3}{x+1} = \frac{x^2+9x+20}{x^2-x-2}$, find x.

2.4 QUADRATIC EQUATIONS

The equation of the form $ax^2 + bx + c = 0\ (a \neq 0)$ containing x^2 as the highest power of x is called an equation of the second degree in x or a quadratic equation.

The quadratic equation has two and only two roots. These two roots may be equal or unequal.

Types of Quadratic Equation

(i) Pure quadratic equation

(ii) Adfected quadratic equation

The form of quadratic equation $ax^2 + bx + c = 0$ when $b = 0$ reduces to $ax^2 + c = 0$. This is called a pure quadratic equation. When $b \neq 0$, the equation is of the form $ax^2 + bx + c = 0$ which is called an Adfected quadratic equation.

Solution of a Pure Quadratic Equation

A pure quadratic can be reduced to the form $ax^2 = d$.

$$\Rightarrow x^2 = \frac{d}{a} \Rightarrow x = \pm\sqrt{\frac{d}{a}}$$

Therefore, the equation has two roots, i.e., x has two values, viz.,

$$\because x = +\sqrt{\frac{d}{a}} \quad \text{and} \ -\sqrt{\frac{d}{a}}$$

Example 13:

Solve $3x^2 - 16x + 5 = 0$.

Solution:

Consider $3x^2 - 16x + 5 = 0$

$$\Rightarrow 3x - x - 15x + 5 = 0$$

$$\Rightarrow x(3x - 1) - 5(3x - 1) = 0$$

$$\Rightarrow (x - 5) \quad (3x - 1) = 0$$

$$\Rightarrow x - 5 = 0 \qquad 3x - 1 = 0$$

$$\therefore \qquad x = 5 \qquad 3x = 1$$

$$x = \frac{1}{3}$$

$$\therefore \ x = 5 \text{ or } \frac{1}{3}$$

Example 14:

Solve $(5x - 3)(5x + 3) = 16$.

Solution:

Consider $(5x - 3)(5x + 3) = 16$

$$\Rightarrow (5x)^2 - 3^2 = 16$$

$$\Rightarrow 25x^2 - 9 = 16$$

$$\Rightarrow 25x^2 = 16 + 9$$

$$\Rightarrow 25x^2 = 25$$

$$\Rightarrow x^2 = 1$$

$$\therefore x = \pm 1$$

Example 15:

Solve $\frac{x}{5}+\frac{5}{x}=\frac{x}{6}+\frac{6}{x}$.

NOTES

Solution:

Consider $\frac{x}{5}+\frac{5}{x}=\frac{x}{6}+\frac{6}{x}$

$$\Rightarrow \frac{x^2+25}{5x}=\frac{x^2+36}{6x}$$

$$i.e., \quad 6\left(x^2+25\right)=5\left(x^2+36\right)$$

$$\Rightarrow 6x^2+150=5x^2+180$$

$$\Rightarrow 6x^2-5x^2=180-150$$

$$\Rightarrow x^2=30$$

$$\therefore x=\pm\sqrt{30}$$

Example 16:

Solve $2\sqrt{(2x+8)(x+5)}=36-3x$.

Solution:

Squaring both sides, we get

$$\left\{2\sqrt{(2x+8)(x+5)}\right\}^2=(36-3x)^2$$

$$\Rightarrow 4(2x+8)(x+5)=1296-216x+9x^2$$

$$\Rightarrow 4\left(2x^2+18x+40\right)=9x^2-216x+1296$$

$$\Rightarrow 8x^2+72x+160=9x^2-216x+1296$$

$$\Rightarrow x^2-288x+1136=0$$

$$\Rightarrow x^2-284x-4x+1136=0$$

$$\Rightarrow x(x-284)-4(x-284)=0$$

$$\Rightarrow (x-4)(x-284)=0$$

$$\therefore \quad x=4 \text{ and } x=284$$

Solution of Adfected Quadratic Equation

An adfected quadratic equation of the form $ax^2 + bx + c = 0$ can be solved by:

1. Factorisation Method
2. Method of Completing the Square
3. Formula Method

NOTES

2.4.1 Factorisation Method

In this method, we factorise the expression $ax^2 + bx + c$ into linear factors as:

$$ax^2 + bx + c = (dx + e)(px \div q)$$

Then $ax^2 + bx + c = 0 \quad or \quad (dx + e)(px \pm q) = 0$

$$\Rightarrow dx + c = 0 \qquad px + q = 0$$

$$\therefore x = \frac{-e}{d} \qquad = \frac{-q}{p}$$

Hence, the two roots are $\frac{-e}{d}$ or $\frac{-q}{p}$.

Example 17:

Solve: $2x^2 - 7x + 3 = 0$.

Solution:

Consider $2x^2 - 7x + 3 = 0$

$$\Rightarrow 2x^2 - 6x - x + 3 = 0$$

$$\Rightarrow 2x(x - 3) - 1(x - 3) = 0 \Rightarrow (2x - 1)(x - 3) = 0$$

$$\Rightarrow 2x = 1 \text{ and } x = 3$$

$$\therefore \quad x = \frac{1}{2} \text{ and } x = 3$$

Example 18:

Solve: $4x^2 = 11x + 3$.

Solution:

Consider $4x^2 = 11x + 3$

$$\Rightarrow 4x^2 - 11x - 3 = 0 \qquad \Rightarrow 4x^2 - 12x + x - 3 = 0$$

$$\Rightarrow 4x(x - 3) + 1(x - 3) = 0 \qquad \Rightarrow (x - 3)(4x + 1) = 0$$

$\therefore$ either $x - 3 = 0$ $\quad 4x + 1 = 0$

$\therefore$ $x = 3$ $\qquad x = \frac{-1}{4}$

NOTES

Hence, the roots are $x = 3$ and $x = \frac{-1}{4}$.

Example 19:

Solve: $x^2 + 6x + 8 = 0$.

Solution:

Consider $x^2 + 6x + 8 = 0$

$$\Rightarrow x^2 + 4x + 2x + 8 = 0$$

$$\Rightarrow x(x + 4) + 2(x + 4) = 0$$

$$\Rightarrow (x + 4)(x + 2) = 0$$

$$\Rightarrow x + 4 = 0 \text{ and } x + 2 = 0 \quad \left| \begin{array}{l} x = -4 \\ x = -2 \end{array} \right.$$

$\therefore$ The roots are $x = -4$ and $x = -2$.

2.4.2 Method of Completing the Square

Let us take an equation for solving under this method.

Take $x^2 - 5x - 7 = 0$.

Write this equation by transposition of the constants on the RHS and the terms containing the unknown quantities on the LHS.

$\therefore x^2 - 3x = 5$

Now, adding to both sides the square of the coefficient of 'x' as divided by 2, we get

$$\Rightarrow x^2 - 3x + \left(\frac{3}{2}\right)^2 = 5 + \left(\frac{3}{2}\right)^2 \qquad \Rightarrow \left(x - \frac{3}{2}\right)^2 = 5 + \frac{9}{4}$$

$$\Rightarrow x + \frac{3}{2} = \pm\sqrt{\frac{29}{4}} \qquad \Rightarrow x - \frac{3}{2} = \pm\sqrt{\frac{29}{2}}$$

$$\Rightarrow x = \frac{3}{2} - \pm\sqrt{\frac{29}{2}}$$

$$\therefore x = \frac{3 + \sqrt{29}}{2} \text{ or } \frac{3 - \sqrt{29}}{2}$$

Example 20:

Solve $4x^2 + 4x - 3 = 0$.

Solution:

NOTES

Consider $4x^2 + 4x - 3 = 0$.

Step 1: Shift the constant to RHS.

$$\Rightarrow 4x^2 + 4x = 3$$

Step 2: Divided the equation by the co-efficient of 'a', i. e., 4.

$$\Rightarrow x^2 + x = \frac{3}{4}$$

Step 3: Now, complete the square.

$$\Rightarrow x^2 + x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \left(\frac{1}{2}\right)^2 \quad \Rightarrow \left(x + \frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4}$$

Step 4: Take square root on both sides.

$$\Rightarrow x + \frac{1}{2} = \sqrt{1} = \pm 1 \quad \Rightarrow x = -\frac{1}{2} \pm 1$$

$$\therefore x = \frac{1}{2} \; or \; \frac{-3}{2}$$

2.4.3 Formula Method

This method is similar to the method proposed by ***Sridhara Acharya***, an ancient Indian Mathematician and Astronomer. Therefore, it is also known as ***Sridhara Acharya*** Method.

Consider the quadratic equation $ax^2 + bx + c = 0$.

The roots of the equation can be obtained by using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the two roots of the equation are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \; or \; \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example 21:

Solve $9x^2 - 3x - 2 = 0$ by using the formula method.

Solution:

Here, $a = 9, b = -3$ and $c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

NOTES

$$\Rightarrow x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(9)(-2)}}{2(9)}$$

$$\Rightarrow x = \frac{3 \pm \sqrt{9 + 72}}{18}$$

$$\Rightarrow x = \frac{3 \pm \sqrt{81}}{18} = \frac{3 \pm 9}{18}$$

$$\Rightarrow x = \frac{3 + 9}{18} = \frac{12}{18} = \frac{2}{3} \text{ or } x = \frac{3 - 9}{18} = -\frac{6}{18} = -\frac{1}{3}$$

$\therefore$ The roots are $\frac{2}{3}$ and $-\frac{1}{3}$.

Example 22:

Solve $x^2 + x - 6 = 0$.

Solution:

Here, $a = 1, b = 1$ and $c = -6$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-6)}}{2(1)}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 24}}{2} \quad \Rightarrow x = \frac{-1 \pm 5}{2}$$

$$x = \frac{-1 + 5}{2} = \frac{4}{2} \text{ and } x = \frac{-1 - 5}{2} = -\frac{6}{2}$$

$\therefore$ The roots are $x = 2$ and $x = -3$.

Example 23:

Solve $5(x - 2)^2 - 6 = -13(x - 2)$.

Solution:

Consider $5(x - 2)^2 - 6 = -13(x - 2)$

$$\Rightarrow 5(x^2 - 4x + 4) - 6 = -13x + 26$$

$$\Rightarrow 5x^2 - 20x + 20 - 6 + 13x - 26 = 0$$

$$\therefore 5x^2 - 7x - 12 = 0$$

Here, $a = 5$, $b = -7$ and $c = -12$

NOTES

$$\Rightarrow x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(-12)}}{2(5)} = \frac{7 \pm \sqrt{49 + 240}}{10}$$

$$\Rightarrow x = \frac{7 \pm \sqrt{289}}{10} \Rightarrow \frac{7 \pm 17}{10}$$

$$x = \frac{7+17}{10} = \frac{24}{10} \text{ and } x = \frac{7-17}{10} = \frac{-10}{10}$$

$\therefore$ The roots are $x = 2.4$ and $x = -1$.

Example 24:

Solve $\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$.

Solution:

Given $\frac{1}{x-2} + \frac{2}{x-1} = \frac{6}{x}$

$$\text{i.e., } \frac{x - 1 + 2x - 4}{(x-2)(x-1)} = \frac{6}{x}$$

$$(3x - 5)x = 6(x-2)(x-1)$$

$$3x^2 - 5x = 6x^2 - 18x + 12 \quad \Rightarrow 3x^2 - 13x + 12 = 0$$

Here, $a = 3$, $b = -13$ and $c = 12$

$$\Rightarrow x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(3)(12)}}{2(3)} = \frac{13 \pm \sqrt{25}}{6}$$

$$\Rightarrow \frac{13 \pm 5}{6}$$

Thus, $x = \frac{13+5}{6} \text{ or } \frac{13-5}{6}$

$$\therefore x = 3 \text{ or } x = \frac{4}{3}$$

Relation between the Roots and Coefficients of a Quadratic Equation

Let α and β be are the roots of the equation $ax^2 + bx + c = 0$, where a, b and c are constant.

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad ....(i)$$

NOTES

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ....(ii)$$

Adding (i) and (ii), we get $\alpha + \beta$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$\therefore$ Sum of the roots, $\alpha + \beta = \dfrac{-b}{a}$

$\therefore$ Sum of the roots $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Now, multiplying (i) and (ii), we get

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$i.e., \ \alpha\beta = \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore \alpha\beta = \dfrac{c}{a}$

Hence, product of the roots $= \dfrac{\text{constant term}}{\text{coefficient of } x}$

To find the equation when the roots are known:

If α and β are the roots of same equation, then the equation is

$x^2 - (\alpha + \beta)x + \alpha\,\beta = 0$

i.e., $x^2 -$ (sum of the roots) . x + product of roots = 0

If α and β are the roots of the equation, and if $(\alpha + \beta)$ and $\alpha\beta$ are given, then the required equation is given by:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Example 25:

If α and β are the roots of equation $x^2 - 5x + 6 = 0$, then find out the value of

(i) $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$ (ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (iii) $\alpha^2 - \beta^2$.

NOTES

Solution:

On comparing $x^2 - 5x + 6 = 0$ with $ax^2 + bx + c = 0$, we get $a = 1, b = -5$ and $c = 6$.

Consider

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-5)}{1} = 5$$

and $\alpha.\beta = \frac{c}{a} = \frac{6}{a} = 6$

Now,

$$\text{(i) } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{5^2 - 2.6}{6} = \frac{25 - 12}{6} = \frac{13}{6}$$

$$\text{(ii) } \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{5^3 - 3(6)(5)}{6} = \frac{125 - 90}{6} = \frac{35}{6}$$

Check Your Progress

4. Solve $2\sqrt{(2x + 8)(x + 5)} = 36 - 3x$.
5. If α and β are the roots of equation $x^2 - 5x + 6 = 0$, then find out the value of: (i) $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$, (ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ and (iii) $\alpha^2 - \beta^2$.

2.5 SIMULTANEOUS EQUATIONS

A first degree equation in two variables, say 'x' and 'y', is called a linear equation in two variables. Two or more such linear equations in two variables, 'x' and 'y', are called linear simultaneous equations or simple simultaneous equations.

Method of Simultaneous Equations

There are three methods of solving simple simultaneous equations in two variables:

NOTES

(a) Elimination Method

(b) Substitution Method

(c) Cross Multiplication Method

2.5.1 Elimination Method

In this method, the given equations are transformed to equivalent equations, so that coefficient of any one of the variables in both the transformed equations become numerically equal. Solving the equation either by adding or subtracting, so that the terms in the transformed equation must cancel.

Example 26:

Find two numbers whose sum is 9 and difference is 1

Solution:

Let the numbers be x and y.

Given that $x + y = 9$ ....(i)

$x - y = 1$ ....(ii)

Adding (i) and (ii), we get

$$\begin{array}{l} x + y = 9 \\ \underline{x - y = 1} \\ 2x \quad = 10 \end{array}$$

$$\Rightarrow x = 10/2 \qquad \therefore \boxed{x = 5}$$

Subtracting (i) and (ii), we get

$$\begin{array}{l} x + y = 9 \\ \underline{x - y = 1} \\ \quad 2y = 8 \end{array}$$

$$\Rightarrow y = 8/2 \qquad \therefore \boxed{y = 4}$$

Example 27:

Given $2m + 3n = 8$, $3m + 2n = 7$ and $4m - 3n = 5p - 17$. Find the value of p.

Solution:

Consider $2m + 3n = 8$ ....(i)

$3m + 2n = 7$..................(ii)

Multiplying equation (i) × 3 and equation (ii) × 2, we get

$$\begin{array}{r} 6m + 9n = 24 \\ 6m + 4n = 14 \\ \hline 5n = 10 \end{array} \quad \text{(Subtract)}$$

$\Rightarrow n = 10/5 \quad \therefore \quad n = 2$

Multiplying equation (i) × 2 and equation (ii) × 3, we get

$$\begin{array}{r} 4m + 6n = 16 \\ 9m + 6n = 21 \\ (-) \quad (-) \quad (-) \\ \hline -5m \quad = -5 \end{array} \quad \text{(Subtract)}$$

$\therefore \quad m = 1 \qquad \Rightarrow m = \frac{-5}{-5}$

Now, substituting the value of *m* and *n* in the given equation, we get

$4m - 3n = 5p - 17$

$\Rightarrow (4 \times 1) - (3 \times 2) = 5p - 17 \quad \Rightarrow 4 - 6 = 5p - 17$

$\Rightarrow -2 + 17 = 5p \quad \Rightarrow 15 - 5p = 0 \quad \Rightarrow p = \frac{15}{5} = 3$

Hence, the value of $p = 3$.

Example 28:

Solve $3x + 7y = 13$ and $5x - 2y = 8$ by Elimination method.

Solution:

Consider $3x + 7y = 13$..................(i)

$5x - 2y = 8$..................(ii)

Multiplying (*i*) by 2 and (*ii*) by 7, we get

$$\begin{array}{r} 6x + 14y = 26 \\ 35x - 14y = 56 \\ \hline 41x \quad = 82 \end{array} \quad \text{(Add)}$$

$\Rightarrow x = 82/41 \quad \therefore x = 2$

Multiplying (i) by 5 and (ii) by 3, we get

$$\begin{array}{r} 15x + 35y = 65 \\ 15x - 6y = 24 \\ (-) \quad (+) \quad (-) \\ \hline -41y \quad = 41 \end{array} \quad \text{(Subtract)}$$

$\therefore\ y = 1 \qquad \Rightarrow y = \frac{41}{41}$

Hence, $x = 2$ and $y = 1$

NOTES

Example 29:

Solve x and y, $6x + 2y = 8xy$ and $3x + 8y = 30xy$.

Solution:

Consider $\quad 6x + 2y = 18xy$..................(i)

$3x + 8y = 30xy$..................(ii)

$$\begin{array}{r} \cancel{6x} + \ \ 2y = 18xy \\ \cancel{6x} + 16y = 60xy \\ (-) \quad (-) \quad (-) \\ \hline -14y = -42xy \end{array}$$

(i) – (ii) × 2 $\Rightarrow$

$\Rightarrow \quad 42xy = 14y$

$\Rightarrow \quad x = \frac{14y}{42y} \quad \Rightarrow x = \frac{1}{3}$

The value of x in equation (*i*) $\Rightarrow x = \frac{1}{3}$

$\Rightarrow \quad 6\left(\frac{1}{3}\right) + 2y = 18\left(\frac{1}{3}\right)y$

$\Rightarrow \quad 2 + 2y = 6y$

$\Rightarrow \quad 6y - 2y = 2 \ \Rightarrow \quad 4y = 2 \ \Rightarrow y = \frac{2}{4}$

$\therefore\ y = \frac{1}{2}$

2.5.2 Substitution Method

In this method, the value of y (*or* x) is found in terms of x (*or* y) from an equation, and substituting this value in the other equation, we get a linear equation of one variable. Solving this equation, we get the value of x (*or* y). Putting this value in any of the equations given, we get the value of y (*or* x).

Example 30:

Solve $x + 2y = 7$ and $2x - y = 4$.

Solution:

Let $\quad x + 2y = 7$..................(i)

$2x - y = 4$..................(ii)

From equation (i), we have $x = 7 - 2y$

Substituting for x in (ii), we get $2x - y = 4 \Rightarrow 2(7 - 2y) - y = 4$

i.e., $14 - 4y - y = 4$

$\therefore -5y = -14 + 4 \quad \Rightarrow y = \frac{-10}{-5} = 2. \quad \therefore y = 2$

Substituting the value of y in (i), we get

$x = 7 - 2(2) = 7 - 4 = 3 \quad \therefore x = 3$

NOTES

Example 31:

Solve by substitution method: $4x - y = 2$ and $-3x + 2y = 1$.

Solution:

$4x - y = 2$............................(i)

$-3x + 2y = 1$....................(ii)

Now, $4x - y = 2$

$\Rightarrow -y = 2 - 4x$

$\therefore y = 4x - 2$

Substituting the value of y in (ii), we get

$-3x + 2(4x - 2) = 1$

$\Rightarrow -3x + 8x - 4 = 1 \quad \Rightarrow 5x - 4 = 1 \quad \Rightarrow 5x = 5$

$\therefore x = 1$

Substituting the value of x in (i), we get

$4(1) - y = 2 \Rightarrow -y = 2 - 4$

$\therefore y = 2$

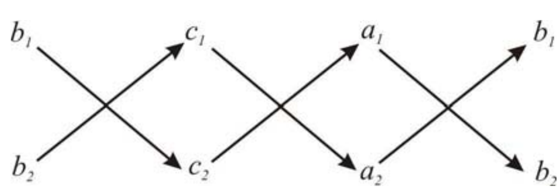
Hence, $x = 1$ and $y = 2$.

2.5.3 Cross Multiplication Method

Consider the two equations

$a_1x + b_1y + c_1 = 0$....................(i)

$a_2x + b_2y + c_2 = 0$....................(ii)



$$\text{i.e., } \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \text{ ....(iii)}$$

NOTES

From equation (iii), we get

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \qquad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Again, from equation (iii), we get

$$\frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$\therefore \qquad y = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ is called Method of Cross Multiplication.

Example 32:

Solve $x + 2y - 4 = 0$ and $3x + y - 7 = 0$.

Solution:

The given equations are:

$$x + 2y - 4 = 0$$

$$3x + y - 7 = 0$$

$$\frac{x}{-14 + 4} = \frac{y}{-12 + 7} = \frac{1}{1 - 6}$$

$$i.e., \frac{x}{-10} = \frac{y}{-5} = \frac{1}{-5} \qquad \therefore x = \frac{-10}{-5} = 2$$

$$\therefore y = \frac{-5}{-5} = 1$$

Hence, $x = 2$ and $y = 2$.

Example 33:

Solve $10x - 9y = 12$ and $3x - 9y = 17$.

Solution:

Consider the equations

$$10x - 9y - 12 = 0$$

$$3x - 9y - 17 = 0$$

NOTES

Now, $$\frac{x}{-153+108} = \frac{y}{-36+170} = \frac{1}{-90+27}$$

$$\Rightarrow \frac{x}{45} = \frac{y}{134} = \frac{1}{-63}$$

Equating $\frac{x}{45} = \frac{1}{-63} \Rightarrow -63x = 45$

i.e., $x = \frac{-45}{63}$ and $\frac{y}{134} = \frac{1}{-63}$ $\therefore x = \frac{-5}{7}$

i.e., $-63y = 134$ $\therefore y = -\frac{134}{63}$

Hence, $x = -\frac{5}{7}$ and $y = -\frac{134}{63}$.

Application of Equation to Business

Example 34:

The sum of 6 times a number and 5 times a number is 66. Which is that number?

Solution:

Let the number be x.

$\therefore$ 6 times the number is $6x$, and similarly, 5 times is $5x$

Given, $6x + 5x = 66$

$$\Rightarrow 11x = 66 \quad \therefore x = \frac{66}{11} = 6$$

So, the number is 6.

Example 35:

The age of the father is 4 times that of his son. Five years ago, father was 7 times as old as his son. Find their present age.

Solution:

Let the son's age before 5 years $= x$ and father's age before 5 years $= 7x$

$\therefore$ Present age of son $= x + 5$

Present age of father $= 7x + 5$

Consider, $7x + 5 = 4(x + 5)$

$$\Rightarrow 7x + 5 = 4x + 20 \Rightarrow 7x - 4x = 20 - 5 \Rightarrow 3x = 15$$

$\therefore$ $x = 5$

Hence, present age of son = 5 + 5 = 10

Present age of father = (7 × 5) + 5 = 40

Example 36:

NOTES

Form an equation whose roots are 4 and –6.

Solution:

Roots of the equation are 4 and –6.

$SOR = m + n$

Sum of the roots = 4 + (– 6)

= –2

$POR = m \times n$

Product $= 4 \times (-6) = -24$

Sum of the roots $\frac{b}{a}$ and product of the roots 24

Equation $x^2 + 2x - 24 = 0$

Example 37:

Five years ago, father was 5 times as old as his son, and in 3 years, it will 3 times as old as his son. Find their present age.

Solution:

Let the age of the father be x years and the age of the son be y years.

Condition I : $x - 5 = 5(y - 5)$

$\Rightarrow x - 5 = 5y - 25$

$\Rightarrow x - 5y = -20$....................(i)

Condiion II : $x + 3 = 3(y + 3)$

$\Rightarrow x + 3 = 3y + 9$

$\Rightarrow x - 3y = 6$....................(ii)

Subtracting (i) from (ii), we get

$-2y = -26$

$y = \frac{-26}{-2} \Rightarrow y = 13$

Putting $y = 13$ in (i), we get

$x = 6 + 39 = 45$

Hence, father's age is 45 years and son's age is 13 years.

Example 38:

In a boarding house of 50 members, the total monthly miscellaneous expenses were increased by ₹ 76. When the number of boarders increased by 14, the average monthly miscellaneous expenses were therefore reduced by one rupee per head. Find the original rate of miscellaneous expenses per head per month.

Solution:

Let the average monthly expenditure be x.

Total expenditure = $50x$

In a boarding house, there are 50 members. As per the given condition, if the number of boarder increases (50 + 14) = 64, the expenses reduces by rupee 1.

We have $50x + 76 = (x-1)64$

$$\Rightarrow 64x - 64 = 50x + 76 \Rightarrow \quad 64x - 50x = 76 + 64$$

$$\Rightarrow 14x = 140$$

$$\therefore \quad x = 10$$

Therefore, original rate is ₹ 10.

Example 39:

The age of the father is four times that of the son. Five years ago, the age of the father was 7 times that of his son. Find their present age.

Solution:

Let the present age of the son be x

$\therefore$ The present age of the father be $4x$.

Five years ago, the age of the son was $x - 5$ and that of the father was $4x - 5$

Then, we have $4x - 5 = 7(x - 5)$

$$i.e., 4x - 5 = 7x - 35 \quad \Rightarrow 3x = 30 \quad \Rightarrow x = \frac{30}{3} = 10$$

$$\therefore 4x = 4 \times 10 = 40 \text{ years}$$

The present age of the father is 40 years and that of his son is 10 years.

Example 40:

The weekly wages of 30 person consisting men and women amounts to ₹ 190. Each man receives ₹ 7 and each women ₹ 5. Find the number of men and women.

Solution:

Let the number of men be x.

$\therefore$ Number of women $= 30 - x$

NOTES

Given, $7x + (30 - x)5 = 190$

i.e., $7x + 150 - 5x = 190$

$\therefore 2x = 40 \quad \therefore x = 20$

$\therefore$ Number of women = 30 – 20 = 10

Hence, the number of men is 20 and number of women is 10.

Example 41:

Divide ₹ 110 into two parts so that 5 times of one part together with 6 times of the other part will be equal to ₹ 610.

Solution:

Let x and y be the two parts.

Then the number will be $x + y$.

$$\therefore x + y = 110$$

and also, it is given $5x + 6y = 610$

Consider $(x + y = 110) \times 5$

$$5x + 6y = 610,\ 5x + 5y = 550$$

Subtracting $-y = -60$, we get

$$y = 60$$

Putting $y = 60$ in $x + y = 110$, we get

$$\Rightarrow x + 60 = 110 \quad \Rightarrow x = 110 - 60 \quad \therefore x = 50$$

Hence, the two parts are ₹ 50 and ₹ 60.

Example 42:

Income of A and B are in the ratio of 7 : 5 and their expenditure is in the ratio of 9 : 8. An increase of A's expenditure by ₹ 100 and decrease of B's expenditure by ₹ 200 will make their savings of ₹ 400 each. Find their income and expenditure.

Solution:

Let the income of A and B be $7x$ and $5x$ respectively.

The expenditure of A and B be $9y$ and $8y$ respectively.

According to the question,

For A:

$$7x - (9y + 100) = 400$$

$$7x - 9y - 100 = 400$$

$$7x - 9y = 500 \text{....................(i)}$$

For B:

$$5x - (8y - 200) = 400$$

$$5x - 8y + 200 = 400$$

$$5x - 8y = 200 \text{....................(ii)}$$

Multiply equation (i) 5 and equation (ii) by 7, we get

$$35x - 45y = 2500$$

$$35x - 56y = 1400$$

$$\underline{(-) \quad (+) \quad (-)}$$

$$11y = 1100$$

$$\Rightarrow y = 1100/11 = ₹\ 100$$

Put the value of y in equation (ii), we get

$$5x - 8y = 200$$

$$\Rightarrow 5x - 8 \times 100 = 200 \qquad \Rightarrow 5x = 200 + 800$$

$$\therefore x = 1000/5 = ₹\ 200$$

Hence, Income of $A = 7x = 7 \times 200 = ₹\ 1,400$

$B = 5x = 5 \times 200 = ₹\ 1,000$

Expenditure of $A = 9y = 9 \times 100 = ₹\ 900$

$B = 8y = 8 \times 100 = ₹\ 800$

Example 43:

A man sells 7 horses and 8 cows at ₹ 29,400 and 5 horses and 6 cows at ₹ 21,500. What is the selling price of each?

Solution:

Let the selling price of horses = x

and the selling price of cow = y

$$\therefore 7x + 8y = 29,400 \text{....................(i)}$$

$$5x + 6y = 21,500 \text{....................(ii)}$$

Multiplying equation (i) by 3 and (ii) by 4, we get

$$21x + 24y = 88,200 \text{....................(iii)}$$

$$20x + 24y = 86,500 \text{....................(iv)}$$

Subtracting equation (iv) from equation (iii), we get

$x =$ ₹ 2,200

Now, putting $x = 2{,}200$ in equation (i), we get

NOTES

$$7(2{,}200) + 8y = 29{,}400$$

$$8y = 14{,}000 \Rightarrow y = ₹\ 1{,}750$$

$\therefore$ Price of a horse ₹ 2,200 and price of a cow ₹ 1,750

Example 44:

Two years ago, a man was six times as old as his son. In 18 years, he will be as twice as old as his son. Determine their present ages.

Solution:

Let the present ages of the father and the son be x and y respectively.

Given that $(x-2) = 6(y-2)$

$$\Rightarrow \quad x - 2 = 6y - 12$$

$$\Rightarrow \quad x - 6y = -10 \text{....................(i)}$$

Also, $x + 18 = 2(y + 18)$

$$\Rightarrow \quad x + 18 = 2y + 36$$

$$\Rightarrow \quad x - 2y = 18 \text{....................(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$-4y = -28 \quad \Rightarrow y = 7$$

Substituting $y = 7$ in equation (i), we get

$$x - 42 = -10 \quad \text{or } x = 32$$

$$\therefore \quad x = 32 \ \text{ and } \ y = 7$$

Hence, the age of father is 32 and that of his son is 7 years.

Example 45:

Two persons A and B whose salaries together amount of ₹ 100. A spends 75% and B 70% of their salaries. If their savings are 5 : 4, find their respective salaries.

Solution:

Let x and y be the salaries of A and B.

$\therefore$ Given, $x + y = 100$....................(i)

$\therefore$ A's saving $= x - 75\% \text{ of } x = x - \frac{3}{4}x \text{ or } \frac{1}{4}x$

B's savings $= y - 70\%$ of $y = y - \frac{7}{10}y$ or $\frac{3}{10}y$

Also, $\frac{1}{4}x : \frac{3}{10}y = 5 : 4$

NOTES

$$\therefore \quad 4.\frac{1}{4}x = 5.\frac{3}{10}y$$

$$\Rightarrow \quad \therefore x = \frac{3}{2}y \text{....................(ii)}$$

Putting $x = \frac{3}{2}y$ in equation (i), we get

$$\frac{3}{2}y + y = 100 \text{ or } 5y = 200 \quad \Rightarrow \quad y = \frac{200}{5}$$

$$\therefore y = 40$$

Hence, $x = \frac{3 \times 40}{2} = 60$

$\therefore$ A's income ₹ 60 and B's income ₹ 40.

Example 46:

Form an equation whose roots are 2 and – 3.

Solution:

Root of the equation = 2 and –3

$SOR = m + n$

Sum of the roots = 2 + (–3) = –1

$POR = m \times n$

Product = 2(–3) = –6

$x^2 - (m + n)x + mn = 0$

$x^2 + x - 6 = 0$

Example 47:

Form an equation whose roots are 7 and –4.

Solution:

Roots of the equation = 7 and –4

Sum of the roots $= 7 + (-4) = 3$

Product = 7(–4) = –28

$x^2 - (m + n)x + mn = 0$

Equation $x^2 + 3x - 28 = 0$

NOTES

Example 48:

Find the roots of the equation $x^2 - 3x - 10 = 0$.

Solution:

$x^2 - 3x - 10 = 0$

$\Rightarrow x^2 - 5x + 2x - 10 = 0 \quad \Rightarrow x(x-5) + 2(x-5) = 0$

$\Rightarrow (x+2)(x-5) = 0 \quad \therefore x + 2 = 0 \text{ or } x - 5 = 0$

$\therefore x = -2, x = 5 \quad \therefore x = -2 \text{ or } 5$

Example 49:

Solve for x and y, $\frac{3}{x} + \frac{1}{y} = 9$ and $\frac{1}{x} - \frac{2}{y} = 10.$

Solution:

$$\frac{3}{x} + \frac{1}{y} = 9 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\frac{1}{x} - \frac{2}{y} = 10 \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Here, $(i) \times 2 + (ii) \Rightarrow$

$$\frac{6}{x} + \frac{2}{y} = 18$$

$$\frac{1}{x} - \frac{2}{y} = 10$$

$$\frac{6}{x} + \frac{1}{x} = 18 + 10$$

$$\Rightarrow \frac{7}{x} = 28 \quad \Rightarrow 28x = 7 \quad \Rightarrow x = \frac{7}{28} = \frac{1}{4}$$

Putting the value of x in (ii), we get

$$\frac{1}{\frac{1}{4}} - \frac{2}{y} = 10 \quad \Rightarrow 4 - \frac{2}{y} = 10$$

$$\Rightarrow 4y - 2 = 10y \quad \Rightarrow 10y - 4y = -2$$

$$\Rightarrow 6y = -2 \quad \Rightarrow y = \frac{-2}{6} = \frac{-1}{3}$$

$$\therefore \quad x = \frac{1}{4} \text{ and } y = -\frac{1}{3}$$

Example 50:

Solve for $B, \frac{B+1}{2} - \frac{B-2}{3} = \frac{B+4}{5} + \frac{7}{15}$.

NOTES

Solution:

$$\frac{B+1}{2} - \frac{B-2}{3} = \frac{B+4}{5} + \frac{7}{15}$$

$$\Rightarrow \frac{3(B+1) - 2(B-2)}{6} = \frac{3(B+4)+7}{15}$$

$$\Rightarrow \frac{3B+3-2B+4}{6} = \frac{3B+12+7}{15}$$

$$\Rightarrow \frac{B+7}{6} = \frac{3B+19}{15} \Rightarrow 15(B+7) = 6(3B+19)$$

$$\Rightarrow 15B + 105 = 18B + 114 \quad \Rightarrow 18B - 15B = 105 - 114$$

$$\Rightarrow 3B = -9 \quad \Rightarrow B = -\frac{9}{3}$$

$$\therefore B = -3$$

Example 51:

If $\frac{3x+5}{4x+2} = \frac{3x+4}{4x+7}$, find x.

Solution:

$$\frac{3x+5}{4x+2} = \frac{3x+4}{4x+7}$$

$$\Rightarrow (3x+5)(4x+7) = (4x+2)(3x+4)$$

$$\Rightarrow 12x^2 + 21x + 20x + 35 = 12x^2 + 16x + 6x + 8$$

$$\Rightarrow 41x + 35 = 22x + 8$$

$$\Rightarrow 41x - 22x = 8 - 35$$

$$\Rightarrow 19x = -27 \quad \therefore \quad x = -\frac{27}{19}$$

Example 52:

Solve the following equation in the method of completing the square:

(i) $x^2 + 9x + 7 = 0$

(ii) $4x^2 + 4x - 3 = 0$

(iii) $3x^2 - 5x - 8 = 0$

Solution:

NOTES

(i) Now, $x^2 + 9x + 7 = 0$

$$\Rightarrow x^2 + 9x = -7 \quad \Rightarrow x^2 + 9x + \left(\frac{9}{2}\right)^2 = -7 + \left(\frac{9}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{9}{2}\right)^2 = -7 + \frac{81}{4} \quad \Rightarrow \left(x + \frac{9}{2}\right)^2 = \frac{53}{4}$$

Taking square root both sides, we get

$$x + \frac{9}{2} = \pm\sqrt{\frac{53}{4}} \Rightarrow x = \pm\sqrt{\frac{53}{4}} - \frac{9}{2} \quad \therefore x = \frac{-9 \pm \sqrt{53}}{2}$$

Hence, $x = \dfrac{-9 + \sqrt{53}}{2}$ or $\dfrac{-9 - \sqrt{53}}{2}$

(ii) Now, $4x^2 + 4x - 3 = 0$

$$\Rightarrow 4x^2 + 4x = 3$$

Dividing the equation by the coefficient of i.e., 4, we get

$$\Rightarrow x^2 + x = \frac{3}{4} \quad \Rightarrow x^2 + x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} \quad \Rightarrow \left(x + \frac{1}{2}\right)^2 = \frac{3+1}{4}$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 = \frac{4}{4} \quad \Rightarrow \left(x + \frac{1}{2}\right)^2 = \sqrt{1}$$

$$\Rightarrow x + \frac{1}{2} = \pm 1 \quad \Rightarrow x = -\frac{1}{2}, \pm 1$$

Hence, $x = \dfrac{1}{2}$ or $-\dfrac{3}{2}$

(iii) $3x^2 - 5x - 8 = 0 \quad \Rightarrow 3x^2 - 5x = 8$

Dividing both the sides by 3, we get

$$\Rightarrow x^2 - \frac{5}{3}x = \frac{8}{3}$$

Now complete the square,

$$\Rightarrow x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = \frac{8}{3} + \left(\frac{5}{6}\right)^2$$

$$\Rightarrow \left(x - \frac{5}{6}\right)^2 = \frac{8}{3} + \frac{25}{36}$$

$$\Rightarrow \left(x - \frac{5}{6}\right)^2 = \frac{96 + 25}{36} \quad \Rightarrow \left(x - \frac{5}{6}\right)^2 = \frac{121}{36}$$

Taking square root of both the sides, we get

$$\Rightarrow x - \frac{5}{6} = \pm\sqrt{\frac{121}{36}} \quad \Rightarrow x - \frac{5}{6} = \pm\frac{11}{6} \quad \Rightarrow x = \frac{5}{6} \pm \frac{11}{6}$$

Hence, $x = \frac{5 + 11}{6} = \frac{16}{6}$ or $\frac{5 - 11}{6} = \frac{-6}{6} = -1$

$$\therefore \ x = -1 \text{ or } \frac{8}{3}$$

Example 53:

Solve the following equation by using formula method.

(i) $9x^2 - 3x - 2 = 0$

(ii) $2x^2 - 7x + 3 = 0$

(iii) $12x^2 - 23x - 24 = 0$

Solution:

(i) $9x^2 - 3x - 2 = 0$

Here, $a = 9, b = -3 \ and \ c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(9)(-2)}}{2(9)}$$

$$\Rightarrow x = \frac{3 \pm \sqrt{9 + 72}}{18} \Rightarrow x = \frac{3 \pm \sqrt{81}}{18}$$

$$\Rightarrow x = \frac{3 \pm 9}{18}$$

$$\Rightarrow x = \frac{3 + 9}{18} \quad \text{or} \quad x = \frac{3 - 9}{18}$$

$\Rightarrow x = \frac{12}{18}$ or $x = \frac{-6}{18}$

$\Rightarrow x = \frac{2}{3}$ or $x = \frac{-1}{3}$

NOTES

$\therefore$ The two roots are $\frac{2}{3}$ or $\frac{-1}{3}$.

(ii) $2x^2 - 7x + 3 = 0$

Here, $a = 2$, $b = -7$ and $c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{(-)7 \pm \sqrt{(-7)^2 - 4\,.\,2\,.\,3}}{2(2)} \Rightarrow \quad x = \frac{7 \pm \sqrt{49 - 24}}{4}$$

$$\Rightarrow \quad x = \frac{7 \pm 5}{4}$$

$$\Rightarrow \quad x = \frac{7 \pm 5}{4} \quad \text{or} \quad x = \frac{7 - 5}{4}$$

$$\Rightarrow \quad x = \frac{12}{4} = 3 \quad \text{or} \quad x = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ The two roots are 3 or $\frac{1}{2}$.

(iii) $12x^2 - 23x - 24 = 0$

Here, $a = 12$, $b = -23$ and $c = -24$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \quad x = \frac{-(-23) \pm \sqrt{(-23)^2 - 4(12)(-24)}}{2(12)}$$

$$\Rightarrow x = \frac{23 \pm \sqrt{529 + 1152}}{24} \quad \Rightarrow x = \frac{23 \pm \sqrt{1681}}{24}$$

$$\Rightarrow x = \frac{23 \pm 41}{24}$$

$$\Rightarrow x = \frac{-23 + 41}{24} \quad \text{or} \quad x = \frac{-23 - 41}{24}$$

$$\Rightarrow x = \frac{18}{24} \quad x = \frac{-64}{24} \quad \Rightarrow x = \frac{3}{4} \,\Big|\, x = \frac{8}{3}$$

$\Rightarrow x = 0.75 \quad x = -2.66$

$\therefore$ The two roots are 0.75 or –2.66.

NOTES

Example 54:

Solve the equation by formula method $6x^2 = 12 + 11x$.

Solution:

$$6x^2 - 11x - 12 = 0$$

Here, $a = 6$, $b = -11$ and $c = -12$

$$x = \frac{-6 \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(6)(-12)}}{2 \times 6}$$

$$\Rightarrow \quad x = \frac{11 \pm \sqrt{121 - 288}}{12}$$

$$\Rightarrow \quad x = \frac{11 \pm \sqrt{169}}{12}$$

$$\Rightarrow \quad x = \frac{11 \pm 13}{12}$$

$$x = \frac{11 - 13}{12}$$

$$\Rightarrow \quad x = \frac{24}{12} \quad \text{or} \quad x = \frac{-2}{12}$$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = 0.166$$

$\therefore$ The two roots are 2 or 0.166.

Example 55:

Solve for x: $3(x - 3)(x + 4) + 3(x - 2)(x - 4) = 19(x - 4)(x - 3)$.

Solution:

$$3(x-3)(x+4) + 3(x-2)(x-4) = 19(x-4)(x-3)$$

$$\Rightarrow 3(x^2 + 4x - 3x - 12) + 3(x^2 - 4x - 2x + 8) = 19(x^2 - 3x - 4x + 12)$$

$$\Rightarrow 3(x^2 + x - 12) + 3(x^2 - 6x + 8) = 19(x^2 - 7x + 12)$$

$$\Rightarrow 3x^2 + 3x - 36 + 3x^2 - 18x + 24 = 19x^2 - 133x + 228$$

$$\Rightarrow -15x + 133x = 19x^2 + 228 - 24 + 36 - 6x^2$$

$$\Rightarrow 118x = 13x^2 + 240$$

$$\Rightarrow 13x^2 - 118x + 240 = 0$$

NOTES

Here, $a = 13$, $b = -118$ and $c = 240$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(-118) \pm \sqrt{(-118)^2 - 4(13)(240)}}{2(13)}$$

$$\Rightarrow x = \frac{118 \pm \sqrt{13924 - 12480}}{26}$$

$$\Rightarrow x = \frac{118 \pm \sqrt{1444}}{26}$$

$$\Rightarrow x = \frac{118 \pm 38}{26}$$

$$\Rightarrow x = \frac{118 + 38}{26} \quad \text{or} \quad x = \frac{118 - 38}{26}$$

$$\Rightarrow x = \frac{156}{26} \text{ or } x = \frac{80}{26}$$

$$\Rightarrow x = 6 \quad \text{or} \quad x = \frac{40}{13}$$

$\therefore$ The two roots are 6 or $\frac{40}{13}$.

Example 56:

Using formula method, solve $5(x-2)^2 - 6 = -13(x-2)$.

Solution:

$$5(x-2)^2 - 6 = -13(x-2) \quad \Rightarrow 5(x^2 - 4x + 4) - 6 = -13x + 26$$

$$\Rightarrow 5x^2 - 20x + 20 - 6 + 13x - 26 \;\Rightarrow 5x^2 - 7x - 12 = 0$$

Hence, $a = 5, b = -7, c = -12$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4.5.(-12)}}{2.5} \qquad \Rightarrow x = \frac{7 \pm \sqrt{49 + 240}}{10}$$

NOTES

$$\Rightarrow x = \frac{7 \pm \sqrt{289}}{10} \qquad \Rightarrow x = \frac{7 \pm 17}{10}$$

$$\Rightarrow x = \frac{7 + 17}{10} \text{ or } x = \frac{7 - 17}{10}$$

$$\Rightarrow x = \frac{24}{10} \text{ or } x = \frac{-10}{10}$$

$$\Rightarrow x = \frac{12}{5} \text{ or } x = -1$$

The two roots are $\frac{12}{5}$ or -1.

Example 57:

Solve the following equation: $\frac{2p}{p+1} + \frac{p+2}{p-1} = 3$.

Solution:

$$\frac{2p}{p+1} + \frac{p+2}{p-1} = 3 \quad \Rightarrow \frac{2p(p-1) + (p+2)(p+1)}{(p+1)(p-1)} = 3$$

$$\Rightarrow 2p^2 - 2p + p^2 + p + 2p + 2 = 3(p^2 - p + p - 1)$$
$$\Rightarrow 3p^2 + p + 2 = 3p^2 - 3$$
$$\Rightarrow 3p^2 - 3p^2 + p = -3 - 2$$
$$\Rightarrow p = -5$$
$$\therefore p = -5$$

Example 58:

Solve $\frac{2}{x-1} + \frac{3}{x+4} = \frac{5}{x+3}$.

Solution:

$$\frac{2}{x-1} + \frac{3}{x+4} = \frac{5}{x+3}$$

$$\Rightarrow \frac{2}{x-1} + \frac{3}{x+4} = \frac{2}{x+3} + \frac{3}{x+3}$$

$$\Rightarrow \frac{2}{x-1} - \frac{2}{x+3} = \frac{3}{x+3} - \frac{3}{x+4}$$

NOTES

$$\Rightarrow \frac{2(x+3)-2(x-1)}{(x-1)(x+3)} = \frac{3(x+3)-3(x+4)}{(x+3)(x+4)}$$

$$\Rightarrow \frac{2x+6-2x+2}{(x-1)(x+3)} = \frac{3x+12-3x-9}{(x+3)(x+4)}$$

$$\Rightarrow \frac{8}{(x-1)} = \frac{3}{(x+4)}$$

$$\Rightarrow 8(x+4) = 3(x-1)$$
$$\Rightarrow 8x + 32 = 3x - 3$$
$$\Rightarrow 8x - 3x = -3 - 32$$

$$\Rightarrow 5x = -35 \Rightarrow x = \frac{-35}{5}$$

$$\therefore x = -7$$

Example 59:

Solve by elimination method: $x + y = 15$ and $3x - y = 21$.

Solution:

Let $x + y = 15$ ....(i)

$3x - y = 21$ ....(ii)

Adding (i) and (ii), we get

$$\begin{array}{l} x + y = 15 \\ \underline{3x - y = 21} \\ 4x \quad\quad = 36 \end{array}$$

$$\Rightarrow x = \frac{36}{4}$$

$$\Rightarrow x = 9$$

Subtracting (i) × 3 and (ii) × 1, we get

$$\begin{array}{r} 3x + 3y = 45 \\ \underline{3x - \ \ y = 21} \\ 4y = 24 \end{array}$$

$$\Rightarrow y = \frac{24}{4}$$

$$\Rightarrow y = 6$$

$$\therefore x = 9,\ y = 6$$

NOTES

Example 60:

Solve by elimination method: $4x - 2y = 12$ and $5x + 2y - 104 = 0$.

Solution:

Let $4x - 2y = 12$ ....(i)

$5x + 2y = 104$ ....(ii)

Subtracting equations (i) and (ii), we get

$$\begin{array}{l} 4x - 2y = 12 \\ \underline{5x + 2y = 104} \\ 9x \qquad = 116 \end{array}$$

$$\Rightarrow x = \frac{116}{9} \qquad \therefore x = 12.88$$

Subtracting equations (i) × 5 and (ii) × 4, we get

$$\begin{array}{l} 20x - 10y = 60 \\ 20x + \ 8y = 416 \\ \underline{(-) \quad (-) \quad (-)} \\ \quad -18y = -356 \end{array}$$

$$\Rightarrow y = \frac{-356}{-18} \qquad \therefore y = 19.77$$

$$\therefore x = 12.88 \text{ and } y = 19.77$$

Example 61:

Solve by elimination method $4x - 3y = 8$ and $3x - 4y = -1$.

Solution:

$4x - 3y = 8$ ....(i)

$3x - 4y = -1$ ....(ii)

Subtracting equation (i) × 3 and equation (ii) × 4, we get

$$\begin{array}{l} 12x - \ 9y = 24 \\ 12x - 16y = -4 \\ \underline{(-) \quad (+) \quad (+)} \\ \qquad 7y = 28 \end{array}$$

$\Rightarrow y = \dfrac{28}{7} \qquad \therefore y = 4$

Multiplying equations (i) × 4 and (ii) × 3, we get

NOTES

$$\begin{array}{l} 16x - 12y = 32 \\ 9x - 12y = -3 \\ \underline{(-)\quad(+)\quad(+)} \qquad \text{(Subtract)} \\ 7x \qquad = 35 \end{array}$$

$$\Rightarrow x = \frac{35}{7}$$

$$\therefore x = 5$$

$$\therefore x = 5,\ y = 4$$

Example 62:

Solve for x and y using method of substitution: $3x + 3y = 12$ and $2x + 4y = 12$.

Solution:

$$3x + 3y = 12 \quad ....(i)$$

$$2x + 4y = 12 \quad ....(ii)$$

From (i), we get

$$3x + 3y = 12$$

$$\Rightarrow 3x = 12 - 3y$$

$$\Rightarrow x = \frac{12 - 3y}{3} \quad \Rightarrow x = \frac{3(4 - y)}{3}$$

$$\Rightarrow x = 4 - y \quad ....(iii)$$

Putting the value of x in equation (ii), we get

$$\Rightarrow 2x + 4y = 12$$

$$\Rightarrow 2(4 - y) + 4y = 12 \quad \Rightarrow 8 - 2y + 4y = 12$$

$$\Rightarrow 2y = 12 - 8 \quad \Rightarrow 2y = 4$$

$$\Rightarrow y = \frac{4}{2} \quad \Rightarrow y = 2$$

Putting the value of y in equation (ii), we get

$$\Rightarrow x = 4 - y$$

$$\Rightarrow x = 4 - 2 \quad \Rightarrow x = 2$$

$$\therefore \quad x = 2,\ \ y = 2$$

Example 63:

Solve by substitution method: $4x - y = 2$ and $-3x + 2y = 1$.

Solution:

NOTES

$4x - y = 2$ ....(i)

$-3x + 2y = 1$ ....(ii)

From equation (i), we get

$4x - y = 2$

$\Rightarrow -y = 2 - 4x$

$\Rightarrow y = 4x - 2$ ....(iii)

Putting the value of y in equation (ii), we get

$-3x + 2y = 1$

$\Rightarrow -3x + 2(4x - 2) = 1 \quad \Rightarrow -3x + 8x - 4 = 1$

$\Rightarrow 5x = 1 + 4 \quad \Rightarrow 5x = 5$

$\Rightarrow x = \frac{5}{5} \qquad \Rightarrow x = 1$

Putting the value of x in equation (ii), we get

$y = 4x - 2$

$\Rightarrow y = 4(1) - 2 \quad \Rightarrow y = 4 - 2 \quad \Rightarrow y = 2$

$\therefore x = 1, \;\; y = 2$

Example 64:

Two years ago, a man was six times as old as his son. In 18 years, he will be twice as old as his son. Determine their present ages.

Solution:

Let the age of a man be x and age of a son be y.

Given, $(x - 2) = 6(y - 2)$

$\Rightarrow x - 2 = 6y - 12$

$\Rightarrow x - 6y = -12 + 2$

$\Rightarrow x - 6y = -10$ ....(i)

NOTES

Also, $x + 18 = 2\left(y + 18\right)$

$\Rightarrow x + 18 - 2y + 36$

$\Rightarrow x - 2y = 36 - 18$

$\Rightarrow x - 2y = 18$ ....(ii)

Subtracting equations (i) and (ii), we get

$$\begin{array}{l} x - 6y = -10 \\ x - 2y = \;\; 18 \\ \underline{(-)\;(+)\quad (-)} \\ \quad -4y = -28 \end{array}$$

$$\Rightarrow y = \frac{28}{4}$$

$\therefore y = 7$

Substituting $y = 7$ in equation (i), we get

$x - 6y = -10$

$\Rightarrow x - 6 \times 7 = -10 \Rightarrow x - 42 = -10$

$\Rightarrow x = -10 + 42 \qquad \Rightarrow x = 32$

$\therefore\; x = 32,\; y = 7$

Example 65:

The age of the father is 4 times that of his son. Five years ago, father was 7 times as old as his son. Find their present age.

Solution:

Let the age of a son before 5 years = x and the age of father before 5 years = $7x$

$\therefore$ Present age of Son $= x + 5$

Present age of Father = $7x + 5$

Consider, $\quad 7x + 5 = 4(x + 5)$

$\Rightarrow 7x + 5 = 4x + 20$

$\Rightarrow 7x - 4x = 20 - 5$

$\Rightarrow 3x = 15$

$$\Rightarrow x = \frac{15}{3} \qquad \therefore x = 5$$

Hence, Present age of Son = $x + 5 = 5 + 5 = 10$

Present age of Father = $7x + 5 = 7 \times 5 + 5 = 40$

Example 66:

NOTES

Five years ago, father was 5 times as old as his son, and after 3 years, he will be 3 times as old as his son. Find their present ages.

Solution:

Let the age of the Father be x years and the age of the Son be y years.

Condition I:

$$x - 5 = 5(y - 5)$$

$$\Rightarrow x - 5 - 5y - 25$$

$$\Rightarrow x - 5y = 5 - 25$$

$$\Rightarrow x - 5y = -20 \quad \text{..................(i)}$$

Condition II:

$$x + 3 = 3(y + 3)$$

$$\Rightarrow x + 3 = 3y + 9$$

$$\Rightarrow x - 3y = 9 - 3$$

$$\Rightarrow x - 3y = 6 \quad \text{................(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$\begin{array}{r} x - 5y = -20 \\ x - 3y = \quad 6 \\ \underline{(-)\ (+)\quad (-)} \\ -2y = -26 \end{array}$$

$$\Rightarrow y = \frac{-26}{-2} \quad \Rightarrow y = 13$$

Putting $y = 13$ in equation (ii), we get

$x - 3y = 6$

$\Rightarrow x - 3(13) = 6$

$\Rightarrow x - 39 = 6$

$\Rightarrow x = 39 + 6$

$\Rightarrow x = 45$

Hence, father's age is 45 years and son's age is 13 years.

NOTES

Example 67:

A father is 25 years older than his daughter. In three years, the father's age will be 5 years more than that of his daughter. Find their present age.

Solution:

Let the present age of the daughter be 'x'

And the father's present age be $x + 25$

After three years,

Daughter's age will be $x + 3$

And Father's age will be $2(x + 3) + 5$

$$= 2x + 6 + 5$$

$$= 2x + 11$$

Taking Father age: Present and after 3 years

$$(x + 25) + 3 = 2x + 11$$

$$\Rightarrow x + 28 = 2x + 11$$

$$\Rightarrow 2x - x = 28 - 11$$

$$\Rightarrow x = 17$$

$\therefore$ Daughter's age = 17

and Father's age $= 17 + 25 = 42$

Check Your Progress

6. Given are $2m + 3n = 8$, $3m + 2n = 7$ and $4m - 3n = 5p - 17$. Find the value of p.
7. Solve $10x - 9y = 12$ and $3x - 9y = 17$.
8. Solve by elimination method: $4x - 3y = 8$ and $3x - 4y = -1$.

2.6 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Form an equation whose roots are 2 and –3.

Solution:

Let the equation be $x^2 + x(a + b) + ab = 0$.

Consider $x^2 + x(a + b) + ab = 0$

$$\Rightarrow x^2 + x(2 - 3) + \{2 \times (-3)\} = 0$$

$$\Rightarrow x^2 + x(-1) + (-6) = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

NOTES

$\therefore$ The equation is $x^2 - x - 6 = 0$ whose roots are 2 and –3.

2. Solve for x : $\dfrac{3x-1}{2} + \dfrac{x+2}{3} = \dfrac{9x+12}{5} - 2.$

Solution:

Consider $\dfrac{3x-1}{2} + \dfrac{x+2}{3} = \dfrac{9x+12}{5} - 2$

$$\frac{3(3x-1) + 2(x+2)}{6} = \frac{9x+12-5(2)}{5}$$

$$\Rightarrow \frac{9x-3+2x+4}{6} = \frac{9x+12-10}{5}$$

$$\Rightarrow \frac{11x+1}{6} = \frac{9x+2}{5}$$

Cross multiplying, we get

$$5(11x+1) = 6(9x+2)$$

$$\Rightarrow 55x + 5 = 54x + 12$$

$$\Rightarrow 55x - 54x = 12 - 5$$

$$\therefore x = 7$$

3. If $\dfrac{x+3}{x+1} = \dfrac{x^2+9x+20}{x^2-x-2}$, find x.

Solution:

Consider $\dfrac{x+3}{x+1} = \dfrac{x^2+9x+20}{x^2-x-2}$

$$\frac{x+3}{x+1} = \frac{x^2+5x+4x+20}{x^2-2x+x-2}$$

$$\Rightarrow \frac{x+3}{x+1} = \frac{x(x+5)+4(x+5)}{x(x-2)+1(x-2)} \quad \Rightarrow \frac{x+3}{x+1} = \frac{(x+4)(x+5)}{(x+1)(x-2)}$$

Cancelling $(x+1)$ both sides, we get

$$\therefore (x+3)(x-2) = (x+4)(x+5)$$

$$\Rightarrow x^2 - 2x + 3x - 6 = x^2 + 5x + 4x + 20$$

$$\Rightarrow x^2 + x - 6 = x^2 + 9x + 20$$

NOTES

$$\Rightarrow x - 9x = 20 + 6$$

$$\Rightarrow -8x = 26 \quad \Rightarrow x = \frac{26}{-8}$$

$$\therefore \quad x = -\frac{13}{4}$$

4. Solve $2\sqrt{(2x+8)(x+5)} = 36 - 3x$.

Solution:

Squaring both sides, we get

$$\left\{2\sqrt{(2x+8)(x+5)}\right\}^2 = (36-3x)^2$$

$$\Rightarrow 4(2x+8)(x+5) = 1296 - 216x + 9x^2$$

$$\Rightarrow 4\left(2x^2 + 18x + 40\right) = 9x^2 - 216x + 1296$$

$$\Rightarrow 8x^2 + 72x + 160 = 9x^2 - 216x + 1296$$

$$\Rightarrow x^2 - 288x + 1136 = 0$$

$$\Rightarrow x^2 - 284x - 4x + 1136 = 0$$

$$\Rightarrow x(x-284) - 4(x-284) = 0$$

$$\Rightarrow (x-4)(x-284) = 0$$

$$\therefore \quad x = 4 \text{ and } x = 284$$

5. If α and β are the roots of equation $x^2 - 5x + 6 = 0$, then find out the value of: (i) $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$, (ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ and (iii) $\alpha^2 - \beta^2$.

Solution:

On comparing $x^2 - 5x + 6 = 0$ with $ax^2 - bx + c = 0$, we get

$$a = 1, b = -5 \text{ and } c = 6$$

Consider $\alpha + \beta = \frac{-b}{a} = \frac{-(-5)}{1} = 5$

and $$\alpha.\beta = \frac{c}{a} = \frac{6}{a} = 6$$

Now,

(i) $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{5^2 - 2.6}{6} = \frac{25 - 12}{6} = \frac{13}{6}$$

(ii) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{5^3 - 3(6)(5)}{6} = \frac{125 - 90}{6} = \frac{35}{6}$$

6. Given $2m + 3n = 8$, $3m + 2n = 7$ and $4m - 3n = 5p - 17$. Find the value of p.

Solution:

Consider, $2m + 3n = 8$ ....(i)

$3m + 2n = 7$..................(ii)

Multiplying equation (i) × 3 and equation (ii) × 2, we get

$6m + 9n = 24$ (Subtract)

$\underline{6m + 4n = 14}$

$5n = 10$

$\Rightarrow n = 10/5 \quad \therefore \quad n = 2$

Multiplying equation (i) × 2 and equation (ii) × 3, we get

$4m + 6n = 16$ (Subtract)

$9m + 6n = 21$

$\underline{(-) \quad (-) \quad (-)}$

$-5m \quad = -5$

$\Rightarrow m = \frac{-5}{-5} \quad \therefore \quad m = 1$

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Now, substituting the value of m and n in the given equation, i.e.,

$$4m - 3n = 5p - 17$$

$$\Rightarrow (4 \times 1) - (3 \times 2) = 5p - 17$$
$$\Rightarrow 4 - 6 = 5p - 17$$
$$\Rightarrow -2 + 17 = 5p$$
$$\Rightarrow 15 - 5p = 0$$
$$\Rightarrow p = \frac{15}{5} = 3$$

Hence, the value of $p = 3$.

7. Solve $10x - 9y = 12$ and $3x - 9y = 17$.

Solution:

Consider the equations

$$10x - 9y - 12 = 0$$

$$3x - 9y - 17 = 0$$

Now, $$\frac{x}{-153 + 108} = \frac{y}{-36 + 170} = \frac{1}{-90 + 27}$$

$$\Rightarrow \frac{x}{45} = \frac{y}{134} = \frac{1}{-63}$$

Equating $\frac{x}{45} = \frac{1}{-63} \Rightarrow -63x = 45$, we get

i.e., $x = \frac{-45}{63}$ and $\frac{y}{134} = \frac{1}{-63}$ $\therefore x = \frac{-5}{7}$

i.e., $-63y = 134$ $\therefore y = -\frac{134}{63}$

Hence, $x = -\frac{5}{7}$ and $y = -\frac{134}{63}$

8. Solve by elimination method: $4x - 3y = 8$ and $3x - 4y = -1$.

Solution:

$4x - 3y = 8$ ....(i)

$3x - 4y = -1$ ....(ii)

Multiplying equation (i) × 3 and equation (ii) × 4, we get

NOTES

$$
\begin{array}{l}
12x - \;\;9y = 24 \\
12x - 16y = -4 \\
\underline{(-)\quad(+)\quad(+)} \qquad \text{(Subtract)} \\
\qquad 7y = 28
\end{array}
$$

$$\Rightarrow y = \frac{28}{7} \qquad \therefore y = 4$$

Multiplying equation (i) × 4 and equation (ii) × 3, we get

$$
\begin{array}{l}
16x - 12y = 32 \\
9x - 12y = -3 \\
\underline{(-)\quad(+)\quad(+)} \qquad \text{(Subtract)} \\
7x \qquad = 35
\end{array}
$$

$$\Rightarrow x = \frac{35}{7} \qquad \therefore x = 5$$

$$\therefore x = 5,\ y = 4$$

2.7 SUMMARY

- An equation is a relation between two variables (two or more) and holds good only for certain values of the variables. Thus, it is clear that in an equation, the equality holds for certain values of the variables. However, in case of the identities, the equality holds for any value of variables.
- The equation of the form containing x^2 as the highest power of x is called an equation of the second degree in x or a quadratic equation.
- The quadratic equation has two and only two roots. These two roots may be equal or unequal.
- Linear equations of degree 'one' are called linear equations. For example, $x = 1$, $5x + 8 = 0$, $10x + 13 = 0$, etc. are all examples of Linear Equations in one variable.
- The quadratic equation has two and only two roots. These two roots may be equal or unequal.
- In Substitution method, the value of y (or x) is found in terms of x (or y) from an equation, and substituting this value in the other equation, we get a linear equation of one variable.

NOTES

2.8 KEY TERMS

- **Equation:** An equation is a relation between two variables (two or more) and holds good only for certain values of the variables.
- **Linear Equations:** The equations of degree 'one' are called linear equations. For example, $x = 1$, $5x + 8 = 0$, $10x + 13 = 0$, etc. are all examples of Linear Equations in one variable.
- **Quadratic Equation:** The quadratic equation has two and only two roots. These two roots may be equal or unequal.
- **Substitution Method:** In this method, the value of y (or x) is found in terms of x (or y) from an equation, and substituting this value in the other equation, we get a linear equation of one variable.

2.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What do you mean by equation?
2. State the types of equation.
3. Define linear equations.
4. What do you mean by simultaneous linear equations?
5. What do you mean by quadratic equations?

Long Answer Questions

1. Give the general form of quadratic equation.
2. How many methods are available to solve simultaneous equations?
3. Form an equations whose roots are 2 and –3.
4. Form an equation whose roots are 7 and –4.
5. Find the roots of the equation $x^2 - 3x - 10 = 0$.
6. Solve for x and y. Given $\frac{3}{x} + \frac{1}{y} = 9$ and $\frac{1}{x} - \frac{2}{y} = 10$.
7. Solve for B. Given $\frac{B+1}{2} - \frac{B-2}{3} = \frac{B+4}{5} + \frac{7}{15}$.
8. If $\frac{3x+5}{4x+2} = \frac{3x+4}{4x+7}$, find x.

9. Solve the following equation in the method of completing the square:

   (i) $x^2 + 9x + 7 = 0$

   (ii) $4x^2 + 4x - 3 = 0$

   (iii) $3x^2 - 5x - 8 = 0$

NOTES

10. Solve the following equation by using formula method:

    (i) $9x^2 - 3x - 2 = 0$

    (ii) $2x^2 - 7x - 3 = 0$

    (iii) $12x^2 - 23x - 24 = 0$

11. Solve the equation $6x^2 = 12 + 11x$ by formula method.
12. Solve for $x : 3(x - 3)(x + 4) + 3(x + 2)(x - 4) = 19(x - 4)(x - 3)$.
13. Using formula method ,solve the following: $5(x - 2)^2 - 6 = -13(x - 2)$.
14. Solve the following equation: $\frac{2p}{p+1} + \frac{p+2}{p-1} = 3$.
15. Solve for x if $\frac{2}{x-1} + \frac{3}{x+4} = \frac{5}{x+3}$.
16. Solve the equations $x + y = 15$ and $3x - y = 21$ in elimination method.
17. Solve by elimination method $4x - 2y = 12,\ 5x + 2y - 104 = 0$.
18. Solve by elimination method $4x - 3y = 8,\ 3x + 4y = -1$.
19. Solve for x and y using method of substitution: $3x + 3y = 12,\ 2x + 4y = 12$.
20. Solve by Substitution method: $4x - y = 2,\ -3x + 2y = 1$.
21. Two years ago, a man was six times as old as his son. In 18 years, he will be twice as old as his son. Determine their present ages.
22. The age of the father is 4 time that of his son. Five years ago, father was 7 times as old as his son. Find their present age.
23. Five years ago, father was 5 times as old as his son, and in 3 years, he will be 3 times as old as his son. Find their present ages.
24. A father is 25 years older than his daughter. In three years, the father’s age will be 5 years more than that of his daughter. Find their present ages.

2.10 FURTHER READING

NOTES

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UNIT 3 ELEMENTARY MATRICES

NOTES

Structure

3.0 INTRODUCTION

In the middle of the 19th century, Arthur Cayley, a British Mathematician introduced the concept of Matrices. Gradually, the importance of this concept were realized by other scholars. Also and as of now, theory of matrices has come to stay as a watch word, not only in mathematics, but also in many other disciplines of science and arts both. Presently, apart from Mathematicians, it is being used by economists, business mathematicians, demographers, sociologists, physicists and statisticians. It finds applications in communication theory, in electrical engineering and also in quantum mechanics. They study of matrices was originated while solving different types of linear problems. One of them deals with the determination of the set solutions of a system of linear equations.

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3.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the concept of Elementary Matrices
- Describe the types of Matrices
- Discuss the addition and subtraction of Matrices
- Examine the determinants of Matrices

3.2 MATRIX

Arrangement of elements in horizontal and vertical lines is called a Matrix.

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \end{bmatrix}$$

The horizontal and vertical lines are mathematically called rows and columns.

$$\begin{bmatrix} 2 & 0 & 5 \end{bmatrix} \to Rows$$

$$\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix} \downarrow Columns$$

Order of Matrix

The number of rows and columns of a matrix is called Order of Matrix.

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{(2 \times 2)} \text{ is matrix order 2 by 2.}$$

Notations

A matrix is usually denoted by capital letters. The elements of a matrix are shown enclosed in ordinary brackets ( ) or square brackets [ ] or a pair of bar $\left|\;\right| \left|\;\right|$.

Example:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix}, A = \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} or\ A = \begin{Vmatrix} 2 & 3 \\ 4 & 8 \end{Vmatrix}$$

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General Format of Matrix

A general matrix with '*m*' rows and '*n*' columns is below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots. & \ldots. \\ a_{31} & a_{32} & a_{33} \ldots\ldots\ldots\ldots & a_{3n} \end{bmatrix} = (a_n)$$

Two suffixes are attached to each element in the matrix, the first suffix indicating row and the second suffix indicating the column in which the element is located. Thus, it denotes the element in the i^{th} row and in the j^{th} column.

Remarks

(i) In a matrix, the number of rows need not be equal to the number of columns.

(ii) A matrix has no value.

3.3 TYPES OF MATRICES

Any matrix is broadly classified into two types, i.e., Rectangular Matrix and Square Matrix.

3.3.1 Rectangular Matrix

A matrix in which the number of rows is not equal to the number of columns is called rectangle matrix.

Example:

$$\begin{bmatrix} 8 & 1 & 5 \\ 2 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

3.3.2 Square Matrix

Any matrix with on equal number of rows and columns is called Square Matrix. These are of two types:

(i) Row Matrix

(ii) Column Matrix.

Examples:

$[3 \;\; 6 \;\; 7 \;\; -4 \;\; 0]$ is a row matrix of order (1×5)

$[0 \;\; 0 \;\; 0 \;\; 0]$ is a row matrix of order (1×4)

(i) Row Matrix: A matrix is said to be a row matrix, if it has only one row and any number of column, i.e., if it is of the type $1 \times n$.

NOTES

Examples:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{(2 \times 2)}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -2 & -3 \end{bmatrix} \text{is a square matrix of order } (3 \times 3)$$

(ii) Column Matrix: A matrix is said to be a column matrix, if it has only one column and any number of row, i.e., if it is of the type $m \times 1$.

Examples:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{is a column matrix of order } (4 \times 1)$$

$$\text{and } \begin{bmatrix} 0 \\ -3 \\ -4 \end{bmatrix} \text{is a column matrix of order } (3 \times 1)$$

3.3.3 Principal Diagonal

In a square matrix, the diagonal which starts from left hand top corner and ends at to right hand bottom corner is called the principal diagonal or leading diagonal elements.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{Principal diagonal}$$

3.3.4 Diagonal Matrix

A diagonal matrix is a matrix in which all the elements except the elements in the principal diagonal are zeroes.

Example:

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{is diagonal matrix of dimension } (3 \times 3)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{is diagonal matrix of dimension } (2 \times 2)$$

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix} \text{is not a diagonal matrix}$$

NOTES

3.3.5 Scalar Matrix

It is a diagonal matrix in which all the elements in the principal diagonal elements are equal.

Example:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$ (Scalar matrix is of order 2 and 3 respectively.)

Remarks:

(i) Every scalar matrix is a diagonal matrix.

(ii) Every diagonal matrix is a square matrix.

(iii) Every unit matrix in a scalar matrix as well as a diagonal matrix. But the converse is not true.

(iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are not unit matrices.

3.3.6 Unit Matrix (Identity Matrix)

A scalar matrix in which all the principal diagonal elements are called a "Unit or Identity Matrix". It is always denoted by I.

Example:

$$I = [1]_{1 \times 1} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

Unit matrices of order 1, 2, 3

Any square matrix in which all the elements above or below the principal diagonal elements are zeroes.

3.3.7 Triangular Matrix

(a) Upper Triangular Matrix

A square matrix, all of whose elements below the principal diagonal are zero, is called an "Upper Triangular Matrix".

(b) Lower Triangular Matrix

A square matrix, all of whose elements above the leading diagonal are zero, is called a "Lower Triangular Matrix"

NOTES

Thus, $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$ and upper and lower triangular matrices respectively.

3.3.8 Transpose of a Matrix

The matrix obtained from a given matrix A by interchanging its row and columns is called transpose of a matrix and is denoted by A^I or A^T.

Example:

Consider $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix}$ or order (2×3)

If we interchange its rows and columns, then we get

$$A^1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$$

This matrix is called the Transpose of the Matrix A.

3.3.9 Symmetric Matrix

Any square matrix A is said to be symmetric matrix, if satisfied the condition $A = A^1$.

Example:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 2 & 6 \\ 3 & 6 & 4 \end{bmatrix} \qquad A^1 = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 2 & 6 \\ 3 & 6 & 4 \end{bmatrix}$$

3.3.10 Skew Symmetric Matrix

As above, condition $A^1 = -A$.

Example:

$$A = \begin{bmatrix} 0 & 5 & -6 \\ -5 & 0 & 7 \\ 6 & -7 & 0 \end{bmatrix}$$

3.3.11 Zero Matrix or Null Matrix

A matrix of any order, in which all the elements are zeroes, is called a Zero Matrix or Null Matrix (Void Matrix). It is always denoted by '0'.

Example:

$A = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all Null Matrices.

3.3.12 Sub-matrix

The matrix formed by deleting or omitting either some rows or columns from a Matrix. A is known as the sub-matrix of A.

Example:

$$A = \begin{bmatrix} 2 & 6 & 7 & 8 \\ 3 & 1 & 2 & 3 \\ 4 & 2 & 3 & 5 \end{bmatrix}$$

Sub-matrices of 'A' are $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 & 7 & 8 \\ 1 & 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Algebra of Matrices

The algebra of matrices includes equality matrices, addition of matrices, subtraction matrices and multiplication matrices.

Equality of Matrices

Two matrices of the same order are said to be equal only when the corresponding elements (the elements in the respective of the two matrices) are equal.

$$A = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 4 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -4 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 4 & 7 \end{bmatrix}$$

Here, $A = C$ but $A \neq B \neq C$

3.4 ADDITION AND SUBTRACTION

The two matrices can be added or subtracted only when their orders are same. The sum and difference is obtained by adding or subtracting the corresponding elements of the matrices.

Examples:

(a) If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3 \times 2}$ and $B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix}_{3 \times 2}$

then $A + B = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ c_3 & b_3 \end{bmatrix} \pm \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 \pm c_1 & b_1 \pm d_1 \\ a_2 \pm c_2 & b_2 \pm d_2 \\ a_3 \pm c_3 & b_3 \pm d_3 \end{bmatrix}_{3 \times 2.}$

(b) If $\begin{bmatrix} 5 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} -3 & 1 & 2 \\ 7 & 4 & -5 \end{bmatrix}_{2 \times 3}$,

$$\text{then } A + B = \begin{bmatrix} 5-3 & 2+1 & 0+2 \\ 3+7 & 1+4 & 4-5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 10 & 5 & -1 \end{bmatrix}_{2\times 3}$$

NOTES

(c) *If* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$, then $A + B$ is not defined as they are not of the same order.

Properties of Matrix Addition

Example 1:

Matrix Addition is commutative: If A and B are two matrices of the same order, then $A + B = B + A$.

Solution:

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$, then $A + B = \begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix}$ and

$B + A = \begin{bmatrix} 7 & 1 \\ -2 & 10 \end{bmatrix}$, i.e., $A + B = B + A$

Example 2:

Matrix Addition is associative: Let A, B and C be three matrices of the same order, then $(A + B) + C = A + (B + C)$.

Solution:

$A = \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix}$, then $A + B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$

$(A + B) + C = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 6 & 3 \end{bmatrix}$ ....(i)

Consider $B + C = \begin{bmatrix} 5 & 8 \\ 5 & 3 \end{bmatrix}$

then $A + (B + C) = \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 6 & 3 \end{bmatrix}$ ....(ii)

$\therefore$ From (i) and (ii), $(A + B) + C = A + (B + C)$

Example 3:

If $A = \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 9 & 1 \end{bmatrix}$, find out $2A + 3B$.

Solution:

$$2A = 2\begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 6 & 8 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 27 & 3 \end{bmatrix}$$

Consider, $2A + 3B = \begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 18 & 24 \\ 27 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 10+18 & 6+24 \\ 8+27 & 12+3 \end{bmatrix} = \begin{bmatrix} 28 & 30 \\ 35 & 15 \end{bmatrix}$$

Example 4:

If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find a matrix X such that $2A + 5B + 2X = 0$.

Solution:

Given $2A + 5B + 2X = 0$ $\quad\quad \therefore 2X = -2A - 5B$

$$2X = -2\begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$$

$$2X = -\begin{bmatrix} 18 & 2 \\ 8 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 25 \\ 35 & 60 \end{bmatrix} = \begin{bmatrix} -23 & -27 \\ -43 & -66 \end{bmatrix}$$

$$X = -\frac{1}{2}\begin{bmatrix} 23 & 27 \\ 43 & 66 \end{bmatrix}$$

Example 5:

Evaluate $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$.

Solution:

Given $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 21 \\ 18 & 15 & 12 \end{bmatrix}$$

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$$= \begin{bmatrix} 1+0-27 & 3+2-24 & 5+4-21 \\ 2+6-18 & 4+8-15 & 6+10-12 \end{bmatrix}$$

$$= \begin{bmatrix} -26 & -19 & -12 \\ -10 & -3 & 4 \end{bmatrix}$$

NOTES

Example 6:

Solve for A and B if $A - 2B = \begin{bmatrix} 4 & 6 & -10 \\ 6 & -4 & 2 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix}$.

Solution:

Now, $A - 2B = \begin{bmatrix} 4 & 6 & -10 \\ 6 & -4 & 2 \end{bmatrix}$ ....(i)

$2A - B = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix}$ ....(ii)

$No.\ (i) \times 2 \Rightarrow 2A - 4B = \begin{bmatrix} 8 & 12 & -20 \\ 12 & -8 & 4 \end{bmatrix}$ ....(iii)

$$No.\ (ii) - No.\ (iii) \Rightarrow -B + 4B = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 12 & -20 \\ 12 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -20 & 22 \\ -8 & 8 & -2 \end{bmatrix}$$

$$\therefore \quad 3B = \begin{bmatrix} -4 & -20 & 22 \\ -8 & 8 & -2 \end{bmatrix} \qquad \therefore \quad \begin{bmatrix} -4/3 & -20/3 & 22/3 \\ -8/3 & 8/3 & -2/3 \end{bmatrix}$$

Substituting the value of matrix B in (ii), we get

$$2A - \begin{bmatrix} -4/3 & -20/3 & 22/3 \\ -8/3 & 8/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} 4-4/3 & -8-20/3 & 2+22/3 \\ 4-8/3 & 0+\ 8/3 & 2-2/3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 8/3 & -44/3 & 28/3 \\ 4/3 & 8/3 & 4/3 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 4/3 & -22/3 & 14/3 \\ 2/3 & 4/3 & 2/3 \end{bmatrix}$$

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Example 7:

Solve for A and B if $A + 2B = \begin{bmatrix} 21 & 16 \\ 21 & 2 \end{bmatrix}$ and $2A + 3B = \begin{bmatrix} -12 & -11 \\ 1 & -16 \end{bmatrix}$.

Solution:

$$\text{Now,} \quad A + 2B = \begin{bmatrix} 21 & 16 \\ 21 & 2 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$2A + 3B = \begin{bmatrix} -12 & -11 \\ 1 & -16 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$\Rightarrow$ (i) × 2 – (ii), we get

$$2A + 4B = \begin{bmatrix} 42 & 32 \\ 42 & 4 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} -12 & -11 \\ 1 & -16 \end{bmatrix}$$

$$(-) \quad (-) \quad (-)$$

$$B = \begin{bmatrix} 54 & 43 \\ 41 & 20 \end{bmatrix}$$

Example 8:

If $A = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 9 & 0 & 8 \\ 7 & 6 & 5 \end{bmatrix}$, find $5A + 3B$.

Solution:

$$5A + 3B = 5\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 0 \\ 1 & 2 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 2 & 1 \\ 9 & 0 & 8 \\ 7 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 30 & 35 \\ 40 & 45 & 0 \\ 5 & 10 & 15 \end{bmatrix} + \begin{bmatrix} 9 & 6 & 3 \\ 27 & 0 & 24 \\ 21 & 18 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 34 & 36 & 38 \\ 67 & 45 & 24 \\ 26 & 28 & 30 \end{bmatrix}$$

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Example 9:

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ 4 & 5 \end{bmatrix}$, find $A^1 + B^1$.

Solution:

$$A^1 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \text{ and } B^1 = \begin{bmatrix} 2 & 4 \\ 7 & 5 \end{bmatrix}$$

$$\therefore \quad A^1 + B^1 = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 7 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 9 & 8 \end{vmatrix}$$

Check Your Progress

1. Solve for A and B if $A - 2B = \begin{bmatrix} 4 & 6 & -10 \\ 6 & -4 & 2 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 4 & -4 & 2 \\ 4 & 0 & 2 \end{bmatrix}$.
2. Matrix Addition is Associative: Let A, B and C be three matrices of the same order, then $(A + B) + C = A + (B + C)$.

3.5 MULTIPLICATION OF MATRICES

Two matrices can be multiplied only when the number of columns of the first matrix is equal to the number of rows of second matrix. Hence, product is defined, i.e., if A is a matrix of order $(m \times p)$ and B is of order $(p \times n)$, then AB is defined as a matrix of order $(m \times n)$, such that the elements of AB in the i^{th} row and j^{th} column is obtained by multiplying the elements of the i^{th} row of A by the corresponding elements of the j^{th} columns of B, and adding the result.

For example, the following diagrams show how to obtain the elements in the product of two matrices.

$$\text{Step 1: } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}_{3 \times 2} = \begin{bmatrix} ax + by + cz & * \\ * & * \end{bmatrix}$$

$$\text{Step 2: } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}_{3 \times 2} = \begin{bmatrix} * & au + by + cw \\ * & * \end{bmatrix}$$

$$\text{Step 3: } \overline{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}}_{2\times 3} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}_{3\times 2} = \begin{bmatrix} * & * \\ dx + dy + fz & * \end{bmatrix}$$

$$\text{Step 4: } \overline{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}}_{2\times 3} \begin{pmatrix} x & u \\ y & v \\ z & w \end{pmatrix} = \begin{bmatrix} * & * \\ dx + dy + fz & * \end{bmatrix}$$

Combining the above four steps gives us

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} ax + by + cz & au + by + cw \\ dx + cy + fz & du + cv + fw \end{bmatrix}_{2\times 2}$$

Then,

$$AB = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ \downarrow 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 1 + 5\times 2 + 1\times 3 & 2\times 4 + 5\times 3 + 1\times 5 \\ 3\times 1 + 4\times 2 + 2\times 3 & 3\times 4 + 4\times 3 + 2\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 28 \\ 17 & 34 \end{bmatrix}$$

Scalar Multiplication

Multiplication of any matrix by a real number is called Scalar multiplication.

Example 10:

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \text{ then } 2A = 2\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 4 \end{bmatrix}.$$

Multiplication of a Matrix by a Scalar

The product of a matrix A by a scalar number K is a matrix whose each element is K times the corresponding elements of A. Thus,

$$\text{if } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \text{ then } KA = \begin{bmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \end{bmatrix}.$$

$$\text{Similarly, } -2\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ -6 & -12 \end{bmatrix} \text{ and } -1\begin{bmatrix} -1 & -0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$$

Thus, KA is called the scalar multiplication of the Matrix A.

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Example 11:

If $A = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ and $C = \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$, form as many product as you can – AB, BC, CA and AC.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = [\, 1 \times 2 + 2 \times 4 \quad 1 \times 3 + 2 \times 5]$$

$$= [2 + 8 \quad 3 + 10] = [10 \quad 13]$$

$$BC = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\times 3 & + & 3\times 5 \\ 4\times 3 & + & 5\times 5 \end{bmatrix} = \begin{bmatrix} 6 + 15 \\ 12 + 25 \end{bmatrix} = \begin{bmatrix} 21 \\ 37 \end{bmatrix}$$

$$CA = \begin{bmatrix} 3 \\ 5 \end{bmatrix}[1 \quad 2] = \begin{bmatrix} 3\times 1 & + & 3 \times 2 \\ 5 \times 1 & + & 5 \times 2 \end{bmatrix} = \begin{bmatrix} 3 + 6 \\ 5 + 10 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 2 \times 5 \end{bmatrix} = \begin{bmatrix} 3 + 10 \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix}$$

Example 12:

If $A = \begin{bmatrix} 3 & 4 & 7 \\ 9 & 0 & 5 \\ -6 & -8 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 9 \\ 3 & -2 & -7 \end{bmatrix}$, find $4A + 2B$.

Solution:

$$4A + 2B = 4\begin{bmatrix} 3 & 4 & 7 \\ 9 & 0 & 5 \\ -6 & -8 & -5 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 9 \\ 3 & -2 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 16 & 28 \\ 36 & 0 & 20 \\ -24 & -32 & -20 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 10 \\ 6 & 14 & 18 \\ 6 & -4 & -14 \end{bmatrix} = \begin{bmatrix} 14 & 20 & 38 \\ 42 & 14 & 38 \\ -18 & -36 & -34 \end{bmatrix}$$

Example 13:

Given that $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix}$, verify $(5A) = 5A$.

Solution:

$$5A = 5\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix} \quad \Rightarrow 5A = \begin{bmatrix} 5 & 10 & 5 \\ 10 & 20 & 10 \\ 15 & 30 & 15 \end{bmatrix}$$

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$$\therefore \quad (5A) = \begin{bmatrix} 5 & 10 & 5 \\ 10 & 20 & 10 \\ 15 & 30 & 15 \end{bmatrix}$$

$$\therefore \quad (5A) = 5A$$

Example 14:

If $A = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 4 \\ 6 & 2 \end{bmatrix}$, show that $(AB) = B'A'$.

Solution:

$$\text{Consider } AB = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 32+12 & 16+4 \\ 48+24 & 24+8 \end{bmatrix} = \begin{bmatrix} 44 & 20 \\ 72 & 32 \end{bmatrix}$$

$$\therefore (AB) = \begin{bmatrix} 44 & 72 \\ 20 & 32 \end{bmatrix}$$

$$\text{Now, } BA = \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 32+12 & 48+24 \\ 16+\ 4 & 24+\ 8 \end{bmatrix} = \begin{bmatrix} 44 & 72 \\ 20 & 32 \end{bmatrix}$$

$$\therefore \quad (AB) = BA$$

Example 15:

Solve for a, b and c, given that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8+15+24 \\ 16+25+36 \\ 28+40+54 \end{bmatrix} = \begin{bmatrix} 47 \\ 77 \\ 122 \end{bmatrix}$$

Hence, $a = 47$, $b = 77$ and $c = 122$

Example 16:

Keerthi buys 8 dozen of pens, 10 dozens of pencils and 4 dozens of rubber. Pens cost ₹ 18 per dozen, pencils ₹ 9 per dozen and rubber ₹ 6 per dozen represents the quantities bought by a row matrix and prices by a column matrix and hence obtain the total cost.

Solution:

Quantities are expressed as row matrix $\begin{bmatrix} 8 & 10 & 4 \end{bmatrix}$

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Prices are expressed as a column matrix $\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$

$$\text{Total Cost} = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix} = \left[(8\times 18)+(10\times 9)+(4\times 6)\right]$$

$$= 144 + 90 + 24 = 258$$

Example 17:

If $A = \begin{bmatrix} 1 & 2 & -3 \\ 6 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 3 \\ 6 & 3 & 10 \\ 2 & 0 & 3 \end{bmatrix}$, find AB and BA.

Solution:

$$AB = A \times B$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 6 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 & 3 \\ 6 & 3 & 10 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+12-6 & -1+6+0 & 3+20-9 \\ 24+\ 0+6 & -6+0+0 & 18+\ 0+9 \\ 8-\ 6+2 & -2-3+0 & 6-10+3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 & 14 \\ 30 & -6 & 27 \\ 4 & -5 & 1 \end{bmatrix}$$

$$BA = B \times A$$

$$= \begin{bmatrix} 4 & -1 & 3 \\ 6 & 3 & 10 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 6 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\ 6+\ 6 & 8+0-\ 3 & -12-3+\ 3 \\ 6+18+20 & 12+0-10 & -18+9+10 \\ 2+\ 0+\ 6 & 4+0-\ 3 & -6+0+\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & -12 \\ 44 & 2 & 1 \\ 8 & 1 & -3 \end{bmatrix}$$

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Example 18:

If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 15 \end{bmatrix}$, find AB.

Solution:

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-15 \\ -2-18+20 & -4-36+40 & -2-18+60 \\ -3-12+15 & -6-24+30 & -3-12+45 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 0 & 0 & -10 \\ 0 & 0 & 40 \\ 0 & 0 & 30 \end{bmatrix}$$

Example 19:

If $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$, find $(AB)^1 = B^1 A^1$.

Solution:

Given, $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$

$$A^1 = \begin{bmatrix} 4 & 3 \\ 2 & -7 \\ -1 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 2 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6+1 & 12+0-5 \\ 6+21-1 & 9+0+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 26 & 14 \end{bmatrix}$$

$$(AB)^1 = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

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$$B^1 A^1 = \begin{bmatrix} 2 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & -7 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6+1 & 6+21-1 \\ 12+0-5 & 9+\ 0+5 \end{bmatrix} = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

$$\therefore (AB)^1 = B^1 A^1$$

Example 20:

Prove that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A + 5I = 0$ where I is the identity matrix and 0 is the zero matrix.

Solution:

Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Consider L.H.S. $= A^2 - 4A + 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ L.H.S. = R.H.S. (proved)

Example 21:

If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find a matrix X such that $2A + 5B + 3X = 0$.

NOTES

Solution:

$2A + 5B + 3X = 0$

$$\Rightarrow \; 2\begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} + 3X = 0$$

$$\Rightarrow \; \begin{bmatrix} 18 & 2 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 25 \\ 35 & 60 \end{bmatrix} + 3X = 0$$

$$\Rightarrow \; \begin{bmatrix} 18+5 & 2+25 \\ 8+35 & 6+60 \end{bmatrix} + 3X = 0 \quad \Rightarrow \begin{bmatrix} 23 & 27 \\ 43 & 66 \end{bmatrix} + 3X = 0$$

$$\Rightarrow \; 3X = -\begin{bmatrix} 23 & 27 \\ 43 & 66 \end{bmatrix} \quad \therefore X = -\frac{1}{3}\begin{bmatrix} 23 & 27 \\ 43 & 66 \end{bmatrix}$$

Example 22:

If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, calculate $A^2 - 5A + 9I$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 + 1(-1) + 1.1 & 2.1 + 1.0 + 1.3 & 2.1 + 1.1 + 1(-1) \\ (-1)2 + 0(-1) + 1.1 & (-1)1 + 0.0 + 1.3 & (-1)1 + 0.1 + 1(-1) \\ 1.2 + 3(-1) + (-1)1 & 1.1 + 3.0 + (-1)3 & 1.1 + 3.1 + (-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4-1+1 & 2+3 & 2+1-1 \\ -2+1 & -1+3 & -1-1 \\ 2-3-1 & 1-3 & 1+3+1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 5 \\ -5 & 0 & 5 \\ 5 & 15 & -5 \end{bmatrix}$$

$$9I = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

NOTES

Consider $A^2 - 5A + 9I$

$$= \begin{bmatrix} 4 & 5 & 2 \\ -1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ -5 & 0 & 5 \\ 5 & 15 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 & -3 \\ 4 & 2 & -7 \\ -7 & -17 & 10 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 4 & 11 & -7 \\ -7 & -17 & 19 \end{bmatrix}$$

Check Your Progress

3. If $A = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1 \times 2}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2 \times 2}$ and $C = \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2 \times 1}$, form as many product as you can – AB, BC, CA and AC.

4. If $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$, find $(AB)^1 = B^1 A^1$.

5. If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, calculate $A^2 + 5A + 9I$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3.6 DETERMINANTS OF MATRICES

The working knowledge of determinants is a basic necessity for a student of commerce as a large number of physical phenomena are expressed in terms of linear simultaneous equations which depend for their solution on the knowledge of various methods of solving a system of linear equations. A brief sketch of the important properties of determinants with applications to the solution of a system of linear equations is given in this topic. This will enable the student to avail himself of the advantages of the determinants in Business Mathematics.

Determinant of a Square Matrix

Every square matrix A is associated with a real number called its determinant and is denoted by $|A|$ or ΔA.

There are two methods to find the determinants of a matrix. They are: 1. expansion method and 2. Matrix method

Determinants of order 2×2:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } |A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \times d - c \times b = ad - cb.$$

Determinants of order 3×3:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And its value is the number

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

This is called the expansion of the determinant along its first row. To obtain this expansion, we multiply each element of the first row by the determinant of the second order which is obtained by leaving the row and the column passing through the element.

NOTES

Example 23:

Evaluate $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution:

Consider $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= 3abc - a^3 - b^3 - c^3$$

NOTES

Example 24:

If $\begin{vmatrix} 3 & 4 & x \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$, find x.

Solution:

$$\text{Given, } \begin{vmatrix} 3 & 4 & x \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$$

$$\Rightarrow 3(2+3) - 4(4+15) + x(-2+5) = -40$$

$$\Rightarrow 3 \times 5 - 4 \times 19 + x \times 3 = -40$$

$$\Rightarrow 15 - 76 + 3x = -40$$

$$\Rightarrow 3x = -40 + 61 = 21 \Rightarrow x = \frac{21}{3} \qquad \therefore x = 7$$

Example 25:

Evaluate $\begin{vmatrix} x & 1 & 2 \\ 2 & x & 2 \\ 3 & 1 & x \end{vmatrix}$.

Solution:

$$\text{Consider } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= x \begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 2 \begin{vmatrix} 2 & x \\ 3 & 1 \end{vmatrix}$$

$$= x(x^2 - 2) - (2x - 6) + 2(2 - 3x)$$

$$= x^3 - 2x - 2x + 6 + 4 - 6x$$

$$= x^3 - 4x + 10$$

Example 26:

Solve for x : $\begin{vmatrix} 3 & x & 1 \\ -4 & 6 & 7 \\ 2 & -1 & 4 \end{vmatrix} = 0.$

NOTES

Solution:

$$\text{Let} \begin{vmatrix} 3 & x & 1 \\ -4 & 6 & 7 \\ 2 & -1 & 4 \end{vmatrix} = 3\begin{vmatrix} 6 & 7 \\ -1 & 4 \end{vmatrix} - x\begin{vmatrix} -4 & 7 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} -4 & 6 \\ 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(24+7) - x(-16-14) + 1(4-12) = 0$$

$$\Rightarrow \quad 93 + 30x - 8 = 0$$

$$\Rightarrow \quad 30x + 85 = 0$$

$$\therefore \quad x = \frac{85}{30} = \frac{17}{6}$$

Example 27:

$$\text{Find } 'y' \text{ if } \begin{bmatrix} -3 & -6 & 1 \\ 5 & y & -2 \\ 2 & -3 & 5 \end{bmatrix} = 7.$$

Solution:

$$\text{Let } A = \begin{bmatrix} -3 & -6 & 1 \\ 5 & y & -2 \\ 2 & -3 & 5 \end{bmatrix}$$

$$\Rightarrow \quad -3\begin{vmatrix} y & -2 \\ -3 & 5 \end{vmatrix} + 6\begin{vmatrix} 5 & -2 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 5 & y \\ 2 & -3 \end{vmatrix} = 7$$

$$\Rightarrow \quad -3(5y+6) + 6(25+4) + 1(-15+2y) = 7$$

$$\Rightarrow \quad -15y + 18 + 150 + 24 - 15 + 2y = 7$$

$$\Rightarrow \quad -17y + 177 = 7$$

$$\Rightarrow \quad -17y = 7 - 177$$

$$\Rightarrow \quad -17y = -170$$

$$\Rightarrow \quad y = \frac{170}{17}$$

$$\therefore \quad y = 10$$

Example 28:

$$\text{Find the value of } `a\text{' if } \begin{vmatrix} 6 & -2 & -4 \\ a & 2 & -1 \\ -5 & 3 & a \end{vmatrix} = 0.$$

Solution:

$$\Rightarrow 6\begin{vmatrix} 2 & -1 \\ 3 & a \end{vmatrix} + 2\begin{vmatrix} a & -1 \\ -5 & a \end{vmatrix} - 4\begin{vmatrix} a & 2 \\ -5 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 6(2a+3) + 2(a^2-5) - 4(3a+10) = 0$$

$$\Rightarrow 12a + 18 + 2a^2 - 10 - 12a - 40 = 0$$

$$\Rightarrow 2a^2 - 32 = 0 \Rightarrow 2a^2 = 32 \Rightarrow a^2 = 16$$

$$\therefore a = \pm 4$$

Example 29:

Solve for x and y if $\begin{bmatrix} x & 3 & y \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = 16$ and $\begin{bmatrix} 1 & 2 & -1 \\ -x & 1 & 2 \\ y & -1 & 1 \end{bmatrix} = 28.$

Solution:

Consider $x\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + y\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 16$

$$\Rightarrow x(12-2) - 3(16-1) + y(8-3) = 16$$

$$\Rightarrow 10x - 45 + 5y = 16$$

$$\Rightarrow 10x + 5y = 61 \qquad \text{...............(i)}$$

Again, $1\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -x & 2 \\ y & 1 \end{vmatrix} - 1\begin{vmatrix} -x & 1 \\ y & -1 \end{vmatrix} = 28$

$$\Rightarrow 1(1+2) - 2(-x-2y) - 1(x-y) = 28$$

$$\Rightarrow 3 + 2x + 4y - x + y = 28$$

$$\Rightarrow x + 5y = 25 \qquad \text{..............(ii)}$$

Consider (i) and (ii)

Now, putting the value of x in (ii), we get

$$\begin{array}{rcl} 10x + 5y = 61 & & x + 5y = 25 \\ x + 5y = 25 & \Rightarrow & 4 + 5y = 25 \\ (-)\ \ (-)\ \ \ (-) & \Rightarrow & 5y = 25 - 4 \\ \hline 9x = 36 & \Rightarrow & 5y = 21 \\ \Rightarrow x = \dfrac{36}{9} & \Rightarrow & y = \dfrac{21}{5} \\ \Rightarrow x = 4 & \therefore & x = 4,\ y = \dfrac{21}{5} \end{array}$$

3.7 CRAMER'S RULE

NOTES

Finding the solution of simultaneous equation using determinants is called Cramer's Rule.

Consider the system of equations

$$a_1 x + b_1 y = c_1 \quad ....(i)$$

$$a_2 x + b_2 y = c_2 \quad ....(ii)$$

in two variables x and y.

Method: To sum three equity using Cramer's rule.

Let $\Delta_1 \; or \; \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ be the determinant formed by taking the coefficients of 'x' and 'y'.

Consider $\Delta_1 \; or \; \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

which is formed by taking the coefficient of x and y from the given let of equation.

Hence,

$$A_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

which is obtained by problems 1st column element by R.H.S.D. Δ_2 Similarly, for A system of three equation,

$$x = \frac{\Delta_x}{\Delta}, \; y = \frac{\Delta_y}{\Delta}, \; z = \frac{\Delta_z}{\Delta}$$

Example 30:

Solve by Cramer's Rule:

$$6x + 5y = 2$$

$$4x - 3y = 14$$

Solution:

According to Cramer's Rule, $6x + 5y = 2$

$$4x - 3y = 14$$

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}$$

$$\Delta = \begin{vmatrix} 6 & 5 \\ 4 & -3 \end{vmatrix} = -18 - 20 = -38$$

NOTES

$$\Delta x = \begin{vmatrix} 2 & 5 \\ 14 & -3 \end{vmatrix} = -6 - 70 = -76$$

$$\Delta y = \begin{vmatrix} 6 & 2 \\ 4 & 14 \end{vmatrix} = 84 - 8 = 76$$

$$\therefore\ x = \frac{\Delta x}{\Delta} = \frac{-76}{-38} = 2$$

$$\therefore\ y = \frac{\Delta y}{\Delta} = \frac{76}{-38} = -2$$

Example 31:

Solve by Cramer's rule:

$$x + y + z = 11$$

$$2x - 6y - z = 0$$

$$3x + 4y + 2z = 0$$

Solution:

By Cramer's Rule,

Let $x = \dfrac{\Delta x}{\Delta},\ y = \dfrac{\Delta x}{\Delta},\ z = \dfrac{\Delta z}{\Delta}.$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -6 \\ 3 & 4 \end{vmatrix}$$

$$= 1(-12 + 4) - 1(4 + 3) + 1(8 + 18)$$

$$= -8 - 7 + 26$$

$$\Delta = 11$$

$$\Delta x = \begin{vmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -6 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & -6 \\ 0 & 4 \end{vmatrix}$$

$$= 11(-12 + 4) - 1(0 + 0) + 1(0 + 0)$$

$$= 11(-8) - 0 + 0$$

$$\Delta x = -88$$

$$\Delta y = \begin{vmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}$$

$$= 1(0 + 0) - 11(4 + 3) + 1(0 + 0)$$

$$= 0 - 11(-7) + 0$$

$$\Delta y = -77$$

$$\Delta z = \begin{vmatrix} 1 & 1 & 11 \\ 2 & -6 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & 0 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} + 11\begin{vmatrix} 2 & -6 \\ 3 & 4 \end{vmatrix}$$

$$= 1(0 + 0) - 1(0 - 0) + 11(8 + 18)$$

$$= 0 - 0 + 11(26)$$

$$\Delta z = 286$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{-88}{11} = -8,\ y = \frac{\Delta y}{\Delta} = \frac{-77}{11} = -7 \text{ and } z = \frac{286}{11} = 26$$

Example 32:

Using Cramer's rule, solve $2x + y - z = 6$, $x + 3y + 2z = 3$ and $3x - y = 5$.

Solution:

By Cramer's rule,

$$x = \frac{\Delta x}{\Delta},\ y = \frac{\Delta y}{\Delta} \text{ and } z = \frac{\Delta z}{\Delta}$$

$$2x + y - z = 6,\ x + 3y + 2z = 3,\ 3x - y = 5$$

NOTES

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= 2(0+2) - 1(0-6) - 1(-1-9)$$

$$= 4 + 6 + 10$$

$$\Delta = 20$$

$$\Delta x = \begin{vmatrix} 6 & 1 & -1 \\ 3 & 3 & 2 \\ 5 & -1 & 0 \end{vmatrix}$$

$$= 6(0+2) - 1(0-10) - 1(-3-15)$$

$$= 12 + 10 + 18$$

$$\Delta x = 40$$

$$\Delta y = \begin{vmatrix} 2 & 6 & -1 \\ 1 & 3 & 2 \\ 3 & 5 & 0 \end{vmatrix}$$

$$= 2(0-10) - 6(0-6) - 1(5-9)$$

$$= -20 + 36 + 4$$

$$\Delta y = 20$$

$$\Delta z = \begin{vmatrix} 2 & 1 & 6 \\ 1 & 3 & 3 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 2(15+3) - 1(5-9) + 6(-1-9)$$

$$= 36 + 4 - 60$$

$$\Delta z = -20$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{40}{20} = 20,\ y = \frac{\Delta y}{\Delta} = \frac{20}{20} = 1 \text{ and } z = \frac{-20}{20} = -1$$

$$\therefore x = 20,\ y = 1 \text{ and } z = -1$$

Example 33:

Using Cramer's rule, solve $x + y + z = 6$, $2x + 3y - z = 5$ and $6x - 2y - 3z = -7$.

Solution:

NOTES

Here, $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= 1(-9-2) - 1(-6+6) + 1(-4-18)$$

$$= -11 - 0 - 22$$

$$\Delta = -33$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 5 & 3 & -1 \\ -7 & -2 & -3 \end{vmatrix}$$

$$= 6(-9-2) - 1(-15-7) + 1(-10+21)$$

$$= -66 + 22 + 11$$

$$\Delta_x = -33$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 5 & -1 \\ 6 & -7 & -3 \end{vmatrix}$$

$$= 1(-15-7) - 6(-6+6) + 1(-14-30)$$

$$= -22 + 0 - 44$$

$$\Delta_y = -66$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 6 \\ 2 & 3 & 5 \\ 6 & -2 & -7 \end{vmatrix}$$

$$= 1(-21+10) - 1(-14-30) + 6(-4-18)$$

$$= -11 + 44 - 132$$

$$\Delta_z = -99$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-33}{-33} = 1,\ y = \frac{\Delta_y}{\Delta} = \frac{-66}{-33} = 2 \text{ and } z = \frac{-99}{-33} = 3$$

$\therefore x = 1$, $y = 2$ and $z = 3$

NOTES

Example 34:

For a creation crop to get more yield, it is necessary to use 19 units area, 17 units of potash and 12 units of nitrogen. One quintal of the mixture '*P*' contains 1, 4 and 2 units of the chemicals; one quintal of the mixture '*Q*' contains 3, 2, 2 units and one quintal of the mixture '*R*' contains 4, 3, 2 units of these chemicals respectively. How much of each type of mixture should be used to get the yield more? Solve by using Cramer's rule.

Solution:

According to Cramer's rule :

$$P + 3Q + 4R = 19$$

$$4P + 2Q + 3R = 17$$

$$2P + 2Q + 2R = 12$$

$$P = \frac{\Delta_P}{\Delta},\ Q = \frac{\Delta_Q}{\Delta} \text{ and } R = \frac{\Delta_R}{\Delta}$$

$$\Delta = \begin{vmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{vmatrix}$$

$$= 1(4-6) - 3(8-6) + 4(8-4)$$

$$= -2 - 6 + 16$$

$$\Delta = 8$$

$$\Delta_P = \begin{vmatrix} 19 & 3 & 4 \\ 17 & 2 & 3 \\ 12 & 2 & 2 \end{vmatrix}$$

$$= 19(4-6) - 3(34-36) + 4(34-24)$$

$$= -38 + 6 + 40$$

$$\Delta_P = 8$$

$$\Delta_Q = \begin{vmatrix} 1 & 19 & 4 \\ 4 & 17 & 3 \\ 2 & 12 & 2 \end{vmatrix}$$

$$= 1(34-36) - 19(8-6) + 4(48-34)$$

$$= -2 - 38 + 56 = -40 + 56$$

$$\Delta_Q = 16$$

$$\Delta_R = \begin{vmatrix} 1 & 3 & 19 \\ 4 & 2 & 17 \\ 2 & 2 & 12 \end{vmatrix}$$

$$= 1(24 - 34) - 3(48 - 34) + 19(8 - 4)$$

$$= -10 - 42 + 76 = -52 + 76$$

$$\Delta_R = 24$$

$$P = \frac{\Delta_P}{\Delta} = \frac{8}{8} = 1, Q = \frac{\Delta_Q}{\Delta} = \frac{16}{8} = 2 \text{ and } R = \frac{24}{8} = 3$$

$\therefore P = 1, Q = 2$ and $R = 3$

NOTES

Example 35:

The price of four Mathematics books, two Accounting books and three Computer books is ₹ 134. The cost of one Mathematics book, three Accounting books and two Computer books is ₹ 81; and the cost of two Mathematics books, one Accounting book and five Computer books is ₹ 130. Find the rate per book of each.

Solution:

Let M, A and C denote the rate per book of Mathematics, Accounting and Computer respectively. The above data can be in the form of simultaneous equations.

$$4M + 2A + 3C = 134, \; M + 3A + 2C = 81, \; 2M + A + 5C = 130$$

This can be solved using Cramer's rule as following determinants:

Here, $M = \frac{\Delta_1}{\Delta}, A = \frac{\Delta_2}{\Delta}$ and $C = \frac{\Delta_3}{\Delta}$

$$\Delta = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \end{vmatrix}$$

$$= 4(13) - 2(1 + 3) + 3(-5)$$

$$= 52 - 17$$

$$\Delta = 35$$

$$\Delta_M = \begin{vmatrix} 134 & 2 & 3 \\ 81 & 3 & 2 \\ 130 & 1 & 5 \end{vmatrix}$$

$$= 134(15-2) - 2(405-260) + 3(81-390)$$

$$= 1742 - 290 - 927$$

NOTES

$$= 1742 - 1217$$

$$\Delta_M = 525$$

$$\Delta_A = \begin{vmatrix} 4 & 134 & 3 \\ 1 & 81 & 2 \\ 2 & 130 & 5 \end{vmatrix}$$

$$= 4(405-260) - 134(5-4) + 3(130-162)$$

$$= 580 - 134 - 96$$

$$= 580 - 230$$

$$\Delta_A = 350$$

$$\Delta_C = \begin{vmatrix} 4 & 2 & 134 \\ 1 & 3 & 81 \\ 2 & 1 & 130 \end{vmatrix}$$

$$= 4(390-81) - 2(130-162) + 134(1-6)$$

$$= 1236 + 64 - 670$$

$$\Delta_C = 630$$

$$M = \frac{\Delta_M}{\Delta} = \frac{525}{35} = 15,$$

$$A = \frac{\Delta_A}{\Delta} = \frac{350}{35} = 10 \text{ and}$$

$$C = \frac{\Delta C}{\Delta} = \frac{630}{35} = 18$$

Hence, the cost of Mathematics books is ₹ 15, Accounting is ₹ 10 and Computer books is ₹ 18.

3.8 MINORS AND CO-FACTORS

The minor of an element a_{ij} is the determinant of the submatrix obtained by deleting the i^{st} row and j^{th} column of matrix.

$$\text{Consider } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The minor of the element a_{ij} is the second order determinant obtained by deleting the i^{st} row and i^{st} column.

NOTES

$$\text{Thus, minor of } a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

$$\text{minor of } b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix},$$

$$\text{minor of } c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

$$\text{minor of } a_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \text{ and so on.}$$

Example: Let $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

$$\text{minor of } 1 = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \qquad \text{minor of } 5 = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$$

Co-factors of the Elements of a Square Matrix

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Co-factor: If the minors are multiply by proper sings which are given below, then it is called co-factor of the element.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & - \end{bmatrix}$$

The co-factor of an element *a* is defined $(-1)^{i+j} A_{ij}$ where A_{ij} is the minor of the element a_{ij} and is denoted C_{iij}. Thus, C_{ij} = co-factor of A_{ij}.

Example: The co-factor of an element

$$a_{12} = c_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = A_{ij}$$

$$\text{and factor of } a_{12} = c = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and so on}$$

3.9 ADJOINT OF A SQUARE MATRIX

NOTES

Adjoint of square matrix 'A' is the transpose of the matrix formed by the co-factors of A. It is denoted by Adj A.

$$A = \text{Adj } A = |A|$$

where I is the Unit Matrix.

Example: $A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

Now, $a_{11} = 3, a_{12} = 4, a_{21} = 5, a_{22} = 8$

$\therefore$ Co-factor of 3 = $c_{11} = (-1)^{1+1} |8| = 8$

Co-factor of 4 = $c_{12} = (-1)^{1+2} |5| = -5$

Co-factor of 5 = $c_{12} = (-1)^{2+1} |4| = -4$

Co-factor of 8 = $c_{22} = (-1)^{2+2} |3| = 3$

$\therefore$ Co-factor of the elements of $A = \begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}$

Adjoint of A = Adj $A = \begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

Example 36:

If $A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$, then find Adjoint A.

Solution:

Co-factor of 2 $= c_{11} = (-1)^{1+1} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

Co-factor of 1 $= c_{12} = (-1)^{1+2} - \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$

Co-factor of 4 $= c_{13} = (-1)^{1+3} + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$

Co-factor of 2 $= c_{21} = (-1)^{2+1} - \begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix} = 4$

$$\text{Co-factor of } 1 = c_{22} = (-1)^{2+2} + \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = -4$$

$$\text{Co-factor of } 2 = c_{23} = (-1)^{2+3} - \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -2$$

$$\text{Co-factor of } 3 = c_{31} = (-1)^{3+1} - \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = -2$$

$$\text{Co-factor of } 2 = c_{32} = (-1)^{3+2} + \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4$$

$$\text{Co-factor of } 4 = c_{33} = (-1)^{3+3} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

Co-factor of the elements of Matrix A

$$A_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 4 & -4 & -2 \\ -2 & 4 & 0 \end{bmatrix}$$

$$\text{Adjoint of } A = \begin{bmatrix} 0 & 4 & -2 \\ -2 & -4 & 4 \\ 1 & -2 & 0 \end{bmatrix}$$

Adj A = Transpose of Co-factor matrix

Example 37:

Find the adjoint of $\begin{bmatrix} -5 & 7 \\ -2 & 3 \end{bmatrix}$ and hence show that $A(\text{Adj } A) = |A| \,.\, I$.

Solution:

Co-factor of –5 = + (3) = 3

Co-factor of 7 = – (–2) = 2

Co-factor of –2 = –(7) = –7

Co-factor of 3 = +(–5) = –5

$$\text{Co-factor matrix } \begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix}$$

Adj A = Transpose of Co-factor matrix

$$\therefore \text{ Adj } A = \begin{bmatrix} 3 & -7 \\ 2 & -5 \end{bmatrix}$$

NOTES

$$A\,.\,\text{Adj}\,A = \begin{bmatrix}-5 & 7\\ -2 & 3\end{bmatrix}\begin{bmatrix}3 & -7\\ 2 & -5\end{bmatrix} = \begin{bmatrix}-15+14 & 35-35\\ -6+6 & 14-15\end{bmatrix}$$

NOTES

$$|A| = (-15+14) = -1 \quad I = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$|A|.I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.\,(-1) = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

L.H.S. = R.H.S. (Proved)

Example 38:

Find the adjoint of the square matrix $A = \begin{bmatrix}-2 & 1 & 0\\ 3 & -1 & 4\\ 2 & 5 & 3\end{bmatrix}$.

Solution:

The co-factors are:

$$A_{11} = \begin{vmatrix}-1 & 4\\ 5 & 3\end{vmatrix} = (-3-20) = -23$$

$$A_{12} = -\begin{vmatrix}3 & 4\\ 2 & 3\end{vmatrix} = -(9-8) = -1$$

$$A_{13} = \begin{vmatrix}3 & -1\\ 2 & 5\end{vmatrix} = (15+2) = 17$$

$$A_{21} = -\begin{vmatrix}1 & 0\\ 5 & 3\end{vmatrix} = -(3-0) = -3$$

$$A_{22} = \begin{vmatrix}-2 & 0\\ 2 & 3\end{vmatrix} = (-6-0) = -6$$

$$A_{23} = -\begin{vmatrix}-2 & 1\\ 2 & 5\end{vmatrix} = -(-10-2) = 12$$

$$A_{31} = \begin{vmatrix}1 & 0\\ -1 & 3\end{vmatrix} = (4-0) = 4$$

$$A_{32} = -\begin{vmatrix}-2 & 0\\ 3 & 4\end{vmatrix} = -(-8-0) = 8$$

$$A_{33} = \begin{vmatrix}-2 & 1\\ 3 & -1\end{vmatrix} = (2-3) = -1$$

$$\text{Co-factor matrix} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

$$= \begin{bmatrix} -23 & -1 & 17 \\ -3 & -6 & 12 \\ 4 & 8 & -1 \end{bmatrix}$$

NOTES

Adj A = Transpose of Co-factor matrix

$$\therefore \text{ Adj } A = \begin{bmatrix} -23 & -3 & 4 \\ -1 & -6 & 8 \\ 17 & 12 & -1 \end{bmatrix}$$

3.10 INVERSE OF A SQUARE MATRIX

A matrix X is said to be inverse of the matrix A if $XA = AX = 1$ where I is the identity matrix. X is denoted by A^{-1} (called the inverse of A).

For the square matrix A, A^{-1} is given by:

$$\frac{1}{|A|} \text{ Adj } A \text{ provided } |A| \neq 0$$

$$\text{Thus, } A^{-1} = \frac{1}{|A|} \text{ Adj } A$$

Remarks:

(i) Inverse of a matrix is unique.

(ii) A^{-1} exists and only if A is non-singular.

(iii) The inverse of the inverse of a matrix A is A itself, i.e. , $\left(A^{-1}\right)^{-1} = A$.

(iv) If $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, then $|A| = 12 - 12 = 0$. Hence, A^{-1} is not done.

Example 39:

Find inverse of $\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$.

Solution:

$$\text{Given, } A = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$|A| = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 6 + 1 = 7$$

NOTES

$$\text{Co-factor of matrix } (A) = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{bmatrix}$$

Example 40:

Find inverse of $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$.

Solution:

$$\text{Let} \quad A = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

Calculatiom of Adjoint A:

Co-factor of 2 = 5

Co factor of $-3 = -(-4)$

Co-factor of $-4 = -(-3) = 3$

Co factor of 5 = 2

$$\text{Co-factor of Matrix } \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}$$

Adj A = Transpose of co-factor matrix

$$\text{Adj } A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Calculation:

$$|A| = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix} = 10 - 12 = -2$$

$$\text{Inverse } (A^{-1}) = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}}{-2} = -\begin{bmatrix} \frac{5}{2} & 2 \\ \frac{3}{2} & 1 \end{bmatrix}$$

NOTES

Example 41:

Find inverse of $\begin{bmatrix} 12 & 15 \\ 2 & 3 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 12 & 15 \\ 2 & 3 \end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|}$$

Now, calculate $|A| = (3) \times 12 - (2) \times 15$

$= 36 - 30 = 6$

Again, calculate Adj A:

Co-factor of 12 = 3

Co-factor of 15 = –(–2) = 2

Co-factor of 2 = –15

Co-factor of 3 = 12

Co-factor Matrix $= \begin{bmatrix} 3 & 2 \\ 15 & 12 \end{bmatrix}$

Adj A = Transpose of co-factor matrix

Adj $A = \begin{bmatrix} 3 & 15 \\ 2 & 12 \end{bmatrix}$

Now, $A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} 3 & 15 \\ 2 & 12 \end{bmatrix}}{6} = \begin{bmatrix} \frac{3}{6} & \frac{15}{6} \\ \frac{2}{6} & \frac{12}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{3} & 2 \end{bmatrix}$

Example 42:

Find inverse of $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

NOTES

$$\therefore\ |A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 1(1 + 2) - 2(-1 - 4) - 1(1 - 2)$$

$$= 3 + 10 + 1$$

$$= 14$$

Calculation of Co-factor Matrix:

$$A_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3, \qquad A_{12} = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -(-1 - 4) = 5$$

$$A_{13} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 1 - 2 = -1, \quad A_{21} = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -(2 - 1) = -1$$

$$A_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3, \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -(-1 - 4) = 5$$

$$A_{31} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5, \qquad A_{32} = -\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = -(2 - 1) = -1$$

$$\text{Co-factor of Matrix (A)} = \begin{bmatrix} 3 & 5 & -1 \\ 1 & 3 & 5 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\therefore\ \text{Adj } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{vmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} \cdot \text{Adj } A \qquad \therefore\ A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$$

Example 43:

Find the inverse of $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 8 \\ -1 & 5 & 7 \end{bmatrix}$.

NOTES

Solution:

Given $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 8 \\ -1 & 5 & 7 \end{bmatrix}$

We know, $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

Calculate $|A| = 1\,(14 - 40) - (-3)\,(0 + 8) + 2(0 + 2)$

$= -26 + 24 + 4$

$= -26 + 28 = 2$

Again, calculate

$$\text{Co-factor of } A = \begin{vmatrix} -\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 8 \\ -1 & 7 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 5 \end{vmatrix} \\ -\begin{vmatrix} -3 & 2 \\ 5 & 7 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -1 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ -1 & 5 \end{vmatrix} \\ +\begin{vmatrix} -3 & 2 \\ 2 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} 14 - 40 & -(0 + 8) & 0 + 2 \\ -(-21 - 10) & 7 + 2 & -(5 - 3) \\ -24 - 4 & -8(8 - 0) & (2 - 0) \end{vmatrix}$$

$$= \begin{vmatrix} -26 & -8 & 2 \\ 31 & 9 & -2 \\ -28 & -8 & 2 \end{vmatrix}$$

Adj A = Transpose of Co-factor Matrix

$$\therefore \quad \text{Adj } A = \begin{vmatrix} -26 & 31 & -28 \\ -8 & 9 & -8 \\ 2 & -2 & 2 \end{vmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{= \begin{vmatrix} -26 & 31 & -28 \\ -8 & 9 & -8 \\ 2 & -2 & 2 \end{vmatrix}}{2} = \begin{bmatrix} -13 & \dfrac{31}{2} & -14 \\ -4 & \dfrac{9}{2} & -4 \\ 1 & -1 & 1 \end{bmatrix}$$

3.11 SOLUTION OF SIMULTANEOUS EQUATION USING MATRIX METHOD

NOTES

Let $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$ be a set of simultaneous equations in x, y and z.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow AX = D \Rightarrow X = A^{-1}.D, \quad \text{if } |A| \neq 0$$

Using these matrices above, the equation can be written as:

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \text{Adj } A \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

These equations can be expressed in the matrix from as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now, the given set of equations can be put in the form $AX = D$.

Example 44:

Solve by matrix method $2x + 3y = 8$ and $3x - y = 1$.

Solution:

Let the given equations in the matrix notation be $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

$$\Rightarrow AX = B \Rightarrow X = A^{-1}.B$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Now, $|A| = (-2 - 9) = -11 \neq 0$

So, A^{-1} exists and $A^{-1} = \frac{1}{|A|} \text{Adj } A$

The co-factors of the elements of A are:

Co-factor of 2 = –1

Co-factor of 3 = –3

Co-factor of 3 = –3

Co-factor of 1 = 2

NOTES

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}^1 = \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{-1}{11}\begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \quad X = \frac{-1}{11}\begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -11 \\ -22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence, $x = 1$ and $y = 2$

Example 45:

Solve by matrix method $x + y + z = 6$, $x + 2y + 3z = 14$ and $-x + y - z = -2$.

Solution:

Let the given equations in the matrix notations be

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

$$\Rightarrow AX = B \Rightarrow X = A^{-1}.B$$

Now, $|A| = 1(-2-3) - 1(-1+3) + 1(1+2)$

$= -5 - 2 + 3$

$= -4$

So, A^{-1} exists and $A^{-1} = \dfrac{1}{|A|}\text{Adj } A$

NOTES

The co-factors of the elements of A are:

Co-factor of 1 = (–2 – 3) = –5

Co-factor of 1 = (–1 + 3) = –2

Co-factor of 1 = (1 + 2) = 3

Co-factor of 1 = (–1 – 1) = 2

Co-factor of 2 = (–1 + 1) = 0

Co-factor of 3 = (1 + 1) = 2

Co-factor of –1 = (3 – 2) = 1

Co-factor of 1 = (3 – 1) = 2

Co-factor of –1 = (2 – 1) = 1

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -5 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{-1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{-1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1$, $y = 2$ and $x = 3$.

Example 46:

Solve by matrix method $x + y + z = 6$, $x + 2y + 3z = 15$ and $-x + y - z = -2$.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 15 \\ -2 \end{bmatrix}$$

$$\Rightarrow AX = B \Rightarrow X = A^{-1}.B$$

NOTES

Now, $|A| = 1(-2-3) - 1(-1+3) + 1(1+2)$

$= 1(-5) - 1(2) + 1(3)$

$= -5 - 2 + 3$

$= -4$

The co-factors of the elements of A are:

$$\text{Co-factor of } 1 = +\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (-2-3) = -5$$

$$\text{Co-factor of } 1 = -\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} - (-1+3) = -2$$

$$\text{Co-factor of } 1 = +\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = +(1+2) = 3$$

$$\text{Co-factor of } 1 = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = +(-1-1) = 2$$

$$\text{Co-factor of } 2 = +\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = +(-1+1) = 0$$

$$\text{Co-factor of } 3 = -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -(1+1) = -2$$

$$\text{Co-factor of } -1 = +\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = +(3-2) = 1$$

$$\text{Co-factor of } 1 = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$\text{Co-factor of } -1 = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = +(2-1) = 1$$

$$\text{Matrix formed by co-factor} = \begin{bmatrix} -5 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{Adj } A = \begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adjA}{|A|} = \frac{\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}}{-4} = \frac{1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$= -\frac{1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 15 \\ -2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -30+30-2 \\ -12+\ \ 0+4 \\ 18-\ 30-2 \end{bmatrix}$$

NOTES

$$= \frac{1}{4}\begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{2}{4} \\ \frac{8}{4} \\ \frac{14}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{7}{2} \end{bmatrix}$$

$\therefore x = \frac{1}{2}, y = 2$ and $z = \frac{7}{2}$

Example 47:

A real estate business company has the following record of sales of sites during three months for three dimensions of sites $30' \times 40'$, $30' \times 50'$ and $40' \times 60'$ Which have different rates of profits?

Months	**Sales of Sites**			**Total Profit**
	30' × 40'	**30' × 50'**	**40' × 60'**	**in ₹**
April	5	3	7	509
May	4	26	2	964
June	7	2	10	655

Find out the rate of profit on each of the items.

Solution:

Let x, y and z denote the rates profits in ₹ for each of the items $30' \times 40'$, $30' \times 50'$ and $40' \times 60'$.

The given data can be expressed as a system of linear equations:

$$5x + 3y + 7z = 509$$

$$4x + 26y + 2z = 964$$

$$7x + 2y + 10z = 655$$

In matrix notation, we have

$$\begin{bmatrix} 5 & 3 & 7 \\ 4 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 509 \\ 964 \\ 655 \end{bmatrix} \quad \text{or} \quad AX = B$$

Now, $$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 4 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = -16$$

Since $|A| \neq O A^{-1}$ exists and

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 509 \\ 964 \\ 655 \end{bmatrix}$$

NOTES

Now, $$\text{Adj } A = \begin{bmatrix} 256 & -26 & -174 \\ -16 & 1 & 11 \\ -176 & 18 & 118 \end{bmatrix}$$

But $$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{-1}{16} \begin{bmatrix} 256 & -26 & -174 \\ -16 & 1 & 11 \\ -176 & 18 & 118 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16} \begin{bmatrix} 256 & -26 & -174 \\ -16 & 1 & 11 \\ -176 & 18 & 118 \end{bmatrix} \begin{bmatrix} 509 \\ 964 \\ 655 \end{bmatrix}$$

$$= \frac{-1}{16} \begin{bmatrix} 130304 & -154246 & -115280 \\ -13234 & 964 & 11790 \\ -88566 & 10604 & 77290 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16} \begin{bmatrix} -400 \\ -480 \\ 672 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 42 \end{bmatrix}$$

$\therefore x = 25.\ y = 30$ and $z = 42$

Application Problems

Example 48:

A trust fund has ₹ 50,000 that must be invested in two different types of bonds. The first band pays 6% interest per year and the second bond pays 9% interest per year. Using matrix multiplication, determine how to divide ₹ 50,000 amongst the two types of bonds if the trust fund obtained an annual total interest of ₹ 4,050.

Solution:

Let the investment in the first bond be ₹ X. Then the investment in the second band is ₹ 50,000 – X.

Now, express the data in the matrix form as

Therefore, investment in first band = ₹ 15,000 and investment in second band

NOTES

₹ 50,000 – 15,000 = ₹ 35,000

$$\begin{bmatrix} X & 50000 - X \end{bmatrix}\begin{bmatrix} 6\% \\ 9\% \end{bmatrix} = \begin{bmatrix} 4050 \end{bmatrix}$$

$$\begin{bmatrix} X & 50000 - X \end{bmatrix}\begin{bmatrix} .06 \\ .09 \end{bmatrix} = \begin{bmatrix} 4050 \end{bmatrix}$$

$$\begin{bmatrix} .06X + 4{,}500 - .09X \end{bmatrix} = \begin{bmatrix} 4050 \end{bmatrix}$$

$$\begin{bmatrix} -.03X + 4500 - .09X \end{bmatrix} = \begin{bmatrix} 4050 \end{bmatrix}$$

$$= 4500 - 4050 = .03X$$

$$= 450 = .03X$$

Therefore, investment in first band = ₹ 15,000 and investment in second bond = ₹ 50,000 – 15,000 = ₹ 35,000.

Example 49:

A company is to employ 60 labourers from either of the party *A* or *B*, comprising persons in different age groups as under:

	Category I (20 – 25 years)	Category II (26 – 30 years)	Category III (31 – 40 years)
Party A	25	20	15
Party B	20	30	10

Rate of Labour applicable to categories I, II and III are ₹ 1,200, ₹ 1,000 and ₹ 600 respectively. Using matrices, find which party is economically preferable over the others.

Solution:

Here, matrix of various categories of labourers of the two parties is:

$$L = \begin{matrix} A \\ B \end{matrix}\begin{bmatrix} 25 & 20 & 15 \\ 20 & 30 & 10 \end{bmatrix}_{2 \times 3}$$

Also, matrix of salaries to be paid to these categories is:

$$S = \begin{bmatrix} 1200 \\ 1000 \\ 600 \end{bmatrix}\begin{matrix} I \\ II \\ III_{3 \times 1} \end{matrix}$$

Therefore, labour charges to each party are given by the matrix:

$$L \times S = \begin{matrix} A \\ B \end{matrix}\begin{bmatrix} 25 & 30 & 15 \\ 20 & 30 & 10 \end{bmatrix}\begin{bmatrix} 1200 \\ 1000 \\ 600 \end{bmatrix}$$

$$= 600\begin{bmatrix} 30000 + 20000 + 9000 \\ 24000 + 30000 + 6000 \end{bmatrix} = \begin{bmatrix} 59000 \\ 60000 \end{bmatrix}$$

Therefore, labour charges of Party A = ₹ 59,000

labour charges of Party B = ₹ 60,000

Party A is more economical than Party B.

NOTES

Example 50:

Three firms A, B and C supplied 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively to a contractor. If the cost of stone and sand are ₹ 12,00 and ₹ 500 per truck load respectively, find the total amount paid by the contractor to each of these firms by using matrix method.

Solution:

The matrix Q for the material supplied firms is given by:

$$\begin{array}{c} \\ Q = \begin{array}{c} A \\ B \\ B \end{array} \end{array} \begin{array}{c} \text{Stone} \quad \text{Sand} \\ \begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix} \end{array}$$

Also, the matrix P for the cost of Stone and Sand is given by:

$$P = \begin{bmatrix} 1200 \\ 500 \end{bmatrix} \begin{array}{l} \text{Stone} \\ \text{Sand} \end{array}$$

Now, the amount payable by contractor to each of these firms is given by the matrix

$$Q \times P = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix} \begin{bmatrix} 1200 \\ 500 \end{bmatrix} = \begin{bmatrix} 53000 \\ 44500 \\ 34000 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array}$$

Therefore, payment made by the contractor is:

Firm A = 53,000, Firm B = 44,500 and Firm C = 34,000

Example 51:

At Bangalore, merchant 'A' has 300 bags of Rice, 600 bags of Wheat and 800 bags of Ragi and another merchant 'B' has 250 bags ,700 bags and 1000 bags of same foodgrains. The price (in ₹) at three cities are:

$$\begin{array}{r} \\ \text{Mysore} \\ \text{Mangalore} \\ \text{Kolar} \end{array} \begin{array}{c} \text{Rice} \quad \text{Wheat} \quad \text{Ragi} \\ \begin{bmatrix} 100 & 90 & 80 \\ 110 & 80 & 70 \\ 120 & 70 & 80 \end{bmatrix} \end{array}$$

To which city, each merchant will send his supply in order to get maximum gross receipts? Solve by matrix multiplication method.

NOTES

Solution:

$$\begin{array}{l} \\ \text{Merchant } A \\ \text{Merchant } B \end{array}\begin{array}{c} \begin{array}{ccc}\text{Rice} & \text{Wheat} & \text{Ragi}\end{array} \\ \begin{bmatrix} 300 & 600 & 800 \\ 250 & 700 & 1000 \end{bmatrix}\end{array}$$

Prices are:

$$\begin{array}{l} \\ \text{Mysore} \\ \text{Mangalore} \\ \text{Kolar} \end{array}\begin{array}{c} \begin{array}{ccc}\text{Rice} & \text{Wheat} & \text{Ragi}\end{array} \\ \begin{bmatrix} 100 & 90 & 80 \\ 110 & 80 & 70 \\ 120 & 70 & 80 \end{bmatrix}\end{array}$$

Gross Receipts = Quantity of goods sold × Price

$$= \begin{bmatrix} 300 & 600 & 800 \\ 250 & 700 & 1000 \end{bmatrix} \times \begin{bmatrix} 100 & 90 & 80 \\ 110 & 80 & 70 \\ 120 & 70 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 30000 + 54000 + 64000 & 33000 + 48000 + 56000 & 36000 + 42000 + 64000 \\ 25000 + 63000 + 80000 & 27500 + 56000 + 70000 & 30000 + 49000 + 80000 \end{bmatrix}$$

$$= \begin{array}{c} A \\ B \end{array}\begin{bmatrix} 148000 & 137000 & 142000 \\ 168000 & 153500 & 159000 \end{bmatrix}$$

$\therefore$ Both A and B gets receipts from Mysore.

Example 52:

A company is considering which of three methods of production it should use in producing three products A, B and C. The amount of each product produced by each method is as shown below.

$$\begin{array}{l} \\ \text{Method I} \\ \text{Method II} \\ \text{Method III} \end{array}\begin{array}{c} \begin{array}{ccc}\text{Product A} & \text{Product B} & \text{Product C}\end{array} \\ \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 3 & 3 & 9 \end{bmatrix}\end{array}$$

Further information relating to profit per unit is as under:

Product	**Profit/ Unit**
A	10
B	4
C	6

Using matrix multiplication, find which method maximizes total profit?

NOTES

Solution:

The given problem in matrix notation

$$\begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 3 & 3 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 40 + 32 + 12 \\ 50 + 28 + \ \ 6 \\ 31 + 12 + 54 \end{bmatrix} = \begin{bmatrix} 84 \\ 84 \\ 96 \end{bmatrix}$$

From this, we have

Method I yields the total profit = ₹ 84

Method II yields the total profit = ₹ 84

Method III yields the total profit = ₹ 96

Hence, **Method III** maximizes total profit.

Example 53:

Two oil merchants have the following stock of oil (in kgs):

	Groundnut oil	Sunflower oil	Coconut oil
Merchant A	250	300	150
Merchant B	400	350	100

The approximate prices (in ₹ per kg) of three types of oil in three markets:

	Groundnut oil	Sunflower oil	Coconut oil
Market X	70	50	150
Market Y	60	55	110
Market Z	55	60	132

In which market each of the above, businessman has to sell their stocks to get maximum receipt? Solve by matrix multiplication method.

Solution:

Let P =

	Groundnut oil	Sunflower oil	Coconut oil
Merchant A	250	300	150
Merchant B	400	350	100

Let Q =

	Market X	Market Y	Market Z
Groundnut oil	70	60	55
Sunflower oil	50	55	60
Coconut oil	150	140	132

$$PQ = \begin{bmatrix} 250 & 300 & 150 \\ 400 & 350 & 100 \end{bmatrix} \begin{bmatrix} 70 & 60 & 55 \\ 50 & 55 & 60 \\ 150 & 140 & 132 \end{bmatrix}$$

$$= \begin{bmatrix} 17500+15000+22500 & 15000+16500+21000 & 13750+18000+19800 \\ 28000+17500+15000 & 24000+19250+14000 & 22000+21000+13200 \end{bmatrix}$$

NOTES

$$PQ = \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 55000 & 52500 & 51550 \\ 60500 & 57250 & 56200 \end{bmatrix} \end{array}$$

Both merchants A and B should their stock in market X in order to get maximum receipt.

Example 54:

Two businessman are trading in shares have three banking company shares as shown in the following table:

Merchant	Vijaya Bank	Canara Bank	Corporation Bank
Mr. Jain	200	100	300
Mr. Gupta	250	150	100

The approximate prices (in ₹) three banking company shares in three stock exchange market are given below:

	Vijaya Bank	Canara Bank	Corporation Bank
Bangalore	39	40	38
Bombay	40	50	45
New Delhi	35	45	42

In which market, each of the above businessmen has to sell their stocks to get maximum receipt? Solve by matrix multiplication method.

Solution:

Let $A =$

	Vijaya Bank	Canara Bank	Corporation Bank
Mr. Jain	200	100	300
Mr. Gupta	250	150	100

Let $B =$

	Bangalore	Bombay	New Delhi
Vijaya Bank	39	40	35
Canara Bank	40	50	45
Corporation Bank	38	35	42

Multiplying the two matrices, we get

$$AB = \begin{bmatrix} 200 & 100 & 300 \\ 250 & 150 & 100 \end{bmatrix} \begin{bmatrix} 39 & 40 & 35 \\ 40 & 50 & 45 \\ 38 & 35 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 7800+4000+11400 & 8000+5000+10500 & 7000+4500+12600 \\ 9750+6000+\ 3800 & 10000+7500+\ 3500 & 8750+8750+\ 4200 \end{bmatrix}$$

$$PQ = \begin{array}{l} \\ \text{Mr. Jain} \\ \text{Mr. Gupta} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Bangalore} & \text{Bombay} & \text{New Delhi} \end{array} \\ \begin{bmatrix} 23200 & 23500 & 24100 \\ 19550 & 21000 & 19700 \end{bmatrix} \end{array}$$

Mr. Jain will receive maximum receipt from New Delhi market. Hence, he should sell his shares in the New Delhi market.

Mr. Gupta will receive maximum receipt from Bombay market. Hence, he should sell his shares in the Bombay market.

NOTES

Check Your Progress

6. Find $'y'$ if $\begin{vmatrix} -3 & -6 & 1 \\ 5 & y & -2 \\ 2 & -3 & 5 \end{vmatrix} = 7.$

7. Solve by Cramer's rule:

   $x + y + z = 11$

   $2x - 6y - z = 0$

   $3x + 4y + 2z = 0$

8. A company is to employ 60 labourers from either of the party A or B, comprising persons in different age groups as under:

	Category I (20 – 25 years)	Category II (26 – 30 years)	Category III (31 – 40 years)
Party A	25	20	15
Party B	20	30	10

   Rate of Labour applicable to categories I, II and III are ₹ 1,200, ₹ 1,000 and ₹ 600 respectively. Using matrices, find which party is economically preferable over the others.

3.12 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Solve for A and B if $A - 2B = \begin{bmatrix} 4 & 6 & -10 \\ 6 & -4 & 2 \end{bmatrix}$ and

$$2A - B = \begin{bmatrix} 4 & -4 & 2 \\ 4 & 0 & 2 \end{bmatrix}.$$

NOTES

Solution:

$$A - 2B = \begin{bmatrix} 4 & 6 & -10 \\ 6 & -4 & 2 \end{bmatrix} \qquad \text{....................(i)}$$

$$2A - B = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix} \qquad \text{.....................(ii)}$$

$$\text{No. (i)} \times 2 \Rightarrow 2A - 4B = \begin{bmatrix} 8 & 12 & -20 \\ 12 & -8 & 4 \end{bmatrix} \qquad \text{.....................(iii)}$$

$$\text{No. (ii)} - \text{No. (iii)} \Rightarrow -B + 4B = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 12 & -20 \\ 12 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -20 & 22 \\ -8 & 8 & -2 \end{bmatrix}$$

$$\therefore \quad 3B = \begin{bmatrix} -4 & -20 & 22 \\ -8 & 8 & -2 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} -4/3 & -20/3 & 22/3 \\ -8/3 & 8/3 & -2/3 \end{bmatrix}$$

Substituting the value of the matrix B in (ii), we get

$$2A - \begin{bmatrix} -4/3 & -20/3 & 22/3 \\ -8/3 & 8/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 2 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} 4 - 4/3 & -8 - 20/3 & 2 + 22/3 \\ 4 - 8/3 & 0 + 8/3 & 2 - 2/3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 8/3 & -44/3 & 28/3 \\ 4/3 & 8/3 & 4/3 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 4/3 & -22/3 & 14/3 \\ 2/3 & 4/3 & 2/3 \end{bmatrix}$$

2. Matrix Addition is Associative: Let A, B and C be three matrices in the same order, then $(A + B) + C = A + (B + C)$.

Solution:

If $A = \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix}$, then $A + B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$.

$$(A + B) + C = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 6 & 3 \end{bmatrix} \qquad \text{..............(i)}$$

Consider if $B + C = \begin{bmatrix} 5 & 8 \\ 5 & 3 \end{bmatrix}$, then

$$A + (B + C) = \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ 6 & 3 \end{bmatrix} \quad \text{...............(ii)}$$

$\therefore$ From (i) and (ii), $(A + B) + C = A + (B + C)$.

NOTES

3. If $A = [1 \quad 2]_{1 \times 2}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2 \times 2}$ and $C = \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2 \times 1}$, then form as many product as you can AB, BC, CA and AC.

Solution:

$$AB = [1 \quad 2] \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = [\, 1 \times 2 + 2 \times 4 \quad 1 \times 3 + 2 \times 5] = 0$$

$$= [2 + 8 \quad 3 + 10] = [10 \quad 13]$$

$$BC = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 3 & + & 3 \times 5 \\ 4 \times 3 & + & 5 \times 5 \end{bmatrix} = \begin{bmatrix} 6 + 15 \\ 12 + 25 \end{bmatrix} = \begin{bmatrix} 21 \\ 37 \end{bmatrix}$$

$$CA = \begin{bmatrix} 3 \\ 5 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} 3 \times 1 & + & 3 \times 2 \\ 5 \times 1 & + & 5 \times 2 \end{bmatrix} = \begin{bmatrix} 3 + 6 \\ 5 + 10 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$AC = [1 \quad 2] \begin{bmatrix} 3 \\ 5 \end{bmatrix} = [1 \times 3 + 2 \times 5] = [3 + 10] = [13]$$

4. If $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$, find $(AB)^1 = B^1 A^1$.

Solution:

Given, $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$

$$A^1 = \begin{bmatrix} 4 & 3 \\ 2 & -7 \\ -1 & 1 \end{bmatrix} \qquad B^1 = \begin{bmatrix} 2 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 6 + 1 & 12 + 0 - 5 \\ 6 + 21 - 1 & 9 + 0 + 5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 26 & 14 \end{bmatrix}$$

$$(AB)^1 = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

NOTES

$$B^1A^1 = \begin{bmatrix} 2 & -3 & -1 \\ 3 & 0 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & -7 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6+1 & 6+21-1 \\ 12+0-5 & 9+\ 0+5 \end{bmatrix} = \begin{bmatrix} 3 & 26 \\ 7 & 14 \end{bmatrix}$$

$$\therefore (AB)^1 = B^1A^1$$

5. If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, calculate $A^2 - 5A + 9I$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2+1(-1)+\ 1.1 & 2.1+1.0+\ 1.3 & 2.1+1.1+\ 1\ (-1) \\ (-1)2+0(-1)+\ 1.1 & (-1)1+0.0+\ 1.3 & (-1)1+0.1+\ 1\ (-1) \\ 1.2+3(-1)+(-1)1 & 1.1+3.0+(-1)3 & 1.1+3.1+(-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4-1+1 & 2+3 & 2+1-1 \\ -2+1 & -1+3 & -1-1 \\ 2-3-1 & 1-3 & 1+3+1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 5 \\ -5 & 0 & 5 \\ 5 & 15 & -5 \end{bmatrix}$$

$$9I = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Consider $A^2 - 5A + 9I$

$$= \begin{bmatrix} 4 & 5 & 2 \\ -1 & 2 & -2 \\ -2 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ -5 & 0 & 5 \\ 5 & 15 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 & -3 \\ 4 & 2 & -7 \\ -7 & -17 & 10 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 4 & 11 & -7 \\ -7 & -17 & 19 \end{bmatrix}$$

6. Find $'y'$ if $\begin{bmatrix} -3 & -6 & 1 \\ 5 & y & -2 \\ 2 & -3 & 5 \end{bmatrix} = 7.$

Solution:

Let $A = \begin{bmatrix} -3 & -6 & 1 \\ 5 & y & -2 \\ 2 & -3 & 5 \end{bmatrix}$

$$\Rightarrow \quad -3\begin{vmatrix} y & -2 \\ -3 & 5 \end{vmatrix} + 6\begin{vmatrix} 5 & -2 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 5 & y \\ 2 & -3 \end{vmatrix} = 7$$

$$\Rightarrow \quad -3(5y+6) + 6(25+4) + 1(-15+2y) = 7$$

$$\Rightarrow \quad -15y + 18 + 150 + 24 - 15 + 2y = 7$$

$$\Rightarrow \quad -17y + 177 = 7$$

$$\Rightarrow \quad -17y = 7 - 177$$

$$\Rightarrow \quad -17y = -170$$

$$\Rightarrow \quad y = \frac{170}{17}$$

$$\therefore \quad y = 10$$

7. Solve by Cramer's rule:

$$x + y + z = 11$$

$$2x - 6y - z = 0$$

$$3x + 4y + 2z = 0$$

Solution:

By Cramer's Rule,

Let $x = \frac{\Delta x}{\Delta}, y = \frac{\Delta x}{\Delta}, z = \frac{\Delta z}{\Delta}.$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -6 \\ 3 & 4 \end{vmatrix}$$

$$= 1(-12+4) - 1(4+3) + 1(8+18)$$

$$= -8 - 7 + 26$$

NOTES

$$\Delta = 11$$

$$\Delta x = \begin{vmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -6 & -1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & -6 \\ 0 & 4 \end{vmatrix}$$

$$= 11(-12+4) - 1(0+0) + 1(0+0)$$

$$= 11(-8) - 0 + 0$$

$$\Delta x = -88$$

$$\Delta y = \begin{vmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}$$

$$= 1(0+0) - 11(4+3) + 1(0+0)$$

$$= 0 - 11(-7) + 0$$

$$\Delta y = -77$$

$$\Delta z = \begin{vmatrix} 1 & 1 & 11 \\ 2 & -6 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & 0 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} + 11\begin{vmatrix} 2 & -6 \\ 3 & 4 \end{vmatrix}$$

$$= 1(0+0) - 1(0-0) + 11(8+18)$$

$$= 0 - 0 + 11(26)$$

$$\Delta z = 286$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{-88}{11} = -8,\ y = \frac{\Delta y}{\Delta} = \frac{-77}{11} = -7 \text{ and } z = \frac{286}{11} = 26$$

8. A company is to employ 60 labourers from either of the party A or B, comprising persons in different age groups as under.

Category	*I* (20 – 25 *years*)	*II* (26 – 30 *years*)	*III* (31 – 40 *years*)
Party A	25	20	15
Party B	20	30	10

Rate of Labour applicable to categories I, II and III are ₹ 1,200, ₹ 1000 and ₹ 600 respectively. Using matrices, find which party is economically preferable over the others.

Solution:

Here, matrix of various categories of labourers of the two parties is;

$$L = \begin{matrix} A \\ B \end{matrix} \begin{bmatrix} 25 & 20 & 15 \\ 20 & 30 & 10 \end{bmatrix}_{2 \times 3}$$

Also matrix of salaries to be paid to these categories is

$$S = \begin{bmatrix} 1200 \\ 1000 \\ 600 \end{bmatrix} \begin{matrix} I \\ II \\ III \end{matrix}_{3 \times 1}$$

Therefore, labour charges to each party are given by the matrix.

$$L \times S = \begin{matrix} A \\ B \end{matrix} \begin{bmatrix} 25 & 30 & 15 \\ 20 & 30 & 10 \end{bmatrix} \begin{bmatrix} 1200 \\ 1000 \\ 600 \end{bmatrix}$$

$$600 \begin{bmatrix} 30000 + 20000 + 9000 \\ 24000 + 30000 + 6000 \end{bmatrix} = \begin{bmatrix} 59000 \\ 60000 \end{bmatrix}$$

Therefore, labour charges of party A = ₹ 59000

labour charges of party B = ₹ 60000

Party A is more economical than party B.

3.13 SUMMARY

- Arrangement of elements in horizontal and vertical lines is called a Matrix. The number of rows and columns of a matrix is called Order of Matrix. A matrix in which the number of rows is not equal to the number of columns is called rectangle matrix. In a square matrix, the diagonal which starts from left-hand top corner and ends at to right-hand bottom corner is called the principal diagonal or leading diagonal are called principal diagonal elements.
- A diagonal matrix is a matrix in which all the elements except the elements in the principal diagonal are zeroes. A scalar matrix is a diagonal matrix in which all the elements in the principal diagonal elements are equal.
- Transpose matrix obtained from a given matrix A by interchanging its row and columns is called transpose of a matrix and is denoted by A^I or A^T. A matrix of any order, in which all the elements are zeroes, is called a Zero Matrix or Null Matrix (Void Matrix). It is always denoted by '0'. Two matrices of the same order are said to be equal only when the corresponding elements (the elements in the respective of the two matrices) are equal.

NOTES

3.14 KEY TERMS

- **Matrix:** Arrangement of elements in horizontal and vertical lines is called a Matrix.
- **Order of Matrix:** The number of rows and columns of a matrix is called Order of Matrix. Two matrices can be multiplied only when the number of columns of the first matrix is equal to the number of rows of second matrix. Hence, product is defined.
- **Diagonal Matrix:** A diagonal matrix is a matrix in which all the elements except the elements in the principal diagonal are zeroes.
- **Scalar Matrix:** It is a diagonal matrix in which all the elements in the principal diagonal elements are equal.
- **Transpose of a Matrix:** The matrix obtained from a given matrix A by interchanging its row and columns is called transpose of a matrix and is denoted by A^I or A^T .
- **Null Matrix:** A matrix of any order, in which all the elements are zeroes, is called a Zero Matrix or Null Matrix (Void Matrix). It is always denoted by '0'.
- **Equality of Matrices:** Two matrices of the same order are said to be equal only when the corresponding elements (the elements in the respective of the two matrices) are equal.

3.15 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Define matrix.
2. What do you mean by row matrix?
3. What is a scalar matrix? Give example.
4. What is an square matrix?
5. What do you mean by zero matrix or null matrix?
6. Give the meaning of diagonal matrix.
7. Give the meaning of equal matrices.
8. What do you mean by unit matrix?
9. What is sub-matrix?
10. What do you mean by transpose of matrix?
11. Give the meaning of column matrix.
12. Give the rules of inverse of matrix.
13. State the name of different types of matrices.

14. What do you mean by equality of matrix?
15. Give the meaning of triangular matrix.
16. Define determinants.
17. What is minor of determinants?
18. Write properties of determinants.
19. Define inverse of matrix.
20. Define adjoint of matrix.
21. Write the properties of Cramer's rule.
22. Discuss adjoint of matrix.

NOTES

Long Answer Questions

1. If $A = \begin{bmatrix} 3 & 6 & 0 & 9 \\ 4 & 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 3 & 0 & 9 \\ 3 & -3 & 6 & 9 \end{bmatrix}$, find $B + A$.

2. If $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, find B^2.

3. If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}$, find AB.

4. If $A = \begin{bmatrix} 3 & 6 & 1 \\ 9 & -9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, find $A + B$.

5. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$, find $A^1 + B^1$.

6. If $B = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$, find $\frac{1}{2} B^2$.

7. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -1 & 3 \end{bmatrix}$, find $A + B$.

8. If $\begin{bmatrix} a & 5 \\ -8 & 4 \end{bmatrix} = 0$, find a.

9. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$, find $A + B$.

NOTES

3.16 FURTHER READING

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UNIT 4 LOGARITHMS AND ANTILOGARITHMS

NOTES

Structure

4.0 INTRODUCTION

Logarithms were quickly adopted by scientists because of various useful properties that simplified long, tedious calculations. In particular, scientists could find the product of two numbers m and n by looking up each number's logarithm in a special table, adding the logarithms together, and then consulting the table again to find the number with that calculated logarithm (known as its antilogarithm). Expressed in terms of common logarithms, this relationship is given by $\log mn = \log m + \log n$.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the concept of Logarithms
- Discuss the Antilogarithms
- Describe the Principles and Calculations
- Explain the concept of Simple Interest andCompound Interest

4.2 LOGARITHM

The logarithm of a number to the given base is the index or power to which base must be raised to produce that number.

Logarithmic Form

(a) $2^2 = 4$ $\quad\therefore\quad \log_2^4 = 2$

(b) $3^2 = 27$ $\quad\therefore\quad \log_3^{27} = 3$

(c) $a^x = n$ $\quad\therefore\quad \log_a^n = x$

4.3 INDEX FORM

(a) $\log_3^{81} = 4$ $\quad\therefore\quad 3^4 = 81$

(b) $\log_2^{32} = 5$ $\quad\therefore\quad 2^5 = 32$

(c) $\log_a^n = m$ $\quad\therefore\quad a^m = n$

Note:

(a) The logarithm of a number to the base "e" is called natural logarithm.
$e = 2.71828$

(b) The logarithm of a number to the base "10" is called common logarithm.

(c) The logarithm of number 1 to any base is zero.

Examples:

$\log_2^1 = 0$ $\qquad \log_{10}^1 = 0$ $\qquad \log_{100}^1 = 0$

(d) The logarithm of a number to the same base is 1.

$\log_2^2 = 1$ $\qquad \log_4^4 = 1$ $\qquad \log_a^a = 1$

4.4 LAWS OF LOGARITHM

I. Law: Logarithm of product is equal to sum of logarithms for the same base.

$$\log_a (mn) = \log_a m + \log_a n$$

II. Law: Logarithm of a division is equal to difference of logarithms to the same base.

$$log_a \left(\frac{m}{n}\right) = log_a m - log_a n$$

III. Law: $\log_a (m^n) = n\,(\log_a m)$

IV. Law: Change of base

$$\log_n^m = \frac{\log_a^m}{\log_a^m} \quad \text{where } a \text{ is the new base}$$

Example 1:

Find the value of $\log_3 243$.

Solution:

Let $\log_3 243 = x$

$\therefore$ $243 = 3^x$ (Write in index form)

$3^5 = 3^x$ ($3^5 = 243$)

$\therefore$ $x = 5$ (When the base are equal, the powers are also equal)

Example 2:

Find m if $\log_3 m = 3$.

Solution:

Consider $\log_3 m = 3$

$\therefore$ $3^3 = m$ (Write in Index form)

$27 = m$

$\therefore$ $m = 27$

Example 3:

Find the value of $\log_b a$, $\log_c b$, $\log_a c$, $\log_e d$ and $\log_a e$ if change of the base is 10.

Solution:

$$\frac{\log_{10} a}{\log_{10} b} \times \frac{\log_{10} b}{\log_{10} c} \times \frac{\log_{10} c}{\log_{10} d} \times \frac{\log_{10} d}{\log_{10} e} \times \frac{\log_{10} e}{\log_{10} a} = 1$$

Example 4:

Prove that $\log \frac{26}{33} - \log \frac{65}{69} + \log \frac{55}{46} = 0.$

Solution:

$$\text{L.H.S.} = \log\left(\frac{26}{33}\right) - \log\left(\frac{65}{69}\right) + \log\left(\frac{55}{46}\right)$$

(By II Law) $\left(\left(\log \frac{m}{n}\right) = \log_a m - \log_a n\right)$

$= (\log 26 - \log 33) - (\log 65 - \log 69) + (\log 55 - \log 46)$

$= \log 26 - \log 33 - \log 65 + \log 69 + \log 11 \times \log 55 - \log 46$

$= \log 13 \times 2 - \log 11 \times 3 - \log 13 \times 5 + \log 23 \times 3 + \log 11 \times 5 - \log 23 \times 2$

$= (\log 13 + \log 2) - (\log 11 + \log 3) - \log(13 + 5) + \log (23 + 3) + \log (11 + 5) - \log (23 + 2)$

$= \log 13 + \log 2 - \log 11 - \log 3 - \log 13 - \log 5 + \log 23 + \log 3 + \log 11 + \log 5 - \log 23 - \log 2$

$= 0 = \text{R.H.S.}$ (All values are cancelled)

NOTES

Example 5:

Show that $\log_2 [\log_2 (\log_2 16)] = 1$.

Solution:

Let $\log_2 16 = x$ . . . . (1)

$\therefore \log_2 (\log_2 x) = 1$

Let $\log_2 x = y$ . . .. .(2)

$\therefore \log_2 y = 1$

$\therefore y = 2^1$ (In index form)

$\log_2 x = 2$ $\rightarrow$ by (2)

$\therefore x = 2^2$ (In index form)

$x = 4$ . . . . (3)

By (1),

$\log_2 16 = x$

$\log_2 16 = 4$ $\rightarrow$ by (3)

$\therefore 2^4 = 16$

$\therefore 2^4 = 16 \rightarrow \log_2 16 = 1 = R.H.S.$

Example 6:

Prove that $2 \log \frac{11}{13} - \log \frac{55}{91} + \left(\frac{130}{77}\right) = \log 2.$

Solution:

$$\text{L.H.S.} = 2\log\left(\frac{11}{13}\right) - \log\left(\frac{55}{91}\right) + \left(\frac{130}{77}\right) = \log 2$$

$= 2(\log 11 - \log 13) - (\log 55 - \log 91) + (\log 130 - \log 77)$ (By Law II)

$= 2 \log 11 - 2 \log 13 - \log 55 + \log 91 + \log 130 - \log 77$

$= 2 \log 11 - 2 \log 13 - \log (11 \times 5) + \log (13 \times 7) + \log (13 \times 10) - \log (11 \times 7)$

$= 2 \log 11 - 2 \log 13 - (\log 11 + \log 5) + (\log 13 + \log 7) + (\log 13 + \log 10) - (\log 11 + \log 7)$

$= 2 \log 11 - 2 \log 13 - \log 11 - \log 5 + \log 13 + \log 7 + \log 13 + \log 10 - \log 11 - \log 7$

(All values are cancelled except log 5 and log 10)

$= -\log 5 + \log 10$

$= \log 10 - \log 5$

$= \log\left(\frac{10}{5}\right) = \log 2 = \text{R.H.S.}$

NOTES

Example 7:

Prove that $\log\left(\frac{a^2}{bc}\right) + \log\left(\frac{b^2}{ca}\right) + \log\left(\frac{c^2}{ab}\right) = 0.$

Solution:

$$\text{L.H.S.} = \log\left(\frac{a^2}{bc}\right) + \log\left(\frac{b^2}{ca}\right) + \log\left(\frac{c^2}{ab}\right)$$

$= 2 \log a - (\log b + \log c) + 2 \log b - (\log c + \log a) +$
$2 \log c - (\log a + \log b)$ (By I and II Law)

$= 2 \log a - \log b - \log c + 2\log b - \log c - \log a +$
$2 \log c - \log a - \log b$

$= 0 = \text{R.H.S.}$ (All values are cancelled)

Example 8:

Prove that $\log_a b \times \log_c b \times \log_c a = 1.$

Solution:

L.H.S. $= \log_a b \times \log_c b \times \log_c a = 1$

Change to the base 10

$$= \frac{\log_{10} b}{\log_{10} a} \times \frac{\log_{10} c}{\log_{10} b} \times \frac{\log_{10} a}{\log_{10} c}$$ (All values are cancelled)

$= 1 = \text{R.H.S.}$

4.5 USE OF COMMON LOGARITHM TABLES

Note 1: The logarithms of number greater than 1 consists of two parts, namely, integral part and fractional part.

Integral part is characteristic and fractional part is called Mantissa.

For a number greater than 1, characteristic is always one less than the number of digits to the left of decimal point. Mantissa can be obtained from logarithm table.

NOTES

Example:

(a) log (3471) = 3.5404

In given number 3471, there are 4 digits, characteristic is 3. Observe the logarithmic table for 34th row and 7th column, then you get 5403. Find mean difference in the same table, you get 1.

$\therefore$ 5403 + 1.5404   This is called Mantissa

$\therefore$ log 3471 = 3.5404

Similarly,

(b) log (347.1) = 2.5404

(c) log (34.71) = 1.5404

(d) log (3.471) = 0.5404

(e) log (34710) = 4.5404

(f) log (3471235) = 6.5404

Note:

Logarithm of a number less than 1 consists of two parts,called characteristic and Mantissa. The characteristic is always one more than the number of zeros to the right of decimal point and Mantissa obtained from logarithmic table.

Example:

(a) log (0.3426) = $\bar{1}.5348$

To the right of decimal point, no zero is present. Therefore, characteristic is one bar $(\bar{1})$, then observe the logarithmic table to obtain the Mantissa, i.e., 34th row and 2nd column is 5340 and Mean difference for 6 is 8.

$\therefore$ log (0.3426)= $\bar{1}.5348$

Similarly,

(b) log (0.03426) = $\bar{2}.5348$

$\therefore$ To the right of decimal point, one zero is present and characteristic is two bar $(\bar{2})$.

(c) log (0.003426) = $\bar{3}.5348$

$\therefore$ To the right of decimal point, two zero is present and characteristic is three bar $(\bar{3})$.

(d) log (0.0003426) = $\bar{4}.5348$

(e) log (0.00003426) = $\bar{5}.5348$

(f) log (0.000003426) = $\bar{7}.5348$

NOTES

Check Your Progress

1. Prove that $\log \frac{26}{33} - \log \frac{65}{69} + \log \frac{55}{46} = 0$.

2. Prove that $2 \log \frac{11}{13} - \log \frac{55}{91} + \left(\frac{130}{77}\right) = \log 2$.

4.6 ANTILOGARITHM OF A NUMBER

Example:

(a) Antilog (0.5379) = 7.307 7

7307

(Read the Antilog table, then put the decimal point after one digit from left to right because the characteristic is 0.)

When characteristic is 1, put the decimal point after 2 digits.

When characteristics is 2, put the decimal point after 3 digits.

(b) Antilog (1.5379) = 73.07

(c) Antilog (2.5379 = 730.7

(d) Antilog (3.5379) = 7307.0

(e) Antilog (4.5379) = 73070.0

(f) Antilog $(\bar{1}.5379)$ = 0.7307

(g) Antilog $(\bar{2}.5379)$ = 0.07307

(h) Antilog $(\bar{3}.5379)$ = 0.007307

(i) Antilog $(\bar{5}.5379)$ = 0.00007307

Example 9:

If it is given that log 2 = 0.3010 and log 3 = 0.4771, then find log 8, log 6 and log 24.

Solution:

(a) log 8

$\log 2^3 = 3 \log 2$ (By III Law)

log 8 = 3(0.3010)

log 8 = 0.9030

(b) log 6

log(2, 3) = log 2 + log 3 (By I Law)

= 0.3010 + 0.4771

log 6 = 0.7781

(c) log 24

log (8 × 3) = log 8 + log 3

log 24 = 0.9030 + 0.4771

log 24 = 1.3801

Example 10:

Find the number of digits in: (a) 2^{50} and (b) $3^{25.}$

Solution:

(a) log 2^{50}

Taking logarithm on both sides,

$\log 2^{50} = \log x$

$50 \log 2 = \log x$

$50(0.3010) = \log x$

$15.0500 = \log x$

$\therefore x = \text{Antilog}\,(15.05)$

There are 16 digits in 2^{50}.

(b) 3^{25}

Let $3^{25} = x$

Taking logarithm on both sides,

$\log 3^{25} = \log x$

$25 \log 3 = \log x$

$25(0.4771) = \log x$

$11.9275 = \log x$

$\therefore x = \text{Antilog}\,(11.9275)$

$\therefore$ There are 12 digits in 3^{25}.

Example 11:

Find the number of zeros immediately after decimal point in $(0.05)^{50}$.

Solution:

(a) $(0.05)^{50}$

Let $(0.05)^{50}$

Taking log on both sides,

$\log x = \log(0.05)^{40}$

NOTES

$\log x = 40 \log(0.05)$

$\log x = 40\,(\bar{2}.6990)$

$\log x = 40(-2 + 0.6990)$

$= -80 + 40 \times (0.6990)$

$= -80 + 27.9600$

$= -53 + 0.9660$

$\log x = -53 + 0.9660$

There are 52 zeros after decimal.

Example 12:

Find the number of zero's immediately after decimal point in $(0.2)^{55}$

Solution:

$\mathbf{(0.2)^{55}}$

Let $x = (0.2)^{55}$

$\therefore\ \log x = \log (0.2)^{55}$

$\log x = 55 \log (0.2)$

$\log x = 55\,(\bar{1}.3010)$

$\log x = -55 + 16.5550$

$\log x = -39 + 0.0550$

$\log x = (\overline{39}.5550)$

$x = \text{Antilog of } (39.5550)$

Number of zeros to right of decimal point are 38 (i.e., 39 – 1 = 38)

Example 13:

Find the value using logarithmic table: $\dfrac{1}{(0.8931)^4}$.

Solution:

Let $x = \dfrac{1}{(0.8931)^4}$

Taking log on both sides,

$\log x = \log \dfrac{1}{(0.8931)^4}$

$\log x = \log 1 - \log (0.8931)^4$

$\log x = 0 - 4 \log (0.8931)$ $\quad (\because \log 1 = 0 \text{ and by II Law})$

$\log x = -4\ (\bar{1}.9509)$

$\log x = 4 - 4\ (0.9509)$

$\log x = 4 - 3.8036$

$\therefore \quad x = \text{Antilog}\ (0.1964)$

NOTES

$x = 1.571$

Example 14:

With the help of log tables, find the value of $\dfrac{1}{(1.045)^{10}}$.

Solution:

Let $x = \dfrac{1}{(1.045)^{10}}$

Taking log on both sides,

$\log x = \log \dfrac{1}{(1.045)^{10}}$

$\log x = \log 1 - \log (1.045)^{10}$ (by II Law)

$\log x = 0 - 10 \log (1.045)$ ($\because$ log 1 = 0 and by III Law)

$\log x = -10(0.0191)$

$\log x = -0.1910$

$\log x = 1 - 0.1910 - 1$ (Add and subtract 1 because there is no Antilog for negative number)

$\log x = \bar{1} - (1 - 0.8090)$

$\log x = 1 + (0.8090)$

$\log x = 1.8090$

$\therefore\ x = \text{Antilog}\ (1.8090)$

$\boxed{x = 0.6442}$

Example 15:

Given log 2 = 0.30103 and log 3 = 0.47712. Find the number of digits in 3^{10}.

Solution:

Let $x = 3^{10}$

Taking log on both sides,

$\log x = \log 3^{10}$

$\log x = 10 \log 3$

$\log x = 10(0.47712)$

$\log x = 4.77120$

$x = \text{Antilog}\ (4.77120)$

$\therefore$ The number of digits are 5.

Example 16:

How many zeros are there between the decimal point and first significant figure in $(0.5)^{100}$? (Assume that $\log 0.5 = \bar{1}.6990$)

NOTES

Solution:

Let $x = (0.5)^{100}$

Taking log on both sides,

$\log x = \log (0.5)^{100}$

$\log x = 100 \log 0.5$

$\log x = 100(\bar{1}.6990)$

$\log x = 100(-1 + 0.6990)$

$\log x = -100 + 69.9000$

$\log x = -30.10$

$\log x = 31 - 30.10 - 31$ (Add and subtract 31)

$\log x = \overline{31} + (31 - 30.10)$

$\log x = \overline{31} + 0.90$

$x = \text{Anti log}\left(\overline{31}.90\right)$

$\therefore$ There are 30 zeros between decimal point and first significant digits. ($\because$ 31 – 1 = 30)

Example 17:

If $\log_4 x - \log_2 x = 6$, find x.

Solution:

Let $\log_4 x - \log_2 x = 6$

Change to base 2

$$\frac{\log_2^x}{\log_2^4} - \frac{\log_2^x}{1} = 6$$

$$\frac{\log_2 x}{\log_2 2} - \frac{\log_2 x}{1} = 6$$

$$\frac{\log_2 x}{2.1} - \frac{\log_2 x}{1} = 6$$

$$\log_2 x\left(\frac{1}{2} - \frac{1}{1}\right) = 6$$

$$\log_2 x\left(\frac{-1}{2}\right) = 6$$

$$-\log_2 x = 6 \times 2$$

$$-\log_2 x = 12$$

$$\therefore \quad \log_2 x = -12 \qquad \text{(By adding } -1\text{)}$$

$$\therefore \quad x = 2^{-12} \qquad \boxed{\therefore x = \frac{1}{2^{12}}}$$

NOTES

Example 18:

Find the value of $\frac{71.26 \times 18.52}{21.62 \times 17.45}$.

Solution:

Let $x = \frac{71.26 \times 18.52}{21.62 \times 17.45}$

Taking log on both sides,

$$\log x = \log\left(\frac{71.26 \times 18.52}{21.62 \times 17.45}\right)$$

$$\log x = \log(71.26 \times 18.52) - \log(21.62 \times 17.45) \quad \text{(by I Law)}$$

$$\log x = 1.8529 + 1.2677 - 1.3349 - 1.2417$$

$$\log x = 3.1206 - 2.5766$$

$$\log x = 0.5440$$

$$\therefore \quad x = \text{Antilog}(0.5440) \qquad \boxed{x = 3.499}$$

Example 19:

If $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}$ (log a + log b), show that $a^2 + b^2 = 7ab$.

Solution:

Consider $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log(ab) \qquad \text{(By I Law)}$$

$$\log\left(\frac{a+b}{3}\right) = \log(ab)^{1/2}$$

$\therefore \quad \frac{a+b}{3} = (ab)^{1/2}$ (Since log is present on both side)

$a + b = 3\ (ab)^{1/2}$ (By cross Multiplication)

NOTES

Squaring on both sides, we get

$(a+b)^2 = 3^2 \left((ab)^{1/2}\right)^2$ (2 cancel)

$(a+b)^2 = 9(ab)$ (Apply $(a+b)^2$ formula)

$a^2 + b^2 + 2ab - 9ab = 0$

$a^2 + b^2 - 7ab = 0$

$\therefore \quad a^2 + b^2 = 7ab$

Example 20:

If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}$ (log a + log b), then prove that $a = b$.

Solution:

Consider $\log\frac{a+b}{2} = \frac{1}{2}(\log\ a + \log\ b)$

$\log\frac{a+b}{2} = \frac{1}{2}\log(ab)$ (By I Law)

$\log\frac{a+b}{2} = \log(ab)^{1/2}$ (By II Law)

$\therefore \quad \frac{a+b}{2} = (ab)^{1/2}$

$a + b = 2\ (ab)^{1/2}$ (By Cross multiplication)

Squaring on both sides, we get

$(a+b)^2 = 2^3 \left((ab)^{1/2}\right)^2$ (2 is cancelled)

$(a+b)^2 = 4(ab)$ (Making square root)

$a^2 + b^2 + 2ab = 4ab$ $\quad \therefore \sqrt{(a-b)^2} = 0$

$a^2 + b^2 + 2ab - 4ab = 0$ $\quad a - b = 0$

$a^2 + b^2 - 2ab = 0$

$(a+b)^2 = 0$

NOTES

Example 21:

Find the number of digits in 7^{15}.

Solution:

$\mathbf{7^{15}}$

Let $7^{15} = x$

Taking logarithm on both sides, we get

$$\log 7^{15} = \log x$$
$$\log x = \log 7^{15}$$
$$\log x = 15 \log 7$$
$$\log x = 15\,(0.8451)$$
$$\log x = 12.6765$$
$$x = \text{Antilog of } (12.6765)$$

There are 13 digits in 7^{15}.

Example 22:

Find the number of zeros immediately after decimal point in $(0.004)^{36}$.

Solution:

$\mathbf{(0.004)^{36}}$

Let $x = (0.004)^{36}$

$$\log x = \log(0.004)^{36}$$
$$\log x = 36 \log(0.004)$$
$$= 36\,(\bar{3}.6021)$$
$$= 36(-3 + 0.6021)$$
$$= -108 + 36(0.6021)$$
$$= -108 + 21.6756$$
$$= -108 + 21 + 06756$$
$$= -87 + 0.6756$$
$$x = \text{Antilog}\,(87.6756)$$

$\therefore$ There are 86 zeros to the right of decimal point ($\because$ $87 - 1 = 86$).

Example 23:

Find 4^{th} root of $\dfrac{71.2 \times 13.92}{21.62 \times 4.9}$.

Solution:

Let $x = \left(\dfrac{71.2 \times 13.92}{21.62 \times 4.9}\right)^{1/4}$ fourth root means raise to the powers $\dfrac{1}{4}$

Taking log on both sides, we get

$$\log x = \log\left(\frac{71.2 \times 13.92}{21.62 \times 4.9}\right)^{1/4}$$

$$\log x = \frac{1}{4} \log \frac{71.2 \times 13.92}{21.62 \times 4.9} \qquad \text{(By II Law)}$$

$$\log x = \frac{1}{4}\left[\log(17.2 \times 1392) - \log(21.62 \times 4.9)\right]$$

$$\log x = \frac{1}{4}(\log 17.2 + \log 13.92 - \log 21.62 - \log 4.9)$$

$$\log x = \frac{1}{4}(2.3791 - 2.0251)$$

$$\log x = \frac{1}{4}[0.354] = 0.0885$$

$$x = \text{Antilog } (0.0885)$$

$$\boxed{x = 1.226}$$

4.7 SIMPLE INTEREST

Interest is a charge paid for the use of borrowed money. It is paid by borrower to the lender. Hence, it is the compensation received by the lender of money from the borrower at a particular rate and for a specified period.

When money is borrowed, the borrower is called the debtor and the lender is called the creditor. The sum of money borrowed is called the principal, the time for which it is the borrowed is called the period and the sum returned by the borrower with interest is called the amount. Symbolically,

$A = P + I$

where, A = Amount, P = Principal and I = Interest.

Simple Interest

If interest is calculated only on the principal, then it is called simple interest, i.e., principal alone produces interest. Simple Interest depends upon three factors such as the principal, the rate of interest and the term of the loan.

Formula used in calculating Simple Interest

1. Amount = Principal + Simple Interest
2. Simple Interest = Amount – Principal $\qquad$ SI = Simple interest
3. Principal = Amount – Simple Interest $\qquad$ P = Principal amount

NOTES

4. $SI = \frac{PRT}{100}$ $\quad$ R = Interest rate per annum

5. $P = \frac{SI \times 100}{R \times T}$ $\quad$ T = Term/period in (years)

A = Amount

To Find Simple Interest

Example 24:

Find the simple interest on ₹ 2,276 for 2 years 6 months at 12.5% p.a.

Solution:

Given, P = 2,276, R = 12.5% and T = 2 years 6 months = 2.5 years

$$SI = \frac{P \times R \times T}{100} = \frac{2{,}276 \times 12.5 \times 2.5}{100} = \frac{71{,}125}{100}$$

$$SI = ₹\ 711.25$$

Example 25:

If simple interest on a certain sum is ₹ 360 for 2 years at 6% p.a., then find the sum.

Solution:

Given, SI = ₹ 360, R = 6% p.a., T = 2 years and P = ?

$$SI = \frac{P \times R \times T}{100} \Rightarrow 360 = \frac{P \times 6 \times 2}{100} \Rightarrow 36{,}000 = 12P$$

$$\Rightarrow \quad P = \frac{36{,}000}{12} = 3000 \quad \Rightarrow \therefore P = 3000$$

The sum is ₹ 3,000.

To Find the Rate

Example 26:

A person deposited ₹ 6,200 on June 20, 2018. It amounted to ₹ 6,250 on September 1, 2018 at the rate of simple interest. Find the rate of interest.

Solution:

Here, P = ₹ 6,200 and A = ₹ 6,250

∴ SI = ₹ 6,250 – ₹ 62,00 = ₹ 50

T = June + July + Aug + Sep = 10 + 31 + 31 + 1 = 73 days = $\frac{1}{5}$ years

$$R = \frac{SI \times 100}{P \times T} = \frac{50 \times 100}{6200 \times \frac{1}{5}} = \frac{50 \times 100 \times 5}{6200} = \frac{25000}{6200}$$

$$= 4.032\%$$

Example 27:

At what rate of simple interest will ₹ 800 amount to ₹ 836 in 9 months?

Solution:

Here, $P =$ ₹ 800 and $A =$ ₹ 836

$\therefore \quad SI = A - P =$ ₹ 836 – ₹ 800 = ₹ 36

$$T = \frac{9}{12} = \frac{3}{4} \text{ years}, \qquad R = \frac{SI \times 100}{P \times T} = \frac{36 \times 100}{800 \times \frac{3}{4}} = \frac{3600}{600} = 6\%$$

To Find the Time

Example 28:

In what time will a sum ₹ 2,000 amounts to ₹ 2,240 at the rate of 4% p.a. simple interest?

Solution:

Given, $P =$ ₹ 2,000, $A =$ ₹ 2,240, $SI = (2{,}240 - 2{,}000) =$ ₹ 240, $R = 4\%$ and $T = ?$

$$T = \frac{SI \times 100}{P \times R} = \frac{240 \times 100}{2000 \times 4} = 3 \text{ years}$$

Example 29:

In what period will ₹ 750 amount to ₹ 975 at 5% p.a. simple interest?

Solution:

Given, $P = 750$, $A = 975$, $R = 5\%$, $SI = 975 - 750 = 225$ and $T = ?$

$$T = \frac{SI \times 100}{P \times R} = \frac{225 \times 100}{750 \times 5} = 6 \text{ years}$$

Example 30:

In what time will ₹ 800 amounts to ₹ 896 at 6% p.a. simple interest?

Solution:

Let Time, $T = ?$

Principal, $P =$ ₹ 800,

Interest = 896 – 800 = 96,

Rate $R = 6\%$ p.a.

$$\frac{PRT}{100} \Rightarrow 96 = \frac{800 \times 6 \times T}{100} \Rightarrow 96 = 48 \Rightarrow T = \frac{96}{48}$$

$T = 2$ years

NOTES

Example 31:

In what time will ₹ 1,250 amount to ₹ 2,150 at 9% p.a. simple interest?

Solution:

Given, $P = 1{,}250$, $R = 9$ and $SI = 2{,}150 - 1{,}250 = 900$

Consider,

$$T = \frac{SI \times 100}{PR} \quad \Rightarrow T = \frac{900 \times 100}{1250 \times 9}$$

$$\Rightarrow T = \frac{900 \times 100}{1250 \times 9} \quad \Rightarrow T = \frac{90000}{11250} = 8 \text{ years}$$

Example 32:

Find simple interest if $P =$ ₹ 800, $T = 4$ years and $R = 5$.

Solution:

Given, $P = 800$, $T = 4$ years and $R = 5$

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{800 \times 5 \times 4}{100} = \frac{16000}{100} = 160$$

$$\therefore SI = 160.$$

Example 33:

Find the simple interest on ₹ 300 at 8% p.a. for 14 weeks.

Solution:

Given, $P = 300$ and $R = 8$

Consider,

$$SI = \frac{P \times R \times N}{100} = \frac{300 \times 8 \times 14}{100 \times 52} = \frac{33{,}600}{5{,}200} = 6.46$$

Example 34:

Find the simple interest on ₹ 2,276 for 2 years 6 months at 12.5% p.a.

Solution:

Given, $P =$ ₹ 2,276, $R = 12.5\%$ and $T = 2$ years 6 months

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{2276 \times 12.5 \times 2.5}{100} = \frac{71125}{100}$$

$$= ₹\ 711.25$$

Example 35:

In what time will ₹ 800 amount to ₹ 896 at 6% p.a. simple interest?

Solution:

Let Time = T

Interest = 896 – 800 = 96

Rate = 6 p.a.

Principal = ₹ 800

NOTES

$$SI = \frac{P \times R \times T}{100}$$

$$\Rightarrow 96 = \frac{800 \times 6 \times T}{100}$$

$$\Rightarrow 96 = \frac{4800 \times T}{100}$$

$$\Rightarrow 96 = 48 \times T$$

$$\Rightarrow 96 = 48T$$

$$\Rightarrow 48T = 96$$

$$\Rightarrow T = \frac{96}{48}$$

$$\therefore T = 2 \text{ years}$$

Example 36:

A man deposits ₹ 5,000 in a Savings Account which pays a simple interest at a rate of 4.5% for the first two years and then at the rate of 5% for the next three years. Find the amount due at the end of five years.

Solution:

Hence, for the first two years,

Given, $P = 5{,}000,\ T = 2,\ R = 4.5\% = \dfrac{4.5}{100} = 0.045$

$$\therefore \quad SI = \frac{P \times R \times T}{100}$$

$$\Rightarrow \frac{5{,}000 \times 4.5 \times 2}{100}$$

$$\Rightarrow \frac{45{,}000}{100} = 450$$

NOTES

For the next three years,

Given, $P = 5{,}000$, $T = 3$ and $R = 5\% = \frac{5}{100} = 0.05$

$$\therefore \quad SI = \frac{P \times R \times T}{100}$$

$$\Rightarrow \frac{5{,}000 \times 5 \times 3}{100}$$

$$\Rightarrow \frac{75{,}000}{100} = 750$$

4.8 COMPOUND INTEREST

When interest at the end of each fixed period is added to the principal and the amount thus obtained is taken as the principal for the next period, the interest obtained is called compound interest.

$$\text{Compound Interest} = P\left(1 + \frac{R}{100}\right)^T - P$$

When varying rate of interest is given,

$$A = P\left(1 + \frac{\frac{R_1}{m}}{100}\right)^{T_1 \times m} \left(1 + \frac{\frac{R_2}{m}}{100}\right)^{T_2 \times m} \left(1 + \frac{\frac{R_3}{m}}{100}\right)^{T_3 \times m} \text{-----------(i)}$$

Here, P is the principal

A is amount

R_1 is the rate interest for T_1 year

R_2 is the rate interest for T_2 year

R_3 is the rate interest for T_3 year and so on.

Example 37:

Find compound interest on ₹ 2,560 for 3 years at 8% p.a.

Solution:

Given, $P = 2{,}560$, $R = 8\%$ and $T = 3$ years

$$CI = P\left(1 + \frac{R}{100}\right)^T - P$$

$$= 2560\left(1 + \frac{8}{100}\right)^3 - 2560$$

$$= 2560 \times 1.26 - 2560 = 3225.6 - 2560$$

$$CI = 665.6$$

Example 38:

Mrs. Sheela borrowed ₹ 30,000 for 6 years. Calculate compound interest @ 12% p.a. reckoned quarterly.

Solution:

Given, $A = 30{,}000$, $R = 12\%$ p.a. and $T = 6$ years

Now, interest has to be calculated quarterly.

$\therefore$ Rate of Interest = 12%

$$CI = P\left(1 + \frac{\frac{R}{m}}{100}\right)^{T\times m} - P$$

$$= 30{,}000\left(1 + \frac{\frac{12}{4}}{100}\right)^{6\times 4} - 30{,}000$$

$$= 30{,}000 \times (1.04)^{24} - 30{,}000$$

$CI = 30000 \times 2.03 - 3000$

$= 60{,}900 - 30{,}900$

Compound interest = 30,900

Example 39:

Find the compound interest on ₹ 2,500 for 2 years at 12% p.a.

Solution:

Given, $P =$ ₹ 2,500, $R = 12\%$ and $T = 2$ years

$$CI = P\left(1 + \frac{\frac{R}{m}}{100}\right)^{T\times m} - P$$

$$= 30{,}000\left(1 + \frac{\frac{12}{4}}{100}\right)^{6\times 4} - 30{,}000$$

$= (30{,}000 \times 2.03) - 30{,}000 = 60{,}983.82 - 30{,}000 = 30{,}983.82$

$= 3{,}125 - 2{,}500 = 625$

$\therefore$ Compound interest = ₹ 625

Example 40:

At what rate per cent, compound interest per annum with ₹ 640 amounts to ₹ 774.40 in 2 years?

NOTES

Solution:

Compound Interest = ₹ 774.4 – ₹ 640 = ₹ 134.4

$R = ?$

$T = 2$ years

$P =$ ₹ 640

$$\text{Compound Interest} = P\left(1 + \frac{R}{100}\right)^T - P$$

$$\Rightarrow 134.4 = 640\left(1 + \frac{R}{100}\right)^2 - 640$$

$$\Rightarrow 134.4 = 640\left(\frac{100 + R}{100}\right)^2 - 640$$

$$\Rightarrow 640\left(\frac{(100 + R)^2}{10000} - 1\right) = 134.4$$

$$\Rightarrow 0.21 = \left(\frac{(100 + R)^2}{10000} - 1\right)$$

$$\Rightarrow 2100 = (100 + R)^2 - 10000$$

$$\Rightarrow (100 + R)^2 = 12100 \qquad \Rightarrow 100 + R = 110 \qquad \Rightarrow R = 10\%$$

$\therefore$ Rate = 10% per annum

Example 41:

In what time will ₹ 1,200 amount to ₹ 1,323 at 5% compound interest?

Solution:

Here, $P = 1{,}200$, $A = 1323$, $R = 5$ and $T = ?$

$$\frac{A}{P} = \left(1 + \frac{R}{100}\right)^T;$$

$$\Rightarrow \frac{1323}{1200} = \left(1 + \frac{5}{100}\right)^T$$

$$\frac{441}{400} = \left(1 + \frac{1}{20}\right)^T = \left(\frac{21}{20}\right)^T$$

$$\Rightarrow \left(\frac{21}{20}\right)^T = \left[\frac{21}{21}\right]$$

(When the bases are equal, powers are equal.)

$\therefore$ $T = 2$ years

To Find the Difference between Simple Interest and Compound Interest

Example 42:

Find the difference between simple interest and compound interest on ₹ 5,000 for 5 years, charging half-yearly at 6% p.a.

Solution:

Given, P = ₹ 5,000, R = 6% and T = 5 years

Simple Interest:

$$SI = \frac{P \times \frac{R}{m} \times T \times m}{100} = \frac{5000 \times \frac{12}{2} \times 5 \times 2}{100} = 1,500$$

SI = ₹ 1,500

Compound Interest:

$$CI = P\left(1 + \frac{\frac{R}{m}}{100}\right)^{T \times m} - P$$

$$= 5000\left(1 + \frac{\frac{12}{2}}{100}\right)^{5 \times 2} - 5000 = 5000\left(\frac{106}{100}\right)^{5\times2} - 5000$$

$$= 5000(1.34) - 5000 = 6,719.58 - 5,000$$

CI = ₹ 1,719.58

∴ Difference between CI and SI is (1,719.58 – 1,500) = ₹ 219.58

Example 43:

Find the difference between simple interest and compound interest for 3 years at 5% p.a. on ₹ 12,000 (interest being charged half-yearly).

Solution:

Given, P = ₹ 12,000, R = 5%, T = 3 years

Simple Interest:

$$SI = \frac{P \times \frac{R}{m} \times T \times m}{100} = \frac{12,000 \times \frac{5}{2} \times 3 \times 2}{100} = 1800$$

SI = ₹ 1,800

Compound Interest:

$$CI = P\left(1 + \frac{\frac{R}{m}}{100}\right)^{T \times m} - P$$

$$= 12,000\left(1 + \frac{\frac{5}{2}}{100}\right)^{3 \times 2} - 12,000$$

$$= 12{,}000\left(\frac{102.5}{100}\right)^6 - 12{,}000$$

NOTES

$$= 13{,}916.34 - 12{,}000$$

CI = ₹ 1,916.34

∴ Difference between CI and SI is (19,16.34 – 1,800) = ₹ 116.34

Varying Rate of Interest

Example 44:

Find the compound interest on ₹ 3,550 for 3 years if the rate of interest is 5% for first year, 6% for second year and 7% for third year.

Solution:

When the variable rate of interest is given, the formula for the amount is

$$A = P\left[\left(1+\frac{R_1}{100}\right)^{T_1}\left(1+\frac{R_2}{100}\right)^{T_2}\left(1+\frac{R_3}{100}\right)^{T_3}\ldots\ldots\ldots\ldots\ldots\right]$$

$$= 3550\left(1+\frac{5}{100}\right)\left(1+\frac{6}{100}\right)\left(1+\frac{7}{100}\right)$$

= 3,550 (1.05) (1.06) (1.07)

= 4,228

$CI = A - P$ = ₹ 4,228 – ₹ 3,550 = ₹ 678

∴ CI = ₹ 678

Example 45:

A man borrowed ₹ 4,500 for 9 years at CI. If the rate of interest is 4% p.a. for the first two years, 4.5% p.a. for the next three years and 5% p.a. for the last 4 years. How much does he repay at the end of 9 years?

Solution:

Let $$A = P\left[\left(1+\frac{R_1}{100}\right)^{T_1}\left(1+\frac{R_2}{100}\right)^{T_2}\left(1+\frac{R_3}{100}\right)^{T_3}\ldots\ldots\ldots\ldots\ldots\right]$$

$$= 4500\left(1+\frac{4}{100}\right)^2\left(1+\frac{4.5}{100}\right)^3\left(1+\frac{5}{100}\right)^4$$

= 4,500 $(1.04)^2$ $(1.45)^3$ $(1.05)^4$

= 6,751

∴ A man repays a sum of ₹ 6,751 at the end of 9 years.

Example 46:

Find the compound interest on ₹ 30,000 at 6% p.a. for 3 years. What is the difference between simple interest and compound interest on the same?

Solution:

$$CI = P\left[1 + \frac{R}{100}\right]^T - P = 30,000\left[1 + \frac{6}{100}\right]^3 - 30,000$$

$$= 30,000(1.06)^3 - 30,000$$

$$= 30,000(1.191) - 30,000$$

$$= 35,730 - 30,000 = 5,730$$

$$SI = \frac{PTR}{100} = \frac{30,000 \times 6 \times 3}{100} = 5,400$$

$$CI - SI = 5,730 - 5,400 = 330$$

Example 47:

Find the rate per cent for ₹ 200 to earn ₹ 80 interest for 5 years.

Solution:

Given, $P =$ ₹ 200, $T = 5$ years and $SI =$ ₹ 80

Consider,

$$SI = \frac{P \times R \times T}{100} \quad \Rightarrow 80 = \frac{200 \times R \times 5}{100}$$

$$\Rightarrow 200 \times R \times 5 = 80 \times 100$$

$$\Rightarrow R = \frac{80 \times 100}{200 \times 5} \quad \Rightarrow R = \frac{80}{2 \times 5} \quad \Rightarrow R = 8$$

$\therefore$ Rate = 8%

Example 48:

Find the compound interest of ₹ 3,000 for 7 years at 14% p.a.?

Solution:

Given, $P =$ ₹ 3,000, $T = 7$ years and $R = 14$

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{3,000 \times 7 \times 14}{100} = \frac{2,94,000}{100} = ₹\ 2,940$$

Example 49:

Find the compound interest on ₹ 3,000 for 3 years at 4% p.a.

Solution:

NOTES

Given, $P =$ ₹ 3,000,

$R = 4\%$ and

$T = 3$ years

Consider,

$$A = P\left(1 + \frac{R}{100}\right)^T = 3,000\left(1 + \frac{4}{100}\right)^3 = 3,000\left(\frac{100 + 4}{100}\right)^3$$

$$= 3,000(1.04)^3 = 3,000(1.124864)$$

$=$ ₹ 3,374.542

Example 50:

Find the compound interest on ₹ 2,500 for 2 years at 12% p.a.

Solution:

Given, $P =$ ₹ 2,500, $R = 12\%$ and $T = 2$ years

$$CI = P\left(1 + \frac{R}{100}\right)^T - P = 2500\left(1 + \frac{12}{100}\right)^2 - 2500$$

$$= 2500\left(\frac{100 + 12}{100}\right)^2 - 2500$$

$$= 2500\left(\frac{112}{100}\right)^2 - 2500$$

$$= 2500(1.12)^2 - 2500$$

$$= (2500 \times 1.25) - 2500$$

$$= 3125 - 2500 = 625$$

Example 51:

Find the compound interest on ₹ 2,560 for 3 years at 8% p.a.

Solution:

Given, $P =$ ₹ 2,560, $R = 8\%$ and $T = 3$ years.

$$CI = P\left(1 + \frac{R}{100}\right)^T - P$$

$$= 2560\left(1 + \frac{8}{100}\right)^3 - 2560 = 2560\left(\frac{100 + 8}{100}\right)^3 - 2560$$

$$= 2560\left(\frac{108}{100}\right)^3 - 2560$$

$$= 2560(1.08)^3 - 2560 = (2560 \times 1.25) - 2560 = 3200 - 2560$$

$$= 640$$

NOTES

Example 52:

Find the compound interest on ₹ 10,000 for 2 years at the rate of 4% p.a. payable half-yearly. What will be the simple interest in the above case?

Solution:

When the interest is payable half-yearly, the interest to be taken as half and double the time.

Here, $P =$ ₹ 10,000

$r = 4/2 = 2$

$T = 2 \times 2 = 4$

$$A = P\left(1 + \frac{r}{100}\right)^T \quad i.e., \quad A = 10,000\left(1 + \frac{2}{100}\right)^4$$

$$A = 10,000(1.02)^4 \quad i.e., \quad A = 10,000(1.0824) \Rightarrow A = 10,824$$

$$CI = A - P = ₹\ 10,824 - ₹\ 10,000 = ₹\ 824$$

$$SI = \frac{PRT}{100} = 10,000 \times \frac{2}{100} = ₹\ 800$$

Example 53:

A merchant borrowed ₹ 62,500 and paid ₹ 67,600 in full settlement after 2 years. Find the ratio of compound interest.

Solution:

Given, $A =$ ₹ 67,600,

$P =$ ₹ 62,500,

$n = 2,$

$r = ?$

$$A = \left(1 + \frac{P}{100}\right)^n$$

NOTES

$$\Rightarrow \quad \frac{67600}{62500} = \left(1 + \frac{P}{100}\right)^2$$

$$\Rightarrow \quad \left(1 + \frac{P}{100}\right)^2 = \frac{676}{625}$$

$$\Rightarrow \quad \left(1 + \frac{P}{100}\right)^2 = \left(\frac{26}{25}\right)^2 \Rightarrow 1 + \frac{P}{100} = \frac{26}{25}$$

$$\Rightarrow \quad \frac{100 + P}{100} = \frac{26}{25} \quad \Rightarrow 25(100 + P) = 26 \times 100$$

$$\Rightarrow \quad 2500 + 25P = 2600 \quad \Rightarrow 25P = 2600 - 2500$$

$$\Rightarrow \quad 25P = 100 \quad \Rightarrow P = \frac{100}{25}$$

$$\Rightarrow \quad P = 4 \quad \therefore P = 4\%$$

Example 54:

Find the difference between Simple and Compound interest on ₹ 3,000 in 3 years at 4% p.a.

Solution:

Given, $P = 3{,}000$,

$R = 4$ and

$T = 3$.

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{3{,}000 \times 4 \times 3}{100} = 360$$

$$CI = P\left(1 + \frac{R}{100}\right)^T - P$$

$$= 3{,}000\left(1 + \frac{4}{100}\right)^3 - 3{,}000 = 3{,}000\left(\frac{100 + 4}{100}\right)^3 - 3{,}000$$

$$= 3{,}000\left(\frac{104}{100}\right)^3 - 3{,}000$$

$$= 3{,}000 \times (1.04)^3 - 3{,}000 = 3{,}000 \times 1.1248 - 3{,}000$$

$$= 3374 - 3000 = 374$$

$\therefore CI =$ ₹ 374

Difference between CI and $SI = 374 - 360 =$ ₹ 14

Example 55:

Find the compound interest on ₹ 20,000 at 6% p.a. for 4 years. What is the simple interest in the same amount?

Solution:

NOTES

Given, $P = 20{,}000,$

$R = 6\%$ and

$T = 4$

Consider,

$$CI = P\left[1 + \frac{R}{100}\right]^T - P$$

$$= 20{,}000\left[1 + \frac{6}{100}\right]^4 - 20000$$

$$= 20{,}000\left[\frac{100 + 6}{100}\right]^4 - 20000$$

$$= 20{,}000\left[\frac{106}{100}\right]^4 - 20000$$

$$= 20{,}000(1.06)^4 - 20000$$

$$= 20{,}000 \times 1.2625 - 20000 = 25250 - 20{,}000$$

$$= 5{,}250$$

Now, $$SI = \frac{P \times R \times T}{100} = \frac{20000 \times 4 \times 36}{100} \Rightarrow \frac{4{,}80{,}000}{100} \Rightarrow 4800$$

Example 56:

The difference between Simple Interest and Compound Interest on certain sum at money for 5 years at 3% p.a. is ₹ 54.90. Find the sum.

Solution:

Given, $P = 100$

$R = 3\%$ and

$T = 5$ years

$$SI = \frac{P \times R \times T}{100} \Rightarrow \frac{100 \times 3 \times 5}{100} \Rightarrow \frac{15000}{100}$$

$$A = P\left(1 + \frac{R}{100}\right)^T = 100\left(1 + \frac{3}{100}\right)^T = 100\left(\frac{100 + 3}{100}\right)^5$$

$$= 100\left(\frac{103}{100}\right)^5 = 100(1.03)^5 = 100(1.1593) = 115.93$$

$$\therefore A = 115.93$$

$$CI = A - P = 115.93 - 100 = 15.93$$

Difference between CI and SI = 15.93 – 15 = 0.93

If the difference is 0.93, then principle = 100

If the difference is 54.90, then principle $= \frac{100 \times 54.9}{0.93} = 5{,}903$

Example 57:

The simple interest on a certain sum of money for 2 years is ₹ 1,550 and the compound interest is ₹ 1,588.75. Find the sum and rate of interest.

Solution:

Compound Interest = ₹ 1588.75

Simple Interest $= \frac{1550.00}{38.75}$

Simple Interest for 1 year $= \frac{1550}{2} = 775$

Now, P = 775, t = 1 year and interest, r = ?

$$\therefore \quad P = \frac{SI \times 100}{R \times T} = \frac{38.75 \times 100}{775 \times 1} = \frac{3875}{775} = 5\%$$

SI = 775, P = 5%, T = 1 year and R = ?

$$\therefore \quad P = \frac{SI \times 100}{R \times T} = \frac{775 \times 100}{5 \times 1} = \frac{77{,}500}{5}$$

$\therefore$ Rate = 5% and Sum of Interest = ₹ 15,500.

Example 58:

Find the difference between CI and SI on ₹ 600 for 5 years at 10% p.a.

Solution:

Given, P = 6000,
n = 5 years and
R = 10%

$$CI = P\left(1 + \frac{P}{100}\right)^n - R = 6{,}000\left(1 + \frac{10}{100}\right)^5 - 6{,}000$$

$$= 6{,}000(1 + 0.1)^5 - 6{,}000 = 6{,}000(1.01)^5 - 6{,}000$$

$$= 6,000 \times 1.61051 - 6,000 = 9663.06 - 6,000$$

$$= 3,663.06$$

$$SI = \frac{P \times R \times T}{100} = \frac{6000 \times 10 \times 5}{100} = \frac{3,00,000}{100} = 3000$$

$$\therefore CI - SI = 3663.06 - 3,000 = 663.06$$

NOTES

Example 59:

Manohar borrowed ₹ 12,650 from moneylender at 18% p.a. simple interest. After 3 years, he paid ₹ 10,381 and gave a buffalo to clean debt. Find the cost of buffalo.

Solution:

Given, $P = 12,650$

$R = 18\%$ and

$T = 3$ years.

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{12,650 \times 18 \times 3}{100} = \frac{683100}{100} \Rightarrow 6,831$$

Let, Buffalo's Cost = x

$$\Rightarrow 10,381 + x = 12,650 + 6,831$$

$$\Rightarrow x = 19481 - 10,381 = 9,100$$

$$x = 9,100$$

Hence Bufflo's cost ₹ 9,100.

Example 60:

Find the difference between the *SI* and *CI* on ₹ 3,000 in 3 years at 4%.

Solution:

Given, $P =$ ₹ 3,000, $R = 4\%$ and $T = 3$ years.

Consider,

$$SI = \frac{P \times R \times T}{100} = \frac{3000 \times 4 \times 3}{100} = \frac{36000}{1000} = 360$$

$$CI = P\left(1 + \frac{R}{100}\right)^T - P$$

$$= 3,000\left(1 + \frac{4}{100}\right)^3 - 3,000$$

$$= 3,000\left(\frac{100 + 4}{100}\right)^3 - 3,000$$

$$= 3,000(1.04)^3 - 3,000$$
$$= 3,000(1.124864) - 3,000$$
$$= 3,374.59 - 3,000$$

NOTES

$$CI = 374.59$$

Difference between CI and SI = 374.59 – 360 = 14.59

Example 61:

Find the Compound Interest of ₹ 9,600 at 12% p.a. in 4 years payable half-yearly.

Solution:

Given, P = 9,600,

$$R = \frac{12}{2} = 6\%,$$

$$T = 4 \times 2 = 8 \text{ years}$$

Consider,

$$A = P\left(1 + \frac{R}{100}\right)^T = 9,600\left(1 + \frac{6}{100}\right)^8$$

$$= 9,600\left(\frac{100 + 6}{100}\right)^8 = 9,600\left(\frac{106}{100}\right)^8 = 9600(1.06)^8$$

$$= 9,600(1.5938) = 15,300$$

$$\therefore A = ₹\ 15,300$$

Check Your Progress

3. If $\log_4 x - \log_2 x = 6$, then find x.
4. If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}$ (log a + log b), then prove that $a = b$.
5. Find the difference between CI and SI on ₹ 600 for 5 years at 10% p.a.

4.9 ANSWERS TO 'CHECK YOUR PROGRESS'

NOTES

1. Prove that $\log \frac{26}{33} - \log\frac{65}{69} + \log\frac{55}{46} = 0$.

Solution:

L.H.S. = $\log\left(\frac{26}{33}\right) - \log\left(\frac{65}{69}\right) + \log\left(\frac{55}{46}\right)$

(By II Law) $\left(\left(\log\frac{m}{n}\right) = \log_a m - \log_a n\right)$

= (log 26 – log 33) – (log 65 – log 69) + (log 55 – log 46)

= log 26 – log 33 – log 65 + log 69 + log 11 × log 55 – log 46

= log 13 × 2 – log 11 × 3 – log 13 × 5 + log 23 × 3 + log 11 × 5 – log 23 × 2

= (log 13 + log 2) – (log 11 + log 3) – log(13 + 5) + log (23 + 3) + log (11 + 5) – log (23 + 2)

= log 13 + log 2 – log 11 – log 3 – log 13 – log 5 + log 23 + log 3 + log 11 + log 5 – log 23 – log 2

= 0 = R.H.S. (All values are cancelled)

2. Prove that $2\log\frac{11}{13} - \log\frac{55}{91} + \left(\frac{130}{77}\right) = \log 2$.

Solution:

L.H.S. = $2\log\left(\frac{11}{13}\right) - \log\left(\frac{55}{91}\right) + \left(\frac{130}{77}\right) = \log 2$

= 2(log 11 – log 13) – (log 55 – log 91) + (log 130 – log 77)
(By Law II)

= 2 log 11 – 2 log 13 – log 55 + log 91 + log 130 – log 77

= 2 log 11 –2 log 13 – log(11 × 5) + log (13 × 7) + log(13 × 10) – log(11 × 7)

= 2 log 11 – 2 log 13 – (log 11 + log 5) + (log 13 + log 7) + (log 13 + log 10) – (log 11 + log 7)

= 2 log 11 – 2 log 13 – log 11 – log 5 + log 13 + log 7 + log 13 + log 10 – log 11 – log 7

(All values cancel Except log 5 & log 10)

= – log 5 + log 10

$= \log 10 - \log 5$

$= \log\left(\frac{10}{5}\right) = \log 2 = \text{R.H.S.}$

NOTES

3. If $\log_4 x - \log_2 x = 6$, then find x.

Solution:

Let $\log_4 x - \log_2 x = 6$

Change to base 2

$$\frac{\log_2^x}{\log_2^4} - \frac{\log_2^x}{1} = 6$$

$$\frac{\log_2 x}{\log_2 2} - \frac{\log_2 x}{1} = 6$$

$$\frac{\log_2 x}{2.1} - \frac{\log_2 x}{1} = 6$$

$$\log_2 x\left(\frac{1}{2} - \frac{1}{1}\right) = 6$$

$$\log_2 x\left(\frac{-1}{2}\right) = 6$$

$-\log_2 x = 6 \times 2$

$-\log_2 x = 12$ (By –1)

$\therefore \quad \log_2 x = -12$

$\therefore \quad x = 2^{-12}$ $\boxed{\therefore x = \frac{1}{2^{12}}}$

4. If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log a + \log b)$, then prove that $a = b$.

Solution:

Consider, $\log\frac{a+b}{2} = \frac{1}{2}(\log a + \log b)$

$\log\frac{a+b}{2} = \frac{1}{2}\log(ab)$ (By I Law)

$\log\frac{a+b}{2} = \log(ab)^{1/2}$ (By II Law)

$$\therefore \qquad \frac{a+b}{2} = (ab)^{1/2}$$

$$a + b = 2(ab)^{1/2} \qquad \text{(By Cross multiplication)}$$

Squaring on both sides, we get

$$(a+b)^2 = 2^3\left((ab)^{1/2}\right)^2 \qquad \text{(2 is cancelled)}$$

$$(a+b)^2 = 4(ab) \qquad \text{Taking square root}$$

$$a^2 + b^2 + 2ab = 4ab \qquad \therefore \sqrt{(a-b)^2} = 0$$

$$a^2 + b^2 + 2ab - 4ab = 0 \qquad a - b = 0$$

$$a^2 + b^2 - 2ab = 0$$

$$(a+b)^2 = 0$$

5. Find the difference between *CI* and *SI* on ₹ 600 for 5 years at 10% p.a.

Solution:

Given, $P = 6000$, $n = 5$ year and $R = 10\%$

$$CI = P\left(1 + \frac{P}{100}\right)^n - R = 6{,}000\left(1 + \frac{10}{100}\right)^5 - 6{,}000$$

$$= 6{,}000(1 + 0.1)^5 - 6{,}000 = 6{,}000(1.01)^5 - 6{,}000$$

$$= 6{,}000 \times 1.61051 - 6{,}000 = 9663.06 - 6{,}000$$

$$= 3{,}663.06$$

$$SI = \frac{P \times R \times T}{100} = \frac{6000 \times 10 \times 5}{100} = \frac{3{,}00{,}000}{100} = 3000$$

$$\therefore CI - SI = 3663.06 - 3{,}000 = 663.06$$

4.10 SUMMARY

- The logarithm of a number to the given base is the index or power to which base must be raised to produce that number.
- Interest is a charge paid for the use of borrowed money. It is paid by borrower to the lender. Hence, it is the compensation received by the lender of money from the borrower at a particular rate and for a specified period.
- If interest is calculated only on the principal, then it is called simple interest i.e., principal alone produces interest. Simple interest depends upon three factors such as the principal, the rate of interest and the term of the loan.

4.11 KEY TERMS

- **Logarithm:** The logarithm of a number to the given base is the index or power to which base must be raised to produce that number.
- **Simple Interest:** Interest is a charge paid for the use of borrowed money. It is paid by borrower to the lender. Hence, it is the compensation received by the lender of money from the borrower at a particular rate and for a specified period.
- **Compound interest:** When interest at the end of each fixed period is added to the principal and the amount thus obtained is taken as the principal for the next period, the interest obtained is called compound interest.

4.12 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is Logarithm?
2. What are Logarithmic Form?
3. What are Index Form?
4. State the Laws of Logarithm.
5. What are the uses of Common Logarithm Tables?
6. What is interest?
7. What is rate of interest?
8. Define simple interest.
9. Define compound interest.

Long Answer Questions

1. Explain the Antilogarithm of a number.
2. If $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$, show that $a^2 + b^2 = 7ab$.
3. If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log a + \log b)$, prove that $a = b$.
4. Find the number of digits in 7^{15}.
5. Find the number of zeros immediately after decimal point in $(0.004)^{36}$.
6. In what time will ₹ 1,250 amount to ₹ 2,150 at 9% p.a. simple interest?
7. Find simple interest if P = ₹ 800, T = 4 years and R = 5.
8. Find the simple interest on ₹ 300 at 8% p.a. for 14 weeks.

9. Find the compound interest on ₹ 30,000 at 6% p.a. for 3 years. What is the difference between simple interest and compound interest on the same?
10. Find the rate per cent for ₹ 200 to earn ₹ 80 interest for 5 years.
11. Find the compound interest of ₹ 3,000 for 7 years at 14% p.a.
12. Find the simple interest on ₹ 3,000 for 3 years at 4% p.a.
13. Find the simple interest on ₹ 2,276 for 2 years 6 months at 12.5% p.a.
14. Find the compound interest on ₹ 2,500 for 2 years at 12% p.a.
15. Find the compound interest on ₹ 2,560 for 3 years at 8% p.a.
16. In what time will ₹ 800 amount ₹ 896 at 6% p.a. simple interest?
17. Find the compound interest on ₹ 10,000 for 2 years at the rate of 4% p.a. payable half-yearly. What will be the simple interest in the above case?
18. A merchant borrowed ₹ 62,500 and paid ₹ 67,600 in full settlement after 2 years. Find the ratio of compound interest.
19. Find the difference between Simple and Compound Interest on ₹ 3,000 in 3 years at 4% p.a.
20. A man deposit ₹ 5,000 in a savings account which pays a simple interest ata rate of 4.5% for the first two years and then at the rate of 5% for the next 3 years. Find the amount due at the end of five years.
21. Find the compound interest on ₹ 20,000 at 6% p.a. for 4 years. What is the simple interest in the same amount?
22. The difference between simple interest and compound interest on certain sum at money for 5 years at 3% p.a.is ₹ 54.90. Find the sum.
23. The simple interest on a certain sum of money for 2 years is ₹ 1,550 and the compound interest is ₹ 1,588.75. Find the sum and rate of interest.
24. Find the difference between *CI* and *SI* on ₹ 600 for 5 years at 10% p.a.
25. Manohar borrowed ₹ 12,650 for moneylender at 18% p.a. simple interest. After 3 years, he paid ₹ 10,381 and gave a buffalo to clean the debt. Find the cost of buffalo.
26. Find the difference between *SI* and *CI* on ₹ 3,000 in 3 years at 4% p.a.
27. Find the compound interest of ₹ 9,600 at 12% p.a. in 4 years payable half-yearly.

NOTES

4.13 FURTHER READING

1. Dr. Nirmala Satyaprasad and Dr. R.G. Saha (2021), *Business Mathematics*, Himalaya Publishing House Pvt. Ltd.
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UNIT 5 AVERAGES

Structure

5.0 INTRODUCTION

In the words of Croxton and Cowden, "An average is a single value within the range of the data that is used to represent all of the values in the series. Since an average is somewhere within the range of the data, it is called a measure of cultural value."

Sometimes, some observations got relatively more importance than other observations. The weight for such observation should be given on the basis of their relative importance.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the concept of Simple, Weighted and Statistical Averages
- Describe the Arithmetic Mean
- Discuss the Harmonic Mean
- Examine the Geometric Mean
- Explain the concept of Profit and Loss

5.2 AVERAGES

Average

An average is a single figure which sums up the characteristics of a whole group of figures. In the words of Clark, "average is an attempt to find one single figure to describe whole of figures."

An average is described as a measure of central tendency as it is more or less a central value around which various values cluster.

In the words of Croxton and Cowden, "An average is a single value within the range of the data that is used to represent all of the values in the series. Since an average is somewhere within the range of the data, it is called a measure of cultural value."

NOTES

Objectives Served by Averages

Averages serve the following purposes:

1. To obtain a clear and concise picture of a large number of numerical data.
2. To compare different groups by means of averages.
3. To obtain a clear picture of a whole group by the study of sample data.
4. To give a definite rates to the relationship between different groups.

Characteristics or Requisites or Properties of Good Average

1. It is rigidly defined. Its value is always definite.
2. It is easy to calculate and understand. Hence, it is very popular.
3. It is based on all the observations so that it becomes a good representative.
4. It can be easily used for comparison.
5. It is capable of further algebraic treatment such as finding the sum of the values of the observations when the mean and the total number of the observations are given; finding the combined arithmetic mean when different groups are given, etc.
6. It is not affected much by sampling fluctuations. Hence, sampling stability

Essentials of a Good Average

The essentials of a good average are as follows:

1. It should be rigidly defined.
2. It should be based on all the observations of the data.
3. It should be readily comprehensible or understandable.
4. It should be capable of being calculated with reasonable ease and rapidity.
5. It should be affected as little as possible by fluctuations of sampling.
6. It should be readily amenable to arithmetic or algebraic treatment.

Types of Averages

1. Arithmetic mean
2. Median
3. Mode

4. Geometric mean
5. Harmonic mean

NOTES

5.3 SIMPLE AND WEIGHTED STATISTICAL AVERAGES

Simple Arithmetic Mean

In simple arithmetic mean, it is assumed that all the items are of equal importance. For finding simple average, total values are divided by number of observations.

Weighted Arithmetic Mean

Sometime, some observations get relatively more importance than other observation. The weight for such observation should be given on the basis of their relative importance. The value of each observations is multiplied by its weight and the product are divided by number of weights for finding average is called weight arithmetic mean. Symbolically,

$$\bar{X} = \frac{\sum WX}{\sum W}$$

Example 1:

A contractor employs three types of workers, skilled, semi-skilled and unskilled. To a skilled worker, he pays ₹ 300 per day; to a semi-skilled worker, he pays ₹ 200 per day; and to a unskilled worker, he pays ₹ 100 per day. The number of skilled, semi-skilled and unskilled workers are 20, 15 and 10 respectively. What is the average wage paid by the contractor?

Solution:

An Appropriate Average is Weighted AM

Types of workers	*Pay per days (in ₹) (x)*	*No. of workers (w)*	*wx*
Skilled	300	20	6,000
Semi-skilled	200	15	3,000
Unskilled	100	10	1,000
		$\sum w = 45$	$\sum wx = 10{,}000$

Weighted Arithmetic Mean

$$\bar{X} = \frac{\sum wx}{\sum w} = \frac{10000}{45} = 222.22$$

NOTES

Example 2:

The following table gives the marks of two candidates. Find the weighted average marks of each candidate. By which figure, the second candidate have had to increase in subject B all other marks remaining the same in order that both candidates have same place?

Subjects	*Weights*	*Marks of 2 candidates*	
A	1	70	80
B	2	65	64
C	3	58	56
D	4	63	60

Solution:

Subjects	*Weights*	*Marks of 2 candidates*			
	(w)	*x*	*y*	*wx*	*wy*
A	1	70	80	70	80
B	2	65	64	130	128
C	3	58	56	174	168
D	4	63	60	252	240
	$\sum w = 10$			$\sum wx = 626$	$\sum wy = 616$

Weighted Arithmetic Mean

$$\bar{X}_w = \frac{\sum wx}{\sum w} = \frac{626}{10} = 62.6$$

$$\bar{Y}_w = \frac{\sum wy}{\sum w} = \frac{616}{10} = 61.6$$

The difference in marks of 2 candidates = $\sum wx - \sum wy$, i.e., 626 – 616 = 10 marks.

Example 3:

Calculate the average marks scored by a student in five different subjects. The details of the marks are given below:

Kannada 85, English 70, Mathematics 90, Science 60 and Social 55.

The weights given for these subjects are:

Mathematics 5, Science 4, Social 3, English 2 and Kannada 1.

Solution:

Subject	*Weight (w)*	*Marks (x)*	*wx*
Mathematics	5	90	450
Science	4	60	240
Social	3	55	165

English	2	70	140
Kannada	1	85	85
	$\sum w = 15$		$\sum wx = 1080$

NOTES

Weighted Arithmetic Mean

$$\bar{X}_w = \frac{\sum wx}{\sum w} = \frac{1080}{15} = 72 \text{ marks}$$

Example 4:

A train runs 25 km at a speed of 30 kmph, another 50 km at a speed of 40 kmph. Due to repairs of the track, it travels for 6 minutes at a speed of 10 kmph, and finally, covers the remaining distance of 24 km at a speed of 24 kmph, what is the average speed in kmph?

Solution:

Let the speed in kmph be variable (x) and time taken in minutes be weight (w).

$$\text{Time taken in minutes} = \frac{\text{Distance travelled}}{\text{Speed}}$$

$$= \frac{25}{30} \times 60 = 50 \text{ units}$$

$$= \frac{50}{40} \times 60 = 75 \text{ minutes}$$

Speed	*Time taken (x)*	*wx (w)*
30	50	1500
40	75	3000
10	6	60
24	60	1440
	$\sum w = 191$	$\sum wx = 6000$

Weighted Arithmetic Mean

$$\bar{X}_w = \frac{\sum wx}{\sum w} = \frac{6000}{191} = 31.41 \text{ kmph}$$

NOTES

Example 5:

Calculate simple and weighted average from the following data:

Month	*Price per tonne (in '000)*	*No. of tonnes purchased*
January	42.5	25
February	51.25	30
March	50	40
April	52	50
May	44.25	10
June	54	45

Solution:

Month	*Price per tonne (₹ in '000) (x)*	*No. of tonnes purchased (w)*	*wx*
January	42.5	25	1,062.5
February	51.25	30	1,537.5
March	50	40	2,000.0
April	52	50	2,600.0
May	44.25	10	442.5
June	54	45	2,430.0
$n = 6$	$\sum x = 294$	$\sum w = 200$	$\sum wx = 10027.5$

Simple Arithmetic Mean

$$\bar{X} = \frac{\sum x}{n} = \frac{294}{6} = 49$$

Weighted Arithmetic Mean

$$\bar{X}_w = \frac{\sum wx}{\sum w} = \frac{10072.5}{200} = 50.3625$$

The correct average price paid is ₹ 50.36 and not ₹ 49, i.e., weighted arithmetic mean is correct than simple arithmetic mean.

Combined Arithmetic Mean

Arithmetic mean and number of observations of two or more related groups are known as combined mean of the entire group. The combined average of two series can be calculated by the given formula:

$$\bar{x}_{12} = \frac{n_1.\bar{x}_1 + n_2.\bar{x}_2}{n_1 + n_2}$$

where, n_1 = Number of observations of the first group, n_2 = Number of observations of the second group $\overline{x}_1$ = AM of the first group, $\overline{x}_2$ = AM of the second group, $\overline{x}_{12}$ = Combined AM

NOTES

Example 6:

From the following data, ascertain the combined mean of a factory consisting of two branches, namely Branch A and Branch B. In Branch A, the number of workers is 500, and their average salary of ₹ 300. In Branch B, the number of workers is 1,000 and their average salary is ₹ 250.

Solution:

Let the number of workers in Branch A be n_1 = 500 and average salary $\overline{X}_1$ be ₹ 300.

Let the number of workers in Branch B be n_2 = 1,000 and average salary $\overline{X}_2$ be ₹ 250.

$$\overline{X}_{12} = \frac{n_1.\overline{X}_1 + n_2.\overline{X}_2}{n_1 + n_2}$$

$$= \frac{500(300) + 1000(250)}{500 + 1,000}$$

$$= \frac{1,50,000 + 2,50,000}{1,500}$$

$$= \frac{4,00,000}{1,500}$$

$$= 266.67$$

Thus, the combined average salary of the factory is ₹ 266.67.

Corrected Arithmetic Mean

$$Correct\ \overline{x} = \frac{Correct \sum x}{N}$$

$$Correct \sum x = Incorrect\ \sum x - Wrong\ items + Correct\ \ items$$

Example 7:

The mean of 20 values is 45. If one of these values is taken as 64 instead of 46, find the corrected mean.

Solution:

$\overline{x}$ = 45, N = 20, wrong items = 64 and correct items = 46

$$Correct \sum x = Incorrect\ \sum x - Wrong\ items + Correct\ \ items$$

$$= (45 \times 20) - 64 + 46$$

$$= 900 - 18$$

Correct $\sum x = 882$

Correct $\bar{x} = \frac{882}{20} = 44.1$

NOTES

Example 8:

Te mean wages of 50 labourers in a factory is ₹ 380 and the mean wages of 30 labourers working in morning shift is ₹ 400. Find the mean wages of workers working in evening shift.

Solution:

$n_1 = 30,\ \bar{X}_1 = ₹\ 400,\ n_2 = 20$, i.e., $(50 - 30)$, $\bar{X}_{12} = 380$ and $\bar{X}_2 = ?$

$$\bar{X}_{12} = \frac{n_1.\bar{X}_1 + n_2.\bar{X}_2}{n_1 + n_2}$$

$$380 = \frac{30(400) + 20.\bar{X}_2}{30 + 20}$$

$$380 = \frac{12000 + 20\bar{X}_2}{50}$$

$$(380 \times 50) = 12000 + 20\bar{X}_2$$

$$19000 = 12000 + 20\bar{X}_2$$

$$20\bar{X}_2 = 19000 - 12000$$

$$\bar{X}_2 = \frac{7000}{20} = 350$$

The mean wages of workers working in evening shift is ₹ 350.

Example 9:

The average salary paid to all the employees of a factory was ₹ 50,000. The mean annual salary paid to male and female employees are ₹ 52,000 and ₹ 42,000 respectively. Determine the number of males and females.

Solution:

Male $(n_1) = ?$

$\bar{X}_1 = 52{,}000$

Female $(n_2) = ?$

$\bar{X}_2 = 42{,}000$

$\bar{X}_{12} = 50{,}000$

$$\bar{X}_{12} = \frac{n_1.\bar{X}_1 + n_2.\bar{X}_2}{n_1 + n_2}$$

NOTES

$$50{,}000 = \frac{n_1(52{,}000) + n_2(42{,}000)}{n_1 + n_2}$$

$50{,}000(n_1 + n_2) = n_1(52000) + n_2(42{,}000)$

$50{,}000n_1 + 50{,}000n_2 = n_1\,52{,}000 + n_2\,42{,}000$

$50{,}000n_1 + 50{,}000n_2 - n_1\,52{,}000 - n_2\,42{,}000 = 0$

$-2{,}000n_1 + 8{,}000n_2 = 0$

$8000n_2 = 2000n_1$ (transposition to right side minus sign become plus sign)

Therefore, $n_1 = 8000$, $n_2 = 2000$ or 8000 : 2000 or 8 : 2 or 80% and 20% (8/10 × 100)

Factory employees 80% of male employees and 20% of female employees or for every 8 male employees, there are 2 female employees in a factory.

Example 10:

In a certain examination, average grade of all the students in a Class *A* is 68.4 and students in Class *B* is 71.2. If the average of both class combined is 70, find the number of students in Class *A* to the number of students in Class *B*.

Solution:

Class A (n_1) = ?

$\overline{X}_1 = 68.4$,

Class B (n_2) = ?

$\overline{X}_2 = 71.2$

$\overline{X}_{12} = 70$

$$\overline{X}_{12} = \frac{n_1.\overline{X}_1 + n_2.\overline{X}_2}{n_1 + n_2}$$

$$70 = \frac{n_1(68.40) + n_2(71.2)}{n_1 + n_2}$$

$70\,(n_1 + n_2) = n_1(68.4) + n_2(71.2)$

$70n_1 + 70n_2 = n_1\,68.4 + n_2\,71.2$ or $70n_1 + 70n_2 - n_1\,68.4 - n_2\,71.2 = 0$

$1.6n_1 - 1.2n_2 = 0$ or $1.6n_1 = 1.2n_2$ (to avoid decimal multiply with 10)

Therefore, $n_1 = 12$, $n_2 = 16$ or 12 : 16 or 3 : 4. For every 3 students in Class *A*, there are 4 students in Class *B*.

Example 11:

The average weight of 10 different balls was 25.686 gms. The average of the first four balls was 25.680 gms and that of last three was 25.686 gms. If the average of the fifth and sixth ball was 0.042 gms greater than the weight of the seventh ball, what was the weight of the seventh ball?

NOTES

Solution:

$\bar{X}_{1234}$ = 25.686 (average of 10 balls)

$\bar{X}_1$ = 25.680 (average of first 4 balls) (First 4 balls) $N_1 = 4$

$\bar{X}_2 = x + 0.042$ (average of 5th and 6th ball) (5th & 6th balls) $N_2 = 2$

$\bar{X}_3 = x$ (average of 7th ball be x) (7th ball) $N_3 = 1$

$\bar{X}_4$ = 25.686 (average of last 3 balls) (Last 3 balls) $N_4 = 3$

4 groups (1234) Total balls = 10

$$\bar{X}_{1234} = \frac{N_1.\bar{X}_1 + N_2.\bar{X}_2 + N_3.\bar{X}_3 + N_4.\bar{X}_4}{N_1 + N_2 + N_3 + N_4}$$

$$25.686 = \frac{4(25.680) + 2(X + 0.042) + 1(X) + 3(25.686)}{4 + 2 + 1 + 3}$$

$$25.686 = \frac{102.72 + 0.084 + 2x + x + 77.058}{10}$$

$$256.86 = 179.862 + 3x$$

$$-3x = 179.826 - 256.86$$

$$-3x = -76998$$

$$x = \frac{76.998}{3} = 25.666 \text{ gms}$$

The weight of 7th ball is 25.666 gms.

Example 12:

The mean age of 100 persons is 30 years. If the mean age of group of men is 32 years and the group of women age is 27 years, then find the number of men and women.

Solution:

$\bar{X}_1 = 32,$

$\bar{X}_2 = 27,$

$\bar{X}_{12} = 30,$

$n_1 + n_2 = 100$

$$\bar{X}_{12} = \frac{n_1.\bar{X}_1 + n_2.\bar{X}_2}{n_1 + n_2}$$

$$30 = \frac{n_1(32) + 27(100 - n_1)}{100}$$

$3000 = 32n_1 + 2700 - 27n_1$

$3000 = 5n_1 + 2700$

$5n_1 = 300$

$n_1 = \frac{300}{5}$

$n_1 = 60$

$n_1 + n_2 = 100$

$60 + n_2 = 100$

$n_2 = 100 - 60$

$n_2 = 40$

NOTES

Example 13:

The mean age of 40 persons is 30 years. The mean age of group a person is 22. The combined arithmetic mean is 25. Find the number of persons.

Solution:

$\overline{X}_{12} = 25$

$n_1 = 40$

$\overline{X}_1 = 30$

$n_2 = ?$

$\overline{X}_2 = 22$

$$\overline{X}_{12} = \frac{n_1.\overline{X}_1 + n_2.\overline{X}_2}{n_1 + n_2}$$

$$25 = \frac{40(30) + n_2(22)}{40 + n_2}$$

Cross multiplying $25(40 + n_2) = 1200 + 22n_2$

$1000 + 25n_2 = 1200 + 22n_2$

$25n_2 - 22n_2 = 1200 - 1000$

$3n_2 = 200$

$n_2 = \frac{200}{3}$

$n_2 = 67$

The number of persons is 67 + 40 = 107.

NOTES

Missing Frequency or Values

Example 14:

If the average wage paid to 25 workers is 796, find the missing numbers.

Wages	*No. of workers*
500	1
600	3
700	–
800	–
900	6
1,000	2
1,100	1

Solution:

$N = 25, \quad AM = 796$

Let the missing number of workers be X and Y.

Wages ₹ *(x)*	*No. of workers (f)*	*fx (x*f)*
500	1	500
600	3	1,800
700	x	700x
800	y	800y
900	6	5,400
1,000	2	2,000
1,100	1	1,100
	N = 25	$\sum fx$ = 10,800 + 700x + 880y

Step 1: 1 + 3 + x + y + 6 + 2 + 1 = 25

$13 + x + y = 25$

$x + y = 25 - 13$

$x + y = 12$ .... (1)

Step 2: $\overline{X} = \frac{\sum fx}{N}$, 796 = (10,800 + 700$x$ + 800y) / 25

Cross multiplying , we get

$796 \times 25 = 10{,}800 + 700x + 800y$

$19{,}900 - 10{,}800 = 700x + 800y$

$700x + 800y = 9100$ .... (2)

Step 3: Multiplying Equation (1) with 700, we get

$x + y = 12 \times 700,$

$700x + 700y = 8,400$ .... (3)

Step 4: Subtracting Equation (3) from Equation (2), we get

$700x + 800y = 9,100$ (Equation 2)

$700x + 700y = 8,400$ (Equation 3)

$100y = 700$

$y = 700/100$

$y = 7$

Step 5: Substituting the value of y in Equation (1), we get

$x + y = 12$

$x + 7 = 12$

$x = 12 - 7$

$x = 5$

The missing frequencies are $x = 5$ and $y = 7$ respectively.

Check Your Progress

1. Mean wages of 50 labourers in a factory is ₹ 380 and the mean wages of 30 labourers working in morning shift is ₹ 400. Find the mean wages of workers working in evening shift.
2. If the average wage paid to 25 workers is 796, find the missing numbers

Wages	*No. of workers*
500	1
600	3
700	–
800	–
900	6
1,000	2
1,100	1

5.4 ARITHMETIC MEAN

Arithmetic Mean

Arithmetic mean is defined as the value obtained by dividing the total values of all items in the series by their number.

In other words, it is defined as the sum of the given observations divided by the number of observations, i.e., add values of all items together and divide this sum by the number of observations. Symbolically,

NOTES

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Merits of Arithmetic Mean

1. It is rigidly defined. Its value is always definite.
2. It is easy to calculate and understand. Hence, it is very popular.
3. It is based on all the observations so that it becomes a good representative.
4. It can be easily used for comparison.
5. It is capable of further algebraic treatment such as finding the sum of the values of the observations when the mean and the total number of the observations are given; finding the combined arithmetic mean when different groups are given, etc.
6. It is not affected much by sampling fluctuations.

Demerits of Arithmetic Mean

1. Arithmetic mean is affected very much by extreme values.
2. It cannot be determined by inspection nor it can be located graphically.
3. It cannot be obtained if a single observation is missing or lost.
4. We cannot calculate it when open-end class intervals are present in the data.

Properties of Arithmetic Mean

1. The sum of the deviations of all the values of x from their arithmetic mean is zero.
2. The product of the arithmetic mean and the number of items gives the total of all items.
3. Find the combined arithmetic mean when different groups are given.

Arithmetic Mean for Ungrouped Data

A. Individual Series

Direct Method

The following steps are involved in calculating arithmetic mean under individual series using direct method:

1. Add up all the values of all the observations in the series.
2. Divide the sum of the values by the number of observations. The result is the arithmetic mean.

The following formula is used:

$$\overline{X} = \frac{\Sigma x}{n}$$

where, $\overline{X}$ = Arithmetic mean

$\sum x$ = Sum of the values

n = Number of items

NOTES

Example 15:

Value (x): 125 128 132 135 140 148 155 157 159 161

Solution:

Calculation of Arithmetic Mean

Sl. No.	*Value (x)*
A	125
B	128
C	132
D	135
E	140
F	148
G	155
H	157
I	159
J	161
n = 10	$\sum x$ = 1,440

$$\overline{X} = \frac{\Sigma x}{n} = \frac{1{,}440}{10} = 144$$

Example 16:

The monthly income of ten families of a certain locality are given in rupees as below:

Family	*Income (in ₹)*
A	850
B	700
C	840
D	750
E	500
F	800

NOTES

G	420
H	2500
I	2300
J	1500

Calculate the arithmetic mean.

Solution:

Calculation of Arithmetic Mean

Family	*Income (in ₹)*
A	850
B	700
C	840
D	750
E	500
F	800
G	420
H	2,500
I	2,300
J	1,500
$n = 10$	$\sum x = 11{,}160$

$$\overline{X} = \frac{\Sigma x}{n} = \frac{11{,}160}{10} = 1{,}116$$

The average income of 10 families is ₹ 1,116.

Short-cut Method or Indirect Method

The following steps are involved in calculating arithmetic mean under individual series using short-cut or indirect method:

1. Assume one of the values in the series as an average. It is called as working mean or assumed average.
2. Find out the deviation of each value from the assumed average.
3. Add up the deviations.
4. Apply the following formula:

$$\overline{X} = A + \frac{\Sigma dx}{n}$$

where,

$\overline{X}$ = Arithmetic mean

A = Assumed average

$\sum dx$ = Sum of the deviations

n = Number of items

Example 17:

Calculate the arithmetic average of the data given below using short–cut method.

Roll No.	*Marks obtained*
1	43
2	48
3	65
4	57
5	31
6	60
7	37
8	48
9	78
10	59

NOTES

Solution:

Calculation of Arithmetic Mean

Roll No.	*Marks obtained (x)*	*dx = (x – A)*
1	43	–17
2	48	–12
3	65	+5
4	57	–3
5	31	–29
6	60 (A)	0
7	37	–23
8	48	–12
9	78	+18
10	59	–1
		$\sum dx = -74$

$$\overline{X} = A + \frac{\Sigma dx}{n}$$

$$= 60 + \left(-\frac{74}{10}\right)$$

$$= 60 - 7.4$$

$$= 52.6 \text{ marks}$$

NOTES

Example 18:

Calculate arithmetic mean of the weight of 10 students in a class.

SI. No.	*Weight (in kg)*
1	44
2	56
3	49
4	50
5	48
6	52
7	50
8	47
9	51
10	53

Solution:

Calculation of Arithmetic Mean

(a) Direct Method				**(b) Indirect Method**	
SI.	*Weight No.*	*SI. (x)*	*Weight No.*	*(dx = x – A) (x)*	
1	44	1	44	(44 – 48)	–4
2	56	2	56	(56 – 48)	8
3	49	3	49	(49 – 48)	1
4	50	4	50	(50 – 48)	2
5	48	5	48(A)	(48 – 48)	0
6	52	6	52	(52 – 48)	4
7	50	7	50	(50 – 48)	2
8	47	8	47	(47 – 48)	–1
9	51	9	51	(51 – 48)	3
10	53	10	53	(53 – 48)	5
$n = 10$	$\sum x = 500$	$n = 10$		$\sum dx = 20$	

(a) Direct Method

$$\overline{X} = \frac{\Sigma x}{n} = \frac{500}{10} = 50 \text{ kgs}$$

(b) Indirect Method

$$\overline{X} = A + \frac{\Sigma x}{n} = 48 + \frac{20}{10} = 48 + 2 = 50 \text{ kgs}$$

Example 19:

The earning of Mr. N. for the past week were:

Days	*Earnings (x)*
Monday	450
Tuesday	375
Wednesday	500
Thursday	350
Friday	270

Find his average earning per day.

NOTES

Solution:

Calculation of Arithmetic Mean

(a) Direct Method		**(b) Indirect Method**		
Days	*(x)*	*(x)*	*dx = (x – A)*	
Monday	450	450	(450 –500)	–50
Tuesday	375	375	(375 – 500)	–125
Wednesday	500	500 (A)	(500 – 500)	0
Thursday	350	350	(350 – 500)	–150
Friday	270	270	(270 – 500)	–230
$n = 5$	$\sum x = 1945$		$\sum dx = -555$	

(a) Direct Method

$$\overline{X} = \frac{\Sigma x}{n} = \frac{1945}{5} = 389$$

(b) Indirect Method

$$\overline{X} = A + \frac{\Sigma x}{n} = 500 + \frac{-555}{5} = 500 - 111 = 389$$

B. Discrete Series

In the discrete series, every term (i.e., value of x) is multiplied by its corresponding frequency (fx) and then their total (sum) Σfx is divided by the total frequency (N) or Σf .

Direct Method

Formula: $\overline{X} = \frac{\Sigma fx}{N}$

Example 20:

Following are the marks obtained by students of a class in statistics. Calculate arithmetic mean.

NOTES

Marks	*No. of students*
35	3
40	8
45	12
50	9
55	4
60	7
65	15
70	5
75	10
80	7
85	5
90	3
95	2

Solution:

Calculation of Arithmetic Mean

Marks	*No. of students* (*x*)	*fx* (*f*)
35	3	105
40	8	320
45	12	540
50	9	450
55	4	220
60	7	420
65	15	975
70	5	350
75	10	750
80	7	560
85	5	425
90	3	270
95	2	190
	$N = 90$	$\Sigma fx = 5,575$

$$\overline{X} = \frac{\Sigma fx}{N} = \frac{5,575}{90} = 61.94 \text{ marks}$$

Example 21:

The coins are tossed 1024 times. The theoretical frequency of 10 heads upto '0' heads are given below:

NOTES

No. of heads	*Frequency*
0	1
1	10
2	45
3	120
4	210
5	252
6	210
7	120
8	45
9	10
10	1

Calculate the mean number of heads per tossing.

Solution:

Calculation of Mean Number of Heads per Tossing

No. of heads (*x*)	*Frequency* (*f*)	*Product* (*fx*)
0	1	0
1	10	10
2	45	90
3	120	360
4	210	840
5	252	1260
6	210	1260
7	120	840
8	45	360
9	10	90
10	1	10
	$N = 1024$	$\sum fx = 5120$

$$\overline{X} = \frac{\Sigma fx}{N} = \frac{5,120}{1,024} = 5 \text{ heads}$$

Indirect Method or Short-cut Method

Formula: $\overline{X} = A + \frac{\sum fdx}{N}$

NOTES

Example 22:

Following are the marks obtained by students of a class in statistics. Calculate arithmetic mean.

Marks	*No. of students*
35	3
40	8
45	12
50	9
55	4
60	7
65	15
70	5
75	10
80	7
85	5
90	3
95	2

Solution:

Calculation of Arithmetic Mean

Marks (x)	*No. of students (f)*	*(x – A) = dx*	*fdx*
35	3	–30	–90
40	8	–25	–200
45	12	–20	–240
50	9	–15	–135
55	4	–10	–40
60	7	–5	–35
65 (A)	15	0	0
70	5	5	25
75	10	10	100
80	7	15	105

85	5	20	100
90	3	25	75
95	2	30	60
	$N = 90$		$\sum fdx = -275$

NOTES

$$\overline{X} = A + \frac{\sum fdx}{N} = 65 + \frac{-275}{90} = 65 - 3.06 = 61.94 \text{ marks}$$

Example 23:

The following data relates to sizes of shoes (in inches) sold at a store during a given week. Find the average size by the short-cut method.

Size of shoes in inches	*No. of pairs*
4.5	1
5	2
5.5	4
6	5
6.5	15
7	30
7.5	60
8	82
8.5	82
9	75
9.5	44
10	25
10.5	15
11.0	4

Solution:

Calculation of Arithmetic Mean

Size of shoes	*No. of pairs*	*dx = (x – A)*	*fdx*
4.5	1	–3.5	–3.5
5	2	–3	–6
5.5	4	–2.5	–10
6	5	–2	–10
6.5	15	–1.5	–22.5
7	30	–1	–30
7.5	60	–0.5	–30
8 (A)	82	0	0

NOTES

8.5	82	+0.5	+41
9	75	+1	+75
9.5	44	+1.5	+66
10	25	+2	+50
10.5	15	+2.5	+37.5
11.0	4	+3	+12
	$N = 444$		$\sum fdx = 169.5$

$$\overline{X} = A + \frac{\sum fdx}{N} = 8 + \frac{169.5}{444} = 8 + 0.38 = 8.38$$

The average size of shoes sold is 8.4 inches.

Step Deviation Method

Formula: $\overline{X} = A + \frac{\sum fdx'}{N} \times c$

The following steps are involved in computing mean under step deviation method:

1. Find out the mid-value of each group or class.
2. Assume one of the mid-values as an average.
3. Find out the deviation of each mid-value from the assumed average in terms of class interval.
4. Multiplying the deviation of each class by its frequency.
5. Add up the products.
6. Apply the following formula

$$\overline{X} = A + \frac{\sum fdx'}{N} \times c$$

where $\overline{X}$ = Arithmetic mean

A = Assumed average

$\sum fdx'$ = Sum of the deviations in terms of class interval

N = Total frequency

c = Class interval

Discrete Series

Example 24:

Following of the marks obtained by students of a class in statistics. Calculate arithmetic mean.

NOTES

Marks	*No. of students*
35	3
40	8
45	12
50	9
55	4
60	7
65	15
70	5
75	10
80	7
85	5
90	3
95	2

Solution:

Calculation of Arithmetic Mean

x	f	$dx' = \frac{x - A}{c} (c = 5)$	fdx'
35	3	–30/5 = –6	–18
40	8	–25/5 = –5	–40
45	12	–20/5 = –4	–48
50	9	–15/5 = –3	–27
55	4	–10/5 = –2	–8
60	7	–5/5 = –1	–7
65	15	0/5 = 0	0
70	5	5/5 = 1	5
75	10	10/5 = 2	20
80	7	15/5 = 3	21
85	5	20/5 = 4	20
90	3	25/5 = 5	15
95	2	30/5 = 6	12
	$N = 90$		$\sum fdx' = \mathbf{-55}$

$$\overline{X} = A + \frac{\sum fdx'}{N} \times c = 65 + \frac{-55}{90} \times 5 = 65 + \frac{-275}{90} = 65 - 3.05$$

$= 61.95$ marks

NOTES

Continuous Series

In continuous series, variables are represented by class interval. Each class intervals has its own frequency. Midpoint (class marks) of each class should be ascertained first. Then the procedure of finding the arithmetic mean is the same as used in the discrete series.

Exclusive Class Interval

Direct Method

Example 25:

From the following figures, find the mean using indirect method.

Marks	*No. of persons*
0-10	5
10-20	10
20-30	20
30-40	40
40-50	30
50-60	20
60-70	10
70-80	4

Solution:

Calculation of Mean

Marks	*x*	*f*	$dx' = (x - A)$	fdx'
0-10	5	5	–40	–200
10-20	15	10	–30	–300
20-30	25	20	–20	–400
30-40	35	40	–10	–400
40-50	45 (A)	30	0	0
50-60	55	20	+10	+200
60-70	65	10	+20	+200
70-80	75	4	+30	+120
		$N = 139$		$\sum fdx' = -780$

$$\overline{X} = A + \frac{\sum fdx'}{N} \times c = 45 + \left(-\frac{780}{139}\right) = 45 + (-5.6) = 39.4 \text{ marks}$$

Example 26:

Calculate Arithmetic Mean from the following data:

Production in tonnes	*No. of factories*
10-20	5
20-30	4
30-40	7
40-50	12
50-60	10
60-70	8
70-80	4

NOTES

Solution:

Calculation of Arithmetic Mean

Production in tonnes (x)	*No. of factories (f)*	*Midpoint (x)* (*LL* + *UL*)/2	*fx*
10-20	5	(10 + 20)/2 = 15	(5 × 15) = 75
20-30	4	(20 + 30)/2 = 25	(4 × 25) = 100
30-40	7	(30 + 40)/2 = 35	(7 × 35) = 245
40-50	12	(40 + 50)/2 = 45(A)	(12 × 45) = 540
50-60	10	(50 + 60)/2 = 55	(10 × 55) = 550
60-70	8	(60 + 70)/2 = 65	(8 × 65) = 520
70-80	4	(70 + 80)/2 = 75	(4 × 75) = 300
	N = 50		$\sum fx$ = 2,330

$$\overline{X} = \frac{\sum fx}{N} = \frac{2{,}330}{50} = 46.6 \text{ tonnes}$$

Example 27:

Calculate Arithmetic Mean from the following data:

Production in tonnes	*No. of factories*
10-20	5
20-30	4
30-40	7
40-50	12
50-60	10
60-70	8
70-80	4

NOTES

Solution:

Calculation of Arithmetic Mean

Production in tonnes (x)	*f*	*Midpoint (x) (LL + UL)/2*	$dx' = x - A / c$ *(A = 45) c = 10*	*fdx'*
10-20	5	(10 + 20)/2 = 15	–30/10 = –3	(5 × –3) = –15
20-30	4	(20 + 30)/2 = 25	–20/10 = –2	(4 × –2) = –8
30-40	7	(30 + 40)/2 = 35	–10/10 = –1	(7 × –1) = –7
40-50	12	(40 + 50)/2 = 45(A)	0/10 = 0	(12 × 0) = 0
50-60	10	(50 + 60)/2 = 55	10/10 = 1	(10 × 1) = 10
60-70	8	(60 + 70)/2 = 65	20/10 = 2	(8 × 2) = 16
70-80	4	(70 + 80)/2 = 75	30/10 = 3	(4 × 3) = 12
	$N = 50$			$\sum fdx' = 8$

$$\overline{X} = A + \frac{\sum fdx'}{N} \times c = 45 + \frac{8}{50} \times 10 = 45 + \frac{80}{50} = 45 + 1.6$$

$$= 46.6 \text{ tonnes}$$

Example 28:

Calculate the average marks from the following distribution:

Marks	*No. of students*
0-5	2
5-10	4
10-15	5
15-20	3
20-25	2
25-30	4
30-35	5

Solution:

Marks	*No. of students (CI)*	*Mid-value (f)*	$dx' = x - A / 5$ *(x)*	*fdx'*
0-5	2	2.5	–3	–6
5-10	4	7.5	–2	–8
10-15	5	12.5	–1	–5
15-20	3	17.5 (A)	0	0
20-25	2	22.5	+1	+2
25-30	4	27.5	+2	+8
30-35	5	32.5	+3	+15
	$N = 25$		$\sum dx' = 0$	$\sum fdx' = 6$

Using step deviation formula, Assumed Mean (A) = 17.5

Common factor (c) is 5

$$\overline{X} = A + \frac{\Sigma fdx'}{N} \times c = 17.5 + \frac{6}{25} \times 5 = 17.5 + \frac{30}{25} = 17.5 + 1.2$$

$$= 18.7$$

NOTES

Inclusive Class Interval

Step Deviation Method

Example 29:

The following are the 270 soybean plant heights collected from a particular plot. Find the arithmetic mean of the plants

Plant height (cms)	*No. of plants*
8-12	8
13-17	17
18-22	20
23-27	50
28-32	75
33-37	30
38-42	25
43-47	40
48-52	5

When the length of class is equal, step deviation method is appropriate and when the length of the class is unequal, direct method is appropriate. In this problem, the length of class is equal, i.e., $c = 5$ whether inclusive or exclusive given continuous class converted in to midpoint(x) for calculating arithmetic mean.

Solution:

Calculation of Arithmetic Mean

Plant height (cms) (x)	*No. of plants (f)*	*Midpoint (x) (LL + UL)/2*	$dx' = x - A / c$ $A = 30 \quad i = 5$	$(fx \times dx) = fdx$
8-12	8	(8 + 12)/2 = 10	(10 – 30/5 = –4	(8 × –4) = –32
13-17	17	(13 + 17)/2 = 15	(15 – 30/5) = –3	(17 × –3) = –51
18-22	20	(18 + 22)/2 = 20	(20 – 30/5) = –2	(20 × –2) = –40
23-27	50	(23 + 27)/2 = 25	(25 – 30/5) = –1	(50 × –1) = –50
28-32	75	(28 + 32)/2 = 30(A)	(30 – 30/5) = 0	(75 × 0) = 0
33-37	30	(33 + 37)/2 = 35	(35 – 30/5) = 1	(30 × 1) = 30
38-42	25	(38 + 42)/2 = 40	(40 – 30/5) = 2	(25 × 2) = 50

43-47	40	(43 + 47)/2 = 45	(45 – 30/5) = 3	(40 × 3) = 120
48-52	5	(48 + 52)/2 = 50	(50 – 30/5) = 4	(5 × 4) = 20
	$N = 270$			$\Sigma fdx = 47$

NOTES

$$\overline{X} = A + \frac{\Sigma fdx'}{N} \times c = 30 + \left(\frac{47}{270}\right)5 = 30 + \frac{235}{270} = 30 + 0.870$$

$$= 30.87 \text{ or } 30.9 \text{ cms}$$

Arithmetic Mean for Open-end Classes

When the lower limit of the first class interval and upper limit of the last class interval are not known, it is called open-end classes.

Types of Open-end Classes

Open-end classes are of two types. They are:

(i) Less than open-end class: Given variable (x) must have upper class limit of the continuous series either in ascending or in descending order and given frequency must be in cumulative nature. (Upper class begins with the words bel u or upto or not exceeding or less than).

Procedure:

- *Convert in to equal class interval:* Subtract the class length (*CL*) from the upper limit (*UL*), you will get the lower limit (*LL*) of the first class. Similarly, add the same class length to the lower limit of the last class, you will get the upper limit of the last class. But always notice that the lower limit of the first class (i.e., the lowest class) must not be negative or less than 0. Hence, lower limit = upper limit – class length. For example, class length is 10 (30 – 20), the lower limit (20 – 10 = 10) and then the upper limit of the first class becomes the lower limit of the next class.
- *Convert the cumulative frequencies* into frequencies of the respective classes.

(ii) More than open-ended class: Given variable (x) must have lower class limit of the continuous series either in ascending or in descending order and given frequency must be in cumulative in nature. (Lower class begins with the words above or more than or over.)

Procedure:

- *Lower limit + Class length = Upper limit* (i.e., length of class 30 – 20 = 10. *UL* = *LL* + *CL*. In this problem, the lower limit is 20, 30 and so on. The lower limit of the next must be the upper limit of the previous class. The last class lower limit + class length = upper limit, i.e., 90 + 10 = 100).
- *Convert the cumulative frequencies* into frequencies of the respective classes.

Example 30:

Calculate AM for the following data open-end class:

Marks	Below 20	30	40	50	60	70	80	90	100
No. of students	10	18	25	32	43	61	67	85	100

NOTES

Solution:

Marks (x)	*f*	*Class*	*f*	*Mid (x)* $(UL + LL)/2$	$dx' = x - A/c$, $c = 10$	fdx'
Below 20	10	10-20	10	15	–4	–40
" 30	18	20-30	8	25	–3	–24
" 40	25	30-40	7	35	–2	–14
" 50	32	40-50	7	45	–1	–7
" 60	43	50-60	11	55(A)	0	0
" 70	61	60-70	18	65	+1	18
" 80	67	70-80	6	75	+2	12
" 90	85	80-90	18	85	+3	54
" 100	100	90-100	15	95	+4	60
$N = 100$						$\sum fdx' = 59$

$$\overline{X} = A + \frac{\sum fdx'}{N} \times c = 55 + \left(\frac{59}{100}\right) \times 10 = 55 + \frac{590}{100} = 55 + 5.9$$

$= 60.9$ marks

Example 31:

Calculate *AM* for the following data:

Marks Above	20	30	40	50	60	70	80	90
No. of students	100	95	87	62	43	25	13	2

Solution:

Marks (x)	*f*	*Class*	*f*	*Mid (x)* $(UL + LL)/2$	$dx' = x - A/c$, $c = 10$	fdx'
Above 20	100	20-30	5	25	–3	–15
" 30	95	30-40	8	35	–2	–16
" 40	87	40-50	25	45	–1	–25
" 50	62	50-60	19	55(A)	0	0
" 60	43	60-70	18	65	+1	18
" 70	25	70-80	12	75	+2	24
" 80	13	80-90	11	85	+3	33
" 90	2	90-100	2	95	+4	8
$N = 100$						$\sum fdx' = 27$

$$\overline{X} = A + \frac{\Sigma fdx'}{N} \times c = 55 + \left(\frac{27}{100}\right) \times 10 = 55 + \frac{270}{100} = 55 + 2.7$$

$= 57.7$

NOTES

Miscellaneous Problems

Example 32:

The average rainfall in a city from Monday to Saturday is 0.3 inches. Due to heavy rainfall on Sunday, the average rainfall from Monday to Sunday is 0.5 inches. Find the rainfall on Sunday.

Solution:

Average rainfall from Monday to Saturday (6 days)

Rainfall = 0.3" average rain for the week 0.5"

Total rainfall for the week from Monday to Sunday	= 7 days (7 × 0.5") = 3.5 inches
Less: Total rainfall from Monday to Saturday 6 days	= (6 × 0.3") = 1.8 inches
Rainfall on Sunday	**= 1.7 inches**

Example 33:

In a class of 50 students, 10 have failed and their average mark is 2.5. The total marks secured by the entire class were 281. Find the average marks of pass students.

Solution:

Total marks of the entire class	50 students	= 281 marks
Less: Total marks of failed students	10 (10 × 2.5)	= 25 marks
Total marks of passed students	40	= 256 marks
Average marks of pass students	$\frac{256}{40}$	= 6.4 marks

Example 34:

Rainfall of a month was observed and recorded as follows:

Sunday 87 mm, Monday 63 mm, Tuesday 74 mm, Wednesday 56 mm, Thursday 62 mm, Friday 85 mm, Saturday 73 mm.

There are 5 Monday, Tuesday and Wednesday, and 4 Sunday, Thursday, Friday and Saturday during the month. What was the daily average rainfall during the month?

Solution:

Days (*f*)	*No. of days* (*x*)	*Rainfall (in mm)*	*fx*
Mon	5	63	315
Tue	5	74	370
Wed	5	56	280
Thu	4	62	248
Fri	4	85	340
Sat	4	73	292
Sun	4	87	348
	$N = 31$		$\sum fx = 2193$

NOTES

$$\overline{X} = \frac{\Sigma fx}{N} = \frac{2193}{31} = 70.74$$

The daily average rainfall during the month was 70.74.

Example 35:

The mean height of 50 students is 5 feet 8 inches, the height of 10 students are given below; find the average height of the remaining students

Heights:

5'6" 5'2" 5'4" 5'0" 4'10" 4'8" 6'2" 5'8" 5'9" 5'3"

Solution:

Given data in both feet and inches, it should be converted into either feet or inches here, data converted in to inches (1 feet = 12 inches).

Heights	*Converted into inches*	*Inches*
5'6"	$(5 \times 12) + 6$	66"
5'2"	$(5 \times 12) + 2$	62"
5'4"	$(5 \times 12) + 4$	64"
5'0"	$(5 \times 12) + 0$	60"
4'10"	$(4 \times 12) + 10$	58"
4'8"	$(4 \times 12) + 8$	56"
6'2"	$(6 \times 12) + 2$	74"
5'8"	$(5 \times 12) + 8$	68"
5'9"	$(5 \times 12) + 9$	69"
5'3"	$(5 \times 12) + 3$	63"

Total students 50

Average height (5'8") or 68"

NOTES

Total height of all 50 students (50 × 68")	= 3400"
Less: Total height of 10 students	= 640"
Total height of 40 students	= 2760"

The average height of remaining 40 students $= \frac{2760}{40}$ = 5'9"

Example 36:

Mr. Nanda has appeared for three test of values 20, 50 and 30 marks respectively. In the first test, he scored 75%, and in the second test, he scored 60%. In the entire test, together, he scored an aggregate of 60% marks. Find the percentage of marks in the third test.

Solution:

First test appeared for 20 marks and he scored 75% in first test

(20 × 75%) = 15 marks

Second test appeared for 50 marks and he scored 60% in second test

(50 × 60%) = 30 marks

The entire tests appeared for 100 marks and the aggregate score is 60%

(100 × 60%) = 60 marks

Third test appeared for 30 marks and he scored in third test

(60 – 30 + 15) = 15 marks

$$\text{Third test per cent} = \frac{15}{30} \times 100 = 50\%$$

Example 37:

Karnataka Government decided to declare old-age monthly pension to the pension aged 65 and above. Calculate total old age pension (OAP) payable and its average.

Age in years of 30 persons: 65, 68, 91, 93, 73, 85, 69, 73, 74, 78, 83, 93, 74, 73, 78, 69, 70, 71, 83, 84, 94, 95, 98, 99, 93, 69, 82, 84, 83, 87

Age	:	60-70	70-80	80-90	90-100
OAP	:	200	300	400	500

Solution:

Age	*OAP* *(x)*	*Tally Bar* *(f)*	*No. of persons*	*fx*
60-70	200	*1111*	5	1,000
70-80	300	*1111 1111*	9	2,700
80-90	400	*1111 111*	8	3,200
90-100	500	*1111 111*	8	4,000
		$N = 30$		$\sum fx = 10,900$

Total old age pension payable = ₹ 10,900

Average old age pension payable = $\overline{X} = \frac{\Sigma fx}{N} = \frac{10900}{30} = 363.33$

NOTES

Example 38:

In a factory, there are 100 skilled, 250 semi-skilled and 150 unskilled workers. It has been observed that, on average, a unit length of a particular fabric is woven by a skilled worker in 3 hours, by a semiskilled worker in 4 hours and by an unskilled worker in 5 hours. After a training of 2 years, the semi-skilled workers are expected to become skilled and unskilled workers to become semi-skilled. How much less time will be required after 2 years of training for weaving the unit length of fabric by an average worker?

Solution:

Skilled $N_1 = 100 \ \bar{X}_1 = 3$

$$\bar{X}_{123} = \frac{N_1.\bar{X}_1 + N_2.\bar{X}_2 + N_3.\bar{X}_3}{N_1 + N_2 + N_3}$$

Before training:

Semi-skilled $N_2 = 250, \ \bar{X}_2 = 4$

$$\bar{X}_{123} = \frac{100(3) + 250(4) + 150(5)}{100 + 250 + 150}$$

Unskilled $N_3 = 150, \ \bar{X}_5 = 5$

$$\bar{X}_{123} = \frac{300 + 1000 + 750}{500} = \frac{2050}{500} = 4.1 \text{ hours}$$

After training:

Skilled $N_1 = 100 + 250 = 350, \ \bar{X} = 3$

$$\bar{X}_{12} = \frac{N_1.\bar{X}_1 + N_2.\bar{X}_2}{N_1 + N_{23}}$$

Semi-skilled $N_2 = 150, \ \bar{X}_2 = 4$

$$\bar{X}_{123} = \frac{350(3) + 150(4)}{350 + 150} = \frac{1050 + 600}{500} = \frac{1650}{500} = 3.3 \text{ hours}$$

The less time taken after 2 years training is (4.1 – 3.3) = 0.8 hours or (.8 × 60) = 48 minutes.

Example 39:

Find the *CI* if the *AM* 30.1 and assumed mean is 31.5 of the following data:

Class	:	7-14	14-21	21-28	28-35	35-42	42-49
f	:	5	10	25	30	20	10

NOTES

Solution:

CI	*Distribution Step Deviation (x)*	*f*	*fdx*
7-14	–3	5	–15
14-21	–2	10	–20
21-28	–1	25	–25
28-35	0	30	0
35-42	1	20	20
42-49	2	10	20
		$N = 100$	$\sum fdx = -20$

$$\overline{X} = A + \left(\frac{\sum fdx}{N}\right)$$

$$30.1 = 31.5 + \left(\frac{-20}{100}\right)$$

$$30.1 - 31.5 = \left(\frac{-20c}{100}\right)$$

$$-1.4 = \left(\frac{-20c}{100}\right)$$

Cross multiplying $-1.4 \times 100 = -20c$, we get

$$c = \left(\frac{140}{20}\right) = 7$$

$$\text{Lower limit} = A - \frac{1}{2}c = 31.5 - \frac{1}{2} \times 7 = 28$$

$$\text{Upper limit} = A + \frac{1}{2}c = 31.5 + \frac{1}{2} \times 7 = 35$$

5.5 HARMONIC MEAN

Harmonic Mean

Harmonic Mean '*HM*' between two numbers *a* and *b* is such a number that $1/HM - 1/a = 1/b - 1/HM$. Thus, if we are given these two numbers, the harmonic mean

$$HM = 2ab/a + b$$

Properties of Harmonic Mean

- If all the observations taken by a variable are constants, say *k*, then the harmonic mean of the observations is also *k*.

- The harmonic mean has the least value when compared to the geometric mean and the arithmetic mean.

Advantages of Harmonic Mean

- A harmonic mean is rigidly defined.
- It is based upon all the observations.
- The fluctuations of the observations do not affect the harmonic mean.
- More weight is given to smaller items.

Disadvantages of Harmonic Mean

- Not easily understandable.
- Difficult to compute.

Calculation of Harmonic Mean for Individual Series:

Formula: $H = \dfrac{n}{\sum \frac{1}{x}}$

Example 40:

Calculate the harmonic mean of 6, 8, 12 and 16.

Solution:

x	$\frac{1}{x}$
6	0.167
8	0.125
12	0.083
16	0.063
Total	**0.438**

Therefore, $H = \dfrac{n}{\sum \frac{1}{x}}$

$= \dfrac{4}{0.438}$

$= 9.1324$

NOTES

Example 41:

Find out harmonic mean for the following data: 14, 16, 87, 0.06, 98, 46.

Solution:

x	$\frac{1}{x}$
14	0.0714
16	0.0625
87	0.0115
0.06	16.6667
98	0.0102
46	0.0217
Total	**16.8440**

$$\text{Therefore, } H = \frac{n}{\sum \frac{1}{x}}$$

$$= \frac{6}{16.844}$$

$$= 0.3562$$

Example 42:

The profits of ABC Ltd. for the last five years are 20%, 25%, 28%, 30% and 26% respectively. Find out the average profit of the concern.

Solution:

x	$\frac{1}{x}$
20	0.05
25	0.04
28	0.035714
30	0.033333
26	0.038462
Total	**0.197509**

$$\text{Therefore, } H = \frac{n}{\sum \frac{1}{x}}$$

$$= \frac{5}{0.197509}$$

$$= 25.35\%$$

Computation of Harmonic Mean for Discrete and Continuous Series

NOTES

Formula: $H = \dfrac{N}{\sum \dfrac{f}{x}}$

Example 43:

From the following information, calculate Harmonic mean:

Marks	*No. of students*
20	12
40	18
60	25
80	35

Solution:

x	20	40	60	80	**Total**
f	12	18	25	35	$N = 90$
$\dfrac{f}{x}$	0.6	0.45	0.42	0.44	$\sum \dfrac{f}{x} = 1.904167$

$$\text{Therefore, } H = \frac{N}{\sum \frac{f}{x}} = \frac{90}{1.904167} = 47.26477$$

Example 44:

The runs scored by a batsman in various matches are given below. Calculate the mean runs scored by the batsman by using Harmonic mean

Marks	*No. of students*
20	30
50	45
75	54
90	36
120	8
150	7

Solution:

x	20	50	75	90	120	150	**Total**
f	30	45	54	36	8	7	$N = 180$
$\dfrac{f}{x}$	1.5	0.9	0.72	0.4	0.066667	0.046667	$\sum \dfrac{f}{x} = 3.633333$

$$\text{Therefore, } H = \frac{N}{\sum \frac{f}{x}} = \frac{180}{3.6333} = 49.54$$

NOTES

Example 45:

A profit earned by 100 BSE listed companies is given below. Find out the harmonic mean.

Profit ('000 ₹)	*No. of companies*
Below 10	6
Below 20	18
Below 30	36
Below 40	44
Below 50	52
Below 60	63
Below 70	77
Below 80	84
Below 90	100

Solution:

Profits	*No. of ('000 ₹)*	*Class companies*	*f interval*	*Mid- point*	*f/x (x)*
Below 10	6	0-10	6	5	1.2
Below 20	18	20-30	12	15	0.8
Below 30	36	20-30	18	25	0.72
Below 40	44	30-40	8	35	0.228571
Below 50	52	40-50	8	45	0.177778
Below 60	63	50-60	11	55	0.2
Below 70	77	60-70	14	65	0.215385
Below 80	84	70-80	7	75	0.093333
Below 90	100	80-90	16	85	0.188235
	Total		**100**		**3.823302**

$$\text{Therefore, } H = \frac{N}{\sum \frac{f}{x}} = \frac{100}{3.8233} = 26.155$$

Check Your Progress

3. Calculate the arithmetic average of the data given below using short-cut method.

Roll No.	*Marks obtained*
1	43
2	48
3	65
4	57
5	31
6	60
7	37
8	48
9	78
10	59

4. Following are the marks obtained by students of a class in statistics. Calculate arithmetic mean.

Marks	*No. of students*
35	3
40	8
45	12
50	9
55	4
60	7
65	15
70	5
75	10
80	7
85	5
90	3
95	2

NOTES

5.6 GEOMETRIC MEAN

NOTES

A geometric mean is a mean or average which shows the central tendency of a set of numbers by using the product of their values. For a set of n observations, a geometric mean is the nth root of their product. The geometric mean (GM) for a set of numbers $x_1, x_2, \dots, x_n$ is given as:

$$GM = (x_1, x_2, \dots, x_n)1/n$$

Properties of the Geometric Mean

The main properties of the geometric mean are:

(a) The geometric mean is less than the arithmetic mean, $GM < AM$.

(b) The product of the items remains unchanged if each item is replaced by the geometric mean.

(c) The geometric mean of the ratio of corresponding observations in two series is equal to the ratios of their geometric means.

(d) The geometric mean of the products of corresponding items in two series is equal to the product of their geometric mean.

Advantages of Geometric Mean

(a) A geometric mean is based upon all the observations.

(b) It is rigidly defined.

(c) The fluctuations of the observations do not affect the geometric mean.

(d) It gives more weight to small items.

Disadvantages of Geometric Mean

(a) A geometric mean is not easily understandable by a non-mathematical person.

(b) If any of the observations is zero, the geometric mean becomes zero.

(c) If any of the observations is negative, the geometric mean becomes imaginary.

Calculation of Geometric Mean for Individual Series

Formula: $GM = \sqrt[n]{x_1 x_2 x_3 x_4 \ldots\ldots\ x_n}$ or

$$GM = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$$

Example 46:

Calculate Geometric Mean for the following data: 45, 65, 76, 89, 104, 129, 86, 216.

NOTES

Solution:

x	$\log x$
45	1.653
65	1.813
76	1.881
89	1.949
104	2.017
129	2.111
86	1.934
216	2.334
Total	**15.693**

$$GM = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$$

$$= \text{Antilog}\left(\frac{\Sigma \log 15.693}{8}\right)$$

$$= \text{Antilog}(1.9616)$$

$$= 91.54$$

Example 47:

From the following data, find out Geometric Mean: 102, 23, 59, 154, 27, 6, 88.

Solution:

x	$\log x$
102	2.0086
23	1.3617
59	1.7709
154	2.1875
28	1.4472
28	0.7782
88	1.9445
Total	**11.4985**

$$GM = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$$

$$= \text{Antilog}\left(\frac{\Sigma \log 11.4985}{7}\right)$$

$$= \text{Antilog}(1.6426)$$

NOTES

$$= 43.918$$

Example 48:

A deposit of ₹ 1,00,000 grows at the rate of 5%, 6.2%, 6.8% and 7% in four subsequent years. Find the average annual growth rate.

Solution:

Year	*Growth rate of deposit*	*Value of deposit at the end of the year (x)*	*log x*
1	5%	105	2.021189
2	6.2%	106.2	2.026125
3	6.8%	106.8	2.028571
4	7%	107	2.029384
Total			**8.105269**

$$\text{GM} = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$$

$$= \text{Antilog}\left(\frac{\Sigma \log 8.1053}{4}\right)$$

$$= \text{Antilog}(2.0263)$$

$$= 106.24$$

Hence, the average growth rate of deposit is (106.24 – 100) = 6.24%.

Example 49:

Find the average growth rate of population which in the first year grows at 4%, second year at 5%, third year at 5.5%, fourth year at 6% and in the fifth year at 6.32%.

Solution:

Year	*Growth rate of population*	*Value of deposit at the end of the year (x)*	*log x*
1	4%	104	2.017033
2	5%	105	2.021189
3	5.5%	105.5	2.023252
4	6%	106	2.025306
5	6.32%	106.32	2.026615
Total			**10.1134**

$$GM = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$$

$$= \text{Antilog}\left(\frac{\Sigma \log 10.1134}{5}\right)$$

$$= \text{Antilog}(2.022679)$$

$$= 105.36$$

Hence, the average growth rate of population is (105.36 – 100) = 5.36%.

NOTES

Calculation of Geometric Mean for Discrete and Continuous Series

Formula: $GM = \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$

Example 50:

Calculate Geometric Mean for the following data:

x	f
102	6
23	9
59	8
154	12
28	16
6	9
88	15
45	16
34	18
78	23

Solution:

x	f	$\log x$	$f \log x$
102	6	2.0086	12.0516
23	9	1.361728	12.25555
59	8	1.770852	14.16682
154	12	2.187521	26.25025
28	16	1.447158	23.15453
6	9	0.778151	7.003361
88	15	1.944483	29.16724
45	16	1.653213	26.4514

34	18	1.531479	27.56662
78	23	1.892095	43.51818
	$\sum f = 132$		$\sum f \log x = 221.5855$

NOTES

$$GM = \text{Antilog}\left(\frac{\sum f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{\sum 221.5855}{132}\right)$$

$$= \text{Antilog}(1.678678)$$

$$= 47.7175$$

Example 51:

Calculate Geometric Mean for the following data:

x	f
89	16
4	19
119	18
203	21
208	16
67	19
58	15
45	6
43	8
108	32

Solution:

X	f	$log\ x$	$f\ log\ x$
89	16	1.94939	31.19024
4	19	0.60206	11.43914
119	18	2.075547	37.35985
203	21	2.307496	48.45742
208	16	2.318063	37.08901
67	19	1.826075	34.69542
58	15	1.763428	26.45142
45	6	1.653213	9.919275
43	8	1.633468	13.06775
108	32	2.033424	65.06956
	$\sum f = 170$		$\sum f \log x = 314.74$

$$GM = \text{Antilog}\left(\frac{\sum f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{\sum 314.74}{170}\right)$$

$= \text{Antilog}(1.8514)$
$= 71.025$

NOTES

Example 52:

Calculate Geometric Mean for the following distribution:

x	*f*
0-10	18
10-20	9
20-30	8
30-40	14
40-50	18
50-60	24
60-70	6
70-80	11
80-90	19

Solution:

Class interval	*x*	*f*	*log x*	*f log x*
0-10	5	18	0.69897	12.58146
20-30	15	9	1.176091	10.58482
20-30	25	8	1.39794	11.18352
30-40	35	14	1.544068	21.61695
40-50	45	18	1.653213	29.75783
50-60	55	24	1.740363	41.7687
60-70	65	6	1.812913	10.87748
70-80	75	11	1.875061	20.62567
80-90	85	19	1.929419	36.65896
		$\sum f = 127$		$\sum f \log x = 195.6554$

$$GM = \text{Antilog}\left(\frac{\sum f \log x}{N}\right)$$

NOTES

$$= \text{Antilog}\left(\frac{\sum 195.6554}{127}\right)$$

$$= \text{Antilog}(1.5406)$$

$$= 34.72$$

Example 53:

Calculate Geometric Mean from the following:

Marks	*No. of students*
0-10	16
10-20	7
20-30	12
30-40	18
40-50	18
50-60	18
60-70	12
70-80	15
80-90	14

Solution:

Class Interval	*x*	*f*	*log x*	*f log x*
0-10	5	16	0.69897	11.18352
10-20	15	7	1.176091	8.232639
20-30	25	12	1.39794	16.77528
30-40	35	18	1.544068	27.79322
40-50	45	18	1.653213	29.75783
50-60	55	18	1.740363	31.32653
60-70	65	12	1.812913	21.75496
70-80	75	15	1.875061	28.12592
80-90	85	14	1.929419	27.01186
		$\sum f = 130$		$\sum f \log x = 201.9618$

$$GM = \text{Antilog}\left(\frac{\sum f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{\sum 201.9618}{130}\right)$$

$$= \text{Antilog}(1.5536)$$

$$= 35.77$$

NOTES

5.7 PROFIT AND LOSS

Cost Price (*CP*): This is the price at which an article is purchased.

Selling Price (*SP*): This is the price at which an article is sold.

Profit or Gain: If the selling price is more than the cost price, the difference between them is the profit incurred.

Profit or Gain = Selling Price – Cost Price

Loss: If the selling price is less than the cost price, the difference between them is the loss incurred.

Loss = Cost Price – Selling Price.

Profit or Loss is always calculated on the cost price.

Marked Price: This is the price marked as the selling price on an article, also known as the listed price.

Discount or Rebate: This is the reduction in price offered on the marked or listed price.

The basic formulas used in profit and loss:

- Gain % = (Gain / *CP*) * 100
- Loss % = (Loss / *CP*) * 100
- *SP* = [(100 + Gain%) / 100] * *CP*
- *SP* = [(100 – Loss %) / 100] * *CP*

The above two formulas can be stated as:

If an article is sold at a gain of 10%, then *SP* = 110% of *CP*.

If an article is sold at a loss of 10%, then *SP* = 90% of *CP*.

- *CP* = [100 / (100 + Gain%)] * *SP*
- *CP* = [100 / (100 – Loss%)] * *SP*

Example 54:

An article is purchased for ₹ 550 and sold for ₹ 600. Find the gain per cent.

Solution:

Gain = *SP* – *CP* = 600 – 550 = 50

$$\text{Gain in \%} = \left(\frac{50}{550} \times 100\right) = 9\%$$

NOTES

Example 55:

A man sold a fan for ₹ 750. Find the cost price if he incurred a loss of 7%.

Solution:

$$CP = \left[\frac{100}{100 - Loss\ \%} \right] \times SP$$

Therefore, the cost price of the fan = $\left(\frac{100}{93} \right) \times 750 = 806$

Example 56:

In a transaction, the profit percentage is 80% of the cost. If the cost further increases by 20% but the selling price remains the same, how much is the decrease in profit percentage?

Solution:

Let's assume CP = ₹ 100.

Then Profit = ₹ 80 and SP = ₹ 180.

The cost increases by 20% $\rightarrow$ New CP = ₹ 120, SP = ₹ 180.

Profit % = $\left(\frac{60}{120} \times 100 \right) = 50\%$

Therefore, profit decreases by 30%.

Example 57:

A man bought 20 toys for ₹ 400 and sold 15 of them for ₹ 350. Find his gain or loss per cent.

Solution:

CP of 20 toys = ₹ 400 ?

CP of 1 toy = ₹ 20

SP of 15 toys = ₹ 350 $\rightarrow$ SP of 1 toy = ₹ 23.33

Therefore, Gain = 23.33 – 20 = 3.33

Gain per cent = $\left(\frac{3.33}{20} \times 100 \right) = 16.65\%$

Example 58:

The cost price of 10 pens is the same as the selling price of n pens. If there is a loss of 40%, approximately what is the value of n?

Solution:

Let the price of each pen be ₹ 1.

Then the cost price of n pens is ₹ n and the selling price of n pens is ₹ 10.

Loss = $n - 10$.

Loss of 40% $\rightarrow$ (Loss/CP) $\times$ 100 = 40

Therefore, $\left[\frac{n - 10}{n}\right] \times 100 = 40 \rightarrow n = 17$ (approx.)

NOTES

Percentage

"Per cent" comes from the Latin *Per Centum*. The Latin word Ce*ntum* means 100. For example, a Century is 100 years.

My Dictionary says "Percentage" is "the result obtained by multiplying a quantity by a per cent". So, 10 per cent of 50 apples is 5 apples; 5 apples is the percentage.

But in practice, people use both words the same way.

Meaning of Percentage

In mathematics, a percentage is a number or ratio that represents a fraction of 100. It is often denoted by the symbol "%" or simply as "per cent" or "pct." For example, 35% is equivalent to the decimal 0.35, or the fraction $\frac{35}{100}$.

Percentage Formula

Although the percentage formula can be written in different forms, it is essentially an algebraic equation involving three values.

$$P \times V_1 = V_2$$

where, P is the percentage, V_1 is the first value that the percentage will modify, and V_2 is the result of the percentage operating on V_1. The calculator provided automatically converts the input percentage into a decimal to compute the solution. However, if solving for the percentage, the value written will be the actual percentage, not its decimal representation.

Example: $P \times 30 = 1.5$

$$P = \frac{1.5}{30} = 0.05 \times 100 = 5\%$$

Percentage Difference Formula

The percentage difference between two values is calculated by dividing the absolute value of the difference between two numbers by the average of those two numbers. Multiplying the result by 100 will yield the solution in per cent, rather than decimal form.

$$\text{Percentage Difference} = \frac{|V_1 - V_2|}{(V_1 + V_2)/2} \times 100$$

Example: $\frac{|12 - 8|}{(12 + 8) / 2} = \frac{4}{10} = 0.4 = 40\%$

NOTES

Percentage Change Formula

Percentage increase and decrease are calculated by computing the difference between two values and comparing that difference to the initial value. Mathematically, this involves using the absolute value of the difference between two values, and dividing the result by the initial value, essentially calculating how much the initial value has changed.

$$\text{Percentage change} = \frac{|\text{New value} - \text{Initial value}|}{\text{Initial value}} \times 100$$

The percentage increase calculator above computes an increase or decrease of a specific percentage of the input number. It basically involves converting a per cent into its decimal equivalent, and either subtracting or adding the decimal equivalent from and to 1, respectively. Multiplying the original number by this value will result in either an increase or decrease of the number by the given per cent.

Example: 600 increased by 10% (0.1)

$600 \times (1 + 0.1) = 660$

600 decreased by 10%

$600 \times (1 - 0.1) = 540$

Example 59:

Calculate 25% of 90.

Solution:

$$25\% = \frac{25}{100}$$

and $\frac{25}{100} \times 90 = 22.5$

So, 25% of 90 is 22.5.

Example 60:

15% of 300 oranges are bad. How many oranges are bad?

Solution:

$$15\% = \frac{15}{100}$$

and $\frac{15}{100} \times 300 = 45$

So, 45 oranges are bad.

Example 61:

A whiteboard is reduced 20% in price in a sale. The old price was ₹ 150. Find the new price.

Solution:

NOTES

First, find 20% of ₹ 150:

$$20\% = \frac{20}{100}$$

$$\text{and } \frac{20}{100} \times 150 = 30$$

20% of 150 is 30.

So, the reduction is 30 from original price, i.e., (150 – 30) = 120.

Check Your Progress

5. Calculate Geometric Mean for the following data:

x	f
102	6
23	9
59	8
154	12
28	16
6	9
88	15
45	16
34	18
78	23

6. Calculate Geometric Mean for the following data:

x	f
89	16
4	19
119	18
203	21
208	16
67	19
58	15

NOTES

45	6
43	8
108	32

7. In a transaction, the profit percentage is 80% of the cost. If the cost further increases by 20% but the selling price remains the same, how much is the decrease in profit percentage?

5.8 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Mean wages of 50 labourers in a factory is ₹ 380. The mean wages of 30 labourers working in morning shift is ₹ 400. Find the mean wages of workers working in evening shift.

Solution:

$n_1 = 30$, $\overline{X}_1 = ₹\ 400$, $n_2 = 20$, i.e., $(50 - 30)$, $\overline{X}_{12} = 380$ and $\overline{X}_2 = ?$

$$\overline{X}_{12} = \frac{n_1.\overline{X}_1 + n_2.\overline{X}_2}{n_1 + n_2}$$

$$380 = \frac{30(400) + 20.\overline{X}_2}{30 + 20}$$

$$380 = \frac{12000 + 20\overline{X}_2}{50}$$

$$(380 \times 50) = 12000 + 20\,\overline{X}_2$$

$$19000 = 12000 + 20\,\overline{X}_2$$

$$20\,\overline{X}_2 = 19000 - 12000$$

$$\overline{X}_2 = \frac{7000}{20} = 350$$

The mean wages of workers working in evening shift is ₹ 350.

2. If the average wage paid to 25 workers is 796, find the missing numbers.

Wages	*No. of workers*
500	1
600	3
700	-
800	-
900	6
1,000	2
1,100	1

NOTES

Solution:

$N = 25, \quad AM = 796$

Let the missing number of workers be X and Y.

Wages ₹ (x)	*No. of workers (f)*	*fx (x*f)*
500	1	500
600	3	1,800
700	x	$700x$
800	y	$800y$
900	6	5,400
1,000	2	2,000
1,100	1	1,100
	$N = 25$	$\sum fx = 10,800 + 700x + 880y$

Step 1: $1 + 3 + x + y + 6 + 2 + 1 = 25$

$13 + x + y = 25$

$x + y = 25 - 13$

$x + y = 12$ .... (1)

Step 2: $\overline{X} = \frac{\sum fx}{N}$, $796 = (10,800 + 700x + 800y) / 25$

Cross multiplying , we get

$796 \times 25 = 10,800 + 700x + 800y$

$19,900 - 10,800 = 700x + 800y$

$700x + 800y = 9100$ .... (2)

Step 3: Multiplying Equation (1) with 700, we get

$x + y = 12 \times 700,$

$700x + 700y = 8,400$ .... (3)

Step 4: Subtracting Equation (3) from Equation (2), we get

$700x + 800y = 9,100$ (Equation 2)

$700x + 700y = 8,400$ (Equation 3)

$100y = 700$

$y = 700/100$

$y = 7$

Step 5: Substituting the value of y in Equation (1), we get

$x + y = 12$

$x + 7 = 12$

$x = 12 - 7$

$x = 5$

The missing frequencies are $x = 5$ and $y = 7$ respectively.

NOTES

3. Calculate the arithmetic average of the data given below using short-cut method.

Roll No.	Marks obtained
1	43
2	48
3	65
4	57
5	31
6	60
7	37
8	48
9	78
10	59

Solution:

Calculation of Arithmetic Mean

Roll No.	*Marks obtained (x)*	*dx = (x – A)*
1	43	–17
2	48	–12
3	65	+5
4	57	–3
5	31	–29
6	60 (A)	0
7	37	–23
8	48	–12
9	78	+18
10	59	–1
		$\sum dx = -74$

$$\overline{X} = A + \frac{\Sigma dx}{n}$$

$$= 60 + \left(-\frac{74}{10}\right)$$

$$= 60 - 7.4$$

$$= 52.6 \text{ marks}$$

4. Following are the marks obtained by students of a class in statistics. Calculate arithmetic mean.

Marks	*No. of students*
35	3
40	8
45	12
50	9
55	4
60	7
65	15
70	5
75	10
80	7
85	5
90	3
95	2

NOTES

Solution:

Calculation of Arithmetic Mean

Marks *(x)*	*No. of students* *(f)*	*fx*
35	3	105
40	8	320
45	12	540
50	9	450
55	4	220
60	7	420
65	15	975
70	5	350
75	10	750
80	7	560
85	5	425
90	3	270
95	2	190
	$N = 90$	$\Sigma fx = 5,575$

$$\overline{X} = \frac{\Sigma fx}{N} = \frac{5,575}{90} = 61.94 \text{ marks}$$

NOTES

5. Calculate Geometric Mean for the following data:

x	f
102	6
23	9
59	8
154	12
28	16
6	9
88	15
45	16
34	18
78	23

Solution:

x	f	$\log x$	$f \log x$
102	6	2.0086	12.0516
23	9	1.361728	12.25555
59	8	1.770852	14.16682
154	12	2.187521	26.25025
28	16	1.447158	23.15453
6	9	0.778151	7.003361
88	15	1.944483	29.16724
45	16	1.653213	26.4514
34	18	1.531479	27.56662
78	23	1.892095	43.51818
	$\sum f = 132$		$\sum f \log x = 221.5855$

$$GGM = \text{Antilog}\left(\frac{\sum f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{\sum 221.5855}{132}\right)$$

$$= \text{Antilog}(1.678678)$$

$$= 47.7175$$

6. Calculate Geometric Mean for the following data:

x	f
89	16
4	19

119	18
203	21
208	16
67	19
58	15
45	6
43	8
108	32

NOTES

Solution:

X	*f*	*log x*	*f log x*
89	16	1.94939	31.19024
4	19	0.60206	11.43914
119	18	2.075547	37.35985
203	21	2.307496	48.45742
208	16	2.318063	37.08901
67	19	1.826075	34.69542
58	15	1.763428	26.45142
45	6	1.653213	9.919275
43	8	1.633468	13.06775
108	32	2.033424	65.06956
	$\Sigma f = 170$		$\Sigma f \log x = 314.74$

$$GM = \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{\Sigma\ 314.74}{170}\right)$$

$$= \text{Antilog}(1.8514)$$
$$= 71.025$$

7. In a transaction, the profit percentage is 80% of the cost. If the cost further increases by 20% but the selling price remains the same, how much is the decrease in profit percentage?

Solution:

Let's assume *CP* = ₹ 100.

Then Profit = ₹ 80 and *SP* = ₹ 180.

The cost increases by 20% $\rightarrow$ New *CP* = ₹ 120, *SP* = ₹ 180.

$$\text{Profit \%} = \left(\frac{60}{120} \times 100\right) = 50\%$$

Therefore, profit decreases by 30%.

5.9 SUMMARY

NOTES

- An average is described as a measure of central tendency as it is more or less a central value around which various values cluster. In the words of Croxton and Cowden, "An average is a single value within the range of the data that is used to represent all of the values in the series. Since an average is somewhere within the range of the data, it is called a measure of cultural value."
- Arithmetic mean and number of observations of two or more related groups are known as combined mean of the entire group. the sum of the given observations divided by the number of observations, i.e., add values of all items together and divide this sum by the number of observations.
- Harmonic Mean '*HM*' between two numbers *a* and *b* is such a number that $1/HM - 1/a = 1/b - 1/HM$. Thus, if we are given these two numbers, the harmonic mean $HM = 2ab/a + b$. If all the observations taken by a variable are constants, say *k*, then the harmonic mean of the observations is also *k*. The harmonic mean has the least value when compared to the geometric mean and the arithmetic mean.
- If the selling price is more than the cost price, then the difference between them is the profit incurred. Profit or Gain = Selling Price – Cost Price.
- If the selling price is less than the cost price, then the difference between them is the loss incurred. Loss = Cost Price – Selling Price. Profit or Loss is always calculated on the cost price.
- Per cent comes from the Latin *Per Centum*. The Latin word *Centum* means 100. For example, a Century is 100 years. My Dictionary says "Percentage" is "the result obtained by multiplying a quantity by a per cent". So, 10 per cent of 50 apples is 5 apples; 5 apples is the percentage.
- The percentage difference between two values is calculated by dividing the absolute value of the difference between two numbers by the average of those two numbers. Multiplying the result by 100 will yield the solution in per cent, rather than decimal form.
- Percentage increase and decrease are calculated by computing the difference between two values and comparing that difference to the initial value. Mathematically, this involves using the absolute value of the difference between two values, and dividing the result by the initial value, essentially calculating how much the initial value has changed.

4.10 KEY TERMS

- **Average:** An average is a single figure which sums up the characteristics of a whole group of figures. In the words of Clark, "average is an attempt to find one single figure to describe whole of figures."

NOTES

- **Simple Arithmetic Mean:** In simple arithmetic mean, it is assumed that all the items are of equal importance. For finding simple average, total values are divided by number of observations.
- **Arithmetic Mean:** Arithmetic mean is defined as the value obtained by dividing the total values of all items in the series by their number.
- **Harmonic Mean:** Harmonic Mean '*HM*' between two numbers *a* and *b* is such a number that $1/HM - 1/a = 1/b - 1/HM$. Thus, if we are given these two numbers, the harmonic mean $HM = 2ab/a + b$.
- **Geometric Mean:** A geometric mean is a mean or average which shows the central tendency of a set of numbers by using the product of their values. For a set of *n* observations, a geometric mean is the *n*th root of their product.
- **Cost Price (CP):** This is the price at which an article is purchased.
- **Selling Price (SP):** This is the price at which an article is sold.
- **Profit or Gain:** If the selling price is more than the cost price, the difference between them is the profit incurred. Profit or Gain = Selling Price – Cost Price
- **Loss:** If the selling price is less than the cost price, the difference between them is the loss incurred. Loss = Cost Price – Selling Price. Profit or Loss is always calculated on the cost price.

5.11 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Define the term mean.
2. What do you mean by arithmetic mean?
3. What are the merits of arithmetic mean?
4. State four objectives of statistical averages.
5. What are the types of statistical average?
6. Give any four requisites of a good average.
7. Write the limitations of statistical averages.
8. Define median.
9. Under what circumstances would it be appropriate to use median?
10. What do you mean by mode?

Long Answer Questions

1. What are the merits and demerits of arithmetic mean?
2. Explain the merits and demerits of mean.
3. Explain merits and demerits of Mode.

NOTES

4. Calculate of arithmetic mean:

Value (x):	122	120	138	135	142	147	150	158	150	164

5. Calculate the arithmetic average of the data given below using short-cut method.

Roll No.	:	1	2	3	4	5	6	7	8	9	10
Marks obtained	:	21	24	35	27	15	30	16	24	36	28

6. Following are the marks obtained by students of a class in statistics. Calculate the arithmetic mean.

Marks	:	35	40	45	50	55	60	65	70	75	80	85
No. of students	:	8	5	12	9	6	7	15	8	10	6	9

7. Calculate the average marks from the following distribution:

Marks	:	0-5	5-10	10-15	15-20	20-25	25-30	30-35
No. of students	:	4	8	10	6	4	8	10

8. Calculate *AM* for the following data open-end class:

Marks	:	Below 20	30	40	50	60	70	80	90	100
No. of students	:	15	23	30	37	48	66	72	90	100

9. Calculate the average marks scored by a student in five different subjects. The details of the marks are given below:

   Kannada 75, English 80, Mathematics 85, Science 75 and Social 60.

   The weights given for these subjects are:

   Mathematics 5, Science 4, Social 3, English 2 and Kannada 1

10. The mean of 10 values is 25. If one of these values is taken as 34 instead of 26, then find the corrected mean.

11. The average salary paid to all the employees of a factory was ₹ 25,000. The mean annual salary paid to male and female employees are ₹ 26,000 and ₹ 21,000 respectively. Determine the number of males and females.

12. Calculate the Geometric Mean for the following data:

    45, 65, 76, 89, 104, 129, 86, 216

13. Calculate the harmonic mean of 8, 11, 15 and 18.

14. Calculate the median for the following data:

Size of item	:	4	6	8	10	12	14	16
Frequency	:	3	5	4	6	3	2	5

15. Calculate the median marks from the following frequency distribution:

Marks	*No. of students*
0-10	4
0-20	12
0-30	18

0-40	28
0-50	58
0-60	75
0-70	80

NOTES

16. Calculate the modal wages.

Daily wages in ₹ (x) :	20-25	25-30	30-35	35-40	40-45	45-50
No. of workers (f) :	1	4	7	11	6	3

17. Calculate the mean from the following data:

X less than :	10	20	30	40	50	60	70	80
Frequency :	4	16	40	76	96	112	120	125

18. Calculate arithmetic mean from the following data:

Height (in cms)	*No. of students*
0-10	2
10-20	4
20-30	6
30-40	8
40-50	10
50-60	12
60-70	14

19. Find the missing frequency from the following distribution if its mean is 15.25:

X :	10	12	14	16	18	20
Y :	3	7	(?)	20	8	5

20. You are given the following incomplete information and its mean is 25. Find out the missing frequencies.

Class interval :	0-10	10-20	20-30	30-40	40-50	Total
No. of frequencies :	5	–	15	–	5	45

21. Find the median value of the following data:

CI :	4-7	8-11	12-15	16-19	20-23	24-27
f :	12	23	40	65	17	3

5.12 FURTHER READING

NOTES

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